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Decay of Solutions of a Nonlinear Viscoelastic Hyperbolic Equation

سلوك حلول المعادلات التفاضلية الجزئية الزائدية غير الخطية بوجود دالة الذاكرة

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Abstract

In this work, we are going to study under some conditions on *p, m* and suitable conditions on *g*, the decay of solutions of the nonlinear viscoelastic hyperbolic equation in problem (P) as t $\to +\infty$:

$$\begin{cases} u_{tt} - \Delta u - w\Delta u_t + \int_0^t g(t-s)\Delta u(x,s)ds + a|u_t|^{m-2}u_t = b|u|^{p-2}u, x \in \Omega, t > 0 \\ u(0,x) = u_0(x), x \in \Omega \\ u_t(0,x) = u_1(x), x \in \Omega \\ u(t,x) = 0, x \in \Gamma, t > 0. \end{cases}$$

(P)

Where Ω is a bounded domain in R^N (N>1), with smooth boundary Γ, and *a, b, w* are positive constants, *m≥2, p≥2*, and the function *g(t)* satisfying some conditions. We show that the energy of solutions decays exponentially if *m=2* and polynomial if *m>2*, provided that the initial data are small enough.

Keywords and phrases: Nonlinear Damping, Strong Damping, Viscoelastic, Exponential Decay, Polynomial Decay.

ملخص

في هذا البحث، نحن بصدد دراسة سلوك حلول المعادلات من الشكل (P) عندما الزمن t يؤول إلى المالانهاية، تحت بعض الشروط على دالة الذاكرة *g* والوسيطين *m ,p* . سنجد أن هناك تناقص على شكل أسي للطاقة (سلوك الحلول) عندما *m=2* وتناقص على شكل كثير الحدود عندما *m>2* بشرط نأخذ المعطيات الابتدائية صغيرة ما أمكن.

1. Introduction

In this article we consider the problem (P), where the function *g(t)* satisfying some conditions. In the physical point of view, this type of problems arise usually in viscoelasticity. This type of problems have been considered first by Dafermos [8], in 1970, where the general decay was discussed. A related problems to (P) have attracted a great deal of attention in the last decades, and many results have been appeared on the existence and long time behavior of solutions. See in this directions [2], [3], [4]-[7], [13], [20], [22], [23], [26] and references therein.

In the absence of the strong damping Δu_t, that is for w =0, and when the function g vanishes identically (i.e: g = 0), then problem (P) can be reduced to the following initial boundary damped wave equation with nonlinear damping and nonlinear sources terms.

$$u_{tt} - \Delta u + a|u_t|^{m-2} u_t = b|u|^{p-2} u \tag{1}$$

Some special cases of equation (1) arise in quantum field theory which describe the motion of charged mesons in an electromagnetic field. Equation (1) together with initial and boundary conditions of Dirichlet type, has been extensively studied and results concerning existence, blow up and asymptotic behavior of smooth, as well as weak solutions have been established by several authors.

For b = 0, that is in the absence of the source term, it is well known that the damping term $|u_t|^{m-2} u_t$ assures global existence and decay of the solution energy for arbitrary initial data (see [12] and [16]).

For a = 0, the source term causes finite-time blow-up of solutions with a large initial data (negative initial energy). That is to say, the norm of our solution $u(t,x)$ in the energy space reaches $+\infty$ when the time t approaches certain value T* called " *the blow up time*", (see [1] and [15] for more details). The interaction between the damping term $|u_t|^{m-2}u_t$ and the source term $|u|^{p-2}u$ makes the problem more interesting. This situation was first considered by Levine [17], [18] in the linear damping case (m = 2), where he showed that solutions with negative initial energy blow up in finite time T*. The main ingredient used in [17] and [18] is the "*concavity method*" where the basic idea of this method is to construct a positive function L(t) of the solution and show that for some $\gamma > 0$, the function L (t) is a positive concave function of t. In order to find such γ, it suffies to verify that:

$$\frac{d^2L^{-\gamma}(t)}{dt^2} = -\gamma L^{-\gamma-2}(t)\left[LL''-(1+\gamma)L'^2(t)\right] \leq 0, \forall t \geq 0.$$

This is equivalent to prove that L(t) satisfies the differential inequality

$$LL''-(1+\gamma)L'^2(t) \geq 0, \forall t \geq 0.$$

Unfortunately, this method fails in the case of nonlinear damping term (m>2). Georgiev and Todorova in their famous paper [10], extended Levine's result to the nonlinear damping case (m> 2). More precisely, in [10] and by combining the Galerkin approximation with the contraction mapping theorem, the authors showed that problem (1) in a bounded domain with initial and boundary conditions of Dirichlet type has a unique solution in the interval [0,T) provided that T is small enough. Also, they proved that the obtained solutions continue to exist globally in time if $m \geq p$ and the initial data are small enough. Whereas for $p>m$ the unique solution of problem (1) blows up in finite time provided that the initial data are large enough. (i. e: the initial energy is sufficiently negative). This later result has been pushed by Messaoudi in [24] to the situation where the initial energy *E(0)* < 0. For more general result in this direction, we refer the interested reader to the works of Vitillaro [28], Levine [19] and Messaoudi [21].

