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NUMERICAL SOLUTIONS OF SYSTEM OF VOLTERRA INTEGRAL EQUATIONS

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Dedication

I dedicate this thesis to my mother (Zainab) and my father (Isa), to my husband (Ali) who helped me in every step, to my beloved children (Sarah, Salah al-Din, Maryam) to all my family and to my husband's family and to my friends, to everyone who supported and encouraged me, thank you very much to all of you.

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I would also like to extend my thanks and gratitude to all those who contributed to my success in my MSC thesis.

Declaration

I, the undersigned, declare that I submitted the thesis entitled:

NUMERICAL SOLUTIONS OF SYSTEM OF VOLTERRA INTEGRAL EQUATIONS

I declare that the work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's Name: تسنيم عيسى محمد مغربي

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NUMERICAL SOLUTIONS OF SYSTEM OF VOLTERRA INTEGRAL EQUATIONS

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Abstract

In this thesis, we focus on the numerical treatment of system of Volterra integral equations. The numerical techniques to be considered are Galerkin method with Laguerre polynomials, Chebyshev collocation method, and Bernstein's approximation method.

Moreover, some illustrative examples to show the validity and applicability of these techniques are solved. A comparison between these methods is carried out.

Numerical results have shown that Bernstein's approximation method is the most accurate and efficient numerical technique for solving linear system of Volterra integral equations in comparison with its counterparts.

Keywords: Volterra integral equations; Galerkin method with Laguerre polynomials; Chebyshev collocation method; Bernstein's approximation method; and numerical techniques.

Introduction

System of integral equations is encountered in various applications in many fields including mechanics, potential theory, geophysics, electricity, magnetism, electrostatics, diffusion problems, quantum mechanics, heat radiation, optimization, fluid mechanics, fracture mechanics, optimal control systems and phenomena in physics and biology

[1] [2] [3] [4] [5] [6].

In recent years, there has been a growing interest in integral equations and systems of integral equations due to their wide range of application as missioned above. Moreover, many initial and boundary value problems associated with the ordinary and partial differential equations can be cast into a system of integral equations.

Various numerical methods, for solving system of Volterra equations using various polynomials have been developed by many researchers. Very recently, Shahsavaran [7] used the Block-pulse functions and Taylor expansion methods. Taylor polynomials were also used by Wang [8] with computer algebra. Maleknejad et al [9] studied first kind of Volterra integral equation by using a recursive scheme. Bernstein polynomials were used for the solution of second order linear and first order nonlinear differential equations of second kind by Bhatti [10] . These polynomials have also been used for solving Fredholm integral equations of second kind by Shirin [11] . Moreover, Qatanani et al. [12] [13] have used numerical schemes to solve systems of Volterra integro-differential equations and linear fractional Volterra integral equations. A comparison between these numerical methods have been carried out. Kazem Nouri [14] used the Chebyshev collocation method for solving systems of Volterra integral equations. Very recently Buranay et al. [15] used modified Bernstein–Kantorovich operators to solve Fredholm and Volterra integral equations. Jafarian et al [16] used Bernstein collocation method to approximate the exact solution of the linear second kind Fredholm and Volterra integral equations systems.

In this work, some numerical techniques were implemented to approximate the solution of system of linear Volterra integral equations, namely, Galerkin method with Laguerre polynomials, Chebyshev collocation method and Bernstein's approximation method

This system of Volterra integral equations has the general form:

$$f_i(x) = g_i(x) + \int_a^x k_{ij}(x,t) f_j(t) dt , \qquad a \le x \le b$$

The kernels $k_{ij}(x,t)$ and the function $g_i(x)$ are given real valued functions. The unknown functions $f_i(x)$ are to be determined. A comparison between these methods is carried out by solving some numerical examples.

In this thesis, we focus on the numerical treatment of system of Volterra integral equations. The numerical techniques to be considered are Galerkin method with Laguerre polynomials, Chebyshev collocation method, and Bernstein's approximation method.

Moreover, some illustrative examples to show the validity and applicability of these techniques are solved. A comparison between these methods is carried out.

Numerical results shown that Bernstein's approximation method is the most accurate and efficient numerical technique for solving linear system of Volterra integral equations in comparison with its counterparts.

This thesis is organized as follows:

In chapter 1, some basic concepts of systems of integral equations together with their solvability is presented. Three numerical techniques, namely the Galerkin method with Laguerre polynomials, Chebyshev collocation method and Bernstein’s approximation method are addressed in chapter 2. In chapter 3, the proposed methods are implemented using three numerical examples with known analytical solution by applying MATLAB software, conclusion are included.

Chapter One
Mathematical Preliminaries

In this chapter, we will review some of the most important concepts of systems of integral equations.

Definition 1.1 [17] [18] [19] [20]

A system of integral equations is a set of two or more integral equations in two or more unknown functions.

A system of integral equations can be written in the general form as

$$C(x)F(x) = G(x) + \lambda \int_{u(x)}^{h(x)} K(x,t)F(t)dt \qquad (1.1)$$

where $u(x)$ and $h(x)$ are the limits of integration, the limits of integration may both be variables, constants, or mixed. λ is a constant parameter such that $\lambda \neq 0$. Also,

$G(x) = [g_i(x)]_{n\times 1}, \qquad i = 1, 2, \dots, n$

$C(x) = [c_i(x)]_{n\times n}, \qquad i = 1, 2, \dots, n$

$F(x) = [f_i(x)]_{n\times 1}, \qquad i = 1, 2, \dots, n$

$\lambda K(x,t) = \left[\lambda_{ij} k_{ij}(x,t)\right]_{n\times n}, \quad i = 1,2, \dots, n \quad j = 1,2, \dots, n$

where $g_i(x)$,$c_i(x)$ and $k_{ij}(x,t)$ are known functions and $f_i(x)$ are unknown functions.

1.1 Classification of System of Integral Equations

1- System of Volterra Integral Equations

This system has the standard form

$$C(x)F(x) = G(x) + \lambda \int_a^x K(x,t)F(t)dt \quad , \quad a \le x \le b \qquad (1.2)$$

where $\lambda, a, b \in \mathbb{R}, \lambda \neq 0$ and x is a variable. Also,

$G(x) = [g_i(x)]_{n\times 1}, \qquad i = 1, 2, \dots\ n$

$C(x) = [c_i(x)]_{n\times n}, \qquad i = 1, 2, \dots, n$

$F(x) = [f_i(x)]_{n\times 1}, \qquad i = 1, 2, \dots n$

$\lambda K(x,t) = \left[\lambda_{ij} k_{ij}(x,t)\right]_{n\times n}, \quad i = 1, 2, \dots, n, \qquad j = 1, 2, \dots, n$

In system (1. 2) $k_{ij}(x,t)$, $c_i(x)$ and $g_i(x)$ are known functions and $f_i(x)$ are unknown functions [21] [22] [4] [23]. There are two kinds of system of Volterra integral equations, known as the first kind and the second kind, the first kind of system has $F(x)$ present only under the integral sign, otherwise will be the second kind.

2- System of Fredholm Integral Equations

This system has the standard form

$$C(x)F(x) = G(x) + \lambda \int_a^b K(x,t)F(t)dt \ , \quad a \le x \le b \qquad (1.3)$$

where $\lambda, a, b \in \mathbb{R}, \lambda \neq 0$ and x is a variable. Also,

$G(x) = [g_i(x)]_{n\times 1}, \qquad i = 1, 2, \dots n$

$C(x) = [c_i(x)]_{n\times n}, \qquad i = 1, 2, \dots, n$

$F(x) = [f_i(x)]_{n\times 1}, \qquad i = 1, 2, \dots n$

$\lambda K(x,t) = \left[\lambda_{ij} k_{ij}(x,t)\right]_{n\times n}, \qquad i = 1,2, \dots n \ , \qquad j = 1, 2, \dots n$

In system (1.3) $k_{ij}(x,t)$,$c(x)$ and $g_i(x)$ are known functions and $f_i(x)$ are unknown functions [21] [22] [4] [23].

There are two kinds of system of Fredholm integral equations, known as the first kind and the second kind, the first kind of system has $F(x)$ present only under the integral sign, otherwise will be the second kind.

1.2 Linearity [4]

The system (1. 1) of integral equations

$$C(x)F(x) = G(x) + \lambda \int_{u(x)}^{h(x)} K(x,t)F(t)dt$$

is said to be linear if the exponent of the unknown functions $F(x)$ under the integral sign is one and all equations do not contain nonlinear functions of $F(x)$. Otherwise, the system is called nonlinear.

1.3 Homogeneity [4]

The system (1. 1) of integral equations

$$C(x)F(x) = G(x) + \lambda \int_{u(x)}^{h(x)} K(x,t)F(t)dt$$

is said to be homogenous if $G(x)$ is identically zero. otherwise, it is called nonhomogeneous.

1.4 Existence and uniqueness of solution

For convenience, we consider the existence and uniqueness of a solution for linear Volterra integral equation in Banach space [24].

Definition 1.2: (Banach Space) [25]

A Banach space is a normed vector space $(X, \|.\|)$ complete relative to the norm metric, i.e., such that every Cauchy sequence (or fundamental sequence)

$$\{x_n\}_{n=1}^{\infty} \subseteq X : p(x_n,x_m) = \|x_n - x_m\| \to 0, \quad n,m \to \infty$$

converges to an element $x \in X$:

$$\lim_{n\to\infty} x_n = x \Leftrightarrow p(x_n,x) = \|x_n - x\| \to 0, \qquad n \to \infty$$

The linear Volterra integral equation is written as:

$$c(x)f(x) = g(x) + \lambda \int_a^x k(x,t)f(t)dt \quad , \quad a \le x \le b \tag{1.4}$$

where $g(x)$ *and* $c(x)$ are known continuous functions on interval $[a, b]$, the kernel $k(x, t)$ is a given continuous function on the region

$D = \{(x, t): a \leq t \leq x \leq b\}$, $\lambda \neq 0$, and $f(x)$ is unknown continuous function in $[a, b]$ that must be determined i.e.,$f(x) \in C[a, b]$.

