An-Najah National University
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On A mathematical Design System: Maximum Reliability, Minimum Cost

By

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I present this work to whom I loved, parents, brothers, sisters, my wife, my children, all of my family, and my friends.

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Contents

Abstract

In this thesis, reliability concepts, measures of reliability and static models have been studied. Also, comparisons between exponential and logistic distributions have been discussed.

We used two methods; dynamic programming and heuristic approach to maximize the reliability of an electronic device systems, where the optimum structure of components and units of the assumed system have been determined.

Examples are given to show the optimum structure of the system with the maximum reliability and minimum cost.

Comparison between dynamic programming and heuristic approach shows that the dynamic programming results are better than the results obtained by the heuristic approach.

Finally, our objective is to characterize marginal cost and minimize cost capacity plans for a typical service delivery system. Results indicate that marginal costs are convex with respect to reliability of service, while

changes in the demand distribution's variability may impact optimal capacity by either increasing or decreasing required capacity.

Two demand distributions are assumed: uniform and logistic distributions. The results show that the logistic demand distribution gives an optimum criterions which are more realistic. Also, the optimum capacity using logistic is greater under the condition $\frac{b}{B} > 0.5$.

Introduction

Reliability

Reliability is a word with many different meanings. A reliable person is one who has the abilities to do certain tasks according to a specified standard. A similar meaning is given when the word is applied to a piece of equipment; namely the ability of that equipment to fulfill what is required from it. The original use of the word reliability was purely qualitative but as used today, reliability is always a quantitative concept.

Reliability engineering helps to ensure the success of space missions, maintain the national security, deliver a steady supply of electric power, provide a reliable transportation, and so on. "Reliability is therefore an important concepts in the design of engineering systems. Reliability may be defined as the probability that an item or a piece of equipment will carry out its specified function satisfactorily for the state period when used under the designed conditions".

The application of reliability concepts to electric power generation goes back to the 1930's. However, World War II is generally regarded as the beginning of the reliability field, when the Germans introduced the reliability concepts to improve their V-1 and V-2 rockets.

During the period between 1945-1950, the United States Navy, Air force, and Army conducted various studies on the failure of electronic equipments, equipment repair and maintance costs, etc. As a result of this effort, in 1950, the Department of Defense formed committee on reliability, which later became known as the advisory group on the reliability of Electronic Equipment (AGREE). This group published a report in 1957 that led to specifications for the reliability of military electronic equipments. Furthermore because of the awareness of the reliability problem in the United States, the early 1950's witnessed the appearance of the Institute of Electrical, and Electronic Engineers. Transaction on Reliability and proceeding of the National Symposium on Reliability and Quality control. Since 1950s thousands of publications have appeared in this field.

Reliability of an equipment is related to the life time (time of successful operation before failure) of the equipment under specified conditions. It follows that the life time to failure T of an equipment is a non-negative random variable. The corresponding distribution function and probability density function of T which denoted by F(t) and f(t) respectively. With this set up we can define the reliability R(t), of an equipment at time t under specific conditions to be the "probability that the equipment was working without failure in the interval [0,t) symbolically".

$$R(t) = p(T>t) = 1-F(t).$$

Chapter (1)

Design Reliability

1.1 Introduction

This chapter is concerned with the development of fundamental definitions, and concepts for reliability. Any reliability analysis of a system must be based on precisely defined concepts. Since it is readily accepted that a population of supposedly identical systems, operating under similar conditions fail at different points in time, it follows that a failure phenomenon can only be described in probability terms. These concepts provide the basis for quantifying the reliability of a system. The basic definitions are presented in Sec. 1.2.

Section 1.3 is concerned with reliability measures. In section 1.4 we make comparison between exponential function and logistic function. In section 1.5 we use methods to demonstrate reliability analysis. We represent the series, parallel, and complex system reliability models.

In section 1.6 we represent the method used to calculate reliability. There are two methods used to calculate the reliability, the heuristic and the dynamic programming methods.

1.2 Selected Terms:

Various terms are specific to the field of reliability engineering. Some of the most common terms are as follows:

1. Reliability

The probability that an item will perform its specified function satisfactorily for the stated period when used under the designed condition is called reliability. [6]

2. Hazard rate:

The rate of change of the number of failed parts divided by the number of survived parts at time t is called Hazard rate. [6,7]

3. Failure:

The inability of an item to operate within the initially stated guidelines is called failure.[9,10]

4. Active redundancy:

The term indicated that all redundant items are operating simultaneously is called active redundancy.[9]

5. Parallel system:

A system that is not considered to have failed unless all components have failed.[7,9]

6. Series System:

In a series system all subsystems must operate successfully if the system is to function.

1.3 Reliability Measures:

1.3.1 Time - dependent Reliability:

The probability of failure as a function of time can be defined by

$$p(T \le t) = F(t), \qquad t \ge 0 \tag{1.1}$$

where T is a random variable denoting the failure time.F(t) is the probability that the system will fail by time t. F(t) is a failure distribution function.

We can write

$R(t)$ is the time dependent reliability distribution function.

$f(t)$ is the failure density function for the item.

t: is time.

$$R(t) = 1 - F(t) = 1 - \int_0^t f(x)dx \tag{1.2}$$

or $$R(t) = \int_{t}^{\infty} f(x)dx \qquad 1.3$$

or $$R(t) = e^{-\int_{0}^{t} Z(x)dx} \qquad 1.4$$

where Z(x) is the hazard rate, on the time dependent failure rate, of the item

1.3.2 The Expected Life:

The expected life [8] is defined as

$$E(t) = \int_{0}^{\infty} t f(t)dt \qquad 1.5$$

A useful alternate form is:

$$E(t) = \int_{0}^{\infty} R(t)dt \qquad 1.6$$

Another method for determing the expected life

$$E(t) = \lim_{s \to 0} R(s) \qquad 1.7$$

where s is Laplace transform variable.

$R(s)$ is Laplace transform of the reliability function.

1.3.3 Hazard Rate:

The hazard rate [8,10] of an item or system may be obtained by using either of the following two relationships:

$$Z(t)=\frac{f(t)}{R(t)}=\frac{f(t)}{1-F(t)} \qquad 1.8$$

or

$$Z(t)=\frac{-1}{R(t)}\left[\frac{d}{dt}R(t)\right] \qquad 1.9$$

1.4 The Reliability, Hazard function, and Expected life for exponential, and logistic distribution functions.

1.4.1 Exponential Distribution:

The probability density function of exponential distribution is given by:

$$f(t)=\begin{cases}\frac{1}{\lambda}e^{\frac{-1}{\lambda}t} & t\rangle 0,\lambda\rangle 0\\ 0 & otherwise\end{cases}$$

The exponential distribution is widely used in reliability. For this distribution, we merely compute equations 1.2, 1.5 and 1.8.

i. Reliability function:

$$R(t) = p(T > t) = \int_t^{\infty} f(x)dx = e^{\frac{-1}{\lambda}t}$$

The graph is shown in Fig. 1.1

ii. Hazard function denoted by $Z(t)$

$$Z(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\lambda}e^{\frac{-1}{\lambda}t}}{e^{\frac{-1}{\lambda}t}} = \frac{1}{\lambda}$$

The graph is shown in Fig. 1.2

iii. Mean time to failure: denoted by $MTTF$

$$MTTF = \int_0^{\infty} R(t)dt = \int_0^{\infty} e^{\frac{-1}{\lambda}t}dt = \lambda$$

Some authors called ($MTTF$) as an Expected life and is denoted by $E(t)$.

The graph is shown in Fig. (1.3).

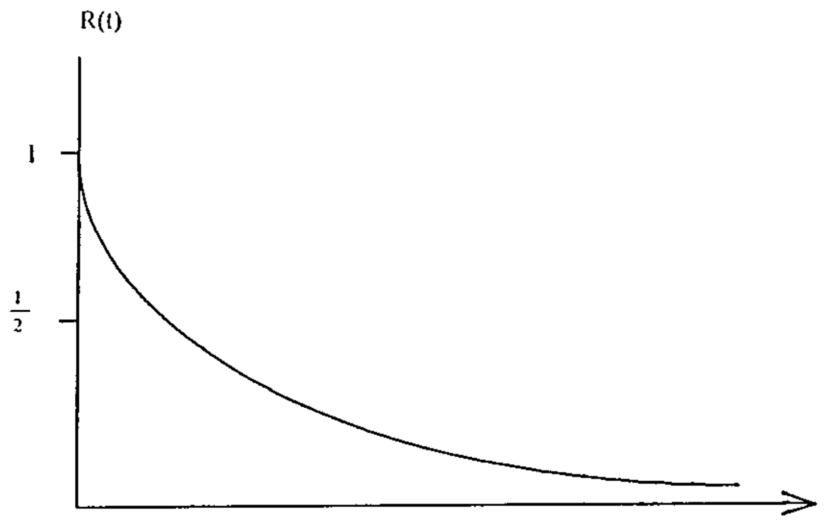

Fig. (1.1)
Reliability

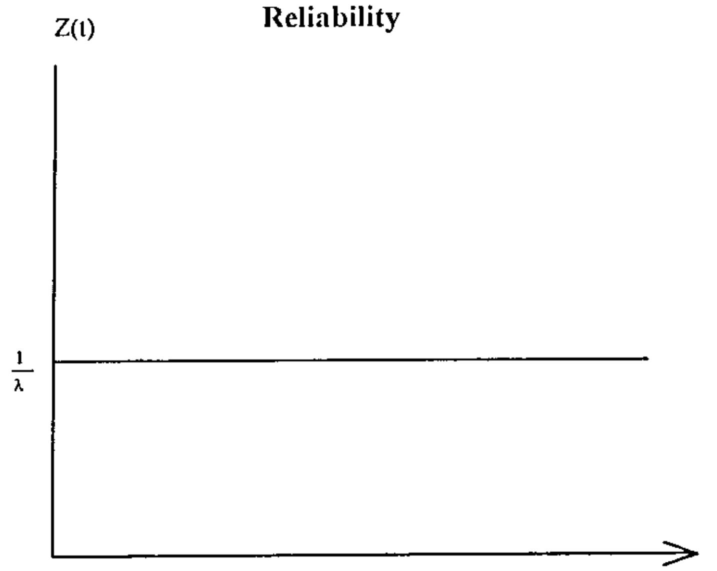

Fig. (1.2)
Hazard function

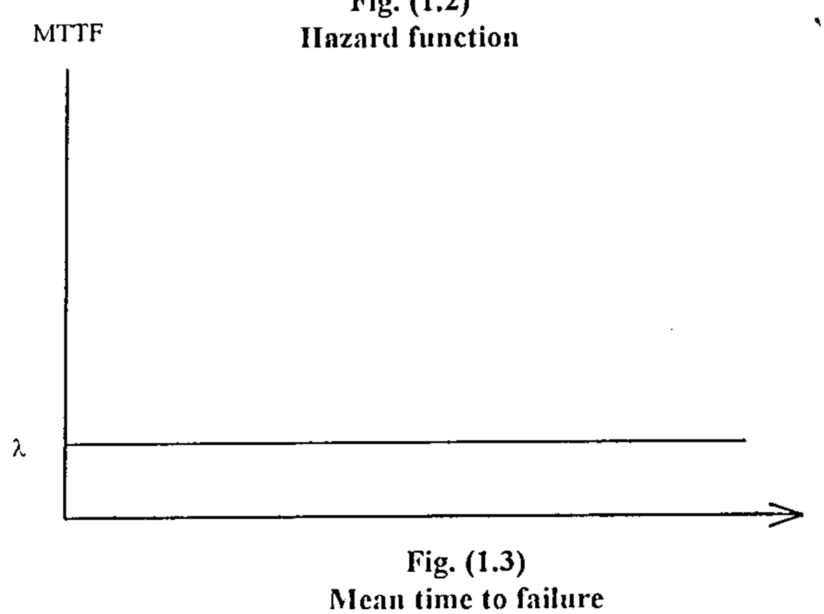

Fig. (1.3)
Mean time to failure

1.4.2 Logistic Distribution:

The probability density function of logistic or sech square density function is given by:

$$f(x,\lambda,\sigma) = \begin{cases} \dfrac{\frac{a}{\sigma} e^{\frac{-a}{\sigma}(x-\lambda)}}{\left[1+e^{-\frac{a}{\sigma}(x-\lambda)}\right]^2} & -\infty<x<\infty \\ 0 & ,othewise \end{cases} \quad , \quad \sigma = \sqrt{\lambda} \quad , \quad a = \frac{\pi}{\sqrt{3}}$$

