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Cantor Set in Measure Theory

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Abstract

This thesis is a survey for the using of Cantor sets $C_{1/3}$ and $C_{1/2}$ in measure theory. It is proved that $C_{1/3}$ and $C_{1/2}$ are measurable and have zero measure. Following that it is shown that the measure of $C_{1/3}+C_{1/3}$ is positive and the measure of $C_{1/2}+C_{1/2}$ is zero. Also it is shown that there exists a subset A of $C_{1/3}$ such that $A+A$ is non measurable. At the end of this thesis it is shown that there is no subset $B\subset C_{1/2}\cup\frac{1}{2}C_{1/2}$ such that $B+B$ is Bernstein in $[0,1.5]$.

Introduction

This thesis will concentrate on the study of the behavior of some sets in measure theory. Measure theory is a basis of modern theories of integration. Lebesgue measure is a special case of it. Four main sets will be discussed in this thesis:

(i) Uncountable sets with zero measure.

(ii) Set with zero measure but its algebraic sum has a positive measure.

(iii) Set with zero measure, and also its algebraic sum has zero measure.

(iv) Measure zero set with non-measurable sum.

Thus algebraic sum can't characterize measure zero sets.

To achieve this study we need to look at surprising sets which are the Cantor sets. The Cantor set which was defined by Cantor is a set of length zero which contains uncountably many points. A perfect set does not have to contain an open set. Therefore, the Cantor set shows that closed subsets of the real line can be more complicated than intuition might at first suggest. It is in fact often used to construct difficult, counter-intuitive objects in analysis. For example the measure of the Cantor middle third set $C_{1/3}$ is zero and its sum has positive measure. While the measure of the Cantor middle half set $C_{1/2}$ is zero and its sum also has zero measure. Moreover there exists a set $A \subseteq C_{1/3}$ such that $A + A$ is non-measurable.

The contents of this thesis are divided into three chapters. In the first one we give some basic definitions and preliminary results that are used in subsequent chapters. In the beginning of the second chapter we will study, in detail, the construction of the Cantor middle third and half sets. Finally we will discuss the properties of the cantor middle third $C_{1/3}$ sets.

In chapter three we will discuss the measure of the cantor sets and will be introduced to the Cantor sets has positive measure. Following that we will define a set A, subset of the $C_{1/3}$, which has zero measure, but $A+A$ is non-measurable. Finally we will present an application of the $C_{1/2}$ such that $C_{1/2}+C_{1/2}=[0,1.5]$, furthermore there is no subset $B\subset C_{1/2}\cup\frac{1}{2}C_{1/2}$ such that $B+B$ is Bernstein in $[0,1.5]$.

Chapter One

Preliminaries

Chapter One

Preliminaries

In this chapter, we shall give necessary facts and definitions of cardinal numbers, dense sets, Borel sets, lebesgue measure, and connected space. The purpose of this chapter is to clarify terminology and notations that we shall use throughout this thesis.

1.1 Cardinality

In this section we shall give required definitions and facts a bout cardinality of sets.

1.1.1 Definition

Two sets A and B are equivalent if and only if there exists a one-to-one function from A onto B. A and B are also said to be in one-to-one correspondence, and we write $A \approx B$. (See [4], p. 93)

We shall use the symbol N_k to denote the set $\{1,2,3,\cdots,k\}$. Each N_k may be thought of as the standard set with k elements since we shall compare the sizes of other sets with them.

1.1.2 Definition

A set S is finite if and only if $S=\phi$ or S is equivalent to N_k for some natural number k. In the case $S=\phi$, we say ϕ has cardinal number zero and write $|\phi|=0$. If S is equivalent to N_k, then S has cardinal number k and we write $|S|=k$.

A set S is infinite if and only if it is not finite. (See [4], p. 93)

1.1.3 Definition

A set S is denumerable if and only if it is equivalent to the set of natural numbers N. A denumerable set S has cardinal number $\aleph_0$ and write $|S| = \aleph_0$. If a set is finite or denumerable, it is countable, otherwise the set is uncountable. The symbol $\aleph_0$ is the first infinite cardinal number. Other infinite cardinal numbers are associated with uncountable sets.

The interval $(0,1)$ is an example of an uncountable set. The cardinal number of $(0,1)$ is defined to be **c** (which stands for continuum).

(See [4], p.97, 99)

1.1.4 Remark

2^N where N is a set of all natural numbers is the set of all functions $f : N \to \{0,1\}$. So the cardinal number of 2^N is the cardinal number of the set of functions $f : N \to \{0,1\}$. Therefore $|2^N|$ = **c**. (See [2], p. 9)

1.1.5 Fact

It is will known that $n < \aleph_0 <$ **c** for all $n \in N$. And there are no sets A for which $\aleph_0 < |A| <$ **c**. (See [2], p. 8, and 9)

1.1.6 Definition

A set Λ is a directed set if and only if there is a relation $\leq$ on Λ satisfying:

$a)$ $\lambda \leq \lambda$ for each $\lambda \in \Lambda$,

$b)$ if $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$ then $\lambda_1 \leq \lambda_3$,

c) if $\lambda_1, \lambda_2 \in \Lambda$ then there is some $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$, $\lambda_2 \leq \lambda_3$.

The relation $\leq$ is referred to as directed on Λ. (See [2], p.73)

1.1.7 Definition

A net in a set X is a function $P : \Lambda \to X$ where Λ is some directed set. The point $P(\lambda)$ is denoted x_λ, and we denote the net as (x_λ) .(See[2],p.73)

1.2 Topological Properties

The topology in our thesis is the standard topology for the real numbers R and in this section we will give basic definitions of topological properties.

1.2.1 Definition

If X is a topological space and $E \subset X$, the closure of E in X is the set $\overline{E} = cl(E) = \cap\{K \subset X : K$ is closed and $E \subset K\}$, also the interior of E in X is the set $E^\circ = Int(E) = \cup\{G \subset X : G$ is open and $G \subset E\}$.

(See [2], p. 25, 27)

1.2.2 Definition

Let (X, τ) be topological space. A set D is dense in X if and only if $cl_X D = X$. Also a set $E \subset X$ is said to be nowhere dense, if $(cl_X E)^\circ = \phi$. That is $cl_X E$ has empty interior. (See [2, 5], p.109, 306)

1.2.3 Definition

A point $x \in R$ is a cluster point (or a point of accumulation) of a subset $S \subseteq R$ if each ε-neighborhood $v_\varepsilon = (x-\varepsilon, x+\varepsilon)$ of x contains at least one point of S distinct from x.

S' is the set which contains all cluster points of S. (See [6], p.59)

1.2.4 Note

If $S' \neq \phi$, then S is not a finite set.

1.2.5 Definition

A space X is disconnected if and only if there are disjoint non-empty open sets H and K in X such that $X = H \cup K$. We then say that X is disconnected by H and K.

When no such disconnected exists, X is connected. (See [2], p.191)

1.2.6 Definition

If $x \in X$, the largest connected subset C_x of X containing x is called a component of x. It exists being just the union of all connected subsets of X containing x. (See [2], p.194)

1.2.7 Definition

A space X is totally disconnected if and only if the component in X are the points. Equivalently X is totally disconnected if and only if the only nonempty connected subsets of X are the one point sets.

(See [2], p. 210)

1.2.8 Definition

A space X is compact if and only if each open cover of X has a finite subcover. (See [2], p.116)

1.2.9 Proposition

A subset S of real numbers is compact if and only if it is closed and bounded. (See [6], p. 186)

1.2.10 Proposition

Suppose $\{A_j\}$ is a collection of sets such that each A_j is non-empty, compact, and $A_{j+1} \subset A_j$. Then $A = \cap A_j$ is non-empty. (See [7], p.2)

1.2.11 Definition

A metric space is an ordered pair (M, ρ) consisting of a set M together with a function $\rho : M \times M \to R$ satisfying for all $x, y, z \in M$:

a) $\rho(x, y) \geq 0$,

b) $\rho(x, x) = 0$; $\rho(x, y) = 0$ implies $x = y$,

c) $\rho(x, y) = \rho(y, x)$,

d) $\rho(x, y) + \rho(y, z) \geq \rho(x, y)$

The function ρ is called a metric on M . (See [2], p. 16)

1.3 Algebra of Sets

1.3.1 Definition

A collection $\mathcal{C}$ of subsets of X is called an algebra of sets or a Boolean algebra if

(i) $A \cup B$ is in $\mathcal{C}$ whenever A and B are in $\mathcal{C}$.