Before we study the existence and uniqueness of a continuous solution of the linear Volterra integral equation in Banach space based on fixed-point theory, several definitions and theorems are given:

Definition 1.3: [25]
For a metric space (M, d) , let $M^o \subseteq M$ with a map $f: M^o \to M$

A point $p \in M^o$ is said to be a fixed point of f if $f(p) = p$.

Definition 1.4: [25]
Let (M, d) be a complete matric space (Banach space), a mapping $f: M \to M$ is said to be contraction if $\exists a \in \mathbb{R}$ for $0 \leq a < 1$ such that

$$d\big(f(u), f(v)\big) \leq a d(u, v) \quad , \qquad \forall u, v \in M$$

Theorem 1.1: (Fixed- Point Theorem): [25]
If the mapping $f: M \to M$ is the contraction on a complete metric space (M, d), then f has a unique fixed point $\tilde{x} \in M$.

We use the fixed-point method to create a diagram that is used to solve the linear Volterra integral equation, by starting with an initial approximation that is used in a recurrence relation to find other approximate solutions.

Next, we introduce an operator T defined as follows:

$$C(x)F(x) = T(f) = G(x) + \lambda \int_a^x K(x,t)F(t)dt \, , \qquad \text{where } f = f_1, f_2, \dots, f_n \qquad (1.5)$$

and the solution of the system is a fixed-point of $T = [T_1, T_2, \dots, T_n]^t$.

Choose $f_i^0(x) \in C[a, b]$ as an initial function and the following fixed-point iteration will be generated:

$$C(x)f_i^r(x) = T_i(f^{r-1}) = g_i(x) + \sum_{j=0}^{n} \lambda_{ij} \int_a^x k_{ij}(x,t) f_j^{r-1}(t) \quad dt \tag{1. 6}$$

We determine multiple approximations f_i^r, for $i = 1,2, \dots n$, $r \geq 1$ and the sequence $\{f_i^n\}$ converges to $F(x)$ as $n \to \infty$. It is just the contractive property which is responsible for clustering the sequence $\{f_i^r\}$ towards a limit point.

Then the major concepts that are required to the fixed-point theorem are contraction mapping and complete metric space, T becomes a contractive mapping under some assumptions, based on this theory

Theorem 1.2: [25]

For a complete matric space $(C[a, b], \|.\|_\infty)$, and the continuous function $G \in C[a, b]$ and $K \in C([a, b] \times [a, b])$, if the following condition holds

$$\lambda_i < \frac{1}{n(b-a)^2 N_i}, \quad \forall i = 1,2, \dots, n$$

where N_i is any positive constants, then the mapping T in equation (1. 5) becomes a contractive mapping.

Now we want to show that there is only one solution for linear Volterra integral equation, so we have to prove that T has a unique fixed-point and the generated sequence $\{f_i^n\}_{n=0}^{\infty}$ in equation (1. 6) converges to this fixed-point.

The following theorem explains the convergence of $f_i^r(x)$.

Theorem 1.3: [25]

Let $(C[a, b], \|.\|_\infty)$ be a complete metric space and T_i be n contraction mapping on the linear Volterra integral equation, then for each $i = 1,2, \dots, n$ we get:

1- T_i has a unique fixed-point $f_i^* \in C[a, b]$ such that $f_i^* = T_i(f^*)$.

2- For any $f_i^0 \in C[a, b]$, the sequence $\{f_i^r(x)\} \subset C[a, b]$ defined by

$f_i^r(x) = T_i(f^{r-1})$, for $r = 1, \dots$, converges to f_i^*.

Chapter Two

Numerical Techniques for Solving the Linear System of Volterra Integral Equations

There are many numerical techniques available for solving system of Volterra integral equations. In this chapter we will present some of these important numerical techniques namely:

- Galerkin Method With Laguerre Polynomials

- Chebyshev Collocation Method

- Bernstein's Approximation Method

2.1 Galerkin Method With Laguerre Polynomials

We start by considering the Laguerre polynomials

2.1.1 Laguerre Polynomials: [26] [27] [28]

We can get Laguerre polynomials by solving the Laguerre differential equation

$$xy'' + (1-x)y' + ny = 0 \qquad (2.1)$$

where n is a positive integer, we notice that $x = 0$ is a regular singular point of equation (2.1). The solution to this equation is

$$y = c_0 \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!\,(k!)^2} x^k$$

Definition 2. 1: [30]

Let $x = x_0$ be a singular point of $y'' + p(x)y' + q(x)y = 0$, $x = x_0$ is a regular singular point of the equation if both $(x-x_0)p(x)$ and $(x-x_0)^2 q(x)$ are analytic at $x = x_0$.

If we let c_0 =1, we get the standard solution denoted by $L_n(x)$, called the Laguerre polynomials of order n, and is given by

$$L_n = \sum_{k=1}^{n} \frac{(-1)^k n!}{(n-k)!\,(k!)^2} x^k$$

We can list few of these Laguerre polynomials corresponding to $n = 0, 1, 2, 3$ respectively

$$L_0(x) = \; 1$$

$$L_1(x) = 1 - x$$

$$L_2(x) = \frac{1}{2!}(2 - 4x + x^2)$$

$$L_3(x) = \frac{1}{3!}(6 - 18x + 9x^2 - x^3)$$

2.1.2 Galerkin Method With Laguerre Polynomials: [29] [30] [27] [31]

Now we can start using Galerkin method with Laguerre polynomial, to solve system of Volterra integral equations

$$C(x)F(x) = G(x) + \lambda \int_a^x K(x,t)F(t)dt \quad , \quad a \le x \le b \qquad (2.\ 2)$$

where $\lambda, a, b \in \mathbb{R}, \lambda \neq 0$ and x is a variable. Also,

$G(x) = [g_i(x)]_{n\times 1}, \qquad i = 1,2, \dots, n$

$C(x) = [c_i(x)]_{n\times n}, \qquad i = 1,2, \dots, n$

$F(x) = [f_i(x)]_{n\times 1}, \qquad i = 1,2, \dots, n$

$\lambda K(x,t) = \left[\lambda_{ij} k_{ij}(x,t)\right]_{n\times n} \; , \quad i = 1,2, \dots, n \, , \qquad j = 1,2, \dots, n$

In system (2. 2) $k_{ij}(x,t)$, $c_i(x)$ and $g_i(x)$ are known functions and $f_i(x)$ are unknown functions.

We take the linear combination of Laguerre Polynomials to find an approximate solution $F(x)$, so we assume that

$$f_{im}(x) = \sum_{r=0}^{m} a_{ir} L_r(x) \qquad i = 1, \dots, n \;\; r = 0,1, \dots m \tag{2. 3}$$

where, $L_r(x)$ are Laguerre polynomials of degree r, m is the number of Laguerre polynomials, and a_{ir} are unknown parameters.

Equation (2. 3) can be written in the matrix form as

$$f_i(x) = L(x)A_i \quad , \qquad i = 1,2, \dots, n$$

where

$$L(x) = [L_0(x) \quad L_1(x) \quad \dots \quad \dots \quad L_m(x)]$$

$$A_i = [a_{i0} \quad a_{i1} \quad \cdots \quad \cdots \quad a_{im}]^T$$

Hence, the matrix $F(x)$ defined as a column matrix of the unknown functions that can be expressed by

$$F(x) = A\mathcal{L}(x)$$

so that

$$\mathcal{L}(x) = \begin{bmatrix} L(x) & 0 & \dots & 0 \\ 0 & L(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L(x) \end{bmatrix}_{n\times n} \quad , \quad A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}_{n\times 1}$$

similarly, the integral part

$$\int_a^x k_{ij}(x,t) \sum_{r=0}^{m} a_{ir} L_r(t) \; dt$$

becomes

$$A \int_a^x K(x,t)\mathcal{L}(t)dt$$

then we have

$$C(x)A\mathcal{L}(x) = G(x) + \lambda A \int_a^x K(x,t)\mathcal{L}(t)dt$$

$$G(x) = C(x)A\mathcal{L}(x) - \lambda A \int_a^x K(x,t)\mathcal{L}(t)dt \qquad (2.4)$$

Multiplying both sides of equation (2. 4) by $\mathcal{L}(x)$ and integrating the resulting equation with respect to x over the interval $[a, b]$, to get

$$\int_a^b G(x)^T\mathcal{L}(x)\,dx = A\int_a^b \left(C(x)\mathcal{L}(x)\right)^T\mathcal{L}(x)dx - \lambda A\int_a^b \left(\int_a^x K(x,t)\mathcal{L}(t)dt\right)^T \mathcal{L}(x)dx$$

this leads to

$$AK_{rs} = G_s \qquad (2.5)$$

where

$$A = \begin{bmatrix} a_{10} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n0} & \cdots & a_{nm} \end{bmatrix}_{n\times(m+1)} \quad , G_s = \int_a^b G(x)^T\mathcal{L}(x)\,dx$$

$$K_{rs} = \int_a^b \left(C(x)\mathcal{L}(x)\right)^T\mathcal{L}(x)dx - \lambda \int_a^b \left(\int_a^x K(x,t)\mathcal{L}(t)dt\right)^T \mathcal{L}(x)dx$$

G_s is a matrix with dimension $n \times (m + 1)$, and K_{rs} is a square matrix with order $n(m + 1)$.