In the presence of the viscoelastic term $(g \neq 0)$ and for *w*=0, our problem

(P) becomes

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + a|u_t|^{m-2}u_t = b|u|^{p-2}u, x \in \Omega, t > 0 \\ u(0,x) = u_0(x), x \in \Omega \\ u_t(0,x) = u_1(x), x \in \Omega \\ u(t,x) = 0, x \in \Gamma, t > 0. \end{cases} \quad (2)$$

For a=0, problem (2) has been investigated by Berrimi and Messaoudi [3]. They established the local existence result by using the Galerkin method together with the contraction mapping theorem. Also, they showed that for a suitable initial data, then the local solution is global in time and in addition, they showed that the dissipation given by the viscoelastic integral term is strong enough to stabilize the oscillations of the solution with the same rate of decaying (exponential or polynomial) of the kernel g. Also their result has been obtained under weaker conditions than those used by Cavalcanti et al[6], in which a similar problem has been addressed.

Messaoudi in [20], showed that under appropriate conditions between *m, p* and *g* ablow up and global existence result, of course his work generalizes the result by Georgiev and Todorova [10]. One of the main direction of the research in this field it seems to find the minimal dissipation such that the solutions of problems similar to (2) decay uniformly to zero, as time goes to infinity. Consequently, several authors introduced different types of dissipative mechanisms to stabilize these problems. For example, a localized frictional linear damping of the form $a(x)u_t$ acting in sub-domain $\overline{w} \subset \Omega$ has been considered by Cavalcanti et al[6]. More precisely the authors in [5] looked in to the following problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + a(x)u_t + |u|^{\gamma} u = 0. \quad (3)$$

for $\gamma > 0$, g a positive function and $a : \Omega \to R^+$ a function, which may be null on a part of the domain Ω. By assuming $a(x) \geq a_0 > 0$ on the sub-domain $\overline{w} \subset \Omega$, the authors showed a decay result of an exponential rate, provided that the kernel g satisfies

$$-\zeta_1 g(t) \leq g'(t) \leq -\zeta_2 g(t), t \geq 0, \quad (4)$$

and $\|g\|_{L^1(0,\infty)}$ is small enough.

This later result has been improved by Berrimi and Messaoudi [2], in which they showed that the viscoelastic dissipation alone is strong enough to stabilize the problem even with an exponential rate. In many existing works on this field, the following conditions on the kernel

$$g'(t) \leq -\zeta \ g^p(t), t \geq 0, p \geq 1, \quad (5)$$

is crucial in the proof of the stability.

For a viscoelastic systems with oscillating kernels, we mention the work by Rivera etal[25]. In that work the authors proved that if the kernel satisfies *g(0)* > 0 and decays exponentially to zero, that is for *p* =1 in (5), then the solution also decays exponentially to zero. On the other hand, if the kernel decays polynomially, i.e. (*p*>1) in the inequality (5), then the solution also decays polynomially with the same rate of decay. In the presence of the strong damping (*w*> 0) and in the absence of the viscoelastic term (*g* =0), the problem (P) takes the following form

$$\begin{cases} u_{tt} - \Delta u - w\Delta u_t + a|u_t|^{m-2} u_t = b|u|^{p-2} u, x \in \Omega, t > 0 \\ u(0,x) = u_0(x), x \in \Omega \\ u_t(0,x) = u_1(x), x \in \Omega \\ u(t,x) = 0, x \in \Gamma, t > 0. \end{cases} \quad (6)$$

Problem (6) represents the wave equation with a strong damping Δu_t. When *m* =2, this problem has been studied by Gazzola and Squassina[9]. In their work, the authors proved some results on well posedness and asymptotic be havior of solutions. They showed the global existence and polynomial decay property of solutions provided that the initial data is in the potential well. The proof in [9] is based on a method used in [14]. Unfortunately their decay rate is not optimal, and their result has been improved by Gerbi and Said-Houari [11], by using an appropriate modification of the energy method and some differential and integral inequalities. Introducing a strong damping term Δu_t makes the problem from that considered in [10], for this reason less results where known for the wave equation with strong damping and many problem remain unsolved.

In this paper, we investigated problem (P), in which all the damping mechanism have been considered in the same time (i. e. *w>0, g =0*, and $m \geq 2$), these assumptions make our problem diffierent form those studied in the literatur, we show that the energy of solutions decays exponentially if m=2 and polynomial if m>2, provided that the initial data are small enough, using the arguments in Rivera [27].

2. Preliminaries

In our work, we consider a viscoelastic wave equation, with strong damping, polynomial nonlinear damping and source term. Namely we looked in to the following problem

$$u_{tt} - \Delta u - w\Delta u_t + \int_0^t g(t-s)\Delta u(x,s)ds + a|u_t|^{m-2} u_t = b|u|^{p-2} u, x \in \Omega, t > 0 \quad (7)$$

subjected to the following initial and boundary conditions

$$u(0,x)=u_0(x), u_t(0,x)=u_1(x), x\in\Omega \tag{8}$$

$$u(t,x)=0, x\in\Gamma, t>0, \tag{9}$$

where Ω is a bounded domainin R^N (N > 1), with smooth boundary Γ, and *a, b* and *w* are positive constants, $m\geq 2$, $p\geq 2$, and g is a nonnegative nonincreasing function. This type of problems are not only important from the theoretical point of view, but also arise in many physical applications and describe a great deal of models in applied science. One of the most important field of such problems arise in the models of nonlinear viscoelasticity. Many authors studied these types of problems, and several results appeared in the literature.