(ii) $X \setminus A$ is in $\mathcal{C}$ whenever A is .(See [1], p. 17)

1.3.2 Definition

An algebra $\mathcal{C}$ of sets is called a σ-algebra if every union of a countable collection of sets in $\mathcal{C}$ is again in $\mathcal{C}$. That is if $< A_i >$ is a sequence of sets, then $\bigcup_{i=1}^{\infty} A_i$ must again belong to $\mathcal{C}$. (See [1], p.18)

1.3.3 Definition

The smallest σ-algebra, which contains all of the open sets, is called Borel algebra. And the Borel set is an element of a Borel algebra $\mathcal{B}$.

(See [1], p. 52)

1.4 Measure and Measurable Space

1.4.1 Definition

By a measurable space we mean a couple $(X, \mathcal{B})$ consisting of a set X and a σ-algebra $\mathcal{B}$ of subsets of X. A subset A of X is called measurable (or measurable with respect to $\mathcal{B}$) if $A \in \mathcal{B}$. (See [1], p. 253)

1.4.2 Definition

A set function is a function that associates an extended real number to each set in some collection of sets. (See [1], p.54)

1.4.3 Definition

A set function m that assigns to each set E in some collection M of sets of real numbers a nonnegative extended real number mE called the measure of E. (See [1], p. 54)

1.4.4 Definition

For any set A of real numbers consider the countable collection $\{I_n\}$ of open intervals that cover A, we define the Lebesgue outer measure $\mu^* A$ by $\mu^* A = \inf\{\sum L(I_n)$ such that $A \subseteq \cup I_n\}$.(See [1], p. 56)

1.4.5 Definition

A set E of real numbers is said to be lebesgue measurable if for each set A of real numbers we have $\mu^* A = \mu^*(A \cap E) + \mu^*(A \cap E^c)$.

(See [1], p. 58)

For any sets A, B, E and $E_i : i = 1,2,3,\cdots$ we have the following properties:

i) $\mu^* \varphi = 0$.

ii) If $A \subset B$ then $\mu^* A \le \mu^* B$.

iii) If $E \subset \bigcup_{i=1}^{\infty} E_i$ then $\mu^* E \le \sum_{i=1}^{\infty} \mu^* E$. (See [1], p. 288)

1.4.6 Definition

Let μ be a measure on an algebra $\mathcal{C}$ and μ^* the induced outer measure. We define the inner measure μ_* induced by μ by setting

$\mu_* A = \sup[\mu A - \mu^*(A \setminus E)]$, Where the supremum is taken over all sets $A \in \mathcal{C}$ for which $\mu^*(A \setminus E) < \infty$.(See [1], p.317)

1.4.7 Definition

A bounded set E is said to be measurable if $\mu_* E = \mu^* E$.

(See [1], p. 318)

1.4.8 Lemma

Let E be any bounded subset in the real numbers R, then $\mu_* E \leq \mu^* E$. If E belongs to an algebra $\mathcal{C}$, then $\mu_* E = \mu^* E$.

(See [1], p. 318)

1.4.9 Proposition

Let E and F be disjoint sets, then

$$\mu_* E + \mu_* F \leq \mu_*(E \cup F) \leq \mu_* E + \mu^* F \leq \mu^*(E \cup F) \leq \mu^* E + \mu^* F$$

(See [1], p. 320)

1.4.10 Note

Let E be any bounded subset in the real numbers, and μ is Lebesgue measure on R,

then $\mu_* E = \sup\{\ \mu F : F \subset E, F$ closed$\}$ (See [1], p. 323)

1.4.11 Proposition

If A is countable, then $\mu(A) = 0$. (See [1], p.58)

Chapter Tow

Cantor Sets

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Cantor Sets

2.1 Introduction to Cantor Sets

In the years 1871-1884 Georg Cantor invented the theory of infinite sets. In the process Cantor constructed a set which is called a "Cantor" set.

To construct the Cantor set, take a line and remove the middle third. There are two line segments left. Take the remaining two pieces and remove their middle thirds. Repeat this process infinite number of times. The resulting collection of points is called a "Cantor" set. Indeed repeatedly removing the middle third of every piece, we could also keep removing any other fixed percentage (other than 0 % and 100 %) from the middle. The resulting sets are all homeomorphic to the Cantor set, i.e. these sets are topologically the same.

The Cantor set is an unusual object. The deletion process produces an infinite set of points. On the other hand these points are uncountable, also it has no interior point.(See [10], [11])

2.1.1 Remark

The Cantor set C is a totally disconnected compact metric space. (See [2], p. 217)

Our study will be concentrated on Cantor middle third set $C_{1/3}$ and Cantor middle half set $C_{1/2}$.

Cantor Middle Third set $C_{1/3}$:

Beginning with the unit interval $I=[0,1]$, define closed subsets $A_1 \supset A_2 \supset \cdots$ in I as follows: we obtain A_1 by removing the interval $(1/3,2/3)$ from I, A_2 is then obtained by removing from A_1 the open intervals (1/9,2/9) and (7/9,8/9). In general, having A_{n-1}, A_n is obtained by removing the open middle thirds from each of the 2^{n-1} closed intervals that make up A_{n-1}. The cantor middle third set is the subspace $C_{1/3} = \cap A_n$ of I. (See [2], p. 121)

Cantor mMiddle Half Set $C_{1/2}$:

Start with the unit interval $F_0 = [0,1]$. Remove the (open) middle half-resulting in $F_1 = [0,1/4] \cup [3/4,1]$. Then repeat the process removing the middle half of each of the intervals that remain. At stage n we get a set F_n that is the union of 2^n intervals each being of length 4^{-n}. These are nested: $F_0 \supset F_1 \supset F_2 \supset \cdots$, so their intersection $C_{1/2} = \bigcap_{n=0}^{\infty} F_n$, is called the cantor middle half set. (See [8], p.315)

2.2 Ternary Representation of Numbers

2.2.1 Definition

For any $x \in [0,1]$, x can be represented in the scale of some integer $b>1$ as $x=(0.a_1a_2a_3\cdots)_b$, where every a_i is one of the integers $0,\cdots,b-1$. Also x can be represented by a convergent series as:

$$x=\sum_{i=1}^{\infty}\frac{a_i}{b^i} \ : a_i \in \{0,\cdots,b-1\} \text{ for every } i=1,2,\cdots.$$

The integer b is called the base of the scale. For $b=2$ it is called a binary expansion; and for $b=3$ it is called a ternary expansion (See [9], p.941)

2.2.2 Definition

Let $x, y \in [0,1]$ in base b expansion. Then x, y will be called equivalent with respect to base b expansion and we write $x \sim_b y$ if and only if there is a b expansion of x and a b expansion of y such that the two expansions disagree on only finitely many digits.

(See [3], p.788)

Rational with Respect to the Base b:

Let $x \in [0,1]$, then there exist a sequence $< x_n >$, where $x_n \in \{0,1,\cdots,b-1\}$, and $x = \sum_{i=1}^{\infty} \frac{x_i}{b^i}$. If the expansion of x ends in a

sequence of zeros, then there exists $m \in N$ such that $x_i = 0$ for all $i > m$, hence $x = \sum_{i=1}^{m} \frac{x_i}{b^i}$: $x_i \in \{0,1,\cdots,b-1\}$.