For this distribution, we merely reduce equations 1.2, 1.5 and 1.8.

i. Reliability function of R(t)

$$R(t) = P(T>t) = \int_t^{\infty} f(x)dx$$

$$= \int_t^{\infty} \frac{\frac{a}{\sigma} e^{\frac{-a}{\sigma}(x-\lambda)}}{\left[1+e^{\frac{-a}{\sigma}(x-\lambda)}\right]^2} dx = \left[\frac{1}{1+e^{-\frac{a}{\sigma}(x-\lambda)}}\right]_t^{\infty}$$

$$= \left[1-\frac{1}{1+e^{\frac{-a}{\sigma}(t-\lambda)}}\right] =$$

$$R(t) = \frac{e^{\frac{-a}{\sigma}(t-\lambda)}}{1+e^{\frac{-a}{\sigma}(t-\lambda)}}$$

See Fig. (1.4)

ii. Hazard function: $Z(t)$

$$Z(t) = \frac{f(t)}{R(t)} = \frac{\frac{a}{\sigma}e^{\frac{-a}{\sigma}(t-\lambda)}}{\left[1+e^{\frac{-a}{\sigma}(t-\lambda)}\right]^2} \Bigg/ \frac{e^{\frac{-a}{\sigma}(t-\lambda)}}{\left[1+e^{\frac{-a}{\sigma}(t-\lambda)}\right]}$$

$$= \frac{a/\sigma}{\left[1+e^{\frac{-a}{\sigma}(t-\lambda)}\right]} = \frac{a}{\sigma\left[1+e^{\frac{-a}{\sigma}(t-\lambda)}\right]}$$

See Fig. (1.5)

iii. Mean time to failure: denoted by $MTTF$

$$MTTF = \int_0^\infty R(t)dt = \int_0^\infty \frac{e^{\frac{-a}{\sigma}(x-\lambda)}}{1+e^{\frac{-a}{\sigma}(x-\lambda)}}dx$$

Integrate by substitution we will have

$$u = 1+e^{\frac{-a}{\sigma}(x-\lambda)}$$

$$du = \frac{-a}{\sigma}e^{\frac{-a}{\sigma}(x-\lambda)}dx$$

$$MTTF = \frac{-\sigma}{a}\int_0^{\infty}\frac{du}{u}$$

$$= \frac{-\sigma}{a}\ln u\Big]_0^{\infty} = \frac{-\sigma}{a}\ln\left|1+e^{\frac{-a}{\sigma}(x-\lambda)}\right|\Bigg]_0^{\infty}$$

Therefore:

$$MTTF = \frac{\sigma}{a}\ln\left(1+e^{\frac{a}{\sigma}\lambda}\right)$$

If λ is very large,

$$MTTF = \frac{\sigma}{a}\ln e^{\frac{a}{\sigma}\lambda} = \frac{\sigma}{a}\bullet\frac{a}{\sigma}\lambda = \lambda$$

See Fig. (1.6)

1.5 Static Reliability Models (Steady State):

1.5.1 Series Structure

The simplest and perhaps the most common structure in Reliability analysis is the series configuration.

In the series case the functional operation of the system depends on the proper operation of all system components. [7]

The reliability block diagram for series system is shown in Fig. 1.7.

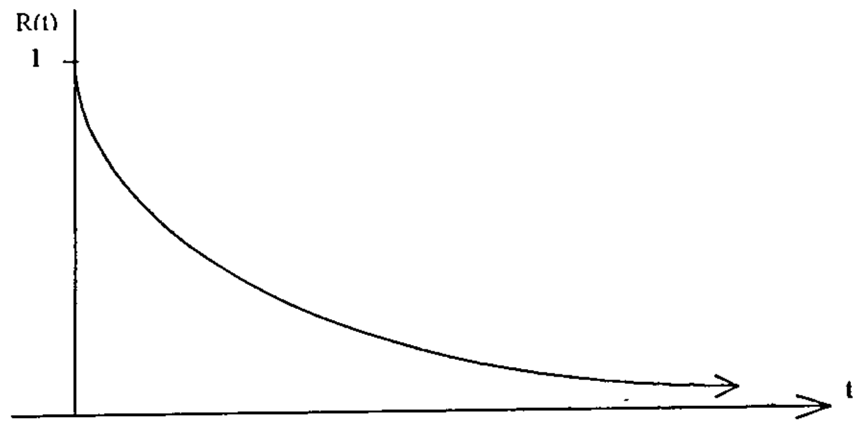

Fig. (1.4)
Reliability

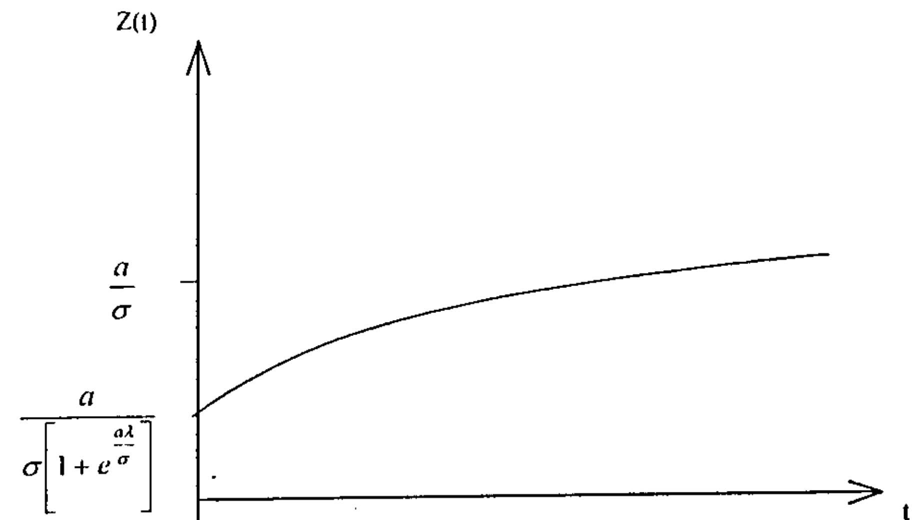

Fig. (1.5)
Hazard function

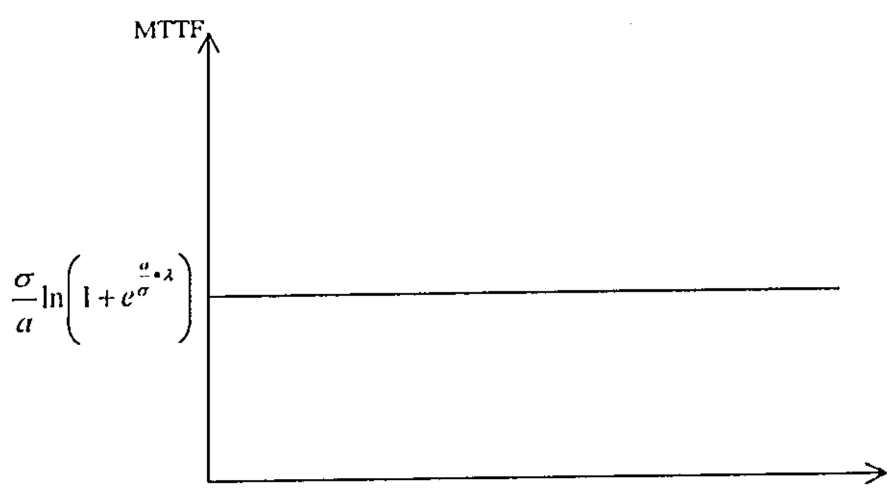

Fig. (1.6)
Mean time to failure

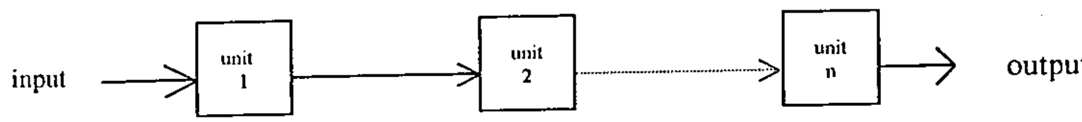

Fig. (1.7): Series Reliability Structure

The probability of the system success is denoted by Pss, and because they are independent events it becomes:

$$Pss = p(x_1, x_2, \dots x_n)$$

$$= p(x_1)p(x_2)\dots p(x_n)$$

The reliability of the system in series; denoted by Rss is given by

$$Rss = \prod_{i=1}^{n} R_i$$

Where R_i is called the reliability of component i at time t for i=1,2,…,n.

Assume that all components have the same reliability; R, then the reliability of the system is:

$$Rss = R^n$$

1.5.2 Parallel – Systems :

In many systems several signal paths perform the same operation at the same time. If the system structure is operated such that one or more paths will allow the remaining path or paths to perform properly, the system can be represented by a parallel model. The reliability block diagram for a parallel system is shown in Fig. (1.8).

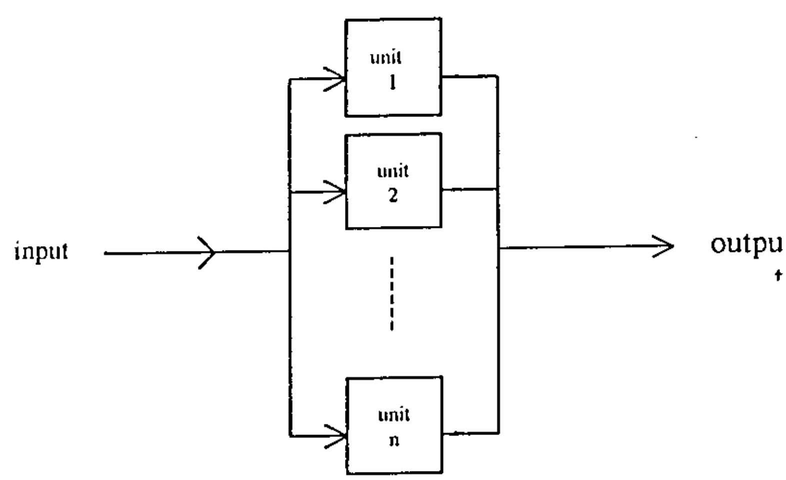

Fig. (1.8) Parallel reliability structure

The reliability of the system in parallel is denoted by *Rsp*, and the equivalent parallel component reliability is:

$$Rsp = 1 - \prod_{i=1}^{n}(1-R_i)$$

where

R_i is called the reliability of component i at time t for i= 1,2, ..., n.

Assume that all components have identical reliabilities, R, then the equivalent parallel component reliabilities are:

$$Rsp = 1 - (1 - R)^n$$

1.5.3 Parallel and Series Combination:

In this section we can either provide redundant components [10], which gives a system block diagram as show in Fig. (1.9). The equivalent reliability for a parallel banks is the reliability of the jth component, R_j is:

$$R_j = 1 - \prod_{i=1}^{m}(1 - R_i), \qquad j = 1,2, \ldots ,n$$

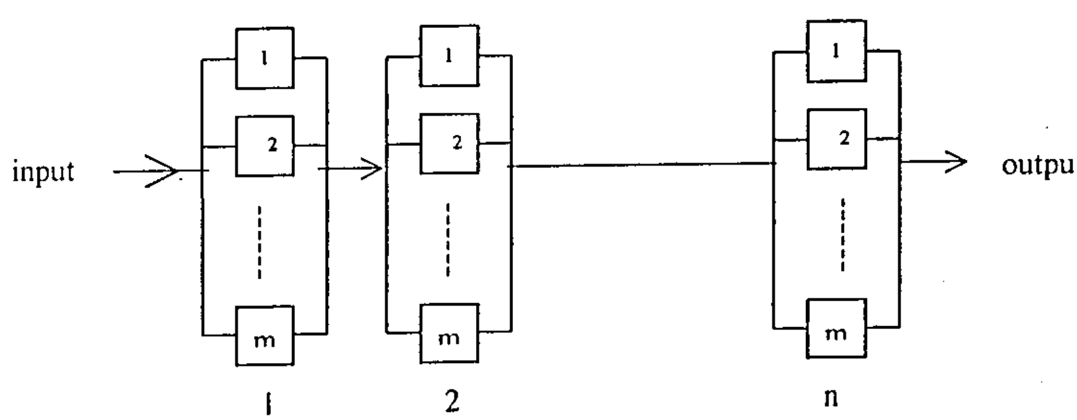

Fig. (1.9) Series-Parallel System

Then the equivalent of the reliabilities of the components are in series, are

$$Rs\text{-}p = \prod_{j=1}^{n} R_j$$

Assume that all units have the same reliability $\dot{R}$ then the equivalent reliability of the system is:

$$Rs\text{-}p = (1-(1-R)^m)^n$$

A second arrangement is shown in Figure (1.10), where the banks of series system are parallel.