The energy related to problem (P) is $2E(t)=\|u_t(t)\|_2^2+2J(t)$,

where

$$J(t)=\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{b}{p}\|u(t)\|_p^p, \tag{10}$$

and

$$I(t)=\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2+(g\circ\nabla u)(t)-b\|u(t)\|_p^p,$$

We assume that the kernel *g* satisfies the following conditions:

(G1) $g:R^+\to R^+$ is a bounded C^1-function such that $g(0)>0, 1-\int_0^t g(s)ds=l>0.$

(G2) $g(t)\geq 0, g'(t)\leq 0, g(t)\leq -\xi g'(t), \forall t\geq 0.$

Let us denote by $(g\circ u)(t)=\int_0^t g(t-s)\int_\Omega |u(s)-u(t)|^2 dxds.$

We first state a local existence theorem that can be established by combining arguments of Georgiev and Todorova [10]. In fact this depends on the parameters values of the coefficients and the exponents *m* and *p*.

Let us introduce the following complete metric space

$$Y_T = \{u : u \in C([0,T], H_0^1(\Omega)), u_t \in C([0,T], H_0^1(\Omega)) \cap L^m([0,T]\times\Omega)\}. \quad (11)$$

Theorem 1. *Let* $(u_0, u_1) \in (H_0^1(\Omega))^2$ *be given. Suppose that* $m \geq 2, p \geq 2$ *be such that*

$$\max\{m, p\} \leq \frac{2(n-1)}{n-2}, n \geq 3. \quad (12)$$

Then, under the conditions (G1) and (G2), the problem (P) has a unique local solution $u(x,t) \in Y_T$, *for T small enough.*

Now, we will state the global existence, for this purpose it suffices to prove that the norm of the solution is bounded, independently of t, in the following theorem. The existence of the source term $\left(|u|^{p-2}u\right)$ forces us to use the potential well depth method in which the concept of so called stable set appears. Let us introduce the stable set as:

$$W = \{u \in H_0^1(\Omega) : J(u) < d, I(u) > 0\} \cup \{0\} \quad (13)$$

where

$$d = \underbrace{\inf}_{u \in H_0^1(\Omega)/\{0\}} \underbrace{\sup}_{\lambda \geq 0} J(\lambda u). \quad (14)$$

Theorem 2. *Suppose that (G1), (G2) and (12) hold. If* $u_0 \in W, u_1 \in H_0^1(\Omega)$ *and*

$$\frac{bc_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}}<1, \tag{15}$$

where c_* *is the best Poincaré's constant. Then the local solution u(t,x) is global in time.*

The following Lemma du to Nakao will play a decisive role in the proof of our aim result.

Lemma 3 [27]**.** *Let* $\Phi(t)$ *be a nonincreasing and nonnegative function defined on* $[0,T], T>1,$ *satisfying*

$$\Phi^{1+r}(t)\le k_0\left(\Phi(t)-\Phi(t+1)\right), t\in[0,T],$$

for $k_0>1$ *and* $r\ge 0$. *Then we have, for each* $t\in[0,T]$,

$$\begin{cases}\Phi(t)\le\Phi(0)\exp\left(-k\ [t-1]^+\right), r=0\\ \Phi(t)\le\left\{\Phi(0)^{-r}+k\ r[t-1]^+\right\}^{\frac{-1}{r}}, r>0\end{cases}, \tag{16}$$

where $[t-1]^+=\max\{t-1,0\}$, *and* $k=\ln\left(\frac{k_0}{k_0-1}\right)$.

The following technical lemma will play an important role in the sequel.

Lemma 4 [5]**.** *For any* $v\in C^1\left(0,T,H^2(\Omega)\right)$ *we have*

$$\int_\Omega\int_0^t g(t-s)\Delta v(s)v'(t)dsdx=\frac{1}{2}\frac{d}{dt}(g\circ\nabla v)(t)-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)\int_\Omega|\nabla v(t)|^2dxds\right]$$

$$-\frac{1}{2}(g'\circ\nabla v)(t)+\frac{1}{2}g(t)\int_\Omega|\nabla v(t)|^2dxds$$

3. Main result

The following lemma is very useful

Lemma 5. *Suppose that (12) and (15) hold. Then*

$$b\|u(t)\|_p^p \le (1-\eta)\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 \tag{17}$$

where $\eta = 1-\beta$.

We can now state and prove the asymptotic behavior of the solution of our problem.

Theorem 6. *Suppose that (G1), (G2) and (12) hold. Assume further that* $u_0 \in W$ *and* $u_1 \in H_0^1(\Omega)$ *satisfying (15). Then the global solution satisfies*

$$E(t) \le E(0)\exp(-\lambda t), \forall t \ge 0 \ \textit{if } m = 2, \tag{18}$$

or

$$E(t) \le \left(E(0)^{-r} + k_0 rt\right)^s, \forall t \ge 0 \ \textit{if } m > 2, \tag{19}$$

where λ *and* k_0 *are constants independent of t,* $r = \frac{m}{2} - 1$ *and* $s = \frac{2}{2-m}$..