Since for all r we have $\sum_{i=r+1}^{\infty} \frac{b-1}{b^i} = \frac{1}{b^r}$, then

$$x = (\sum_{i=1}^{m-1} \frac{x_i}{b^i}) + (\sum_{i=m+1}^{\infty} \frac{b-1}{b^i}) + (\frac{x_m - 1}{b^m})$$

Therefore x has two possible base b expansions one ending in asserting of 0's and the other ending in asserting of $(b-1)$'s. In the case $b=4$ we will say that x is quaternary rational, and according to the (2.2.2 Definition) all such quaternary rationals are equivalents. (See [3], p.788)

Ternary Rational:

Let $x \in [0,1]$, then there exist a sequence $< x_n >$, where $x_n \in \{0,1,2\}$, and $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$. If the expansion of x ends in a sequence of zeros, then there exists $m \in N$ such that $x_i = 0$ for all $i > m$, hence $x = \sum_{i=1}^{m} \frac{x_i}{3^i}$: $x_i \in \{0,1,2\}$. So $x = (\sum_{i=1}^{m-1} \frac{x_i}{3^i}) + (\sum_{i=m+1}^{\infty} \frac{2}{3^i}) + (\frac{x_m - 1}{3^m})$

Therefore x has two possible ternary expansions one ending in asserting of 0's and the other ending in asserting of 2's. In this case we will say that x

is *ternary rational*, and according to the (2.2.2 Definition) all such ternary rationals are equivalents. (See [3], p. 6)

2.2.3 Remark

Let $x \in [0,1]$. If we represent x as a quaternary expansion, then x has a unique representation except when x is quaternary rational. (See [6], p. 60)

2.2.4 Proposition

The Cantor middle third set is precisely the set of points in the interval I having a ternary expansion without 1's. (See [2], p. 121)

Proof

Let's focus on the ternary representations of the decimals between 0 and 1. Since, in base three, 1/3 is equivalent to 0.1, and 2/3 is equivalent to 0.2. We see that in the first stage of the construction (when we removed the middle third of the unit interval) we actually removed all of the real numbers whose ternary decimal representation have a 1 in the first decimal place, except for 0.1 itself. (Also, 0.1 is equivalent to 0.0222... in base three, so if we choose this representation we are removing all the ternary decimals with 1 in the first decimal place.) In the same way, the second stage of the construction removes all those ternary decimals that have a 1 in the second decimal place. The third stage removes those with a 1 in the third decimal place, and so on. (By noticing that 1/9 is equivalent to 0.01 and 2/9 is equivalent to 0.02 in base three.) Thus, after every thing has been removed, the numbers that are left – that is, the numbers making up the Cantor set – are precisely those whose ternary decimal representations consist entirely of 0's and 2's. Then the

Cantor middle third set $C_{1/3}$ is precisely the set of points in the interval I having a ternary expansion without 1's i.e. $x=\sum_{i=1}^{\infty}\frac{x_i}{3^i}\ :\ x_i=0,2$ for all i.

2.2.5 Proposition

The Cantor set $C_{1/2}$ is precisely the set of points in the interval I having a quaternary expansion without 1's and 2's. (See [8], p.316)

Proof

Since in base four expansion, $1/4$ is equivalent to 0.1 and $3/4$ is equivalent 0.3. We see that in the first stage of construction (when we removed the middle half of the unit interval) we actually remove all elements $x\in[0,1]$ such that $0.1<x<0.3$, that is we remove all of the real numbers whose four decimal representation is 1 and 2 in first decimal place, except for 0.1 itself. (Also 0.1 is equivalent to 0.0333 . . . in base four, so we choose the representation in which we are removing all the four decimals with 1 and 2 in the first decimal place). In the same way, the second stage of the construction removes all those fourth decimals that have a 1 and 2 in the second decimal place. The third stage removes those with a 1 and 2 in the third decimal place, and so on. (By noticing that 1/16 is equivalent to 0.01 and 3/16 is equivalent to 0.03 in base four expansion). finally all numbers left, making up the Cantor middle half set $C_{1/2}$ are precisely those whose four decimal representations which consist entirely of 0's and 3's. Thus $C_{1/2}$ is the set of points, x, in the unit interval such that there is a base four expansion of x that uses only zeros and threes. That is $x = \sum x_i / 4^i : x_i = 0,3 \forall i = 1,2,3,\cdots$. □

The Cantor middle third set $C_{1/3}$ at least contains the endpoints of all of the intervals that make up each of the sets A_n, that is since by removing open middle thirds, then for every $n \in N$, $0 \in A_n$ and hence $0 \in C_{1/3}$. The same argument shows that $1 \in C_{1/3}$. In fact, if y is the endpoint of some closed interval of some particular set A_n, then it is also an endpoint of one of the intervals of A_{n+1} for all n.

2.2.6 Proposition

Each of the Cantor middle third set $C_{1/3}$ and Cantor middle half set $C_{1/2}$ is 1) Closed, 2) Dense in it self, 3) and of no interior.

(See [2], p. 217)

Proof

It is enough to prove it for the Cantor middle third set $C_{1/3}$ because the other proof for $C_{1/2}$ is similar.

1) Cantor middle third set is closed

From the construction of the Cantor middle third set

$C_{1/3} = \cap A_n$, since each sets A_n can be written as a finite union of 2^n closed intervals, each of which has a length of $1/3^n$, as follows:

- $A_0 = [0,1]$
- $A_1 = [0,1/3] \cup [2/3,1]$
- $A_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$
- ...

Since A_n is a finite union of closed sets, A_n is a closed set for all $n \in N$, then $C_{1/3}$ is an intersection of closed sets, therefore $C_{1/3}$ is a closed set.

2) Cantor middle third set is dense in itself

All endpoints of every subinterval will be contained in $C_{1/3}$. Take any $x \in C_{1/3} = \cap A_n$ then x is in A_n for all n, so x must be contained in one of the 2^n intervals that comprise the set A_n. Define x_n to be the left endpoint of that subinterval (if x is equal to that endpoint, then let x_n be the right endpoint of that subinterval). Since each subinterval has length $1/3^n$, we have:

$|x - x_n| < 1/3^n$. Hence, the sequence (x_n) converges to x, and since all endpoints of the subintervals are contained in the Cantor set, we have found a sequence of numbers not equal to x contained in $C_{1/3}$ that converges to x. Therefore, x is a limit point of $C_{1/3}$. But since x was arbitrary, every point of $C_{1/3}$ is a limit point of it. Thus $C_{1/3}$ is dense in itself.

3) $C_{1/3}$ has no interior point

Assume that there exists $x \in Int(C_{1/3})$, then there exists an $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset C_{1/3}$. Choose $n \in N$ such that $3^{-n} < \varepsilon$, then $(x-\varepsilon, x+\varepsilon) \not\subset A_n$. Therefore $(x-\varepsilon, x+\varepsilon) \not\subset C_{1/3}$, and this contradicts that $x \in IntC_{1/3}$. Therefore $C_{1/3}$ has no interior point.

2.2.7 Corollary

$C_{1/3}$ does not contain any open set

2.3 Properties of Cantor Middle Third Set $C_{1/3}$

2.3.1 Proposition

The Cantor middle third sets $C_{1/3}$, and $C_{1/2}$ are compact

(See [2], p. 216)

Proof

From (2.2.5 Proposition part 1) $C_{1/3}$ is closed, also it is bounded.

Since by (1.2.9 Proposition) every closed bounded subset of the real numbers is compact, then $C_{1/3}$ is compact

And similarly $C_{1/2}$ is also compact.

2.3.2 Proposition

The Cantor middle third sets $C_{1/3}$, and $C_{1/2}$ are uncountable.

(See [2], p. 217)

Proof

By using Binary expansion for all $y \in [0,1]$, and ternary expansion for all $x \in C_{1/3}$ define the function

$f : C_{1/3} \to [0,1]$

$\sum \frac{x_i}{3^i} \to \sum \frac{y_i}{2^i}$, where $y_i = x_i$ if $x_i = 0$ and $y_i = 1$ if $x_i = 2$

We have f is one-to-one and onto. Since $[0,1]$ is uncountable, then $C_{1/3}$ is also uncountable. Similarly, $C_{1/2}$ is uncountable also.