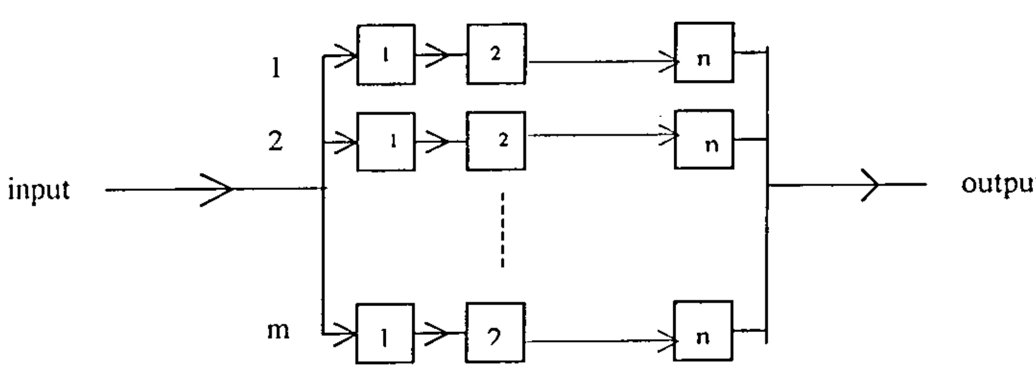

Fig. (1.10) Parallel- Series System

The equivalent reliability for each parallel path is:

$$R_j = \prod_{i=1}^{n} R_i \qquad j = 1,2, \ldots ,n$$

Thus, the system reliability is:

$$Rp\text{-}s = 1 - \prod_{j=1}^{m}\left(1 - R_j\right)$$

Assume that all components have the same reliability, R, then the reliability of the system is:

$$Rp\text{-}s = (1 - (1 - R)^n)^m$$

1.5.4 Complex Systems:

Certain design configurations of complex failure modes may produce systems in which pure parallel or series configurations are not appropriate [15]. As an example consider the system shown in Fig. (1.11). In this system the failure of the subsystem E drops out both E, D and E, C paths. We don't have a pure parallel arrangement. Several ways to handle such situation have been proposed. The extensive calculations required by this method can be simplified by the use of computers to evaluate all combinations.

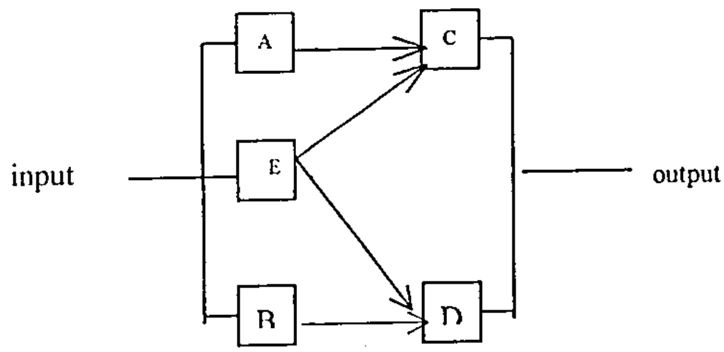

Fig. (1.11) Complex System Structure

1.6 Methods to Calculate Reliability:

1.6.1. Heuristic Method:

Many techniques have been applied to obtain the solution of optimization problems, however, several heuristic approaches are very attractive for solving the redundancy allocation problems.

In this section, the Sharma and Venkateswaran approach [15] developed an intuitive procedure for allocating redundancy among subsystem. The sequential steps involved in solving the problem are as follows:

Step 1: Assign $x_j = 1$ for $j = 1,2, \ldots, N$, because this is (casada system) there must be at least one component in each stage and the system should not violate any constraints.

2.3 Design of a system for maximum reliability:

Reliability of a component or of a system of components is simply its probability of survival. Systems are classified as either series or parallel. A series system is one that survives only if all its components survive, a parallel system survives if any one of its components survives.

With constraints, say on cost or configuration, then is a limit to the number of system that could be included in parallel. Thus in practice the attainable reliability has an upper bound. The next examples show how to use dynamic programming to design a constrained system for maximum reliability.

Examples 2.3.1

We want to design a device consisting of three main components arranged in series. Reliability may be improved by putting them paralleled on each component. Each component should include no more than three units in parallel. The total money allowed for this device is $11,000.

The data for reliability R_i and cost C_i for a unit on the ith component are shown in table 2.1. Costs are given is dollars and time is not a factor, reliabilities of units remain essentially constant.

Let m_i, i=1,2,3 denote the number of units i, to be put in parallel on the ith component. We determine m_1, m_2, m_3 so that total reliability of the system is maximized without exceeding the total money available.

Table (2.1): Reliability and cost per unit

Unit 1		Unit 2		Unit 3	
R_1	C_1	R_2	C_2	R_3	C_3
0.5	2	0.7	3	0.6	1

A present system is shown in Figure 2.1. Here we choose one unit in component one, two units in component two, and three units in component three.

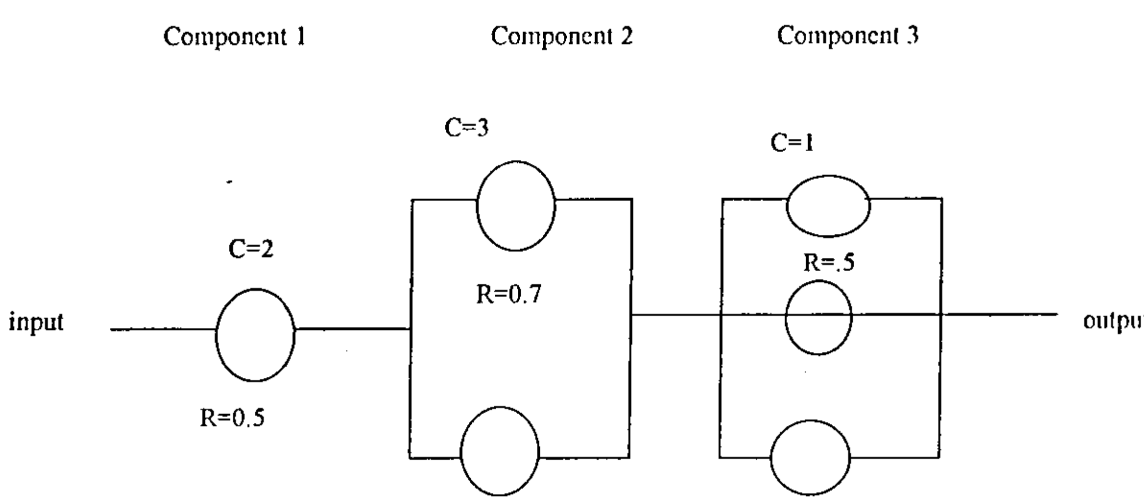

Figure (2.1) A non optimal, $ 11,000 system

The reliability of the system is:

$$R_s = [1-(1-R_1)]\cdot[1-\underset{i=1}{\overset{2}{\pi}}(1-R_{i,2})]\cdot[1-\underset{i=1}{\overset{3}{\pi}}(1-R_{i,3})]$$

$$= [1 - 0.5] . [1 - (1\text{-}0.7)(1\text{-}0.7) [1\text{-}(1\text{-}0.6)(1\text{-}0.6)(1\text{-}0.6)]$$

$$= 0.5 \times 0.91 \times 0.936 = 0.42588.$$

The cost of this system is $ 11,000 and three units are the maximum to be included in any components.

So the constraints are achieved. We can change the number of units in each component to m_1=2, m_2=1 and m_3=2 we reduce the cost to $9,000, but we yield a reliability 0.441. We seek, the optimal such system subject to the constraints.

Costs are additive convenient for references in Table 2.2. It shows the reliability and cost of component consisting of m_i units. Here $R_i m_i$ is the reliability of component consisting of m_i units, and $C_i m_i$ unit is the corresponding cost of the component. The results in table 2.3 come from preliminary calculation of reliability and the cost of all possible parallel systems could be components in the final system.

Let $\bar{R}_{imi}$ (C_{imi}) denote the reliability of the i th component as a function of a cost C_{imi}.

Let x_i be capital allocated to all i components and let $f_i(x_i)$ be the reliability of the system of i components. Then the states used in dynamic programming are represented by x_i, while the components are the stages in the problem. Thus $f_1(x_1) = \max_{0 \le C_1 m_1 \le x_1} [R_1 m_1 (C_1 m_1)]$ and

$f_i(x_i) = \max [R_i m_i (C_i m_i) f_{i-1}(x_i - C_i m_i)]\ i = 2,3$ if 7 is allocated to two components one and two, then $C_2 m_2$ is the cost for component two, and the cost for component one is $7 - C_2 m_2$.

Thus $f_2(7)$ is the maximum two-component system reliability. These calculations appear in Tables 2.3, 2.4, and 2.5.

From table 2.5 the optimal reliability is 0.5145, occurring with $m_3 = 2$, this leaves 11 - 2 = 9 for the first and second component.

Table (2.2) Reliability and cost of Parallel Components

	i = 1		i = 2		i = 3	
m_i	R_{1m1}	C_{1m1}	R_{2m2}	C_{2m2}	R_{3m3}	C_{3m3}
1	0.5	2	0.7	3	0.6	1
2	0.75	4	0.91	6	0.84	2
3	0.875	6	0.973	9	0.936	3

Table 2.4 shows that the optimal m_2 is 1, the cost of the unit in component two is 3, this leaves 9-3 = 6 unit of capital for component one, and from table 2.3 the optimal m_1 is 3 units. The result optimal system is shown in Figure 2.2.

