



**An-Najah National University**  
**Faculty of Graduate Studies**

# **NUMERICAL SOLUTION FOR FREDHOLM INTEGRAL EQUATIONS**

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**This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of  
Master of Computerized Mathematics, Faculty of Graduate Studies, An-Najah National  
University, Nablus - Palestine.**

**2024**

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By

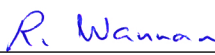
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## **Dedication**

In the name of Allah, the Most Gracious, the Most Merciful.

All praise is due to Allah, and thanks be to Him, before and after. May peace and blessings be upon our Master, Muhammad, and his family and companions.

I dedicate this thesis to my beloved daughter, Zeina, the light of my life, who has granted me boundless love and support, and who has been my source of inspiration and happiness at every moment.

To my dear sister, Sahar, who has been my constant support and aid throughout every stage of my journey, and who has always been by my side in every challenge. May Allah bless you.

To my dear parents, to whom I owe the highest gratitude and appreciation for all the love, care, and sincere prayers they have given me. I can never thank you enough.

To my dear brothers and sisters: Fadl, Osama, Wissam, Diaa, Thaer, Ayat, and Qais, who have always been my source of strength and motivation, pushing me to move forward.

I pray to Allah to grant you all health, happiness, and success, and to make us all among the people of Paradise.

With my sincere love and gratitude to each one of you

## Declaration

I, the undersigned, declare that I submitted the thesis entitled:

### NUMERICAL SOLUTION FOR FREDHOLINTEGRAL EQUATIONS

I declare that the work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

**Student's Name:** Faten Farooq Youssef Khalaf

**Signature:** *Faten Khataf*

**Date:** 8/9/2024

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# NUMERICAL SOLUTION FOR FREDHOLM INTEGRAL EQUATIONS

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## Abstract

The integral Fredholm-Hammerstein integral equation has many applications in various areas of mathematical physics, including heat transfer problems fluid dynamics and quantum mechanics.

In this thesis, we focus on the numerical treatment of the Fredholm-Hammerstein equations using the Galerkin method and the shifted Chebyshev Polynomial method.

To test the efficiency and the accuracy of this method, one numerical example with known exact solution is presented. Numerical results show that the conveyance of this method is in good agreement with the exact solution. Moreover, we conclude that the in Galerkin method and shifted Chebyshev method provides very accurate results.

**Keyword:** Hammerstein Equations; Nonlinear Fredholm Integral Equation; Shifted Chebyshev Polynomials; Taylor Expansion; Galerkin Method.

# Chapter One

## Introduction and Review of Integral Equations

### 1.1 Introduction

The integral equation method is widely used for solving many problems in mathematical physics and engineering such as potential theory, Dirichlet problems, electrostatics, contact problems, astrophysics problems and radiative heat transfer problems [14-23]. these equations appear naturally in inverse problems, fluid dynamics, potential theory and spread of interdependent epidemics, elasticity [1-3].

Integral equations have many types including the Hammerstein-type equation. This type is one of the principal tools in various areas of applied mathematics, physics and engineering that is encountered in a variety of applications in many fields [4]. it arises from the reformulation of boundary value problem associated with partial or ordinary differential equations.

Hammerstein-type integral equations play an important role in physical problems and are often used to reformulate or rewrite mathematical problems. For example, the propagation of mono-frequency acoustic or electromagnetic waves over flat nonhomogeneous terrain [5]. In addition, these equations appear in nonlinear physical phenomena such as electromagnetic fluid dynamics, reformulation of boundary value problems with a nonlinear boundary condition [6-8]. For instance, one of the most important domains of applications of the ideas and methods of nonlinear functional analysis and also the theory of nonlinear operators of monotone type is integral equations of the Fredholm–Hammerstein [7-8].

This equation is defined as following

$$u(t) - \int_a^b k(t,y)[u(y)]^m dy = f(t) \quad (1.1)$$

where  $f(t) \in L^2 [a, b]$  and  $k(t, y) \in L^2 [a, b] \times [a, b]$  are known functions,  $u(t)$  is the unknown function to be determined,  $t \in I = [a, b]$  and  $m > 1$  is a positive integer [9]. Since we use the shifted Chebyshev polynomials and these polynomials are an orthogonal set on the interval  $[0,1]$ , throughout this article, without loss of generality. Numerous numerical methods have been proposed for approximating the solution of

above Fredholm– Hammerstein integral equations [10].Kumar and Sloan [11] have considered the solution of the Fredholm–Hammerstein integral equations and presented a collocation-type method to solve it. Brunner [12] used this method for the numerical solution of nonlinear Volterra integral and integro-differential equations. Elnagar and Kazemi [13] investigated the Chebyshev spectral method to an equivalent equation of nonlinear Volterra–Hammerstein integral equations and discussed some convergence results.

This thesis is organized as follows:

Chapter one defines the integral equations, classifies their types including Volterra integral equations, Fredholm integral equations and Singular integral equations. It also defines the Linearity of integral equations and provides examples of their types. In addition, the chapter touches upon the difference between the homogeneous and non-homogeneous integral equations.

Chapter two contains the types of the Chebyshev Polynomials and provides the shifted Chebyshev Polynomials. It also provides the power of  $x$  in terms of Chebyshev Polynomials. Then, the chapter concludes with the Galerkin method.

Chapter three provides two numerical examples of Fredholm integral equations. The first example assumes  $M = 2, 4, 6$  while the second example assumes  $M = 2, 4$ . Then, the chapter shows the use of Chebyshev Polynomials and the Galerkin method to get a matrix system. This system is solved by Python programming language to get constant values of  $c$  to find the numerical solutions. Then, the difference is found between the exact solutions and the numerical solutions to get the errors.

Chapter four shows the results of the difference between the exact solutions and the numerical solutions.

Many applications in applied mathematics can be represented by an integral equation.

## 1.2 Classifications of Integral Equations

Definition (1.2) [15]: An integral equation is an equation in which the unknown function  $u(x)$  appears under an integral sign.

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} g(x, t)u(t)dt \quad (1.2)$$

Where  $u(x)$  is unknown function,  $f(x)$  is a given function,  $\beta(x)$  and  $\alpha(x)$  are the limits of integration that can either be constants or variables or mixed,  $\lambda$  is a constant parameter,  $g(x, t)$  called the kernel of the integral equation is a function of two variables  $x$  and  $t$ .

### 1.2.1 Types of Integral Equation

#### a. Volterra integral equation [16]:

We introduce the Fredholm integral equation and following formula represent the equation.

$$v(x)u(x) = f(x) + \lambda \int_s^x g(x, t)u(t)dt \quad (1.3)$$

Where  $g(x, t)$  called the kernel of the integral equation linear or nonlinear is a function of two variables  $x$  and  $t$ , limits of integration of variable  $x$  and constant  $s$ .

Formulas of Volterra integral equations take many forms, including:

#### 1. Volterra Integral Equation of The First Kind:

Assum  $v(x) = 0$ , the equation (1.2) written as:

$$f(x) + \lambda \int_s^x g(x, t)u(t)dt = 0 \quad (1.4)$$

The unknown function  $u(x)$  appears only under the integral.

#### 2. Volterra Integral Equation of The Second Kind:

Assum  $v(x) = 1$ , the equation (1.2) written as:

$$u(x) = f(x) + \lambda \int_s^x g(x, t)u(t)dt \quad (1.5)$$

The unknown function  $u(x)$  appears under the integral and outside it as well.

### **b. Fredholm Integral Equation**

We introduce the Fredholm integral equation and following formula represent the equation

$$v(x)u(x) = f(x) + \lambda \int_s^c g(x,t)u(t)dt \quad , s \leq x , t \leq c \quad (1.6)$$

Where  $g(x,t)$  called the kernel of the integral equation linear or nonlinear is a function of two variables  $x$  and  $t$ , Integration limits fixed numbers  $s$  and  $c$ .

### **a. Fredholm Integral Equation of The First Kind**

Assum  $v(x) = 0$ , the equation (1.5) written as:

When  $v(x) = 0$ , then (1.5) becomes

$$f(x) + \lambda \int_s^c g(x,t)u(t)dt = 0 \quad , s \leq x , t \leq c \quad (1.7)$$

The unknown function  $u(x)$  appears only under the integral.

### **3. Fredholm Integral Equation of The Second Kind**

Assum  $v(x) = 1$ , the equation (1.b) written as:

$$u(x) = f(x) + \lambda \int_s^c g(x,t)u(t)dt \quad , s \leq x , t \leq c \quad (1.8)$$

The unknown function  $u(x)$  appears under the integral and outside it as well.

### **c. Volterra-Fredholm Integral Equations**

Formula of Volterra-Fredholm integral equations is given as:

$$u(x) = f(x) + \lambda_1 \int_0^x g_1(x,t)u(t)dt + \lambda_2 \int_s^c g_2(x,t)u(t)dt \quad (1.9)$$

Which is a combination of (1.2) and (1.5), appears in one integral equation.

### **c. Integro-Differential Equations [15, 16]**

The derivative of the unknown function  $u(x)$  appears on one side, where  $r$  is the number of times the derivation and  $u(x)$  is under the integral and appears on the other side, the formula of the previous Integral Equations is given as follows:

### 1. Volterra Integro-Differential Equations

$$u^{(r)}(x) = f(x) + \lambda \int_0^x g(x, t)u(t)dt \quad (1.10)$$

### 2. Fredholm Integro-Differential Equations

$$u^{(r)}(x) = f(x) + \lambda \int_s^c g(x, t)u(t)dt \quad , s \leq x , t \leq c \quad (1.11)$$

### 3. Volterra-Fredholm Integro-Differential Equations

$$u^{(r)}(x) = f(x) + \lambda_1 \int_0^x g_1(x, t)u(t)dt + \lambda_2 \int_s^c g_2(x, t)u(t)dt \quad (1.12)$$

Example of the Volterra-Fredholm integro-differential equations are given by:

$$u'(x) = 24x + x^4 + 3 - \int_0^x (x - t)u(t)dt - \int_0^1 tu(t)dt , u(0) = 0$$

### e. Fredholm–Hammerstein integral equation

A Fredholm-Hammerstein Intergral equation has the standard formula

$$u(t) - \int_a^b k(t, y)[u(y)]^m dy = f(t) \quad (1.13)$$

where  $f(t) \in L^2 [a, b]$  and  $k(t, y) \in L^2 [a, b] \times [a, b]$  are known functions,  $u(t)$  is the unknown function to be determined,  $t \in I = [a, b]$  and  $m > 1$  is a positive integer [9]

#### 1.2.2 Singular Integral Equations

Definition (1.2) [16,15]: An Integral equation is called singular if one of the limits of integration  $\alpha(x)$ ,  $\beta(x)$  or both are infinite or the kernel has singularities within the range of integration, Examples of such singular Integral Equations are given as:

the limits of integration  $\alpha(x)$ ,  $\beta(x)$  both are infinite

$$u(x) = 1 + x^2 + \frac{1}{6} \int_{-\infty}^{\infty} (x + t)u(t)dt \quad (1.14)$$

the lower limit  $\alpha(x)$  of integration are infinite

$$u(x) = x + \frac{1}{3} \int_{-\infty}^0 \cos(x+t) u(t) dt \quad (1.15)$$

The upper limit of integration are Infinite

$$u(x) = 2x + 6 \int_0^{\infty} \sin(x-t) u(t) dt \quad (1.16)$$

weakly singular integral equations

$$f(x) = \int_0^x \frac{1}{(x-t)^\tau} u(t) dt, 0 < \tau < 1 \quad (1.17)$$

### 1.2.3 Linearity of Integral Equations

**Definition** (1.3)[17,18]: The integral is linear if an unknown function  $u(x)$  under the integral is raised to the power of 1, and nonlinear if the unknown function  $u(x)$  is raised to a power other than 1 or a nonlinear function such as  $\sinh u, \cos u, \ln(1 + u), u^2(x), \sin(u(x)), e^{u(x)}$ .

#### Examples of linear integral

$$u(x) = 1 - \int_0^1 (x-t)u(t) dt \quad (1.18)$$

$$u(x) = 1 - \int_0^x (x-t)u(t) dt \quad (1.19)$$

#### Examples of nonlinear integral

$$u(x) = 2x - \int_0^1 (x-t)u^2(t) dt \quad (1.20)$$

$$u(x) = 1 + \int_0^x \cos(x-t)u(t) dt \quad (1.21)$$

### 1.2.4 Homogeneity of integral equations [17,16,15]

The integral equation is called homogeneous if  $f(x) = 0$  otherwise it is nonhomogeneous

#### homogeneous integral equation

$$u(x) = \int_0^1 xu(t) dt \quad (1.22)$$

$$u(x) = \int_0^x (x+t)u(t) dt$$

### nonhomogeneous integral equation

$$u(x) = 2x - \int_0^1 (x-t)u^2(t)dt \quad (1.23)$$

#### 1.2.5 The Existence of a Unique Solution []

Some integral equations has a solution and some other has no solution or that it has an infinite number of solutions, the following theorems state the existence and uniqueness among the solution of Fredholm integral equation of the second kind.

Note: It is important to say that we will discuss the analytical methods in the space  $x = [a, b]$  with  $\|\cdot\|_\infty$ .

#### Theorem 1.1 (Fredholm Alternative Theorem)

If the homogeneous Fredholm integral equation

$$f(x) = \beta \int_a^b K(x, t)f(t)dt \quad (1.24)$$

has always a unique solution. This theorem is known by the Fredholm alternative theorem.

has only the trivial solution  $f(x) = 0$  then the corresponding nonhomogeneous Fredholm equation

$$f(x) = g(x) + \beta \int_a^b K(x, t)f(t)dt \quad (1.25)$$

has always a unique solution. This theorem is known by the Fredholm alternative theorem.

#### Theorem 1.2 (Unique Solution)

If the kernel  $k(x, t)$  in Fredholm integral equation (1.23) is continuous, real valued function, bounded in the square  $a \leq x \leq b$  and  $a \leq t \leq b$ , and if  $g(x)$  is a continuous real valued function, then a necessary condition for the existence of a unique solution for Fredholm integral equation (2.3) is given by

$$|\beta|N(b-a) < 1 \quad (1.26)$$

where

$$|K(x, t)| \leq N \in R \quad (1.27)$$

On the contrary, if the necessary condition (2.3) does not hold, then a continuous solution may exist for Fredholm integral equation. To illustrate this, we consider the Fredholm integral equation

$$f(x) = 2 - 3x + \int_0^1 (3x + t)f(t)dt \quad (1.28)$$

It is clear that  $\beta = 1$ ,  $|K(x, t)| \leq 4$  and  $(b - a) = 1$ . This gives

$$|\beta|N(b - a) = 4 \not\leq 1 \quad (1.29)$$

However, the Fredholm equation (1.26) has an exact solution given by

$$f(x) = 6x \quad (1.30)$$

## Chapter Two

### Numerical methods for solving Fredholm- Hammerstein Integral Equations

#### 2.1 Chebyshev Polynomials

Chebyshev Polynomials are important in numerical analysis and approximation theory in some quadrature rules on these Polynomials [ 28]. There are four types of Chebyshev polynomials.

##### 2.1.1 The First-Kind Polynomial $E_m$

**Definition 2.1** [20.26] The Chebyshev polynomial  $E_m(x)$  of the first kind is a polynomial in  $x$  of degree  $m$ , defined by the relation

$$E_m(x) = \cos m\Phi \text{ when } x = \cos \Phi \quad (2.1)$$

If the variable  $x$  belongs to the period  $[-1,1]$ , then the corresponding variable  $\Phi$  belongs to the period  $[0, \pi]$ , elementary equations:

$$\cos(0\Phi) = 1$$

$$\cos(1\Phi) = \cos \Phi$$

$$\cos(2\Phi) = 2 \cos^2 \Phi - 1$$

$$\cos(3\Phi) = 4 \cos^3 \Phi - 3 \cos \Phi$$

$$\cos(4\Phi) = 8 \cos^4 \Phi - 8 \cos^2 \Phi + 1$$

$$\cos(5\Phi) = 16 \cos^5 \Phi - 20 \cos^3 \Phi + 5 \cos \Phi$$

$$\cos(6\Phi) = 32 \cos^6 \Phi - 48 \cos^4 \Phi + 18 \cos^2 \Phi - 1$$

⋮

We may immediately deduce from equation (2.1), that the first few Chebyshev polynomials are

Let  $m = 0$

$$E_0(x) = \cos(0\Phi) = \cos 0 = 1$$

Let  $m = 1$

$$E_1(x) = \cos(1\Phi)$$

But  $x = \cos\Phi$

$$\therefore E_1(x) = x$$

Let  $m = 2$

$$E_2(x) = \cos(2\Phi) = 2 \cos^2\Phi - 1$$

But  $x = \cos\Phi$

$$E_2(x) = 2x^2 - 1$$

Let  $m = 3$

$$E_3(x) = \cos(3\Phi) = 4 \cos^3\Phi - 3 \cos\Phi$$

But  $x = \cos\Phi$

$$E_3(x) = 4x^3 - 3x$$

Let  $m = 4$

$$\begin{aligned} E_4(x) &= \cos(4\Phi) \\ &= 8 \cos^4\Phi - 8 \cos^2\Phi + 1 \end{aligned}$$

But  $x = \cos\Phi$

$$E_4(x) = 8x^4 - 8x^2 + 1$$

Let  $m = 5$

$$\begin{aligned} E_5(x) &= \cos(5\Phi) \\ &= 16 \cos^5\Phi - 20 \cos^3\Phi + 5 \cos\Phi \end{aligned}$$

But  $x = \cos\Phi$

$$E_5(x) = 16x^2 - 20x^3 + 5x$$

In the same way we find that

$$E_0(x) = 1$$

$$E_1(x) = x$$

$$E_2(x) = 2x^2 - 1$$

$$E_3(x) = 4x^3 - 3x$$

$$E_4(x) = 8x^4 - 8x^2 + 1$$

$$E_5(x) = 16x^2 - 20x^3 + 5x$$

$$E_6(x) = 32x^6 - 48x^4 + 18x^2 - 1 \quad (2.2)$$

⋮

By combining the trigonometric identity we find the general formula for  $E_m(x)$

$$\begin{aligned} \cos(m\Phi) &= \frac{e^{im\Phi} + e^{-im\Phi}}{2} \\ &= \frac{e^{i(m-1)\Phi} + e^{-i(m-1)\Phi}}{2} \cdot (e^{i\Phi}) \\ &= \frac{e^{i(m-1)\Phi} + e^{-i(m-1)\Phi}}{2} \cdot (e^{i\Phi} + e^{i\Phi} - e^{-i\Phi}) \\ &= \frac{e^{i(m-1)\Phi} + e^{-i(m-1)\Phi}}{2} \cdot (e^{i\Phi} + e^{i\Phi}) - \frac{e^{i(m-2)\Phi} + e^{-i(m-2)\Phi}}{2} \end{aligned}$$

$$\cos(m\Phi) = 2 \cos\Phi \cos(m-1)\Phi - \cos(m-2)\Phi$$

With definition (2.1) we find

$$\cos m\Phi = E_m(x)$$

$$\cos \Phi = x$$

For  $m = m - 1$

$$\cos (m - 1)\Phi = E_{m-1}(x)$$

For  $m = m - 2$

$$\cos (m - 2)\Phi = E_{m-2}(x)$$

The general form of the basic recurrence relation

$$E_m(x) = 2xE_{m-1}(x) - E_{m-2}(x), m = 2, 3, \dots, \quad (2.3)$$

which together with the initial conditions

$$E_0(x) = 1$$

$$E_1(x) = x \quad (2.4)$$

### 2.1.2 The Second-Kind Polynomial $Q_m$

**Definition 2.1.2** [21] The Chebyshev polynomial  $Q_m(x)$  of the second kind is a polynomial of degree  $m$  in  $x$  defined by

$$Q_m(x) = \frac{\sin(m + 1)\Phi}{\sin \Phi}, \quad \text{when } x = \cos \Phi \quad (2.5)$$

If the range of the variable  $x$  is the interval  $[-1, 1]$ , then the range of the corresponding variable  $\Phi$  can be taken as  $[0, \pi]$ , elementary formulae:

$$\sin (1\Phi) = \sin \Phi$$

$$\sin (2\Phi) = 2 \sin \Phi \cos \Phi$$

$$\sin (3\Phi) = \sin \Phi (4 \cos^2 \Phi - 1)$$

$$\sin(4\Phi) = \sin \Phi (8 \cos^3 \Phi - 4 \cos \Phi)$$

⋮

So that the ratio of sine functions (2.5) is indeed a polynomial in  $\cos \Phi$ , and we may immediately deduce that

let  $m = 0$

$$Q_0(x) = \frac{\sin(0 + 1)\Phi}{\sin \Phi} = \frac{\sin \Phi}{\sin \Phi} = 1$$

$m = 1$

$$\begin{aligned} Q_1(x) &= \frac{\sin(1 + 1)\Phi}{\sin \Phi} \\ &= \frac{\sin 2\Phi}{\sin \Phi} = \frac{2 \sin \theta \cos \Phi}{\sin \Phi} = 2 \cos \Phi \end{aligned}$$

But  $x = \cos \Phi$

$$\therefore Q_1(x) = 2x$$

In the same way we find that

$$Q_0(x) = 1$$

$$Q_1(x) = 2x$$

$$Q_2(x) = 4x^2 - 1$$

$$Q_3(x) = 8x^3 - 4x$$

$\vdots$

(2.6)

By combining the trigonometric identity we find the general formula for  $Q_m(x)$

$$\sin(m + 1)\Phi = \sin(m\Phi) \cos \Phi + \sin \Phi \cos m\Phi$$

$$\sin(m - 1)\Phi = \sin(m\Phi) \cos \Phi - \sin \Phi \cos m\Phi$$

combine the ends of the two equations to get

$$\sin(m + 1)\Phi + \sin(m - 1)\Phi = 2 \cos \Phi \sin m\Phi \quad (2.7)$$

with Definition (2.1.2), we find the general formula for  $Q_m(x)$

$$x = \cos \Phi$$

$$\sin(m + 1)\Phi = \sin \Phi Q_m(x)$$

For  $m = m - 1$

$$\sin(m)\Phi = \sin \Phi Q_{m-1}(x)$$

For  $m = m - 2$

$$\sin(m - 1)\Phi = \sin \Phi Q_{m-2}(x)$$

We substitute in the trigonometric congruence (2.7)

$$\sin \Phi Q_m(x) + \sin \Phi Q_{m-2}(x) = 2 \cos \Phi \sin \Phi Q_{m-1}(x)$$

general formula for  $Q_m(x)$

$$Q_m(x) = 2xQ_{m-1}(x) - Q_{m-2}(x)$$

$$m = 2, 3, \dots \quad (2.8.1)$$

which together with the initial conditions

$$Q_0(x) = 1$$

$$Q_1(x) = 2x \quad (2.8.2)$$

provides an efficient procedure for generating the polynomials. A similar trigonometric identity

$$\sin(m + 1)\Phi - \sin(m - 1)\Phi = 2 \sin \Phi \cos(m\Phi)$$

leads us to a relationship

$$Q_m(x) - Q_{m-2}(x) = 2E_m(x) \quad (2.9)$$

$$m = 2, 3, \dots$$

### 2.1.3 The Third-Kind Polynomials $N_m$

Definition 2.1.3 [21] The Chebyshev polynomial  $N_m(x)$  of the third type is a polynomial which are related to  $E_m$ , but which have trigonometric definitions involving the half angle  $\frac{\phi}{2}$  (where  $x = \cos \phi$  as before) defined by the relation

$$N_m(x) = \frac{\cos\left(m + \frac{1}{2}\right)\phi}{\cos \frac{1}{2}\phi} \quad (2.10)$$

when  $x = \cos \phi$

If the range of the variable  $x$  is the interval  $[-1, 1]$ , then the range of the corresponding variable  $\phi$  can be taken as  $[0, \pi]$ , formulae :

Let  $m = 0$

$$N_0(x) = \frac{\cos\left(0 + \frac{1}{2}\right)\phi}{\cos \frac{1}{2}\phi} = \frac{\cos \frac{1}{2}\phi}{\cos \frac{1}{2}\phi} = 1$$

$m = 1$

$$\begin{aligned} N_1(x) &= \frac{\cos\left(1 + \frac{1}{2}\right)\phi}{\cos \frac{1}{2}\phi} \\ &= \frac{4\cos^3 \frac{1}{2}\phi - 3\cos \frac{1}{2}\phi}{\cos \frac{1}{2}\phi} \\ &= 4\cos^2 \frac{1}{2}\phi - \phi \\ &= 2\cos\phi - 1 \end{aligned}$$

But  $x = \cos\phi$  become

$$N_1(x) = 2x - 1$$

In the same way we find that

$$N_0(x) = 1$$

$$N_1(x) = 2x - 1$$

$$N_2(x) = 4x^2 - 2x - 1$$

$$N_3(x) = 8x^3 - 4x^2 - 4x + 1$$

⋮

(2.11)

By combining the trigonometric identity we find the general formula for  $N_m(x)$

$$\cos\left(m + \frac{1}{2}\right)\Phi + \cos\left(m - 2 + \frac{1}{2}\right)\Phi = 2 \cos \Phi \cos\left(m - 1 + \frac{1}{2}\right)\Phi$$

with Definition (2.1.3), we find the general formula for  $N_m(x)$

$$\cos\left(m + \frac{1}{2}\right)\Phi = \cos \frac{1}{2} \Phi N_m(x)$$

$$x = \cos \Phi$$

For  $m = m - 1$

$$\cos\left(m - 1 + \frac{1}{2}\right)\Phi = \cos \frac{1}{2} \Phi N_{m-1}(x)$$

For  $m = m - 2$

$$\cos\left(m - 2 + \frac{1}{2}\right)\Phi = \cos \frac{1}{2} \Phi N_{m-2}(x)$$

$$\cos \frac{1}{2} \Phi N_m(x) + \cos \frac{1}{2} \Phi N_{m-2}(x) = 2x \cos \frac{1}{2} \Phi N_{m-1}(x)$$

it immediately follows that

$$V_m(x) = 2xV_{m-1}(x) - V_{m-2}(x), \quad m = 2, 3, 4, 5, \dots$$

which together with the initial conditions

$$N_0(x) = 1$$

$$N_1(x) = 2x - 1$$

### 2.1.4 The Fourth-Kind Polynomials $R_m$

**Definition 2.1.4** [21] The Chebyshev polynomial  $R_m(x)$  of fourth-kind is a polynomial which are related to  $Q_m$ , but which have trigonometric definitions involving the half angle  $\frac{\phi}{2}$  (where  $x = \cos \phi$  as before) defined by the relation

$$R_m(x) = \frac{\sin\left(m + \frac{1}{2}\right)\phi}{\sin \frac{1}{2}\phi} \quad (2.12)$$

If the range of the variable  $x$  is the interval  $[-1, 1]$ , then the range of the corresponding variable  $\phi$  can be taken as  $[0, \pi]$ , formulae :

Let  $m = 0$

$$\begin{aligned} R_0(x) &= \frac{\sin\left(0 + \frac{1}{2}\right)\phi}{\sin \frac{1}{2}\phi} \\ &= \frac{\sin \frac{1}{2}\phi}{\sin \frac{1}{2}\phi} \end{aligned}$$

$$R_0(x) = 1$$

$m = 1$

$$\begin{aligned} R_1(x) &= \frac{\sin\left(1 + \frac{1}{2}\right)\phi}{\sin \frac{1}{2}\phi} \\ &= \frac{3 \sin \frac{1}{2}\phi - 4 \sin^3 \frac{1}{2}\phi}{\sin \frac{1}{2}\phi} \\ &= 3 - 4 \sin^2 \frac{1}{2}\phi \end{aligned}$$

$$= 2 \cos \phi + 1$$

But  $x = \cos\Phi$  become

$$= 2x + 1$$

In the same way we find that

$$R_0(x) = 1$$

$$R_1 = 2x + 1$$

$$R_2(x) = 4x^2 + 2x - 1$$

$$R_3(x) = 8x^3 + 4x^2 - 4x - 1$$

$$\vdots \tag{2.13}$$

By combining the trigonometric identity we find the general formula for  $R_m(x)$

$$\sin\left(m + \frac{1}{2}\right)\Phi + \sin\left(m - 2 + \frac{1}{2}\right)\Phi = 2 \cos \Phi \sin\left(m - 1 + \frac{1}{2}\right)\Phi$$

with Definition (2.1.4), we find the general formula for  $R_m(x)$

$$\sin\left(m + \frac{1}{2}\right)\Phi = \sin\frac{1}{2}\Phi R_m(x)$$

$$x = \cos \Phi$$

For  $m = m - 1$

$$\sin\left(m - 1 + \frac{1}{2}\right)\Phi = \sin\frac{1}{2}\Phi R_{m-1}(x)$$

For  $m = m - 2$

$$\sin\left(m - 2 + \frac{1}{2}\right)\Phi = \sin\frac{1}{2}\Phi R_{m-2}(x)$$

$$\sin\frac{1}{2}\Phi R_m(x) + \sin\frac{1}{2}\Phi R_{m-2}(x) = 2x \sin\frac{1}{2}\Phi R_{m-1}(x)$$

it immediately follows that

$$R_m(x) = 2xR_{m-1}(x) - R_{m-2}(x) , \quad m = 2,3,4,5, \dots$$

which together with the initial conditions

$$R_0(x) = 1$$

$$R_1(x) = 2x + 1$$

## 2.2 Shifted Chebyshev Polynomials

### 2.2.1 The Shifted Polynomials $E_m^*$ , $Q_m^*$ , $N_m^*$ , $R_m^*$

[19,22,28,29] Since the range  $[0, 1]$  is quite often more convenient to use than the range  $[-1, 1]$ , we sometimes map the independent variable  $x$  in  $[0, 1]$  to the variable  $a$  in  $[-1, 1]$  by the transformation

$$a = 2x - 1 \text{ or } x = \frac{1}{2}(1 + a) \quad (2.14)$$

From the previous (2.14) leads to shifted Chebyshev polynomial (of the first kind)

$E_m^*(x)$  of degree  $m$  in  $x$  on  $[0, 1]$  given by

$$E_m^*(x) = E_m(a) = E_m(2x - 1) \quad (2.15)$$

Thus we have new polynomials of equations (2.2)

Let  $m = 0$

$$E_0^*(x) = 1$$

Let  $m = 1$

$$E_1^*(x) = E_1(a) = E_1(2x - 1)$$

$$= 2x - 1$$

Let  $m = 2$

$$E_2^*(x) = E_2(a) = E_2(2x - 1)$$

$$= 2(2x - 1)^2 - 1$$

$$2(4x^2 - 4x + 1) - 1$$

$$= 8x^2 - 8x + 1$$

Let  $m = 3$

$$E_3^*(x) = E_3(a) = E_3(2x - 1)$$

$$= 4(2x - 1)^3 - 3(2x - 1)$$

$$= 4(8x^3 - 12x^2 + 6x - 1) - 6x + 3$$

$$= 32x^3 - 48x^2 + 18x - 1$$

Let  $m = 4$

$$E_4^*(a) = E_4(2x - 1)$$

$$= 8(2x - 1)^4 - 8(2x - 1)^2 + 1$$

$$= 128x^4 - 256x^3 + 160x^2 - 32x + 1$$

In the same way we find that

$$E_0^*(x) = 1$$

$$E_2^*(x) = 8x^2 - 8x + 1$$

$$E_3^*(x) = 32x^3 - 48x^2 + 18x - 1$$

$$E_4^*(x) = 128x^4 - 256x^3 + 160x^2 - 32x + 1$$

$$\vdots \tag{2.16}$$

From the equations (2.16) and (2.15) we deduce the relationship  $E_m^*$  in the form:

$$E_m^*(x) = 2(2x - 1)E_{m-1}^*(x) - E_{m-2}^*(x) \tag{2.16.1}$$

with initial conditions

$$E_0^*(x) = 1$$

$$E_1^*(x) = 2x - 1 \tag{2.16.2}$$

Let  $m = 5$

$$E_5^*(x) = 2(2x - 1)E_{5-1}^*(x) - E_{5-2}^*(x)$$

$$E_5^*(x) = 2(2x - 1)E_4^*(x) - E_3^*(x)$$

$$E_5^*(x) = 2(2x - 1)(128x^4 - 256x^3 + 160x^2 - 32x + 1) - (32x^3 - 48x^2 + 18x - 1)$$

$$E_5^*(x) = 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1$$

Let  $m = 6$

$$E_6^*(x) = 2(2x - 1)E_{6-1}^*(x) - E_{6-2}^*(x)$$

$$E_6^*(x) = 2(2x - 1)E_5^*(x) - E_4^*(x)$$

$$E_6^*(x) = 2(2x - 1)(512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1) - (128x^4 - 256x^3 + 160x^2 - 32x + 1)$$

$$E_6^*(x) = 2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1$$

Let  $m = 7$

$$E_7^*(x) = 2(2x - 1)E_{7-1}^*(x) - E_{7-2}^*(x)$$

$$E_7^*(x) = 2(2x - 1)E_6^*(x) - E_5^*(x)$$

$$E_7^*(x) = (4x - 2)(2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1) - (512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1)$$

$$E_7^*(x) = 8192x^7 - 28672x^6 + 39424x^5 - 26880x^4 + 9408x^3 - 1568x^2 + 98 - 1$$

Let  $m = 8$

$$E_8^*(x) = 2(2x - 1)E_{8-1}^*(x) - E_{8-2}^*(x)$$

$$E_8^*(x) = 2(2x - 1)E_7^*(x) - E_6^*(x)$$

$$E_8^*(x) = (4x - 2)(8192x^7 - 28672x^6 + 39424x^5 - 26880x^4 + 9408x^3 - 1568x^2 + 98x - 1) - (2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1)$$

$$E_8^*(x) = 32768x^8 - 131072x^7 + 212992x^6 - 180224x^5 + 84480x^4 - 21504x^3 + 2688x^2 - 128x + 1$$

Let  $m = 9$

$$E_9^*(x) = 2(2x - 1)E_{9-1}^*(x) - E_{9-2}^*(x)$$

$$E_9^*(x) = 2(2x - 1)E_8^*(x) - E_7^*(x)$$

$$E_9^*(x) = (4x - 2)(32768x^8 - 131072x^7 + 212992x^6 - 180224x^5 + 84480x^4 - 21504x^3 + 2688x^2 - 128x + 1) - (8192x^7 - 28672x^6 + 39424x^5 - 26880x^4 + 9408x^3 - 1568x^2 + 98x - 1)$$

$$= 131072x^9 - 589824x^8 + 1105920x^7 - 1118208x^6 + 658944x^5 - 228096x^4 + 44352x^3 - 4320x^2 + 162x - 1$$

Let  $m = 10$

$$E_{10}^*(x) = 2(2x - 1)E_{10-1}^*(x) - E_{10-2}^*(x)$$

$$E_{10}^*(x) = 2(2x - 1)E_9^*(x) - E_8^*(x)$$

$$E_{10}^*(x) = (4x - 2)(131072x^9 - 589824x^8 + 1105920x^7 - 1118208x^6 + 658944x^5 - 228096x^4 + 44352x^3 - 4320x^2 + 162x - 1) - (32768x^8 - 131072x^7 + 212992x^6 - 180224x^5 + 84480x^4 - 21504x^3 + 2688x^2 - 128x + 1)$$

$$E_{10}^*(x) = 524288x^{10} - 2621440x^9 + 5570560x^8 - 6553600x^7 + 4659200x^6 - 2050048x^5 + 549120x^4 - 84480x^3 + 6600x^2 - 200x + 1$$

Let  $m = 11$

$$E_{11}^*(x) = 2(2x - 1)E_{11-1}^*(x) - E_{11-2}^*(x)$$

$$E_{11}^*(x) = 2(2x - 1)E_{10}^*(x) - E_9^*(x)$$

$$\begin{aligned}
E_{11}^*(x) &= (4x - 2)(524288x^{10} - 2621440x^9 + 5570560x^8 - 6553600x^7 \\
&\quad + 4659200x^6 - 2050048x^5 + 549120x^4 - 84480x^3 + 6600x^2 \\
&\quad - 200x + 1) - (131072x^9 - 589824x^8 + 1105920x^7 - 1118208x^6 \\
&\quad + 658944x^5 - 228096x^4 + 44352x^3 - 4320x^2 + 162x - 1)
\end{aligned}$$

$$\begin{aligned}
E_{11}^*(x) &= 2097152x^{11} - 11534336x^{10} + 27394048x^9 - 36765696x^8 \\
&\quad + 30638080x^7 - 16400384x^6 + 5637632x^5 - 1208064x^4 \\
&\quad + 151008x^3 - 9680x^2 + 242x - 1
\end{aligned}$$

Let  $m = 12$

$$E_{12}^*(x) = 2(2x - 1)E_{12-1}^*(x) - E_{12-2}^*(x)$$

$$E_{12}^*(x) = 2(2x - 1)E_{11}^*(x) - E_{10}^*(x)$$

$$\begin{aligned}
E_{12}^*(x) &= (4x - 2)(2097152x^{11} - 11534336x^{10} + 27394048x^9 - 36765696x^8 \\
&\quad + 30638080x^7 - 16400384x^6 + 5637632x^5 - 1208064x^4 \\
&\quad + 151008x^3 - 9680x^2 + 242x - 1) \\
&\quad - (524288x^{10} - 2621440x^9 + 5570560x^8 - 6553600x^7 \\
&\quad + 4659200x^6 - 2050048x^5 + 549120x^4 - 84480x^3 + 6600x^2 \\
&\quad - 200x + 1)
\end{aligned}$$

$$\begin{aligned}
E_{12}^*(x) &= 8388608x^{12} - 50331648x^{11} \\
&\quad + 132120576x^{10} - 199229440x^9 + 190513152x^8 - 120324096x^7 \\
&\quad + 50692096x^6 - 14057472x^5 + 2471040x^4 - 256256x^3 \\
&\quad + 13728x^2 - 288x + 1
\end{aligned}$$

By the equations (2.1) and (2.13) we derive another special property of the polynomial

$E_m^*(x)$

$$E_{2m}(x) = \cos 2m\Phi = \cos m(2\Phi)$$

$$= E_m(\cos 2\Phi)$$

$$= E_m(2x^2 - 1) = E_m^*(x^2)$$

so that

$$E_{2m}(x) = E_m^*(x^2) \quad (2.17)$$

By the equations (2.1) and (2.17) we get the form

$$E_m^*(x) = \cos 2m\phi \quad \text{when } x = \cos^2 \phi \quad (2.18)$$

We assume that  $\phi = \frac{\varphi}{2}$  to become this relationship in the following form

$$E_m^*(x) = \cos 2m \frac{\varphi}{2}, \text{ when } x = \cos^2 \left( \frac{\varphi}{2} \right)$$

$$E_m^*(x) = \cos m\varphi, \text{ when } x = \cos^2 \frac{\varphi}{2} = \frac{1}{2}(1 + \cos \varphi). \quad (2.19)$$

From the equations (2.15) and (2.19) we get the final Formula

$$E_m^*(x) = E_m(a) = \cos m\varphi$$

$$\text{when } a = 2 \left( \frac{1}{2}(1 + \cos \varphi) \right) - 1$$

$$\therefore a = \cos \varphi \quad (2.20)$$

in precisely analogous ways may be defined Shifted polynomials  $Q_m^*, N_m^*, R_m^*$  of the second, third and fourth kinds [21]

If from (2.14) and (2.6) we get  $Q_m^*$

$$Q_m^*(x) = Q_m(a) = Q_m(2x - 1)$$

let  $m = 0$

$$Q_0^*(x) = Q_0(a)$$

$$= Q_0(2x - 1) = 1$$

let  $m = 1$

$$Q_1^*(x) = Q_1(a) = Q_1(2x - 1)$$

$$= 2(2x - 1) = 4x - 2$$

let  $m = 2$

$$\begin{aligned}Q_2^*(x) &= Q_2(a) = Q_2(2x - 1) \\&= 4(2x - 1)^2 - 1 \\&= 4(4x^2 - 4x - 1) - 1 \\&= 16x^2 - 16x - 5\end{aligned}$$

In the same way we find that

$$Q_0^*(x) = 1$$

$$Q_1^*(x) = 4x - 4$$

$$Q_2^*(x) = 16x^2 - 16x - 5$$

$$Q_3^* = 64x^3 - 96x^2 + 40x - 4$$

$\vdots$

(2.21)