Proof of Theorem 6.

Multiplying the first equation in (P), by u_t and integrate over Ω to obtain

$$\frac{d}{dt}E(t) + w\|\nabla u_t\|_2^2 + a\|u_t\|_m^m = \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2 .$$

Then, integrate the last equality over [t, t +1] to get

$$E(t+1)-E(t)+w\int_t^{t+1}\|\nabla u_t\|_2^2ds+a\int_t^{t+1}\|u_t\|_m^m ds=\frac{1}{2}\int_t^{t+1}(g'\circ\nabla u)(s)ds-\frac{1}{2}\int_t^{t+1}g(s)\|\nabla u\|_2^2ds.$$

Therefore,

$$E(t)-E(t+1)=F^m(t)-\frac{1}{2}\int_t^{t+1}(g'\circ\nabla u)(s)ds+\frac{1}{2}\int_t^{t+1}g(s)\|\nabla u\|_2^2ds, \qquad (20)$$

where

$$F(t)^m = w\int_t^{t+1}\|\nabla u_t\|_2^2ds+a\int_t^{t+1}\|u_t\|_m^m ds. \qquad (21)$$

Using Poincaré's inequality to find

$$\int_t^{t+1}\|u_t\|_2^2ds\le c(\Omega)\int_t^{t+1}\|u_t\|_m^2ds \qquad (22)$$

Exploiting Holder's inequality, we obtain

$$\int_t^{t+1}\|u_t\|_m^2ds\le\left(\int_t^{t+1}ds\right)^{\frac{m-2}{m}}\left(\int_t^{t+1}\left(\|u_t\|_m^2\right)^{\frac{m}{2}}ds\right)^{\frac{2}{m}}$$

$$\le\left(\int_t^{t+1}\left(\|u_t\|_m^2\right)^{\frac{m}{2}}ds\right)^{\frac{2}{m}} \qquad (23)$$

Combining (21), (22) and (23) we obtain, for a constant c_1, depending on Ω

$$\int_t^{t+1}\|u_t\|_2^2ds\le c_1F(t)^2, c_1>0. \qquad (24)$$

By applying the mean value theorem, we get for some $t_1 \in \left[t, t+\frac{1}{4}\right]$, $t_2 \in \left[t+\frac{3}{4}, t+1\right]$

$$\|u_t(t_i)\|_2 \le 2c(\Omega)^{1/2} F(t), i = 1,2. \tag{25}$$

Hence, by (G2) and since

$$\int_t^{t+1} \|\nabla u_t\|_2^2 ds \le c_2 F(t)^2, c_2 > 0. \tag{26}$$

there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$, $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$\|\nabla u_t(t_i)\|_2^2 \le 4c(\Omega) F(t)^2, i = 1,2. \tag{27}$$

Next, we multiply the first equation in (P) by u and integrate over $\Omega \times [t_1, t_2]$ to obtain

$$\int_{t_1}^{t_2} \left[\left(1 - \int_0^t g(r)dr\right)\|\nabla u\|_2^2 ds - b\|u\|_p^p\right] ds = -\int_{t_1}^{t_2}\int_\Omega u u_{tt} dxds - w\int_{t_1}^{t_2}\int_\Omega \nabla u \nabla u_t dxds - a\int_{t_1}^{t_2}\int_\Omega u|u_t|^{m-2} \; u_t dxds$$
$$+ \int_{t_1}^{t_2}\int_0^s g(s-r)\int_\Omega \nabla u(s)[\nabla u(r) - \nabla u(s)] \; dxdrds$$

Obviously,

$$\int_{t_1}^{t_2} I(s)ds = -\int_{t_1}^{t_2}\int_\Omega u u_{tt} dxds - w\int_{t_1}^{t_2}\int_\Omega \nabla u \nabla u_t dxds - a\int_{t_1}^{t_2}\int_\Omega u|u_t|^{m-2} \; u_t dxds$$
$$+ \int_{t_1}^{t_2}\int_0^s g(s-r)\int_\Omega \nabla u(s)[\nabla u(r) - \nabla u(s)] \; dxdrds + \int_{t_1}^{t_2} (g \circ \nabla u)(s)ds. \tag{28}$$

Note that by integrating by parts, to obtain

$$\left|\int_{t_1}^{t_2}\int_{\Omega} uu_{tt}dxds\right| = \left|\left[\int_{\Omega} uu_t dx\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\Omega} u_t^{\ 2}dxds\right| = \left|\int_{\Omega} u(t_2)u_t(t_2)dx - \int_{\Omega} u(t_1)u_t(t_1)dx - \int_{t_1}^{t_2}\int_{\Omega} u_t^{\ 2}dxds\right|$$

,

Using Holder's and Poincaré's inequalities, we get

$$\left|\int_{t_1}^{t_2}\int_{\Omega} uu_{tt}dxds\right| \leq c_*^2 \sum_{i=1}^{2} \left\|\nabla u_t(t_i)\right\|_2 \left\|\nabla u\ (t_i)\right\|_2 + c_*^2 \int_{t_1}^{t_2} \left\|\nabla u_t\right\|_2^2 \ dt. \tag{29}$$