Finally, by solving the linear algebraic system (2. 5), we get the approximate solution of the system of Volterra integral equations (2. 2).

Error and Convergence:

Theorem 2.1: [16]

Suppose that $f_{im}(x)$, $i = 1,2, \dots, n$ are the linear combination of Laguerre Polynomials of degree m such that their coefficients have been produced by solving the generalized linear system (2. 5), then the given polynomials converge to the exact solution of the system of Volterra integral equations, when $m \to +\infty$.

Proof:

The error concerning the approximation method can be written as

$$e_m(x) = \sum_{i=1}^{n} e_{im}(x)$$

where

$$e_{im} = \sum_{i=1}^{n}\big(f_i(x) - f_{im}(x)\big) - \sum_{i=1}^{n} \lambda_{ij} \int_a^x k_{ij}(x,t)(\, f_i(t) - f_{im}(t))\, dt$$

In virtue of Theorem (2. 1), the error function $e_m(x)$ must tend to zero, when

$m \to \infty$, Hence, we have :

$$\begin{aligned} \|e_m\| &\le \sum_{i=1}^{n} \|e_{im}\| \\ &\le \sum_{i=1}^{n} \big\|\big(f_i(x) - f_{im}(x)\big)\big\| \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{ij} \int_a^x \big\|k_{ij}(x,t)\big\|(\|f_i(t) - f_{im}(t)\|)ds \end{aligned}$$

Since $\big\|k_{ij}(x,t)\big\|$ is bounded and $\lambda \neq 0$, therefore, $\|f_i(x) - f_{im}(x)\| \to 0$

implies that $\|e_m\| \to 0$.

2.2 Chebyshev Collocation Method

We start by considering the Chebyshev Polynomials

2.2.1 Chebyshev Polynomials: [32] [33] [26] [28]

Chebyshev Polynomials are solutions of Chebyshev differential equation

$$(1 - x^2)y'' - xy' + n^2 y = 0,\ n = 0,1,2, \ldots \tag{2. 6}$$

Let $x = \cos\theta$ we get

$$\frac{d^2y}{d\theta^2} + n^2 y = 0 \tag{2.7}$$

Equation (2. 7) has the general solution

$$y = A\ cos\ n\theta\ + B \sin n\theta$$

or

$$y = A \cos(n\ cos^{-1}x) + B \sin(n\ cos^{-1}x) \qquad |x| \leq 1$$

which is equivalent to

$$y = A\ T_n(x)\ + B\ U_n(x) \qquad |x| \leq 1$$

where $T_n(x)$ is called Chebyshev polynomials of first kind of degree n, and $U_n(x)$ is called the Chebyshev polynomials of second kind of degree n.

Since $x = cos\ \theta$, $T_n(x)$ can be defined as

$$T_n(x) = T_n(\cos\theta) = \cos n\ \theta$$

we conclude that

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2\ x^2 - 1$$

we get a more general formula of $T_n(x)$ by using the recurrence relation

$$T_{n+1}(x) = 2x\ T_n(x) - T_{n-1}(x)$$

we list few of Chebyshev polynomials:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2\ (x) = 2\ x^2 - 1$$

$$T_3(x) = 4\ x^3 - 3x$$

$$T_4(x) = 8\,x^4 - 8\,x^2 + 1$$

Chebyshev polynomials of the second kind are:

$$U_n(x) = U_n(\cos\theta) = \frac{\sin(\,(n+1)\theta)}{\sin\theta}\,, \qquad n = 0, 1, \ldots$$

so, we conclude that

$$U_0(x) = 1$$

$$U_1(x) = 2x$$

$$U_2(x) = 4x^2 - 1$$

we get a more general formula of $U_n(x)$ by using the recurrence relation

$$U_{n+1}(x) = 2x\,U_n(x) - U_{n-1}(x)$$

we list few of Chebyshev polynomials of second kind

$$U_3(x) = 8x^3 - 4x$$

$$U_4(x) = 16\,x^4 - 12x^2 + 1$$

$$U_5 = 32x^5 - 32x^3 + 6x$$

2.2.2 Chebyshev Collocation Method: [32] [34]

To implement this method for solving the system of Volterra integral equations (2. 2) we let

$$f_i(x) \cong \sum_{r=0}^{m} a_{ir} T_r(x), \quad i = 1, \ldots, n \qquad -1 \le x \le 1 \tag{2.8}$$

where $T_r(x)$ denotes the Chebyshev polynomials of the first kind, a_{ir} are the unknown coefficients and n is any positive integer.

Assuming that the integrals on the system (2. 2) are bounded in the range $[0, b]$, then a solution can be optimal by means of the shifted Chebyshev polynomials

$$s_r = b(x_r + 1)/2$$

where x_r are Chebyshev Gauss points of order m in interval $[-1,1]$ i.e., $x_r = cos\left(\frac{r\pi}{m}\right)$, $r = 0,1,2 \dots \dots m$

Since the interval $[-1, 1]$, is the domain of the Chebyshev polynomials of the first kind, we define the shifted Chebyshev polynomials of degree r as $T_r\left(\frac{2}{b}x - 1\right)$ in the interval $[0, b]$.

We suppose that the kernel functions and solutions of system (2.2) can be expressed as a truncated Chebyshev series, then equation (2. 8) can be written in the matrix form:

$$f_i(x) = T(x)A_i, \qquad i = 1, \dots, n \qquad (2.9)$$

where

$$T(x) = [T_0(x) \quad T_1(x) \quad \dots \quad \dots \quad T_m(x)]$$

$$A_i = [a_{i0} \quad a_{i1} \quad \cdots \quad \cdots \quad a_{im}]^T$$

Hence, the matrix $F(x)$ defined as a column matrix of unknown functions can be expressed by

$$F(x) = A\mathcal{T}(x) \qquad (2.10)$$

so that

$$\mathcal{T}(x) = \begin{bmatrix} T(x) & 0 & \dots & 0 \\ 0 & T(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T(x) \end{bmatrix}_{n\times n} , \quad A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}_{n\times 1}$$

Similarly, kernel functions $k_{ij}(x, t)$ can be expressed as a truncated Chebyshev series

$$k_{ij}(x_s, t) = \sum_{r=0}^{m} k_r^{ij}(x_s)\, T_r(t) \tag{2.11}$$

where x_s are the Chebyshev collocation points defined by

$$x_s = \cos\left(\frac{s\pi}{m}\right), \qquad s = 0,1,\dots,m$$

and the Chebyshev coefficients $k_r^{ij}(x_s)$ are determined by means of the relation

$$k_r^{ij}(x_s) = \frac{2}{n}\sum_{r=0}^{m} k_{ij}(x_s,t_r)T_r(t_r)\,, \qquad t_r = \cos\left(\frac{r\pi}{m}\right)$$

$$K(x_s) = K\,\mathcal{T}(t_r)^{-1}$$

then the matrix representation of $k_{ij}(x_s,t)$ in equation (2. 11) becomes

$$k_{ij}(x_s,t) = k_{ij}(x_s)T(t)^T$$

where

$$k_{ij}(x_s) = \left[\frac{1}{2}k_0^{ij}(x_s) \quad k_1^{ij}(x_s) \quad k_2^{ij}(x_s) \quad \dots \quad k_{m-1}^{ij}(x_s) \quad \frac{1}{2}k_m^{ij}(x_s)\right]_{1\times(m+1)}$$

Then we define the integral part of system (2. 2) by

$$D(x_s) = \int_{-1}^{x_s} K(x_s)T(t)^T T(t)\,A\,\,dt$$

we get

$$C\,\mathcal{T}\,A = G + \lambda\,D \tag{2.12}$$

where

$$C = \begin{bmatrix} c(x_0) & 0 & \dots & 0 \\ 0 & c(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c(x_m) \end{bmatrix}$$

Using

$$Y(x_s) = \int_{-1}^{x_s} T(t)^T T(t)\, dt = \left[\int_{-1}^{x_s} T_i(t)T_j(t)\, dt\right] = [y_{ij}(x_s)], \qquad i,j = 0,1,\dots,m$$

then we have

$$d_i(x_s) = \sum_{j=1}^{n} k_{ij}(x_s)\ Y(x_s)\ A_j, \quad i = 1,2,\dots,n$$

in compact notation

$$D(x_s) = K(x_s)Y(x_s)A, \qquad s = 0,1,\dots,m$$

in which $K(x_s)$ and A are defined above, and

$$D(x_s) = \begin{bmatrix} d_1(x_s) \\ d_2(x_s) \\ \vdots \\ d_n(x_s) \end{bmatrix}_{n\times 1}, \qquad Y(x_s) = \begin{bmatrix} y(x_s) & 0 & \dots & 0 \\ 0 & y(x_s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y(x_s) \end{bmatrix}_{(m+1)\times(m+1)}$$

The matrix D can be written in the form

$$D = \bar{K}\ Y\ A \tag{2. 13}$$

where

$$\bar{K} = \begin{bmatrix} K(x_0) & 0 & \dots & 0 \\ 0 & K(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & K(x_m) \end{bmatrix}_{n(m+1)\times(m+1)^n},$$

$$Y = \begin{bmatrix} Y(x_0) & 0 \\ 0 & Y(x_0) \\ \vdots & \vdots \\ Y(x_m) & 0 \\ 0 & Y(x_m) \end{bmatrix}_{(m+1)^n\times n(m+1)}$$

Substituting the equation (2. 13) into equation (2. 12), then we get the matrix equation for the system of Volterra integral equations, namely

$$C\mathcal{T}\ A = G + \lambda\bar{K}\ Y\ A$$

or

$$G = \left(C\mathcal{T} - \lambda\bar{K}\, Y \right) A$$

Define $W = C\mathcal{T} - \lambda\bar{K}\, Y$ so we have

$$G = WA \tag{2. 14}$$

where W is a square matrix with dimension $n(m+1)$.The unknown Chebyshev coefficient matrix A is simply computed from the linear algebraic system (2. 14), to get the approximate solution of the system of Volterra integral equations.