2.3.3 Lemma

Let $x \in (0,2)$ then x has **c** many representations

$x = a + b$ such that $a, b \in (0,1)$.

Proof

Let $x \in (0,2)$ be arbitrary. Let $\delta = \min \{\left|\frac{x}{2} - 1\right|, \frac{x}{2}\}$.

Then for all $0 < \varepsilon < \delta / 2$, we have $[x/2 + \varepsilon,\ x/2 - \varepsilon\ ] \subset (0,1)$ and $x/2 + \varepsilon + x/2 - \varepsilon = x$ .Since we have uncountably number of ε, So x has **c** many representations.

2.3.4 Proposition

The Cantor set $C_{1/3}$ when added to itself gives the interval $[0,2]$. (See [1], p.783)

Proof

Since $C_{1/3} \subseteq [0,1]$, then $C_{1/3} + C_{1/3} \subseteq [0,2]$ … … … … … (*)

Let $x \in [0,2]$ be arbitrary and let $c = x/2$, therefore $c \in [0,1]$, so by using ternary representation of c there exists a sequence $< c_i >$ such that $c = \sum_{i=1}^{\infty} \frac{c_i}{3^i} \quad : c_i = 0,1,2$.

Since $x \in [0,2]$, then $x = 0$, 2 or $0 < x < 2$

If $x = 0$, or 2 then we are done because 0 and 1 $\in C_{1/3}$ and

0+ 0= 0 also 1+1=2.

Now if $x \in (0,2)$, then by (2.3.3 Lemma) we have **c** many representation of x.

Let $x = a + b = c + c$ such that $a, b \in (0,1)$,

Using the ternary representation for $a, b,$ and c we have $a = \sum a_i / 3^i, b = \sum b_i / 3^i \ : a_i, b_i = 0,1,2 \ \forall \ i = 1,2,3,\cdots$

Then $x = \sum a_i/3^i + \sum b_i/3^i = \sum c_i/3^i + \sum c_i/3^i$

$$= \sum (c_i + c_i)/3^i$$

If $c_i = 0$ then $a_i = b_i = 0$

If $c_i = 1$ then $a_i = 0$ and $b_i = 2$ or

$a_i = 2$ and $b_i = 0$

If $c_i = 2$ then $a_i = b_i = 2$.

Then we get

$x = \sum a_i/3^i + \sum b_i/3^i$, Such that a_i and $b_i \neq 1$ for all i, So $\sum a_i/3^i$ and $\sum b_i/3^i$ are in the Cantor middle third set.

That is $\forall x \in [0,2]$ we have $x = a + b$ such that a and b in the Cantor middle third set.

Therefore $[0,2] \subseteq C_{1/3} + C_{1/3}$ … … … … … … … … … (**)

From (∗) and (∗∗) we get $C_{1/3}+C_{1/3} = [0,2]$.

2.3.5 Definition

A subset P of R is perfect set if it is closed and dense in itself:

i.e., each point of P is an accumulation point of P; that is: P is perfect in R if and only if $P = P'$. (See [2], p. 216)

2.3.6 Remark

Every closed interval $[a,b]$, $a \neq b$ is a perfect set.

2.3.7 Proposition

Every non-empty perfect set is uncountable. (See [7], p. 165)

Proof

If P is perfect, it consists of accumulation points, and therefore can not be finite. Therefore it is either countable or uncountable. Suppose P was countable and could be written as $P = \{x_1, x_2, x_3, \cdots\}$

The interval $U_1 = (x_1 - 1, x_1 + 1)$ is a neighborhood of x_1. Since x_1 must be an accumulation point of P, Therefore there are infinitely many elements of P contained in U_1.

Take one of those elements, say x_2 and take a neighborhood U_2 of x_2 such that $U_2 = (x_2 - \delta, x_2 + \delta)$ where $\delta = 1/2 \min\{|x_2 - x_1|, |x_2 - (x_1 + 1)|, |x_2 - (x_1 - 1)|\}$. Then the closure (U_2) is contained in U_1 and $x_1 \notin clU_2$. Again, x_2 is an accumulation point of P, so that the neighborhood U_2 contains infinitely many elements of P.

Select an element, say x_3, and by the same argument take a neighborhood U_3 of x_3 such that closure (U_3) is contained in U_2 and $x_2 \notin clU_3$ so x_1 and x_2 are not contained in closure (U_3).

Continue in that fashion to find sets U_n and points x_n such that:

- Closure $(U_{n+1}) \subset U_n$
- x_j is not contained in U_n for all $0 < j < n$
- x_n is contained in U_n

Now consider the set $V = \cap($ closure $(U_n) \cap P)$

Since each set (closure $(U_n) \cap P)$ is closed and bounded, it is compact.

Also, by construction, (closure $(U_{n+1}) \cap P) \subset ($ closure $(U_n) \cap P)$.

Therefore, by (1.2.10 proposition), V is not empty. Also $x_n \notin V$ because $x_n \notin U_{n+1}$ for all integer numbers n. But V is non-empty, therefore P is not countable.

2.3.8 Note

The Cantor middle third set $C_{1/3}$ as well as $C_{1/2}$ is closed and dense in itself, then it is a perfect set, hence is uncountable.

Chapter Three

Measure and Cantor Sets

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Measure and Cantor Sets

Different measurable sets can be seen in mathematics, measurable sets, non-measurable sets, and measure zero sets. We sow that $C_{1/3}$ is uncountable with zero measure but $(C_{1/3} + C_{1/3})$ has positive measure. While $C_{1/2}$ and $(C_{1/2} + C_{1/2})$ have zero measures. Another Cantor set with positive measure will be studied. Also we will define a set A, subset of $C_{1/3}$ with measure zero, but $A + A$ is non-measurable. Finally we will see applications of $C_{1/2}$.

3.1 Measure of Cantor Sets

3.1.1 Proposition

The Cantor set $C_{1/3}$ is measurable and has measure zero.

(See [1], p. 64)

Proof

Since $C_{1/3} = \cap A_n$, then $C_{1/3}$ is a countable intersection of closed sets, therefore $C_{1/3}$ is a Borel set, so $C_{1/3}$ is measurable.

From the construction of $C_{1/3}$ we have for every stage $n > 0$ we remove 2^{n-1} disjoint intervals from each previous set each having length $1/3^n$. Therefore we will removed a total length of

$$\sum_{n=1}^{\infty} 2^{n-1} \frac{1}{3^n} = \frac{1}{3} \sum_{n=1}^{\infty} (2/3)^{n-1}$$

$$= 1/3 \sum_{n=0}^{\infty} (2/3)^n$$

$$= \frac{1}{3}\left(\frac{1}{1-2/3}\right) = 1$$

from the unit interval [0,1].

Then $C_{1/3}$ is obtained by removing a total length 1 from the unit interval [0,1], so $\mu(I \setminus C_{1/3}) = 1$. Since $\mu(I) = \mu(C_{1/3}) + \mu(I \setminus C_{1/3})$, then $\mu(C_{1/3}) = \mu(I) - \mu(I \setminus C_{1/3}) = 1 - 1 = 0$

Therefore the set $C_{1/3}$ is measurable and has zero measure.

3.1.2 Remark

The measure of the $C_{1/3}$ is zero, but $\mu(C_{1/3} + C_{1/3}) \neq 0$, since $C_{1/3} + C_{1/3} = [0,2]$.

3.1.3 Proposition

The Cantor $C_{1/2}$ is measurable and has zero measure.