By using Holder's inequality once again, we have

$$\left|\int_{t_1}^{t_2}\int_{\Omega} \nabla u \nabla u_t dxds\right| \leq \int_{t_1}^{t_2} \left\|\nabla u\ \right\|_2 \left\|\nabla u_t\right\|_2 \ ds. \tag{30}$$

Furthermore, by equation (27), we have

$$\left\|\nabla u_t(t_i)\right\|_2 \left\|\nabla u\ (t_i)\right\|_2 \leq c_3\left(c(\Omega)\right)^{1/2} F(t) \underbrace{\sup}_{t_1 \leq s \leq t_2} E(s)^{1/2}, \tag{31}$$

where, $c_3 = 2\left(\frac{2p}{l(p-2)}\right)^{1/2}$. We have by Holder's inequality

$$\int_{t_1}^{t_2} \left\|\nabla u_t\right\|_2 \left\|\nabla u\ \right\|_2 dt \leq \int_{t_1}^{t_2} E(s)^{1/2}\left(\frac{2p}{l(p-2)}\right)^{1/2} \left\|\nabla u_t\right\|_2 ds \leq \frac{1}{2} c_3 \underbrace{\sup}_{t_1 \leq s \leq t_2} E(s)^{1/2} \int_{t_1}^{t_2} \left\|\nabla u_t\right\|_2 ds,$$

Which implies

$$\int_{t_1}^{t_2} \left\|\nabla u_t\right\|_2 dt \leq \left(\int_{t_1}^{t_2} 1dt\right)^{1/2} \left(\int_{t_1}^{t_2} \left\|\nabla u_t\right\|_2^2 dt\right)^{1/2} \leq \frac{\sqrt{3c_2}}{2} F(t).$$

Then,

$$\int_{t_1}^{t_2} \|\nabla u_t\|_2 \|\nabla u\|_2 dt \le c_4 F(t) \underbrace{\sup}_{t_1 \le s \le t_2} E(s)^{1/2}, \tag{32}$$

where $c_4 = \frac{c_3\sqrt{3c_2}}{4}$. Therefore equation (29), becomes

$$\left| \int_{t_1}^{t_2}\int_{\Omega} u u_{tt} dxds \right| \le 2c_*^2 c_3 F(t) \underbrace{\sup}_{t_1 \le s \le t_2} E(s)^{1/2} + c_*^2 c_2 F(t)^2. \tag{33}$$

We then exploit Young's inequality to estimate

$$\int_{t_1}^{t_2}\int_{\Omega}\int_0^t g(s-\tau)\nabla u(t)\left[\nabla u(s)-\nabla u(t)\right]d\tau dxdt \le \delta \int_{t_1}^{t_2}\int_0^t g(s-\tau)\|\nabla u\|_2^2 d\tau ds + \frac{1}{4\delta}\int_{t_1}^{t_2}(g \circ \nabla u)dt, \forall \delta > 0.$$

Now, the third term in the right-hand side of equation (28), can be estimated as follows

$$\int_{t_1}^{t_2}\int_{\Omega} |u_t|^{m-2} u_t u dxds \le \int_{t_1}^{t_2}\int_{\Omega} |u||u_t|^{m-1} dxds.$$

By Holder's inequality, we find

$$\int_{t_1}^{t_2}\int_{\Omega} |u||u_t|^{m-1} dxds \le \int_{t_1}^{t_2}\left[\left(\int_{\Omega}|u_t|^m\right)^{\frac{m-1}{m}}\left(\int_{\Omega}|u|^m dx\right)^{\frac{1}{m}}\right]ds = \int_{t_1}^{t_2}\|u_t\|_m^{m-1}\|u\|_m ds.$$

By Sobolev-Poincaré's inequality, we have

$$\int_{t_1}^{t_2}\|u_t\|_m^{m-1}\|u\|_m ds \le c(\Omega)\int_{t_1}^{t_2}\|u_t\|_m^{m-1}\|\nabla u\|_2 ds,$$

for $2 < m \le \frac{2n}{n-2}, n \ge 3$ or $2 < m < \infty, n = 1,2$.

Using Holder's inequality, and since $t_1, t_2 \in [t, t+1]$ and E(t) decreasing in time, we conclude from the last inequality and (21), that

$$\int_{t_1}^{t_2} \|u_t\|_m^{m-1} \|u\|_m ds \le c(\Omega)\left(\frac{2p}{l(p-2)}\right)^{1/2} \int_{t_1}^{t_2} \|u_t\|_m^{m-1} (J(t))^{1/2} ds$$

$$\le c(\Omega)\left(\frac{2p}{l(p-2)}\right)^{1/2} \int_{t_1}^{t_2} \|u_t\|_m^{m-1} (E(t))^{1/2} ds$$

$$\le c(\Omega)\left(\frac{2p}{l(p-2)}\right)^{1/2} \underbrace{\sup}_{t_1 \le s \le t_2} E(s)^{1/2} \left(\int_{t_1}^{t_2} \|u_t\|_m^m ds\right)^{\frac{m-1}{m}} \left(\int_{t_1}^{t_2} ds\right)^{\frac{1}{m}} \tag{34}$$

$$\le \left(\frac{1}{a}\right)^{\frac{m-1}{m}} c(\Omega) \underbrace{\sup}_{t_1 \le s \le t_2} E(s)^{1/2} \left(\frac{2p}{l(p-2)}\right)^{1/2} F(t)^{m-1}$$