Error and convergence:

Theorem 2.2: [16]

Suppose that $f_{im}(x)$,$i = 1,2, \dots, n$ are the Chebyshev polynomial of degree m such that their coefficients have been produced by solving the generalized linear system (2.14), then the given polynomials converge to the exact solution of the system of Volterra integral equations, when $m \to +\infty$.

Proof:

The error of the approximation method can be written as

$$e_m(x) = \sum_{i=1}^{n} e_{im}(x)$$

where

$$e_{im} = \sum_{i=1}^{n} \left(f_i(x) - f_{im}(x) \right) - \sum_{i=1}^{n} \lambda_{ij} \int_a^x k_{ij}(x,t)(f_i(t) - f_{im}(t))\, dt$$

In virtue of (2. 2), the error function $e_m(x)$ must tend to zero, when

$m \to \infty$, Hence, we have:

$$\|e_m\| \le \sum_{i=1}^{n} \|e_{im}\|$$
$$\le \sum_{i=1}^{n} \left\| \left(f_i(x) - f_{im}(x)\right) \right\|$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{ij} \int_a^x \left\|k_{ij}(x,t)\right\| (\|f_i(t) - f_{im}(t)\|) ds$$

Since $\left\|k_{ij}(x,t)\right\|$ is bounded and $\lambda \neq 0$,

therefore, $\|f_i(x) - f_{im}(x)\| \to 0$ implies that $\|e_m\| \to 0$.

2.3 Bernstein's Approximation Method

2.3.1 Bernstein's Polynomials [9] [35]

The $(n+1)$ Bernstein basis polynomials of degree n are defined as

$$p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \qquad i = 0,1,2, \dots n$$

The basis polynomials $p_{n,i}$ have several properties:

1. Non-negativity $p_{n,i}(x) \ge 0, \quad 0 \le x \le 1, \quad i = 0,1,2, \dots . n$
2. Partition of unity $\sum_{i=0}^{n} p_{n,i}(x) = 1$
3. Symmetry $p_{n,i}(x) = p_{n,n-i}(1-x)$

Theorem 2.3: [11]

For all functions $f \in C[0,1]$, the sequence $\{B_n(f); n = 1,2,3, \dots\}$ converges uniformly to f, where $B_n(f)$ converges to a function $f : [0,1] \to \mathcal{R}$ is defined by

$$B_n\big(f(x)\big) = \sum_{i=0}^{n} a_i\, p_{ni}(x) \tag{2.15}$$

It follows that, for any $f \in C[0,1]$ and for any $\epsilon > 0$, there exists n such that the inequality $\|B_n(f) - f\| < \epsilon$ holds.

2. Bernstein's Approximation Method [36] [37] [16] [38] [39]

To solve the system of Volterra integral equations (2. 2) using Bernstein's approximation method, we introduce $f_{im}(x)$ defined as

$$f_{im}(x) = \sum_{r=0}^{m} a_{ir} P_{mr}(x), \qquad i = 1,2, \dots, n$$

which are the Bernstein's expansions of degree m for the unknown functions $f_i(x)$ for $i = 1,2, \dots, n$. After substituting these polynomials into system (2.2), we get

$$\begin{cases} \sum_{j=1}^{n} c_{1j}(x) \sum_{r=0}^{m} a_{1r} p_{mr}(x) = g_1(x) + \lambda \sum_{r=0}^{m} \int_a^x k_{1,j}(x,t) p_{mr}(t)\, dt \\ \vdots \\ \sum_{j=1}^{n} c_{ij}(x) \sum_{r=0}^{m} a_{ir}\, p_{mr}(x) = g_i(x) + \lambda \sum_{r=0}^{m} \int_a^x k_{i,j}(x,t) p_{mr}(t)\, dt \\ \vdots \\ \sum_{j=1}^{n} c_{mj}(x) \sum_{r=0}^{m} a_{nr}\, p_{mr}(x) = g_n(x) + \lambda \sum_{r=0}^{m} \int_a^x k_{n,j}(x,t) p_{mr}(t)\, dt \end{cases}$$

By replacing the variable x with $x_r = a + \frac{r(b-a)}{m}$ $for\ r = 0,1, \dots, m$, we get the generalized linear system

$$\sum_{i=1}^{n} \sum_{r=0}^{m} Y_{qr}^{ij}\, a_{ir} = g_i(x_r) + \lambda \sum_{i=1}^{n} \sum_{r=0}^{m} T_{qr}^{i,j}\ a_{ir}\ , q = 0,1, \dots, m$$

where

$$T_{qr}^{i,j} = \int_a^{x_q} k_{i,j}(x_q, t)\, p_{r,m}(t)\, dt, \qquad Y_{qr}^{ij} = c_{ij}(x_q)\, p_{r,m}(x_q)$$

It can be summarized in a matrix form as follows:

$$ZA = G + SA \qquad (2.\ 16)$$

where

$$A = [a_{ir}] = [a_{10} \quad \dots \quad a_{1m} \quad \dots \quad a_{n0} \quad \dots \quad a_{nm}\ ]^T$$

$$G = [g_i(x_r)] = [g_1(x_0) \quad \dots \quad g_1(x_m) \quad \dots \quad g_n(x_0) \quad \dots \quad g_n(x_m)]^T$$

$Z = \begin{bmatrix} z^{11} & \dots & z^{1n} \\ \vdots & \ddots & \vdots \\ z^{n1} & \dots & z^{nn} \end{bmatrix}$, $S = \begin{bmatrix} s^{11} & \dots & s^{1n} \\ \vdots & \ddots & \vdots \\ s^{n1} & \dots & s^{nn} \end{bmatrix}$, and the matrices z^{ij} , s^{ij} for $i,j = 1,2,\dots,n$ are given respectively as follows

$$z^{ij} = \begin{bmatrix} z_{00}^{ij} & \dots & z_{0m}^{ij} \\ \vdots & \ddots & \vdots \\ z_{m0}^{ij} & \dots & z_{mm}^{ij} \end{bmatrix}, \quad \text{with } z_{qr}^{ij} = Y_{qr}^{ij}$$

$$s^{ij} = \begin{bmatrix} s_{00}^{ij} & \dots & s_{0m}^{ij} \\ \vdots & \ddots & \vdots \\ s_{m0}^{ij} & \dots & s_{mm}^{ij} \end{bmatrix}, \quad \text{with } s_{qr}^{ij} = T_{qr}^{ij}$$

system (2. 16) can be written as

$$G = (Z - S)A \tag{2. 17}$$

Finally, by solving the linear algebraic system (2. 17), we get the approximate solution of the system of Volterra integral equations (2. 2).

Error and Convergence:

Theorem 2.4: [16]

Suppose that $f_{im}(x)$,$i = 1,2,\dots,n$ are the Bernstein's expansions of degree m such that their coefficients have been produced by solving the generalized linear system (2. 17), then the given polynomials converge to the exact solution of the Volterra integral equations, when $m \to +\infty$.

Proof:

The error concerning the approximation method can be written as

$$e_m(x) = \sum_{i=1}^{n} e_{im}(x)$$

where

$$e_{im} = \sum_{i=1}^{n}\big(f_i(x) - f_{im}(x)\big) - \sum_{i=1}^{n} \lambda_{ij} \int_a^x k_{ij}(x,t)(\, f_i(t) - f_{im}(t))\; dt$$

In virtue of (2. 4), the error function $e_m(x)$ must tend to zero, when

$m \to \infty$, Hence, we have :

$$\begin{aligned} \|e_m\| &\le \sum_{i=1}^{n} \|e_{im}\| \\ &\le \sum_{i=1}^{n} \big\|\big(f_i(x) - f_{im}(x)\big)\big\| \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{ij} \int_a^x \big\|k_{ij}(x,t)\big\|(\|f_i(t) - f_{im}(t)\|)ds \end{aligned}$$

Since $\big\|k_{ij}(x,t)\big\|$ is bounded and, $\lambda \neq 0$

therefore, $\|f_i(x) - f_{im}(x)\| \to 0$ implies that $\|e_m\| \to 0$.

Chapter Three

Numerical Examples and results

In this chapter, some numerical examples are presented to show the validity of the proposed numerical methods presented in chapter 2. In addition, the numerical results are compared with the exact solution.

3.1 Example:

Consider the system of Volterra integral equations:

$$(2x^2+3)f_1(x) = g_1(x) + \int_0^x (x^2-2t)f_1(t) + (t^2-x)f_2(t) + 2tf_3(t)\ dt$$

$$(1-3x^2)f_2(x) = g_2(x) + \int_0^x t(x+1)f_1(t) + tx\ (x^2+1)f_2(t) + (2t^2+x^3)f_3(t)dt$$

$$(3x^2+6)f_3(x) = g_3(x) + \int_0^x (t-x)f_1(t) + (t^2-x^3)f_2(t) + (2xt+t^2)f_3(t)dt$$

(3.1)

where

$$g_1(x) = \frac{-1}{15}(6x^5+35x^4-95x^3-270x^2-225x-360)$$

$$g_2(x) = \frac{-1}{30}(15x^7+10x^6-33x^5+275x^4+115x^3-390x^2+150)$$

$$g_3(x) = \frac{1}{60}(40x^6-66x^5-175x^4+250x^3+780x^2+360x+360)$$

$x \in [0,2]$, the exact solution of system (3. 1) is $f_1(x) = 5x+8$, $f_2(x) = 2x^2-5$,

and $f_3(x) = x^2+x+1$.