If from (2.14) and (2.11) we get  $N_m^*$

$$N_m^*(x) = N_m(a) = N_m(2x - 1)$$

let  $m = 0$

$$N_0^*(x) = N_0(a) = N_0(2x - 1) = 1$$

let  $m = 1$

$$\begin{aligned}N_1^*(x) &= N_1(a) = N_1(2x - 1) \\&= 2(2x - 1) - 1 \\&= 4x - 3\end{aligned}$$

In the same way we find that

$$N_0^*(x) = 1$$

$$\begin{aligned}
N_1^*(x) &= 4x - 3 \\
N_2^*(x) &= 16x^2 - 20x - 3 \\
&\vdots
\end{aligned} \tag{2.22}$$

If from (2.14) and (2.13) we get  $F_m^*$

$$F_m^*(x) = F_m(a) = F_m(2x - 1)$$

let  $m = 0$

$$F_0^*(x) = F_0(a) = F_0(2x - 1) = 1$$

let  $m = 1$

$$\begin{aligned}
F_1^*(x) &= F_1(a) = F_1(2x - 1) \\
&= 2(2x - 1) + 1 \\
&= 4x - 1
\end{aligned}$$

In the same way we find that

$$\begin{aligned}
F_0^*(x) &= 1 \\
F_1^*(x) &= 4x - 1 \\
F_2^*(x) &= 16x^2 - 12x + 1 \\
&\vdots
\end{aligned} \tag{2.23}$$

## 2.3 Powers of $x$ in terms of Chebyshev Polynomials

### 2.3.1 Powers of $x$ in terms of $\{E_m(x)\}$ [21]

In the section we wanted to refer to  $x$  and  $x^m$  and to derive for formula we refer to

To express  $x = \cos\Phi$  using the equations (2.24.2) and (2.25.2) we find that

$$2 \cos \Phi = e^{i\Phi} + e^{-i\Phi}$$

We add the power  $m$  to both sides of the previous equation to become

$$(2 \cos \Phi)^m = (e^{i\Phi} + e^{-i\Phi})^m$$

Or

$$(2)^m (\cos \Phi)^m = (e^{i\Phi} + e^{-i\Phi})^m \quad (2.26)$$

We find  $((e^{i\Phi} + e^{-i\Phi})^m)$  using the equation (2.24.2)

$$(e^{i\Phi} + e^{-i\Phi})^m = \sum_{n=0}^m \binom{m}{n} (e^{i\Phi})^{m-n} (e^{-i\Phi})^n \quad (2.27)$$

For  $n = 0$

$$\begin{aligned} &= \binom{m}{0} (e^{i\Phi})^m (e^{-i\Phi})^0 \\ &= e^{im\Phi} \end{aligned}$$

For  $n = 1$

$$\begin{aligned} &= \binom{m}{1} (e^{i\Phi})^{m-1} (e^{-i\Phi})^1 \\ &= \binom{m}{1} e^{(m-2)i\Phi} \end{aligned}$$

For  $n = 2$

$$\begin{aligned} \sum_{n=2}^m \binom{m}{n} (e^{i\Phi})^{m-n} (e^{-i\Phi})^n &= \binom{m}{2} (e^{i\Phi})^{m-2} (e^{-i\Phi})^2 \\ &= \binom{m}{2} e^{(m-4)i\Phi} \end{aligned}$$

For  $n = 3$

$$\begin{aligned} &= \binom{m}{3} (e^{i\Phi})^{m-3} (e^{-i\Phi})^3 \\ &= \binom{m}{3} e^{(m-6)i\Phi} \end{aligned}$$

⋮

For  $n = m - 3$

$$\begin{aligned} &= \binom{m}{m-3} (e^{i\Phi})^{m-(m-3)} (e^{-i\Phi})^{m-3} \\ &= \binom{m}{m-3} e^{3i\Phi} e^{-(m-3)i\Phi} \\ &= \binom{m}{m-3} (e^{-(m-6)i\Phi}) \end{aligned}$$

For  $n = m - 2$

$$\begin{aligned} &= \binom{m}{m-2} (e^{i\Phi})^{m-(m-2)} (e^{-i\Phi})^{(m-2)} \\ &= \binom{m}{m-2} e^{2i\Phi} e^{(2-m)i\Phi} \\ &= \binom{m}{m-2} e^{-(m-4)i\Phi} \end{aligned}$$

For  $n = m - 1$

$$\begin{aligned} &= \binom{m}{m-1} (e^{i\Phi})^{m-(m-1)} (e^{-i\Phi})^{(m-1)} \\ &= \binom{m}{m-1} e^{i\Phi} e^{-(m-1)i\Phi} \\ &= \binom{m}{m-1} e^{-(m-2)i\Phi} \end{aligned}$$

For  $n = m$

$$\begin{aligned} &= \binom{m}{m} (e^{i\Phi})^{m-m} (e^{-i\Phi})^m \\ &= e^{-mi\Phi} \end{aligned}$$

By summing up the previous terms, the equation becomes (2.27)

$$(e^{i\Phi} + e^{-i\Phi})^m = e^{im\Phi} + \binom{m}{1} e^{(m-2)i\Phi} + \binom{m}{2} e^{(m-4)i\Phi}$$

$$\begin{aligned}
& + \binom{m}{3} (e^{(m-6)i\Phi}) + \dots + \binom{m}{m-3} (e^{-(m-6)i\Phi}) \\
& + \binom{m}{m-2} e^{-(m-4)i\Phi} + \binom{m}{m-1} e^{-(m-2)i\Phi} + e^{-mi\Phi} \quad (2.28)
\end{aligned}$$

From the equations (2.24.4) and (2.28), we find that

$$\binom{m}{1} = \binom{m}{m-1}$$

$$\binom{m}{2} = \binom{m}{m-2}$$

$$\binom{m}{3} = \binom{m}{m-3}$$

⋮

By grouping terms with equal coefficients

$$\begin{aligned}
(e^{i\Phi} + e^{-i\Phi})^m &= e^{im\Phi} + \binom{m}{1} e^{(m-2)i\Phi} + \binom{m}{2} e^{(m-4)i\Phi} \\
&+ \binom{m}{3} (e^{(m-6)i\Phi}) + \dots + \binom{m}{3} (e^{-(m-6)i\Phi}) + \binom{m}{2} e^{-(m-4)i\Phi} + \binom{m}{1} e^{-(m-2)i\Phi} + e^{-mi\Phi}
\end{aligned}$$

or

$$\begin{aligned}
(e^{i\Phi} + e^{-i\Phi})^m &= (e^{im\Phi} + e^{-mi\Phi}) + \binom{m}{1} (e^{(m-2)i\Phi} + e^{-(m-2)i\Phi}) + \binom{m}{2} (e^{(m-4)i\Phi} \\
&+ e^{-(m-4)i\Phi}) + \binom{m}{3} (e^{(m-6)i\Phi} + e^{-(m-6)i\Phi}) + \dots \\
&= (e^{i\Phi} + e^{-i\Phi})^m + \binom{m}{1} (e^{i\Phi} + e^{-i\Phi})^{(m-2)} + \binom{m}{2} (e^{i\Phi} + e^{-i\Phi})^{(m-4)} \\
&+ \binom{m}{3} (e^{i\Phi} + e^{-i\Phi})^{(m-6)} + \dots \quad (2.29)
\end{aligned}$$

The number of such terms will be

$$\frac{m}{2} + 1$$

We deduce from (2.26) and, (2.29)

$$2^{m-1} \cos \Phi^m = \sum_{n=0}^{\frac{m}{2}} \binom{m}{n} \cos(m-2n)\Phi$$

Hence, from the definition (2.1.1) of  $E_m(x)$

$$x^m = 2^{-(m-1)} \sum_{n=0}^{\frac{m}{2}} \binom{m}{n} E_{(m-2n)}(x) \quad (2.30)$$

The first few powers are

For  $m = 0$

$$x^0 = 1 = E_0(x)$$

For  $m = 1$

$$x = E_1(x)$$

For  $m = 2$

$$\begin{aligned} x^2 &= 2^{-(2-1)} \sum_{n=0}^{\frac{2}{2}} \binom{2}{n} E_{(2-2n)}(x) \\ &= 2^{-1} \sum_{n=0}^1 \binom{2}{n} E_{(2-2n)}(x) \\ &= \frac{1}{2} \left( \binom{2}{0} E_2(x) + \binom{2}{1} E_0(x) \right) \\ &= \frac{1}{2} E_2(x) + E_0(x) \end{aligned}$$

For  $m = 3$

$$\begin{aligned}
x^3 &= 2^{-(3-1)} \sum_{n=0}^{\frac{3}{2}} \binom{3}{n} E_{(3-2n)}(x) \\
&= 2^{-2} \sum_{n=0}^{\frac{1}{2}} \binom{3}{n} E_{(3-2n)}(x) \\
&= \frac{1}{4} \left( \binom{2}{0} E_3(x) + \binom{3}{1} E_1(x) \right) \\
&= \frac{1}{4} E_3(x) + \frac{3}{4} E_1(x)
\end{aligned}$$

For  $m = 4$

$$\begin{aligned}
x^4 &= 2^{-(4-1)} \sum_{n=0}^{\frac{4}{2}} \binom{4}{n} E_{(4-2n)}(x) \\
&= 2^{-3} \sum_{n=0}^2 \binom{4}{n} E_{(4-2n)}(x) \\
&= 2^{-3} \left( \binom{4}{0} E_4(x) + \binom{4}{1} E_2(x) + \frac{1}{2} \binom{4}{2} E_0(x) \right) \\
&= \frac{1}{8} (E_4(x) + 4E_2(x) + 3E_0(x))
\end{aligned}$$

### 2.3.2 Powers of $x$ in terms of $\{E_m^*(x)\}$

[25,26,27] We have the following formula in terms of shifted Chebyshev polynomials for  $x^n$

$$x^m = 2^{-(2m-1)} \left\{ \frac{1}{2} \binom{2m}{m} E_0^*(x) + \sum_{n=0}^{m-1} \binom{2m}{n} E_{(m-n)}^*(x) \right\}, m \geq 1 \quad (2.31)$$

The first few powers are

For  $m = 1$

$$x^1 = 2^{-(2(1)-1)} \left\{ \frac{1}{2} \binom{2(1)}{1} E_0^*(x) + \sum_{n=0}^{1-1} \binom{2(1)}{n} E_{(1-n)}^*(x) \right\}$$

$$x^1 = 2^{-(1)} \left\{ \frac{1}{2} \binom{2}{1} E_0^*(x) + \sum_{n=0}^0 \binom{2}{n} E_{(1-n)}^*(x) \right\}$$

$$= \frac{1}{2} \left( \binom{2}{0} E_1^*(x) + \binom{2}{1} E_0^*(x) \right)$$

$$= \frac{1}{2} E_1^*(x) + \frac{1}{2} E_0^*(x)$$

For  $m = 2$

$$x^2 = 2^{-(2(2)-1)} \left\{ \frac{1}{2} \binom{2(2)}{2} E_0^*(x) + \sum_{n=0}^{2-1} \binom{2(2)}{n} E_{(2-n)}^*(x) \right\}$$

$$x^2 = 2^{-(3)} \left\{ \frac{1}{2} \binom{4}{2} E_0^*(x) + \sum_{n=0}^1 \binom{4}{n} E_{(2-n)}^*(x) \right\}$$

$$= \frac{1}{8} \left( \binom{4}{0} E_2^*(x) + \binom{4}{1} E_1^*(x) + \frac{1}{2} \binom{4}{2} E_0^*(x) \right)$$

$$= \frac{1}{8} E_2^*(x) + \frac{4}{8} E_1^*(x) + \frac{3}{8} E_0^*(x)$$

For  $m = 3$

$$x^3 = 2^{-(2(3)-1)} \left\{ \frac{1}{2} \binom{2(3)}{4} E_0^*(x) + \sum_{n=0}^{3-1} \binom{2(3)}{n} E_{(3-n)}^*(x) \right\}$$

$$x^3 = 2^{-(6-1)} \left\{ \frac{1}{2} \binom{6}{3} E_0^*(x) + \sum_{n=0}^2 \binom{6}{n} E_{(3-n)}^*(x) \right\}$$

$$= \frac{1}{32} \left( \binom{6}{0} E_3^*(x) + \binom{6}{1} E_2^*(x) + \binom{6}{2} E_1^*(x) + \frac{1}{2} \binom{6}{3} E_0^*(x) \right)$$

$$= \frac{1}{32} E_3^*(x) + \frac{6}{32} E_2^*(x) + \frac{15}{32} E_1^*(x) + \frac{10}{32} E_0^*(x)$$

For  $m = 4$

$$x^4 = 2^{-(2(4)-1)} \left\{ \frac{1}{2} \binom{2(4)}{4} E_0^*(x) + \sum_{n=0}^{4-1} \binom{2(4)}{n} E_{(4-n)}^*(x) \right\}$$

$$x^4 = 2^{-(7)} \left\{ \frac{1}{2} \binom{8}{4} E_0^*(x) + \sum_{n=0}^3 \binom{8}{n} E_{(4-n)}^*(x) \right\}$$

$$= \frac{1}{128} \left( \binom{8}{0} E_4^*(x) + \binom{8}{1} E_3^*(x) + \binom{8}{2} E_2^*(x) + \binom{8}{3} E_1^*(x) + \frac{1}{2} \binom{8}{4} E_0^*(x) \right)$$

$$= \frac{1}{128} E_4^*(x) + \frac{8}{128} E_3^*(x) + \frac{28}{128} E_2^*(x) + \frac{56}{128} E_1^*(x) + \frac{35}{128} E_0^*(x)$$

For  $m = 5$

$$x^5 = 2^{-(2(5)-1)} \left\{ \frac{1}{2} \binom{2(5)}{5} E_0^*(x) + \sum_{n=0}^{5-1} \binom{2(5)}{n} E_{(6-n)}^*(x) \right\}$$

$$x^5 = 2^{-(10-1)} \left\{ \frac{1}{2} \binom{10}{5} E_0^*(x) + \sum_{n=0}^4 \binom{10}{n} E_{(5-n)}^*(x) \right\}$$

$$= \frac{1}{512} \left( \binom{10}{0} E_5^*(x) + \binom{10}{1} E_4^*(x) + \binom{10}{2} E_3^*(x) + \binom{10}{3} E_2^*(x) + \binom{10}{4} E_1^*(x) \right. \\ \left. + \frac{1}{2} \binom{10}{5} E_0^*(x) \right)$$

$$= \frac{1}{512} E_5^*(x) + \frac{10}{512} E_4^*(x) + \frac{45}{512} E_3^*(x) + \frac{120}{512} E_2^*(x) + \frac{210}{512} E_1^*(x) + \frac{126}{512} E_0^*(x)$$

For  $m = 6$

$$x^6 = 2^{-(2(6)-1)} \left\{ \frac{1}{2} \binom{2(6)}{6} E_0^*(x) + \sum_{n=0}^{6-1} \binom{2(6)}{n} E_{(6-n)}^*(x) \right\}$$

$$x^6 = 2^{-(11)} \left\{ \frac{1}{2} \binom{12}{6} E_0^*(x) + \sum_{n=0}^5 \binom{12}{n} E_{(6-n)}^*(x) \right\}$$

$$\begin{aligned}
&= \frac{1}{2048} \left( \binom{12}{0} E_6^*(x) + \binom{12}{1} E_5^*(x) + \binom{12}{2} E_4^*(x) + \binom{12}{3} E_3^*(x) + \binom{12}{4} E_2^*(x) \right. \\
&\quad \left. + \binom{12}{5} E_1^*(x) + \frac{1}{2} \binom{12}{6} E_0^*(x) \right) \\
&= \frac{1}{2048} E_6^*(x) + \frac{12}{2048} E_5^*(x) + \frac{66}{2048} E_4^*(x) + \frac{220}{2048} E_3^*(x) + \frac{495}{2048} E_2^*(x) \\
&\quad + \frac{792}{2048} E_1^*(x) + \frac{462}{2048} E_0^*(x)
\end{aligned}$$

For  $m = 7$

$$\begin{aligned}
x^7 &= 2^{-(2(7)-1)} \left\{ \frac{1}{2} \binom{2(7)}{7} E_0^*(x) + \sum_{n=0}^{7-1} \binom{2(7)}{n} E_{(7-n)}^*(x) \right\} \\
x^7 &= 2^{-(13)} \left\{ \frac{1}{2} \binom{14}{7} E_0^*(x) + \sum_{n=0}^6 \binom{14}{n} E_{(7-n)}^*(x) \right\} \\
&= \frac{1}{8192} \left( \binom{14}{0} E_7^*(x) + \binom{14}{1} E_6^*(x) + \binom{14}{2} E_5^*(x) + \binom{14}{3} E_4^*(x) + \binom{14}{4} E_3^*(x) \right. \\
&\quad \left. + \binom{14}{5} E_2^*(x) + \binom{14}{6} E_1^*(x) + \frac{1}{2} \binom{14}{7} E_0^*(x) \right) \\
&= \frac{1}{8192} E_7^*(x) + \frac{14}{8192} E_6^*(x) + \frac{91}{8192} E_5^*(x) + \frac{364}{8192} E_4^*(x) + \frac{1001}{8192} E_3^*(x) \\
&\quad + \frac{2002}{8192} E_2^*(x) + \frac{3003}{8192} E_1^*(x) + \frac{1716}{8192} E_0^*(x)
\end{aligned}$$

For  $m = 8$

$$\begin{aligned}
x^8 &= 2^{-(2(8)-1)} \left\{ \frac{1}{2} \binom{2(8)}{8} E_0^*(x) + \sum_{n=0}^{8-1} \binom{2(8)}{n} E_{(8-n)}^*(x) \right\} \\
x^8 &= 2^{-(15)} \left\{ \frac{1}{2} \binom{16}{8} E_0^*(x) + \sum_{n=0}^7 \binom{16}{n} E_{(8-n)}^*(x) \right\} \\
&= \frac{1}{32768} \left( \binom{16}{0} E_8^*(x) + \binom{16}{1} E_7^*(x) + \binom{16}{2} E_6^*(x) + \binom{16}{3} E_5^*(x) + \binom{16}{4} E_4^*(x) \right. \\
&\quad \left. + \binom{16}{5} E_3^*(x) + \binom{16}{6} E_2^*(x) + \binom{16}{7} E_1^*(x) + \frac{1}{2} \binom{16}{8} E_0^*(x) \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{32768} E_8^*(x) + \frac{16}{32768} E_7^*(x) + \frac{120}{32768} E_6^*(x) + \frac{560}{32768} E_5^*(x) + \frac{1820}{32768} E_4^*(x) \\
&\quad + \frac{4368}{32768} E_3^*(x) + \frac{8008}{32768} E_2^*(x) + \frac{11440}{32768} E_1^*(x) + \frac{6435}{32768} E_0^*(x)
\end{aligned}$$

For  $m = 9$

$$x^9 = 2^{-(2(9)-1)} \left\{ \frac{1}{2} \binom{2(9)}{9} E_0^*(x) + \sum_{n=0}^{9-1} \binom{2(9)}{n} E_{(9-n)}^*(x) \right\}$$

$$x^9 = 2^{-(17)} \left\{ \frac{1}{2} \binom{18}{9} E_0^*(x) + \sum_{n=0}^8 \binom{18}{n} E_{(9-n)}^*(x) \right\}$$

$$\begin{aligned}
&= \frac{1}{131072} \left( \binom{18}{0} E_9^*(x) + \binom{18}{1} E_8^*(x) + \binom{18}{2} E_7^*(x) + \binom{18}{3} E_6^*(x) + \binom{18}{4} E_5^*(x) \right. \\
&\quad \left. + \binom{18}{5} E_4^*(x) + \binom{18}{6} E_3^*(x) + \binom{18}{7} E_2^*(x) + \binom{18}{8} E_1^*(x) \right. \\
&\quad \left. + \frac{1}{2} \binom{18}{9} E_0^*(x) \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{131072} E_9^*(x) + \frac{18}{131072} E_8^*(x) + \frac{153}{131072} E_7^*(x) + \frac{816}{131072} E_6^*(x) \\
&\quad + \frac{3060}{131072} E_5^*(x) + \frac{8560}{131072} E_4^*(x) + \frac{18564}{131072} E_3^*(x) + \frac{31824}{131072} E_2^*(x) \\
&\quad + \frac{43758}{131072} E_1^*(x) + \frac{24310}{131072} E_0^*(x)
\end{aligned}$$

For  $m = 10$

$$x^{10} = 2^{-(2(10)-1)} \left\{ \frac{1}{2} \binom{2(10)}{10} E_0^*(x) + \sum_{n=0}^{10-1} \binom{2(10)}{n} E_{(10-n)}^*(x) \right\}$$

$$x^{10} = 2^{-(19)} \left\{ \frac{1}{2} \binom{20}{10} E_0^*(x) + \sum_{n=0}^9 \binom{20}{n} E_{(10-n)}^*(x) \right\}$$

$$\begin{aligned}
&= \frac{1}{524288} \left( \binom{20}{0} E_{10}^*(x) + \binom{20}{1} E_9^*(x) + \binom{20}{2} E_8^*(x) + \binom{20}{3} E_7^*(x) + \binom{20}{4} E_6^*(x) \right. \\
&\quad + \binom{20}{5} E_5^*(x) + \binom{20}{6} E_4^*(x) + \binom{20}{7} E_3^*(x) + \binom{20}{8} E_2^*(x) \\
&\quad \left. + \binom{20}{9} E_1^*(x) + \frac{1}{2} \binom{20}{10} E_0^*(x) \right) \\
&= \frac{1}{524288} E_{10}^*(x) + \frac{20}{524288} E_9^*(x) + \frac{190}{524288} E_8^*(x) + \frac{1140}{524288} E_7^*(x) \\
&\quad + \frac{4845}{524288} E_6^*(x) + \frac{15504}{524288} E_5^*(x) + \frac{38760}{524288} E_4^*(x) + \frac{77520}{524288} E_3^*(x) \\
&\quad + \frac{125970}{524288} E_2^*(x) + \frac{167960}{524288} E_1^*(x) + \frac{92378}{524288} E_0^*(x)
\end{aligned}$$

For  $m = 11$

$$\begin{aligned}
x^{11} &= 2^{-(2(11)-1)} \left\{ \frac{1}{2} \binom{2(11)}{11} E_0^*(x) + \sum_{n=0}^{11-1} \binom{2(11)}{n} E_{(11-n)}^*(x) \right\} \\
x^{11} &= 2^{-(21)} \left\{ \frac{1}{2} \binom{22}{11} E_0^*(x) + \sum_{n=0}^{10} \binom{22}{n} E_{(11-n)}^*(x) \right\} \\
&= \frac{1}{2097152} \left( \binom{22}{0} E_{11}^*(x) + \binom{22}{1} E_{10}^*(x) + \binom{22}{2} E_9^*(x) + \binom{22}{3} E_8^*(x) + \binom{22}{4} E_7^*(x) \right. \\
&\quad + \binom{22}{5} E_6^*(x) + \binom{22}{6} E_5^*(x) + \binom{22}{7} E_4^*(x) + \binom{22}{8} E_3^*(x) \\
&\quad \left. + \binom{22}{9} E_2^*(x) + \binom{22}{10} E_1^*(x) + \frac{1}{2} \binom{22}{11} E_0^*(x) \right) \\
&= \frac{1}{2097152} E_{11}^*(x) + \frac{22}{2097152} E_{10}^*(x) + \frac{231}{2097152} E_9^*(x) + \frac{1540}{2097152} E_8^*(x) \\
&\quad + \frac{7315}{2097152} E_7^*(x) + \frac{26334}{2097152} E_6^*(x) + \frac{74613}{2097152} E_5^*(x) \\
&\quad + \frac{17544}{2097152} E_4^*(x) + \frac{319770}{2097152} E_3^*(x) + \frac{497420}{2097152} E_2^*(x) \\
&\quad + \frac{646646}{2097152} E_1^*(x) + \frac{352716}{2097152} E_0^*(x)
\end{aligned}$$

For  $m = 12$

$$\begin{aligned}
x^{12} &= 2^{-(2(12)-1)} \left\{ \frac{1}{2} \binom{2(12)}{12} E_0^*(x) + \sum_{n=0}^{12-1} \binom{2(12)}{n} E_{(12-n)}^*(x) \right\} \\
x^{12} &= 2^{-(23)} \left\{ \frac{1}{2} \binom{24}{12} E_0^*(x) + \sum_{n=0}^{11} \binom{24}{n} E_{(12-n)}^*(x) \right\} \\
&= \frac{1}{8388608} \left( \binom{24}{0} E_{12}^*(x) + \binom{24}{1} E_{11}^*(x) + \binom{24}{2} E_{10}^*(x) + \binom{24}{3} E_9^*(x) + \binom{24}{4} E_8^*(x) \right. \\
&\quad + \binom{24}{5} E_7^*(x) + \binom{24}{6} E_6^*(x) + \binom{24}{7} E_5^*(x) + \binom{24}{8} E_4^*(x) \\
&\quad \left. + \binom{24}{9} E_3^*(x) + \binom{24}{10} E_2^*(x) + \binom{24}{11} E_1^*(x) + \frac{1}{2} \binom{24}{12} E_0^*(x) \right) \\
&= \frac{1}{8388608} E_{12}^*(x) + \frac{24}{8388608} E_{11}^*(x) + \frac{276}{8388608} E_{10}^*(x) + \frac{2024}{8388608} E_9^*(x) \\
&\quad + \frac{10626}{8388608} E_8^*(x) + \frac{42504}{8388608} E_7^*(x) + \frac{134596}{8388608} E_6^*(x) \\
&\quad + \frac{346104}{8388608} E_5^*(x) + \frac{735471}{8388608} E_4^*(x) + \frac{1307504}{8388608} E_3^*(x) \\
&\quad + \frac{1961256}{8388608} E_2^*(x) + \frac{2496144}{8388608} E_1^*(x) + \frac{1352078}{8388608} E_0^*(x)
\end{aligned}$$

## 2.4 Galerkin Method [25]

The remaining weighted method, the basic idea of which is can be approximated the unknown function  $u(t)$  in the sum of  $m + 1$  trail functions  $b_i(t)$  If applied to Equation (1.e) we get

$$u(t) \approx u_{app} = \sum_{i=0}^m c_i b_i(t) \quad (2.32)$$

And by assuming that the function  $u_{app}$  is a linear combination of  $b_i(t)$ , its xpansion coefficients  $c_i$  are to be determined uniquely. Substituting the approximate solution given by equation (2.32) into equation (1.e), the result is the residual function defined by

$$E(t, c_i) = L[u_{app}] - f(t) \quad (2.33)$$

Where

$$L[u_{app}] = u_{app} - \int_a^b k(t, y)[u(y)]^n dy \quad (2.34)$$

Since the residual function is identically equal to zero for the exact solution, the challenge is to choose the coefficients  $c_i$  so that the residual function is minimized. In integral form, this can be achieved with the following condition

$$\int_a^b w_i(t)E(t, c_i)dt = 0 \quad (2.35)$$

where  $w_i(t)$  is the weight function. The Galerkin approach makes the residual  $E(t, c_i)$  orthogonal to  $m + 1$ , given the independent function on the domain  $[a, b]$ . In this approach, the weighting function is chosen to be identical to the trial functions.

Main results

Consider the approximate solution given by equation (2.32) which is  $u_i(t) = t^i$ , that is

$$u_{app} = \sum_{i=0}^m c_i t^i \quad (2.36)$$

Then, by equation (1.e), we have

$$\kappa(y, u(y)) = [u_{app}]^n = \left[ \sum_{i=0}^m c_i y^i \right]^n \quad (2.37)$$

Can be written by Taylor expansion from  $\kappa(t, y)$  about zero

$$\begin{aligned} \kappa(y, u(y)) &= \kappa(y=0, u(y)) + \left( \frac{\partial}{\partial y} \kappa(y=0, u(y)) \right) y \\ &+ \left( \frac{1}{2!} \frac{\partial^2}{\partial y^2} \kappa(y=0, u(y)) \right) y^2 + \dots + \left( \frac{1}{m!} \frac{\partial^m}{\partial y^m} \kappa(y=0, u(y)) \right) y^m \\ &+ \dots + \left( \frac{1}{\mathfrak{D}!} \frac{\partial^{\mathfrak{D}}}{\partial y^{\mathfrak{D}}} \kappa(y=0, u(y)) \right) y^{\mathfrak{D}} \end{aligned}$$

Where  $\mathfrak{D} = m.n$  then, we have

$$\begin{aligned} \kappa(y, u(y)) &= \\ &([\sum_{i=0}^m c_i y^i]^n)_{y=0} + \left( \frac{\partial}{\partial y} [\sum_{i=0}^m c_i y^i]^n \right)_{y=0} y + \left( \frac{1}{2!} \frac{\partial^2}{\partial y^2} [\sum_{i=0}^m c_i y^i]^n \right)_{y=0} y^2 + \end{aligned}$$

$$\begin{aligned} & \dots \left( \frac{1}{m!} \frac{\partial^m}{\partial y^m} \left[ \sum_{i=0}^m c_i y^i \right]^n \right)_{y=0} y^m + \dots + \\ & \left( \frac{1}{\mathfrak{D}!} \frac{\partial^{\mathfrak{D}}}{\partial y^{\mathfrak{D}}} \left[ \sum_{i=0}^m c_i y^i \right]^n \right)_{y=0} y^{\mathfrak{D}} \end{aligned} \quad (2.38)$$

Moreover, since

$$\left( \frac{\partial^a}{\partial y^a} \left[ \sum_{i=a+1}^m c_i y^i \right]^n \right)_{y=0} = 0 \quad (2.39)$$

We obtain the following expansion for the above relation

$$\begin{aligned} \mathfrak{n}(y, u(y)) &= ([c_0]^n) + \left( \frac{\partial}{\partial y} [c_0 + c_1 y]^n \right)_{y=0} y + \left( \frac{1}{2!} \frac{\partial^2}{\partial y^2} \left[ \sum_{i=0}^2 c_i y^i \right]^n \right)_{y=0} y^2 \\ &+ \left( \frac{1}{3!} \frac{\partial^3}{\partial y^3} \left[ \sum_{i=0}^3 c_i y^i \right]^n \right)_{y=0} y^3 + \left( \frac{1}{4!} \frac{\partial^4}{\partial y^4} \left[ \sum_{i=0}^4 c_i y^i \right]^n \right)_{y=0} y^4 \\ &+ \dots \left( \frac{1}{m!} \frac{\partial^m}{\partial y^m} \left[ \sum_{i=0}^m c_i y^i \right]^n \right)_{y=0} y^m \\ &+ \left( \frac{1}{(m+1)!} \frac{\partial^{m+1}}{\partial y^{m+1}} \left[ \sum_{i=0}^m c_i y^i \right]^n \right)_{y=0} y^{m+1} + \dots \\ &+ \left( \frac{1}{\mathfrak{D}!} \frac{\partial^{\mathfrak{D}}}{\partial y^{\mathfrak{D}}} \left[ \sum_{i=0}^m c_i y^i \right]^n \right)_{y=0} y^{\mathfrak{D}} \end{aligned}$$

Therefore, we can write

$$[u_{app}]^n = \sum_{i=0}^{\mathfrak{D}} v_i y^i$$

Where

$$v_a = \left\{ \begin{array}{l} \left( \frac{1}{a!} \frac{\partial^a}{\partial y^a} \left[ \sum_{i=0}^a c_i y^i \right]^n \right)_{y=0}, \text{ for } a = 0, 1, \dots, m \\ \left( \frac{1}{a!} \frac{\partial^a}{\partial y^a} \left[ \sum_{i=0}^m c_i y^i \right]^n \right)_{y=0}, \text{ for } a = m + 1, \dots, \mathfrak{D} \end{array} \right\} \quad (2.40)$$

Now, by transformations of orthogonal polynomials based on formula (2.31), we will obtain an efficient method to solve equation (1.e). This method is as follows

Algorithm

Step 1. Choose  $u_{app} = \sum_{i=0}^m c_i t^i$  and  $[u_{app}]^n = \sum_{i=0}^{\mathfrak{D}} v_i y^i$

Step 2. Use formula (2.31) and set  $u_{app} = \sum_{i=0}^m \alpha_i E_i^*(t)$

And  $[u_{app}]^n = \sum_{i=0}^{\mathfrak{D}} \beta_i E_m^*(y)$

Step 3 Apply Galerkin method and solve the following nonlinear:

$$\gamma_j \alpha_j - \sum_{i=0}^{\mathfrak{D}} \beta_i K_{i,j} = f_j, \quad j = 0, \dots, m, \quad i = 0, \dots, \mathfrak{D} \quad (2.41)$$

Where

$$f_j = \int_0^1 \omega(t) f(t) E_j^*(t) dt \quad (2.42)$$

$$K_{i,j} = \int_0^1 \int_0^1 \omega(t) k(t, y) E_i^*(y) E_j^*(t) dy dt \quad (2.43)$$

$$\gamma_j = \begin{cases} \pi, & j = 0 \\ \frac{\pi}{2}, & j \neq 0 \end{cases} \quad (2.44)$$

$$\omega(t) = \frac{1}{\sqrt{x-x^2}}, \text{ (weight function on the interval } [0,1]) \quad (2.45)$$

## Chapter Three

### Numerical Examples and Results

In this chapter, two numerical examples of Fredholm-Hammerstein integral equations are provided. The first example assumes  $M=2, 4, 6$  while the second example assumes  $M=2, 4$ . Then, Chebyshev Polynomials and the Galerkin method are used to get a numerical solution through using Python programming language that. Finally, the difference is found between the exact solutions and the numerical solutions to get the errors. These errors are represented in tables and figures based on using Python.

Examples (3.1):

Consider the following nonlinear Fredholm-Hammerstein integral equation

$$u(t) = -\frac{1}{4}(t - 4e^t + te^2) + \int_0^1 ty[u(y)]^2 dy \quad (3.1)$$

With exact solution  $(t) = e^t$ .

To solve the above problem using the First Kind shifted Chebyshev Polynomials method, we assume  $M = 2, 4, 6$  Then, we construct trial space as follows

- 1) when  $M = 2$ ,  $n = 2$ ,  $\mathcal{D} = 4$

$$u_{app} = \sum_{i=0}^2 c_i t^i$$

$$u_{app} = c_0 + c_1 t + c_2 t^2 \quad (3.2)$$

Similarly

$$[u_{app}]^2 = \sum_{i=0}^4 v_i y^i$$

$$[u_{app}]^2 = v_0 + v_1 y + v_2 y^2 + v_3 y^3 + v_4 y^4 \quad (3.3)$$

We find  $v_i$  from (2.40)

$$v_0 = c_0^2$$

$$v_1 = \frac{1}{1!} \frac{\partial}{\partial y} (c_0 + c_1 y)^2 \Big|_{y=0}$$

$$v_1 = \frac{\partial}{\partial y} (c_0^2 + 2c_0 c_1 y + c_1^2 y^2) \Big|_{y=0}$$

$$= 2c_0 c_1 + 2c_1^2 y^1 \Big|_{y=0} = 2c_0 c_1$$

$$v_2 = \frac{1}{2!} \frac{\partial}{\partial y} (c_0 + c_1 y + c_2 y^2)^2 \Big|_{y=0}$$

$$= \frac{1}{2!} \frac{\partial^2}{\partial y^2} (c_0 + c_1 y + c_2 y^2)^2 \Big|_{y=0}$$

$$= \frac{1}{2!} \frac{\partial^2}{\partial y^2} ((c_0 + c_1 y)^2 + 2(c_0 + c_1 y)c_2 y^2 + (c_2 y^2)^2) \Big|_{y=0}$$

$$= \frac{1}{2!} \frac{\partial^2}{\partial y^2} (c_0^2 + 2c_0 c_1 y + c_1^2 y^2 + 2c_0 c_2 y^2 + 2c_1 c_2 y^3 + c_2^2 y^4) \Big|_{y=0}$$

$$= \frac{1}{2!} \frac{\partial}{\partial y} (2c_0 c_1 + 2c_1^2 y + 4c_0 c_2 y + 6c_1 c_2 y^2 + 4c_2^2 y^3) \Big|_{y=0}$$

$$= \frac{1}{2!} (2c_1^2 + 4c_0 c_2 + 12c_1 c_2 y + 12c_2^2 y^2) \Big|_{y=0} = c_1^2 + 2c_0 c_2$$

$$v_3 = \frac{1}{3!} \frac{\partial^3}{\partial y^3} (c_0 + c_1 y + c_2 y^2)^2 \Big|_{y=0}$$

$$= \frac{1}{3!} \frac{\partial^3}{\partial y^3} ((c_0 + c_1 y) + c_2 y^2)^2 \Big|_{y=0}$$

$$= \frac{1}{3!} \frac{\partial^3}{\partial y^3} ((c_0 + c_1 y)^2 + 2(c_0 + c_1 y)c_2 y^2 + (c_2 y^2)^2) \Big|_{y=0}$$

$$= \frac{1}{3!} \frac{\partial^3}{\partial y^3} (c_0^2 + 2c_0 c_1 y + c_1^2 y^2 + 2c_0 c_2 y^2 + 2c_1 c_2 y^3 + c_2^2 y^4) \Big|_{y=0}$$

$$= \frac{1}{3!} \frac{\partial^2}{\partial y^2} (2c_0 c_1 + 2c_1^2 y + 4c_0 c_2 y + 6c_1 c_2 y^2 + 4c_2^2 y^3) \Big|_{y=0}$$

$$\begin{aligned}
&= \frac{1}{3!} \frac{\partial}{\partial y} (2c_1^2 + 4c_0c_2 + 12c_1c_2y + 12c_2^2y^2)_{y=0} \\
&= \frac{1}{3!} (12c_1c_2 + 24c_2^2y^1)_{y=0} \\
&= \frac{1}{6} (12c_1c_2) = 2c_1c_2 \\
v_4 &= \frac{1}{4!} \frac{\partial^4}{\partial y^4} (c_0 + c_1y + c_2y^2)^2_{y=0} \\
&= \frac{1}{3!} \frac{\partial^4}{\partial y^4} ((c_0 + c_1y) + c_2y^2)^2_{y=0} \\
&= \frac{1}{4!} \frac{\partial^4}{\partial y^4} ((c_0 + c_1y)^2 + 2(c_0 + c_1y)c_2y^2 + (c_2y^2)^2)_{y=0} \\
&= \frac{1}{4!} \frac{\partial^4}{\partial y^4} (c_0^2 + 2c_0c_1y + c_1^2y^2 + 2c_0c_2y^2 + 2c_1c_2y^3 + c_2^2y^4)_{y=0} \\
&= \frac{1}{4!} \frac{\partial^3}{\partial y^3} (2c_0c_1 + 2c_1^2y + 4c_0c_2y + 6c_1c_2y^2 + 4c_2^2y^3)_{y=0} \\
&= \frac{1}{4!} \frac{\partial^2}{\partial y^2} (2c_1^2 + 4c_0c_2 + 12c_1c_2y + 12c_2^2y^2)_{y=0} \\
&= \frac{1}{4!} \frac{\partial}{\partial y} (12c_1c_2 + 24c_2^2y^1)_{y=0} \\
&= \frac{1}{4!} (24c_2^2)_{y=0} = c_2^2 \tag{3.4}
\end{aligned}$$

and using formula (2.31), we have  $u_{app} = \sum_{i=0}^2 \alpha_i E_i^*(t)$

$$\begin{aligned}
\alpha_0 &= c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2 \\
\alpha_1 &= \frac{1}{2}c_1 + \frac{4}{8}c_2 \\
\alpha_2 &= \frac{1}{8}c_2 \tag{3.5}
\end{aligned}$$

and using formula (2.31), we have

$$[u_{app}]^2 = \sum_{i=0}^4 \beta_i E_i^*(y)$$

Where,  $\sum_{i=0}^4 \beta_i$

$$\beta_0 = v_0 + \frac{1}{2}v_1 + \frac{3}{8}v_2 + \frac{10}{32}v_3 + \frac{35}{128}v_4$$

$$\beta_1 = \frac{1}{2}v_1 + \frac{4}{8}v_2 + \frac{15}{32}v_3 + \frac{56}{128}v_4$$

$$\beta_2 = \frac{1}{8}v_2 + \frac{6}{32}v_3 + \frac{28}{128}v_4$$

$$\beta_3 = \frac{1}{32}v_3 + \frac{8}{128}v_4$$

$$\beta_4 = \frac{1}{128}v_4 \tag{3.6}$$

So, we have

$$\sum_{i=0}^2 \alpha_i E_i^*(t) - \int_0^1 ty [\sum_{i=0}^4 \beta_i E_i^*(y)] dy = -\frac{1}{4}(t - 4e^t + te^2) \tag{3.7}$$