Table (2.3) One-Component System

x_1	$m_1 = 1$		$m_1 = 2$		$m_1 = 3$		m_1^*	$f_1^*(x_1)$
	$R_{1m1}=0.5$	$C_{1m1}=2$	$R_{1m1}=0.75$	$C_{1m1}=4$	$R_{1m1}=0.875$	$C_{1m1}=6$		
0	-		-		-		-	-
1	-		-		-		-	-
2	0.5		-		-		1	0.5
3	0.5		-		-		1	0.5
4	0.5		0.75		-		2	0.75
5	0.5		0.75		-		2	0.75
6	0.5		0.75		0.875		3	0.875
7	0.5		0.75		0.875		3	0.875
8	0.5		0.75		0.875		3	0.875
9	0.5		0.75		0.875		3	0.875
10	0.5		0.75		0.875		3	0.875
11	0.5		0.75		0.875		3	0.875

Table (2.4) Two-Component System

x_1	$m_2=1$		$m_2=2$		$m_2=3$		m_2^*	$f_2^*(x_2)$
	$R_{2m2}=0.7$	$C_{2m2}=3$	$R_{2m2}=0.91$	$C_{1m1}=6$	$R_{1m1}=0.973$	$C_{1m1}=9$		
0	-		-		-		-	-
1	-		-		-		-	-
2	-		-		-		-	
3	-		-		-			
4	-		-		-		-	-
5	(0.7)(0.5)=0.35		-		-		1	0.35
6	(0.7)(0.5)=0.35		-		-		1	0.35
7	(0.7)(0.75)=0.525		-		-		1	0.525
8	(0.7)(0.75)=0.525		(0.91)(0.5)=0.455		-		1	0.525
9	(0.7)(0.875)=0.6125		(0.91)(0.5)=0.455		-		1	0.6125
10	(0.7)(0.875)=0.6125		(0.91)(0.75)=0.6825		-		2	0.6825
11	(0.7)(0.875)=0.6125		(0.91)(0.75)=0.6825		(0.973)(0.5)=0.4865		2	0.6825

Table (2.5) Three-Component System

x_1	$m_3=1$		$m_3=2$		$m_3=3$		m_3^*	$f_3^*(x_3)$
	$R_{3m3}=0.6$	$C_{3m3}=1$	$R_{3m3}=0.84$	$C_{3m3}=2$	$R_{3m3}=0.936$	$C_{3m3}=3$		
0	-		-		-		-	-
1	-		-		-		-	-
2	-		-		-		-	
3	-		-		-			
4	-		-		-		-	-
5	-		-		-		-	-
6	(0.6)(0.35)=0.21		-		-		1	0.21
7	(0.6)(0.35)=0.21		(0.84)(0.35)=0.294		-		2	0.29
8	(0.6)(0.525)=0.315		(0.84)(0.35)=0.294		(0.936)(0.35)=0.3276		3	0.3276
9	(0.6)(0.525)=0.315		(0.84)(0.525)=0.441		(0.936)(0.35)=0.3276		2	0.441
10	(0.6)(0.6125)=0.3675		(0.84)(0.525)=0.441		(0.936)(0.525)=0.4914		3	0.4914
11	(0.6)(0.6825)=0.4095		(0.84)(0.6125)=0.5145		(0.936)(0.525)=0.4914		2	0.5145

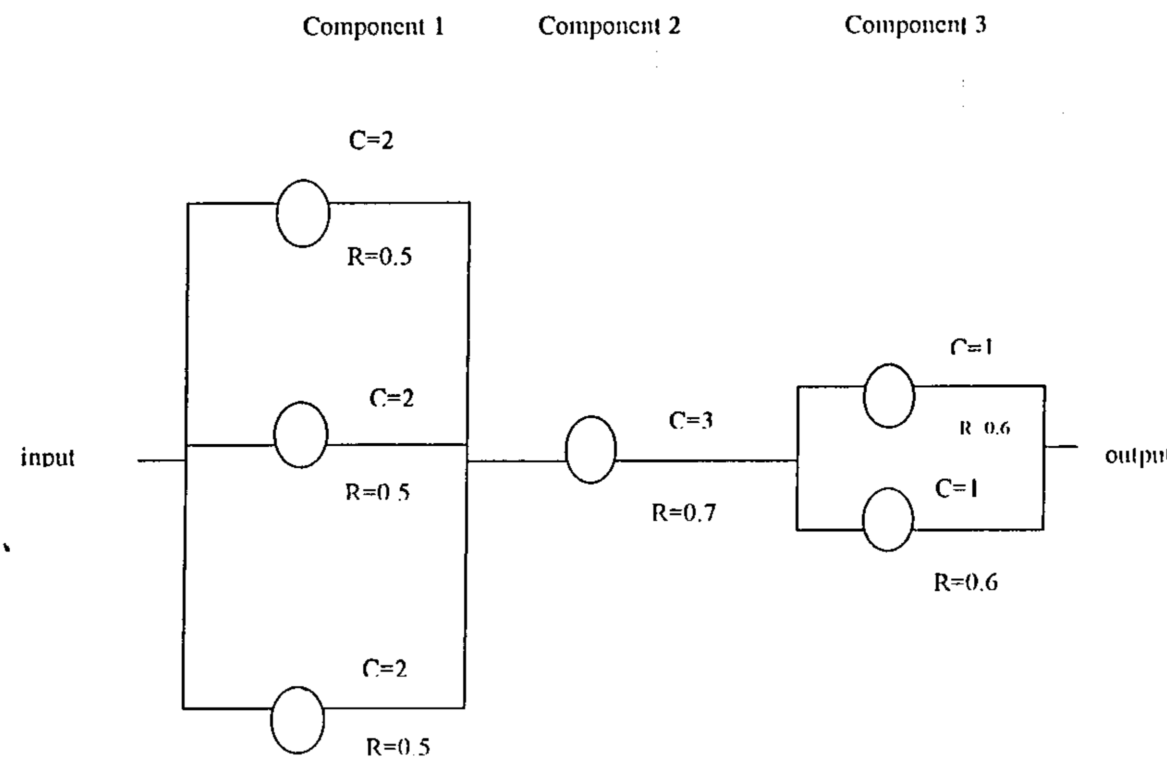

Fig.(2.2) The optimal solution for example 2.3.1

The optimal system attainable for 11,000 with maximum reliablity of 0.5145.

Example 2.3.2

We want to design a device consisting of three main components arranged in series. Reliability may be improved by installing parallel units on each component. Each component may include no more than four units in parallel. The total capital available for the device is $20,000. Data for reliability R_i and Cost C_i for a unit on the ith component are given in Table

2.6. Costs are in thousands of dollars, and time is not a factor reliabilities of units remain essentially constant.

Table (2.6): Reliability and cost per unit

Unit 1		Unit 2		Unit 3	
R_1	C_1	R_2	C_2	R_3	C_3
0.5	2	0.7	3	0.6	1

Let m_i denote the number of units i=1,2,3,4 to put in parallel on the ith component. We need to determine m_1,m_2, and m_3 so that total reliability of the system is maximized without exceeding the total available capital.

We are assuming that costs are additive, then table 2.7 becomes convenient for reference. It shows that reliabilities and costs of components consisting of m_i units. Here R_{imi} is the reliability each ith component consisting of m_i units, and C_{imi}, is the corresponding cost of each component. The result in table 2.7 come from preliminary calculations of reliability and cost of all possible parallel systems that could be the components in final system.

Table (2.7) Reliability and cost of Parallel Components

m_i	i = 1		i = 2		i = 3	
	R_{1m1}	C_{1m1}	R_{2m2}	C_{2m2}	R_{3m3}	C_{3m3}
1	0.5	2	0.7	3	0.6	1
2	0.75	4	0.91	6	0.84	2
3	0.875	6	0.973	9	0.936	3
4	0.9375	8	0.9919	12	0.99744	4

Let $R_{imi}(C_{imi})$ denote the reliability of the ith component as a function of the cost C_{imi} that is allocated to it. Let x_i be the capital allocated to all i components and let $f_i(x_i)$ be the reliability of the system of i components. Then the stage used in dynamic programming can be represented by x_i while the components are the stages in the problem.

Thus $f_1(x_1) = \max_{\substack{m1 \\ 0 \le C_{1m1} \le X_1}} [R_{1m1}(C_{1m1})]$

and

$f_i(x_i) = \max_{0 \le C_{im} \le x_i} [R_i m_i(C_{mii}) f_{i-1}(x_i - C_{mi})]$ i=2,3,4

The calculations appear in Tables 2.8, 2.9 and 2.10. From table 2.10 we see that the optimal reliability is 0.8537, occurring with $m_3 = 3$, This leaves 17 units of capital for the other three components.

Table 2.9 shows that the optimal m_2 is three, the cost of one unit in component 2 is 3, this leaves 8 units of capital for component 1.

Table (2.8) One-Component System

x_1	$m_1=1$ ($R_{1m_1}=0.5$, $C_{1m_1}=2$)	$m_1=2$ ($R1_{m_1}=0.75$, $C_{1m_1}=4$)	$m_1=3$ ($R_{1m_1}=0.875$, $C_{1m_1}=6$)	$m_1=4$ ($R_{1m_1}=09375$, $C_{1m_1}=8$)	m_1^*	$f_1^*(x_1)$
0	-	-	-	-	-	-
1	-	-	-	-	-	-
2	0.5	-	-	-	1	0.5
3	0.5	-	-	-	1	0.5
4	0.5	0.75	-	-	2	0.75
5	0.5	0.75	-	-	2	0.75
6	0.5	0.75	0.875	-	3	0.875
7	0.5	0.75	0.875	-	3	0.875
8	0.5	0.75	0.875	0.9375	4	0.9375
9	0.5	0.75	0.875	0.9375	4	0.9375
10	0.5	0.75	0.875	0.9375	4	0.9375
11	0.5	0.75	0.875	0.9375	4	0.9375
12	0.5	0.75	0.875	0.9375	4	0.9375
13	0.5	0.75	0.875	0.9375	4	0.9375
14	0.5	0.75	0.875	0.9375	4	0.9375
15	0.5	0.75	0.875	0.9375	4	0.9375
16	0.5	0.75	0.875	0.9375	4	0.9375
17	0.5	0.75	0.875	0.9375	4	0.9375
18	0.5	0.75	0.875	0.9375	4	0.9375
19	0.5	0.75	0.875	0.9375	4	0.9375
20	0.5	0.75	0.875	0.9375	4	0.9375

Table (2.9) Two-Component System

x_2	$m_2=1$ ($R_{2m2}=0.7$, $C_{1m21}=3$)	$m_2=2$ ($R_{2m2}=091$, $C_{1m1}=6$)	$m_2=3$ ($R_{2m2}=0973$, $C_{1m1}=9$)	$m_2=4$ ($R_{2m2}=.9919$, $C_{1m1}=12$)	m_2^*	$f_2^*(x_2)$
0	-	-	-	-	-	-
1	-	-	-	-	-	-
2	-	-	-	-	-	-
3	-	-	-	-	-	-
4	-	-	-	-	-	-
5	(0.5)().7)=0.35	-	-	-	1	0.35
6	(0.5)().7)=0.35	-	-	-	1	0.35
7	(0.75)(0.7)=.525	-	-	-	1	0.525
8	(0.75)(0.7)=.525	(091)(0.5)=0.455	-	-	1	0.525
9	(0.7)(0.875)=0.6125	(091)(0.5)=0.455	-	-	1	0.6125
10	(0.7)(0.875)=0.6125	(0.91)(0.75)=0.6825	-	-	2	0.6825
11	(0.7)(0.9375)=.6562	(0.91)(0.75)=0.6865	(0.973)(0.5)=.4865	-	2	0.6825
12	(0.7)(0.9375)=.6562	(0.91)(0.875)=0.79625	(0.973)(0.5)=.4865	-	2	0.7965
13	(0.7)(0.9375)=.6562	(0.91)(0.875)=0.79625	(0.973)(0.75)=.7297	-	2	0.79625
14	(0.7)(0.9375)=.6562	(0.91)(.9375)=.8531	(0.973)(0.75)=.7297	(0.5)(.9919)=.4959	2	0.8531
15	(0.7)(0.9375)=.6562	(0.91)(.9375)=.8531	(0.875)(.973)=.8513	(0.5)(.9919)=.4959	2	0.8531
16	(0.7)(0.9375)=.6562	(0.91)(.9375)=.8531	(0.875)(.973)=.8513	(0.75)(.9919)=.7439	2	0.8531
17	(0.7)(0.9375)=.6562	(0.91)(.9375)=.8531	(.973)(.9375)=0.9121	(0.75)(.9919)=.7439	3	09121
18	(0.7)(0.9375)=.6562	(0.91)(.9375)=.8531	(.973)(.9375)=0.9121	(0.875)(.9919)=.8679	3	.9121
19	(0.7)(0.9375)=.6562	(0.91)(.9375)=.8531	(.973)(.9375)=0.9121	(0.875)(.9919)=.8679	3	.9121
20	(0.7)(0.9375)=.6562	(0.91)(.9375)=.8531	(.973)(.9375)=0.9121	(0.875)(.9919)=.8679	3	.9121