We implement the aforementioned numerical methods for solving system (3.1).

3.1.1 Galerkin method with Laguerre polynomials:

Take $m = 3$,$n = 3$ then we have

$$f_i(x) = L(x)A_i \quad , \qquad i = 1,2,3$$

where

$$L(x) = [L_0(x) \quad L_1(x) \quad L_2(x) \quad L_3(x)]$$

$$A_i = [a_{i0} \quad a_{i1} \quad a_{i2} \quad a_{i3}]^T$$

and

$$F(x) = A\mathcal{L}(x)$$

such that

$$\mathcal{L}(x) = \begin{bmatrix} L(x) & 0 & 0 \\ 0 & L(x) & 0 \\ 0 & 0 & L(x) \end{bmatrix}_{3\times 3} \quad , \quad A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}_{3\times 1}$$

and the integral part

$$A\int_0^x K(x,t)\mathcal{L}(t)dt$$

so

$$K(x,t) = \begin{bmatrix} x^2 - 2t & t^2 - x & 2t \\ t(x+1) & tx(x^2+1) & 2t^2 + x^3 \\ t - x & t^2 - x^3 & 2tx + t^2 \end{bmatrix}$$

then we have

$$G(x) = C(x)A\mathcal{L}(x) - \lambda A\int_0^x K(x,t)\mathcal{L}(t)dt \qquad (3.\ 2)$$

$$C(x) = \begin{bmatrix} 2x^2 + 3 & 0 & 0 \\ 0 & 1 - 3x^2 & 0 \\ 0 & 0 & 3x^2 + 6 \end{bmatrix}$$

multiplying both sides by $\mathcal{L}(x)$ and integrating the resulting equation (3. 2) with respect to x over the interval $[0, 2]$, we get

$$\int_0^2 G(x)^T \mathcal{L}(x)\,dx = A\int_0^2 \big(C(x)\mathcal{L}(x)\big)^T \mathcal{L}(x)dx - \lambda A \int_0^2 \left(\int_0^x K(x,t)\mathcal{L}(t)\,dt\right)^T \mathcal{L}(x)dx$$

then we get

$$AK_{rs} = G_s \tag{3. 3}$$

where

$$A = \begin{bmatrix} a_{10} & \cdots & a_{13} \\ \vdots & \ddots & \vdots \\ a_{30} & \cdots & a_{33} \end{bmatrix}_{3\times 4} \quad , G_s = \int_0^2 G(x)^T \mathcal{L}(x)\,dx$$

$$G(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{bmatrix}$$

$$K_{rs} = \int_0^2 \big(C(x)\mathcal{L}(x)\big)^T \mathcal{L}(x)dx - \lambda \int_0^2 \left(\int_0^x K(x,t)\mathcal{L}(t)dt\right)^T \mathcal{L}(x)dx$$

By solving the linear algebraic system (3. 3), we have

$$F = \begin{bmatrix} 5x+8 \\ 2x^2-5 \\ x^2+x+1 \end{bmatrix}$$

Table (3. 1) contains both the exact and the numerical solutions of system (3. 1).

Table 3.1

A comparison between the exact and the numerical solutions using the Galerkin method with Laguerre polynomials for system (3.1)

x	Exact solution $f_1(x)$	Exact solution $f_2(x)$	Exact solution $f_3(x)$	approximate solution $f_1(x)$	approximate solution $f_2(x)$	approximate solution $f_3(x)$	Absolute error $e_1(x)$	Absolute error $e_2(x)$	Absolute error $e_3(x)$
0	8	-5.0000	1.0000	8	-5.0000	1.0000	0.0000	0.0000	0.0000
0.2	9	-4.9200	1.2400	9	-4.9200	1.2400	0.0000	0.0000	0.0000
0.4	10	-4.6800	1.5600	10	-4.6800	1.5600	0.0000	0.0000	0.0000
0.6	11	-4.2800	1.9600	11	-4.2800	1.9600	0.0000	0.0000	0.0000
0.8	12	-3.7200	2.4400	12	-3.7200	2.4400	0.0000	0.0000	0.0000
1	13	-3.0000	3.0000	13	-3.0000	3.0000	0.0000	0.0000	0.0000
1.2	14	-2.1200	3.6400	14	-2.1200	3.6400	0.0000	0.0000	0.0000
1.4	15	-1.0800	4.3600	15	-1.0800	4.3600	0.0000	0.0000	0.0000
1.6	16	0.1200	5.1600	16	0.1200	5.1600	0.0000	0.0000	0.0000
1.8	17	1.4800	6.0400	17	1.4800	6.0400	0.0000	0.0000	0.0000
2	18	3.0000	7.0000	18	3.0000	7.0000	0.0000	0.0000	0.0000

Figure (3.1.a) compares the exact solution $\boldsymbol{f_1(x)}$ and the approximate solution with $m = 3$.
Figure (3.1.b) compares the exact solution $\boldsymbol{f_2(x)}$ and the approximate solution with $m = 3$.
Figure (3.1.c) compares the exact solution $\boldsymbol{f_3(x)}$ and the approximate solution with $m = 3$.

As shown in appendix A page 53-54

3.1.2 Chebyshev collocation method:

Take $m = 5, n = 3$, then we have

$$f_i(x) = \sum_{r=0}^{5} a_{ir} T_r(x) \quad , \quad i = 1, 2, 3$$

$$f_i(x) = T(x) A_i$$

were

$$T(x) = [1 \quad x \quad 2x^2-1 \quad 4x^3-3x \quad 8x^4-8x^2+1 \quad 16x^5-20x^3+5x]$$

$$A_i^T = [a_{i0} \quad a_{i1} \quad a_{i2} \quad a_{i3} \quad a_{i4} \quad a_{i5}]$$

then

$$F(x) = \mathcal{T}(x)A$$

$$\mathcal{T}(x) = \begin{bmatrix} T(x) & 0 & 0 \\ 0 & T(x) & 0 \\ 0 & 0 & T(x) \end{bmatrix}, \qquad A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

and

$$\bar{K} = \begin{bmatrix} K(x_0) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & K(x_5) \end{bmatrix}, Y = \begin{bmatrix} Y(x_1) & 0 & 0 \\ 0 & Y(x_1) & 0 \\ 0 & 0 & Y(x_1) \\ \vdots & \vdots & \vdots \\ 0 & 0 & Y(x_5) \end{bmatrix}, \; G = \begin{bmatrix} g_{11}(x_0) \\ g_{21}(x_0) \\ g_{31}(x_0) \\ \vdots \\ g_{11}(x_5) \\ g_{21}(x_5) \\ g_{31}(x_5) \end{bmatrix}$$

$$C = \begin{bmatrix} c(x_0) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & c(x_5) \end{bmatrix}$$

Hence, solving the liner algebraic system $WA = G$ we have

$$F = \begin{bmatrix} 0.0352x^5 - 0.1696x^4 + 0.2650x^3 - 0.1504x^2 + 5.0229x + 8 \\ -0.1463x^5 + 0.7065x^4 - 1.1203x^3 + 2.6520x^2 - 0.1003x - 5 \\ 0.5769x^5 - 2.4425x^4 + 3.3321x^3 - 0.6230x^2 + 1.2037x + 1 \end{bmatrix}$$

Table (3.2) contains both the exact and the numerical solutions of system (3. 1).

Table 3. 2

A comparison between the exact and the numerical solutions using the Chebyshev collocation method for system (3.1)

x	Exact solution $f_1(x)$	Exact solution $f_2(x)$	Exact solution $f_3(x)$	approximate solution $f_1(x)$	approximate solution $f_2(x)$	approximate solution $f_3(x)$	Absolute error $e_1(x)$	Absolute error $e_2(x)$	Absolute error $e_3(x)$
0	8	-5.0000	1.0000	8.0000	-5.0000	1.0000	0	0	0
0.2	9	-4.9200	1.2400	9.0004	-4.9219	1.2388	0.0004	0.0019	0.0012
0.4	10	-4.6800	1.5600	9.9981	-4.6709	1.5384	0.0019	0.0091	0.0216
0.6	11	-4.2800	1.9600	10.9976	-4.2673	1.9460	0.0024	0.0127	0.0140
0.8	12	-3.7200	2.4400	11.9998	-3.7151	2.4589	0.0002	0.0049	0.0189
1	13	-3.0000	3.0000	13.0031	-3.0084	3.0472	0.0031	0.0084	0.0472
1.2	14	-2.1200	3.6400	14.0047	-2.1364	3.6759	0.0047	0.0164	0.0359
1.4	15	-1.0800	4.3600	15.0022	-1.0893	4.3270	0.0022	0.0093	0.0330
1.6	16	0.1200	5.1600	15.9947	0.1359	5.0214	0.0053	0.0159	0.1386
1.8	17	1.4800	6.0400	16.9841	1.5305	5.8415	0.0159	0.0505	0.1985
2	18	3.0000	7.0000	17.9770	3.0674	6.9530	0.0230	0.0674	0.0470

Figure (3.2.a) compares the exact solution $\boldsymbol{f_1(x)}$ and the approximate solution with $m = 5$.
Figure (3.2.b) compares the exact solution $\boldsymbol{f_2(x)}$ and the approximate solution with $m = 5$.
Figure (3.2.c) compares the exact solution $\boldsymbol{f_3(x)}$ and the approximate solution with $m = 5$.
As shown in appendix A page 54-55