Now, we multiply both sides of the above relation with  $\{\omega(t)E_j^*(t)\}_{j=0,1,2}$ ; from the orthogonality of shifted Chebyshev polynomials, we obtain the following nonlinear equation from equation (2.40)

$$\gamma_j \alpha_j - \sum_{i=0}^4 \beta_i K_{i,j} = f_j, \quad j = 0, \dots, 2, \quad i = 0, \dots, 4 \tag{3.8}$$

From (2.44) we find

$$\gamma_j \alpha_j = \begin{bmatrix} \pi \alpha_0 \\ \frac{\pi}{2} \alpha_1 \\ \frac{\pi}{2} \alpha_2 \end{bmatrix} \tag{3.9}$$

From equation (2.43) we find the values of  $K_{i,j}$

$$K_{i,j} = \int_0^1 \int_0^1 \omega(t)k(t,y)E_i^*(y)E_j^*(t)dy dt, j = 0, \dots, 2, \quad i = 0, \dots, 4$$

$$K_{i,j} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_i^*(y)E_j^*(t)dy dt, \quad j = 0, \dots, 2, \quad i = 0, \dots, 4$$

$$K_{0,0} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_0^*(y)E_0^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty dy dt = \frac{\pi}{4}$$

$$K_{0,1} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_0^*(y)E_1^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2t-1)dy dt = \frac{\pi}{8}$$

$$K_{0,2} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_0^*(y)E_2^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8t^2 - 8t + 1)dy dt = 0$$

$$K_{1,0} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_1^*(y)E_0^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2y-1)dy dt = \frac{1}{6} \int_0^1 \frac{1}{\sqrt{t-t^2}} t dt = \frac{\pi}{12}$$

$$K_{1,1} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_1^*(y)E_1^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2t-1)y(2y-1)dy dt = \frac{1}{6} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2t-1)dt = \frac{\pi}{24}$$

$$K_{1,2} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_1^*(y)E_2^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} t(8t^2-8t+1)y(2y-1)dy dt$$

$$= \frac{1}{6} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(8t^2-8t+1)dt = 0$$

$$K_{2,0} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_2^*(y)E_0^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8y^2-8y+1)dy dt$$

$$= -\frac{1}{6} \int_0^1 \frac{1}{\sqrt{t-t^2}} tdt = -\frac{\pi}{12}$$

The rest of the values of  $K_{i,j}$  were founded in the same way [See the appendix] in order to find the following matrix

$$K_{4,2} = \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{12} & -\frac{\pi}{12} & -\frac{\pi}{20} & -\frac{\pi}{60} \\ \frac{\pi}{8} & \frac{\pi}{24} & -\frac{\pi}{24} & -\frac{\pi}{40} & -\frac{\pi}{120} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.10)$$

From equation (2.42) we find the values of  $f_j$

$$f_j = \int_0^1 \omega(t)f(t)E_j^*(t) dt \quad , j = 0, \dots, 2$$

$$= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) E_j^*(t) dt$$

$$f_0 = \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) E_0^*(t) dt$$

$$= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) dt = 2.21406$$

$$f_1 = \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) E_1^*(t) dt$$

$$= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2)(2t - 1) dt = -0.311395$$

$$f_2 = \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) E_2^*(t) dt$$

$$= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2)(8t^2 - 8t + 1) dt = 0.165261$$

$$f_j = \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \end{bmatrix} \quad (2.11)$$

We substitute the equations (3.9),(3.10),(3.11) in equation (3.8) we obtain the following matrix system

$$\begin{bmatrix} \pi\alpha_0 \\ \frac{\pi}{2}\alpha_1 \\ \frac{\pi}{2}\alpha_2 \end{bmatrix} - \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{12} & -\frac{\pi}{12} & -\frac{\pi}{20} & -\frac{\pi}{60} \\ \frac{\pi}{8} & \frac{\pi}{24} & -\frac{\pi}{24} & -\frac{\pi}{40} & -\frac{\pi}{120} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \end{bmatrix} \quad (3.12)$$

We substitute equations (3.5),(3.6) in the system matrices to obtain

$$\begin{bmatrix} \pi(c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2) \\ \frac{\pi}{2}(\frac{1}{2}c_1 + \frac{4}{8}c_2) \\ \frac{\pi}{2}(\frac{1}{8}c_2) \end{bmatrix} - \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{12} & -\frac{\pi}{12} & -\frac{\pi}{20} & -\frac{\pi}{60} \\ \frac{\pi}{8} & \frac{\pi}{24} & -\frac{\pi}{24} & -\frac{\pi}{40} & -\frac{\pi}{120} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_0 + \frac{1}{2}v_1 + \frac{3}{8}v_2 + \frac{10}{32}v_3 + \frac{35}{128}v_4 \\ \frac{1}{2}v_1 + \frac{4}{8}v_2 + \frac{15}{32}v_3 + \frac{56}{128}v_4 \\ \frac{1}{8}v_2 + \frac{6}{32}v_3 + \frac{28}{128}v_4 \\ \frac{1}{32}v_3 + \frac{8}{128}v_4 \\ \frac{1}{128}v_4 \end{bmatrix} = \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \end{bmatrix} \quad (3.13)$$

We substitute equation (3.4) in the previous Matrix system to obtain

$$\begin{bmatrix} \pi(c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2) \\ \frac{\pi}{2}(\frac{1}{2}c_1 + \frac{4}{8}c_2) \\ \frac{\pi}{2}(\frac{1}{8}c_2) \end{bmatrix} - \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{12} & -\frac{\pi}{12} & -\frac{\pi}{20} & -\frac{\pi}{60} \\ \frac{\pi}{8} & \frac{\pi}{24} & -\frac{\pi}{24} & -\frac{\pi}{40} & -\frac{\pi}{120} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_o^2 + \frac{1}{2}(2c_0c_1) + \frac{3}{8}(c_1^2 + 2c_0c_2) + \frac{10}{32}(2c_1c_2) + \frac{35}{128}c_2^2 \\ \frac{1}{2}(2c_0c_1) + \frac{4}{8}(c_1^2 + 2c_0c_2) + \frac{15}{32}(2c_1c_2) + \frac{56}{128}c_2^2 \\ \frac{1}{8}(c_1^2 + 2c_0c_2) + \frac{6}{32}(2c_1c_2) + \frac{28}{128}c_2^2 \\ \frac{1}{32}(2c_1c_2) + \frac{8}{128}c_2^2 \\ \frac{1}{128}c_2^2 \end{bmatrix} = \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \end{bmatrix}$$

Therefore we get

$$c_o = \frac{983}{975}$$

$$c_1 = \frac{533}{628}$$

$$C_2 = \frac{2440}{2899}$$

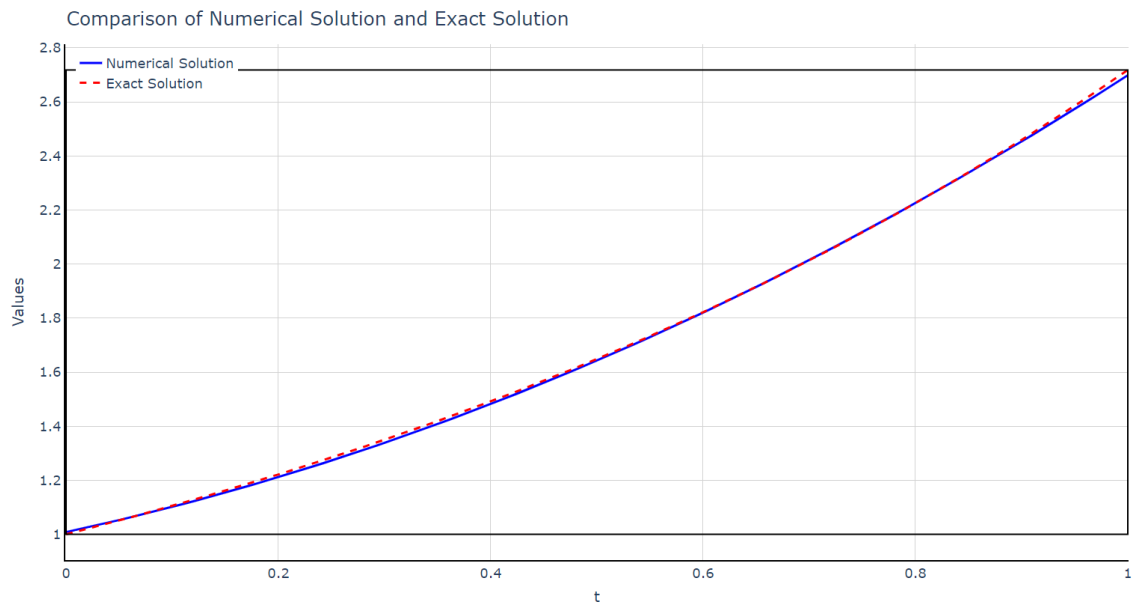
**Table 1**

*The table contains the exact and the numerical solutions assuming  $M = 2$  when  $\mathbf{t} = [0 \ 1]$  and shows the difference between them.*

t	Numerical solutions	Exact solutions	Exact – Numerical
0.00	1.008205128205128	1.000000000000000	0.008205128205128
0.05	1.052745607790665	1.051271096376024	0.001474511414641
0.10	1.101494435082307	1.105170918075648	0.003676482993340
0.15	1.154451610080055	1.161834242728283	0.007382632648228
0.20	1.211617132783909	1.221402758160170	0.009785625376261
0.25	1.272991003193868	1.284025416687741	0.011034413493873
0.30	1.338573221309933	1.349858807576003	0.011285586266070
0.35	1.408363787132103	1.419067548593257	0.010703761461154
0.40	1.482362700660379	1.491824697641270	0.009461996980892
0.45	1.560569961894760	1.568312185490169	0.007742223595409
0.50	1.642985570835247	1.648721270700128	0.005735699864881
0.55	1.729609527481839	1.733253017867395	0.003643490385556
0.60	1.820441831834537	1.822118800390509	0.001676968555972
0.65	1.915482483893341	1.915540829013896	0.000058345120555
0.70	2.014731483658250	2.013752707470477	0.000978776187773
0.75	2.118188831129264	2.117000016612675	0.001188814516589
0.80	2.225854526306385	2.225540928492468	0.000313597813917
0.85	2.337728569189610	2.339646851925991	0.001918282736381
0.90	2.453810959778941	2.459603111156950	0.005792151378008
0.95	2.574101698074378	2.585709659315846	0.011607961241468
1.0	2.698600784075921	2.718281828459045	0.019681044383125

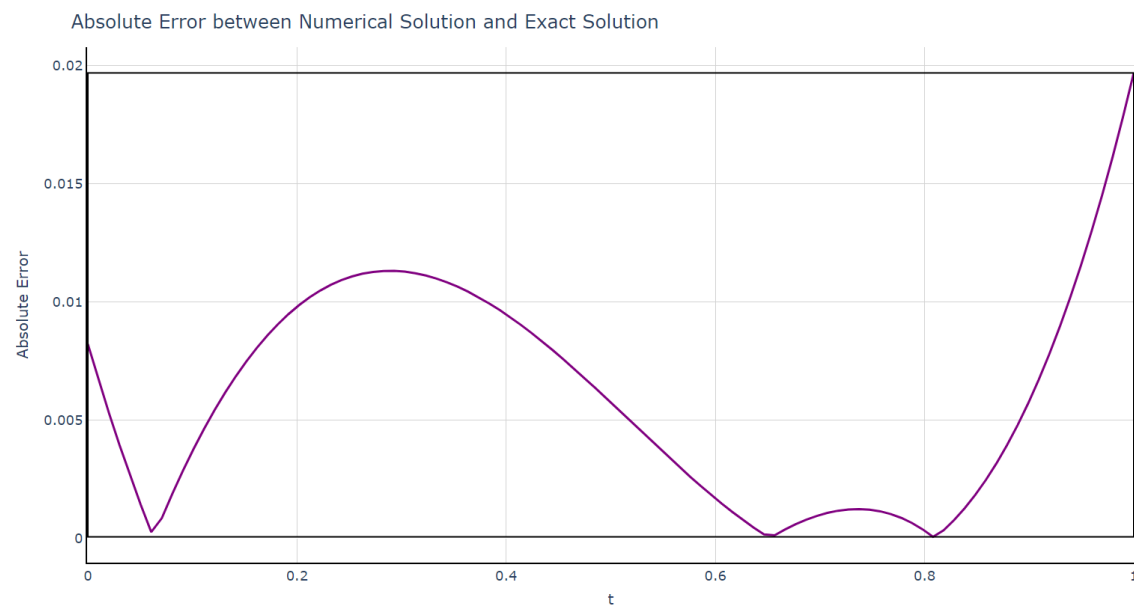
**Figure1**

shows both exact and numerical solutions using  $M = 2$  when  $t = [0, 1]$



**Figure 2**

shows the error between exact and numerical solutions using  $M = 2$  When  $t = [0, 1]$



2) when  $M = 4, n = 2, \mathcal{D} = 8$

$$u_{app} = \sum_{i=0}^4 c_i t^i$$

$$u_{app} = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4$$

Similarly

$$[u_{app}]^2 = \sum_{i=0}^8 v_i y^i$$

We find  $v_i$  from (2.40)

$$v_0 = c_0^2$$

$$v_1 = \frac{\partial}{\partial y} (c_0 + c_1 y)^2 \Big|_{y=0}$$

$$v_1 = \frac{\partial}{\partial y} (c_0^2 + 2c_0 c_1 y + c_1^2 y^2) \Big|_{y=0}$$

$$= 2c_0 c_1 + 2c_1^2 y^1 \Big|_{y=0} = 2c_0 c_1$$

$$v_2 = \frac{1}{2!} \frac{\partial}{\partial y} (c_0 + c_1 y + c_2 y^2)^2 \Big|_{y=0}$$

$$= \frac{1}{2!} \frac{\partial^2}{\partial y^2} (c_0 + c_1 y + c_2 y^2)^2 \Big|_{y=0}$$

$$= \frac{1}{2!} \frac{\partial^2}{\partial y^2} ((c_0 + c_1 y)^2 + 2(c_0 + c_1 y)c_2 y^2 + (c_2 y^2)^2) \Big|_{y=0}$$

$$= \frac{1}{2!} \frac{\partial^2}{\partial y^2} (c_0^2 + 2c_0 c_1 y + c_1^2 y^2 + 2c_0 c_2 y^2 + 2c_1 c_2 y^3 + c_2^2 y^4) \Big|_{y=0}$$

$$= \frac{1}{2!} \frac{\partial}{\partial y} (2c_0 c_1 + 2c_1^2 y + 4c_0 c_2 y + 6c_1 c_2 y^2 + 4c_2^2 y^3) \Big|_{y=0}$$

$$= \frac{1}{2!} (2c_1^2 + 4c_0 c_2 + 12c_1 c_2 y + 12c_2^2 y^2) \Big|_{y=0} = c_1^2 + 2c_0 c_2$$

$$v_3 = \frac{1}{3!} \frac{\partial^3}{\partial y^3} (c_0 + c_1 y + c_2 y^2 + c_3 y^3)^2 \Big|_{y=0}$$

$$= \frac{1}{3!} \frac{\partial^3}{\partial y^3} ((c_0 + c_1 y)^2 + 2(c_0 + c_1 y)(c_2 y^2 + c_3 y^3) + (c_2 y^2 + c_3 y^3)^2) \Big|_{y=0}$$

$$\begin{aligned}
&= \frac{1}{3!} \frac{\partial^3}{\partial y^3} (c_0^2 + 2c_0c_1y + c_1^2y^2 + 2c_0c_2y^2 \\
&\quad + 2c_0c_3y^3 + 2c_1c_2y^3 + 2c_1c_3y^4 + c_2^2y^4 + 2c_2c_3y^5 + c_3^2y^6)_{y=0} \\
&= \frac{1}{3!} \frac{\partial^2}{\partial y^2} (2c_0c_1 + 2c_1^2y + 4c_0c_2y \\
&\quad + 6c_0c_3y^2 + 6c_1c_2y^2 + 8c_1c_3y^3 + 4c_2^2y^3 + 10c_2c_3y^4 + 6c_3^2y^5)_{y=0} \\
&= \frac{1}{3!} \frac{\partial}{\partial y} (2c_1^2 + 4c_0c_2 \\
&\quad + 12c_0c_3y + 12c_1c_2y + 24c_1c_3y^2 + 12c_2^2y^2 + 40c_2c_3y^3 + 30c_3^2y^4)_{y=0} \\
&= \frac{1}{3!} (12c_0c_3 + 12c_1c_2 + 48c_1c_3y^1 + 24c_2^2y + 120c_2c_3y^2 + 120c_3^2y^3)_{y=0} \\
&= \frac{1}{3!} (12c_0c_3 + 12c_1c_2) \\
&= 2c_0c_3 + 2c_1c_2 \\
v_4 &= \frac{1}{4!} \frac{\partial^4}{\partial y^4} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)_{y=0}^2 \\
&= \frac{1}{4!} \frac{\partial^3}{\partial y^3} 2(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)(c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3)_{y=0} \\
&= \frac{1}{4!} \frac{\partial^2}{\partial y^2} (2(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)(2c_2 + 6c_3y^1 + 12c_4y^2) + 2(c_1 + 2c_2y^1 \\
&\quad + 3c_3y^2 + 4c_4y^3)(c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3))_{y=0} \\
&= \frac{1}{4!} \frac{\partial}{\partial y} (2(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)(6c_3 + 24c_4y^1) \\
&\quad + 2(c_1 + 2c_2y^1 + 3c_3y^2 + 4c_4y^3)(2c_2 + 6c_3y^1 + 12c_4y^2) \\
&\quad + 2(c_1 + 2c_2y^1 + 3c_3y^2 + 4c_4y^3)(2c_2 + 6c_3y^1 + 12c_4y^2) + 2(2c_2 \\
&\quad + 6c_3y^1 + 12c_4y^2)(c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3))_{y=0}
\end{aligned}$$

$$= \frac{1}{4!} \frac{\partial}{\partial y} (12c_0c_3 + 48c_0c_4y + 8c_1c_2 + 36c_1c_3y + 96c_1c_4y^2 + 4c_1c_2 + 24c_2^2y + 72c_2c_3y^2 + 160c_2c_4y^3 + 24c_1c_4y^2 + 80c_2c_4y^3 + 180c_3c_4y^4 + 336c_4^2y^5 + 12c_1c_3y + 48c_2c_3y^2 + 120c_3^2y^3 + 240c_3c_4y^4)_{y=0})$$

$$= \frac{1}{4!} (48c_0c_4 + 36c_1c_3 + 192c_1c_4y + 24c_2^2 + 144c_2c_3y + 480c_2c_4y^2 + 48c_1c_4y + 240c_2c_4y^2 + 720c_3c_4y^3 + 1680c_4^2y^4 + 12c_1c_3 + 96c_2c_3y + 360c_3^2y^2 + 960c_3c_4y^3)_{y=0})$$

$$= \frac{1}{4!} (48c_0c_4 + 48c_1c_3 + 24c_2^2) = 2c_0c_4 + 2c_1c_3 + c_2^2$$

$$v_5 = \frac{1}{5!} \frac{\partial^5}{\partial y^5} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)^2_{y=0}$$

$$= \frac{1}{5!} \frac{\partial^4}{\partial y^4} 2(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)(c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3)_{y=0}$$

$$= \frac{1}{5!} \frac{\partial^3}{\partial y^3} (2(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)(2c_2 + 6c_3y^1 + 12c_4y^2) + 2(c_1 + 2c_2y^1 + 3c_3y^2 + 4c_4y^3)(c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3))_{y=0}$$

$$= \frac{1}{5!} \frac{\partial^2}{\partial y^2} (2(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)(6c_3 + 24c_4y^1) + 2(c_1 + 2c_2y^1 + 3c_3y^2 + 4c_4y^3)(2c_2 + 6c_3y^1 + 12c_4y^2) + 2(c_1 + 2c_2y^1 + 3c_3y^2 + 4c_4y^3)(2c_2 + 6c_3y^1 + 12c_4y^2) + 2(2c_2 + 6c_3y^1 + 12c_4y^2)(c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3))_{y=0}$$

$$= \frac{1}{5!} \frac{\partial^2}{\partial y^2} (12c_0c_3 + 48c_0c_4y + 8c_1c_2 + 36c_1c_3y + 96c_1c_4y^2 + 4c_1c_2 + 24c_2^2y + 72c_2c_3y^2 + 160c_2c_4y^3 + 24c_1c_4y^2 + 80c_2c_4y^3 + 180c_3c_4y^4 + 336c_4^2y^5 + 12c_1c_3y + 48c_2c_3y^2 + 120c_3^2y^3 + 240c_3c_4y^4)_{y=0})$$

$$= \frac{1}{5!} \frac{\partial}{\partial y} (48c_0c_4 + 36c_1c_3 + 192c_1c_4y + 24c_2^2 + 144c_2c_3y + 480c_2c_4y^2 + 48c_1c_4y + 240c_2c_4y^2 + 720c_3c_4y^3 + 1680c_4^2y^4 + 12c_1c_3 + 96c_2c_3y + 360c_3^2y^2 + 960c_3c_4y^3)_{y=0})$$

$$\begin{aligned}
&= \frac{1}{5!} (192c_1c_4 + 144c_2c_3 + 960c_2c_4y + 48c_1c_4 + 480c_2c_4y + 2160c_3c_4y^2 \\
&\quad + 6720c_4^2y^3 + 96c_2c_3 + 720c_3^2y + 2880c_3c_4y^2)_{y=0}) \\
&= \frac{1}{5!} (192c_1c_4 + 144c_2c_3 + 48c_1c_4 + 96c_2c_3) \\
&= \frac{1}{120} (240c_1c_4 + 240c_2c_3) = 2c_1c_4 + 2c_2c_3 \\
v_6 &= \frac{1}{6!} \frac{\partial^6}{\partial y^6} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)^2_{y=0} \\
&= \frac{1}{6!} \frac{\partial^5}{\partial y^5} 2(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)(c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3)_{y=0} \\
&= \frac{1}{6!} \frac{\partial^4}{\partial y^4} (2(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)(2c_2 + 6c_3y^1 + 12c_4y^2) + 2(c_1 + 2c_2y^1 \\
&\quad + 3c_3y^2 + 4c_4y^3)(c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3))_{y=0} \\
&= \frac{1}{6!} \frac{\partial^3}{\partial y^3} (12c_0c_3 + 48c_0c_4y + 8c_1c_2 + 36c_1c_3y + 96c_1c_4y^2 + 4c_1c_2 + 24c_2^2y \\
&\quad + 72c_2c_3y^2 + 160c_2c_4y^3 + 24c_1c_4y^2 + 80c_2c_4y^3 + 180c_3c_4y^4 \\
&\quad + 336c_4^2y^5 + 12c_1c_3y + 48c_2c_3y^2 + 120c_3^2y^3 + 240c_3c_4y^4)_{y=0}) \\
&= \frac{1}{6!} \frac{\partial^2}{\partial y^2} (48c_0c_4 + 36c_1c_3 + 192c_1c_4y + 24c_2^2 + 144c_2c_3y + 480c_2c_4y^2 + 48c_1c_4y \\
&\quad + 240c_2c_4y^2 + 720c_3c_4y^3 + 1680c_4^2y^4 + 12c_1c_3 + 96c_2c_3y \\
&\quad + 360c_3^2y^2 + 960c_3c_4y^3)_{y=0}) \\
&= \frac{1}{6!} \frac{\partial}{\partial y} (192c_1c_4 + 144c_2c_3 + 960c_2c_4y + 48c_1c_4 + 480c_2c_4y + 2160c_3c_4y^2 \\
&\quad + 6720c_4^2y^3 + 96c_2c_3 + 720c_3^2y + 2880c_3c_4y^2)_{y=0}) \\
&= \frac{1}{6!} (960c_2c_4 + 480c_2c_4 + 4320c_3c_4y + 20160c_4^2y^2 + 720c_3^2 + 5764c_3c_4y)_{y=0}) \\
&= \frac{1}{720} (1440c_2c_4 + 720c_3^2) = 2c_2c_4 + c_3^2
\end{aligned}$$

$$\begin{aligned}
v_7 &= \frac{1}{7!} \frac{\partial^7}{\partial y^7} (c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4)^2 \Big|_{y=0} \\
&= \frac{1}{7!} \frac{\partial^6}{\partial y^6} 2(c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4)(c_1 + 2c_2 y + 3c_3 y^2 + 4c_4 y^3) \Big|_{y=0} \\
&= \frac{1}{7!} \frac{\partial^5}{\partial y^5} (2(c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4)(2c_2 + 6c_3 y + 12c_4 y^2) + 2(c_1 + 2c_2 y + 3c_3 y^2 + 4c_4 y^3)(c_1 + 2c_2 y + 3c_3 y^2 + 4c_4 y^3)) \Big|_{y=0} \\
&= \frac{1}{7!} \frac{\partial^4}{\partial y^4} (12c_0 c_3 + 48c_0 c_4 y + 8c_1 c_2 + 36c_1 c_3 y + 96c_1 c_4 y^2 + 4c_1 c_2 + 24c_2^2 y + 72c_2 c_3 y^2 + 160c_2 c_4 y^3 + 24c_1 c_4 y^2 + 80c_2 c_4 y^3 + 180c_3 c_4 y^4 + 336c_4^2 y^5 + 12c_1 c_3 y + 48c_2 c_3 y^2 + 120c_3^2 y^3 + 240c_3 c_4 y^4) \Big|_{y=0} \\
&= \frac{1}{7!} \frac{\partial^3}{\partial y^3} (48c_0 c_4 + 36c_1 c_3 + 192c_1 c_4 y + 24c_2^2 + 144c_2 c_3 y + 480c_2 c_4 y^2 + 48c_1 c_4 y + 240c_2 c_4 y^2 + 720c_3 c_4 y^3 + 1680c_4^2 y^4 + 12c_1 c_3 + 96c_2 c_3 y + 360c_3^2 y^2 + 960c_3 c_4 y^3) \Big|_{y=0} \\
&= \frac{1}{7!} \frac{\partial^2}{\partial y^2} (192c_1 c_4 + 144c_2 c_3 + 960c_2 c_4 y + 48c_1 c_4 + 480c_2 c_4 y + 2160c_3 c_4 y^2 + 6720c_4^2 y^3 + 96c_2 c_3 + 720c_3^2 y + 2880c_3 c_4 y^2) \Big|_{y=0} \\
&= \frac{1}{7!} \frac{\partial}{\partial y} (960c_2 c_4 + 480c_2 c_4 + 4320c_3 c_4 y + 20160c_4^2 y^2 + 720c_3^2 + 5760c_3 c_4 y) \Big|_{y=0} \\
&= \frac{1}{7!} (4320c_3 c_4 + 40320c_4^2 y + 5760c_3 c_4) \Big|_{y=0} \\
&= \frac{1}{5040} (10080c_3 c_4) = 2c_3 c_4
\end{aligned}$$

$$\begin{aligned}
v_8 &= \frac{1}{8!} \frac{\partial^8}{\partial y^8} (c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4)^2 \Big|_{y=0} \\
&= \frac{1}{8!} \frac{\partial^7}{\partial y^7} 2(c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4)(c_1 + 2c_2 y + 3c_3 y^2 + 4c_4 y^3) \Big|_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8!} \frac{\partial^6}{\partial y^6} (2(c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4)(2c_2 + 6c_3 y^1 + 12c_4 y^2) + 2(c_1 + 2c_2 y^1 \\
&\quad + 3c_3 y^2 + 4c_4 y^3)(c_1 + 2c_2 y + 3c_3 y^2 + 4c_4 y^3)_{y=0}) \\
&= \frac{1}{8!} \frac{\partial^5}{\partial y^5} (2(c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4)(6c_3 + 24c_4 y^1) \\
&\quad + 2(c_1 + 2c_2 y^1 + 3c_3 y^2 + 4c_4 y^3)(2c_2 + 6c_3 y^1 + 12c_4 y^2) \\
&\quad + 2(c_1 + 2c_2 y^1 + 3c_3 y^2 + 4c_4 y^3)(2c_2 + 6c_3 y^1 + 12c_4 y^2) + 2(2c_2 \\
&\quad + 6c_3 y^1 + 12c_4 y^2)(c_1 + 2c_2 y + 3c_3 y^2 + 4c_4 y^3)_{y=0}) \\
&= \frac{1}{8!} \frac{\partial^5}{\partial y^5} (12c_0 c_3 + 48c_0 c_4 y + 8c_1 c_2 + 36c_1 c_3 y + 96c_1 c_4 y^2 + 4c_1 c_2 + 24c_2^2 y \\
&\quad + 72c_2 c_3 y^2 + 160c_2 c_4 y^3 + 24c_1 c_4 y^2 + 80c_2 c_4 y^3 + 180c_3 c_4 y^4 \\
&\quad + 336c_4^2 y^5 + 12c_1 c_3 y + 48c_2 c_3 y^2 + 120c_3^2 y^3 + 240c_3 c_4 y^4)_{y=0}) \\
&= \frac{1}{8!} \frac{\partial^4}{\partial y^4} (48c_0 c_4 + 36c_1 c_3 + 192c_1 c_4 y + 24c_2^2 + 144c_2 c_3 y + 480c_2 c_4 y^2 + 48c_1 c_4 y \\
&\quad + 240c_2 c_4 y^2 + 720c_3 c_4 y^3 + 1680c_4^2 y^4 + 12c_1 c_3 + 96c_2 c_3 y \\
&\quad + 360c_3^2 y^2 + 960c_3 c_4 y^3)_{y=0}) \\
&= \frac{1}{8!} \frac{\partial^3}{\partial y^3} (192c_1 c_4 + 144c_2 c_3 + 960c_2 c_4 y + 48c_1 c_4 + 480c_2 c_4 y + 2160c_3 c_4 y^2 \\
&\quad + 6720c_4^2 y^3 + 96c_2 c_3 + 720c_3^2 y + 2880c_3 c_4 y^2)_{y=0}) \\
&= \frac{1}{8!} \frac{\partial^2}{\partial y^2} (960c_2 c_4 + 480c_2 c_4 + 4320c_3 c_4 y + 20160c_4^2 y^2 + 720c_3^2 \\
&\quad + 5760c_3 c_4 y)_{y=0}) \\
&= \frac{1}{8!} \frac{\partial}{\partial y} (4320c_3 c_4 + 40320c_4^2 y + 5760c_3 c_4)_{y=0}) \\
&= \frac{1}{8!} (40320c_4^2)_{y=0}) \\
&= \frac{1}{8!} 40320c_4^2 = c_4^2
\end{aligned}$$

We get  $v_i$

$$\begin{aligned}
v_0 &= c_0^2 \\
v_1 &= 2c_0c_1 \\
v_2 &= c_1^2 + 2c_0c_2 \\
v_3 &= 2c_0c_3 + 2c_1c_2 \\
v_4 &= 2c_0c_4 + 2c_1c_3 + c_2^2 \\
v_5 &= 2c_1c_4 + 2c_2c_3 \\
v_6 &= 2c_2c_4 + c_3^2 \\
v_7 &= 2c_3c_4 \\
v_8 &= c_4^2
\end{aligned} \tag{3.14}$$

and using formula (2.31 ),we have  $u_{app} = \sum_{i=0}^4 \alpha_i E_i^*(t)$

$$\begin{aligned}
\alpha_0 &= c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2 + \frac{10}{32}c_3 + \frac{35}{128}c_4 \\
\alpha_1 &= \frac{1}{2}c_1 + \frac{4}{8}c_2 + \frac{15}{32}c_3 + \frac{56}{128}c_4 \\
\alpha_2 &= \frac{1}{8}c_2 + \frac{6}{32}c_3 + \frac{28}{128}c_4 \\
\alpha_3 &= \frac{1}{32}c_3 + \frac{8}{128}c_4 \\
\alpha_4 &= \frac{1}{128}c_4
\end{aligned} \tag{3.15}$$

and using formula (2.31 ),we have

$$[u_{app}]^2 = \sum_{i=0}^8 \beta_i E_m^*(y)$$

Where ,  $\sum_{i=0}^8 \beta_i$

$$\begin{aligned}
\beta_0 &= v_0 + \frac{1}{2}v_1 + \frac{3}{8}v_2 + \frac{10}{32}v_3 + \frac{35}{128}v_4 + \frac{126}{512}v_5 + \frac{462}{2048}v_6 + \frac{1716}{8192}v_7 + \frac{6435}{32768}v_8 \\
\beta_1 &= \frac{1}{2}v_1 + \frac{4}{8}v_2 + \frac{15}{32}v_3 + \frac{56}{128}v_4 + \frac{210}{512}v_5 + \frac{792}{2048}v_6 + \frac{3003}{8192}v_7 + \frac{11440}{32768}v_8 \\
\beta_2 &= \frac{1}{8}v_2 + \frac{6}{32}v_3 + \frac{28}{128}v_4 + \frac{120}{512}v_5 + \frac{495}{2048}v_6 + \frac{2002}{8192}v_7 + \frac{8008}{32768}v_8 \\
\beta_3 &= \frac{1}{32}v_3 + \frac{8}{128}v_4 + \frac{45}{512}v_5 + \frac{220}{2048}v_6 + \frac{1001}{8192}v_7 + \frac{4368}{32768}v_8 \\
\beta_4 &= \frac{1}{128}v_4 + \frac{10}{512}v_5 + \frac{66}{2048}v_6 + \frac{364}{8192}v_7 + \frac{1820}{32768}v_8 \\
\beta_5 &= \frac{1}{512}v_5 + \frac{12}{2048}v_6 + \frac{91}{8192}v_7 + \frac{560}{32768}v_8 \\
\beta_6 &= \frac{1}{2048}v_6 + \frac{14}{8192}v_7 + \frac{120}{32768}v_8 \\
\beta_7 &= \frac{1}{8192}v_7 + \frac{16}{32768}v_8 \\
\beta_8 &= \frac{1}{32768}v_8
\end{aligned} \tag{3.16}$$

We get

$$\sum_{i=0}^4 \alpha_i E_i^*(t) - \int_0^1 ty \left[ \sum_{i=0}^8 \beta_i E_i^*(y) \right] dy = -\frac{1}{4}(t - 4e^t + te^2)$$

Now, we multiply both sides of the above relation with  $\{\omega(t)E_j^*(t)\}_{j=0,1,2,3,4}$ ; from the orthogonality of shifted Chebyshev polynomials, we obtain the following nonlinear equation

$$\gamma_j \alpha_j - \sum_{i=0}^8 \beta_i K_{i,j} = f_j \quad , j = 0, \dots, 4, \quad i = 0, \dots, 8 \tag{3.17}$$

From equation (2.44) we get

$$\gamma_j \alpha_j = \begin{bmatrix} \pi \alpha_0 \\ \frac{\pi}{2} \alpha_1 \\ \frac{\pi}{2} \alpha_2 \\ \frac{\pi}{2} \alpha_3 \\ \frac{\pi}{2} \alpha_4 \end{bmatrix} \quad (3.18)$$

From equation (2.43) we get the values of  $K_{i,j}$

$$K_{i,j} = \int_0^1 \int_0^1 \omega(t) k(t,y) E_i^*(y) E_j^*(t) dy dt, \quad j = 0, \dots, 4, \quad i = 0, \dots, 8 \quad (3.19)$$

$$K_{i,j} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_i^*(y) E_j^*(t) dy dt, \quad j = 0, \dots, 4, \quad i = 0, \dots, 8$$

$$K_{5,0} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_5^*(y) E_0^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty (512y^5 - 1280y^4 + 1120y^3 - 400y^2 + 50y - 1) dy dt$$

$$= \frac{-1}{42} \int_0^1 \frac{1}{\sqrt{t-t^2}} t dt = \frac{-\pi}{84}$$

$$K_{5,1} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_5^*(y) E_1^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty (512y^5 - 1280y^4 + 1120y^3 - 400y^2 + 50y - 1)(2t - 1) dy dt$$

$$= \frac{-1}{42} \int_0^1 \frac{(2t-1)}{\sqrt{t-t^2}} t dt = \frac{-\pi}{168}$$

$$K_{5,2} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_5^*(y) E_2^*(t) dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(512y^5 - 1280y^4 + 1120y^3 - 400y^2 + 50y - 1)(8t^2 - 8t \\
&\quad + 1)dy dt \\
&= \frac{-1}{42} \int_0^1 \frac{(8t^2 - 8t + 1)}{\sqrt{t-t^2}} tdt = 0 \\
K_{5,3} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_5^*(y)E_3^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(512y^5 - 1280y^4 + 1120y^3 - 400y^2 + 50y - 1)(32t^3 - 48t^2 \\
&\quad + 18t - 1)dy dt \\
&= \frac{-1}{42} \int_0^1 \frac{(32t^3 - 48t^2 + 18t - 1)}{\sqrt{t-t^2}} tdt = 0
\end{aligned}$$

The rest of the values of  $K_{i,j}$  were founded in the same way [See the appendix] in order to find the following matrix

From the previous  $K$  values and the  $K$  Values in (3.10) we find the Matrix  $K$

$$K_{8,4} = \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{12} & \frac{-\pi}{12} & \frac{-\pi}{20} & \frac{-\pi}{60} & \frac{-\pi}{84} & \frac{-\pi}{140} & \frac{-\pi}{180} & \frac{-\pi}{252} \\ \frac{\pi}{8} & \frac{\pi}{24} & \frac{-\pi}{24} & \frac{-\pi}{40} & \frac{-\pi}{120} & \frac{-\pi}{168} & \frac{-\pi}{280} & \frac{-\pi}{360} & \frac{-\pi}{504} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.20)$$

From equation (2.42) we get the values of  $f_j$

$$\begin{aligned}
f_j &= \int_0^1 \omega(t)f(t)E_j^*(t) dt \quad , j = 0, \dots, 4 \\
&= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2)E_j^*(t) dt
\end{aligned}$$

$$f_0 = \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) E_0^*(t) dt$$

$$= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) dt = 2.21406$$

$$f_1 = \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) E_1^*(t) dt$$

$$= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2)(2t - 1) dt = -0.311395$$

$$f_2 = \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) E_2^*(t) dt$$

$$= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2)(8t^2 - 8t + 1) dt = 0.165261$$

$$f_3 = \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) E_3^*(t) dt$$

$$= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2)(32t^3 - 48t^2 + 18t - 1) dt = 0.0137007$$

$$f_4 = \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) E_4^*(t) dt$$

$$= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2)(128t^4 - 256t^3 + 160t^2 - 32t + 1) dt$$

$$= 0.000853629$$

$$f_4 = \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \\ 0.0137007 \\ 0.000853629 \end{bmatrix} \quad (3.21)$$

We substitute the equations (3.18),(3.20),(3.21) In Equation (3.17) Let's get the following matrix system

$$\begin{bmatrix} \pi\alpha_0 \\ \frac{\pi}{2}\alpha_1 \\ \frac{\pi}{2}\alpha_2 \\ \frac{\pi}{2}\alpha_3 \\ \frac{\pi}{2}\alpha_4 \end{bmatrix} - \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{12} & \frac{-\pi}{12} & \frac{-\pi}{20} & \frac{-\pi}{60} & \frac{-\pi}{84} & \frac{-\pi}{140} & \frac{-\pi}{180} & \frac{-\pi}{252} \\ \frac{\pi}{8} & \frac{\pi}{24} & \frac{-\pi}{24} & \frac{-\pi}{40} & \frac{-\pi}{120} & \frac{-\pi}{168} & \frac{-\pi}{280} & \frac{-\pi}{360} & \frac{-\pi}{504} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \end{bmatrix} = \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \\ 0.0137007 \\ 0.000853629 \end{bmatrix} \quad (3.22)$$