Table (2.10) Three-Component System

x_3	$m_3=1$ ($R_{3m3}=0.6$, $C_{3m3}=1$)	$m_3=2$ ($R_{3m3}=084$, $C_{3m3}=2$)	$m_3=3$ ($R_{3m3}=.936$, $C_{3m3}=3$)	$m_3=4$ ($R_{3m3}=09744$, $C_{3m3}=4$)	m_3^*	$f_3^*(x_3)$
0	-	-	-	-	-	-
1	-	-	-	-	-	-
2	-	-	-	-	-	-
3	-	-	-	-	-	-
4	-	-	-	-	-	-
5	-	-	-	-	-	-
6	(0.6)(.35)=0.21	-	-	-	1	0.21
7	(0.6)(.35)=0.21	(0.84)(.35)=.294	-	-	2	.294
8	(0.6)(.525)=.315	(0.84)(.35)=.294	(.936)(0.35)=.3276	-	3	.3276
9	(0.6)(.525)=.315	(0.84)(.525)=.378	(.936)(0.35)=.3276	(.9744)(0.35)=.3410	2	0378
10	(0.6)(.6125)=.3675	(0.84)(.525)=.378	(.936)(.525)=.4914	(.9744)(0.35)=.3410	3	.4914
11	(0.6)(.6125)=.3675	(0.84)(.525)=.378	(.936)(.525)=.4914	(.9744)(.525)=.5115	4	.5115
12	(0.6)(.9375)=.5625	(0.84)(.525)=.378	(.936)(.6125)=.5733	(.9744)(.525)=.5115	4	.5733
13	(0.6)(.9375)=.5625	(0.84)(.525)=.378	(.936)(.6825)=.6338	(.9744)(.6125)=.5968	3	.6338
14	(0.6)(.9375)=.5625	(0.84)(.525)=.378	(.936)(.6825)=.6338	(.9744)(.6825)=.6650	2	.6885
15	(0.6)(.9375)=.5625	(0.84)(.525)=.378	(.936)(.796)=.7452	(.9744)(.6825)=.6650	3	.7452
16	(0.6)(.9375)=.5625	(0.84)(.525)=.378	(.936)(.796)=.7452	(.9744)(.79625)=.7758	4	.7758
17	(0.6)(.9375)=.5625	(0.84)(.525)=.378	(.936)(.8531)=.7985	(.9744)(.79625)=.7758	3	.7985
18	(0.6)(.9375)=.5625	(0.84)(.525)=.378	(.936)(.8531)=.7985	(.9744)(.8531)=.8312	4	.8312
19	(0.6)(.9375)=.5625	(0.84)(.525)=.378	(.936)(.8531)=.7985	(.9744)(.8531)=.8312	4	.8312
20	(0.6)(.9375)=.5625	(0.84)(.525)=.378	(.936)(.9121)=.8537	(.9744)(.8531)=.8312	3	.8537

Table 2.8 shows that the optimal m_1 is 4. The resulting optimal system is shown in Figure 2.2.

The solution suggested by Figure 2.3 is the optimal system attainable for $20,000, with maximum reliability of 0.8537.

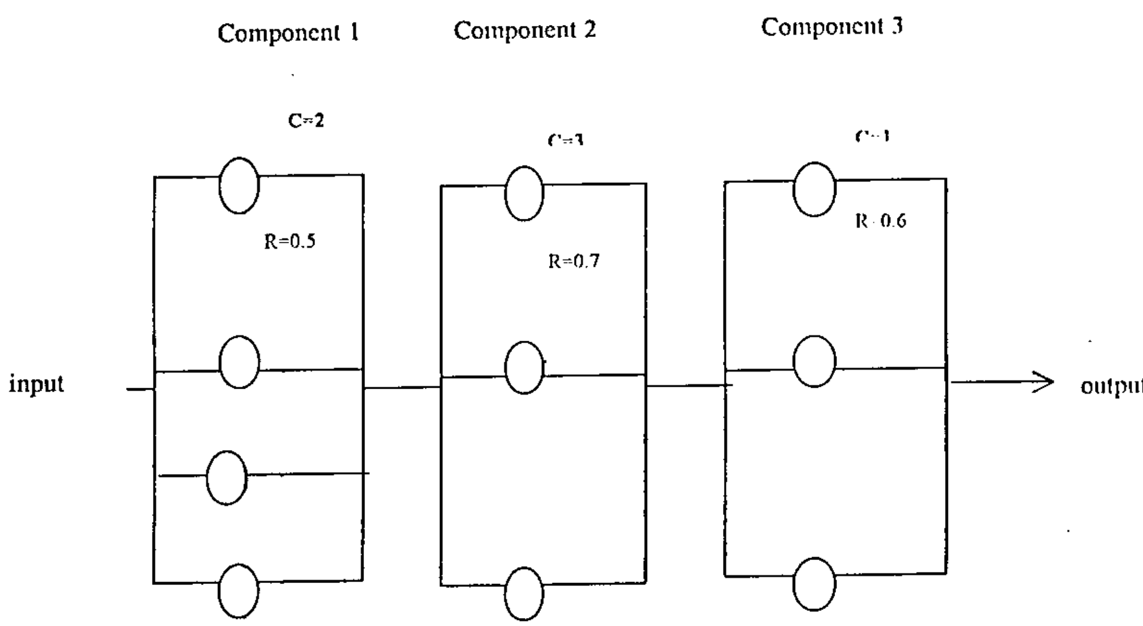

Figure (2.3) The optimal solution for example 2.3.2

2.4 A heuristic Approach: Sharma and venkateswarn's approach

In addition to the general assumptions made for the system reliability optimization problems of N-stage in series with x_j redundant components at stage j, the unreliability of one component at the jth stage, Q_j,j=1,2,3, …,N should be small enough (≤ 0.5) so that

$$Q_s = 1 - \prod_{j=1}^{N} (1 - Q_j^{x_j})$$

can be approximated by

$$Q_s = \sum_{j=1}^{N} Q_j^{x_j}$$

where Q_s is the system unreliability. Therefore the system reliability problem subject to non-linear cost constraints can be formulated as minimize

$$Q_s = \sum_{j=1}^{N} Q_j^{x_j} \qquad \ldots(9)$$

subject to

$$\sum_{j=1}^{N} g_{ij}(x_j) \le b_i \qquad i = 1,2,\ldots,4 \qquad \ldots(10)$$

where $g_{ij}(x_j)$ is the recourse i consumed in stage j, and b_i is the available resource for constraint i.

The objective is to reduce Q_s in successive steps. The procedure at each step is to add one redundant component to the stage with the highest $Q_j^{x_j}$ in eq. (9), if constraints in eq.(10) are not violated. Therefore, the constraints become active only in the neighborhood of the boundary of the feasible region. The sequential steps involved in solving the problems are as follows:

Step 1. Assign $x_j = 1$ for $j=1,2, \ldots ,N$. Because this is a cascade system, there must be at least one component in each stage and the system should not violate any constraints.

Step 2. Find the stage which is the most unreliable. Add a redundant component to that stage.

Step 3. Check the constraints:

a. If any constraint is violated, go to step 4.

b. If no constraint has been violated, go to step 2.

c. If constraint is exactly satisfied, stop. The current x_j's are then the optimum configuration of the system.

Step 4. Remove the redundant component added in step 2. The resulting number is the optimum allocation for that stage. Remove this stage from further consideration.

Step 5. If all stages have been removed from consideration, the current x_j's are the optimum configuration of the system otherwise go to step 2.

The general algorithm can be found in Appendix. As an application of this algorithm we give the following two examples.

Solution of example 2.3.1 by using Sharma and Venkateswarn,s Methods.

we are to design a device of three main components arranged in series. Reliability may be improved by installing parallel units on each component.

Each component may include no more than three units in parallel. The total capital available for the device is \$11,000 Data for the reliability R_i and a cost C_i for a unit in the component is given in Table 2.1. Costs are in thousands of dollars, time is not a factor, reliabilities remain essentially constant.

We need to maximize the reliability in the total capital available is \$11,000.

Minimize

$$Q_s = \sum_{j=1}^{3} Q_j^{x_j}$$

Subject to

$$\sum_{j=1}^{3} C_j X_j \le b$$

where Q_s is the system unreliability

C_j is the cost for a unit on the j th component

X_j is number of units in component j.

$$Q_j^{x} = (1 - R_j)^{xj}$$

where R_j denote the reliability of j component.

Q_j denote the unreliability of j component.

A three-stage problem with a linear constraints. The objective of the problem is to maximize $R_s = \prod_{j=1}^{3}[1-(1-R_j)^{x_j}]$ can be approximated by minimize $Qs = (1-R_1)^{x_1} + (1-R_2)^{x_2} + (1-R_3)^{x_3}$

where $(1-R_1)^{x_1}$, $(1-R_2)^{x_2}$ and $(1-R_3)^{x_3}$ are stage unreliability and are represented by Q_1`, $Q_2$` and Q_3` respectively.

The basic allocation (1,1,1) is assigned to the system. The stage unreliabilities under this configuration are (0.5, 0.3, 0.4) the recourse consumed is 11, and no constraint violated. Since stage one is the most unreliability i.e. $Q_1 = 0.5$. We add one redundancy to this stage to form the new system configuration (2,1,1). If we look at Table 2.11 we see that stage 3 is the highest unreliability. We add one redundancy to this stage to form a new system configuration (2,1,2), and the consume resource is 9. Since Q_2` is the highest unreliability if we add one redundancy to this stage constraint is violate. We removed from $Q_2$` to Q_1`. $Q_1$` is the highest unreliability. So a redundancy component may be added to stage 1 to form a new system configuration (3,1,2) [step 2] we obtain the result presented in Table 2.11 The optimum result for the system is (3,1,2) with the system reliability is 0.5145.

Table (2.11) Results of Example 2.3.1 by Sharma and Venkateswarn's Method

Number of components in stage			Stage unreliability			Constraint
x_1	x_2	x_3	Q'_1	Q'_2	Q'_3	g(x)
1	1	1	0.5^a	0.3	0.4	6
2	1	1	0.25	0.3	0.4^a	8
2	1	2	0.25	0.3^b	0.16	9
2	2	2	-	-	-	12^c
2	1	2	0.25^a	0.3^b	0.16	9
3	1	2	0.0625	0.3^b	0.16	11

a This is the stage to which a redundant components is to be added

b This indicates that the stage has been removed from further consideration.

c The constraint is violated.

Solution of example 2.3.2 by using Sharma and Venkateswarn's methods.

We want to design a device consisting of three main components arranged in series. Reliability may be improved by putting them in paralleled on each component should include no more than four units in parallel. The total capital available for the device is $20,000. Data for reliability R_i and cost C_i for a unit in the i th component are given in Table

2.6. Cost are in thousand dollar, and time is not a factor and reliabilities are constant.

We use Sharma and Venkateswarn's method to maximize the reliability in the total capital available is \$20,000.

Maximize $R_s = \prod_{j=1}^{3}[1-(1-Rj)^{x_j}]$

Minimize $Q_s = \sum_{i=1}^{3} Q_j^{x_j}$

subjected to

$g = \sum_{j=1}^{3} C_j X_j \leq b$

where Q_s is the system unreliability

C_j is the cost for a unit on the j th component

X_j is number of units in j th component.

$Q_j' = (1-R_j)^{xj}$

where Rj denote the reliability of j component

Q_j' denote the unreliability of j component

The basic allocation (1,1,1) is again assigned at each stage for this system. The stage unreliabilities under this configuration are (0.5,0.3,0.4). The resource consume is 20, costs units and no constraint violated. Since stage 1 has the highest unreliability. i.e. Q_1=0.5, we add one redundancy to this stage to form a new system configuration (2,1,1), and check the resource limits. Using the algorithm we obtain the results which are summarized in Table 2.12. The last row of table 2.12 shows the resources 20 units are consumed with the allocation of redundancies (4,3,3). The system of unreliability after calculating unreliability is 0.1462. Hence the system reliability is 0.8537.

Therefore the optimum configuration for the system is (4,3,3).

Table (2.12) Results of Example 2.3.2 by Sharma and Venkateswarn's Method

Number of components in stage			Stage unreliability			Constraint
x_1	x_2	x_3	Q'_1	Q'_2	Q'_3	g(x)
1	1	1	0.5[a]	0.3	0.4	6
2	1	1	0.25	0.3	0.4[a]	8
2	1	2	0.25	0.3[a]	0.16	9
2	2	2	0.25[a]	0.09	0.16	12
3	2	2	0.0625	0.09	0.16[a]	14
3	2	3	0.0625	0.09[a]	0.0256	15
3	3	3	0.0625[a]	0.0081	0.0256	18
4	3	3	0.0036	0.0081	0.0256	20

a This is the stage to which a redundant components is to be added

b This indicates that the stage has been removed from further consideration.

c The constraint is violated.