Proof

Since $C_{1/2} = \cap F_n$, that is $C_{1/2}$ is countable intersection of closed sets, therefore it is Borel set. Hence $C_{1/2}$ is measurable

From the construction of $C_{1/2}$ we have for stage $n > 0$ we remove 2^{n-1} disjoint interval from each previous set each having length $(2/4)^n$. Therefore we will remove a total length of

$$\sum_{n=1}^{\infty} 2^{n-1} \frac{2}{4^n} = \frac{2}{4}\sum_{n=1}^{\infty} (\frac{2}{4})^{n-1}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (\frac{1}{2})^{n}$$

$$= \frac{1}{2}\left(\frac{1}{1-1/2}\right) = 1$$

from the unit interval [0,1].Then $C_{1/2}$ is obtained by removing a total length 1 from the unit interval [0,1]. and $\mu(I \setminus C_{1/2}) = 1$, Then $\mu(C_{1/2}) = \mu(I) - \mu(I \setminus C_{1/2}) = 1 - 1 = 0$.

Thus $C_{1/2}$ is measurable, and has zero measure.

3.1.4 Proposition

If A is the set of points in the unit interval [0,1] having a four expansion without 3's, then A has zero measure.

Proof

Take the representation: $[0,1] = [0,0.1] \cup [0.1,0.2] \cup [0.2,0.3] \cup [0.3,1]$

At first we need to show that, $\mu([0,0.1] \setminus A) = 1/4$

To do so for all $x \in A$, $x = \sum \frac{x_i}{4^i}$ $x_i = 0,1,2$

- Let $A_1 = [0.03,0.1]$, since for all $a \in A_1$, $a = \sum \frac{a_i}{4^i} : a_2 = 3$, then $a \notin A$, so $A \cap A_1 = \varphi$ and $\mu(A_1) = 4^{-2}$

- Let $A_2 = [0.003,0.01] \cup [0.013,0.02] \cup [0.023,0.03]$, since for all $a \in A_2$, $a = \sum \frac{a_i}{4^i} : a_3 = 3$, then $a \notin A$, so $A \cap A_2 = \varphi$ and $\mu(A_2) = 3 * 4^{-3}$

- Let $A_3 = [0.0003,0.001] \cup [0.0013,0.002] \cup [0.0023,0.003] \cup$

$[0.0103,0.011] \cup [0.0113,0.012] \cup [0.0123,0.013] \cup$

$[0.0203,0.021] \cup [0.0213,0.022] \cup [0.0223,0.023]$, since for all $a \in A_3$, $a = \sum \frac{a_i}{4^i} : a_4 = 3$, then $a \notin A$, so $A \cap A_3 = \varphi$, and $\mu(A_3) = 3^2 * 4^{-4}$.

In general we have A_n is the union of 3^{n-1} disjoint closed intervals each have length $4^{-(n+1)}$ and for all $a \in A_n$, $a = \sum \frac{a_i}{4^i} : a_{n+1} = 3$, then $A \cap A_n = \varphi$ and $\mu(A_n) = 3^{n-1} * 4^{-(n+1)}$

Since $\mu(\cup A_n) = \sum \mu(A_n)$,

Then $\mu(A_n)$ $= 1/4^2 + 3/4^3 + 3^2/4^4 + 3^3/4^5 + \cdots$

$$= 1/4^2 + (3/4^3(1 + 3/4 + 3^2/4^2 + 3^3/4^3 + \cdots))$$

$$= 1/4^2 + \left[3/4^3\left(\frac{1}{1-3/4}\right)\right] = 1/4$$

Thus $\mu([0,0.1] \setminus A) = 1/4$, By the same way we see that

$\mu([0.1,0.2] \setminus A) = 1/4$, $\mu([0.2,0.3] \setminus A) = 1/4$ and $\mu([0.3,1] \setminus A) = 1/4$

Therefore $\mu(A) = \mu(I) - \mu(I \setminus A)$

$= 1 - \left[\mu([0,0.1] \setminus A) + \mu([0.1,0.2] \setminus A) + \mu([0.2,0.3] \setminus A) + \mu([0.3,1] \setminus A)\right]$

$= 1 - \left[1/4 + 1/4 + 1/4 + 1/4\right] = 0$.

3.1.5 Proposition

$C_{1/2} + C_{1/2}$ has measure zero. (See [3] p.790)

Proof

If we prove that $1/3(C_{1/2} + C_{1/2})$ has measure zero, then obviously $C_{1/2} + C_{1/2}$ has measure zero. Let $x \in (C_{1/2} + C_{1/2})$, then there exist

$a, b \in C_{1/2}$ such that $x = a + b$. Let $c \in 1/3(C_{1/2} + C_{1/2})$, then $c = \frac{1}{3}x = \frac{1}{3}a + \frac{1}{3}b$. Using base four expansion of a, b and c with all digits of a and b are divisible by 3,then

$$\sum \frac{c_i}{4^i} = \frac{1}{3}(\sum \frac{a_i}{4^i} + \sum \frac{b_i}{4^i}) \; : a_i, b_i = 0,3$$

$$= \frac{1}{3}\sum (a_i + b_i)/4^i : a_i, b_i = 0,3$$, Hence

- $c_i = 1/3\ (0 + 0) = 0$ when $a_i = b_i = 0$
- $c_i = 1/3\ (0 + 3) = 1$ when $a_i = 0$ and $b_i = 3$
- $c_i = 1/3\ (3 + 0) = 1$ when $a_i = 3$ and $b_i = 0$ or
- $c_i = 1/3\ (3 + 3) = 2$ when $a_i = b_i = 3$.

Therefore $c = \frac{1}{3}a + \frac{1}{3}b = \sum \frac{c_i}{4^i} \quad : c_i = 0,1,2$. Hence $c_i \neq 3$ for all $i = 1,2,3,\cdots$. Then unless c is a quaternary rational its expansion will never use the digits three, and (by 2.2.3 remark)the expansion of c is unique. Let S be all quaternary rationals in 1/3 ($C_{1/2} + C_{1/2}$).

Thus for all $x \in 1/3(C_{1/2} + C_{1/2}) \setminus S$, $x = \sum \frac{x_i}{4^i} : x_i = 0,1,2$. Since $x_i \neq 3$ for all $i = 1,2,3,\cdots$Then by (3.1.4 proposition) $1/3(C_{1/2} + C_{1/2}) \setminus S$ has measure zero, and since all elements in S are quaternary rational, then $S \subset Q$, hence S is countable, then $1/3(C_{1/2} + C_{1/2})$ has measured zero. Therefore $C_{1/2} + C_{1/2}$ has zero measure.

3.1.6 Remark

The measure of the Cantor middle third set $C_{1/3}$ is zero and its sum has positive measure, while the measure of the Cantor middle half set and its sum have zero measure.

3.1.7 Remark

There exists a Cantor set with positive measure. This can be constructed as follows:

Start with the unit interval for $I = [0,1]$ and choose a number $0 < p < 1$. Let

$$R_1 = (\frac{2-p}{4}, \frac{2+p}{4}) = (\frac{1}{2} - \frac{p}{4}, \frac{1}{2} + \frac{p}{4})$$

Which has measure $p/2$. Again, define $C_1 = I \setminus R_1$. Now define

$$R_2 = (\frac{2-p}{16}, \frac{2+p}{16}) \cup (\frac{14-p}{16}, \frac{14+p}{16})$$

Which has measure $p/4$; continue as before, such that each R_k has measure $\frac{p}{2^k}$; note again that all the R_k are disjoint. The resulting Cantor set has measure

$$\mu(I \setminus \bigcup_{n=1}^{\infty} R_n) = 1 - \sum_{n=1}^{\infty} \mu(R_n) = 1 - \sum_{n=1}^{\infty} p 2^{-n} = 1 - p > 0$$

Thus we have a continuum many of Cantor sets of positive measures.

(See [12])

3.1.8 Note

Clearly Cantor sets can be constructed for all sorts of "removals", we can remove middle halves, or thirds, or any amount $1/r$, $r > 1$. All of these Cantor sets have measure zero. The key point is that:

If at each stage we remove $1/r$ of each of the remaining intervals, the results is a set of measure zero. The favored examples are the Cantor sets which constructed by Georg Cantor " $C_{1/2}$, and $C_{1/3}$" , are constructed for the case $r = 2$ and $r = 3$ respectively.