We substitute the values  $\alpha$  and  $\beta$  of and from equations (3.15) and (3.16) into system (3.22) to obtain

$$\begin{bmatrix} \pi \left( c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2 + \frac{10}{32}c_3 + \frac{35}{128}c_4 \right) \\ \frac{\pi}{2} \left( \frac{1}{2}c_1 + \frac{4}{8}c_2 + \frac{15}{32}c_3 + \frac{56}{128}c_4 \right) \\ \frac{\pi}{2} \left( \frac{1}{8}c_2 + \frac{6}{32}c_3 + \frac{28}{128}c_4 \right) \\ \frac{\pi}{2} \left( \frac{1}{32}c_3 + \frac{8}{128}c_4 \right) \\ \frac{\pi}{2} \left( \frac{1}{128}c_4 \right) \end{bmatrix} - \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{12} & \frac{-\pi}{12} & \frac{-\pi}{20} & \frac{-\pi}{60} & \frac{-\pi}{84} & \frac{-\pi}{140} & \frac{-\pi}{180} & \frac{-\pi}{252} \\ \frac{\pi}{8} & \frac{\pi}{24} & \frac{-\pi}{24} & \frac{-\pi}{40} & \frac{-\pi}{120} & \frac{-\pi}{168} & \frac{-\pi}{280} & \frac{-\pi}{360} & \frac{-\pi}{504} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_0 + \frac{1}{2}v_1 + \frac{3}{8}v_2 + \frac{10}{32}v_3 + \frac{35}{128}v_4 + \frac{126}{512}v_5 + \frac{462}{2048}v_6 + \frac{1716}{8192}v_7 + \frac{6435}{32768}v_8 \\ \frac{1}{2}v_1 + \frac{4}{8}v_2 + \frac{15}{32}v_3 + \frac{56}{128}v_4 + \frac{210}{512}v_5 + \frac{792}{2048}v_6 + \frac{3003}{8192}v_7 + \frac{11440}{32768}v_8 \\ \frac{1}{8}v_2 + \frac{6}{32}v_3 + \frac{28}{128}v_4 + \frac{120}{512}v_5 + \frac{495}{2048}v_6 + \frac{2002}{8192}v_7 + \frac{8008}{32768}v_8 \\ \frac{1}{32}v_3 + \frac{8}{128}v_4 + \frac{45}{512}v_5 + \frac{220}{2048}v_6 + \frac{1001}{8192}v_7 + \frac{4368}{32768}v_8 \\ \frac{1}{128}v_4 + \frac{10}{512}v_5 + \frac{66}{2048}v_6 + \frac{364}{8192}v_7 + \frac{1820}{32768}v_8 \\ \frac{1}{512}v_5 + \frac{12}{2048}v_6 + \frac{91}{8192}v_7 + \frac{560}{32768}v_8 \\ \frac{1}{2048}v_6 + \frac{14}{8192}v_7 + \frac{120}{32768}v_8 \\ \frac{1}{8192}v_7 + \frac{16}{32768}v_8 \\ \frac{1}{32768}v_8 \end{bmatrix} = \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \\ 0.0137007 \\ 0.000853629 \end{bmatrix}$$



$$\begin{aligned}
& \begin{bmatrix} \pi \left( c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2 + \frac{10}{32}c_3 + \frac{35}{128}c_4 \right) \\ \frac{\pi}{2} \left( \frac{1}{2}c_1 + \frac{4}{8}c_2 + \frac{15}{32}c_3 + \frac{56}{128}c_4 \right) \\ \frac{\pi}{2} \left( \frac{1}{8}c_2 + \frac{6}{32}c_3 + \frac{28}{128}c_4 \right) \\ \frac{\pi}{2} \left( \frac{1}{32}c_3 + \frac{8}{128}c_4 \right) \\ \frac{\pi}{2} \left( \frac{1}{128}c_4 \right) \end{bmatrix} \\
& - \begin{bmatrix} \frac{\pi}{4}v_0 + \frac{\pi}{6}v_1 + \frac{\pi}{8}v_2 + \frac{\pi}{10}v_3 + \frac{\pi}{12}v_4 + \frac{\pi}{14}v_5 + \frac{\pi}{16}v_6 + \frac{\pi}{18}v_7 + \frac{\pi}{20}v_8 \\ \frac{\pi}{8}v_0 + \frac{\pi}{12}v_1 + \frac{\pi}{16}v_2 + \frac{\pi}{20}v_3 + \frac{\pi}{24}v_4 + \frac{\pi}{28}v_5 + \frac{\pi}{32}v_6 + \frac{\pi}{36}v_7 + \frac{\pi}{40}v_8 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
& = \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \\ 0.0137007 \\ 0.000853629 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
& \begin{bmatrix} \pi \left( c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2 + \frac{10}{32}c_3 + \frac{35}{128}c_4 \right) \\ \frac{\pi}{2} \left( \frac{1}{2}c_1 + \frac{4}{8}c_2 + \frac{15}{32}c_3 + \frac{56}{128}c_4 \right) \\ \frac{\pi}{2} \left( \frac{1}{8}c_2 + \frac{6}{32}c_3 + \frac{28}{128}c_4 \right) \\ \frac{\pi}{2} \left( \frac{1}{32}c_3 + \frac{8}{128}c_4 \right) \\ \frac{\pi}{2} \left( \frac{1}{128}c_4 \right) \end{bmatrix} \\
& - \begin{bmatrix} \pi \left( \frac{1}{4}v_0 + \frac{1}{6}v_1 + \frac{1}{8}v_2 + \frac{1}{10}v_3 + \frac{1}{12}v_4 + \frac{1}{14}v_5 + \frac{1}{16}v_6 + \frac{1}{18}v_7 + \frac{1}{20}v_8 \right) \\ \frac{\pi}{2} \left( \frac{1}{4}v_0 + \frac{1}{6}v_1 + \frac{1}{8}v_2 + \frac{1}{10}v_3 + \frac{1}{12}v_4 + \frac{1}{14}v_5 + \frac{1}{16}v_6 + \frac{1}{18}v_7 + \frac{1}{20}v_8 \right) \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
& = \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \\ 0.0137007 \\ 0.000853629 \end{bmatrix}
\end{aligned}$$

We substitute values from equations (3.14) into the previous system to obtain

$$\begin{bmatrix} \pi(c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2 + \frac{10}{32}c_3 + \frac{35}{128}c_4) \\ \frac{\pi}{2}(\frac{1}{2}c_1 + \frac{4}{8}c_2 + \frac{15}{32}c_3 + \frac{56}{128}c_4) \\ \frac{\pi}{2}(\frac{1}{8}c_2 + \frac{6}{32}c_3 + \frac{28}{128}c_4) \\ \frac{\pi}{2}(\frac{1}{32}c_3 + \frac{8}{128}c_4) \\ \frac{\pi}{2}(\frac{1}{128}c_4) \end{bmatrix} - \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{6} & \frac{\pi}{8} & \frac{\pi}{10} & \frac{\pi}{12} & \frac{\pi}{14} & \frac{\pi}{16} & \frac{\pi}{18} & \frac{\pi}{20} \\ \frac{\pi}{8} & \frac{\pi}{12} & \frac{\pi}{16} & \frac{\pi}{20} & \frac{\pi}{24} & \frac{\pi}{28} & \frac{\pi}{32} & \frac{\pi}{36} & \frac{\pi}{40} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0^2 \\ 2c_0c_1 \\ c_1^2 + 2c_0c_2 \\ 2c_0c_3 + 2c_1c_2 \\ 2c_0c_4 + 2c_1c_3 + c_2^2 \\ 2c_1c_4 + 2c_2c_3 \\ 2c_2c_4 + c_3^2 \\ 2c_3c_4 \\ c_4^2 \end{bmatrix} \\
= \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \\ 0.0137007 \\ 0.000853629 \end{bmatrix}$$

From the solution of the previous system we find the values of  $c$

$$c_0 = 1.0000259490980368$$

$$c_1 = 0.9987134401259607$$

$$c_2 = 0.5099581670678115$$

$$c_3 = 0.13998759200049835$$

$$c_4 = 0.06955986064131846$$

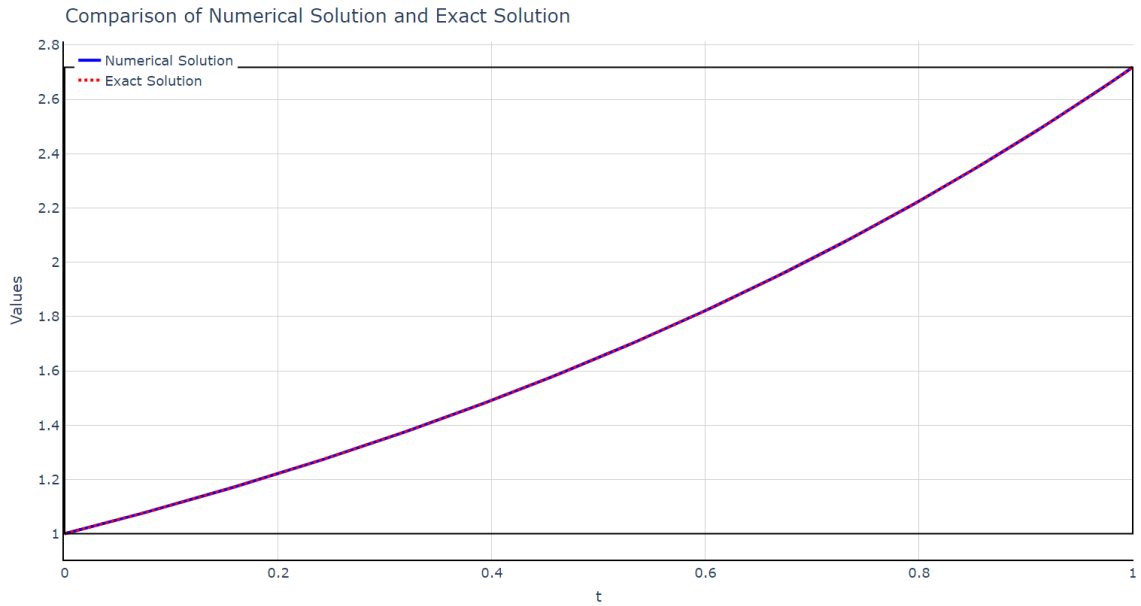
**Table 2**

The table contains the exact and the numerical solutions assuming  $M = 4$  when  $t = [0 \ 1]$  and shows the difference between them.

t	Numerical solutions	Exact solutions	Exact – Numerical
0.0	1.0000259490980368	1.0	2.594909803677048e-05
0.05	1.0512544497201333	1.0512710963760241	1.664665589085068e-05
0.1	1.1051438183593754	1.1051709180756477	2.709971627234964e-05
0.15	1.16181603189719	1.161834242728283	1.954604987508546e-05
0.2	1.2213981603189719	1.2214027581601699	4.597841197995223e-06
0.25	1.2840357189019032	1.2840254166877414	1.030221416176147e-05
0.3	1.349879316027136	1.3498588075760032	2.0508451132883465e-05
0.35	1.4190913292737	1.4190675485932571	2.3780680442841984e-05
0.4	1.4918445701997205	1.4918246976412703	1.9872558450151345e-05
0.45	1.5683222843424196	1.568312185490169	1.0098852250672863e-05
0.5	1.6487181512181146	1.6487212707001282	3.119482013591579e-06
0.55	1.7332362843222207	1.7332530178673953	1.673354517461867e-05
0.6	1.8220912311292474	1.8221188003905089	2.7569261261461975e-05
0.65	1.9155079730928029	1.9155408290138962	3.285592109336655e-05
0.7	2.013721925645588	2.0137527074704766	3.07818248858161e-05
0.75	2.116978938199404	2.117000046612675	2.1078413270814167e-05
0.8	2.225535294145144	2.225540928492468	5.6343473238129604e-06
0.85	2.3396577108528005	2.3396468519259908	1.0858926809742542e-05
0.9	2.4596233396714613	2.45960311115695	2.0228514511444473e-05
0.95	2.585719765929309	2.585709659315846	1.0106613463012337e-05
1.0	2.718245008933626	2.718281828459045	3.681952541922939e-05

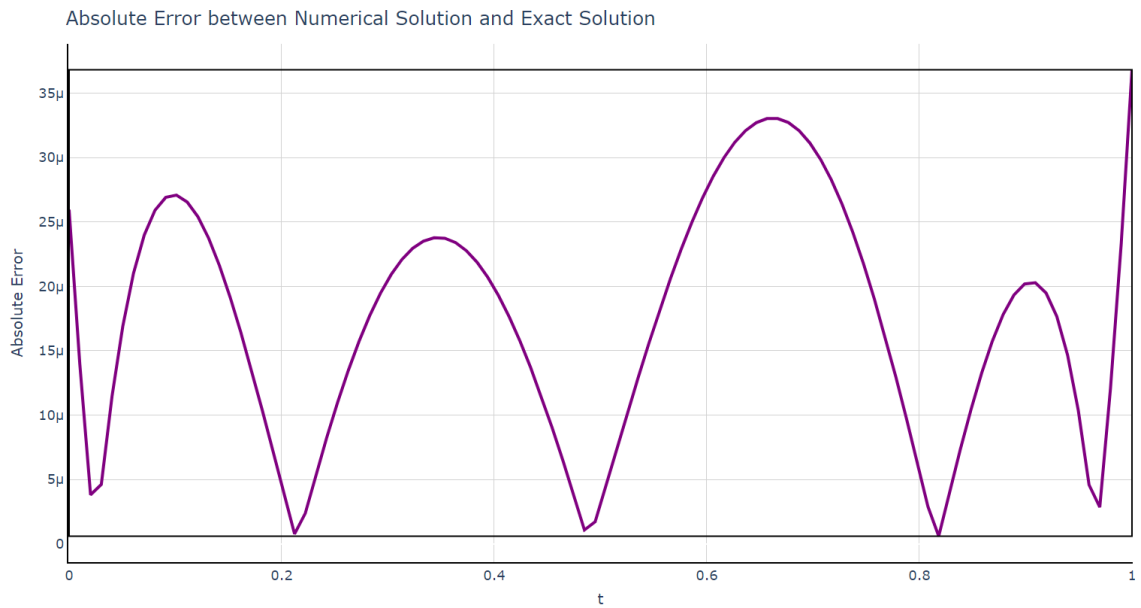
**Figure3**

Shows both exact and numerical solutions using  $M = 4$  when  $t = [0, 1]$



**Figure 4**

Shows the error between exact and numerical solutions using  $M = 4$  When  $t = [0, 1]$



3) when  $M = 6, n=2, \mathcal{D} = 12$

$$u_{app} = \sum_{i=0}^6 c_i t^i$$

$$u_{app} = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + c_6 t^6$$

Similarly

$$[u_{app}]^2 = \sum_{i=0}^{12} v_i y^i$$

We find  $v_i$  from (2.40)

$$v_0 = c_0^2$$

$$v_1 = \frac{\partial}{\partial y} (c_0 + c_1 y)^2 \Big|_{y=0}$$

$$v_1 = \frac{\partial}{\partial y} (c_0^2 + 2c_0 c_1 y + c_1^2 y^2) \Big|_{y=0}$$

$$= 2c_0 c_1 + 2c_1^2 y^1 \Big|_{y=0} = 2c_0 c_1$$

$$v_2 = \frac{1}{2!} \frac{\partial}{\partial y} (c_0 + c_1 y + c_2 y^2)^2 \Big|_{y=0}$$

$$= \frac{1}{2!} \frac{\partial^2}{\partial y^2} (c_0 + c_1 y + c_2 y^2)^2 \Big|_{y=0}$$

$$= \frac{1}{2!} \frac{\partial^2}{\partial y^2} ((c_0 + c_1 y)^2 + 2(c_0 + c_1 y)c_2 y^2 + (c_2 y^2)^2) \Big|_{y=0}$$

$$= \frac{1}{2!} \frac{\partial^2}{\partial y^2} (c_0^2 + 2c_0 c_1 y + c_1^2 y^2 + 2c_0 c_2 y^2 + 2c_1 c_2 y^3 + c_2^2 y^4) \Big|_{y=0}$$

$$= \frac{1}{2!} \frac{\partial}{\partial y} (2c_0 c_1 + 2c_1^2 y + 4c_0 c_2 y + 6c_1 c_2 y^2 + 4c_2^2 y^3) \Big|_{y=0}$$

$$= \frac{1}{2!} (2c_1^2 + 4c_0 c_2 + 12c_1 c_2 y + 12c_2^2 y^2) \Big|_{y=0} = c_1^2 + 2c_0 c_2$$

$$v_3 = \frac{1}{3!} \frac{\partial^3}{\partial y^3} (c_0 + c_1 y + c_2 y^2 + c_3 y^3)^2 \Big|_{y=0}$$

$$= \frac{1}{3!} \frac{\partial^3}{\partial y^3} ((c_0 + c_1 y)^2 + 2(c_0 + c_1 y)(c_2 y^2 + c_3 y^3) + (c_2 y^2 + c_3 y^3)^2) \Big|_{y=0}$$

$$\begin{aligned}
&= \frac{1}{3!} \frac{\partial^3}{\partial y^3} (c_0^2 + 2c_0c_1y + c_1^2y^2 + 2c_0c_2y^2 \\
&\quad + 2c_0c_3y^3 + 2c_1c_2y^3 + 2c_1c_3y^4 + c_2^2y^4 + 2c_2c_3y^5 + c_3^2y^6)_{y=0} \\
&= \frac{1}{3!} \frac{\partial^2}{\partial y^2} (2c_0c_1 + 2c_1^2y + 4c_0c_2y \\
&\quad + 6c_0c_3y^2 + 6c_1c_2y^2 + 8c_1c_3y^3 + 4c_2^2y^3 + 10c_2c_3y^4 + 6c_3^2y^5)_{y=0} \\
&= \frac{1}{3!} \frac{\partial}{\partial y} (2c_1^2 + 4c_0c_2 \\
&\quad + 12c_0c_3y + 12c_1c_2y + 24c_1c_3y^2 + 12c_2^2y^2 + 40c_2c_3y^3 + 30c_3^2y^4)_{y=0} \\
&= \frac{1}{3!} (12c_0c_3 + 12c_1c_2 + 48c_1c_3y^1 + 24c_2^2y + 120c_2c_3y^2 + 120c_3^2y^3)_{y=0} \\
&= \frac{1}{3!} (12c_0c_3 + 12c_1c_2) = 2c_0c_3 + 2c_1c_2 \\
v_4 &= \frac{1}{4!} \frac{\partial^4}{\partial y^4} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)^2_{y=0} \\
&= \frac{1}{4!} \frac{\partial^3}{\partial y^3} 2(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)(c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3)_{y=0} \\
&= \frac{1}{4!} \frac{\partial^2}{\partial y^2} (2(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)(2c_2 + 6c_3y^1 + 12c_4y^2) + 2(c_1 + 2c_2y^1 \\
&\quad + 3c_3y^2 + 4c_4y^3)(c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3))_{y=0} \\
&= \frac{1}{4!} \frac{\partial}{\partial y} (2(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4)(6c_3 + 24c_4y^1) \\
&\quad + 2(c_1 + 2c_2y^1 + 3c_3y^2 + 4c_4y^3)(2c_2 + 6c_3y^1 + 12c_4y^2) \\
&\quad + 2(c_1 + 2c_2y^1 + 3c_3y^2 + 4c_4y^3)(2c_2 + 6c_3y^1 + 12c_4y^2) + 2(2c_2 \\
&\quad + 6c_3y^1 + 12c_4y^2)(c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3))_{y=0} \\
&= \frac{1}{4!} \frac{\partial}{\partial y} (12c_0c_3 + 48c_0c_4y + 8c_1c_2 + 36c_1c_3y + 96c_1c_4y^2 + 4c_1c_2 + 24c_2^2y \\
&\quad + 72c_2c_3y^2 + 160c_2c_4y^3 + 24c_1c_4y^2 + 80c_2c_4y^3 + 180c_3c_4y^4 \\
&\quad + 336c_4^2y^5 + 12c_1c_3y + 48c_2c_3y^2 + 120c_3^2y^3 + 240c_3c_4y^4)_{y=0}
\end{aligned}$$

$$= \frac{1}{4!} (48c_0c_4 + 36c_1c_3 + 192c_1c_4y + 24c_2^2 + 144c_2c_3y + 480c_2c_4y^2 + 48c_1c_4y \\ + 240c_2c_4y^2 + 720c_3c_4y^3 + 1680c_4^2y^4 + 12c_1c_3 + 96c_2c_3y \\ + 360c_3^2y^2 + 960c_3c_4y^3)_{y=0}$$

$$= \frac{1}{4!} (48c_0c_4 + 48c_1c_3 + 24c_2^2) = 2c_0c_4 + 2c_1c_3 + c_2^2$$

$$v_5 = \frac{1}{5!} \frac{\partial^5}{\partial y^5} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5)^2_{y=0}$$

$$= \frac{1}{5!} \frac{\partial^5}{\partial y^5} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 \\ + c_5y^5)(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5)_{y=0}$$

$$= \frac{1}{5!} \frac{\partial^5}{\partial y^5} (c_0^2 + c_0c_1y + c_0c_2y^2 + c_0c_3y^3 + c_0c_4y^4 + c_0c_5y^5 + c_0c_1y + c_1^2y^2 \\ + c_1c_3y^4 + c_1c_4y^5 + c_1c_5y^6 + c_0c_2y^2 + c_2c_1y^3 + c_2^2y^4 + c_2c_3y^5 \\ + c_2c_4y^6 + c_2c_5y^7 + c_0c_3y^3 + c_1c_3y^4 + c_2c_3y^5 + c_3^2y^6 + c_3c_4y^7 \\ + c_3c_5y^8 + c_0c_4y^4 + c_1c_4y^5 + c_2c_4y^6 + c_3c_4y^7 + c_4^2y^8 + c_4c_5y^9 \\ + c_0c_5y^5 + c_1c_5y^6 + c_2c_5y^7 + c_3c_5y^8 + c_4c_5y^9 + c_5^2y^{10})_{y=0}$$

$$= \frac{1}{5!} \frac{\partial^5}{\partial y^5} (c_0^2 + 2c_0c_1y + 2c_0c_2y^2 + 2c_0c_3y^3 + 2c_0c_4y^4 + 2c_0c_5y^5 + c_1^2y^2 \\ + 2c_1c_2y^3 + 2c_1c_3y^4 + 2c_1c_4y^5 + 2c_1c_5y^6 + c_2^2y^4 + 2c_2c_3y^5 \\ + 2c_2c_4y^6 + 2c_2c_5y^7 + c_3^2y^6 + 2c_3c_4y^7 + 2c_3c_5y^8 + c_4^2y^8 + 2c_4c_5y^9 \\ + c_5^2y^{10})_{y=0}$$

$$= \frac{1}{5!} \frac{\partial^4}{\partial y^4} (2c_0c_1 + 4c_0c_2y + 6c_0c_3y^2 + 8c_0c_4y^3 + 10c_0c_5y^4 + 2c_1^2y + 6c_1c_2y^2 \\ + 8c_1c_3y^3 + 10c_1c_4y^4 + 12c_1c_5y^5 + 4c_2^2y^3 + 10c_2c_3y^4 + 12c_2c_4y^5 \\ + 14c_2c_5y^6 + 6c_3^2y^5 + 14c_3c_4y^6 + 16c_3c_5y^7 + 8c_4^2y^7 + 18c_4c_5y^8 \\ + 10c_5^2y^9)_{y=0}$$

$$\begin{aligned}
&= \frac{1}{5!} \frac{\partial^3}{\partial y^3} (4c_0c_2 + 12c_0c_3y + 24c_0c_4y^2 + 40c_0c_5y^3 + 2c_1^2 + 12c_1c_2y \\
&\quad + 24c_1c_3y^2 + 40c_1c_4y^3 + 60c_1c_5y^4 + 12c_2^2y^2 + 40c_2c_3y^3 \\
&\quad + 60c_2c_4y^4 + 84c_2c_5y^5 + 30c_3^2y^4 + 84c_3c_4y^5 + 112c_3c_5y^6 \\
&\quad + 42c_4^2y^6 + 144c_4c_5y^7 + 90c_5^2y^8)_{y=0} \\
&= \frac{1}{5!} \frac{\partial^2}{\partial y^2} (12c_0c_3 + 48c_0c_4y + 120c_0c_5y^2 + 12c_1c_2 + 48c_1c_3y + 120c_1c_4y^2 \\
&\quad + 240c_1c_5y^3 + 12c_2^2y + 120c_2c_3y^2 + 240c_2c_4y^3 + 420c_2c_5y^4 \\
&\quad + 120c_3^2y^3 + 420c_3c_4y^4 + 672c_3c_5y^5 + 336c_4^2y^5 + 1008c_4c_5y^6 \\
&\quad + 720c_5^2y^7)_{y=0} \\
&= \frac{1}{5!} \frac{\partial}{\partial y} (48c_0c_4 + 240c_0c_5y + 48c_1c_3 + 240c_1c_4y + 720c_1c_5y^2 + 12c_2^2 \\
&\quad + 240c_2c_3y + 720c_2c_4y^2 + 1680c_2c_5y^3 + 360c_3^2y^2 + 1680c_3c_4y^3 \\
&\quad + 3360c_3c_5y^4 + 1680c_4^2y^4 + 6048c_4c_5y^5 + 5040c_5^2y^6)_{y=0} \\
&= \frac{1}{5!} (240c_0c_5 + 240c_1c_4 + 1440c_1c_5y + 240c_2c_3 + 1440c_2c_4y + 5040c_2c_5y^2 \\
&\quad + 720c_3^2y + 5040c_3c_4y^2 + 13440c_3c_5y^3 + 6720c_4^2y^3 \\
&\quad + 30240c_4c_5y^4 + 30240c_5^2y^5)_{y=0} \\
&= \frac{1}{5!} (240c_0c_5 + 240c_1c_4 + 240c_2c_3) = 2c_0c_5 + 2c_1c_4 + 2c_2c_3) \\
v_6 &= \frac{1}{6!} \frac{\partial^6}{\partial y^6} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)^2_{y=0} \\
v_6 &= \frac{1}{6!} \frac{\partial^6}{\partial y^6} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 \\
&\quad + c_6y^6)(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)_{y=0}
\end{aligned}$$

$$\begin{aligned}
v_6 &= \frac{1}{6!} \frac{\partial^6}{\partial y^6} ((c_0 c_0) + c_0 c_1 y + c_0 c_2 y^2 + c_0 c_3 y^3 + c_0 c_4 y^4 + c_0 c_5 y^5 \\
&\quad + c_0 c_6 y^6 + c_0 c_1 y + c_1 y (c_1 y) + c_2 y^2 (c_1 y) + c_3 y^3 (c_1 y) + c_4 y^4 (c_1 y) \\
&\quad + c_5 y^5 (c_1 y) + c_6 y^6 (c_1 y) + c_0 c_2 y^2 + c_1 y (c_2 y^2) + c_2 y^2 (c_2 y^2) \\
&\quad + c_3 y^3 (c_2 y^2) + c_4 y^4 (c_2 y^2) + c_5 y^5 (c_2 y^2) + c_6 y^6 (c_2 y^2) + c_0 c_3 y^3 \\
&\quad + c_1 y (c_3 y^3) + c_2 y^2 (c_3 y^3) + c_3 y^3 (c_3 y^3) + c_4 y^4 (c_3 y^3) + c_5 y^5 (c_3 y^3) \\
&\quad + c_6 y^6 (c_3 y^3) + c_0 c_4 y^4 + c_1 y (c_4 y^4) + c_2 y^2 (c_4 y^4) + c_3 y^3 (c_4 y^4) \\
&\quad + c_4 y^4 (c_4 y^4) + c_5 y^5 (c_4 y^4) + c_6 y^6 (c_4 y^4) + c_0 c_5 y^5 + c_1 y (c_5 y^5) \\
&\quad + c_2 y^2 (c_5 y^5) + c_3 y^3 (c_5 y^5) + c_4 y^4 (c_5 y^5) + c_5 y^5 (c_5 y^5) + c_6 y^6 (c_5 y^5) \\
&\quad + c_0 c_6 y^6 + c_1 y (c_6 y^6) + c_2 y^2 (c_6 y^6) + c_3 y^3 (c_6 y^6) + c_4 y^4 (c_6 y^6) \\
&\quad + c_5 y^5 (c_6 y^6) + c_6 y^6 (c_6 y^6))_{y=0} \\
&= \frac{1}{6!} \frac{\partial^6}{\partial y^6} (c_0^2 + c_0 c_1 y + c_0 c_2 y^2 + c_0 c_3 y^3 + c_0 c_4 y^4 + c_0 c_5 y^5 + c_0 c_6 y^6 + c_0 c_1 y \\
&\quad + c_1^2 y^2 + c_1 c_2 y^3 + c_1 c_3 y^4 + c_1 c_4 y^5 + c_1 c_5 y^6 + c_1 c_6 y^7 + c_0 c_2 y^2 \\
&\quad + c_2 c_1 y^3 + c_2^2 y^4 + c_2 c_3 y^5 + c_2 c_4 y^6 + c_2 c_5 y^7 + c_2 c_6 y^8 + c_0 c_3 y^3 \\
&\quad + c_1 c_3 y^4 + c_2 c_3 y^5 + c_3^2 y^6 + c_3 c_4 y^7 + c_3 c_5 y^8 + c_3 c_6 y^9 + c_0 c_4 y^4 \\
&\quad + c_1 c_4 y^5 + c_2 c_4 y^6 + c_3 c_4 y^7 + c_4^2 y^8 + c_4 c_5 y^9 + c_4 c_6 y^{10} + c_0 c_5 y^5 \\
&\quad + c_1 c_5 y^6 + c_2 c_5 y^7 + c_3 c_5 y^8 + c_4 c_5 y^9 + c_5^2 y^{10} + c_5 c_6 y^{11} + c_0 c_6 y^6 \\
&\quad + c_1 c_6 y^7 + c_2 c_6 y^8 + c_3 c_6 y^9 + c_4 c_6 y^{10} + c_5 c_6 y^{11} + c_6^2 y^{12})_{y=0} \\
&= \frac{1}{6!} \frac{\partial^6}{\partial y^6} (c_0^2 + 2c_0 c_1 y + 2c_0 c_2 y^2 + 2c_0 c_3 y^3 + 2c_0 c_4 y^4 \\
&\quad + 2c_0 c_5 y^5 + 2c_0 c_6 y^6 + c_1^2 y^2 + 2c_1 c_2 y^3 + 2c_1 c_3 y^4 + 2c_1 c_4 y^5 \\
&\quad + 2c_1 c_5 y^6 + 2c_1 c_6 y^7 + c_2^2 y^4 + 2c_2 c_3 y^5 + 2c_2 c_4 y^6 + 2c_2 c_5 y^7 \\
&\quad + 2c_2 c_6 y^8 + c_3^2 y^6 + 2c_3 c_4 y^7 + 2c_3 c_5 y^8 + 2c_3 c_6 y^9 + c_4^2 y^8 + 2c_4 c_5 y^9 \\
&\quad + 2c_4 c_6 y^{10} + c_5^2 y^{10} + 2c_5 c_6 y^{11} + c_6^2 y^{12})_{y=0} \\
&= \frac{1}{6!} \frac{\partial^5}{\partial y^5} (2c_0 c_1 + 4c_0 c_2 y + 6c_0 c_3 y^2 + 8c_0 c_4 y^3 + 10c_0 c_5 y^4 + 12c_0 c_6 y^5 \\
&\quad + 2c_1^2 y + 6c_1 c_2 y^2 + 8c_1 c_3 y^3 + 10c_1 c_4 y^4 + 12c_1 c_5 y^5 + 14c_1 c_6 y^6 \\
&\quad + 4c_2^2 y^3 + 10c_2 c_3 y^4 + 12c_2 c_4 y^5 + 14c_2 c_5 y^6 + 16c_2 c_6 y^7 + 6c_3^2 y^5 \\
&\quad + 14c_3 c_4 y^6 + 16c_3 c_5 y^7 + 18c_3 c_6 y^8 + 8c_4^2 y^7 + 18c_4 c_5 y^8 + 20c_4 c_6 y^9 \\
&\quad + 10c_5^2 y^9 + 22c_5 c_6 y^{10} + 12c_6^2 y^{11})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6!} \frac{\partial^4}{\partial y^4} (4c_0c_2 + 12c_0c_3y + 24c_0c_4y^2 + 40c_0c_5y^3 + 60c_0c_6y^4 + 2c_1^2 + 12c_1c_2y \\
&\quad + 24c_1c_3y^2 + 40c_1c_4y^3 + 60c_1c_5y^4 + 84c_1c_6y^5 + 12c_2^2y^2 \\
&\quad + 40c_2c_3y^3 + 60c_2c_4y^4 + 84c_2c_5y^6 + 112c_2c_6y^6 + 30c_3^2y^4 \\
&\quad + 84c_3c_4y^5 + 112c_3c_5y^6 + 144c_3c_6y^7 + 56c_4^2y^6 + 144c_4c_5y^7 \\
&\quad + 180c_4c_6y^8 + 90c_5^2y^8 + 220c_5c_6y^9 + 132c_6^2y^{10})_{y=0} \\
&= \frac{1}{6!} \frac{\partial^3}{\partial y^3} (12c_0c_3 + 48c_0c_4y + 120c_0c_5y^2 + 240c_0c_6y^3 + 12c_1c_2 + 48c_1c_3y \\
&\quad + 120c_1c_4y^2 + 240c_1c_5y^3 + 420c_1c_6y^4 + 24c_2^2y + 120c_2c_3y^3 \\
&\quad + 240c_2c_4y^3 + 420c_2c_5y^4 + 672c_2c_6y^5 + 120c_3^2y^3 + 420c_3c_4y^4 \\
&\quad + 672c_3c_5y^5 + 1008c_3c_6y^6 + 336c_4^2y^5 + 1008c_4c_5y^6 + 1440c_4c_6y^7 \\
&\quad + 720c_5^2y^7 + 1980c_5c_6y^8 + 1320c_6^2y^9)_{y=0} \\
&= \frac{1}{6!} \frac{\partial^2}{\partial y^2} (48c_0c_4 + 240c_0c_5y + 720c_0c_6y^2 + 48c_1c_3 + 240c_1c_4y + 720c_1c_5y^2 \\
&\quad + 1680c_1c_6y^3 + 24c_2^2 + 360c_2c_3y^2 + 720c_2c_4y^2 + 1680c_2c_5y^3 \\
&\quad + 3360c_2c_6y^4 + 360c_3^2y^2 + 1680c_3c_4y^3 + 3360c_3c_5y^4 \\
&\quad + 6048c_3c_6y^5 + 1680c_4^2y^4 + 6048c_4c_5y^5 + 10080c_4c_6y^6 \\
&\quad + 5040c_5^2y^6 + 15840c_5c_6y^7 + 11880c_6^2y^8)_{y=0} \\
&= \frac{1}{6!} \frac{\partial}{\partial y} (240c_0c_5 + 14400c_0c_6y + 240c_1c_4 + 1440c_1c_5y + 5040c_1c_6y^2 \\
&\quad + 720c_2c_3y + 1440c_2c_4y + 5040c_2c_5y^2 + 13440c_2c_6y^3 + 720c_3^2y \\
&\quad + 5040c_3c_4y^2 + 13440c_3c_5y^3 + 30240c_3c_6y^4 + 6720c_4^2y^3 \\
&\quad + 30240c_4c_5y^4 + 60480c_4c_6y^5 + 30240c_5^2y^5 + 110880c_5c_6y^6 \\
&\quad + 95040c_6^2y^7)_{y=0} \\
&= \frac{1}{6!} (1440c_0c_6 + 1440c_1c_5 + 10080c_1c_6y + 720c_2c_3 + 1440c_2c_4 + 10080c_2c_5y \\
&\quad + 40320c_2c_6y^2 + 720c_3^2 + 10080c_3c_4y + 40320c_3c_5y^2 \\
&\quad + 120960c_3c_6y^3 + 20160c_4^2y^2 + 120960c_4c_5y^3 + 302400c_4c_6y^4 \\
&\quad + 151200c_5^2y^4 + 665280c_5c_6y^5 + 665280c_6^2y^6)_{y=0} \\
&= \frac{1}{6!} (1440c_0c_6 + 1440c_1c_5 + 720c_2c_3 + 1440c_2c_4 + 720c_3^2)
\end{aligned}$$

$$= (2c_0c_6 + 2c_1c_5 + c_2c_3 + 2c_2c_4 + c_3^2)$$

$$v_7 = \frac{1}{7!} \frac{\partial^7}{\partial y^7} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)^2_{y=0}$$

$$v_7 = \frac{1}{7!} \frac{\partial^7}{\partial y^7} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)_{y=0}$$

$$v_7 = \frac{1}{7!} \frac{\partial^7}{\partial y^7} ((c_0c_0) + c_0c_1y + c_0c_2y^2 + c_0c_3y^3 + c_0c_4y^4 + c_0c_5y^5 + c_0c_6y^6 + c_0c_1y + c_1y(c_1y) + c_2y^2(c_1y) + c_3y^3(c_1y) + c_4y^4(c_1y) + c_5y^5(c_1y) + c_6y^6(c_1y) + c_0c_2y^2 + c_1y(c_2y^2) + c_2y^2(c_2y^2) + c_3y^3(c_2y^2) + c_4y^4(c_2y^2) + c_5y^5(c_2y^2) + c_6y^6(c_2y^2) + c_0c_3y^3 + c_1y(c_3y^3) + c_2y^2(c_3y^3) + c_3y^3(c_3y^3) + c_4y^4(c_3y^3) + c_5y^5(c_3y^3) + c_6y^6(c_3y^3) + c_0c_4y^4 + c_1y(c_4y^4) + c_2y^2(c_4y^4) + c_3y^3(c_4y^4) + c_4y^4(c_4y^4) + c_5y^5(c_4y^4) + c_6y^6(c_4y^4) + c_0c_5y^5 + c_1y(c_5y^5) + c_2y^2(c_5y^5) + c_3y^3(c_5y^5) + c_4y^4(c_5y^5) + c_5y^5(c_5y^5) + c_6y^6(c_5y^5) + c_0c_6y^6 + c_1y(c_6y^6) + c_2y^2(c_6y^6) + c_3y^3(c_6y^6) + c_4y^4(c_6y^6) + c_5y^5(c_6y^6) + c_6y^6(c_6y^6))_{y=0}$$

$$= \frac{1}{7!} \frac{\partial^7}{\partial y^7} (c_0^2 + 2c_0c_1y + 2c_0c_2y^2 + 2c_0c_3y^3 + 2c_0c_4y^4 + 2c_0c_5y^5 + 2c_0c_6y^6 + c_1^2y^2 + 2c_1c_2y^3 + 2c_1c_3y^4 + 2c_1c_4y^5 + 2c_1c_5y^6 + 2c_1c_6y^7 + c_2^2y^4 + 2c_2c_3y^5 + 2c_2c_4y^6 + 2c_2c_5y^7 + 2c_2c_6y^8 + c_3^2y^6 + 2c_3c_4y^7 + 2c_3c_5y^8 + 2c_3c_6y^9 + c_4^2y^8 + 2c_4c_5y^9 + 2c_4c_6y^{10} + c_5^2y^{10} + 2c_5c_6y^{11} + c_6^2y^{12})_{y=0}$$

$$= \frac{1}{7!} \frac{\partial^6}{\partial y^6} (2c_0c_1 + 4c_0c_2y + 6c_0c_3y^2 + 8c_0c_4y^3 + 10c_0c_5y^4 + 12c_0c_6y^5 + 2c_1^2y + 6c_1c_2y^2 + 8c_1c_3y^3 + 10c_1c_4y^4 + 12c_1c_5y^5 + 14c_1c_6y^6 + 4c_2^2y^3 + 10c_2c_3y^4 + 12c_2c_4y^5 + 14c_2c_5y^6 + 16c_2c_6y^7 + 6c_3^2y^5 + 14c_3c_4y^6 + 16c_3c_5y^7 + 18c_3c_6y^8 + 8c_4^2y^7)_{y=0}$$