Chapter (3)

A service Delivery System with Costs Constraint

3.1 Introduction:

During the eighties the nature of industry was considered monopolistic. But during the nineties the competition policy was followed. Close scruting of scale and scope economics within postal services together with other industries vested with statutory monopoly, revealed many existing operational components within these industries. They do not exhibit sufficient economics to merit monopoly status.

If competitive entry is allowed or not, this is connected with many factors related to the nature of monopoly. Inefficient entry can destroy the industry, and no one will courage it.

The traditional pricing policies have been investigated in detail, while the explicit impact upon optimal policies was not examined as well.

The impact of reliability is investigated by Borinico (1992). Where:

i. the reliability constraint is appended to the standard formulation.

ii. reliability of service both costs and demand for service.

The general form for reliability constraint is

$E\{H_i(X(p,r,z),y,r,z) \leq 0$ i = 1,2, ..., n [1]

where

$E\{H_i\}$ represents the required relationship between the

i. reliability level vector (r).

ii. aggregate demand vector (X).

where p is the price vector as given

iii. y is the local operating variables.

iv z represents a state of a world governed by some common knowledge distribution.

In solving the constrained welfare maximization problem, the following general results obtain:

price: First best prices involve equaling price to reliability constrained marginal cost. The following result for reliability constrained marginal cost for service i obtains.

$$MC_i = \left\{ \frac{\partial C}{\partial X_i} + \sum_{j=1}^{n} \frac{\lambda_j \partial H_j}{\partial X_i} \right\} \qquad \ldots(1)$$

where:

C: represents the expected cost function.

λ_j: represents the lagrange multiplier associated with the j th reliability constraint

Reliability: The expected marginal benefits of increasing reliability should be equated to marginal cost of doing so. [1]

Capacity: Optimal capacities are chosen so as to minimize total expected cost subject to the set of reliability constraints. [1]

Minimize:

$C(X,(p,r,z),y,r)$

subject to : $E_z\{Hi(X(p,r,z),y,r,z) \leq o$ $\quad i = 1,2,\ldots,n$

The purpose of this chapter is to characterize marginal costs and minimum cost capacity plans for a typical service delivery system.

3.2 Mathematical Structure: [1]

In this section we discuss well known example about postal services. This postal system which is provided by one service where the price for that service (p), and the amount of regular capacity *(Q)* must be decided upon prior to witnessing demand. The number of arrival *(X)* which happened at the beginning of the period are covered by cumulative distribution function $F(k) \leq p_r(X \leq k)$.

In this system there are two kinds of units processed ($b|unit) and unprocessed. The unprocessed units must utilize over time or other form of penalty technology ($ B|unit).

A reliability constrain for this time sensitive is assumed to take the following forms.

The expected quantity of arrivals requiring penalty technology must be less than or equal to a prespecified percentage (1-r) of the expected volume of the arrivals of this period.

Small postal facilities employ particular types of technology in processing different forms of preferential mail such as over night express mail.

In general mail is received at the beginning of the tour. For the over night express mail to reach its destination on the time it must be processed in a critical window in the tour in which it arrives if not it must be expedited at additional cost.

For the remainder of this chapter, we give the following definitions apply:

Q: Installed regular capacity, set in advance.

X: Demand for service in a prescribed period, governed by cdf $F(k)=p_r(X\leq k)$.

$c =$ Total cost associated with aparticular realization of demand, $c=c(Q,X)$.

$C =$ Total expected cost function, $C = E\{c(Q,X)\}$,

$b =$ Average cost of processing one unit utilizing regular capacity.

$B =$ Average per unit penalty technology processing cost, $B=b(1+\rho)$.

$\rho =$ over time (or penalty technology) premium, the mark up over normal costs to process one unit not processed utilizing regular capacity

$r =$ the reliability of the system, is given.

We define the following function which is important to the determine the objective function

$bQ =$ represent the direct, cost per unit of regular capacity, scheduled prior to witnessing demand X.

$B(X-Q)^+ =$ represents deferred service and the penalty cost for units which must be processed using penalty technology.

The objective function is given by:

$C(Q,X) = E\{c(Q,X)\} = E\{bQ + B(X - Q)^+\}$.

The following reliability constraint ensures adequate capacity to meet aprespecified percentage of expected demand:

$H(Q,X) = E\{(X-Q)^+ - (1-r)X\} \leq 0$

we can write it as:

$E\{(X-Q)^+ \leq (1-r)E(X) = (1-r)\mu_x$.

The cost minimization formula is given by minimize

$C(Q,X) = E\{hQ+B(X-Q)^{+}\}$ ...(2)

subject to:

$E\{(X-Q)\}^{+} \leq (1-r)E(X) = (1-r)\,\mu_x$ ...(3)

when we solve this model we conclude $C(Q,X)$, and $H(Q,X)$ are convex. The problem Eqs. (2) and (3) has the pleasant structure of minimizing a convex objective function $C(Q,X)$ over a convex region. And we note that:

1. $C(Q,X)$ is un bounded from above.
2. $C(Q,X)$ is bounded from below and $C(Q,X) \geq 0$.
3. $C(Q,X)$ approaches infinity as Q approaches infinity.

We define the lagrangian:

$L(Q,X,\lambda) = C(Q,X) + \lambda H(Q,X)$

Where

$\lambda =$ represent the Lagrange multiplier for $H(Q,X)$

The first order conditions of the lagrangian given by:

$$\frac{\partial L}{\partial Q}(Q,X) = \frac{\partial C(Q,X)}{\partial Q} + \lambda \frac{\partial H(Q,X)}{\partial Q} = 0 \quad \text{...(4)}$$

$$H(Q,X) \leq 0 \quad \text{...(5)}$$

$$\lambda H(Q,X) = 0$$

$$\lambda \geq 0$$

If Eq.(3) is inactive at Q^*, then λ will be zero.

If Eq.(3) is active then λ will be positive we note that H(Q,X) is non increasing in Q which implies that $\frac{\partial H}{\partial Q_i} \le 0$.

If we use the fact $\lambda \ge 0$ and Eq.(4) this implies $\frac{\partial H}{\partial Q_i} \le 0$ at optimum.

If we apply Eq.(4) to Eq.(2) and (3) we obtained

$$\frac{\partial L(Q,X)}{\partial Q} = \frac{\partial C}{\partial Q} E\{bQ + B(X-Q)^+\} + \lambda \frac{\partial H}{\partial Q}$$

$$\frac{\partial C}{\partial Q} = b + B\frac{\partial}{\partial Q}E(X-Q)^+ \qquad \ldots(6)$$

$$\frac{\partial H}{\partial Q} = \frac{\partial}{\partial Q}E(X-Q)^+ - \frac{\partial}{\partial Q}(1-r)E(X)$$

$$= \frac{\partial}{\partial Q}E(X-Q)^+ - O = \frac{\partial}{\partial Q}E(X-Q)^+ \qquad \ldots(7)$$

If we substitute eq.(6) and eq.(7) in eq.(4) we get

$$\frac{\partial L}{\partial Q} = b + B\frac{\partial}{\partial Q}E(X-Q)^+ + \lambda\frac{\partial}{\partial Q}E(X-Q)^+$$

if we choose $\lambda = 0$

$$\frac{\partial L}{\partial Q} = b - T(B+\lambda) = 0 \qquad \ldots(8)$$

where

$$T = \frac{-\partial}{\partial Q}E(X-Q)^+$$

$$T = \Pr(X \ge Q) \qquad \ldots(9)$$

We find the optimal capacity for problem Eqs.(2) and (3) is then determined by solving Eqs.(5) and (8).

In order to solve for optimal price, reliability constrained marginal cost must be determined. Utilizing Eq. (1), together with first partials from Eqs. (2) and (3) we obtain

$$MC = T(B + \lambda) - (1-r)\,\lambda \qquad \ldots(10)$$

We can apply the result to an illustrative example in the following section.

3.3 Implementation of the Model:

In this section we consider two particular demand distributions, the uniform and the logistic distribution, both distributions are bounded on [l,u]. First we study the uniform distribution.

$T = \Pr(X \geq Q)$

We substitute the uniform distribution in eq.(9) we get:

$$T = P(X \geq Q) = \int_Q^u \frac{dx}{u-l} = \left.\frac{x}{u-l}\right]_Q^u = \frac{u-Q}{u-l} \qquad \ldots(11)$$

then we substitute eq. (11) into eq. (8) we get the unconstrained solution:

$$\frac{\partial L(Q,X)}{\partial Q} = b - T(B+\lambda) = 0$$

$$b = T(B+\lambda) = \frac{u-Q}{u-l}(B+\lambda)$$

Suppose ($\lambda = 0$) we get

$b(u\text{-}l) = B(u\text{-}Q)$

$$Q^{*} = u - \frac{b}{B}(u - l) \qquad \ldots(12)$$

Q^{*}:= represents optimal capacity for the unconstrained problem Eq. (2)

if $b \rightarrow B$, then $Q \rightarrow l$

we conclude that the capacity should approach the minimum demand level as over time costs approach regular time cost

Also, we conclude from eq. (12), the optimal capacity always lies between l and u as long as b < B.

In other words, the optimal capacity must be lies in feasible range of demand values.

We substitute eq. (12) into eq.(3) and solving as equality yields the implied reliability obtained when the unconstrained solution applies

$$r_1 = 1 - \frac{b^2(u - l)}{B^2(u + l)}$$

Note that increases in B relative to b increase the implied reliability, as increased overtime cost would require greater capacity, which would the increase system's reliability, r_1.

Solving eq.(3) as an equality, noting that the optimal capacity must be non-negative, the following we obtain the following:

$$r = 1 - \frac{b^2(u-l)}{B^2(u+l)} \qquad \ldots(13)$$

$$\frac{b^2}{B^2} = (1-r)\frac{(u+l)}{(u-l)}$$

$$\frac{b}{B} = \sqrt{(1-r)\frac{(u+l)}{(u-l)}} \qquad \ldots(14)$$

substitute eq.(14) in eq. (12) we get

$$Q^* = u - \sqrt{(1-r)\frac{(u+l)}{(u-l)}}(u-l)$$

$$Q^* = u - \sqrt{(1-r)(u+l)(u-l)} \qquad \ldots(15)$$

It remains to determine reliability constrained marginal cost, in which optimal prices are embodied.

We substitute of eq.(11) and eq.(12) into eq.(10) eventually yields the following result:

$$MC = T(B + \lambda) - (1 - r)\lambda$$

$$MC = \frac{u-Q}{u-l}(B+\lambda) - (1-r)\lambda$$

substitute $\lambda = 0$ we get

$$MC_o = \frac{u-Q}{u-l}B$$

substitute eq.(11) in the last equation we get

$$MC_o = \frac{u - \left(u - \frac{b(u-l)}{B}\right)}{u-l}B$$

$$=\frac{u-u+\frac{b(u-l)}{B}}{u-l}.B=b \qquad \ldots(16)$$

we then find the constrained marginal cost by first solving for λ:

substitute eq. (12) in eq.(8) we obtain:

$$b=\frac{u-Q}{u-l}(B+\lambda)$$

$$B+\lambda=\frac{b(u-l)}{u-Q}$$

$$\lambda=\frac{b(u-l)}{u-Q}-B$$

substitute the regular capacity "Q" in the last equation we get:

$$\lambda=\frac{b(u-l)}{u-\left(u-\sqrt{(1-r)(u-l)(u+l)}\right)}-B$$

$$=\frac{b(u-l)}{\sqrt{(1-r)(u-l)(u+l)}}-B \qquad \ldots(17)$$

then substitute eq. (17) into eq.(10) we will have

$$MC_1 = T(B+\lambda) - (1-r)\lambda$$

$$= TB + T\lambda - (1-r)\lambda$$

$$= TB + (T-(1-r))\lambda$$

$$=TB+(T-(1-r))\left(\frac{b(u-l)}{\sqrt{(1-r)(u-l)(u+l)}}-B\right)$$

substitute eq. (11) in the last eq. We get

$$MC_1=\frac{u-Q}{u-l}B+\left(\frac{u-Q}{u-l}-(1-r)\left(\frac{b(u-l)}{\sqrt{(1-r)(u-l)(u+l)}}-B\right)\right)$$

$$= \frac{u - u + \sqrt{(1-r)(u-l)(u+l)}}{u-l} \bullet B + \left[\frac{u - u + \sqrt{(1-r)(u-l)(u+l)}}{u-l} \bullet (1-r) \right] \bullet \left[\frac{b(u-l)}{\sqrt{(1-r)(u-l)(u+l)}} - B \right]$$

$$= \frac{\sqrt{(1-r)(u+l)(u-l)}}{u-l}.B + \left(\sqrt{\frac{(1-r)(u+l)}{u-l}} - (1-r) \right) \left(\frac{b(u-l)}{\sqrt{(1-r)(u-l)(u+l)}} - B \right)$$

$$= b - \frac{(1-r)b(u-l)}{\sqrt{(1-r)(u-l)(u+l)}} + B(1-r)$$

$$MC_1 = b\left\{ 1 - \sqrt{\frac{(1-r)(u-l)}{u+l}} \right\} + B(1-r) \qquad \ldots(18)$$

we conclude that MC_1 converges to MC_0 as r approaches r_1, and it is convex for $\forall r > r_1$. We conclude for this case that the marginal cost declines, and then increases as $r \to 1$. The shape of marginal cost illustrated in Fig. 3.1. We conclude that the marginal cost is convex but not symmetric about the midpoint $\frac{r_i+1}{2}$. Furthermore as reliability approachs unity, marginal cost provides a close approximation to that which would be obtained from solving deterministic problem.