However it is possible to construct Cantor sets with positive measure as well; the key is to remove less and less as we proceed, for example; remove $1/3$ then $1/3^2$ of each remaining parts then $1/3^3$ of each of remaining parts … and so on. The result is a set of positive measure

These Cantor sets have the same topology as the Cantor set and the same cardinality but different measure.

3.2 Measure Zero Set with Non-Measurable Sum

From previous studies we notice that there exists a measure zero set so that its sum also has zero measure, and a set with measure zero but its sum has positive measure such as $C_{1/2}$ and $C_{1/3}$ respectively. In this section we will discuss a special subset A of $C_{1/3}$ that has measure zero with non-measurable sum.

3.2.1 Definition

We say that $B \subseteq X$ is a Bernstein set (in X) provided B and $X \setminus B$ intersect every non-empty perfect subset of X . (See [3], p.2)

3.2.2 Proposition

Any closed set B in $\mathbb{R}$ can be written as $B = P \cup D$ where P is perfect and D is countable. (See [13], p.3596)

Proof

If B is countable then take the perfect set $P = \varphi$, and $D = B$ then we are done. Otherwise

Let $P = \{x \in B : \forall \varepsilon > 0, (x - \varepsilon, x + \varepsilon) \cap B$ is uncountable$\}$ and $D = B \setminus P$. First need to show that P is closed set,

Let (x_n) be any convergent sequence to x such that $x_n \in P$ for all $n \in N$, since $P \subset B$ and B is closed, then $x \in B$. Let $\varepsilon > 0$ be arbitrary, then there exists $j > 0$ such that $x_j \in (x - \varepsilon, x + \varepsilon) \cap B$, since $x_j \in P$, and $(x - \varepsilon, x + \varepsilon)$ is neighborhood of x_j, therefore $(x - \varepsilon, x + \varepsilon) \cap B$ is uncountable., So $x \in P$, then P is closed.

Now need to prove P is dense in itself,

Let $x \in D$, then we can find two rational numbers a and b such that $a < x < b$ and $(a, b) \cap B$ is countable. Since there are only countably many open intervals $\{O_n\}_{n=1}$ with rational end points, then there exist countable numbers of open intervals O_j such that $O_j \cap B$ is countable, so $D \subset \bigcup_n (O_n \cap B)$. But countable union of countable set is countable, therefore D is countable set. Let $x \in P$ be arbitrary, then for all $\varepsilon > 0$ $(x - \varepsilon, x + \varepsilon) \cap B$ is uncountable, and since D is countable therefore $(x - \varepsilon, x + \varepsilon) \cap B \setminus D$ is also uncountable. Since $P = B \setminus D$, then $(x - \varepsilon, x + \varepsilon) \cap P$ is uncountable and so x is a cluster point, consequently P is dense in itself.

3.2.3 Proposition

Let P be a nonempty perfect set, then for any $y \in P$, $(-\infty, y] \cap P$ or $[y, \infty) \cap P$ is a perfect set.

Proof

Let P be any nonempty perfect set. Let $y \in P$ be arbitrary and define $A = P \cap (-\infty, y]$, $B = P \cap [y, \infty)$, for all $t \in A$ and $t < y$, $t \in A'$ because if $t < y$, then for any $\varepsilon > 0$ $(t - \varepsilon, t + \varepsilon) \cap (-\infty, y) = (a, b)$ where $b = \min\{y, t + \varepsilon\}$ and $a = t - \varepsilon$, then $t \in (a, b)$,

Hence $(a, b) \cap P = (t - \varepsilon, t + \varepsilon) \cap (-\infty, y) \cap P \subset (t - \varepsilon, t + \varepsilon) \cap A$

Since $t \in P'$ then $(a, b) \cap P \setminus \{t\} \neq \phi$, so $t \in A'$

Similarly for all $t \in B$ and $t > y$, $t \in B'$.

If $y \notin A'$ and $y \notin B'$ then there exists $\varepsilon_1, \varepsilon_2 > 0$ such that

$(y - \varepsilon_1, y + \varepsilon_1) \cap A \setminus \{y\} = \phi$ and $(y - \varepsilon_2, y + \varepsilon_2) \cap B \setminus \{y\} = \phi$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ then, $(y - \varepsilon, y + \varepsilon) \cap P \setminus \{y\} = \phi$, but $P = A \cup B$, then $(y - \varepsilon, y + \varepsilon) \cap (A \cup B) \setminus \{y\} = \phi$.

Therefore $(y - \varepsilon, y + \varepsilon) \cap p \setminus \{y\} = \phi$ and this is a contradiction because $y \in P = P'$. Then $y \in A'$ or B'.

If $y \in A'$ then $A = A'$ hence A is perfect and if $y \in B'$ then $B = B'$ hence B is perfect, thus A or B must be perfect.

3.2.4 Corollary

If P is a non empty perfect set, then $\{P' \subset P : P'$ non empty and perfect$\}$ is uncountable. And for every $y \in P$ we can find a perfect subset p_y of P such that for all $x \neq y$ $P_x \neq P_y$.

3.2.5 Lemma

Every Bernstein subset of an interval I of real numbers has inner measure zero and the same outer measure as I.

Proof:

Let I be a non empty open or open half interval in R, and B be any Bernstein subset in I. Since for any non empty open or open half interval I there exists a closed interval $[a,b]$ subset in I where $a \neq b$, and since I is perfect set. Then by (3.2.3 proposition, and 3.2.4 Corollary) there exist uncountably many perfect subsets of I.

Let $\{p_\xi : \xi < \mathbf{c}\}$ be the family of all non-empty perfect subset of I.

If the inner measure of B is not zero then by (1.4.10 note) B must contain a closed set of positive measure, since the measure of any countable set is zero and the closed set with positive measure is uncountable set, so by (3.2.2 proposition) this closed set must have a non empty perfect subsets. Then there exists $\xi < \mathbf{c}$ such that $p_\xi \subset B$, so $p_\xi \cap (I \setminus B) = \phi$ and this contradicts that B is Bernstein subset in I.

Then B has inner measure zero.

Now if B dose not have the same outer measure as I, then by (1.4.9 proposition) $\mu_*(I \setminus B) + \mu^* B \geq \mu_*((I \setminus B) \cup B) = \mu(I)$. Therefore

$\mu_*(I \setminus B) \geq \mu(I) - \mu^*(B) > 0$, that is ($I \setminus B$) has positive inner measure, then there exists a closed subset in ($I \setminus B$) with positive measure. Therefore this closed set contains a non empty perfect subset. Then there exist ξ < **c** such that $p_\xi \subset$ ($I \setminus B$) and $p_\xi \neq \phi$, therefore $p_\xi \cap B = \phi$ and again, this is contradicts that B is Bernstein subset in I, thus B must have the same outer measure as I.

3.2.6 Theorem

Every Bernstein subset of an interval in R is not measurable.

Proof

Let B be any Bernstein subset in an interval I. Then by (3.2.4 lemma) $\mu_*(B) = 0$, and $\mu^*(B) = \mu^*(I)$. Since $\mu^*(I) > 0$, then $\mu_*(B) \neq \mu^*(B)$. Hence B is non-measurable.

3.2.7 Lemma

Let $x \in [0,2]$. If $x/2$ has infinitely many ones in its ternary expansion, then there are **c** many representations of x as the sum of two Cantor-set elements; otherwise, x has only finitely many such representations and all of the elements of $C_{1/3}$ used to represent x are equivalent to $x/2$. (See [3], p.788)

Proof

Since every element of $C_{1/3}$ has a ternary expansion consisting of only even digits, then for any $x \in [0,2] = C_{1/3} + C_{1/3}$ there exists $a, b \in C_{1/3}$ such that $x = a + b$. Let $c = x/2$ then c is the average of a and b. If c_i, a_i, b_i are the i^{th} digits of c , a , b respectively, since $x = c + c = a + b$, then

$$\sum_{i=1}^{\infty} \frac{c_i + c_i}{3^i} = \sum_{i=1}^{\infty} \frac{a_i + b_i}{3^i} .$$

Therefore $\sum_{i=1}^{\infty} \frac{2c_i - (a_i + b_i)}{3^i} = 0$

So $3 | (2c_i - (a_i + b_i) \forall i$

Then for all i we have

- $c_i = 0$ and $a_i = b_i = 0$,
- $c_i = 1$ and $a_i = 0, b_i = 2$,
- $c_i = 1$ and $a_i = 2$, $b_i = 0$, or
- $c_i = 2$ and $a_i = b_i = 2$

If c is ternary rational, then the digits of c must end in either a sequence of zeros or a sequence of two's that is from the property of ternary rational. In either case the digits of a and b must do likewise since when $c_i = 0$ we have $a_i = b_i = 0$ and when $c_i = 2$ we have $a_i = b_i = 2$. And so they are also ternary rationals therefore $a \sim b \sim c$, since all ternary rational are equivalent.