Then, taking in to account (33) and (34), estimate (28) takes the form

$$\int_{t_1}^{t_2} I(t)dt \le \left(2c_*^2 + \frac{\sqrt{3c_2}}{4} w\right) c_3 F(t) \underbrace{\sup}_{t_1 \le s \le t_2} E(s)^{1/2}$$

$$+ c_*^2 c_2 F(t)^2 + \frac{a^{1/m}}{2} c_3 c(\Omega) \underbrace{\sup}_{t_1 \le s \le t_2} E(s)^{1/2} F(t)^{m-1} \tag{35}$$

$$+ \delta \int_{t_1}^{t_2} \int_0^t g(t-s) \|\nabla u\|_2^2 ds dt + \left(\frac{1}{4\delta} + 1\right) \int_{t_1}^{t_2} (g \circ \nabla u)(t) dt.$$

Moreover, we have

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t)$$

$$= \frac{1}{2}\|u_t\|_2^2 + \left(\frac{p-2}{2p}\right)\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \left(\frac{p-2}{2p}\right)(g \circ \nabla u)(t) + \frac{1}{p} I(t) \tag{36}$$

By integrating (36) over $[t_1, t_2]$, we obtain

$$\int_{t_1}^{t_2} E(t)dt = \frac{1}{2}\int_{t_1}^{t_2}\|u_t\|_2^2 dt + \left(\frac{p-2}{2p}\right)\int_{t_1}^{t_2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 dt + \left(\frac{p-2}{2p}\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt + \frac{1}{p}\int_{t_1}^{t_2} I(t)dt,$$
(37)

which implies by exploiting (24)

$$\int_{t_1}^{t_2} E(t)dt = \frac{c_1}{2}(F(t))^2 + \left(\frac{p-2}{2p}\right)\int_{t_1}^{t_2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 dt + \left(\frac{p-2}{2p}\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt + \frac{1}{p}\int_{t_1}^{t_2} I(t)dt.$$
(38)

By using Lemma (5), we see that

$$\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 \le \frac{1}{\eta} I(t). \tag{39}$$

Therefore, (38), takes the form

$$\int_{t_1}^{t_2} E(t)dt \le \frac{c(\Omega)}{2}(F(t))^2 + \left(\frac{p-2}{2p}\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt + \left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\int_{t_1}^{t_2} I(t)dt. \tag{40}$$

Again an integration of $E'(t)$ over $[s,t_2], s\in[0,t_2]$ gives

$$E(s) = E(t_2) + a\int_s^{t_2}\|u_t\|_m^m dt + \frac{1}{2}\int_s^{t_2} g(t)\|\nabla u\|_2^2 dt - \frac{1}{2}\int_s^{t_2}(g'\circ\nabla u)(t)dt + w\int_s^{t_2}\|\nabla u_t\|_2^2 dt \tag{41}$$

By using the fact that $t_2 - t_1 \ge \frac{1}{2}$,we have

$$\int_{t_1}^{t_2} E(s)ds \ge \int_{t_1}^{t_2} E(t_2)ds \ge \frac{1}{2}E(t_2). \tag{42}$$

The fourth term in (35), can be handled as

$$\int_0^t g(t-s)\|\nabla u\|_2^2 ds = \|\nabla u\|_2^2\int_0^t g(t-s)ds \le \frac{2p(1-l)}{l(p-2)}E(t). \tag{43}$$

Thus,

$$\int_{t_1}^{t_2}\int_0^t g(t-s)\|\nabla u\|_2^2\,ds\,dt \le \frac{2p(1-l)}{l(p-2)}\int_{t_1}^{t_2} E(t)dt \le \frac{p(1-l)}{l(p-2)}E(t_1) \le \frac{p(1-l)}{l(p-2)}E(t). \qquad (44)$$

Hence, by (44), we obtain from (35)

$$\int_{t_1}^{t_2} I(t)dt \le \left(2c_*^2+\frac{\sqrt{3C_2}}{4}w\right)c_3F(t)\underbrace{\sup}_{t_1\le s\le t_2}E(s)^{1/2}$$

$$+c_*^2c_2F(t)^2+\frac{a^{1/m}}{2}c_3c(\Omega)\underbrace{\sup}_{t_1\le s\le t_2}E(s)^{1/2}F(t)^{m-1} \qquad (45)$$

$$+\delta\frac{p(1-l)}{l(p-2)}E(t)+\left(\frac{1}{4\delta}+1\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt.$$

From (41) and (42) we have

$$E(t)\le 2\int_{t_1}^{t_2}E(t)dt+a\int_t^{t+1}\|u_t\|_m^m ds+\frac{1}{2}\int_t^{t+1}g(s)\|\nabla u\|_2^2 ds-\frac{1}{2}\int_t^{t+1}(g'\circ\nabla u)(s)ds+w\int_t^{t+1}\|\nabla u_t\|_2^2 ds. \qquad (46)$$