For comparative purposes consider the logistic distribution bounded over the same interval as the uniform distribution, namely $[l,u]$. This distribution is specified by the following:

$$f(x) = \frac{a}{\sigma} \frac{e^{-\frac{a}{\sigma}(x-\theta)}}{\left[1 + e^{-\frac{a}{\sigma}(x-\theta)} \right]^2} \qquad \ldots(19)$$

Utilizing eq. (19) to evaluate eq.(9) yields:

$$T = P_r(X \geq Q) = \int_Q^{\infty} \frac{\frac{a}{\sigma} e^{-\frac{a}{\sigma}(X-\theta)}}{\left[1+e^{-\frac{a}{\sigma}(X-\theta)}\right]^2} dx$$

$$= \frac{e^{-\frac{a}{\sigma}(Q-\theta)}}{1+e^{-\frac{a}{\sigma}(Q-\theta)}} = \frac{1}{1+e^{\frac{a}{\sigma}(Q-\theta)}} \qquad \ldots(20)$$

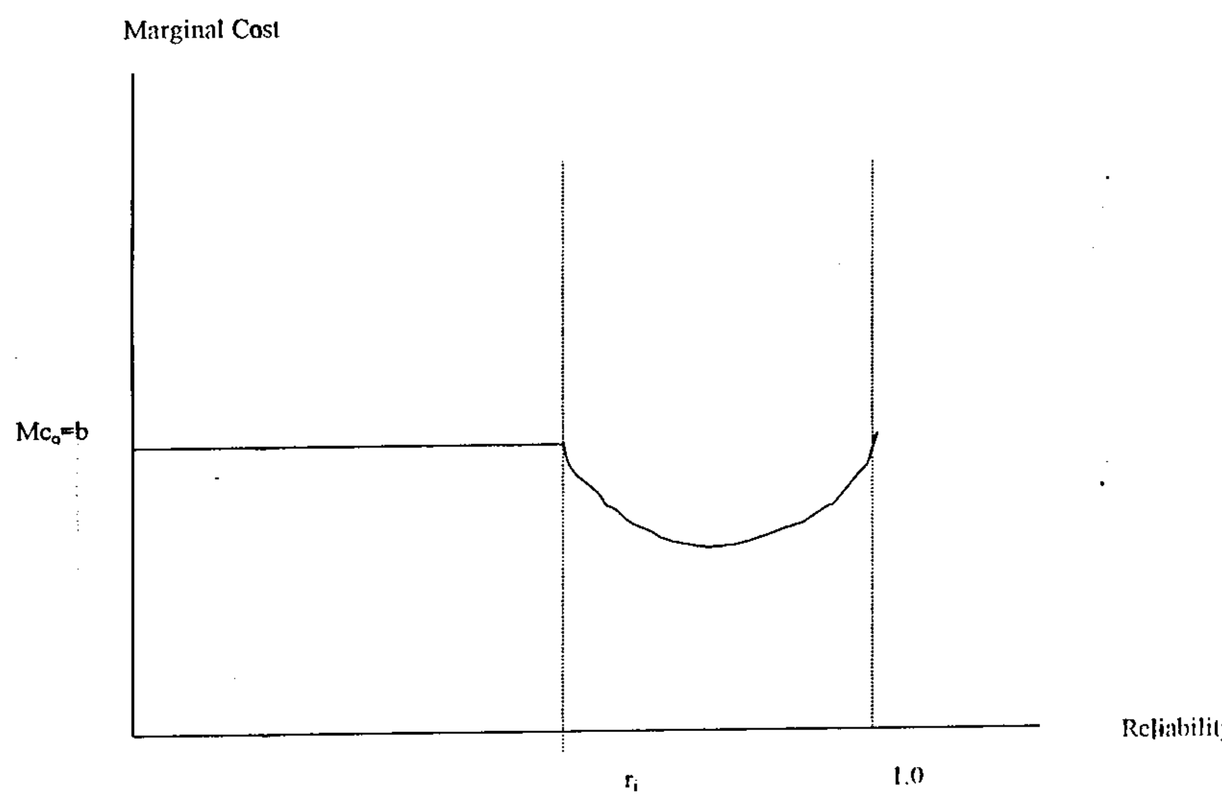

Fig. 3.1 Marginal cost

substitute eq.(20) into eq.(8) the unconstrained solution to eq.(19) and eq.(3)

$$\frac{\partial L}{\partial Q} = b - T(B+\lambda)$$

$$b = T(B+\lambda)$$

$$b = \frac{1}{1+e^{\frac{a}{\sigma}(Q-\theta)}}(B+\lambda)$$

if we choose $\lambda = 0$ we get

$$b = \frac{1}{1+e^{\frac{a}{\sigma}(Q-\theta)}}.B$$

$$1+e^{\frac{a}{\sigma}(Q-\theta)} = \frac{B}{b}$$

$$e^{\frac{a}{\sigma}(Q-\theta)} = \frac{B}{b} - 1$$

$$Q^{*} = \frac{\sigma}{a}\ln\left(\frac{B}{b} - 1\right) + \theta \qquad \ldots(21)$$

To determine the unconstrained marginal cost, substitute eq. (20) into eq.(10) yield

$$MC = \frac{1}{1+e^{\frac{a}{\sigma}}(Q-\theta)}(B+\lambda) - (1-r)\lambda$$

setting $\lambda = 0$ we get.

$$MC_{\rho} = \frac{1}{1+e^{\frac{a}{\sigma}(Q-\theta)}}.B = TB = b \qquad \ldots(22)$$

Constrained marginal cost is found by first solving for λ. Utilizing eq. (4) we obtain the following:

$$\frac{\partial L(X,Q)}{\partial Q} = b - T(B+\lambda) = b - \frac{1}{e^{\frac{a}{\sigma}(Q-\theta)}}(B+\lambda)$$

$$B+\lambda = be^{\frac{a}{\sigma}(Q-\theta)}$$

$$\lambda = be^{\frac{a}{\sigma}(Q-\theta)} - B \qquad \text{...(23)}$$

Substitute eq.(23) in eq. (10) we will have

$$MC = T(B+\lambda) - (1-r)\lambda$$

$$= \frac{1}{e^{\frac{a}{\sigma}(Q-\theta)}}\left(B + be^{\frac{a}{\sigma}(Q-\theta)} - B\right) - (1-r)\left[be^{\frac{a}{\sigma}(Q-\theta)} - B\right] = b - (1-r)\left[be^{\frac{a}{\sigma}(Q-\theta)} - B\right]$$

$$MC_1 = b - (1-r)\left[be^{\frac{a}{\sigma}(Q-\theta)} - B\right] \qquad \text{...(24)}$$

Consider now the comparison between optimal capacity for the uniform and logistic distribution. Both distribution have the same means and limits, but variance of uniform distribution is greater than logistic distribution.

To compare eq.(12) and eq.(21), consider the unconstrained problem Eq.(2): If $\frac{b}{B} < 0.5$ then the amount of required capacity for the uniform distribution exceeds the required capacity for the logistic distribution over the same interval but if $\frac{b}{B} > 0.5$, then the required capacity for the logistic distribution, exceeds the required capacity for the uniform over $[l, u]$.

A graphical representation of the above is provided in Figure 3.2.

This relationship has intuitive appeal when noting that for cases where the penalty cost is very large relative to the regular time cost, a higher variance gives greater levels of installed capacity.

3.4 Conclusion

This chapter deals with a basic model which characterizes the cost of minimization frame work for service such as postal services.

The results:

1. Marginal cost curve is convex
2. Increasing the demand distributions variance, while fixing the mean, may either increase or decrease required capacity.
3. The explicit effect of changing in variance is a function of regular, and over time costs associated with delivery service.

Optimal Capacity

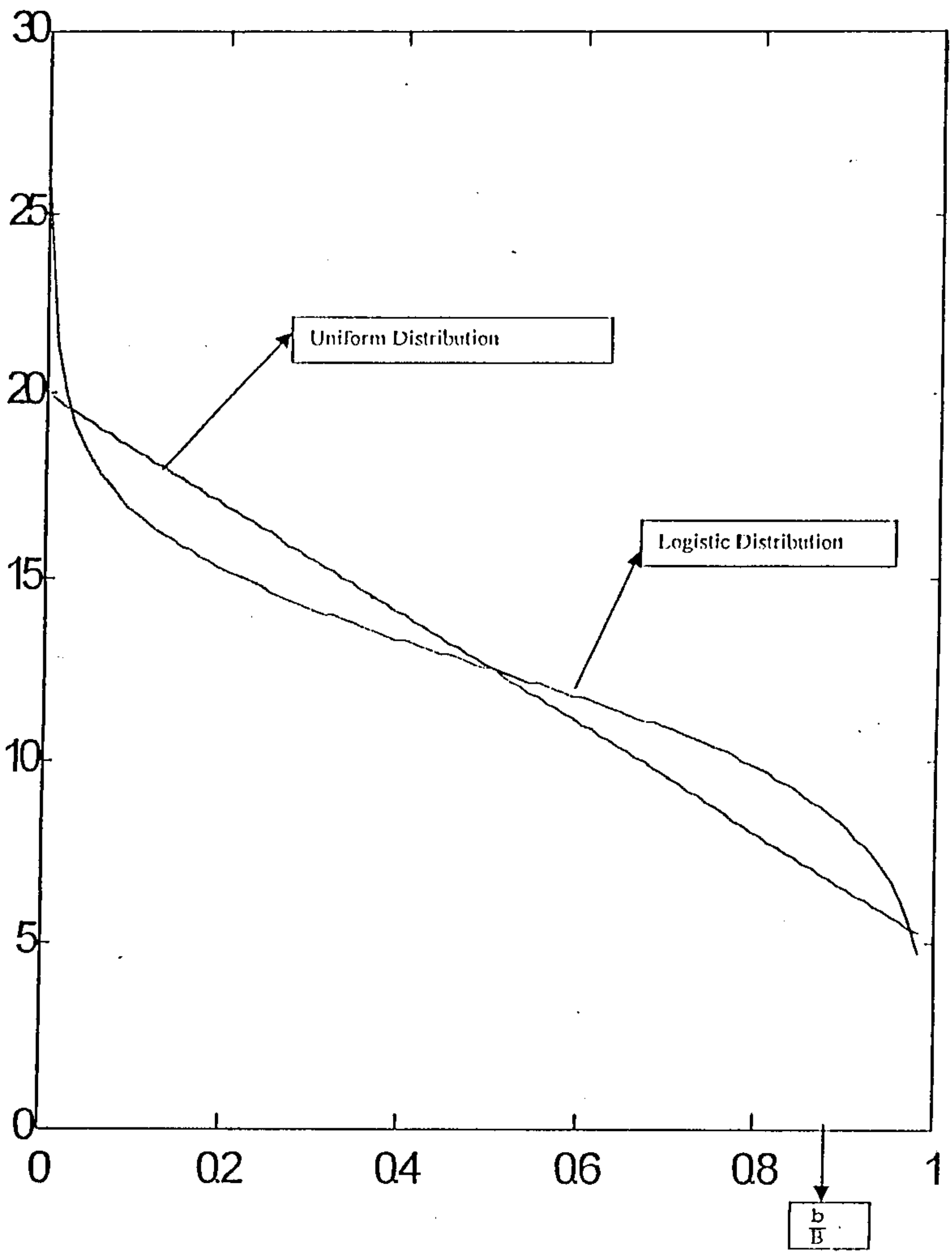

Fig (3.2)

Conclusion

This thesis presents a basic model which characterizes the cost minimization frame work for a service provider offering deferred service, such as a postal service.