Now consider the case c is not a ternary rational, so there is a unique ternary expansion of c.

Let us construct the numbers a and b using only even digits for each c_i that is zero or two, we must have $c_i = a_i = b_i$. But for each $c_i = 1$, we have a choice either $a_i = 0, b_i = 2$ or $a_i = 2, b_i = 0$, two choice for a_i & b_i. Thus if $k \in \{0,1,2,\cdots,n\}$ is the number of digits in c that have the value 1, then there are 2^k possible choices for the pair a,b. In particular if c has infinitely many ones in its expansion then there are $\left|2^N\right| = \mathbf{c}$ many representation for x. If there are only finitely many ones then the digit of a, b, c will all agree on a tail end. That is a, b, c disagreeing on only finitely many digits.

3.2.8 Proposition

There is a set $A \subseteq C_{1/3}$ such that $A + A$ is Bernstein in $[0,2] = C_{1/3} + C_{1/3}$, hence $A + A$ is non-measurable. (See [3], p.789)

Proof

Let R_0 be the set of elements of $[0,2]$ that can be expressed in **c** many ways as the sum of elements of $C_{1/3}$. And R_1 be the elements that can be expressed in only finitely many ways. We can construct R_0 and R_1 by (2.3.7 lemma).

Let $\{ p_\xi : \xi < \mathbf{c} \}$ be the family of all non-empty perfect subsets of $[0,2]$. We will find an $A \subseteq C_{1/3}$ such that each p_ξ intersects both $A + A$ and its complement. Construct a net

$$(< a_\xi, b_\xi, c_\xi, d_\xi > \in C \times C \times p_\xi \times p_\xi : \xi < \mathbf{c})$$

Such that for each $\xi < \mathbf{c}$,

(*) $c_\xi = a_\xi + b_\xi$ and $D_\xi \cap (A_{\xi+1} + A_{\xi+1}) = \varphi$,

Where $A_\xi = \bigcup_{\eta < \xi} \{a_\eta, b_\eta\}$ and $D_\xi = \{d_\eta : \eta \le \xi\}$.

This will ensure that $A = A_c = \bigcup_{\eta<c}\{a_\eta, b_\eta\}$ has the desired properties, since then

$\{c_\xi : \xi < \mathbf{c}\} = \{a_\eta + b_\varsigma : a_\eta, b_\varsigma \in A$ for all $\eta, \varsigma < \mathbf{c}\} \subset A + A$

And $A + A \subseteq [0,2] \setminus D_\xi$, then

$\{c_\xi : \xi < \mathbf{c}\} \subseteq A + A \subseteq R \setminus D_\xi$, therefore

$\{c_\xi : \xi < \mathbf{c}\} \subset p_\xi \cap A + A$ and $D_\xi \subset p_\xi \cap ([0,2] \setminus A + A)$

That is each p_ξ intersects both $A + A$ and its complement. So $A + A$ is a Bernstein set, hence non measurable.

To make an inductive step, assume that for some $\alpha < \mathbf{c}$ we have already constructed a partial net which satisfies $(*)$ for all $\xi < \propto$.

Now we need to show that we can construct a partial net which satisfies $(*)$ for $\xi = \alpha$. We first choose $a_\alpha, b_\alpha, c_\alpha$ such that $a_\alpha + b_\alpha = c_\alpha$ and neither a_α nor b_α is in $D_\alpha - A_\alpha$.

We decide two cases.

Case 1: p_ξ intersects R_0 in a set of cardinality **c**. Since $|(P_\alpha \cap R_0) \setminus D| = \mathbf{c}$, then choose $c_\alpha \in (P_\alpha \cap R_0) \setminus D$. And since $c_\alpha \in R_0$ where R_0 is the set of elements of $C_{1/3} + C_{1/3}$ that can be expressed in **c** many ways as the sum of elements of $C_{1/3}$, and $|D_\alpha \setminus A_\alpha| < \mathbf{c}$, then there exist a_α, b_α in $C_{1/3}$ such that $a_\alpha + b_\alpha = c_\alpha$ and neither a_α nor b_α is in $D_\alpha - A_\alpha$.

Case 2: p_ξ intersect R_1 in a set of cardinality **c**. First choose

$c_\alpha \in p_\alpha \cap R_1 \setminus D_\alpha$ such that $c_\alpha / 2$ is not equivalent to any element of $D_\alpha - A_\alpha$. Then choose $a_\alpha, b_\alpha \in C_{1/3}$ such that $a_\alpha + b_\alpha = c_\alpha$, since by (2.3.7 lemma) both a_α and b_α are equivalent to $c_\alpha / 2$, neither of them in $D_\alpha - A_\alpha$, and since $c_\alpha / 2 \in R_1$ then it is not equivalent to any element of $D_\alpha - A_\alpha$. Our construction is finished by choosing $d_\alpha \in p_\alpha \setminus (A_{\xi+1} + A_{\xi+1})$

That means for each $\xi \le \alpha$ we have $D_\alpha \cap (A_{\xi+1} + A_{\xi+1}) = \varphi$

Then there exists a set $A \subset C_{1/3}$ such that each p_α intersects both $A + A$ and its complement then $A + A$ is Bernstein in $[0,2] = C_{1/3} + C_{1/3}$.

3.3 Applications of the Cantor middle half set $C_{1/2}$

3.3.1 Lemma

Let U be the set of elements of $[0,1]$ that use only zeros and twos in one of its base four expansions, and let V be the set of elements that use only zeros and ones. Then $U + V = [0,1]$.

Proof

Let $c \in [0,1]$, then $c = 0,1$ or $0 < c < 1$. If $0 < c < 1$ then there exists **c** many representations $c = u + v$ such that $u, v \in (0,1)$.

If c_i, u_i, v_i are the i^{th} digits of $c, u,$ and v respectively then

$$\sum_{i=1}^{\infty} \frac{c_i}{4^i} = \sum_{i=1}^{\infty} \frac{u_i}{4^i} + \sum_{i=1}^{\infty} \frac{v_i}{4^i}$$

Therefore $4 \mid c_i - (u_i + v_i)$

Then we can choose u_i and v_i as follows

When $c_i = 0$, take $u_i = v_i = 0$

When $c_i = 1$, take $u_i = 0, v_i = 1$

When $c_i = 2$, take $u_i = 2, v_i = 0$

When $c_i = 3$, take $u_i = 2, v_i = 1$

Then we get $u \in U$ and $v \in V$ that is $[0,1] \subset U + V$ and we have $U + V \subset [0,1]$ directly, Then $U + V = [0,1]$.

3.3.2 Lemma

$C_{1/2} + \frac{1}{2} C_{1/2} = [0,1.5]$. Furthermore, each element in $[0,1.5]$ can be expressed as such a sum in at most two ways, and except for a countable set, each element in $[0,1.5]$ can be expressed in a unique way.