Obviously, (40) and (46) give us

$$E(t)\le 2\left(\frac{c(\Omega)}{2}(F(t))^2+\left(\frac{p-2}{2p}\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt+\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\int_{t_1}^{t_2}I(t)dt,\right)$$

$$+a\int_{t_1}^{t_2}\|u_t\|_m^m ds+\frac{1}{2}\int_{t_1}^{t_2}g(s)\|\nabla u\|_2^2 ds-\frac{1}{2}\int_{t_1}^{t_2}(g'\circ\nabla u)(s)ds+w\int_{t_1}^{t_2}\|\nabla u_t\|_2^2 ds.$$

Consequently, plugging the estimate (45) into the above estimate, we conclude that

$$E(t) \le c(\Omega)\big(F(t)\big)^2 + \left(\frac{p-2}{p}\right)\int_{t_1}^{t_2}\big(g\circ\nabla u\big)(t)dt$$
$$+2\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\left[\left(2c_*^2+\frac{\sqrt{3c_2}}{4}w\right)c_3F(t)\underbrace{\sup}_{t_1\le s\le t_2}E(s)^{1/2}+c_*^2c_2F(t)^2\right]$$
$$+\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\left[a^{1/m}c_3c(\Omega)\underbrace{\sup}_{t_1\le s\le t_2}E(s)^{1/2}F(t)^{m-1}\right] \tag{47}$$
$$+2\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\left[\delta\frac{p(1-l)}{l(p-2)}E(t)+\left(\frac{1}{4\delta}+1\right)\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt\right]$$
$$+F(t)^m+\frac{1}{2}\int_{t_1}^{t_2}g(s)\|\nabla u\|_2^2ds-\frac{1}{2}\int_{t_1}^{t_2}(g'\circ\nabla u)(s)ds+w\int_{t_1}^{t_2}\|\nabla u_t\|_2^2ds.$$

We also have, by the Poincaré's inequality

$$\|u\|_2 \le c\|\nabla u\|_2 \le c\left(\frac{2p}{l(p-2)}\right)^{1/2}E(t)^{1/2} \tag{48}$$

Choosing δ small enough so that

$$1-2\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\delta\frac{p(1-l)}{l(p-2)}>0, \tag{49}$$

we can deduce, from (47) that there exists $k>0,$ such that

$$E(t)\le k\left[F(t)^2+E(t)^{1/2}F(t)+E(t)^{1/2}F(t)^{m-1}+F(t)^m\right]+\frac{1}{2}\int_t^{t+1}g(s)\|\nabla u\|_2^2ds$$
$$-\frac{1}{2}\int_t^{t+1}(g'\circ\nabla u)ds+\left[\left(\frac{p-2}{p}\right)+2\left(\frac{1}{4\delta}+1\right)\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\right]\int_t^{t+1}(g\circ\nabla u)ds \tag{50}$$

Using (G2) again we can write

$$\int_{t_1}^{t_2}(g\circ\nabla u)(t)dt \le -\xi\int_{t_1}^{t_2}(g'\circ\nabla u)(t)dt, \xi>0.$$

Thus, we obtain, from (50),

$$E(t)\le k\left[F(t)^2+E(t)^{1/2}F(t)+E(t)^{1/2}F(t)^{m-1}+F(t)^m\right]+\frac{1}{2}\int_t^{t+1}g(s)\|\nabla u\|_2^2 ds$$
$$-\left(\xi_1+\frac{1}{2}\right)\int_t^{t+1}(g'\circ\nabla u)ds, \tag{51}$$

where $\xi_1=\xi\left[\left(\frac{p-2}{p}\right)+2\left(\frac{1}{4\delta}+1\right)\left(\frac{1}{p}+\frac{p-2}{2p\eta}\right)\right].$

An appropriate use of Young's inequality in (51), we can find $k_1>0$, such

That

$$E(t)\le k_1\left[F(t)^2+F(t)^{2(m-1)}+F(t)^m\right]+\frac{1}{2}\int_t^{t+1}g(s)\|\nabla u\|_2^2 ds-\left(\xi_1+\frac{1}{2}\right)\int_t^{t+1}(g'\circ\nabla u)ds, \tag{52}$$

for k_1 a positive constant. Using (G2) again to get

$$E(t)\le k_1\left[F(t)^2+F(t)^{2(m-1)}+F(t)^m\right]+(1+2\xi_1)\left[\frac{1}{2}\int_t^{t+1}g(s)\|\nabla u\|_2^2 ds-\frac{1}{2}\int_t^{t+1}(g'\circ\nabla u)ds\right], \tag{53}$$

at this end we can distinguish between two cases:

Case 1: *For m =2.* In this case we use (20) and (53), we can find $k_2>0$, such that

$$E(t)\le k_1F(t)^2+(1+2\xi_1)\left[\frac{1}{2}\int_t^{t+1}g(s)\|\nabla u\|_2^2 ds-\frac{1}{2}\int_t^{t+1}(g'\circ\nabla u)ds\right]$$
$$\le k_2\left[E(t)-E(t+1)\right]. \tag{54}$$

Since E(t) is nonincreasing and nonnegative function, an application of Lemma 3 yields

$$E(t) \le k_2 \left[E(t) - E(t+1)\right], t \ge 0, \tag{55}$$

Which implies that

$$E(t) \le E(0)\exp\left(-\lambda [t-1]^{+}\right) \text{ on } [0,\infty) \tag{56}$$

where $\lambda = \ln\left(\frac{k_2}{k_2 - 1}\right)$.