$$\begin{aligned}
&= \frac{1}{7!} \frac{\partial^5}{\partial y^5} (4c_0c_2 + 12c_0c_3y + 24c_0c_4y^2 + 40c_0c_5y^3 + 60c_0c_6y^4 + 2c_1^2 + 12c_1c_2y \\
&\quad + 24c_1c_3y^2 + 40c_1c_4y^3 + 60c_1c_5y^4 + 84c_1c_6y^5 + 12c_2^2y^2 \\
&\quad + 40c_2c_3y^3 + 60c_2c_4y^4 + 84c_2c_5y^6 + 112c_2c_6y^6 + 30c_3^2y^4 \\
&\quad + 84c_3c_4y^5 + 112c_3c_5y^6 + 144c_3c_6y^7 + 56c_4^2y^6 + 144c_4c_5y^7 \\
&\quad + 180c_4c_6y^8 + 90c_5^2y^8 + 220c_5c_6y^9 + 132c_6^2y^{10})_{y=0} + 18c_4c_5y^8 \\
&\quad + 20c_4c_6y^9 + 10c_5^2y^9 + 22c_5c_6y^{10} + 12c_6^2y^{11})_{y=0} \\
&= \frac{1}{7!} \frac{\partial^4}{\partial y^4} (12c_0c_3 + 48c_0c_4y + 120c_0c_5y^2 + 240c_0c_6y^3 + 12c_1c_2 + 48c_1c_3y \\
&\quad + 120c_1c_4y^2 + 240c_1c_5y^3 + 420c_1c_6y^4 + 24c_2^2y + 120c_2c_3y^3 \\
&\quad + 240c_2c_4y^3 + 420c_2c_5y^4 + 672c_2c_6y^5 + 120c_3^2y^3 + 420c_3c_4y^4 \\
&\quad + 672c_3c_5y^5 + 1008c_3c_6y^6 + 336c_4^2y^5 + 1008c_4c_5y^6 + 1440c_4c_6y^7 \\
&\quad + 720c_5^2y^7 + 1980c_5c_6y^8 + 1320c_6^2y^9)_{y=0} \\
&= \frac{1}{7!} \frac{\partial^3}{\partial y^3} (48c_0c_4 + 240c_0c_5y + 720c_0c_6y^2 + 48c_1c_3 + 240c_1c_4y + 720c_1c_5y^2 \\
&\quad + 1680c_1c_6y^3 + 24c_2^2 + 360c_2c_3y^2 + 720c_2c_4y^2 + 1680c_2c_5y^3 \\
&\quad + 3360c_2c_6y^4 + 360c_3^2y^2 + 1680c_3c_4y^3 + 3360c_3c_5y^4 \\
&\quad + 6048c_3c_6y^5 + 1680c_4^2y^4 + 6048c_4c_5y^5 + 10080c_4c_6y^6 \\
&\quad + 5040c_5^2y^6 + 15840c_5c_6y^7 + 11880c_6^2y^8)_{y=0} \\
&= \frac{1}{7!} \frac{\partial^2}{\partial y^2} (240c_0c_5 + 1440c_0c_6y + 240c_1c_4 + 1440c_1c_5y + 5040c_1c_6y^2 + 720c_2c_3 \\
&\quad + 1440c_2c_4y + 5040c_2c_5y^2 + 13440c_2c_6y^3 + 720c_3^2y + 5040c_3c_4y^2 \\
&\quad + 13440c_3c_5y^3 + 30240c_3c_6y^4 + 6720c_4^2y^3 + 30240c_4c_5y^4 \\
&\quad + 60480c_4c_6y^5 + 30240c_5^2y^5 + 110880c_5c_6y^6 + 95040c_6^2y^7)_{y=0} \\
&= \frac{1}{7!} \frac{\partial}{\partial y} (1440c_0c_6 + 1440c_1c_5 + 10080c_1c_6y + 720c_2c_3 + 1440c_2c_4 + 10080c_2c_5y \\
&\quad + 40320c_2c_6y^2 + 720c_3^2 + 10080c_3c_4y + 40320c_3c_5y^2 \\
&\quad + 120960c_3c_6y^3 + 20160c_4^2y^2 + 120960c_4c_5y^3 + 302400c_4c_6y^4 \\
&\quad + 151200c_5^2y^4 + 665280c_5c_6y^5 + 665280c_6^2y^6)_{y=0}
\end{aligned}$$

$$= \frac{1}{7!} (10080c_1c_6 + 10080c_2c_5 + 80640c_2c_6y + 10080c_3c_4 + 80640c_3c_5y \\ + 362880c_3c_6y^2 + 40320c_4^2y^2 + 362880c_4c_5y^2 + 1209600c_4c_6y^3 \\ + 604800c_5^2y^3 + 3326400c_5c_6y^4 + 3991680c_6^2y^5)_{y=0}$$

$$= \frac{1}{7!} (10080c_1c_6 + 10080c_2c_5 + 10080c_3c_4) = 2c_1c_6 + 2c_2c_5 + 2c_3c_4$$

$$v_8 = \frac{1}{8!} \frac{\partial^8}{\partial y^8} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)^2_{y=0}$$

$$v_8 = \frac{1}{8!} \frac{\partial^8}{\partial y^8} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 \\ + c_6y^6)(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)_{y=0}$$

$$v_8 = \frac{1}{8!} \frac{\partial^8}{\partial y^8} ((c_0c_0) + c_0c_1y + c_0c_2y^2 + c_0c_3y^3 + c_0c_4y^4 + c_0c_5y^5 + c_0c_6y^6 \\ + c_0c_1y + c_1y(c_1y) + c_2y^2(c_1y) + c_3y^3(c_1y) + c_4y^4(c_1y) \\ + c_5y^5(c_1y) + c_6y^6(c_1y) + c_0c_2y^2 + c_1y(c_2y^2) + c_2y^2(c_2y^2) \\ + c_3y^3(c_2y^2) + c_4y^4(c_2y^2) + c_5y^5(c_2y^2) + c_6y^6(c_2y^2) + c_0c_3y^3 \\ + c_1y(c_3y^3) + c_2y^2(c_3y^3) + c_3y^3(c_3y^3) + c_4y^4(c_3y^3) + c_5y^5(c_3y^3) \\ + c_6y^6(c_3y^3) + c_0c_4y^4 + c_1y(c_4y^4) + c_2y^2(c_4y^4) + c_3y^3(c_4y^4) \\ + c_4y^4(c_4y^4) + c_5y^5(c_4y^4) + c_6y^6(c_4y^4) + c_0c_5y^5 + c_1y(c_5y^5) \\ + c_2y^2(c_5y^5) + c_3y^3(c_5y^5) + c_4y^4(c_5y^5) + c_5y^5(c_5y^5) + c_6y^6(c_5y^5) \\ + c_0c_6y^6 + c_1y(c_6y^6) + c_2y^2(c_6y^6) + c_3y^3(c_6y^6) + c_4y^4(c_6y^6) \\ + c_5y^5(c_6y^6) + c_6y^6(c_6y^6))_{y=0}$$

$$= \frac{1}{8!} \frac{\partial^8}{\partial y^8} (c_0^2 + c_0c_1y + c_0c_2y^2 + c_0c_3y^3 + c_0c_4y^4 + c_0c_5y^5 + c_0c_6y^6 + c_0c_1y \\ + c_1^2y^2 + c_1c_2y^3 + c_1c_3y^4 + c_1c_4y^5 + c_1c_5y^6 + c_1c_6y^7 + c_0c_2y^2 \\ + c_2c_1y^3 + c_2^2y^4 + c_2c_3y^5 + c_2c_4y^6 + c_2c_5y^7 + c_2c_6y^8 + c_0c_3y^3 \\ + c_1c_3y^4 + c_2c_3y^5 + c_3^2y^6 + c_3c_4y^7 + c_3c_5y^8 + c_3c_6y^9 + c_0c_4y^4 \\ + c_1c_4y^5 + c_2c_4y^6 + c_3c_4y^7 + c_4^2y^8 + c_4c_5y^9 + c_4c_6y^{10} + c_0c_5y^5 \\ + c_1c_5y^6 + c_2c_5y^7 + c_3c_5y^8 + c_4c_5y^9 + c_5^2y^{10} + c_5c_6y^{11} + c_0c_6y^6 \\ + c_1c_6y^7 + c_2c_6y^8 + c_3c_6y^9 + c_4c_6y^{10} + c_5c_6y^{11} + c_6^2y^{12})_{y=0}$$

$$\begin{aligned}
&= \frac{1}{8!} \frac{\partial^8}{\partial y^8} (c_0^2 + 2c_0c_1y + 2c_0c_2y^2 + 2c_0c_3y^3 + 2c_0c_4y^4 + 2c_0c_5y^5 + 2c_0c_6y^6 \\
&\quad + c_1^2y^2 + 2c_1c_2y^3 + 2c_1c_3y^4 + 2c_1c_4y^5 + 2c_1c_5y^6 + 2c_1c_6y^7 \\
&\quad + c_2^2y^4 + 2c_2c_3y^5 + 2c_2c_4y^6 + 2c_2c_5y^7 + 2c_2c_6y^8 + c_3^2y^6 + 2c_3c_4y^7 \\
&\quad + 2c_3c_5y^8 + 2c_3c_6y^8 + c_4^2y^8 + 2c_4c_5y^9 + 2c_4c_6y^{10} + c_5^2y^{10} \\
&\quad + 2c_5c_6y^{11} + c_6^2y^{12})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8!} \frac{\partial^7}{\partial y^7} (2c_0c_1 + 4c_0c_2y + 6c_0c_3y^2 + 8c_0c_4y^3 + 10c_0c_5y^4 + 12c_0c_6y^5 \\
&\quad + 2c_1^2y + 6c_1c_2y^2 + 8c_1c_3y^3 + 10c_1c_4y^4 + 12c_1c_5y^5 + 14c_1c_6y^6 \\
&\quad + 4c_2^2y^3 + 10c_2c_3y^4 + 12c_2c_4y^5 + 14c_2c_5y^6 + 16c_2c_6y^7 + 6c_3^2y^5 \\
&\quad + 14c_3c_4y^6 + 16c_3c_5y^7 + 18c_3c_6y^8 + 8c_4^2y^7 + 18c_4c_5y^8 + 20c_4c_6y^9 \\
&\quad + 10c_5^2y^9 + 22c_5c_6y^{10} + 12c_6^2y^{11})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8!} \frac{\partial^6}{\partial y^6} (4c_0c_2 + 12c_0c_3y + 24c_0c_4y^2 + 40c_0c_5y^3 + 60c_0c_6y^4 \\
&\quad + 2c_1^2 + 12c_1c_2y + 24c_1c_3y^2 + 40c_1c_4y^3 + 60c_1c_5y^4 + 84c_1c_6y^5 \\
&\quad + 12c_2^2y^2 + 40c_2c_3y^3 + 60c_2c_4y^4 + 84c_2c_5y^6 + 112c_2c_6y^6 \\
&\quad + 30c_3^2y^4 + 84c_3c_4y^5 + 112c_3c_5y^6 + 144c_3c_6y^7 + 56c_4^2y^6 \\
&\quad + 144c_4c_5y^7 + 180c_4c_6y^8 + 90c_5^2y^8 + 220c_5c_6y^9 + 132c_6^2y^{10})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8!} \frac{\partial^5}{\partial y^5} (12c_0c_3 + 48c_0c_4y + 120c_0c_5y^2 + 240c_0c_6y^3 \\
&\quad + 12c_1c_2 + 48c_1c_3y + 120c_1c_4y^2 + 240c_1c_5y^3 + 420c_1c_6y^4 + 24c_2^2y \\
&\quad + 120c_2c_3y^3 + 240c_2c_4y^3 + 420c_2c_5y^4 + 672c_2c_6y^5 + 120c_3^2y^3 \\
&\quad + 420c_3c_4y^4 + 672c_3c_5y^5 + 1008c_3c_6y^6 + 336c_4^2y^5 + 1008c_4c_5y^6 \\
&\quad + 1440c_4c_6y^7 + 720c_5^2y^7 + 1980c_5c_6y^8 + 1320c_6^2y^9)_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8!} \frac{\partial^4}{\partial y^4} (48c_0c_4 + 240c_0c_5y + 720c_0c_6y^2 + 48c_1c_3 + 240c_1c_4y + 720c_1c_5y^2 \\
&\quad + 1680c_1c_6y^3 + 24c_2^2 + 360c_2c_3y^2 + 720c_2c_4y^2 + 1680c_2c_5y^3 \\
&\quad + 3360c_2c_6y^4 + 360c_3^2y^2 + 1680c_3c_4y^3 + 3360c_3c_5y^4 \\
&\quad + 6048c_3c_6y^5 + 1680c_4^2y^4 + 6048c_4c_5y^5 + 10080c_4c_6y^6 \\
&\quad + 5040c_5^2y^6 + 15840c_5c_6y^7 + 11880c_6^2y^8)_{y=0}
\end{aligned}$$

$$= \frac{1}{8!} \frac{\partial^3}{\partial y^3} (240c_0c_5 + 14400c_0c_6y + 240c_1c_4 + 1440c_1c_5y + 5040c_1c_6y^2 + 720c_2c_3 + 1440c_2c_4y + 5040c_2c_5y^2 + 13440c_2c_6y^3 + 720c_3^2y + 5040c_3c_4y^2 + 13440c_3c_5y^3 + 30240c_3c_6y^4 + 6720c_4^2y^3 + 30240c_4c_5y^4 + 60480c_4c_6y^5 + 30240c_5^2y^5 + 110880c_5c_6y^6 + 95040c_6^2y^7)_{y=0}$$

$$= \frac{1}{8!} \frac{\partial^2}{\partial y^2} (1440c_0c_6 + 1440c_1c_5 + 10080c_1c_6y + 720c_2c_3 + 1440c_2c_4 + 10080c_2c_5 + 40320c_2c_6y^2 + 720c_3^2 + 10080c_3c_4y + 40320c_3c_5y^2 + 120960c_3c_6y^3 + 20160c_4^2y^2 + 120960c_4c_5y^3 + 302400c_4c_6y^4 + 151200c_5^2y^4 + 665280c_5c_6y^5 + 665280c_6^2y^6)_{y=0}$$

$$= \frac{1}{8!} \frac{\partial}{\partial y} (10080c_1c_6 + 10080c_2c_5 + 80640c_2c_6y + 10080c_3c_4 + 80640c_3c_5y + 362880c_3c_6y^2 + 40320c_4^2y + 362880c_4c_5y^2 + 1209600c_4c_6y^3 + 604800c_5^2y^3 + 3326400c_5c_6y^4 + 3991680c_6^2y^5)_{y=0}$$

$$= \frac{1}{8!} (80640c_2c_6 + 80640c_3c_5 + 725760c_3c_6y + 40320c_4^2 + 725760c_4c_5y + 3628800c_4c_6y^2 + 1814400c_5^2y^2 + 293305600c_5c_6y^3 + 19958400c_6^2y^4)_{y=0}$$

$$= \frac{1}{8!} (80640c_2c_6 + 80640c_3c_5 + 40320c_4^2) = 2c_2c_6 + 2c_3c_5 + c_4^2$$

$$v_9 = \frac{1}{9!} \frac{\partial^9}{\partial y^9} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)^2_{y=0}$$

$$v_9 = \frac{1}{9!} \frac{\partial^9}{\partial y^9} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)_{y=0}$$

$$\begin{aligned}
v_9 = \frac{1}{9!} \frac{\partial^9}{\partial y^9} & ((c_0 c_0) + c_0 c_1 y + c_0 c_2 y^2 + c_0 c_3 y^3 + c_0 c_4 y^4 + c_0 c_5 y^5 \\
& + c_0 c_6 y^6 + c_0 c_1 y + c_1 y (c_1 y) + c_2 y^2 (c_1 y) + c_3 y^3 (c_1 y) + c_4 y^4 (c_1 y) \\
& + c_5 y^5 (c_1 y) + c_6 y^6 (c_1 y) + c_0 c_2 y^2 + c_1 y (c_2 y^2) + c_2 y^2 (c_2 y^2) \\
& + c_3 y^3 (c_2 y^2) + c_4 y^4 (c_2 y^2) + c_5 y^5 (c_2 y^2) + c_6 y^6 (c_2 y^2) + c_0 c_3 y^3 \\
& + c_1 y (c_3 y^3) + c_2 y^2 (c_3 y^3) + c_3 y^3 (c_3 y^3) + c_4 y^4 (c_3 y^3) + c_5 y^5 (c_3 y^3) \\
& + c_6 y^6 (c_3 y^3) + c_0 c_4 y^4 + c_1 y (c_4 y^4) + c_2 y^2 (c_4 y^4) + c_3 y^3 (c_4 y^4) \\
& + c_4 y^4 (c_4 y^4) + c_5 y^5 (c_4 y^4) + c_6 y^6 (c_4 y^4) + c_0 c_5 y^5 + c_1 y (c_5 y^5) \\
& + c_2 y^2 (c_5 y^5) + c_3 y^3 (c_5 y^5) + c_4 y^4 (c_5 y^5) + c_5 y^5 (c_5 y^5) + c_6 y^6 (c_5 y^5) \\
& + c_0 c_6 y^6 + c_1 y (c_6 y^6) + c_2 y^2 (c_6 y^6) + c_3 y^3 (c_6 y^6) + c_4 y^4 (c_6 y^6) \\
& + c_5 y^5 (c_6 y^6) + c_6 y^6 (c_6 y^6))_{y=0}
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{9!} \frac{\partial^9}{\partial y^9} & (c_0^2 + c_0 c_1 y + c_0 c_2 y^2 + c_0 c_3 y^3 + c_0 c_4 y^4 + c_0 c_5 y^5 + c_0 c_6 y^6 + c_0 c_1 y \\
& + c_1^2 y^2 + c_1 c_2 y^3 + c_1 c_3 y^4 + c_1 c_4 y^5 + c_1 c_5 y^6 + c_1 c_6 y^7 + c_0 c_2 y^2 \\
& + c_2 c_1 y^3 + c_2^2 y^4 + c_2 c_3 y^5 + c_2 c_4 y^6 + c_2 c_5 y^7 + c_2 c_6 y^8 + c_0 c_3 y^3 \\
& + c_1 c_3 y^4 + c_2 c_3 y^5 + c_3^2 y^6 + c_3 c_4 y^7 + c_3 c_5 y^8 + c_3 c_6 y^9 + c_0 c_4 y^4 \\
& + c_1 c_4 y^5 + c_2 c_4 y^6 + c_3 c_4 y^7 + c_4^2 y^8 + c_4 c_5 y^9 + c_4 c_6 y^{10} + c_0 c_5 y^5 \\
& + c_1 c_5 y^6 + c_2 c_5 y^7 + c_3 c_5 y^8 + c_4 c_5 y^9 + c_5^2 y^{10} + c_5 c_6 y^{11} + c_0 c_6 y^6 \\
& + c_1 c_6 y^7 + c_2 c_6 y^8 + c_3 c_6 y^9 + c_4 c_6 y^{10} + c_5 c_6 y^{11} + c_6^2 y^{12})_{y=0}
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{9!} \frac{\partial^9}{\partial y^9} & (c_0^2 + 2c_0 c_1 y + 2c_0 c_2 y^2 + 2c_0 c_3 y^3 + 2c_0 c_4 y^4 \\
& + 2c_0 c_5 y^5 + 2c_0 c_6 y^6 + c_1^2 y^2 + 2c_1 c_2 y^3 + 2c_1 c_3 y^4 + 2c_1 c_4 y^5 \\
& + 2c_1 c_5 y^6 + 2c_1 c_6 y^7 + c_2^2 y^4 + 2c_2 c_3 y^5 + 2c_2 c_4 y^6 + 2c_2 c_5 y^7 \\
& + 2c_2 c_6 y^8 + c_3^2 y^6 + 2c_3 c_4 y^7 + 2c_3 c_5 y^8 + 2c_3 c_6 y^9 + c_4^2 y^8 + 2c_4 c_5 y^9 \\
& + 2c_4 c_6 y^{10} + c_5^2 y^{10} + 2c_5 c_6 y^{11} + c_6^2 y^{12})_{y=0}
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{9!} \frac{\partial^8}{\partial y^8} & (2c_0 c_1 + 4c_0 c_2 y + 6c_0 c_3 y^2 + 8c_0 c_4 y^3 \\
& + 10c_0 c_5 y^4 + 12c_0 c_6 y^5 + 2c_1^2 y + 6c_1 c_2 y^2 + 8c_1 c_3 y^3 + 10c_1 c_4 y^4 \\
& + 12c_1 c_5 y^5 + 14c_1 c_6 y^6 + 4c_2^2 y^3 + 10c_2 c_3 y^4 + 12c_2 c_4 y^5 + 14c_2 c_5 y^6 \\
& + 16c_2 c_6 y^7 + 6c_3^2 y^5 + 14c_3 c_4 y^6 + 16c_3 c_5 y^7 + 18c_3 c_6 y^8 + 8c_4^2 y^7 \\
& + 18c_4 c_5 y^8 + 20c_4 c_6 y^9 + 10c_5^2 y^9 + 22c_5 c_6 y^{10} + 12c_6^2 y^{11})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{9!} \frac{\partial^7}{\partial y^7} (4c_0c_2 + 12c_0c_3y + 24c_0c_4y^2 \\
&\quad + 40c_0c_5y^3 + 60c_0c_6y^4 + 2c_1^2 + 12c_1c_2y + 24c_1c_3y^2 + 40c_1c_4y^3 \\
&\quad + 60c_1c_5y^4 + 84c_1c_6y^5 + 12c_2^2y^2 + 40c_2c_3y^3 + 60c_2c_4y^4 \\
&\quad + 84c_2c_5y^6 + 112c_2c_6y^6 + 30c_3^2y^4 + 84c_3c_4y^5 + 112c_3c_5y^6 \\
&\quad + 144c_3c_6y^7 + 56c_4^2y^6 + 144c_4c_5y^7 + 180c_4c_6y^8 + 90c_5^2y^8)_{y=0}
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{9!} \frac{\partial^6}{\partial y^6} (12c_0c_3 + 48c_0c_4y + 120c_0c_5y^2 + 240c_0c_6y^3 \\
&\quad + 12c_1c_2 + 48c_1c_3y + 120c_1c_4y^2 + 240c_1c_5y^3 + 420c_1c_6y^4 + 24c_2^2y \\
&\quad + 120c_2c_3y^3 + 240c_2c_4y^3 + 420c_2c_5y^4 + 672c_2c_6y^5 + 120c_3^2y^3 \\
&\quad + 420c_3c_4y^4 + 672c_3c_5y^5 + 1008c_3c_6y^6 + 336c_4^2y^5 + 1008c_4c_5y^6 \\
&\quad + 1440c_4c_6y^7 + 720c_5^2y^7 + 1980c_5c_6y^8 + 1320c_6^2y^9)_{y=0} \\
&\quad + 220c_5c_6y^9 + 132c_6^2y^{10})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{9!} \frac{\partial^5}{\partial y^5} (48c_0c_4 + 240c_0c_5y + 720c_0c_6y^2 + 48c_1c_3 \\
&\quad + 240c_1c_4y + 720c_1c_5y^2 + 1680c_1c_6y^3 + 24c_2^2 + 360c_2c_3y^2 \\
&\quad + 720c_2c_4y^2 + 1680c_2c_5y^3 + 3360c_2c_6y^4 + 360c_3^2y^2 + 1680c_3c_4y^3 \\
&\quad + 3360c_3c_5y^4 + 6048c_3c_6y^5 + 1680c_4^2y^4 + 6048c_4c_5y^5 \\
&\quad + 10080c_4c_6y^6 + 5040c_5^2y^6 + 15840c_5c_6y^7 + 11880c_6^2y^8)_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{9!} \frac{\partial^4}{\partial y^4} (240c_0c_5 + 14400c_0c_6y + 240c_1c_4 + 1440c_1c_5y \\
&\quad + 5040c_1c_6y^2 + 720c_2c_3y + 1440c_2c_4y + 5040c_2c_5y^2 \\
&\quad + 13440c_2c_6y^3 + 720c_3^2y + 5040c_3c_4y^2 + 13440c_3c_5y^3 \\
&\quad + 30240c_3c_6y^4 + 6720c_4^2y^3 + 30240c_4c_5y^4 + 60480c_4c_6y^5 \\
&\quad + 30240c_5^2y^5 + 110880c_5c_6y^6 + 95040c_6^2y^7)_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{9!} \frac{\partial^3}{\partial y^3} (1440c_0c_6 + 1440c_1c_5 + 10080c_1c_6y + 720c_2c_3 + 1440c_2c_4 \\
&\quad + 10080c_2c_5y + 40320c_2c_6y^2 + 720c_3^2 + 10080c_3c_4y \\
&\quad + 40320c_3c_5y^2 + 120960c_3c_6y^3 + 20160c_4^2y^2 + 120960c_4c_5y^3 \\
&\quad + 302400c_4c_6y^4 + 151200c_5^2y^4 + 665280c_5c_6y^5 + 665280c_6^2y^6)_{y=0}
\end{aligned}$$

$$= \frac{1}{9!} \frac{\partial^2}{\partial y^2} (10080c_1c_6 + 10080c_2c_5 + 80640c_2c_6y + 10080c_3c_4 + 80640c_3c_5y + 362880c_3c_6y^2 + 40320c_4^2y + 362880c_4c_5y^2 + 1209600c_4c_6y^3 + 604800c_5^2y^3 + 3326400c_5c_6y^4 + 3991680c_6^2y^5)_{y=0}$$

$$= \frac{1}{9!} \frac{\partial}{\partial y} (80640c_2c_6 + 80640c_3c_5 + 725760c_3c_6y + 40320c_4^2 + 725760c_4c_5y + 3628800c_4c_6y^2 + 1814400c_5^2y^2 + 293305600c_5c_6y^3 + 19958400c_6^2y^4)_{y=0}$$

$$= \frac{1}{9!} (725760c_3c_6 + 725760c_4c_5 + 7257600c_4c_6y + 3628800c_5^2y + 879916800c_5c_6y^2 + 79833600c_6^2y^3)_{y=0}$$

$$= \frac{1}{9!} (725760c_3c_6 + 725760c_4c_5) = 2c_3c_6 + 2c_4c_5$$

$$v_{10} = \frac{1}{10!} \frac{\partial^{10}}{\partial y^{10}} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)^2_{y=0}$$

$$v_{10} = \frac{1}{10!} \frac{\partial^{10}}{\partial y^{10}} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)_{y=0}$$

$$v_{10} = \frac{1}{10!} \frac{\partial^{10}}{\partial y^{10}} ((c_0c_0) + c_0c_1y + c_0c_2y^2 + c_0c_3y^3 + c_0c_4y^4 + c_0c_5y^5 + c_0c_6y^6 + c_0c_1y + c_1y(c_1y) + c_2y^2(c_1y) + c_3y^3(c_1y) + c_4y^4(c_1y) + c_5y^5(c_1y) + c_6y^6(c_1y) + c_0c_2y^2 + c_1y(c_2y^2) + c_2y^2(c_2y^2) + c_3y^3(c_2y^2) + c_4y^4(c_2y^2) + c_5y^5(c_2y^2) + c_6y^6(c_2y^2) + c_0c_3y^3 + c_1y(c_3y^3) + c_2y^2(c_3y^3) + c_3y^3(c_3y^3) + c_4y^4(c_3y^3) + c_5y^5(c_3y^3) + c_6y^6(c_3y^3) + c_0c_4y^4 + c_1y(c_4y^4) + c_2y^2(c_4y^4) + c_3y^3(c_4y^4) + c_4y^4(c_4y^4) + c_5y^5(c_4y^4) + c_6y^6(c_4y^4) + c_0c_5y^5 + c_1y(c_5y^5) + c_2y^2(c_5y^5) + c_3y^3(c_5y^5) + c_4y^4(c_5y^5) + c_5y^5(c_5y^5) + c_6y^6(c_5y^5) + c_0c_6y^6 + c_1y(c_6y^6) + c_2y^2(c_6y^6) + c_3y^3(c_6y^6) + c_4y^4(c_6y^6) + c_5y^5(c_6y^6) + c_6y^6(c_6y^6))_{y=0}$$

$$\begin{aligned}
&= \frac{1}{10!} \frac{\partial^{10}}{\partial y^{10}} (c_0^2 + c_0 c_1 y + c_0 c_2 y^2 + c_0 c_3 y^3 + c_0 c_4 y^4 + c_0 c_5 y^5 \\
&\quad + c_0 c_6 y^6 + c_0 c_1 y + c_1^2 y^2 + c_1 c_2 y^3 + c_1 c_3 y^4 + c_1 c_4 y^5 + c_1 c_5 y^6 \\
&\quad + c_1 c_7 y^7 + c_0 c_2 y^2 + c_2 c_1 y^3 + c_2^2 y^4 + c_2 c_3 y^5 + c_2 c_4 y^6 + c_2 c_5 y^7 \\
&\quad + c_2 c_6 y^8 + c_0 c_3 y^3 + c_1 c_3 y^4 + c_2 c_3 y^5 + c_3^2 y^6 + c_3 c_4 y^7 + c_3 c_5 y^8 \\
&\quad + c_3 c_6 y^9 + c_0 c_4 y^4 + c_1 c_4 y^5 + c_2 c_4 y^6 + c_3 c_4 y^7 + c_4^2 y^8 + c_4 c_5 y^9 \\
&\quad + c_4 c_6 y^{10} + c_0 c_5 y^5 + c_1 c_5 y^6 + c_2 c_5 y^7 + c_3 c_5 y^8 + c_4 c_5 y^9 + c_5^2 y^{10} \\
&\quad + c_5 c_6 y^{11} + c_0 c_6 y^6 + c_1 c_6 y^7 + c_2 c_6 y^8 + c_3 c_6 y^9 + c_4 c_6 y^{10} + c_5 c_6 y^{11} \\
&\quad + c_6^2 y^{12})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{10!} \frac{\partial^{10}}{\partial y^{10}} (c_0^2 + 2c_0 c_1 y + 2c_0 c_2 y^2 + 2c_0 c_3 y^3 + 2c_0 c_4 y^4 \\
&\quad + 2c_0 c_5 y^5 + 2c_0 c_6 y^6 + c_1^2 y^2 + 2c_1 c_2 y^3 + 2c_1 c_3 y^4 + 2c_1 c_4 y^5 \\
&\quad + 2c_1 c_5 y^6 + 2c_1 c_6 y^7 + c_2^2 y^4 + 2c_2 c_3 y^5 + 2c_2 c_4 y^6 + 2c_2 c_5 y^7 \\
&\quad + 2c_2 c_6 y^8 + c_3^2 y^6 + 2c_3 c_4 y^7 + 2c_3 c_5 y^8 + 2c_3 c_6 y^8 + c_4^2 y^8 + 2c_4 c_5 y^9 \\
&\quad + 2c_4 c_6 y^{10} + c_5^2 y^{10} + 2c_5 c_6 y^{11} + c_6^2 y^{12})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{10!} \frac{\partial^9}{\partial y^9} (2c_0 c_1 + 4c_0 c_2 y + 6c_0 c_3 y^2 + 8c_0 c_4 y^3 + 10c_0 c_5 y^4 + 12c_0 c_6 y^5 \\
&\quad + 2c_1^2 y + 6c_1 c_2 y^2 + 8c_1 c_3 y^3 + 10c_1 c_4 y^4 + 12c_1 c_5 y^5 + 14c_1 c_6 y^6 \\
&\quad + 4c_2^2 y^3 + 10c_2 c_3 y^4 + 12c_2 c_4 y^5 + 14c_2 c_5 y^6 + 16c_2 c_6 y^7 + 6c_3^2 y^5 \\
&\quad + 14c_3 c_4 y^6 + 16c_3 c_5 y^7 + 18c_3 c_6 y^8 + 8c_4^2 y^7 + 18c_4 c_5 y^8 + 20c_4 c_6 y^9 \\
&\quad + 10c_5^2 y^9 + 22c_5 c_6 y^{10} + 12c_6^2 y^{11})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{10!} \frac{\partial^8}{\partial y^8} (4c_0 c_2 + 12c_0 c_3 y + 24c_0 c_4 y^2 + 40c_0 c_5 y^3 + 60c_0 c_6 y^4 + 2c_1^2 \\
&\quad + 12c_1 c_2 y + 24c_1 c_3 y^2 + 40c_1 c_4 y^3 + 60c_1 c_5 y^4 + 84c_1 c_6 y^5 + 12c_2^2 y^2 \\
&\quad + 40c_2 c_3 y^3 + 60c_2 c_4 y^4 + 84c_2 c_5 y^6 + 112c_2 c_6 y^6 + 30c_3^2 y^4 \\
&\quad + 84c_3 c_4 y^5 + 112c_3 c_5 y^6 + 144c_3 c_6 y^7 + 56c_4^2 y^6 + 144c_4 c_5 y^7 \\
&\quad + 180c_4 c_6 y^8 + 90c_5^2 y^8 + 220c_5 c_6 y^9 + 132c_6^2 y^{10})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{10!} \frac{\partial^7}{\partial y^7} (12c_0c_3 + 48c_0c_4y + 120c_0c_5y^2 + 240c_0c_6y^3 \\
&\quad + 12c_1c_2 + 48c_1c_3y + 120c_1c_4y^2 + 240c_1c_5y^3 + 420c_1c_6y^4 + 24c_2^2y \\
&\quad + 120c_2c_3y^3 + 240c_2c_4y^3 + 420c_2c_5y^4 + 672c_2c_6y^5 + 120c_3^2y^3 \\
&\quad + 420c_3c_4y^4 + 672c_3c_5y^5 + 1008c_3c_6y^6 + 336c_4^2y^5 + 1008c_4c_5y^6 \\
&\quad + 1440c_4c_6y^7 + 720c_5^2y^7 + 1980c_5c_6y^8 + 1320c_6^2y^9)_{y=0} \\
&= \frac{1}{10!} \frac{\partial^6}{\partial y^6} (48c_0c_4 + 240c_0c_5y + 720c_0c_6y^2 + 48c_1c_3 \\
&\quad + 240c_1c_4y + 720c_1c_5y^2 + 1680c_1c_6y^3 + 24c_2^2 + 360c_2c_3y^2 \\
&\quad + 720c_2c_4y^2 + 1680c_2c_5y^3 + 3360c_2c_6y^4 + 360c_3^2y^2 + 1680c_3c_4y^3 \\
&\quad + 3360c_3c_5y^4 + 6048c_3c_6y^5 + 1680c_4^2y^4 + 6048c_4c_5y^5 \\
&\quad + 10080c_4c_6y^6 + 5040c_5^2y^6 + 15840c_5c_6y^7 + 11880c_6^2y^8)_{y=0} \\
&= \frac{1}{10!} \frac{\partial^5}{\partial y^5} (240c_0c_5 + 14400c_0c_6y + 240c_1c_4 + 1440c_1c_5y \\
&\quad + 5040c_1c_6y^2 + 720c_2c_3y + 1440c_2c_4y + 5040c_2c_5y^2 \\
&\quad + 13440c_2c_6y^3 + 720c_3^2y + 5040c_3c_4y^2 + 13440c_3c_5y^3 \\
&\quad + 30240c_3c_6y^4 + 6720c_4^2y^3 + 30240c_4c_5y^4 + 60480c_4c_6y^5 \\
&\quad + 30240c_5^2y^5 + 110880c_5c_6y^6 + 95040c_6^2y^7)_{y=0} \\
&= \frac{1}{10!} \frac{\partial^4}{\partial y^4} (1440c_0c_6 + 1440c_1c_5 + 10080c_1c_6y + 720c_2c_3 + 1440c_2c_4 \\
&\quad + 10080c_2c_5y + 40320c_2c_6y^2 + 720c_3^2 + 10080c_3c_4y + 40320c_3c_5y^2 \\
&\quad + 120960c_3c_6y^3 + 20160c_4^2y^2 + 120960c_4c_5y^3 + 302400c_4c_6y^4 \\
&\quad + 151200c_5^2y^4 + 665280c_5c_6y^5 + 665280c_6^2y^6)_{y=0} \\
&= \frac{1}{10!} \frac{\partial^3}{\partial y^3} (10080c_1c_6 + 10080c_2c_5 + 80640c_2c_6y + 10080c_3c_4 + 80640c_3c_5y \\
&\quad + 362880c_3c_6y^2 + 40320c_4^2y + 362880c_4c_5y^2 + 1209600c_4c_6y^3 \\
&\quad + 604800c_5^2y^3 + 3326400c_5c_6y^4 + 3991680c_6^2y^5)_{y=0} \\
&= \frac{1}{10!} \frac{\partial^2}{\partial y^2} (80640c_2c_6 + 80640c_3c_5 + 725760c_3c_6y + 40320c_4^2 + 725760c_4c_5y \\
&\quad + 3628800c_4c_6y^2 + 1814400c_5^2y^2 + 293305600c_5c_6y^3 \\
&\quad + 19958400c_6^2y^4)_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{10!} \frac{\partial}{\partial y} (725760c_3c_6 + 725760c_4c_5 + 7257600c_4c_6y + 3628800c_5^2y \\
&\quad + 879916800c_5c_6y^2 + 79833600c_6^2y^3)_{y=0} \\
&= \frac{1}{10!} (7257600c_4c_6 + 3628800c_5^2 + 1759833600c_5c_6y^1 + 239500800c_6^2y^2)_{y=0} \\
&= \frac{1}{10!} (7257600c_4c_6 + 3628800c_5^2) = 2c_4c_6 + c_5^2
\end{aligned}$$

$$v_{11} = \frac{1}{11!} \frac{\partial^{11}}{\partial y^{11}} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)_{y=0}^2$$

$$\begin{aligned}
v_{11} &= \frac{1}{11!} \frac{\partial^{11}}{\partial y^{11}} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 \\
&\quad + c_6y^6)(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)_{y=0}
\end{aligned}$$