(See [3], p.791)

Proof

We have $C_{1/2} + \frac{1}{2} C_{1/2} \subset [0,1.5]$ directly ... ... (i)

Let $x \in [0,1.5]$ then $\frac{2}{3} x \in [0,1]$, therefore by (3.3.1 lemma) there exists $u \in U$ and $v \in V$ such that $\frac{2}{3} x = u + v$, and if u_i, v_i are the i^{th} digits of u and v respectively, then

$$\frac{2}{3} x = \sum \frac{u_i}{4^i} + \sum \frac{v_i}{4^i} \text{ such that } u_i \in \{0,2\} \text{ and } v_i \in \{0,1\} \text{ for all } i = 1,2,\cdots$$

Therefore $x = \sum \frac{\frac{3}{2} u_i}{4^i} + \frac{1}{2} \sum \frac{3 v_i}{4^i}$ $: u_i \in \{0,2\}$ and $v_i \in \{0,1\}$ for all $i = 1,2,\cdots$

$$= \sum \frac{a_i}{4^i} + \frac{1}{2} \sum \frac{b_i}{4^i} \ : \ a_i, b_i \in \{0,3\} \text{ for all } i = 1,2,\cdots$$

$= a + b$ such that $a \in C_{1/2}$, $b \in \frac{1}{2} C_{1/2}$

Then $x \in (C_{1/2} + \frac{1}{2} C_{1/2})$, and

$[0,1.5] \subset (C_{1/2} + \frac{1}{2} C_{1/2})$ … … … … … … (ii)

Therefore By (i) and (ii) $C_{1/2} + \frac{1}{2} C_{1/2} = [0,1.5]$. This proves the first part of the theorem.

Now fix an x in [0,1.5], then there exist $a \in C_{1/2}$ and $b \in \frac{1}{2} C_{1/2}$ such that $x = a + b$. Let $c = \frac{2}{3} x = \frac{2}{3} a + \frac{2}{3} b$

Since $\frac{2}{3} a = \frac{2}{3} \sum \frac{a_i}{4^i}$: $a_i \in \{0,3\}$, then $\frac{2}{3} a = \sum \frac{a_i}{4^i}$: $a_i \in \{0,2\}$ for all $i = 1,2,\cdots$. Therefore by using the fact that all of digits of a are divisible by three, the computation of $\frac{2}{3} a$ can be carried out digit-wise and results in an element of U

Similarly since $b \in \frac{1}{2} C_{1/2}$, then $\frac{2}{3} b = \frac{2}{3} * \frac{1}{2} \sum \frac{b_i}{4^i}$: $b_i \in \{0,3\}$

Therefore $\frac{2}{3} b = \sum \frac{b_i}{4^i}$: $b_i \in \{0,1\}$ for all $i = 1,2,\cdots$. Hence $\frac{2}{3} b \in V$

Since $c \in [0,1]$, then each such c has at most two such representations and except when c is a quaternary rational "and the quaternary rational set is countable", therefore each such c has a unique representation.

3.3.3 Lemma

Let $A = C_{1/2} \cup \frac{1}{2}C_{1/2}$, then there are two non-empty perfect subsets P and Q of A such that every element of $P+Q$ can be expressed uniquely as the sum of two elements in A. (See [3], p.791)

Proof

$A+A$ is the union of three closed sets : $C_{1/2}+C_{1/2}, \frac{1}{2}C_{1/2}+\frac{1}{2}C_{1/2}$, and $C_{1/2}+\frac{1}{2}C_{1/2}$ by (3.1.5 proposition), The first and second sets both have measure zero. Since $C_{1/2} \subset [0,1/4] \cup [3/4,1]$, then $C_{1/2}+C_{1/2} \subset ([0,1/4]\cup[3/4,1]) + ([0,1/4]\cup[3/4,1])$

$$= [0,2/4] \cup [3/4,5/4] \cup [6/4,2]$$

Therefore $(5/4,6/4) \cap (C_{1/2}+C_{1/2}) = \phi$. And since $(1/2C_{1/2}+1/2C_{1/2}) \subset [0,1]$, then $(5/4,6/4) \cap (1/2C_{1/2}+1/2C_{1/2}) = \phi$

Hence there exists an open interval $I \subseteq [0,1.5]$ that is disjoint from $C_{1/2}+C_{1/2}$ and $\frac{1}{2}C_{1/2}+\frac{1}{2}C_{1/2}$. By (3.3.2 lemma), the third set is the interval $[0,1.5]$.

Furthermore, by (3.3.2 Lemma) we can partition $[0,1.5]$ into two sets X and Y such that X is countable and every element in Y has a unique representation as a sum of two elements, one in $C_{1/2}$ and the other in $\frac{1}{2}C_{1/2}$. Choose an x in $Y \cap I$ and let $x = a + b$ with $a \in C_{1/2}$ and $b \in \frac{1}{2}C_{1/2}$.

Now choose a neighborhood j of a and a neighborhood k of b small enough that the closure $cl(j+k)$ of $j+k$ is a subset of I.

Let R be the intersection of A with $cl(j)$ and S be the intersection of A with $cl(k)$, since A , $cl(j)$, and $cl(k)$ are perfect sets then $A \cap cl(j)$ and $A \cap cl(k)$ are perfect sets. Therefore both R and S are non-empty perfect subsets of A.

Let D be the countable set consisting of all numbers used in the representations of elements of X, and let P and Q be non-empty perfect subsets of $R \setminus D$ and $S \setminus D$ respectively. Fix $x \in P + Q$; we must show that x has a unique representation as a sum of elements of A. Since $x \in (R \setminus D) + (S \setminus D)$ then there exist $a \in R \setminus D$ and $b \in S \setminus D$ with $x = a + b$.

Since $R \subset cl(j)$ and $S \subset cl(k)$, then

$(R \setminus D) + (S \setminus D) \subset cl(j) + cl(k) \subset I$. Therefore $x \in I$

But then x is not in the first two pieces of $A + A$, that is true since I is disjoint from the first two pieces.

So it must be that one of the element a, b is in $C_{1/2}$ and the other in $\frac{1}{2}C_{1/2}$. Since a and b are not in D, x cant be in X.

Therefore $x \in Y$ and we are done.

3.3.4 Theorem

There is no subset $B \subseteq A = C_{1/2} \cup 1/2C_{1/2}$ such that $B + B$ is Bernstein in $[0,1.5]$. (See [3], p.791)

Proof

Suppose that such a set B exists. Let P and Q be as in the previous lemma. B can't contain a non-empty perfect subset, since that would imply $B+B$ also contains a non-empty perfect subset of $[0,1.5]$, and that is contradict $B+B$ is Bernstein in $[0,1.5]$. Therefore there is some element x in $P \setminus B$. Then $x+Q$ is a perfect subset of $P+Q$. And so each element of $x+Q$ has a unique representation as a sum of element in A that is since $x+Q$ is subset in $P+Q$ which has a unique representation as a sum of element in A. But then since $x \notin B$ no element of $x+Q$ is in $B+B$. Then we get a perfect subset $x+Q$ of $[0,1.5]$ which disjoint from $B+B$. So $B+B$ is not Bernstein in $[0,1.5]$, which is a contradiction.

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بسم ا الر ن الرحيم

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قدمت هذه الأطروحة استكمالا لمتطلبات درجة الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس، فلسطين.
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الملخص

هذه الرسالة عبارة عن بحث في استخدام مجموعات الكـانتور $C_{1/3}$ **و** $C_{1/2}$ **في نظرية القياس، تم برهان أن** $C_{1/3}$ **و** $C_{1/2}$ **أنهما لهما قياس وقياسـهم يساوي صفر.**

وبعد ذلك تم اثبات أن القياس لـ $C_{1/3}+C_{1/3}$ **موجب وأن القيـاس لـ ـ** $C_{1/2}+C_{1/2}$ **يساوي صفر، وأيضا تم اثبات أنه يوجد مجموعة A جزية من** $C_{1/3}$ **بحيث تكون المجموعة** $A+A$ **ليس لها قياس.**

وفي نهاية الرسالة تم برهان أنـه لا يوجـد مجموعـة B جزئيـة مـن $B\subset C_{1/2}\cup\frac{1}{2}C_{1/2}$ **بحيث تكون** $B+B$ **بيرنشتاين في** $[0,1.5]$**.**