Case 2: *For m >2.* In this case we, again use (20) and (53) to arrive at

$$F(t)^2 = \left[\left(E(t) - E(t+1)\right) + \frac{1}{2}\int_t^{t+1} \left(g' \circ \nabla u\right) ds - \frac{1}{2}\int_t^{t+1} g(s) \|\nabla u\|_2^2 \, ds \right]^{\frac{2}{m}}. \tag{57}$$

We then use the algebraic inequality

$$\left(a+b\right)^{\frac{m}{2}} \le 2^{\frac{m}{2}} \left(a^{\frac{m}{2}} + b^{\frac{m}{2}} \right), m \ge 2. \tag{58}$$

To infer from (53), and by using (58), that

$$\begin{aligned}
\left[E(t)\right]^{\frac{m}{2}} &\le k_3 \left[1 + F(t)^{2(m-2)} + F(t)^{m-2}\right]^{\frac{m}{2}} F(t)^m \\
&+ 2^{\frac{m}{2}} (1+2\xi_1)^{\frac{m}{2}} \left[-\frac{1}{2}\int_t^{t+1} \left(g' \circ \nabla u\right) ds + \frac{1}{2}\int_t^{t+1} g(s) \|\nabla u\|_2^2 \, ds \right]^{\frac{m}{2}} \\
&\le k_3 \left[1 + F(t)^{2(m-2)} + F(t)^{m-2}\right]^{\frac{m}{2}} \times \left[E(t) - E(t+1)\right] \\
&+ 2^{\frac{m}{2}} (1+2\xi_1)^{\frac{m}{2}} \left[-\frac{1}{2}\int_t^{t+1} \left(g' \circ \nabla u\right) ds + \frac{1}{2}\int_t^{t+1} g(s) \|\nabla u\|_2^2 \, ds \right]^{\frac{m}{2}}
\end{aligned} \tag{59}$$

where $k_3 = 2^{\frac{m}{2}} k_1$. We use (20) to obtain

$$\left[-\frac{1}{2}\int_t^{t+1}(g'\circ\nabla u)ds+\frac{1}{2}\int_t^{t+1}g(s)\|\nabla u\|_2^2\,ds\right]^{\frac{m}{2}}\le\left(E(t)-E(t+1)\right)^{\frac{m}{2}} \tag{60}$$

A combination of (59), (60) yields

$$\begin{aligned}[E(t)]^{\frac{m}{2}} &\le k_3\left[1+F(t)^{2(m-2)}+F(t)^{m-2}\right]^{\frac{m}{2}}\times[E(t)-E(t+1)]\\ &+2^{\frac{m}{2}}(1+2\xi_1)^{\frac{m}{2}}[E(t)-E(t+1)]^{\frac{m}{2}-1}[E(t)-E(t+1)]\\ &\le\left[k_3\left[1+F(t)^{2(m-2)}+F(t)^{m-2}\right]^{\frac{m}{2}}+2^{\frac{m}{2}}(1+2\xi_1)^{\frac{m}{2}}[E(t)-E(t+1)]^{\frac{m}{2}-1}\right]\times[E(t)-E(t+1)]\end{aligned} \tag{61}$$

By using (53), the estimate (61) takes the form

$$\begin{aligned}[E(t)]^{\frac{m}{2}} &\le\left[k_3 2^m\left[1+E(0)^{(m-2)}+E(0)^{\frac{m}{2}-1}\right]+2^{\frac{m}{2}}(1+2\xi_1)^{\frac{m}{2}}E(0)^{\frac{m}{2}-1}\right]\times[E(t)-E(t+1)]\\ &\le k_0[E(t)-E(t+1)].\end{aligned} \tag{62}$$

Again, using Lemma 3, we conclude

$$E(t)\le\left[E(0)^{-r}+k_0 r[t-1]^+\right]^s, \tag{63}$$

with $r=\frac{m}{2}-1>0, s=\frac{2}{2-m}$ and k_0 is some given positive constant.

This completes the proof.

4. Discussion

We can say that this work has enabled us to achieve the desired result, namely the existence in time, uniqueness and the decay of solutions at infinity which extends the recent result of the same author concerning growth exponential of solutions when t → +∞.

Indeed, in this work, we studied a class of hyperbolic problems where it has a number of parameters modeling the vibration problems and wave propagation in objects.

This study was focused specifically on:

A. Existence and uniqueness (local, global) without proofs.

B. Exponential and polynomial decay of solutions at infinity.

To end and complete the study of asymptotic behavior of solutions, it would be interesting to show the blow up result in finite time. Let us mention here that the study of decay of energy according to the function g in unbounded domain R^N stay an open subject.

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