3.1.3 Bernstein's approximation method:

take $m = 4, n = 3$ then we have

$$f_{i4}(x) = \sum_{r=0}^{4} a_{ir} P_{4r}(x)\,, where\ i = 1,2,3$$

$$x_q = 0 + \frac{q(2-0)}{4} = 0.5q \quad ,q = 0,1,2,3,4$$

$$T_{qr}^{ij} = \int_0^{x_q} k_{ij}(x_q,t)\ p_{r4}(t)\ dt\,, \qquad Y_{qr}^{ij} = c_{ij}(x_q)\ p_{r4}(x_q)$$

$$G = [g_i(x_r)] = [g_1(x_0) \quad \dots \quad g_2(x_0) \quad \dots \quad g_3(x_5)]^T$$

$$Z = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}, \qquad S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix},$$

and the matrix z^{ij} , s^{ij} for $i,j = 1,2,3$ are defined with following elements:

$$z^{ij} = \begin{bmatrix} z_{00}^{ij} & \cdots & z_{04}^{ij} \\ \vdots & \ddots & \vdots \\ z_{40}^{ij} & \cdots & z_{44}^{ij} \end{bmatrix} \quad , \quad where \quad z_{qr}^{ij} = Y_{qr}^{ij}$$

$$s^{ij} = \begin{bmatrix} s_{00}^{ij} & \cdots & s_{04}^{ij} \\ \vdots & \ddots & \vdots \\ s_{40}^{ij} & \cdots & s_{44}^{ij} \end{bmatrix} \quad , \quad where \quad s_{qr}^{ij} = T_{qr}^{ij}$$

Hence, solving the liner algebraic system $ZA = G + SA$ we have

$$F = \begin{bmatrix} -0.0003x^4 + \ 0.0012x^3 - \ 0.0018x^2 + 5.0008x + 8 \\ -\ 0.0063x^4 + \ 0.0276x^3 + 1.9620x^2 + 0.0164x - 5 \\ 0.0001x^4 - \ 0.0004x^3 + 1.0002x^2 + x + 1 \end{bmatrix}$$

Table (3.3) contains both the exact and the numerical solutions of system (3. 1).

Table 3. 3

A comparison between the exact and the numerical solutions using the Bernstein's approximation method for system (3.1)

x	**Exact solution $f_1(x)$**	**Exact solution $f_2(x)$**	**Exact solution $f_3(x)$**	**approximate solution $f_1(x)$**	**approximate solution $f_2(x)$**	**approximate solution $f_3(x)$**	**Absolute error $e_1(x)$**	**Absolute error $e_2(x)$**	**Absolute error $e_3(x)$**
0	8	-5.0000	1.0000	8.0000	-5.0000	1.0000	0	0	0
0.2	9	-4.9200	1.2400	9.0001	-4.9180	1.2400	0.0001	0.0020	0.0000
0.4	10	-4.6800	1.5600	10.0001	-4.6779	1.5600	0.0001	0.0021	0.0000
0.6	11	-4.2800	1.9600	11.0001	-4.2787	1.9600	0.0001	0.0013	0.0000
0.8	12	-3.7200	2.4400	12.0000	-3.7196	2.4400	0.0000	0.0004	0.0000
1	13	-3.0000	3.0000	12.9999	-3.0003	2.9999	0.0001	0.0003	0.0001
1.2	14	-2.1200	3.6400	13.9998	-2.1204	3.6398	0.0002	0.0004	0.0002
1.4	15	-1.0800	4.3600	14.9997	-1.0800	4.3597	0.0003	0.0000	0.0003
1.6	16	0.1200	5.1600	15.9996	0.1207	5.1595	0.0004	0.0007	0.0005
1.8	17	1.4800	6.0400	16.9995	1.4812	6.0394	0.0005	0.0012	0.0006
2	18	3.0000	7.0000	17.9992	3.0008	6.9992	0.0008	0.0008	0.0008

Figure (3.3.a) compares the exact solution $f_1(x)$ and the approximate solution with $m = 4$.

Figure (3.3.b) compares the exact solution $f_2(x)$ and the approximate solution with $m = 4$.

Figure (3.3.c) compares the exact solution $f_3(x)$ and the approximate solution with $m = 4$.

Figure 3.3.a

A comparison between the exact solutions f_1 and the numerical solutions using Bernstein's approximation method for system (3.1)

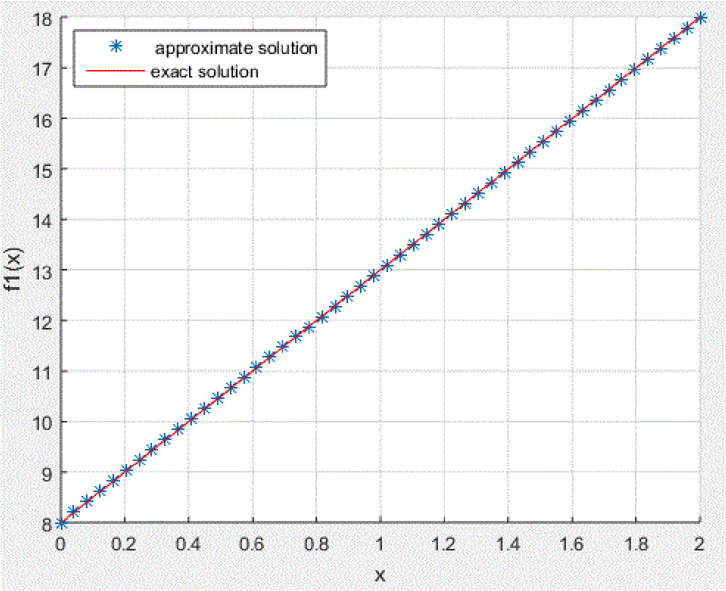


Figure 3.3.b

A comparison between the exact solutions f_2 and the numerical solutions using Bernstein's approximation method for system (3.1)

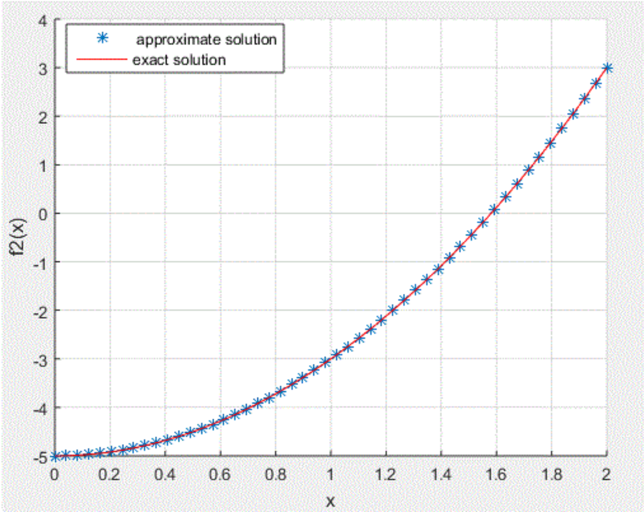
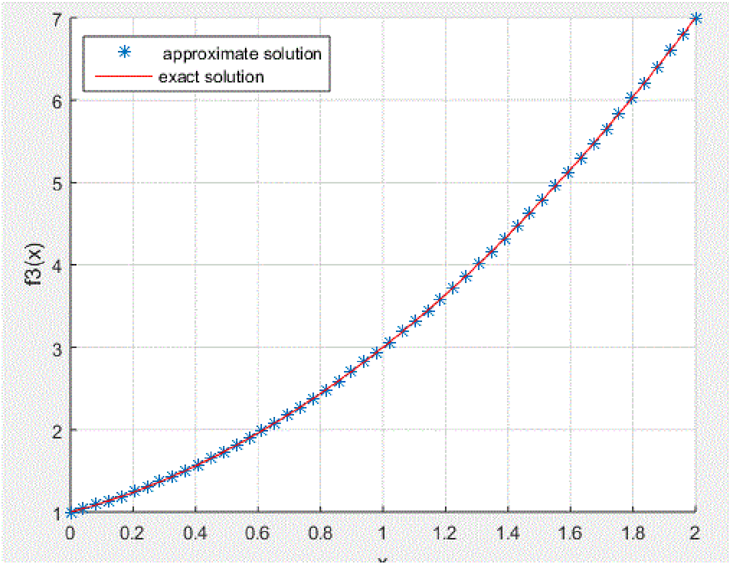


Figure 3.3.c

a comparison between the exact solutions $\boldsymbol{f_3}$ and the numerical solutions using Bernstein's approximation method for system (3.1)



3.2 Example:

In this example, we have the following system:

$$f_1(x) = g_1(x) + \int_{-1}^{x} \sin(x-t)\, f_1(t) + e^{x-t} f_2(t)\, dt$$

$$f_2(x) = g_2(x) + \int_{-1}^{x} \cos(x-t)\, f_1(t) + 3tx^2 f_2(t)\, dt \qquad (3.4)$$

where

$$g_1(x) = \frac{5}{3} - e^{x+1} + \frac{23}{3}x + x^2 + 3\cos(x+1) - \sin(x+1)$$

$$g_2(x) = \frac{1}{12}(48 - 4x - 11x^2 + 4x^5 - 9x^6 - 12\cos(x+1) - 36\sin(x+1))$$

The exact solution of system (3. 4) is $f_1(x) = x^3 + 2x$, and $f_2(x) = x^2 - (x/3)$

We implement the aforementioned numerical methods for solving system (3. 4).