$$\begin{aligned}
v_{11} &= \frac{1}{11!} \frac{\partial^{11}}{\partial y^{11}} ((c_0c_0) + c_0c_1y + c_0c_2y^2 + c_0c_3y^3 + c_0c_4y^4 + c_0c_5y^5 \\
&\quad + c_0c_6y^6 + c_0c_1y + c_1y(c_1y) + c_2y^2(c_1y) + c_3y^3(c_1y) + c_4y^4(c_1y) \\
&\quad + c_5y^5(c_1y) + c_6y^6(c_1y) + c_0c_2y^2 + c_1y(c_2y^2) + c_2y^2(c_2y^2) \\
&\quad + c_3y^3(c_2y^2) + c_4y^4(c_2y^2) + c_5y^5(c_2y^2) + c_6y^6(c_2y^2) + c_0c_3y^3 \\
&\quad + c_1y(c_3y^3) + c_2y^2(c_3y^3) + c_3y^3(c_3y^3) + c_4y^4(c_3y^3) + c_5y^5(c_3y^3) \\
&\quad + c_6y^6(c_3y^3) + c_0c_4y^4 + c_1y(c_4y^4) + c_2y^2(c_4y^4) + c_3y^3(c_4y^4) \\
&\quad + c_4y^4(c_4y^4) + c_5y^5(c_4y^4) + c_6y^6(c_4y^4) + c_0c_5y^5 + c_1y(c_5y^5) \\
&\quad + c_2y^2(c_5y^5) + c_3y^3(c_5y^5) + c_4y^4(c_5y^5) + c_5y^5(c_5y^5) + c_6y^6(c_5y^5) \\
&\quad + c_0c_6y^6 + c_1y(c_6y^6) + c_2y^2(c_6y^6) + c_3y^3(c_6y^6) + c_4y^4(c_6y^6) \\
&\quad + c_5y^5(c_6y^6) + c_6y^6(c_6y^6))_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{11!} \frac{\partial^{11}}{\partial y^{11}} (c_0^2 + c_0c_1y + c_0c_2y^2 + c_0c_3y^3 + c_0c_4y^4 + c_0c_5y^5 + c_0c_6y^6 + c_0c_1y \\
&\quad + c_1^2y^2 + c_1c_2y^3 + c_1c_3y^4 + c_1c_4y^5 + c_1c_5y^6 + c_1c_6y^7 + c_0c_2y^2 \\
&\quad + c_2c_1y^3 + c_2^2y^4 + c_2c_3y^5 + c_2c_4y^6 + c_2c_5y^7 + c_2c_6y^8 + c_0c_3y^3 \\
&\quad + c_1c_3y^4 + c_2c_3y^5 + c_3^2y^6 + c_3c_4y^7 + c_3c_5y^8 + c_3c_6y^9 + c_0c_4y^4 \\
&\quad + c_1c_4y^5 + c_2c_4y^6 + c_3c_4y^7 + c_4^2y^8 + c_4c_5y^9 + c_4c_6y^{10} + c_0c_5y^5 \\
&\quad + c_1c_5y^6 + c_2c_5y^7 + c_3c_5y^8 + c_4c_5y^9 + c_5^2y^{10} + c_5c_6y^{11} + c_0c_6y^6 \\
&\quad + c_1c_6y^7 + c_2c_6y^8 + c_3c_6y^9 + c_4c_6y^{10} + c_5c_6y^{11} + c_6^2y^{12})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{11!} \frac{\partial^{11}}{\partial y^{11}} (c_0^2 + 2c_0c_1y + 2c_0c_2y^2 + 2c_0c_3y^3 + 2c_0c_4y^4 + 2c_0c_5y^5 \\
&\quad + 2c_0c_6y^6 + c_1^2y^2 + 2c_1c_2y^3 + 2c_1c_3y^4 + 2c_1c_4y^5 + 2c_1c_5y^6 \\
&\quad + 2c_1c_6y^7 + c_2^2y^4 + 2c_2c_3y^5 + 2c_2c_4y^6 + 2c_2c_5y^7 + 2c_2c_6y^8 + c_3^2y^6 \\
&\quad + 2c_3c_4y^7 + 2c_3c_5y^8 + 2c_3c_6y^8 + c_4^2y^8 + 2c_4c_5y^9 + 2c_4c_6y^{10} \\
&\quad + c_5^2y^{10} + 2c_5c_6y^{11} + c_6^2y^{12})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{11!} \frac{\partial^{10}}{\partial y^{10}} (2c_0c_1 + 4c_0c_2y + 6c_0c_3y^2 + 8c_0c_4y^3 + 10c_0c_5y^4 + 12c_0c_6y^5 \\
&\quad + 2c_1^2y + 6c_1c_2y^2 + 8c_1c_3y^3 + 10c_1c_4y^4 + 12c_1c_5y^5 + 14c_1c_6y^6 \\
&\quad + 4c_2^2y^3 + 10c_2c_3y^4 + 12c_2c_4y^5 + 14c_2c_5y^6 + 16c_2c_6y^7 + 6c_3^2y^5 \\
&\quad + 14c_3c_4y^6 + 16c_3c_5y^7 + 18c_3c_6y^8 + 8c_4^2y^7 + 18c_4c_5y^8 + 20c_4c_6y^9 \\
&\quad + 10c_5^2y^9 + 22c_5c_6y^{10} + 12c_6^2y^{11})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{11!} \frac{\partial^9}{\partial y^9} (4c_0c_2 + 12c_0c_3y + 24c_0c_4y^2 + 40c_0c_5y^3 + 60c_0c_6y^4 + 2c_1^2 \\
&\quad + 12c_1c_2y + 24c_1c_3y^2 + 40c_1c_4y^3 + 60c_1c_5y^4 + 84c_1c_6y^5 + 12c_2^2y^2 \\
&\quad + 40c_2c_3y^3 + 60c_2c_4y^4 + 84c_2c_5y^6 + 112c_2c_6y^6 + 30c_3^2y^4 \\
&\quad + 84c_3c_4y^5 + 112c_3c_5y^6 + 144c_3c_6y^7 + 56c_4^2y^6 + 144c_4c_5y^7 \\
&\quad + 180c_4c_6y^8 + 90c_5^2y^8 + 220c_5c_6y^9 + 132c_6^2y^{10})_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{11!} \frac{\partial^8}{\partial y^8} (12c_0c_3 + 48c_0c_4y + 120c_0c_5y^2 + 240c_0c_6y^3 + 12c_1c_2 + 48c_1c_3y \\
&\quad + 120c_1c_4y^2 + 240c_1c_5y^3 + 420c_1c_6y^4 + 24c_2^2y + 120c_2c_3y^3 \\
&\quad + 240c_2c_4y^3 + 420c_2c_5y^4 + 672c_2c_6y^5 + 120c_3^2y^3 + 420c_3c_4y^4 \\
&\quad + 672c_3c_5y^5 + 1008c_3c_6y^6 + 336c_4^2y^5 + 1008c_4c_5y^6 + 1440c_4c_6y^7 \\
&\quad + 720c_5^2y^7 + 1980c_5c_6y^8 + 1320c_6^2y^9)_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{11!} \frac{\partial^7}{\partial y^7} (48c_0c_4 + 240c_0c_5y + 720c_0c_6y^2 + 48c_1c_3 + 240c_1c_4y + 720c_1c_5y^2 \\
&\quad + 1680c_1c_6y^3 + 24c_2^2 + 360c_2c_3y^2 + 720c_2c_4y^2 + 1680c_2c_5y^3 \\
&\quad + 3360c_2c_6y^4 + 360c_3^2y^2 + 1680c_3c_4y^3 + 3360c_3c_5y^4 \\
&\quad + 6048c_3c_6y^5 + 1680c_4^2y^4 + 6048c_4c_5y^5 + 10080c_4c_6y^6 \\
&\quad + 5040c_5^2y^6 + 15840c_5c_6y^7 + 11880c_6^2y^8)_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{11!} \frac{\partial^6}{\partial y^6} (240c_0c_5 + 14400c_0c_6y + 240c_1c_4 + 1440c_1c_5y + 5040c_1c_6y^2 \\
&\quad + 720c_2c_3y + 1440c_2c_4y + 5040c_2c_5y^2 + 13440c_2c_6y^3 + 720c_3^2y \\
&\quad + 5040c_3c_4y^2 + 13440c_3c_5y^3 + 30240c_3c_6y^4 + 6720c_4^2y^3 \\
&\quad + 30240c_4c_5y^4 + 60480c_4c_6y^5 + 30240c_5^2y^5 + 110880c_5c_6y^6 \\
&\quad + 95040c_6^2y^7)_{y=0} \\
&= \frac{1}{11!} \frac{\partial^5}{\partial y^5} (1440c_0c_6 + 1440c_1c_5 + 10080c_1c_6y + 720c_2c_3 + 1440c_2c_4 \\
&\quad + 10080c_2c_5y + 40320c_2c_6y^2 + 720c_3^2 + 10080c_3c_4y \\
&\quad + 40320c_3c_5y^2 + 120960c_3c_6y^3 + 20160c_4^2y^2 + 120960c_4c_5y^3 \\
&\quad + 302400c_4c_6y^4 + 151200c_5^2y^4 + 665280c_5c_6y^5 + 665280c_6^2y^6)_{y=0} \\
&= \frac{1}{11!} \frac{\partial^4}{\partial y^4} (10080c_1c_6 + 10080c_2c_5 + 80640c_2c_6y + 10080c_3c_4 + 80640c_3c_5y \\
&\quad + 362880c_3c_6y^2 + 40320c_4^2y + 362880c_4c_5y^2 + 1209600c_4c_6y^3 \\
&\quad + 604800c_5^2y^3 + 3326400c_5c_6y^4 + 3991680c_6^2y^5)_{y=0} \\
&= \frac{1}{11!} \frac{\partial^3}{\partial y^3} (80640c_2c_6 + 80640c_3c_5 + 725760c_3c_6y + 40320c_4^2 + 725760c_4c_5y \\
&\quad + 3628800c_4c_6y^2 + 1814400c_5^2y^2 + 13305600c_5c_6y^3 \\
&\quad + 19958400c_6^2y^4)_{y=0} \\
&= \frac{1}{11!} \frac{\partial^2}{\partial y^2} (725760c_3c_6 + 725760c_4c_5 + 7257600c_4c_6y + 3628800c_5^2y \\
&\quad + 39916800c_5c_6y^2 + 79833600c_6^2y^3)_{y=0} \\
&= \frac{1}{11!} \frac{\partial}{\partial y} (7257600c_4c_6 + 3628800c_5^2 + 79833600c_5c_6y^1 + 239500800c_6^2y^2)_{y=0} \\
&= \frac{1}{11!} (79833600c_5c_6 + 479001600c_6^2)_{y=0} = \frac{1}{11!} 79833600c_5c_6 = 2c_5c_6 \\
v_{12} &= \frac{1}{12!} \frac{\partial^{12}}{\partial y^{12}} (c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + c_5y^5 + c_6y^6)_{y=0}
\end{aligned}$$

$$v_{12} = \frac{1}{12!} \frac{\partial^{12}}{\partial y^{12}} (c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4 + c_5 y^5 + c_6 y^6)(c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4 + c_5 y^5 + c_6 y^6)_{y=0}$$

$$v_{12} = \frac{1}{12!} \frac{\partial^{12}}{\partial y^{12}} ((c_0 c_0) + c_0 c_1 y + c_0 c_2 y^2 + c_0 c_3 y^3 + c_0 c_4 y^4 + c_0 c_5 y^5 + c_0 c_6 y^6 + c_0 c_1 y + c_1 y(c_1 y) + c_2 y^2(c_1 y) + c_3 y^3(c_1 y) + c_4 y^4(c_1 y) + c_5 y^5(c_1 y) + c_6 y^6(c_1 y) + c_0 c_2 y^2 + c_1 y(c_2 y^2) + c_2 y^2(c_2 y^2) + c_3 y^3(c_2 y^2) + c_4 y^4(c_2 y^2) + c_5 y^5(c_2 y^2) + c_6 y^6(c_2 y^2) + c_0 c_3 y^3 + c_1 y(c_3 y^3) + c_2 y^2(c_3 y^3) + c_3 y^3(c_3 y^3) + c_4 y^4(c_3 y^3) + c_5 y^5(c_3 y^3) + c_6 y^6(c_3 y^3) + c_0 c_4 y^4 + c_1 y(c_4 y^4) + c_2 y^2(c_4 y^4) + c_3 y^3(c_4 y^4) + c_4 y^4(c_4 y^4) + c_5 y^5(c_4 y^4) + c_6 y^6(c_4 y^4) + c_0 c_5 y^5 + c_1 y(c_5 y^5) + c_2 y^2(c_5 y^5) + c_3 y^3(c_5 y^5) + c_4 y^4(c_5 y^5) + c_5 y^5(c_5 y^5) + c_6 y^6(c_5 y^5) + c_0 c_6 y^6 + c_1 y(c_6 y^6) + c_2 y^2(c_6 y^6) + c_3 y^3(c_6 y^6) + c_4 y^4(c_6 y^6) + c_5 y^5(c_6 y^6) + c_6 y^6(c_6 y^6))_{y=0}$$

$$= \frac{1}{12!} \frac{\partial^{12}}{\partial y^{12}} (c_0^2 + c_0 c_1 y + c_0 c_2 y^2 + c_0 c_3 y^3 + c_0 c_4 y^4 + c_0 c_5 y^5 + c_0 c_6 y^6 + c_0 c_1 y + c_1^2 y^2 + c_1 c_2 y^3 + c_1 c_3 y^4 + c_1 c_4 y^5 + c_1 c_5 y^6 + c_1 c_6 y^7 + c_0 c_2 y^2 + c_2 c_1 y^3 + c_2^2 y^4 + c_2 c_3 y^5 + c_2 c_4 y^6 + c_2 c_5 y^7 + c_2 c_6 y^8 + c_0 c_3 y^3 + c_1 c_3 y^4 + c_2 c_3 y^5 + c_3^2 y^6 + c_3 c_4 y^7 + c_3 c_5 y^8 + c_3 c_6 y^9 + c_0 c_4 y^4 + c_1 c_4 y^5 + c_2 c_4 y^6 + c_3 c_4 y^7 + c_4^2 y^8 + c_4 c_5 y^9 + c_4 c_6 y^{10} + c_0 c_5 y^5 + c_1 c_5 y^6 + c_2 c_5 y^7 + c_3 c_5 y^8 + c_4 c_5 y^9 + c_5^2 y^{10} + c_5 c_6 y^{11} + c_0 c_6 y^6 + c_1 c_6 y^7 + c_2 c_6 y^8 + c_3 c_6 y^9 + c_4 c_6 y^{10} + c_5 c_6 y^{11} + c_6^2 y^{12})_{y=0}$$

$$= \frac{1}{12!} \frac{\partial^{12}}{\partial y^{12}} (c_0^2 + 2c_0 c_1 y + 2c_0 c_2 y^2 + 2c_0 c_3 y^3 + 2c_0 c_4 y^4 + 2c_0 c_5 y^5 + 2c_0 c_6 y^6 + c_1^2 y^2 + 2c_1 c_2 y^3 + 2c_1 c_3 y^4 + 2c_1 c_4 y^5 + 2c_1 c_5 y^6 + 2c_1 c_6 y^7 + c_2^2 y^4 + 2c_2 c_3 y^5 + 2c_2 c_4 y^6 + 2c_2 c_5 y^7 + 2c_2 c_6 y^8 + c_3^2 y^6 + 2c_3 c_4 y^7 + 2c_3 c_5 y^8 + 2c_3 c_6 y^9 + c_4^2 y^8 + 2c_4 c_5 y^9 + 2c_4 c_6 y^{10} + c_5^2 y^{10} + 2c_5 c_6 y^{11} + c_6^2 y^{12})_{y=0}$$

$$\begin{aligned}
&= \frac{1}{12!} \frac{\partial^{11}}{\partial y^{11}} (2c_0c_1 + 4c_0c_2y + 6c_0c_3y^2 + 8c_0c_4y^3 \\
&\quad + 10c_0c_5y^4 + 12c_0c_6y^5 + 2c_1^2y + 6c_1c_2y^2 + 8c_1c_3y^3 + 10c_1c_4y^4 \\
&\quad + 12c_1c_5y^5 + 14c_1c_6y^6 + 4c_2^2y^3 + 10c_2c_3y^4 + 12c_2c_4y^5 + 14c_2c_5y^6 \\
&\quad + 16c_2c_6y^7 + 6c_3^2y^5 + 14c_3c_4y^6 + 16c_3c_5y^7 + 18c_3c_6y^8 + 8c_4^2y^7 \\
&\quad + 18c_4c_5y^8 + 20c_4c_6y^9 + 10c_5^2y^9 + 22c_5c_6y^{10} + 12c_6^2y^{11})_{y=0} \\
&= \frac{1}{12!} \frac{\partial^{10}}{\partial y^{10}} (4c_0c_2 + 12c_0c_3y + 24c_0c_4y^2 + 40c_0c_5y^3 + 60c_0c_6y^4 + 2c_1^2 + 12c_1c_2y \\
&\quad + 24c_1c_3y^2 + 40c_1c_4y^3 + 60c_1c_5y^4 + 84c_1c_6y^5 + 12c_2^2y^2 \\
&\quad + 40c_2c_3y^3 + 60c_2c_4y^4 + 84c_2c_5y^6 + 112c_2c_6y^6 + 30c_3^2y^4 \\
&\quad + 84c_3c_4y^5 + 112c_3c_5y^6 + 144c_3c_6y^7 + 56c_4^2y^6 + 144c_4c_5y^7 \\
&\quad + 180c_4c_6y^8 + 90c_5^2y^8 + 220c_5c_6y^9 + 132c_6^2y^{10})_{y=0} \\
&= \frac{1}{12!} \frac{\partial^9}{\partial y^9} (12c_0c_3 + 48c_0c_4y + 120c_0c_5y^2 + 240c_0c_6y^3 + 12c_1c_2 + 48c_1c_3y \\
&\quad + 120c_1c_4y^2 + 240c_1c_5y^3 + 420c_1c_6y^4 + 24c_2^2y + 120c_2c_3y^3 \\
&\quad + 240c_2c_4y^3 + 420c_2c_5y^4 + 672c_2c_6y^5 + 120c_3^2y^3 + 420c_3c_4y^4 \\
&\quad + 672c_3c_5y^5 + 1008c_3c_6y^6 + 336c_4^2y^5 + 1008c_4c_5y^6 + 1440c_4c_6y^7 \\
&\quad + 720c_5^2y^7 + 1980c_5c_6y^8 + 1320c_6^2y^9)_{y=0} \\
&= \frac{1}{12!} \frac{\partial^8}{\partial y^8} (48c_0c_4 + 240c_0c_5y + 720c_0c_6y^2 + 48c_1c_3 + 240c_1c_4y + 720c_1c_5y^2 \\
&\quad + 1680c_1c_6y^3 + 24c_2^2 + 360c_2c_3y^2 + 720c_2c_4y^2 + 1680c_2c_5y^3 \\
&\quad + 3360c_2c_6y^4 + 360c_3^2y^2 + 1680c_3c_4y^3 + 3360c_3c_5y^4 \\
&\quad + 6048c_3c_6y^5 + 1680c_4^2y^4 + 6048c_4c_5y^5 + 10080c_4c_6y^6 \\
&\quad + 5040c_5^2y^6 + 15840c_5c_6y^7 + 11880c_6^2y^8)_{y=0} \\
&= \frac{1}{12!} \frac{\partial^7}{\partial y^7} (240c_0c_5 + 1440c_0c_6y + 240c_1c_4 + 1440c_1c_5y + 5040c_1c_6y^2 \\
&\quad + 720c_2c_3y + 1440c_2c_4y + 5040c_2c_5y^2 + 13440c_2c_6y^3 + 720c_3^2y \\
&\quad + 5040c_3c_4y^2 + 13440c_3c_5y^3 + 30240c_3c_6y^4 + 6720c_4^2y^3 \\
&\quad + 30240c_4c_5y^4 + 60480c_4c_6y^5 + 30240c_5^2y^5 + 110880c_5c_6y^6 \\
&\quad + 95040c_6^2y^7)_{y=0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{12!} \frac{\partial^6}{\partial y^6} (1440c_0c_6 + 1440c_1c_5 + 10080c_1c_6y + 720c_2c_3 + 1440c_2c_4 \\
&\quad + 10080c_2c_5y + 40320c_2c_6y^2 + 720c_3^2 + 10080c_3c_4y \\
&\quad + 40320c_3c_5y^2 + 120960c_3c_6y^3 + 20160c_4^2y^2 + 120960c_4c_5y^3 \\
&\quad + 302400c_4c_6y^4 + 151200c_5^2y^4 + 665280c_5c_6y^5 + 665280c_6^2y^6)_{y=0} \\
&= \frac{1}{12!} \frac{\partial^5}{\partial y^5} (10080c_1c_6 + 10080c_2c_5 + 80640c_2c_6y + 10080c_3c_4 + 80640c_3c_5y \\
&\quad + 362880c_3c_6y^2 + 40320c_4^2y + 362880c_4c_5y^2 + 1209600c_4c_6y^3 \\
&\quad + 604800c_5^2y^3 + 3326400c_5c_6y^4 + 3991680c_6^2y^5)_{y=0} \\
&= \frac{1}{12!} \frac{\partial^4}{\partial y^4} (80640c_2c_6 + 80640c_3c_5 + 725760c_3c_6y + 40320c_4^2 + 725760c_4c_5y \\
&\quad + 3628800c_4c_6y^2 + 1814400c_5^2y^2 + 13305600c_5c_6y^3 \\
&\quad + 19958400c_6^2y^4)_{y=0} \\
&= \frac{1}{12!} \frac{\partial^3}{\partial y^3} (725760c_3c_6 + 725760c_4c_5 + 7257600c_4c_6y + 3628800c_5^2y \\
&\quad + 39916800c_5c_6y^2 + 79833600c_6^2y^3)_{y=0} \\
&= \frac{1}{12!} \frac{\partial^2}{\partial y^2} (7257600c_4c_6 + 3628800c_5^2 + 79833600c_5c_6y^1 + 239500800c_6^2y^2)_{y=0} \\
&= \frac{1}{12!} \frac{\partial}{\partial y} (79833600c_5c_6 + 479001600c_6^2y)_{y=0} \\
&= \frac{1}{12!} (479001600c_6^2)_{y=0} \\
&= \frac{1}{12!} 479001600c_6^2 = c_6^2 \tag{3.23}
\end{aligned}$$

and sing formula (2.31 ),we have  $u_{app} = \sum_{i=0}^6 \alpha_i E_i^*(t) u$

$$\alpha_0 = c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2 + \frac{10}{32}c_3 + \frac{35}{128}c_4 + \frac{126}{512}c_5 + \frac{462}{2048}c_6$$

$$\alpha_1 = \frac{1}{2}c_1 + \frac{4}{8}c_2 + \frac{15}{32}c_3 + \frac{56}{128}c_4 + \frac{210}{512}c_5 + \frac{792}{2048}c_6$$

$$\begin{aligned}
\alpha_2 &= \frac{1}{8}c_2 + \frac{6}{32}c_3 + \frac{28}{128}c_4 + \frac{120}{512}c_5 + \frac{495}{2048}c_6 \\
\alpha_3 &= \frac{1}{32}c_3 + \frac{8}{128}c_4 + \frac{45}{512}c_5 + \frac{220}{2048}c_6 \\
\alpha_4 &= \frac{1}{128}c_4 + \frac{10}{512}c_5 + \frac{66}{2048}c_6 \\
\alpha_5 &= \frac{1}{512}c_5 + \frac{12}{2048}c_6 \\
\alpha_6 &= \frac{1}{2048}c_6
\end{aligned} \tag{3.24}$$

and using formula (2.31 ),we have

$$[u_{app}]^2 = \sum_{i=0}^{12} \beta_i E_m^*(y)$$

Where ,  $\sum_{i=0}^{12} \beta_i$

$$\begin{aligned}
\beta_0 &= v_0 + \frac{1}{2}v_1 + \frac{3}{8}v_2 + \frac{10}{32}v_3 + \frac{35}{128}v_4 + \frac{126}{512}v_5 + \frac{462}{2048}v_6 + \frac{1716}{8192}v_7 + \frac{6435}{32768}v_8 \\
&\quad + \frac{24310}{131072}v_9 + \frac{92378}{524288}v_{10} + \frac{352716}{2097152}v_{11} + \frac{1352078}{8388608}v_{12} \\
\beta_1 &= \frac{1}{2}v_1 + \frac{4}{8}v_2 + \frac{15}{32}v_3 + \frac{56}{128}v_4 + \frac{210}{512}v_5 + \frac{792}{2048}v_6 + \frac{3003}{8192}v_7 + \frac{11440}{32768}v_8 \\
&\quad + \frac{43758}{131072}v_9 + \frac{167960}{524288}v_{10} + \frac{646646}{2097152}v_{11} + \frac{2496144}{8388608}v_{12} \\
\beta_2 &= \frac{1}{8}v_2 + \frac{6}{32}v_3 + \frac{28}{128}v_4 + \frac{120}{512}v_5 + \frac{495}{2048}v_6 + \frac{2002}{8192}v_7 + \frac{8008}{32768}v_8 + \frac{31824}{131072}v_9 \\
&\quad + \frac{125970}{524288}v_{10} + \frac{497620}{2097152}v_{11} + \frac{1961256}{8388608}v_{12} \\
\beta_3 &= \frac{1}{32}v_3 + \frac{8}{128}v_4 + \frac{45}{512}v_5 + \frac{220}{2048}v_6 + \frac{1001}{8192}v_7 + \frac{4368}{32768}v_8 + \frac{18564}{131072}v_9 \\
&\quad + \frac{77520}{524288}v_{10} + \frac{319770}{2097152}v_{11} + \frac{1307504}{8388608}v_{12}
\end{aligned}$$

$$\begin{aligned}
\beta_4 &= \frac{1}{128}v_4 + \frac{10}{512}v_5 + \frac{66}{2048}v_6 + \frac{364}{8192}v_7 + \frac{1820}{32768}v_8 + \frac{18568}{131072}v_9 + \frac{38760}{524288}v_{10} \\
&\quad + \frac{17544}{2097152}v_{11} + \frac{735471}{8388608}v_{12} \\
\beta_5 &= \frac{1}{512}v_5 + \frac{12}{2048}v_6 + \frac{91}{8192}v_7 + \frac{560}{32768}v_8 + \frac{3060}{131072}v_9 + \frac{15504}{524288}v_{10} \\
&\quad + \frac{74613}{2097152}v_{11} + \frac{346104}{8388608}v_{12} \\
\beta_6 &= \frac{1}{2048}v_6 + \frac{14}{8192}v_7 + \frac{120}{32768}v_8 + \frac{816}{131072}v_9 + \frac{4845}{524288}v_{10} + \frac{26334}{2097152}v_{11} \\
&\quad + \frac{134596}{8388608}v_{12} \\
\beta_7 &= \frac{1}{8192}v_7 + \frac{16}{32768}v_8 + \frac{153}{131072}v_9 + \frac{1140}{524288}v_{10} + \frac{7315}{2097152}v_{11} \\
&\quad + \frac{42504}{8388608}v_{12} \\
\beta_8 &= \frac{1}{32768}v_8 + \frac{18}{131072}v_9 + \frac{190}{524288}v_{10} + \frac{1540}{2097152}v_{11} + \frac{10626}{8388608}v_{12} \\
\beta_9 &= \frac{1}{131072}v_9 + \frac{20}{524288}v_{10} + \frac{231}{2097152}v_{11} + \frac{2024}{8388608}v_{12} \\
\beta_{10} &= \frac{1}{524288}v_{10} + \frac{22}{2097152}v_{11} + \frac{276}{8388608}v_{12} \\
\beta_{11} &= \frac{1}{2097152}v_{11} + \frac{24}{8388608}v_{12} \\
\beta_{12} &= \frac{1}{8388608}v_{12}
\end{aligned} \tag{3.25}$$

$$\sum_{i=0}^6 \alpha_i E_i^*(t) - \int_0^1 ty \left[ \sum_{i=0}^{12} \beta_i E_i^*(y) \right] dy = -\frac{1}{4}(t - 4e^t + te^2)$$

Now, we multiply both sides of the above relation with  $\{\omega(t)E_j^*(t)\}_{j=0,1,2,3,4}$ ; from the orthogonality of shifted Chebyshev polynomials, we obtain the following nonlinear equation

$$\gamma_j \alpha_j - \sum_{i=0}^{12} \beta_i K_{i,j} = f_j \quad , j = 0, \dots, 6, \quad i = 0, \dots, 12 \quad (3.26)$$

From equation (2.44) we find that

$$\gamma_j \alpha_j = \begin{bmatrix} \pi \alpha_0 \\ \frac{\pi}{2} \alpha_1 \\ \frac{\pi}{2} \alpha_2 \\ \frac{\pi}{2} \alpha_3 \\ \frac{\pi}{2} \alpha_4 \\ \frac{\pi}{2} \alpha_5 \\ \frac{\pi}{2} \alpha_6 \end{bmatrix} \quad (2.27)$$

We find the values of  $k \quad j = 0, \dots, 6, \quad i = 0, \dots, 12$  for from equation (2.43)

$$K_{i,j} = \int_0^1 \int_0^1 \omega(t) k(t, y) E_i^*(y) E_j^*(t) dy dt, \quad j = 0, \dots, 6, \quad i = 0, \dots, 12$$

$$K_{i,j} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_i^*(y) E_j^*(t) dy dt, \quad j = 0, \dots, 6, \quad i = 0, \dots, 12$$

$$K_{9,0} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_9^*(y) E_0^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty (131072y^9 - 589824y^8 + 1105920y^7 - 1118208y^6 + 658944y^5 - 228096y^4 + 44352y^3 - 4320y^2 + 162y - 1) dy dt$$

$$= -\frac{1}{154} \int_0^1 \frac{1}{\sqrt{t-t^2}} t dt = -\frac{\pi}{308}$$

$$K_{9,1} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_9^*(y) E_1^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(131072y^9 - 589824y^8 + 1105920y^7 - 1118208y^6 + 658944y^5 - 228096y^4 + 44352y^3 - 4320y^2 + 162y - 1)(2t - 1)dy dt$$

$$= -\frac{1}{154} \int_0^1 \frac{2t-1}{\sqrt{t-t^2}} t dt = -\frac{\pi}{616}$$

$$K_{9,2} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_9^*(y)E_2^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(131072y^9 - 589824y^8 + 1105920y^7 - 1118208y^6 + 658944y^5 - 228096y^4 + 44352y^3 - 4320y^2 + 162y - 1)(8t^2 - 8t + 1)dy dt$$

$$= -\frac{1}{154} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(8t^2 - 8t + 1)dt = 0$$

$$K_{9,3} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_9^*(y)E_3^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(131072y^9 - 589824y^8 + 1105920y^7 - 1118208y^6 + 658944y^5 - 228096y^4 + 44352y^3 - 4320y^2 + 162y - 1)(32t^3 - 48t^2 + 18t - 1)dy dt$$

$$= -\frac{1}{154} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(32t^3 - 48t^2 + 18t - 1)dt = 0$$

The rest of the values of  $K_{i,j}$  were founded in the same way [See the appendix] in order to find the following matrix

$$K_{12,6} =$$

$$\begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{12} & -\frac{\pi}{12} & -\frac{\pi}{20} & -\frac{\pi}{60} & -\frac{\pi}{84} & -\frac{\pi}{140} & -\frac{\pi}{180} & -\frac{\pi}{252} & -\frac{\pi}{308} & -\frac{\pi}{396} & \frac{\pi}{468} & -\frac{\pi}{572} \\ \frac{\pi}{8} & \frac{\pi}{24} & -\frac{\pi}{24} & -\frac{\pi}{40} & -\frac{\pi}{120} & -\frac{\pi}{186} & -\frac{\pi}{280} & -\frac{\pi}{360} & -\frac{\pi}{504} & -\frac{\pi}{616} & -\frac{\pi}{792} & -\frac{\pi}{936} & -\frac{\pi}{1144} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(3.28)

We find the values of  $f_j$  from equation (2.42)

$$f_j = \int_0^1 \omega(t) f(t) E_j^*(t) dt \quad , j = 0, \dots, 6$$

$$= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) E_j^*(t) dt$$

$$f_5 = \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} E_5^*(t) dt$$

$$= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) (512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dt$$

$$= 0.0000425928$$

$$f_6 = \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} E_6^*(t) dt$$

$$= \int_0^1 -\frac{1}{4} \frac{1}{\sqrt{t-t^2}} (t - 4e^t + te^2) (2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dt = 0.0000425928$$

We find the matrix of  $f_0$  to  $f_6$  values from the previous equations and from equations (3.21)

$$f_6 = \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \\ 0.0137007 \\ 0.000853629 \\ 0.00000425928 \\ 0.00000177207 \end{bmatrix} \quad (3.29)$$

We substitute the equations (3.29),(3.28),(3.27) in equation (2.26) we obtain the following matrix system

$$\begin{bmatrix} \frac{\pi}{2}\alpha_0 \\ \frac{\pi}{2}\alpha_1 \\ \frac{\pi}{2}\alpha_2 \\ \frac{\pi}{2}\alpha_3 \\ \frac{\pi}{2}\alpha_4 \\ \frac{\pi}{2}\alpha_5 \\ \frac{\pi}{2}\alpha_6 \end{bmatrix} - \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{12} & -\frac{\pi}{12} & -\frac{\pi}{20} & -\frac{\pi}{60} & -\frac{\pi}{84} & -\frac{\pi}{140} & -\frac{\pi}{180} & -\frac{\pi}{252} & -\frac{\pi}{308} & -\frac{\pi}{396} & \frac{\pi}{468} & -\frac{\pi}{572} \\ \frac{\pi}{8} & \frac{\pi}{24} & -\frac{\pi}{24} & -\frac{\pi}{40} & -\frac{\pi}{120} & -\frac{\pi}{186} & -\frac{\pi}{280} & -\frac{\pi}{360} & -\frac{\pi}{504} & -\frac{\pi}{616} & -\frac{\pi}{792} & -\frac{\pi}{936} & -\frac{\pi}{1144} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \end{bmatrix} = \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \\ 0.0137007 \\ 0.000853629 \\ 0.00000425928 \\ 0.00000177207 \end{bmatrix} \quad (3.30)$$

Substituting equations (3.24), (3.25) into the system (3.30) we obtain

$$\begin{bmatrix} \pi\alpha_0 \\ \frac{\pi}{2}\alpha_1 \\ \frac{\pi}{2}\alpha_2 \\ \frac{\pi}{2}\alpha_3 \\ \frac{\pi}{2}\alpha_4 \\ \frac{\pi}{2}\alpha_5 \\ \frac{\pi}{2}\alpha_6 \end{bmatrix} = \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{12} & -\frac{\pi}{24} & -\frac{\pi}{40} & -\frac{\pi}{60} & -\frac{\pi}{84} & -\frac{\pi}{140} & -\frac{\pi}{180} & -\frac{\pi}{252} & -\frac{\pi}{308} & -\frac{\pi}{396} & -\frac{\pi}{468} & -\frac{\pi}{572} \\ \frac{\pi}{8} & \frac{\pi}{24} & \frac{\pi}{24} & \frac{\pi}{40} & \frac{\pi}{120} & \frac{\pi}{186} & \frac{\pi}{280} & \frac{\pi}{360} & \frac{\pi}{504} & \frac{\pi}{616} & \frac{\pi}{792} & \frac{\pi}{936} & \frac{\pi}{1144} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_0 + \frac{1}{2}v_1 + \frac{3}{8}v_2 + \frac{10}{32}v_3 + \frac{35}{128}v_4 + \frac{126}{512}v_5 + \frac{462}{2048}v_6 + \frac{1716}{8192}v_7 + \frac{6435}{32768}v_8 + \frac{24310}{131072}v_9 + \frac{92378}{524288}v_{10} + \frac{352716}{2097152}v_{11} + \frac{1352078}{8388608}v_{12} \\ \frac{1}{2}v_1 + \frac{4}{8}v_2 + \frac{15}{32}v_3 + \frac{56}{128}v_4 + \frac{210}{512}v_5 + \frac{792}{2048}v_6 + \frac{3003}{8192}v_7 + \frac{11440}{32768}v_8 + \frac{43758}{131072}v_9 + \frac{167960}{524288}v_{10} + \frac{646646}{2097152}v_{11} + \frac{2496144}{8388608}v_{12} \\ \frac{1}{8}v_2 + \frac{6}{32}v_3 + \frac{28}{128}v_4 + \frac{120}{512}v_5 + \frac{495}{2048}v_6 + \frac{2002}{8192}v_7 + \frac{8008}{32768}v_8 + \frac{31824}{131072}v_9 + \frac{125970}{524288}v_{10} + \frac{497620}{2097152}v_{11} + \frac{1961256}{8388608}v_{12} \\ \frac{1}{32}v_3 + \frac{8}{128}v_4 + \frac{45}{512}v_5 + \frac{220}{2048}v_6 + \frac{1001}{8192}v_7 + \frac{4368}{32768}v_8 + \frac{18564}{131072}v_9 + \frac{77520}{524288}v_{10} + \frac{319770}{2097152}v_{11} + \frac{1307504}{8388608}v_{12} \\ \frac{1}{128}v_4 + \frac{10}{512}v_5 + \frac{66}{2048}v_6 + \frac{364}{8192}v_7 + \frac{1820}{32768}v_8 + \frac{8568}{131072}v_9 + \frac{38760}{524288}v_{10} + \frac{17544}{2097152}v_{11} + \frac{735471}{8388608}v_{12} \\ \frac{1}{512}v_5 + \frac{12}{2048}v_6 + \frac{91}{8192}v_7 + \frac{560}{32768}v_8 + \frac{3060}{131072}v_9 + \frac{15504}{524288}v_{10} + \frac{74613}{2097152}v_{11} + \frac{346104}{8388608}v_{12} \\ \frac{1}{2048}v_6 + \frac{14}{8192}v_7 + \frac{120}{32768}v_8 + \frac{816}{131072}v_9 + \frac{4845}{524288}v_{10} + \frac{26334}{2097152}v_{11} + \frac{134596}{8388608}v_{12} \\ \frac{1}{8192}v_7 + \frac{16}{32768}v_8 + \frac{153}{131072}v_9 + \frac{1140}{524288}v_{10} + \frac{7315}{2097152}v_{11} + \frac{42504}{8388608}v_{12} \\ \frac{1}{32768}v_8 + \frac{18}{131072}v_9 + \frac{190}{524288}v_{10} + \frac{1540}{2097152}v_{11} + \frac{10626}{8388608}v_{12} \\ \frac{1}{131072}v_9 + \frac{20}{524288}v_{10} + \frac{231}{2097152}v_{11} + \frac{2024}{8388608}v_{12} \\ \frac{1}{524288}v_{10} + \frac{22}{2097152}v_{11} + \frac{276}{8388608}v_{12} \\ \frac{1}{2097152}v_{11} + \frac{24}{8388608}v_{12} \\ \frac{1}{8388608}v_{12} \end{bmatrix} = \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \\ 0.0137007 \\ 0.000853629 \\ 0.00000425928 \\ 0.00000177207 \end{bmatrix}$$

$$\begin{bmatrix} \pi\alpha_0 \\ \frac{\pi}{2}\alpha_1 \\ \frac{\pi}{2}\alpha_2 \\ \frac{\pi}{2}\alpha_3 \\ \frac{\pi}{2}\alpha_4 \\ \frac{\pi}{2}\alpha_5 \\ \frac{\pi}{2}\alpha_6 \end{bmatrix} - \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{6} & \frac{\pi}{8} & \frac{\pi}{10} & \frac{\pi}{12} & \frac{\pi}{14} & \frac{\pi}{16} & \frac{\pi}{18} & \frac{\pi}{20} & \frac{\pi}{22} & \frac{\pi}{24} & \frac{\pi}{26} & \frac{\pi}{28} \\ \frac{\pi}{8} & \frac{\pi}{12} & \frac{\pi}{16} & \frac{\pi}{20} & \frac{\pi}{24} & \frac{\pi}{28} & \frac{\pi}{32} & \frac{\pi}{36} & \frac{\pi}{40} & \frac{\pi}{44} & \frac{\pi}{48} & \frac{\pi}{52} & \frac{\pi}{56} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{bmatrix} =$$

$$\begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \\ 0.0137007 \\ 0.000853629 \\ 0.00000425928 \\ -0.00000177207 \end{bmatrix} \quad (3.31)$$

Substituting equation (3.23) into the system (3.30) we obtain

$$\begin{bmatrix} \pi\alpha_0 \\ \frac{\pi}{2}\alpha_1 \\ \frac{\pi}{2}\alpha_2 \\ \frac{\pi}{2}\alpha_3 \\ \frac{\pi}{2}\alpha_4 \\ \frac{\pi}{2}\alpha_5 \\ \frac{\pi}{2}\alpha_6 \end{bmatrix} - \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{6} & \frac{\pi}{8} & \frac{\pi}{10} & \frac{\pi}{12} & \frac{\pi}{14} & \frac{\pi}{16} & \frac{\pi}{18} & \frac{\pi}{20} & \frac{\pi}{22} & \frac{\pi}{24} & \frac{\pi}{26} & \frac{\pi}{28} \\ \frac{\pi}{8} & \frac{\pi}{12} & \frac{\pi}{16} & \frac{\pi}{20} & \frac{\pi}{24} & \frac{\pi}{28} & \frac{\pi}{32} & \frac{\pi}{36} & \frac{\pi}{40} & \frac{\pi}{44} & \frac{\pi}{48} & \frac{\pi}{52} & \frac{\pi}{56} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0^2 \\ 2c_0c_1 \\ c_1^2 + 2c_0c_2 \\ 2c_0c_3 + 2c_1c_2 \\ 2c_0c_4 + 2c_1c_3 + c_2^2 \\ 2c_0c_5 + 2c_1c_4 + 2c_2c_3 \\ 2c_0c_6 + 2c_1c_5 + 2c_2c_4 + c_3^2 \\ 2c_1c_6 + 2c_2c_5 + 2c_3c_4 \\ 2c_2c_6 + 2c_3c_5 + c_4^2 \\ 2c_3c_6 + 2c_4c_5 \\ 2c_4c_6 + c_5^2 \\ 2c_5c_6 \\ c_6^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2.21406 \\ -0.311395 \\ 0.165261 \\ 0.0137007 \\ 0.000853629 \\ 0.00000425928 \\ -0.00000177207 \end{bmatrix}$$