We make comparison between optimal capacity for the uniform and logistic distributions. While both distributions are bounded by the same limits, and posses identical means, the variance for the uniform distribution is greater than logistic. Marginal cost are convex with respect to reliability of service. Increasing the demand distribution's variance while fixing the mean may either increase or decrease required capacity. The explicit effect of this change in variance is a function of regular and over time costs (ρ) associated with the delivery service.

This thesis also provides one such approach by developing a theoretical underpinning for marginal cost determination.

We deal with dynamic programming to maximize reliability of the system, and yield an exact optimal solution. Dynamic programming used to solve variety problems. We try to apply this procedure for the case of complex systems, we found that it could be extended for such system without difficulties, and we recommended the next research for the interrelated students toward such systems.

Finally, Heuristic procedure applied to obtain the solution of optimization problems. We present in this thesis the Sharma and Venkateswarn approach.

We present two examples, and found maximum reliability in the available cost using two methods.

A comparison is made between Dynamic programming and Heuristic Approach, where both give the same results.

Appendix

The following heuristic algorithm and the computer program are to maximize the reliability and minimize the cost.

Algorithm

Given: number of units n, cost (C_i) and reliability (R_i) for each unit and the available cost money (b).

Find: maximum $\{Q_i^{'}$ where $Q_i^{'} = 1 - R_i, i = 1,2,\ldots,n\}$. Find its location

$\rightarrow$ p

SMAX = (max. $Q_p^1)^2$

Add: add one unit where Q_p^1 is maximum.

$Q_p^1 + 1$

Repeat: until total cost > available cost.

Remove : the component where total cost > available cost.

Repeat: until total cost > available cost.

```
//*************************************************************
//
//Simulation For Heuristic Approach.
//Implemented by Mr. Saleh Afaneh.
//
//*************************************************************
import java.util.*;
import java.io.*;
import java.lang.*;
class reliab{

  public static boolean finished=false;
  public static int mNum = 0 ;// no machine yet..

  public static String Tmp= new String();

  public static String nam= new String();
  public static Integer st= new Integer(0);
  public static Float rel= new Float(0);
  public static Float pri= new Float(0);

  public static Vector namV= new Vector();
  public static Vector stV= new Vector();
  public static Vector relV= new Vector();
  public static Vector priV= new Vector();

  public static float Sum;
  public static Float money= new Float(0);

  public static char[] end = new char[1];

public static InputStreamReader is=new
InputStreamReader(System.in);
public static BufferedReader br=new BufferedReader (is);

public static void main(String[] args){
 try{

    System.out.println("\n\tThis is Simple Simulation For
Heuristic Approach. ");
    System.out.println("\t\tImplemented by Mr. Saleh Afaneh. \n");

   do{
   Heuristic();

   mNum = 0 ;
   Tmp= new String();
   nam= new String();
   st= new Integer(0);
   rel= new Float(0);
   pri= new Float(0);
   namV= new Vector();
   stV= new Vector();
   relV= new Vector();
   priV= new Vector();
   Sum=0;
```

```
 money= new Float(0);
 end = new char[1];

  namV.removeAllElements();
  stV.removeAllElements();
  relV.removeAllElements();
  priV.removeAllElements();

   Tmp=new String(br.readLine());
   System.out.print("\nTry again (Y/N): ?");
   br.read(end);

 }while(end[0]=='Y'|| end[0]=='y');

}catch(Exception Exp){System.out.println(Exp.getMessage());}

}//end of main..

public static void Heuristic(){

try{

System.out.println("Input values for machine specifications :\n");

do{
  Tmp=new String(br.readLine());

  System.out.print("Machine Name : ");
  nam=new String(br.readLine());
  namV.addElement(nam);

  st = new Integer(1);
  stV.addElement(st);

  do{
  System.out.print("Machine Reliability (between 0 and 1) : ");
  rel=new Float(br.readLine());
   }while(!(rel.floatValue()>0 && rel.floatValue()<1));

  rel = new Float(1-rel.floatValue());
  rel = rnd (rel.floatValue());

  relV.addElement(rel);

  System.out.print("Machine Price : ");
  pri=new Float(br.readLine());
  priV.addElement(pri);

  mNum++;

  System.out.print("\nMore Machines (Y/N): ?");
  br.read(end);

  if(end[0]=='Y'|| end[0]=='y')
    finished=true;
  else
```

```
    finished=false;

  System.in.read();
  System.in.read();

  }while(finished);

  Tmp=new String(br.readLine());

  System.out.print("Enter the value of available money :");
  money=new Float(br.readLine());

  if(money.floatValue()<calcSum())
    System.out.println("\n Can not buy machine(s) with your money
!!");
  else{
    prntHead();
    print(1);
    print(2);
    print(3);

    while(buyMachine()){
    int ix =maxUnRel();
    Float tempRel= new
Float(((Float)relV.elementAt(ix)).floatValue());

    stV.setElementAt(
    new Integer(((Integer)stV.elementAt(ix)).intValue() + 1)
                ,ix);

    relV.setElementAt(
    rnd (new Float(((Float)relV.elementAt(ix)).floatValue() *
              ((Float)relV.elementAt(ix)).floatValue()
).floatValue())
                                    ,ix);

    if(money.floatValue()<calcSum())
      {
        relV.setElementAt(tempRel,ix);
        stV.setElementAt(
        new Integer((((Integer)stV.elementAt(ix)).intValue() -
1)*(-1))
                ,ix);
        print(1);
        print(4);
        print(3);
           }
   else{
      print(1);
      print(2);
      print(3);

   }
      }//end while..
   }//end else..
```

```
result();

   }catch(Exception ex)
    {
    System.out.println(ex.getMessage());
    }
  }//end of function Heuristic

  public static float calcSum(){
  float sm=0;
  for(int s=0;s<mNum;s++)
  sm=sm+(Math.abs(((Integer)stV.elementAt(s)).intValue()) *
                  ((Float)priV.elementAt(s)).floatValue());

  return sm;
  }//end of function calacSum

  public static boolean buyMachine(){

  float diff= money.floatValue()- calcSum();

    for(int n=0;n<mNum;n++)
      if(diff >= ((Float)priV.elementAt(n)).floatValue() )
      return true;

      return false;

   }//end of function buyMachine

   public static int maxUnRel(){
     float temp=0;
     int idx=0;

         for(int r=0;r<mNum;r++)

           if(((Float)relV.elementAt(r)).floatValue() > temp &&
              ((Integer)stV.elementAt(r)).intValue() > 0){

             temp=((Float)relV.elementAt(r)).floatValue();
             idx=r;

             }

      return idx;
    }// end of function maxUnRel

    public static void prntHead(){
    System.out.println("\n\t\t\t*********** Solution
  ***********\n");

      for(int l=0;l<mNum;l++){
        System.out.print((String)namV.elementAt(l)+"\t");
      }

      for(int l=0;l<mNum;l++){
        System.out.print("Q'"+(l+1)+"\t");
      }
```

```
System.out.println("g(x)\n");

}// end of function prntHead

public static void print(int C){
switch(C) {

 case 1 : for(int l=0;l<mNum;l++)
          System.out.print(
Math.abs(((Integer)stV.elementAt(l)).intValue())+"\t");
           break;
 case 2 : for(int l=0;l<mNum;l++)
          System.out.print(
          ((Float)relV.elementAt(l)).floatValue()+"\t");
           break;
 case 3 : System.out.println(calcSum());
          break;
 case 4 : for(int l=0;l<mNum;l++)
          System.out.print("---"+"\t");
           break;

 }//end switch
 }//end of function print

public static Float rnd(float fl){

 fl = (Math.round(fl*1000));
 fl = fl/1000;

 return new Float(fl);
}//end of function rnd

public static void result(){
System.out.println("\n\t\t\t*********** Result***********\n");

System.out.println("You can buy the following machines :\n ");

 for (int r=0;r<mNum;r++)
   {
   System.out.print("** ");
System.out.print(Math.abs(((Integer)stV.elementAt(r)).intValue())+
"  ");
   System.out.print("( "+namV.elementAt(r).toString()+" )");
   System.out.println(" machine(s) .");

  }
 System.out.print("\nThe remaining money after buying these
machines is : ");
 System.out.print(rnd(money.floatValue()-calcSum()).floatValue());
 System.out.println(" $");
 }//end of function result..

}// end of class reliab..
```

Example : 1
.This is Simple Simulation For Heuristic Approach
.Implemented by Mr. Saleh Afaneh

: Input values for machine specifications

Machine Name : x1
Machine Reliability (between 0 and 1) : 0.5
Machine Price : 2000

More Machines (Y/N):? y

Machine Name : x2
Machine Reliability (between 0 and 1) : 0.7
Machine Price : 3000

More Machines (Y/N):? y

Machine Name : x3
Machine Reliability (between 0 and 1) : 0.6
Machine Price : 1000

More Machines (Y/N):? y

Enter the value of available money :11000

More Machines (Y/N):? n

Enter the value of available money :11000

*********** Solution ***********

x1	x2	x3	Q'1	Q'2	Q'3	g(x)
1	1	1	0.5	0.3	0.4	6000.0
2	1	1	0.25	0.3	0.4	8000.0
2	1	2	0.25	0.3	0.16	9000.0
2	1	2	---	---	---	9000.0
3	1	2	0.063	0.3	0.16	11000.0

***********Result ***********

You can buy the following machines:

**3 (x1) machine(s).
**1 (x2) machine(s).
**2 (x3) machine(s).

The remaining money after buying these machines is : 0.0$

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الخلاصـــــــه

عرضت في هذه الرسالة المفاهيم المتعلقة بالمعولية، ومقاييسها، والطرق المستخدمة لربط الوحدات فيها وهي: ربطها على التوالي، أو ربطها على التوازي أو ربطها على التوالي والتوازي في نفس الوقت أو الربط المعقد.

وقمت بإيجاد مقاييس بالمعولية على اقترانين وهما: الاقتران المنتظم والاقتران الأسي، والمقارنة بين نتائجهما.

كان الهدف هو تعظيم المصداقية وتقليل التكلفة باستخدام طريقتين وهما: البرمجة الديناميكية والمساعد عن الكشف التقريبي.

وقد قدمت أمثلة واستخدمت الطريقتين لتحقيق الهدف، فتبين أن النتائج باستخدام البرامج الديناميكية أفضل من استخدام المساعد عن الكشف التقريبي.

في النهاية كان الهدف هو تحديد التكلفة الحدية وتقليل سعة التكلفة لأنواع مختلفة من الخدمات، وخاصة الخدمات البريدية، وتشير النتائج الى أن منحنى التكلفة الحدية مقعر (التكلفة الحدية مناسبة) مرتبطة بمصداقية الخدمة. وقد استخدمت اقترانين وهما: الاقتران المنتظم والاقتران المنطقي، فكانت النتائج تشير ولأول مرة الى أن الاقتران المنطقـــي أعطـــى ســـعة أعلـــى مـــن الاقتـــران المنتظم عندما كانت $\frac{\text{معدل الفائدة}}{\text{معدل تكلفة الغرامة التكنولوجيه}} < \frac{1}{2}$