3.2.1 Galerkin method with Laguerre polynomials:

Take $m = 5$, $n = 2$ then we have

$$f_i(x) = L(x)A_i \quad , \qquad i = 1,2$$

Where

$$L(x) = [L_0(x) \quad L_1(x) \quad L_2(x) \quad L_3(x) \quad L_4(x) \quad L_5(x)]$$

$$A_i = [a_{i0} \quad a_{i1} \quad a_{i2} \quad a_{i3} \quad a_{i4} \quad a_{i5}]^T$$

and

$$F(x) = A\mathcal{L}(x)$$

such that

$$\mathcal{L}(x) = \begin{bmatrix} L(x) & 0 \\ 0 & L(x) \end{bmatrix}_{2\times 2} \quad , \quad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}_{2\times 1}$$

and the integral part

$$A\int_{-1}^{x} K(x,t)\mathcal{L}(t)dt$$

where

$$K(x,t) = \begin{bmatrix} \sin(x-t) & e^{x-t} \\ \cos(x-t) & 3tx^2 \end{bmatrix}$$

then we have

$$G(x) = C(x)A\mathcal{L}(x) - \lambda A\int_{-1}^{x} K(x,t)\mathcal{L}(t)dt$$

$$C(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

After multiplying both sides by $\mathcal{L}(x)$ and integrating the resulting equation with respect to x over the interval $[-1,1]$, We get

$$\int_{-1}^{1} G(x)^T \mathcal{L}(x)dx = A\int_{-1}^{1} \left(C(x)\mathcal{L}(x)\right)^T \mathcal{L}(x)dx - \lambda A\int_{-1}^{1} \left(\int_{-1}^{x} K(x,t)\mathcal{L}(t)dt\right)^T \mathcal{L}(x)dx$$

then we get

$$AK_{rs} = G_s$$

where

$$A = \begin{bmatrix} a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix}_{2\times 6}$$

$$G_s = \int_{-1}^{1} G(x)^T \mathcal{L}(x)\,dx \quad , \quad G(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}$$

$$K_{rs} = \int_{-1}^{1} \left(C(x)\mathcal{L}(x)\right)^T \mathcal{L}(x)dx - \lambda \int_{-1}^{1} \left(\int_{-1}^{x} K(x,t)\mathcal{L}(t)dt\right)^T \mathcal{L}(x)dx$$

By solving the linear algebraic system $AK_{rs} = G_s$, we have

$$F = \begin{bmatrix} 0.0128x^5 - 0.2489x^4 + 0.9712x^3 + 0.2112x^2 + 2.0129x - 0.0198 \\ 0.0000x^5 - 0.0234x^4 + 0.0099x^3 + 1.0220x^2 - 0.3387x - 0.0021 \end{bmatrix}$$

Table (3.4) contains both the exact and the numerical solutions of system (3. 4).

Table 3.4

A comparison between the exact and the numerical solutions using the Galerkin method with Laguerre polynomials for system (3.4)

x	**Exact solution $f_1(x)$**	**Exact solution $f_2(x)$**	**approximate solution $f_1(x)$**	**approximate solution $f_2(x)$**	**Absolute error $e_1(x)$**	**Absolute error $e_2(x)$**
1	3	0.6667	2.9394	0.6677	0.0606	0.0010
0.75	1.9219	0.3125	1.9427	0.3155	0.0208	0.0030
0.5	1.1250	0.0833	1.1457	0.0838	0.0207	0.0005
0.25	0.5156	-0.0208	0.5108	-0.0228	0.0048	0.0020
0	0	0	-0.0198	-0.0021	0.0198	0.0021
-0.25	-0.5156	0.1458	-0.5260	0.1462	0.0104	0.0004
-0.5	-1.1250	0.4167	-1.1108	0.4201	0.0142	0.0034
-0.75	-1.9219	0.8125	-1.9022	0.8152	0.0197	0.0027
-1	-3	1.3333	-3.0544	1.3253	0.0544	0.0080

Figure (3.4.a) compares the exact solution $\boldsymbol{f_1(x)}$ and the approximate solution with $m = 5$.
Figure (3.4.b) compares the exact solution $\boldsymbol{f_2(x)}$ and the approximate solution with $m = 5$.

As shown in appendix A page 56

3.2.2 Chebyshev collocation method:

Take $m = 4, n = 2$, then we have

$$f_i(x) = \sum_{r=0}^{4} a_{ir} T_r(x) \quad , \quad i = 1,2$$

$$f_i(x) = T(x) A_i$$

where

$$T(x) = [1 \quad x \quad 2x^2 - 1 \quad 4x^3 - 3x \quad 8x^4 - 8x^2 + 1]$$

$$A_i^T = [a_{i0} \quad a_{i1} \quad a_{i2} \quad a_{i3} \quad a_{i4}]$$

then

$$F(x) = \mathcal{T}(x) A$$

$$\mathcal{T}(x) = \begin{bmatrix} T(x) & 0 \\ 0 & T(x) \end{bmatrix} \qquad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

and

$$\bar{K} = \begin{bmatrix} K(x_0) & 0 & \dots & 0 \\ 0 & K(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & K(x_4) \end{bmatrix}, \quad Y = \begin{bmatrix} Y(x_0) & 0 \\ 0 & Y(x_0) \\ Y(x_1) & 0 \\ 0 & Y(x_1) \\ \vdots & \vdots \\ 0 & Y(x_4) \end{bmatrix}, G = \begin{bmatrix} g_{11}(x_0) \\ g_{21}(x_0) \\ \vdots \\ g_{11}(x_4) \\ g_{21}(x_4) \end{bmatrix}$$

Hence, solving the liner algebraic system $WA = G$ we have

$$F \cong \begin{bmatrix} -0.0002x^4 + 0.9995x^3 + 0.0001x^2 + 2.0005x + 0.0002 \\ 0.0005x^4 + 0.0002x^3 + 0.4995x^2 - 0.3333x + 0.0001 \end{bmatrix}$$

and if we take $m = 7$, we will get a more accurate answer

$$F \cong \begin{bmatrix} -0.0072x^7 - 0.0006x^6 + 0.0095x^5 - 0.0005x^4 + 0.9956x^3 - 0.0004x^2 + 1.9987x - 0.0018 \\ -0.0013x^7 - 0.0051x^6 - 0.0008x^5 + 0.0038x^4 - 0.0007x^3 + 0.9982x^2 - 0.3344x - 0.0008 \end{bmatrix}$$

Table (3.5) contains both the exact and the numerical solutions of system (3. 4) with $m = 4$.

As shown in appendix B page 61

Figure (3.5.a) compares the exact solution $\boldsymbol{f_1(x)}$ and the approximate solution with $m = 4$.
Figure (3.5.b) compares the exact solution $\boldsymbol{f_2(x)}$ and the approximate solution with $m = 4$.

Figure 3.5.a

A comparison between the exact solutions $\boldsymbol{f_1}$ and the numerical solutions using Chebyshev collocation method with $\boldsymbol{m = 4}$ for system (3.4)

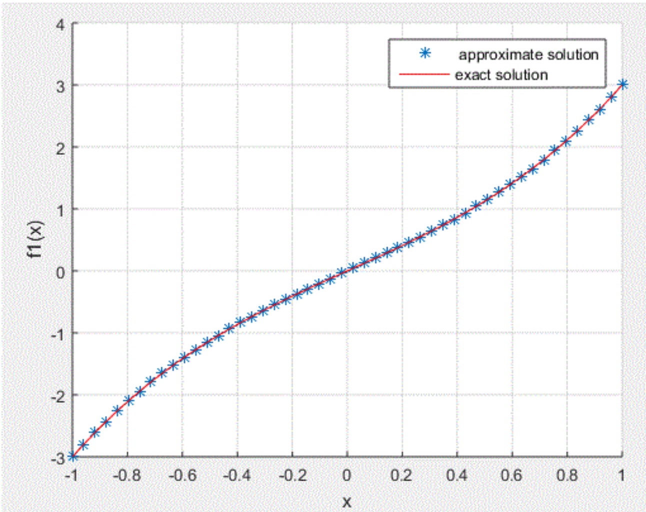


Figure (3.5.b

A comparison between the exact solutions $\boldsymbol{f_2}$ and the numerical solutions using Chebyshev collocation method with $\boldsymbol{m = 4}$ for system (3.4)

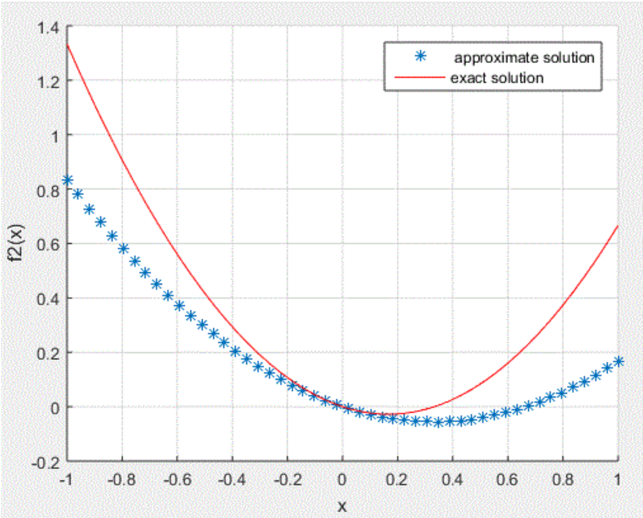


Table (3.6) contains both the exact and the numerical solutions of system (3. 4) with $m = 7$

Table 3.6

A comparison between the exact and the numerical solutions using the Chebyshev collocation method with $\boldsymbol{m = 7}$ for system (3.4)

x	Exact solution $f_1(x)$	Exact solution $f_2(x)$	approximate solution $f_1(x)$	approximate solution $f_2(x)$	Absolute error $e_1(x)$	Absolute error $e_2(x)$
1	3	0.6667	2.9933	0.6589	0.0067	0.0078
0.75	1.9219	0.3125	1.9180	0.3095	0.0038	0.0030
0.5	1.1250	0.0833	1.1221	0.0816	0.0029	0.0017
0.25	0.5156	-0.0208	0.5134	-0.0220	0.0022	0.0012
0	0	0	-0.0018	-0.0008	0.0018	0.0008
-0.25	-0.5156	0.1458	-0.5171	0.1452	0.0014	0.0006
-0.5	-1.1250	0.4167	-1.1260	0.4162	0.0010	0.0004
-0.75	-1.9219	0.8125	-1.9226	0.8124	0.0008	0.0001
-1	-3	1.3333	-2.9999	1.3333	0.0001	0.0000

Figure (3.6.a) compares the exact solution $\boldsymbol{f_1(x)}$ and the approximate solution with $m = 7$.