We get

$$c_0 = 0.9999999618141961$$

$$c_1 = 0.9999958479761861$$

$$c_2 = 0.5000596328436286$$

$$c_3 = 0.16631362575458639$$

$$c_4 = 0.04264979251921536$$

$$c_5 = 0.006951834132615619$$

$$c_6 = 0.002310420070439772$$

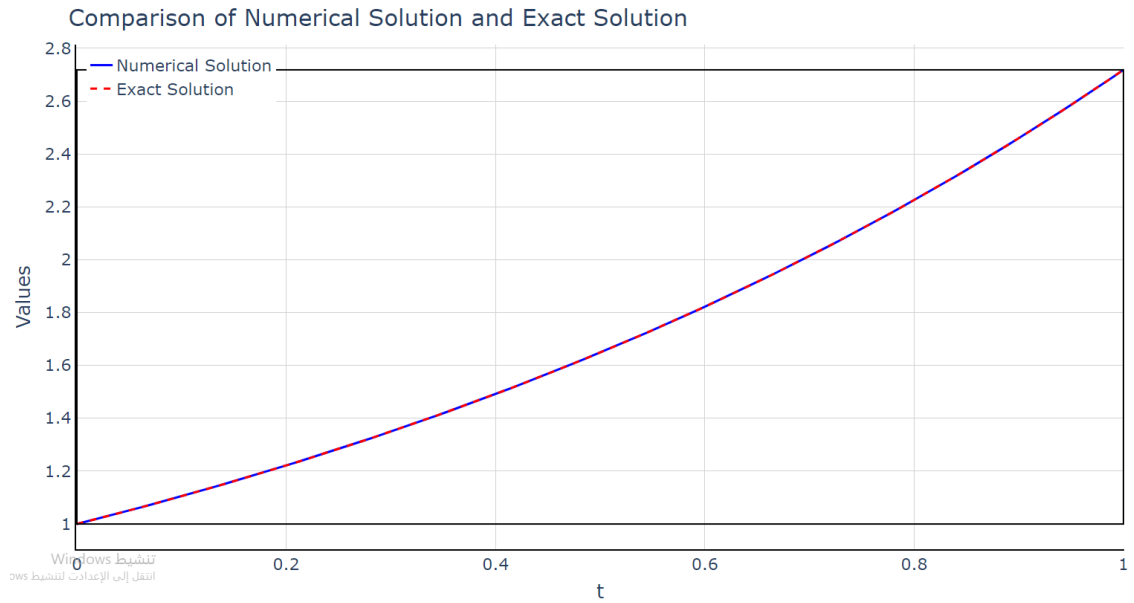
**Table 3**

The table contains the exact and the numerical solutions assuming  $M = 6$  when  $t = [0 \ 1]$  and shows the difference between them.

t	Numerical solutions	Exact solutions	Numerical – Exact
0.0	0.9999999618141961	1.6487212707001282	3.8185803852286426e-08
0.5	1.6487209543952168	1.1051709180756477	3.163049113741323e-07
0.1	1.1051707933740191	1.1051709180756477	1.2470162857880496e-07
0.15	1.1618341349160235	1.161834242728283	1.0781225956435492e-07
0.2	1.221402637851053	1.2214027581601699	1.20309111689109303e-07
0.25	1.2840252549833333	1.2840254166877414	1.6170440808593867e-07
0.3	1.3498585916309316	1.3498588075760032	2.159450716021638e-07
0.35	1.4190672832845534	1.4190675485932571	2.653087036730284e-07
0.4	1.4918243992585634	1.4918246976412703	2.983827069868994e-07
0.45	1.5683118723342337	1.5683121856490169	3.1315593518321805e-07
0.5	1.6487209543952168	1.6487212707001282	3.163049113741323e-07
0.55	1.7332526980552454	1.7332530178673953	3.1981214987375495e-07
0.6	1.8221194642780568	1.8221188003905089	3.3611245209286267e-07
0.65	1.9155404559895446	1.9155408290138962	3.730243516297804e-07
0.7	2.013752277682134	2.0137527074704766	4.297883426929161e-07
0.75	2.116999521011385	2.1170000161612675	4.95601289784986e-07
0.8	2.2255403763848203	2.225540928492468	5.52107647600053e-07
0.85	2.3396462705429797	2.3396468519259908	5.81383011066805e-07
0.9	2.4596025301327002	2.45960311115695	5.810242496018247e-07
0.95	2.5857090712726194	2.585709659315846	5.880432265570334e-07
1.0	2.718281115110909	2.718281828459045	7.13348136116565e-07

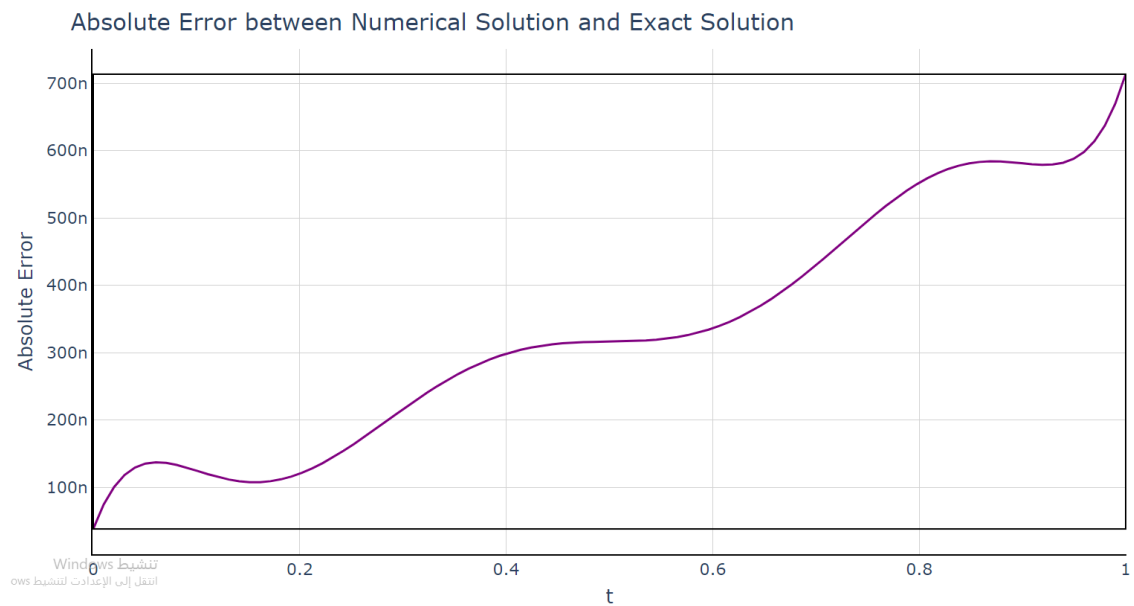
**Figure 5**

Shows both exact and numerical solutions using  $M = 6$  when  $t = [0.1]$



**Figure 6**

Shows the error between exact and numerical solutions using  $M = 6$  When  $t = [0.1]$



**Examples (3.2):**

$$u(t) = te^t - \frac{1}{288}(3 + e^2)t + \frac{1}{36} \int_0^1 ty[u(y)]^2 dy \quad (3.32)$$

With exact solution  $(t) = te^t$ .

To solve the above problem using our method, we assume  $M = 2,3$  then, we construct trail space as follows

1) when  $M = 2$  ,  $n = 2$  ,  $\mathfrak{D} = 4$

$$u_{app} = \sum_{i=0}^2 c_i t^i$$

$$u_{app} = c_0 + c_1 t + c_2 t^2 \quad (3.33)$$

Similarly

$$[u_{app}]^2 = \sum_{i=0}^4 v_i y^i$$

$$[u_{app}]^4 = v_0 + v_1 y + v_2 y^2 + v_3 y^3 + v_4 y^4 \quad (3.34)$$

We find  $v_i$  from (2.40)

$$v_0 = c_0^2$$

$$v_1 = \frac{1}{1!} \frac{\partial}{\partial y} (c_0 + c_1 y)^2 \Big|_{y=0}$$

$$= 2c_0 c_1 + 2c_1^2 y^1 \Big|_{y=0} = 2c_0 c_1$$

$$v_2 = \frac{1}{2!} \frac{\partial}{\partial y} (c_0 + c_1 y + c_2 y^2)^2 \Big|_{y=0}$$

$$= \frac{1}{2!} (2c_1^2 + 4c_0 c_2 + 12c_1 c_2 y + 12c_2^2 y^2) \Big|_{y=0} = c_1^2 + 2c_0 c_2$$

$$v_3 = \frac{1}{3!} \frac{\partial^3}{\partial y^3} (c_0 + c_1 y + c_2 y^2)^2 \Big|_{y=0}$$

$$= \frac{1}{6} (12c_1 c_2) = 2c_1 c_2$$

$$v_4 = \frac{1}{4!} \frac{\partial^4}{\partial y^4} (c_0 + c_1 y + c_2 y^2)^2 \Big|_{y=0}$$

$$= \frac{1}{4!} (24c_2^2)_{y=0} = c_2^2 \quad (3.35)$$

and using formula (2.31), we have  $u_{app} = \sum_{i=0}^2 \alpha_i E_i^*(t)$

$$\alpha_0 = c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2$$

$$\alpha_1 = \frac{1}{2}c_1 + \frac{4}{8}c_2$$

$$\alpha_2 = \frac{1}{8}c_2 \quad (3.36)$$

and using formula (2.31), we have

$$[u_{app}]^2 = \sum_{i=0}^4 \beta_i E_m^*(y)$$

Where,  $\sum_{i=0}^8 \beta_i$

$$\beta_0 = v_0 + \frac{1}{2}v_1 + \frac{3}{8}v_2 + \frac{10}{32}v_3 + \frac{35}{128}v_4$$

$$\beta_1 = \frac{1}{2}v_1 + \frac{4}{8}v_2 + \frac{15}{32}v_3 + \frac{56}{128}v_4$$

$$\beta_2 = \frac{1}{8}v_2 + \frac{6}{32}v_3 + \frac{28}{128}v_4$$

$$\beta_3 = \frac{1}{32}v_3 + \frac{8}{128}v_4$$

$$\beta_4 = \frac{1}{128}v_4 \quad (3.37)$$

We get

$$\sum_{i=0}^2 \alpha_i E_i^*(t) - \int_0^1 \frac{1}{36} ty \left[ \sum_{i=0}^4 \beta_i E_i^*(y) \right] dy = te^t - \frac{1}{288} (3 + e^2)t$$

Now, we multiply both sides of the above relation with  $\{\omega(t)E_j^*(t)\}_{j=0,1,2,3,4}$ ; from the orthogonality of shifted Chebyshev polynomials, we obtain the following nonlinear equation

$$\gamma_j \alpha_j - \sum_{i=0}^4 \beta_i K_{i,j} = f_j \quad , j = 0, \dots, 2, \quad i = 0, \dots, 4 \quad (3.38)$$

From equation (2.44) we get

$$\gamma_j \alpha_j = \begin{bmatrix} \pi \alpha_0 \\ \frac{\pi}{2} \alpha_1 \\ \frac{\pi}{2} \alpha_2 \end{bmatrix} \quad (3.39)$$

From equation (2.43) we get the values of  $K_{i,j}$

$$K_{i,j} = \int_0^1 \int_0^1 \omega(t) k(t,y) E_i^*(y) E_j^*(t) dy dt, \quad j = 0, \dots, 2, \quad i = 0, \dots, 4 \quad (3.40)$$

$$K_{i,j} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} ty E_i^*(y) E_j^*(t) dy dt, \quad j = 0, \dots, 2, \quad i = 0, \dots, 4$$

$$K_{0,0} = \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_0^*(y) E_0^*(t) dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty dy dt = \frac{\pi}{144}$$

$$K_{0,1} = \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_0^*(y) E_1^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} ty (2t-1) dy dt = \frac{\pi}{288}$$

$$K_{0,2} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} ty E_0^*(y) E_2^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} ty (8t^2 - 8t + 1) dy dt = 0$$

$$K_{1,0} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} tyE_1^*(y)E_0^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} ty(2y-1) dy dt = \frac{\pi}{432}$$

$$K_{1,1} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} tyE_1^*(y)E_1^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} ty(2y-1)(2t-1) dy dt = \frac{\pi}{864}$$

$$K_{1,2} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} tyE_1^*(y)E_2^*(t)dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2y-1)(8t^2-8t+1) dy dt = 0$$

$$K_{2,0} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} tyE_2^*(y)E_0^*(t)dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8y^2-8y+1) dy dt = -\frac{\pi}{432}$$

$$K_{2,1} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} tyE_2^*(y)E_1^*(t)dy dt$$

The rest of the values of  $K_{i,j}$  were founded in the same way [See the appendix] in order to find the following matrix

$$K_{4,2} = \begin{bmatrix} \frac{\pi}{144} & \frac{\pi}{432} & -\frac{\pi}{432} & -\frac{\pi}{720} & -\frac{\pi}{2160} \\ \frac{\pi}{288} & \frac{\pi}{864} & -\frac{\pi}{864} & -\frac{\pi}{1440} & -\frac{\pi}{4320} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.41)$$

From equation (2.42) we find the values of  $f_j$

$$\begin{aligned}
 f_j &= \int_0^1 \omega(t) f(t) E_j^*(t) dt \quad , j = 0, \dots, 2 \\
 &= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) E_j^*(t) dt \\
 f_0 &= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) E_0^*(t) dt \\
 &= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) dt = 3.3420952591 \\
 f_1 &= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) E_1^*(t) dt \\
 &= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) (2t-1) dt = 2.03474845557 \\
 f_2 &= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) E_2^*(t) dt \\
 &= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) (8t^2 - 8t + 1) dt = 0.39676382882
 \end{aligned}$$

$$f_j = \begin{bmatrix} 3.3420952591 \\ 2.03474845557 \\ 0.39676382882 \end{bmatrix} \quad (3.42)$$

We substitute the equations (3.39),(3.41),(3.42) in equation (3.8) we obtain the following matrix system

$$\begin{bmatrix} \pi\alpha_0 \\ \frac{\pi}{2}\alpha_1 \\ \frac{\pi}{2}\alpha_2 \end{bmatrix} - \begin{bmatrix} \frac{\pi}{144} & \frac{\pi}{432} & -\frac{\pi}{432} & -\frac{\pi}{720} & -\frac{\pi}{2160} \\ \frac{\pi}{288} & \frac{\pi}{864} & -\frac{\pi}{864} & -\frac{\pi}{1440} & -\frac{\pi}{4320} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} 3.3420952591 \\ 2.03474845557 \\ 0.39676382882 \end{bmatrix} \quad (3.43)$$

We substitute equations (3.36),(3.37) in the system matrices to obtain

$$\begin{bmatrix} \pi(c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2) \\ \frac{\pi}{2}(\frac{1}{2}c_1 + \frac{4}{8}c_2) \\ \frac{\pi}{2}(\frac{1}{8}c_2) \end{bmatrix} - \begin{bmatrix} \frac{\pi}{144} & \frac{\pi}{432} & -\frac{\pi}{432} & -\frac{\pi}{720} & -\frac{\pi}{2160} \\ \frac{\pi}{288} & \frac{\pi}{864} & -\frac{\pi}{864} & -\frac{\pi}{1440} & -\frac{\pi}{4320} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_0 + \frac{1}{2}v_1 + \frac{3}{8}v_2 + \frac{10}{32}v_3 + \frac{35}{128}v_4 \\ \frac{1}{2}v_1 + \frac{4}{8}v_2 + \frac{15}{32}v_3 + \frac{56}{128}v_4 \\ \frac{1}{8}v_2 + \frac{6}{32}v_3 + \frac{28}{128}v_4 \\ \frac{1}{32}v_3 + \frac{8}{128}v_4 \\ \frac{1}{128}v_4 \end{bmatrix} = \begin{bmatrix} 3.3420952591 \\ 2.03474845557 \\ 0.39676382882 \end{bmatrix}$$

We substitute values from equations (3.14) into the previous system to obtain

$$\begin{bmatrix} \pi(c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2) \\ \frac{\pi}{2}(\frac{1}{2}c_1 + \frac{4}{8}c_2) \\ \frac{\pi}{2}(\frac{1}{8}c_2) \end{bmatrix} - \begin{bmatrix} \frac{\pi}{144} & \frac{\pi}{216} & \frac{\pi}{288} & \frac{\pi}{360} & \frac{\pi}{432} \\ \frac{\pi}{288} & \frac{\pi}{432} & \frac{\pi}{576} & \frac{\pi}{720} & \frac{\pi}{864} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0^2 \\ 2c_0c_1 \\ c_1^2 + 2c_0c_2 \\ 2c_1c_2 \\ c_2^2 \end{bmatrix} = \begin{bmatrix} 3.3420952591 \\ 2.03474845557 \\ 0.39676382882 \end{bmatrix}$$

From the solution of the previous system we find the values of  $c$

$$c_0 = 0.02104856$$

$$c_1 = 0.57002061$$

$$c_2 = 2.0207015871$$

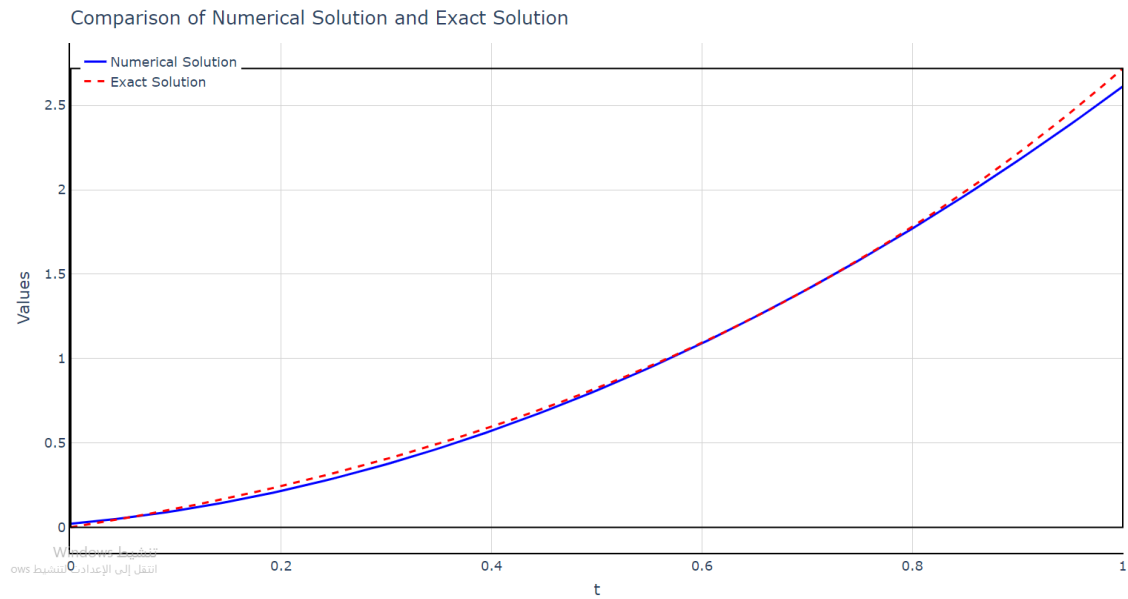
**Table 4**

The table contains the exact and the numerical solutions assuming  $M = 2$  when  $t = [0 \ 1]$  and shows the difference between them.

t	numerical solutions	exact solutions	numerical – exact
0.0	0.021048560000000001	0.0	0.021048560000000001
0.05	0.054601344467750001	0.052563554818801206	0.002037789648948796
0.10	0.098257636871000018	0.110517091807564777	0.012259454936564759
0.15	0.152017437209750028	0.174275136409242498	0.022257699199492470
0.20	0.215880745484000020	0.244280551632033993	0.028399806148033974
0.25	0.289847561693750033	0.321006354171935349	0.031158792478185315
0.30	0.373917885839000097	0.404957642272801011	0.031039756433800914
0.35	0.468091717919750072	0.496673642007640137	0.028581924087890065
0.40	0.572369057936000125	0.596729879056508206	0.024360821120508080
0.45	0.686749905887750089	0.705740483470576052	0.018990577582825963
0.50	0.811234561775000021	0.824360635350064097	0.013126373575064076
0.55	0.945822125597750141	0.953289159827067523	0.007467034229317382
0.60	1.090513497356000228	1.093271280234305642	0.002757782878305415
0.65	1.245308377049750170	1.245101538859032519	0.000206838190717651
0.70	1.410206764679000191	1.409626895229333821	0.000579869449666370
0.75	1.585208660243750067	1.587750012459506088	0.002541352215756021
0.80	1.770314063744000244	1.780432742793974343	0.010118679069974100
0.85	1.965522975179750276	1.988699824137092786	0.023176848957342511
0.90	2.170835394551000164	2.213642800041255132	0.042807405490254968
0.95	2.386251321857750352	2.456424176350054278	0.070172854492303927
1.0	2.611770757099999951	2.718281828459045091	0.106511071359045140

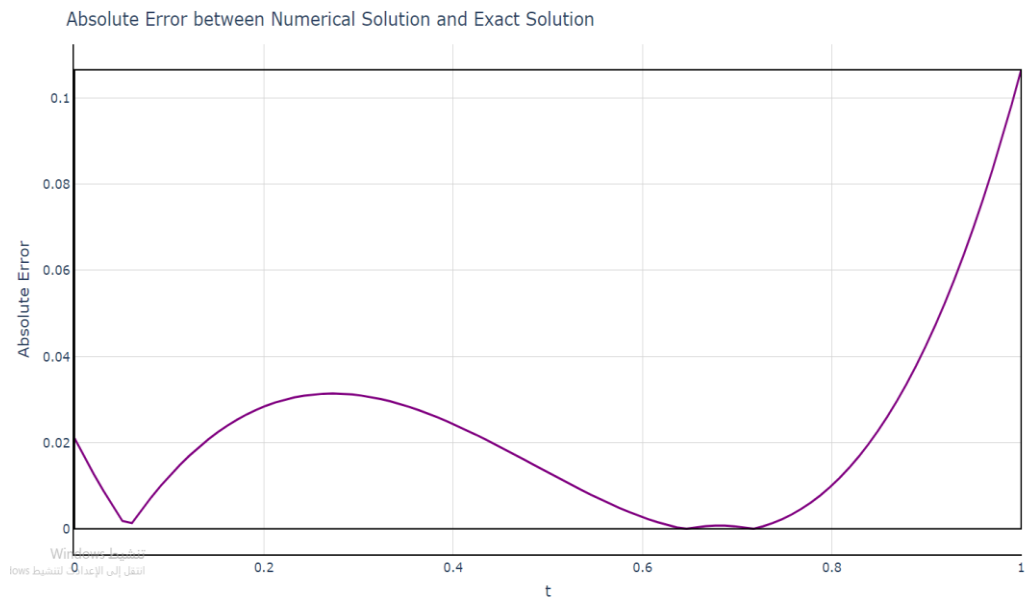
**Figure 7**

Shows both exact and numerical solutions using  $M = 2$  when  $t = [0, 1]$



**Figure 8**

Shows the error between exact and numerical solutions using  $M = 2$  When  $t = [0, 1]$



2) when  $M = 4, n = 2, \mathcal{D} = 8$

$$u_{app} = \sum_{i=0}^4 c_i t^i$$

$$u_{app} = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4$$

Similarly

$$[u_{app}]^2 = \sum_{i=0}^8 v_i y^i$$

We find  $v_i$  from (2.40)

$$v_0 = c_0^2$$

$$v_1 = 2c_0c_1$$

$$v_2 = c_1^2 + 2c_0c_2$$

$$v_3 = 2c_0c_3 + 2c_1c_2$$

$$v_4 = 2c_0c_4 + 2c_1c_3 + c_2^2$$

$$v_5 = 2c_1c_4 + 2c_2c_3$$

$$v_6 = 2c_2c_4 + c_3^2$$

$$v_7 = 2c_3c_4$$

$$v_8 = c_4^2$$

(3.44)

and using formula (2.31), we have  $u_{app} = \sum_{i=0}^4 \alpha_i E_i^*(t)$

$$\alpha_0 = c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2 + \frac{10}{32}c_3 + \frac{35}{128}c_4$$

$$\alpha_1 = \frac{1}{2}c_1 + \frac{4}{8}c_2 + \frac{15}{32}c_3 + \frac{56}{128}c_4$$

$$\alpha_2 = \frac{1}{8}c_2 + \frac{6}{32}c_3 + \frac{28}{128}c_4$$

$$\alpha_3 = \frac{1}{32}c_3 + \frac{8}{128}c_4$$

$$\alpha_4 = \frac{1}{128}c_4$$

(3.45)

and using formula (2.31), we have

$$[u_{app}]^2 = \sum_{i=0}^8 \beta_i E_m^*(y)$$

Where ,  $\sum_{i=0}^8 \beta_i$

$$\beta_0 = v_0 + \frac{1}{2}v_1 + \frac{3}{8}v_2 + \frac{10}{32}v_3 + \frac{35}{128}v_4 + \frac{126}{512}v_5 + \frac{462}{2048}v_6 + \frac{1716}{8192}v_7 + \frac{6435}{32768}v_8$$

$$\beta_1 = \frac{1}{2}v_1 + \frac{4}{8}v_2 + \frac{15}{32}v_3 + \frac{56}{128}v_4 + \frac{210}{512}v_5 + \frac{792}{2048}v_6 + \frac{3003}{8192}v_7 + \frac{11440}{32768}v_8$$

$$\beta_2 = \frac{1}{8}v_2 + \frac{6}{32}v_3 + \frac{28}{128}v_4 + \frac{120}{512}v_5 + \frac{495}{2048}v_6 + \frac{2002}{8192}v_7 + \frac{8008}{32768}v_8$$

$$\beta_3 = \frac{1}{32}v_3 + \frac{8}{128}v_4 + \frac{45}{512}v_5 + \frac{220}{2048}v_6 + \frac{1001}{8192}v_7 + \frac{4368}{32768}v_8$$

$$\beta_4 = \frac{1}{128}v_4 + \frac{10}{512}v_5 + \frac{66}{2048}v_6 + \frac{364}{8192}v_7 + \frac{1820}{32768}v_8$$

$$\beta_5 = \frac{1}{512}v_5 + \frac{12}{2048}v_6 + \frac{91}{8192}v_7 + \frac{560}{32768}v_8$$

$$\beta_6 = \frac{1}{2048}v_6 + \frac{14}{8192}v_7 + \frac{120}{32768}v_8$$

$$\beta_7 = \frac{1}{8192}v_7 + \frac{16}{32768}v_8$$

$$\beta_8 = \frac{1}{32768}v_8 \tag{3.46}$$

We get

$$\sum_{i=0}^4 \alpha_i E_i^*(t) - \int_0^1 \frac{1}{36} ty \left[ \sum_{i=0}^8 \beta_i E_i^*(y) \right] dy = te^t - \frac{1}{288} (3 + e^2)t$$

Now, we multiply both sides of the above relation with  $\{\omega(t)E_j^*(t)\}_{j=0,1,2,3,4}$ ; from the orthogonality of shifted Chebyshev polynomials, we obtain the following nonlinear equation

$$\gamma_j \alpha_j - \sum_{i=0}^8 \beta_i K_{i,j} = f_j \quad , j = 0, \dots, 4, \quad i = 0, \dots, 8 \tag{3.47}$$

From equation (2.44) we get

$$\gamma_j \alpha_j = \begin{bmatrix} \pi \alpha_0 \\ \frac{\pi}{2} \alpha_1 \\ \frac{\pi}{2} \alpha_2 \\ \frac{\pi}{2} \alpha_3 \\ \frac{\pi}{2} \alpha_4 \end{bmatrix} \quad (3.48)$$

From equation (2.43) we get the values of  $K_{i,j}$

$$K_{i,j} = \int_0^1 \int_0^1 \omega(t) k(t,y) E_i^*(y) E_j^*(t) dy dt, \quad j = 0, \dots, 4, \quad i = 0, \dots, 8 \quad (3.49)$$

$$K_{i,j} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} ty E_i^*(y) E_j^*(t) dy dt, \quad j = 0, \dots, 4, \quad i = 0, \dots, 8$$

$$\begin{aligned} K_{5,0} &= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_5^*(y) E_0^*(t) dy dt \\ &= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty (512y^5 - 1280y^4 + 1120y^3 - 400y^2 + 50y - 1) dy dt = \frac{-\pi}{1728} \end{aligned}$$

$$\begin{aligned} K_{5,1} &= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_5^*(y) E_1^*(t) dy dt \\ &= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty (512y^5 - 1280y^4 + 1120y^3 - 400y^2 + 50y - 1) (2t - 1) dy dt \\ &= \frac{-\pi}{6048} \end{aligned}$$

$$\begin{aligned} K_{5,2} &= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_5^*(y) E_2^*(t) dy dt \\ &= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty (512y^5 - 1280y^4 + 1120y^3 - 400y^2 + 50y - 1) (8t^2 - 8t \\ &\quad + 1) dy dt = 0 \end{aligned}$$

$$\begin{aligned}
K_{5,3} &= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_5^*(y)E_3^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(512y^5 - 1280y^4 + 1120y^3 - 400y^2 + 50y - 1)(32t^3 - 48t^2 \\
&\quad + 18t - 1)dy dt = 0
\end{aligned}$$

The rest of the values of  $K_{i,j}$  were founded in the same way [See the appendix] in order to find the following matrix

$$K_{8,4} = \begin{bmatrix} \frac{\pi}{144} & \frac{\pi}{432} & \frac{-\pi}{432} & \frac{-\pi}{720} & \frac{-\pi}{2160} & \frac{-\pi}{3024} & \frac{-\pi}{5050} & \frac{-\pi}{6480} & \frac{-\pi}{9072} \\ \frac{\pi}{288} & \frac{\pi}{864} & \frac{-\pi}{864} & \frac{-\pi}{1440} & \frac{-\pi}{4320} & \frac{-\pi}{6048} & \frac{-\pi}{10080} & \frac{-\pi}{12960} & \frac{-\pi}{18144} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.50)$$

From equation (2.42) we get the values of  $f_j$

$$f_j = \int_0^1 \omega(t)f(t)E_j^*(t) dt \quad , j = 0, \dots, 4$$

$$= \int_0^1 \frac{1}{\sqrt{t-t^2}} (te^t - \frac{1}{288}(3+e^2)t)E_j^*(t) dt$$

$$f_0 = \int_0^1 \frac{1}{\sqrt{t-t^2}} (te^t - \frac{1}{288}(3+e^2)t)E_0^*(t) dt$$

$$= \int_0^1 \frac{1}{\sqrt{t-t^2}} (te^t - \frac{1}{288}(3+e^2)t) dt = 3.3420952591$$

$$f_1 = \int_0^1 \frac{1}{\sqrt{t-t^2}} (te^t - \frac{1}{288}(3+e^2)t)E_1^*(t) dt$$

$$= \int_0^1 \frac{1}{\sqrt{t-t^2}} (te^t - \frac{1}{288}(3+e^2)t)(2t-1) dt = 2.03474845557$$

$$\begin{aligned}
f_2 &= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) E_2^*(t) dt \\
&= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) (8t^2 - 8t + 1) dt = 0.39676382882 \\
f_3 &= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) E_3^*(t) dt \\
f_3 &= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) (32t^3 - 48t^2 + 18t - 1) dt = 0.0483791 \\
f_4 &= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) E_4^*(t) dt \\
&= \int_0^1 \frac{1}{\sqrt{t-t^2}} \left( te^t - \frac{1}{288}(3+e^2)t \right) (128t^4 - 256t^3 + 160t^2 - 32t + 1) dt \\
&= 0.00386263
\end{aligned}$$

$$f_4 = \begin{bmatrix} 3.3420952591 \\ 2.03474845557 \\ 0.39676382882 \\ 0.0483791 \\ 0.00386263 \end{bmatrix} \quad (3.51)$$

We substitute the equations (3.48),(3.50),(3.51) In Equation (3.47) Let's get the following matrix system

$$\begin{bmatrix} \pi\alpha_0 \\ \frac{\pi}{2}\alpha_1 \\ \frac{\pi}{2}\alpha_2 \\ \frac{\pi}{2}\alpha_3 \\ \frac{\pi}{2}\alpha_4 \end{bmatrix} - \begin{bmatrix} \frac{\pi}{144} & \frac{\pi}{432} & \frac{-\pi}{432} & \frac{-\pi}{720} & \frac{-\pi}{2160} & \frac{-\pi}{3024} & \frac{-\pi}{5050} & \frac{-\pi}{6480} & \frac{-\pi}{9072} \\ \frac{\pi}{288} & \frac{\pi}{864} & \frac{-\pi}{864} & \frac{-\pi}{1440} & \frac{-\pi}{4320} & \frac{-\pi}{6048} & \frac{-\pi}{10080} & \frac{-\pi}{12960} & \frac{-\pi}{18144} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \end{bmatrix} = \begin{bmatrix} 3.3420952591 \\ 2.03474845557 \\ 0.39676382882 \\ 0.0483791 \\ 0.00386263 \end{bmatrix} \quad (3.52)$$

We substitute equations (3.46),(3.45) in the system matrices to obtain

$$\begin{bmatrix} \pi\left(c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2 + \frac{10}{32}c_3 + \frac{35}{128}c_4\right) \\ \frac{\pi}{2}\left(\frac{1}{2}c_1 + \frac{4}{8}c_2 + \frac{15}{32}c_3 + \frac{56}{128}c_4\right) \\ \frac{\pi}{2}\left(\frac{1}{8}c_2 + \frac{6}{32}c_3 + \frac{28}{128}c_4\right) \\ \frac{\pi}{2}\left(\frac{1}{32}c_3 + \frac{8}{128}c_4\right) \\ \frac{\pi}{2}\left(\frac{1}{128}c_4\right) \end{bmatrix} - \begin{bmatrix} \frac{\pi}{144} & \frac{\pi}{432} & \frac{-\pi}{432} & \frac{-\pi}{720} & \frac{-\pi}{2160} & \frac{-\pi}{3024} & \frac{-\pi}{5050} & \frac{-\pi}{6480} & \frac{-\pi}{9072} \\ \frac{\pi}{288} & \frac{\pi}{864} & \frac{-\pi}{864} & \frac{-\pi}{1440} & \frac{-\pi}{4320} & \frac{-\pi}{6048} & \frac{-\pi}{10080} & \frac{-\pi}{12960} & \frac{-\pi}{18144} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_0 + \frac{1}{2}v_1 + \frac{3}{8}v_2 + \frac{10}{32}v_3 + \frac{35}{128}v_4 + \frac{126}{512}v_5 + \frac{462}{2048}v_6 + \frac{1716}{8192}v_7 + \frac{6435}{32768}v_8 \\ \frac{1}{2}v_1 + \frac{4}{8}v_2 + \frac{15}{32}v_3 + \frac{56}{128}v_4 + \frac{210}{512}v_5 + \frac{792}{2048}v_6 + \frac{3003}{8192}v_7 + \frac{11440}{32768}v_8 \\ \frac{1}{8}v_2 + \frac{6}{32}v_3 + \frac{28}{128}v_4 + \frac{120}{512}v_5 + \frac{495}{2048}v_6 + \frac{2002}{8192}v_7 + \frac{8008}{32768}v_8 \\ \frac{1}{32}v_3 + \frac{8}{128}v_4 + \frac{45}{512}v_5 + \frac{220}{2048}v_6 + \frac{1001}{8192}v_7 + \frac{4368}{32768}v_8 \\ \frac{1}{128}v_4 + \frac{10}{512}v_5 + \frac{66}{2048}v_6 + \frac{364}{8192}v_7 + \frac{1820}{32768}v_8 \\ \frac{1}{512}v_5 + \frac{12}{2048}v_6 + \frac{91}{8192}v_7 + \frac{560}{32768}v_8 \\ \frac{1}{2048}v_6 + \frac{14}{8192}v_7 + \frac{120}{32768}v_8 \\ \frac{1}{8192}v_7 + \frac{16}{32768}v_8 \\ \frac{1}{32768}v_8 \end{bmatrix} = \begin{bmatrix} 3.3420952591 \\ 2.03474845557 \\ 0.39676382882 \\ 0.0483791 \\ 0.00386263 \end{bmatrix}$$

We substitute values from equations (3.44) into the previous system to obtain

$$\begin{bmatrix} \pi(c_0 + \frac{1}{2}c_1 + \frac{3}{8}c_2 + \frac{10}{32}c_3 + \frac{35}{128}c_4) \\ \frac{\pi}{2}(\frac{1}{2}c_1 + \frac{4}{8}c_2 + \frac{15}{32}c_3 + \frac{56}{128}c_4) \\ \frac{\pi}{2}(\frac{1}{8}c_2 + \frac{6}{32}c_3 + \frac{28}{128}c_4) \\ \frac{\pi}{2}(\frac{1}{32}c_3 + \frac{8}{128}c_4) \\ \frac{\pi}{2}(\frac{1}{128}c_4) \end{bmatrix}$$

$$- \begin{bmatrix} \frac{\pi}{144} & \frac{\pi}{216} & \frac{\pi}{288} & \frac{\pi}{360} & \frac{\pi}{432} & \frac{\pi}{504} & \frac{\pi}{576} & \frac{\pi}{648} & \frac{\pi}{720} \\ \frac{\pi}{288} & \frac{\pi}{432} & \frac{\pi}{576} & \frac{\pi}{720} & \frac{\pi}{864} & \frac{\pi}{1008} & \frac{\pi}{1152} & \frac{\pi}{1296} & \frac{\pi}{1440} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0^2 \\ 2c_0c_1 \\ c_1^2 + 2c_0c_2 \\ 2c_0c_3 + 2c_1c_2 \\ 2c_0c_4 + 2c_1c_3 + c_2^2 \\ 2c_1c_4 + 2c_2c_3 \\ 2c_2c_4 + c_3^2 \\ 2c_3c_4 \\ c_4^2 \end{bmatrix}$$

$$= \begin{bmatrix} 3.3420952591 \\ 2.03474845557 \\ 0.39676382882 \\ 0.0483791 \\ 0.00386263 \end{bmatrix}$$

From the solution of the previous system we find the values of  $c$

$$c_0 = -0.007291503681$$

$$c_1 = 1.081202021308$$

$$c_2 = 0.935789449023$$

$$c_3 = 0.3560601145160$$

$$c_4 = 0.314755408811$$

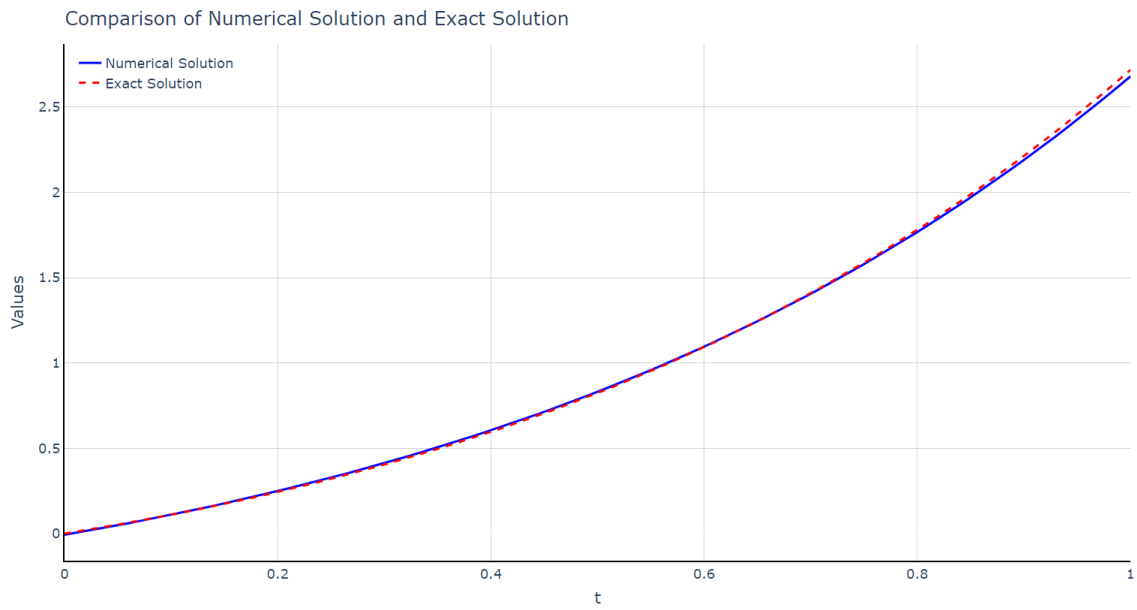
**Table 5**

The table contains the exact and the numerical solutions assuming  $M = 4$  when  $t = [0 \ 1]$  and shows the difference between them.

t	numerical solutions	Exact solutions	numerical – Exact
0.0	-0.007291503681	0.00000000000000	0.007291503681
0.05	0.0491545474481833	0.05256355481880120	0.003409009073982873
0.1	0.1105741286043062	0.11051709180756478	5.7036796741447016e-05
0.15	0.1773051099503333	0.17427513640924244	0.003029973541090858
0.2	0.2497325681470909	0.244280551632034	0.005452016515056973
0.25	0.3282887948700923	0.32100635417193535	0.007282440698156978
0.3	0.4134532951061721	0.40495764227280096	0.00849565283337117
0.35	0.5057527871534871	0.49667364200763997	0.009079145145847178
0.4	0.6057612026215156	0.5967298790565082	0.009031323565007354
0.45	0.7140996864310574	0.7057440483470576	0.008359202960481316
0.5	0.8314365968142343	0.8243606353500641	0.007075961464170222
0.55	0.95848750531449	0.9532891598270675	0.005198345487422484
0.6	1.0960151967865897	1.0932712802343052	0.0027439165522844533
0.65	1.24482966939662	1.2451015388590325	0.0002718694624124218
0.7	1.405788134619898	1.4096268952293336	0.003838760607343783
0.75	1.57979501725143	1.587750012459506	0.007954995208076188
0.8	1.7678019553849926	1.7804327427939743	0.012630787408981714
0.85	1.9708078004340501	1.9886998241370921	0.017892023703041993
0.9	2.1898586171213004	2.213642800041255	0.023784182919954766
0.95	2.4260476834807587	2.4564241763500534	0.03037649286929467
1.0	2.680515490857766	2.718281828459045	0.0377663376012789

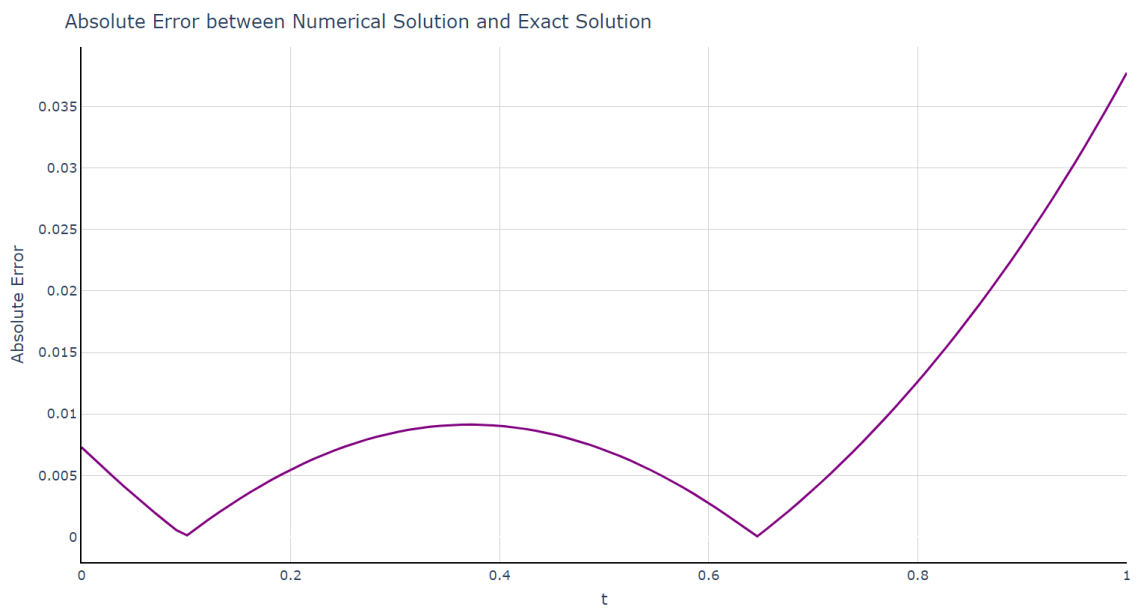
**Figure 9**

Shows both exact and numerical solutions using  $M = 4$  when  $t = [0 \ 1]$



**Figure 10**

Shows the error between exact and numerical solutions using  $M = 4$  When  $t = [0 \ 1]$



## Chapter Four

### Discussion and Conclusion

#### 4.1 Conclusion

In this thesis, we applied the algorithm method for solving the Fredholm– Hammerstein Integral equations using the shifted Chebyshev Polynomials and Galerkin method. Through this method, we got approximate results of numerical and exact solutions. The results show that the more the value of  $M$  increases the less the errors are. Thus, we got more accurate results. In the first example, the results of the errors resulted from the difference between the exact solution and the numerical solution where  $M = 2,4,6$  are apparent in the tables 1, 2, 3 respectively. However, the results of the second example where  $M = 2,4$  are presented in tables 4, 5 respectively. The values of  $M$  are presented in figures 1-10. On the one hand, the curves of the exact and numerical solutions were very approximate in figures 1,3,5,7,9. On the other hand, the curves of the errors showed an inverse relationship between the values of  $M$  and the values of the errors. In other words, when the values of  $M$  increase the values of the errors decrease. Accordingly, we recommend using the algorithm method for solving the Fredholm–Hammerstein integral equations using the shifted Chebyshev Polynomials and Galerkin methods due to its effectiveness and feasibility.

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## Appendices

### Appendix A

#### The values of $K$

$$\begin{aligned} K_{2,2} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_2^*(y)E_2^*(t)dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8y^2 - 8y + 1)(8t^2 - 8t + 1)dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} (8t^3 - 8t^2 + t)(8y^3 - 8y^2 + y)dy dt = 0 \end{aligned}$$

$$\begin{aligned} K_{3,o} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_3^*(y)E_o^*(t)dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32y^3 - 48y^2 + 18y - 1)dy dt = -\frac{\pi}{20} \end{aligned}$$

$$\begin{aligned} K_{3,1} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_3^*(y)E_1^*(t)dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32y^3 - 48y^2 + 18y - 1)(2t - 1)dy dt \\ &= -\frac{1}{10} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2t - 1)dt = -\frac{\pi}{40} \end{aligned}$$

$$\begin{aligned} K_{3,2} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_3^*(y)E_2^*(t)dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32y^3 - 48y^2 + 18y - 1)((8t^2 - 8t + 1)dy dt \end{aligned}$$

$$= -\frac{1}{10} \int_0^1 \frac{1}{\sqrt{t-t^2}} t((8t^2 - 8t + 1)) dt = 0$$

$$\begin{aligned} K_{4,0} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_4^*(y)E_0^*(t) dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(128y^4 - 256y^3 + 160y^2 - 32y + 1) dy dt = -\frac{\pi}{60} \end{aligned}$$

$$\begin{aligned} K_{4,1} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_4^*(y)E_1^*(t) dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(128y^4 - 256y^3 + 160y^2 - 32y + 1)(2t - 1) dy dt = -\frac{\pi}{120} \end{aligned}$$

$$\begin{aligned} K_{4,2} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_4^*(y)E_2^*(t) dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(128y^4 - 256y^3 + 160y^2 - 32y + 1)(8t^2 - 8t + 1) dy dt = 0 \end{aligned}$$

$$\begin{aligned} K_{0,3} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_0^*(y)E_3^*(t) dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32t^3 - 48t^2 + 18t - 1) dy dt = 0 \end{aligned}$$

$$\begin{aligned} K_{0,4} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_0^*(y)E_4^*(t) dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(128t^4 - 256t^3 + 160t^2 - 32t + 1) dy dt = 0 \end{aligned}$$

$$K_{1,3} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_1^*(y)E_3^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2y-1)(32t^3 - 48t^2 + 18t - 1)dy dt$$

$$= \frac{1}{6} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(32t^3 - 48t^2 + 18t - 1)dt = 0$$

$$K_{1,4} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_1^*(y)E_4^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2y-1)(128t^4 - 256t^3 + 160t^2 - 32t + 1)dy dt$$

$$= \frac{1}{6} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(128t^4 - 256t^3 + 160t^2 - 32t + 1)dt = 0$$

$$K_{2,3} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_2^*(y)E_3^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8y^2 - 8y + 1)(32t^3 - 48t^2 + 18t - 1)dy dt$$

$$= -\frac{1}{6} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(32t^3 - 48t^2 + 18t - 1)dt = 0$$

$$K_{2,4} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_2^*(y)E_4^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8y^2 - 8y + 1)(128t^4 - 256t^3 + 160t^2 - 32t + 1)dy dt$$

$$= -\frac{1}{6} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(128t^4 - 256t^3 + 160t^2 - 32t + 1) dt = 0$$

$$K_{3,3} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_3^*(y)E_3^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32y^3 - 48y^2 + 18y - 1)(32t^3 - 48t^2 + 18t - 1) dy dt$$

$$= -\frac{1}{10} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(32t^3 - 48t^2 + 18t - 1) dt = 0$$

$$K_{3,4} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_3^*(y)E_4^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32y^3 - 48y^2 + 18y - 1)(128t^4 - 256t^3 + 160t^2 - 32t + 1) dy dt$$

$$= -\frac{1}{10} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(128t^4 - 256t^3 + 160t^2 - 32t + 1) dt = 0$$

$$K_{4,3} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_4^*(y)E_3^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(128y^4 - 256y^3 + 160y^2 - 32y + 1)(32t^3 - 48t^2 + 18t - 1) dy dt$$

$$= -\frac{1}{20} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(32t^3 - 48t^2 + 18t - 1) dt = 0$$

$$\begin{aligned}
K_{4,4} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_4^*(y)E_4^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(128y^4 - 256y^3 + 160y^2 - 32y \\
&\quad + 1)(128t^4 - 256t^3 + 160t^2 - 32t + 1)dy dt \\
&= -\frac{1}{10} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(128t^4 - 256t^3 + 160t^2 - 32t + 1)dt = 0
\end{aligned}$$

$$\begin{aligned}
K_{5,4} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_5^*(y)E_4^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(512y^5 - 1280y^4 + 1120y^3 - 400y^2 + 50y - 1)(128t^4 - 256t^3 \\
&\quad + 160t^2 - 32t + 1)dy dt \\
&= \frac{-1}{42} \int_0^1 \frac{(128t^4 - 256t^3 + 160t^2 - 32t + 1)}{\sqrt{t-t^2}} tdt = 0
\end{aligned}$$

$$\begin{aligned}
K_{6,0} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_6^*(y)E_0^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048y^6 - 6144y^5 + 6912y^4 - 3584y^3 + 840y^2 - 72y \\
&\quad + 1)dy dt \\
&= \frac{-1}{70} \int_0^1 \frac{1}{\sqrt{t-t^2}} tdt \\
&= \frac{-\pi}{140}
\end{aligned}$$

$$\begin{aligned}
K_{6,1} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_6^*(y)E_1^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048y^6 - 6144y^5 + 6912y^4 - 3584y^3 + 840y^2 - 72y + 1)(2t \\
&\quad - 1)dy dt \\
&= \frac{-1}{70} \int_0^1 \frac{(2t-1)}{\sqrt{t-t^2}} tdt \\
&= \frac{-\pi}{280}
\end{aligned}$$

$$\begin{aligned}
K_{6,2} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_6^*(y)E_2^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048y^6 - 6144y^5 + 6912y^4 - 3584y^3 + 840y^2 - 72y + 1)(8t^2 \\
&\quad - 8t + 1)dy dt \\
&= \frac{-1}{70} \int_0^1 \frac{(8t^2 - 8t + 1)}{\sqrt{t-t^2}} tdt = 0
\end{aligned}$$

$$\begin{aligned}
K_{6,3} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_6^*(y)E_3^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048y^6 - 6144y^5 + 6912y^4 - 3584y^3 + 840y^2 - 72y \\
&\quad + 1)(32t^3 - 48t^2 + 18t - 1)dy dt \\
&= \frac{-1}{70} \int_0^1 \frac{(32t^3 - 48t^2 + 18t - 1)}{\sqrt{t-t^2}} tdt = 0
\end{aligned}$$

$$\begin{aligned}
K_{6,4} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_6^*(y)E_4^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048y^6 - 6144y^5 + 6912y^4 - 3584y^3 + 840y^2 - 72y \\
&\quad + 1)(128t^4 - 256t^3 + 160t^2 - 32t + 1)dy dt \\
&= \frac{-1}{70} \int_0^1 \frac{(128t^4 - 256t^3 + 160t^2 - 32t + 1)}{\sqrt{t-t^2}} tdt = 0
\end{aligned}$$

$$\begin{aligned}
K_{7,0} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_7^*(y)E_0^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8192y^7 - 28672y^6 + 39424y^5 - 26880y^4 + 9408y^3 - 1568y^2 \\
&\quad + 98y - 1)dy dt \\
&= \frac{-1}{90} \int_0^1 \frac{1}{\sqrt{t-t^2}} tdt \\
&= \frac{-\pi}{180}
\end{aligned}$$

$$\begin{aligned}
K_{7,1} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_7^*(y)E_1^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8192y^7 - 28672y^6 + 39424y^5 - 26880y^4 + 9408y^3 - 1568y^2 \\
&\quad + 98y - 1)(2t - 1)dy dt \\
&= \frac{-1}{90} \int_0^1 \frac{(2t - 1)}{\sqrt{t-t^2}} tdt \\
&= \frac{-\pi}{360}
\end{aligned}$$

$$\begin{aligned}
K_{7,2} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_7^*(y)E_2^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8192y^7 - 28672y^6 + 39424y^5 - 26880y^4 + 9408y^3 - 1568y^2 \\
&\quad + 98y - 1)(8t^2 - 8t + 1)dy dt \\
&= \frac{-1}{90} \int_0^1 \frac{(8t^2 - 8t + 1)}{\sqrt{t-t^2}} tdt = 0
\end{aligned}$$

$$\begin{aligned}
K_{7,3} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_7^*(y)E_3^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8192y^7 - 28672y^6 + 39424y^5 - 26880y^4 + 9408y^3 - 1568y^2 \\
&\quad + 98y - 1)(32t^3 - 48t^2 + 18t - 1)dy dt \\
&= \frac{-1}{90} \int_0^1 \frac{(32t^3 - 48t^2 + 18t - 1)}{\sqrt{t-t^2}} tdt = 0
\end{aligned}$$

$$\begin{aligned}
K_{7,4} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_7^*(y)E_4^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8192y^7 - 28672y^6 + 39424y^5 - 26880y^4 + 9408y^3 - 1568y^2 \\
&\quad + 98y - 1)(128t^4 - 256t^3 + 160t^2 - 32t + 1)dy dt \\
&= \frac{-1}{90} \int_0^1 \frac{(128t^4 - 256t^3 + 160t^2 - 32t + 1)}{\sqrt{t-t^2}} tdt = 0
\end{aligned}$$

$$K_{8,0} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_8^*(y)E_0^*(t)dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32768y^8 - 131072y^7 + 212992y^6 - 180224y^5 + 84480y^4 \\
&\quad - 21504y^3 + 2688y^2 - 128y + 1)dy dt \\
&= \frac{-1}{126} \int_0^1 \frac{1}{\sqrt{t-t^2}} t dt \\
&= \frac{-\pi}{252}
\end{aligned}$$

$$K_{8,1} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_8^*(y)E_1^*(t)dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32768y^8 - 131072y^7 + 212992y^6 - 180224y^5 + 84480y^4 \\
&\quad - 21504y^3 + 2688y^2 - 128y + 1)(2t-1)dy dt \\
&= \frac{-1}{126} \int_0^1 \frac{(2t-1)}{\sqrt{t-t^2}} t dt \\
&= \frac{-\pi}{504}
\end{aligned}$$

$$K_{8,2} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_8^*(y)E_2^*(t)dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32768y^8 - 131072y^7 + 212992y^6 - 180224y^5 + 84480y^4 \\
&\quad - 21504y^3 + 2688y^2 - 128y + 1)(8t^2 - 8t + 1)dy dt \\
&= \frac{-1}{126} \int_0^1 \frac{(8t^2 - 8t + 1)}{\sqrt{t-t^2}} t dt = 0
\end{aligned}$$

$$K_{8,3} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_8^*(y)E_3^*(t)dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32768y^8 - 131072y^7 + 212992y^6 - 180224y^5 + 84480y^4 \\
&\quad - 21504y^3 + 2688y^2 - 128y + 1)(32t^3 - 48t^2 + 18t - 1)dy dt \\
&= \frac{-1}{126} \int_0^1 \frac{(32t^3 - 48t^2 + 18t - 1)}{\sqrt{t-t^2}} tdt = 0
\end{aligned}$$

$$K_{8,4} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_8^*(y)E_4^*(t)dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32768y^8 - 131072y^7 + 212992y^6 - 180224y^5 + 84480y^4 \\
&\quad - 21504y^3 + 2688y^2 - 128y + 1)(128t^4 - 256t^3 + 160t^2 - 32t \\
&\quad + 1)dy dt \\
&= \frac{-1}{126} \int_0^1 \frac{(128t^4 - 256t^3 + 160t^2 - 32t + 1)}{\sqrt{t-t^2}} tdt = 0
\end{aligned}$$

$$K_{0,5} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_0^*(y)E_5^*(t)dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1)dy dt \\
&= \frac{1}{2} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1)dt
\end{aligned}$$

$$= 0$$

$$K_{0,6} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_0^*(y)E_6^*(t)dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dy dt \\
&= \frac{1}{2} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dt \\
&= 0
\end{aligned}$$

$$\begin{aligned}
K_{1,5} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_1^*(y)E_5^*(t) dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2y-1)(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dy dt \\
&= \frac{1}{6} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dt \\
&= 0
\end{aligned}$$

$$\begin{aligned}
K_{1,6} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_1^*(y)E_6^*(t) dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2y \\
&\quad - 1)(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dy dt \\
&= \frac{1}{6} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dt = 0
\end{aligned}$$

$$K_{2,5} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_2^*(y)E_5^*(t) dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8y^2 - 8y \\
&\quad + 1)(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1)dy dt \\
&= -\frac{1}{6} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1)dt = 0 \\
K_{2,6} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_2^*(y)E_6^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8y^2 - 8y \\
&\quad + 1)(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1)dy dt \\
&= -\frac{1}{6} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1)dt = 0 \\
K_{3,5} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_3^*(y)E_5^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32y^3 - 48y^2 + 18y \\
&\quad - 1)(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1)dy dt \\
&= -\frac{1}{10} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1)dt = 0 \\
K_{3,6} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_3^*(y)E_6^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32y^3 - 48y^2 + 18y \\
&\quad - 1)(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1)dy dt
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{10} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dt \\
&= 0
\end{aligned}$$

$$K_{4,5} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_4^*(y)E_5^*(t) dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(128y^4 - 256y^3 + 160y^2 - 32y \\
&\quad + 1)(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dy dt
\end{aligned}$$

$$= -\frac{1}{10} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dt = 0$$

$$K_{4,6} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_4^*(y)E_6^*(t) dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(128y^4 - 256y^3 + 160y^2 - 32y \\
&\quad + 1)(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dy dt
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{10} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dt \\
&= 0
\end{aligned}$$

$$K_{5,5} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_5^*(y)E_5^*(t) dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(512y^5 - 1280y^4 + 1120y^3 - 400y^2 + 50y \\
&\quad - 1)(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dy dt
\end{aligned}$$

$$= -\frac{1}{42} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dt = 0$$

$$\begin{aligned} K_{5,6} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_5^*(y)E_6^*(t) dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(512y^5 - 1280y^4 + 1120y^3 - 400y^2 + 50y - 1)(2048t^6 \\ &\quad - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dy dt \\ &= -\frac{1}{42} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dt \\ &= 0 \end{aligned}$$

$$\begin{aligned} K_{6,5} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_6^*(y)E_5^*(t) dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048y^6 - 6144y^5 + 6912y^4 - 3584y^3 + 840y^2 - 72y \\ &\quad + 1)(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dy dt \\ &= -\frac{1}{70} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dt = 0 \end{aligned}$$

$$\begin{aligned} K_{6,6} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_6^*(y)E_6^*(t) dy dt \\ &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048y^6 - 6144y^5 + 6912y^4 - 3584y^3 + 840y^2 - 72y \\ &\quad + 1)(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dy dt \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{70} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dt \\
&= 0
\end{aligned}$$

$$\begin{aligned}
K_{7,5} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_7^*(y)E_5^*(t) dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8192y^7 - 28672y^6 + 39424y^5 - 26880y^4 + 9408y^3 - 1568y^2 \\
&\quad + 98y - 1)(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dy dt \\
&= -\frac{1}{90} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dt = 0
\end{aligned}$$

$$\begin{aligned}
K_{7,6} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_7^*(y)E_6^*(t) dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8192y^7 - 28672y^6 + 39424y^5 - 26880y^4 + 9408y^3 - 1568y^2 \\
&\quad + 98y - 1)(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t \\
&\quad + 1) dy dt \\
&= -\frac{1}{90} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dt \\
&= 0
\end{aligned}$$

$$\begin{aligned}
K_{8,5} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_8^*(y)E_5^*(t) dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32768y^8 - 131072y^7 + 212992y^6 - 180224y^5 + 84480y^4 \\
&\quad - 21504y^3 + 2688y^2 - 128y + 1)(512t^5 - 1280t^4 + 1120t^3 \\
&\quad - 400t^2 + 50t - 1) dy dt
\end{aligned}$$

$$= -\frac{1}{126} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dt = 0$$

$$K_{8,6} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_8^*(y)E_6^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32768y^8 - 131072y^7 + 212992y^6 - 180224y^5 + 84480y^4 - 21504y^3 + 2688y^2 - 128y + 1)(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dy dt$$

$$= -\frac{1}{126} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dt = 0$$

$$K_{9,4} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_9^*(y)E_4^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty((131072y^9 - 589824y^8 + 1105920y^7 - 1118208y^6 + 658944y^5 - 228096y^4 + 44352y^3 - 4320y^2 + 162y - 1)(128t^4 - 256t^3 + 160t^2 - 32t + 1)) dy dt$$

$$= -\frac{1}{154} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(128t^4 - 256t^3 + 160t^2 - 32t + 1) dt = 0$$

$$K_{9,5} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_9^*(y)E_5^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty((131072y^9 - 589824y^8 + 1105920y^7 - 1118208y^6 + 658944y^5 - 228096y^4 + 44352y^3 - 4320y^2 + 162y - 1)(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1)) dy dt$$

$$= -\frac{1}{154} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dt = 0$$

$$K_{9,6} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_9^*(y)E_6^*(t) dy dt$$

$$\begin{aligned} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(131072y^9 - 589824y^8 + 1105920y^7 - 1118208y^6 \\ &\quad + 658944y^5 - 228096y^4 + 44352y^3 - 4320y^2 + 162y \\ &\quad - 1)(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dy dt \\ &= -\frac{1}{154} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dt \\ &= 0 \end{aligned}$$

$$K_{10,0} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{10}^*(y)E_0^*(t) dy dt$$

$$\begin{aligned} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(524288y^{10} - 2621440y^9 + 5570560y^8 - 6553600y^7 \\ &\quad + 4659200y^6 - 2050048y^5 + 549120y^4 - 84480y^3 + 6600y^2 \\ &\quad - 200y + 1) dy dt = -\frac{\pi}{396} \end{aligned}$$

$$K_{10,1} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{10}^*(y)E_1^*(t) dy dt$$

$$\begin{aligned} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(524288y^{10} - 2621440y^9 + 5570560y^8 - 6553600y^7 \\ &\quad + 4659200y^6 - 2050048y^5 + 549120y^4 - 84480y^3 + 6600y^2 \\ &\quad - 200y + 1)(2t - 1) dy dt = -\frac{\pi}{792} \end{aligned}$$

$$\begin{aligned}
K_{10,2} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_{10}^*(y) E_2^*(t) dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty (524288y^{10} - 2621440y^9 + 5570560y^8 - 6553600y^7 \\
&\quad + 4659200y^6 - 2050048y^5 + 549120y^4 - 84480y^3 + 6600y^2 \\
&\quad - 200y + 1)(8t^2 - 8t + 1) dy dt = 0
\end{aligned}$$

$$\begin{aligned}
K_{10,3} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_{10}^*(y) E_3^*(t) dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty (524288y^{10} - 2621440y^9 + 5570560y^8 - 6553600y^7 \\
&\quad + 4659200y^6 - 2050048y^5 + 549120y^4 - 84480y^3 + 6600y^2 \\
&\quad - 200y + 1)(32t^3 - 48t^2 + 18t - 1) dy dt
\end{aligned}$$

$$= -\frac{1}{198} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(32t^3 - 48t^2 + 18t - 1) dt = 0$$

$$\begin{aligned}
K_{10,4} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_{10}^*(y) E_4^*(t) dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty (524288y^{10} - 2621440y^9 + 5570560y^8 - 6553600y^7 \\
&\quad + 4659200y^6 - 2050048y^5 + 549120y^4 - 84480y^3 + 6600y^2 \\
&\quad - 200y + 1)(128t^4 - 256t^3 + 160t^2 - 32t + 1) dy dt
\end{aligned}$$

$$= -\frac{1}{198} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(128t^4 - 256t^3 + 160t^2 - 32t + 1) dt = 0$$

$$K_{10,5} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty E_{10}^*(y) E_5^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(524288y^{10} - 2621440y^9 + 5570560y^8 - 6553600y^7$$

$$+ 4659200y^6 - 2050048y^5 + 549120y^4 - 84480y^3 + 6600y^2$$

$$- 200y + 1)(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1)dy dt$$

$$= -\frac{1}{198} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1)dt = 0$$

$$K_{10,6} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{10}^*(y)E_6^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(524288y^{10} - 2621440y^9 + 5570560y^8 - 6553600y^7$$

$$+ 4659200y^6 - 2050048y^5 + 549120y^4 - 84480y^3 + 6600y^2$$

$$- 200y + 1)(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t$$

$$+ 1)dy dt$$

$$= -\frac{1}{198} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1)dt$$

$$= 0$$

$$K_{11,0} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{11}^*(y)E_0^*(t)dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2097152y^{11} - 11534336y^{10} + 27394048y^9 - 36765696y^8$$

$$+ 30638080y^7 - 16400384y^6 + 5637632y^5 - 1208064y^4$$

$$+ 151008y^3 - 9680y^2 + 242y - 1)dy dt = -\frac{\pi}{468}$$

$$K_{11,1} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{11}^*(y)E_1^*(t)dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2097152y^{11} - 11534336y^{10} + 27394048y^9 - 36765696y^8 \\
&\quad + 30638080y^7 - 16400384y^6 + 5637632y^5 - 1208064y^4 \\
&\quad + 151008y^3 - 9680y^2 + 242y - 1)(2t - 1)dy dt = -\frac{\pi}{936}
\end{aligned}$$

$$K_{11,2} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{11}^*(y)E_2^*(t)dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2097152y^{11} - 11534336y^{10} + 27394048y^9 - 36765696y^8 \\
&\quad + 30638080y^7 - 16400384y^6 + 5637632y^5 - 1208064y^4 \\
&\quad + 151008y^3 - 9680y^2 + 242y - 1)(8t^2 - 8t + 1)dy dt = 0
\end{aligned}$$

$$K_{11,3} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{11}^*(y)E_3^*(t)dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2097152y^{11} - 11534336y^{10} + 27394048y^9 - 36765696y^8 \\
&\quad + 30638080y^7 - 16400384y^6 + 5637632y^5 - 1208064y^4 \\
&\quad + 151008y^3 - 9680y^2 + 242y - 1)(32t^3 - 48t^2 + 18t - 1)dy dt \\
&= -\frac{1}{234} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(32t^3 - 48t^2 + 18t - 1)dt = 0
\end{aligned}$$

$$K_{11,4} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{11}^*(y)E_4^*(t)dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2097152y^{11} - 11534336y^{10} + 27394048y^9 - 36765696y^8 \\
&\quad + 30638080y^7 - 16400384y^6 + 5637632y^5 - 1208064y^4 \\
&\quad + 151008y^3 - 9680y^2 + 242y - 1)(128t^4 - 256t^3 + 160t^2 - 32t \\
&\quad + 1)dy dt
\end{aligned}$$

$$= -\frac{1}{234} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(128t^4 - 256t^3 + 160t^2 - 32t + 1) dt = 0$$

$$K_{11,5} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{11}^*(y)E_5^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2097152y^{11} - 11534336y^{10} + 27394048y^9 - 36765696y^8$$

$$+ 30638080y^7 - 16400384y^6 + 5637632y^5 - 1208064y^4$$

$$+ 151008y^3 - 9680y^2 + 242y - 1)(512t^5 - 1280t^4 + 1120t^3$$

$$- 400t^2 + 50t - 1) dy dt$$

$$= -\frac{1}{234} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1) dt = 0$$

$$K_{11,6} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{11}^*(y)E_6^*(t) dy dt$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2097152y^{11} - 11534336y^{10} + 27394048y^9 - 36765696y^8$$

$$+ 30638080y^7 - 16400384y^6 + 5637632y^5 - 1208064y^4$$

$$+ 151008y^3 - 9680y^2 + 242y - 1)(2048t^6 - 6144t^5 + 6912t^4$$

$$- 3584t^3 + 840t^2 - 72t + 1) dy dt$$

$$= -\frac{1}{234} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1) dt$$

$$= 0$$

$$K_{12,0} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{12}^*(y)E_0^*(t) dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8388608y^{12} - 50331648y^{11} \\
&\quad + 132120576y^{10} - 199229440y^9 + 190513152y^8 - 120324096y^7 \\
&\quad + 50692096y^6 - 14057472y^5 + 2471040y^4 - 256256y^3 \\
&\quad + 13728y^2 - 288y + 1)dy dt = -\frac{\pi}{572}
\end{aligned}$$

$$\begin{aligned}
K_{12,1} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{12}^*(y)E_1^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8388608y^{12} - 50331648y^{11} \\
&\quad + 132120576y^{10} - 199229440y^9 + 190513152y^8 - 120324096y^7 \\
&\quad + 50692096y^6 - 14057472y^5 + 2471040y^4 - 256256y^3 \\
&\quad + 13728y^2 - 288y + 1)dy dt \\
&= -\frac{1}{286} \int_0^1 \frac{2t-1}{\sqrt{t-t^2}} tdt = -\frac{\pi}{1144}
\end{aligned}$$

$$\begin{aligned}
K_{12,2} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{12}^*(y)E_2^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8388608y^{12} - 50331648y^{11} \\
&\quad + 132120576y^{10} - 199229440y^9 + 190513152y^8 - 120324096y^7 \\
&\quad + 50692096y^6 - 14057472y^5 + 2471040y^4 - 256256y^3 \\
&\quad + 13728y^2 - 288y + 1)(8t^2 - 8t + 1)dy dt \\
&= -\frac{1}{286} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(8t^2 - 8t + 1)dt = 0
\end{aligned}$$

$$K_{12,3} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{12}^*(y)E_3^*(t)dy dt$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8388608y^{12} - 50331648y^{11} \\
&\quad + 132120576y^{10} - 199229440y^9 + 190513152y^8 - 120324096y^7 \\
&\quad + 50692096y^6 - 14057472y^5 + 2471040y^4 - 256256y^3 \\
&\quad + 13728y^2 - 288y + 1)(32t^3 - 48t^2 + 18t - 1)dy dt \\
&= -\frac{1}{286} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(32t^3 - 48t^2 + 18t - 1)dt = 0 \\
K_{12,4} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{12}^*(y)E_4^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8388608y^{12} - 50331648y^{11} \\
&\quad + 132120576y^{10} - 199229440y^9 + 190513152y^8 - 120324096y^7 \\
&\quad + 50692096y^6 - 14057472y^5 + 2471040y^4 - 256256y^3 \\
&\quad + 13728y^2 - 288y + 1)(128t^4 - 256t^3 + 160t^2 - 32t + 1)dy dt \\
&= -\frac{1}{286} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(128t^4 - 256t^3 + 160t^2 - 32t + 1)dt = 0 \\
K_{12,5} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{12}^*(y)E_5^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8388608y^{12} - 50331648y^{11} \\
&\quad + 132120576y^{10} - 199229440y^9 + 190513152y^8 - 120324096y^7 \\
&\quad + 50692096y^6 - 14057472y^5 + 2471040y^4 - 256256y^3 \\
&\quad + 13728y^2 - 288y + 1)(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t \\
&\quad - 1)dy dt \\
&= -\frac{1}{286} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1)dt = 0
\end{aligned}$$

$$\begin{aligned}
K_{12,6} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_{12}^*(y)E_6^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8388608y^{12} - 50331648y^{11} \\
&\quad + 132120576y^{10} - 199229440y^9 + 190513152y^8 - 120324096y^7 \\
&\quad + 50692096y^6 - 14057472y^5 + 2471040y^4 - 256256y^3 \\
&\quad + 13728y^2 - 288y + 1)(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 \\
&\quad + 840t^2 - 72t + 1)dy dt \\
&= -\frac{1}{286} \int_0^1 \frac{1}{\sqrt{t-t^2}} t(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1)dt \\
&= 0
\end{aligned}$$

$$K_{2,1} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} ty(8y^2 - 8y + 1)(2t - 1) dy dt = -\frac{\pi}{864}$$

$$K_{2,2} = \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} tyE_2^*(y)E_2^*(t)dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8y^2 - 8y + 1)(8t^2 - 8t + 1) dy dt = 0$$

$$K_{3,0} = \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_3^*(y)E_0^*(t)dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} 4\pi ty(32y^3 - 48y^2 + 18y - 1) dy dt = -\frac{\pi}{720}$$

$$K_{3,1} = \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_3^*(y)E_1^*(t)dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32y^3 - 48y^2 + 18y - 1)(2t - 1) dy dt = -\frac{\pi}{1440}$$

$$K_{3,2} = \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_3^*(y)E_2^*(t) dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32y^3 - 48y^2 + 18y - 1)(8t^2 - 8t + 1) dy dt = 0$$

$$K_{4,0} = \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_4^*(y)E_0^*(t) dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(128y^4 - 256y^3 + 160y^2 - 32y + 1) dy dt = -\frac{\pi}{1440}$$

$$K_{4,1} = \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_4^*(y)E_1^*(t) dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(128y^4 - 256y^3 + 160y^2 - 32y + 1)(2t - 1) dy dt = -\frac{\pi}{4320}$$

$$K_{4,2} = \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_4^*(y)E_2^*(t) dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(128y^4 - 256y^3 + 160y^2 - 32y + 1)(8t^2 - 8t + 1) dy dt = 0$$

$$K_{6,0} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} tyE_6^*(y)E_0^*(t) dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048y^6 - 6144y^5 + 6912y^4 - 3584y^3 + 840y^2 - 72y + 1) dy dt = \frac{-\pi}{5040}$$

$$K_{6,1} = \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_6^*(y)E_1^*(t) dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048y^6 - 6144y^5 + 6912y^4 - 3584y^3 + 840y^2 - 72y + 1)(2t-1) dy dt = \frac{-\pi}{10080}$$

$$K_{6,2} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} tyE_6^*(y)E_2^*(t) dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048y^6 - 6144y^5 + 6912y^4 - 3584y^3 + 840y^2 - 72y + 1)(8t^2 - 8t + 1) dy dt = 0$$

$$K_{6,3} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} tyE_6^*(y)E_3^*(t) dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048y^6 - 6144y^5 + 6912y^4 - 3584y^3 + 840y^2 - 72y + 1)(32t^3 - 48t^2 + 18t - 1) dy dt = 0$$

$$K_{6,4} = \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} tyE_6^*(y)E_4^*(t) dy dt$$

$$= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(2048y^6 - 6144y^5 + 6912y^4 - 3584y^3 + 840y^2 - 72y + 1)(128t^4 - 256t^3 + 160t^2 - 32t + 1) dy dt = 0$$

$$\begin{aligned}
K_{7,0} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_7^*(y)E_0^*(t)dy dt \\
&= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8192y^7 - 28672y^6 + 39424y^5 - 26880y^4 + 9408y^3 \\
&\quad - 1568y^2 + 98y - 1)dy dt = \frac{-\pi}{6480}
\end{aligned}$$

$$\begin{aligned}
K_{7,1} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_7^*(y)E_1^*(t)dy dt \\
&= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8192y^7 - 28672y^6 + 39424y^5 - 26880y^4 + 9408y^3 \\
&\quad - 1568y^2 + 98y - 1)(2t - 1)dy dt = \frac{-\pi}{12960}
\end{aligned}$$

$$\begin{aligned}
K_{7,2} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} tyE_7^*(y)E_2^*(t)dy dt \\
&= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} ty(8192y^7 - 28672y^6 + 39424y^5 - 26880y^4 + 9408y^3 \\
&\quad - 1568y^2 + 98y - 1)(8t^2 - 8t + 1)dy dt = 0
\end{aligned}$$

$$\begin{aligned}
K_{7,3} &= \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} \frac{1}{36} tyE_7^*(y)E_3^*(t)dy dt \\
&= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8192y^7 - 28672y^6 + 39424y^5 - 26880y^4 + 9408y^3 \\
&\quad - 1568y^2 + 98y - 1)(32t^3 - 48t^2 + 18t - 1)dy dt = 0
\end{aligned}$$

$$K_{7,4} = \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_7^*(y)E_4^*(t)dy dt$$

$$\begin{aligned}
&= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(8192y^7 - 28672y^6 + 39424y^5 - 26880y^4 + 9408y^3 \\
&\quad - 1568y^2 + 98y - 1)(128t^4 - 256t^3 + 160t^2 - 32t + 1)dy dt = 0
\end{aligned}$$

$$\begin{aligned}
K_{8,0} &= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_8^*(y)E_0^*(t)dy dt \\
&= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32768y^8 - 131072y^7 + 212992y^6 - 180224y^5 + 84480y^4 \\
&\quad - 21504y^3 + 2688y^2 - 128y + 1)dy dt = \frac{-\pi}{9072}
\end{aligned}$$

$$\begin{aligned}
K_{8,1} &= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_8^*(y)E_1^*(t)dy dt \\
&= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32768y^8 - 131072y^7 + 212992y^6 - 180224y^5 + 84480y^4 \\
&\quad - 21504y^3 + 2688y^2 - 128y + 1)(2t - 1)dy dt = \frac{-\pi}{18144}
\end{aligned}$$

$$\begin{aligned}
K_{8,2} &= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_8^*(y)E_2^*(t)dy dt \\
&= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32768y^8 - 131072y^7 + 212992y^6 - 180224y^5 + 84480y^4 \\
&\quad - 21504y^3 + 2688y^2 - 128y + 1)(8t^2 - 8t + 1)dy dt = 0
\end{aligned}$$

$$\begin{aligned}
K_{8,3} &= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_8^*(y)E_3^*(t)dy dt \\
&= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32768y^8 - 131072y^7 + 212992y^6 - 180224y^5 + 84480y^4 \\
&\quad - 21504y^3 + 2688y^2 - 128y + 1)(32t^3 - 48t^2 + 18t - 1)dy d = 0
\end{aligned}$$

$$\begin{aligned}
K_{8,4} &= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} tyE_8^*(y)E_4^*(t)dy dt \\
&= \frac{1}{36} \int_0^1 \int_0^1 \frac{1}{\sqrt{t-t^2}} ty(32768y^8 - 131072y^7 + 212992y^6 - 180224y^5 + 84480y^4 \\
&\quad - 21504y^3 + 2688y^2 - 128y + 1)(128t^4 - 256t^3 + 160t^2 - 32t \\
&\quad + 1)dy dt = 0
\end{aligned}$$



جامعة النجاح الوطنية

كلية الدراسات العليا

## الحل العددي لمعادلات فريدهولم التكاملية

إعداد

فاتن فاروق يوسف خلف

إشراف

د. عدنان دراغمة

قدمت هذه الرسالة استكمالاً لمتطلبات الحصول على درجة الماجستير في الرياضيات المحوسبة، من كلية الدراسات العليا، في جامعة النجاح الوطنية، نابلس - فلسطين.

2024

# الحل العددي لمعادلات فريدهولم التكاملية

إعداد

فاتن فاروق يوسف خلف

إشراف

د. عدنان دراغمة

## الملخص

معادلات فريدهولم-هامرشتاين التكاملية لها العديد من التطبيقات في مجالات مختلفة من الفيزياء الرياضية، بما في ذلك مشاكل نقل الحرارة وديناميكيات الموائع وميكانيكا الكم.

في هذا البحث، نركز على المعالجة العددية لمعادلات فريدهولم-هامرشتاين باستخدام طريقة جاليركين وطريقة كثيرات الحدود تشيبيشيف المنزاحة.

ولاختبار كفاءة ودقة هذه الطريقة تم تقديم مثالين عدديين مع الحل الدقيق المعروف. تظهر النتائج العددية أن نقل هذه الطريقة يتوافق جيدًا مع الحل الدقيق. علاوة على ذلك، نستنتج أن طريقة تشيبيشيف المتحولة توفر نتائج دقيقة للغاية.

**الكلمات المفتاحية:** معادلات هامرشتاين، معادلة فريدهولم التكاملية غير الخطية، حدود تشيبيشيف المزاحة، توسع تايلور، طريقة جاليركين.