Figure (3.6.b) compares the exact solution $f_2(x)$ and the approximate solution with $m = 7$

Figure 3.6.a

A comparison between the exact solutions f_1 *and the numerical solutions using Chebyshev collocation method with* $m = 7$ *for system (3.4)*

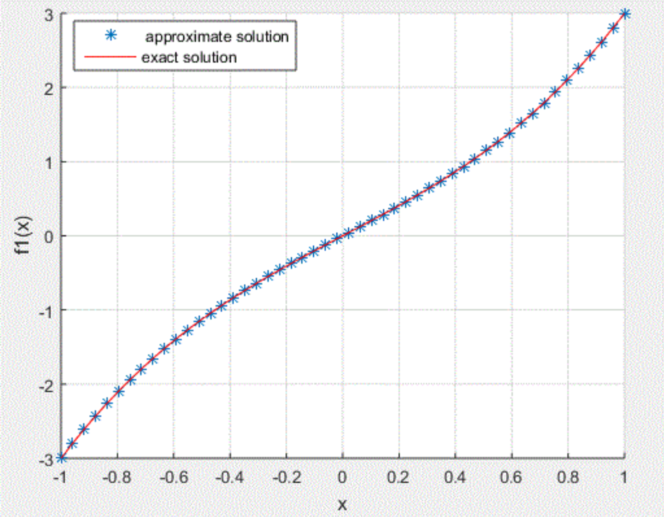
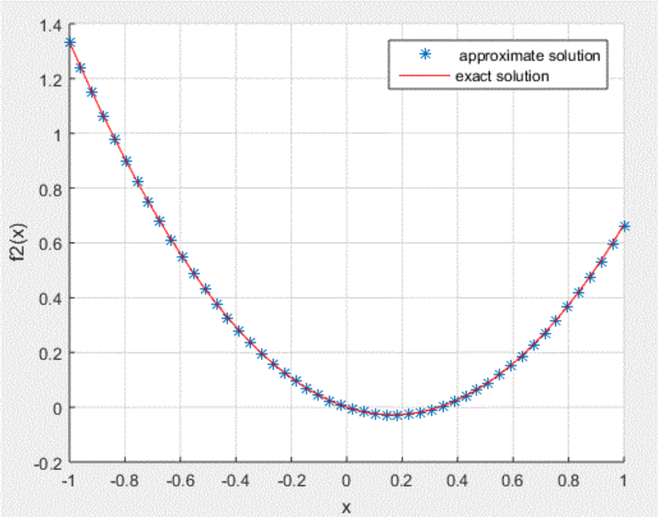


Figure 3.6.b

A comparison between the exact solutions f_2 *and the numerical solutions using Chebyshev collocation method with* $m = 7$ *for system (3.4)*



3.2.3 Bernstein's approximation method:

take $m = 4, n = 2$ then we have

$$f_{i4}(x) = \sum_{r=0}^{4} a_{ir} P_{4r}(x)\,, where\ i = 1,2$$

$$T_{q,r}^{i,j} = \int_{-1}^{x_q} k_{ij}(x_q, t)\ p_{r4}(t)\ dt$$

$$G = [g_i(x_r)] = [g_1(x_0), g_1(x_1), \dots, g_1(x_4) \quad g_2(x_0) \quad \dots \quad g_2(x_4)]^T$$

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} s^{ij} = \begin{bmatrix} s_{00}^{ij} & \dots & s_{04}^{ij} \\ \vdots & \ddots & \vdots \\ s_{40}^{ij} & \dots & s_{44}^{ij} \end{bmatrix} \quad , \quad where \quad s_{qr}^{ij} = T_{qr}^{ij}$$

Hence, solving the liner algebraic system $A = G + SA$ we have

$$F = \begin{bmatrix} 0.0012x^4 + 1.0004x^3 - 0.0012x^2 + 2x + 0.0002 \\ -0.0002x^4 - 0.0004x^3 + 1.0014x^2 - 0.3336x \end{bmatrix}$$

Table (3.7) contains both the exact and the numerical solutions of system (3. 4).

Table 3.7

A comparison between the exact and the numerical solutions using the Bernstein's approximation method for system (3.4)

x	**Exact solution $f_1(x)$**	**Exact solution $f_2(x)$**	**approximate solution $f_1(x)$**	**approximate solution $f_2(x)$**	**Absolute error $e_1(x)$**	**Absolute error $e_2(x)$**
1	3	0.6667	3.0006	0.6672	0.0006	0.0005
0.75	1.9219	0.3125	1.9219	0.3129	0.0000	0.0004
0.5	1.1250	0.0833	1.1250	0.0835	0.0000	0.0002
0.25	0.5156	-0.0208	0.5158	-0.0208	0.0002	0.0000
0	0	0	0.0002	0	0.0002	0.0000
-0.25	-0.5156	0.1458	-0.5155	0.1460	0.0001	0.0002
-0.5	-1.1250	0.4167	-1.1251	0.4172	0.0001	0.0005
-0.75	-1.9219	0.8125	-1.9221	0.8136	0.0003	0.0011
-1	-3	1.3333	-3.0002	1.3352	0.0002	0.0019

Figure (3.7.a) compares the exact solution $\boldsymbol{f_1(x)}$ and the approximate solution with $m = 4$.
Figure (3.7.b) compares the exact solution $\boldsymbol{f_2(x)}$ and the approximate solution with $m = 4$.

As shown in appendix A page 57

3.3 Example:

In this example we consider the following system

$$(3x-8)f_1(x)+(-2x+5)f_2(x)=g_1(x)+\int_0^x (x+t)f_1(t)+xt\,f_2(t)dt$$

$$4xf_1(x)+(x-5)f_2(x)=g_2(x)+\int_0^x (2x-t)f_1(t)+(t+xt)f_2(t)dt$$

(3. 5)

where

$g_1(x)=5e^x-2x-9\sin x-x^2e^x+2xcosx-xe^x+3xsinx$

$g_2(x)=-3x-4e^x+\sin x-x^2e^x+xcosx+xe^x+4xsinx-1$

$x\in[0,2]$

The exact solution of system (3. 3) is $f_1(x)=\sin x$, and $f_2(x)=e^x$

We implement the aforementioned numerical methods for solving system (3.5).

3.3.1 Galerkin method with Laguerre polynomials:

Take $m=4$,$n=2$ then we have

$$f_i(x)=L(x)A_i \quad , \qquad i=1,2$$

where

$$L(x)=[L_0(x)\quad L_1(x)\quad L_2(x)\quad L_3(x)\quad L_4(x)]$$

$$A_i=[a_{i0}\quad a_{i1}\quad a_{i2}\quad a_{i3}\quad a_{i4}]^T$$

and

$$F(x)=A\mathcal{L}(x)$$

such that

$$\mathcal{L}(x) = \begin{bmatrix} L(x) & 0 \\ 0 & L(x) \end{bmatrix}_{2\times2} \quad , \quad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}_{2\times1}$$

and the integral part

$$A\int_0^x K(x,t)\mathcal{L}(t)dt$$

where

$$K(x,t) = \begin{bmatrix} x+t & xt \\ 2x-t & t+xt \end{bmatrix}$$

then we have

$$G(x) = C(x)A\mathcal{L}(x) - \lambda A\int_0^x K(x,t)\mathcal{L}(t)dt$$

$$C(x) = \begin{bmatrix} 3x-8 & -2x+5 \\ 4x & x-5 \end{bmatrix}$$

After multiplying both sides by $\mathcal{L}(x)$ and integrating the resulting equation with respect to x over the interval $[0,2]$, We get

$$\int_0^2 G(x)^T\mathcal{L}(x)\,dx = A\int_0^2 \left(C(x)\mathcal{L}(x)\right)^T\mathcal{L}(x)dx - \lambda A\int_0^2 \left(\int_0^x K(x,t)\mathcal{L}(t)dt\right)^T \mathcal{L}(x)dx$$

then we get

$$AK_{rs} = G_s$$

where

$$A = \begin{bmatrix} a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}_{2\times5}$$

$$G_s = \int_{-1}^{1} G(x)^T\mathcal{L}(x)\,dx \quad , \quad G(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}$$

$$K_{rs} = \int_0^2 \left(C(x)\mathcal{L}(x)\right)^T\mathcal{L}(x)dx - \lambda\int_0^2 \left(\int_0^x K(x,t)\mathcal{L}(t)dt\right)^T \mathcal{L}(x)dx$$

By solving the linear algebraic system $AK_{rs} = G_s$, we have

$$F = \begin{bmatrix} 0.0327x^4 - 0.2166x^3 + 0.0337x^2 + 0.9911x + 0.0005 \\ 0.1138x^4 + 0.0217x^3 + 0.6127x^2 + 0.9678x + 1.0022 \end{bmatrix}$$

Table (3.8) Contains both the exact and the numerical solutions of system (3. 5)

Table 3.8

A comparison between the exact and the numerical solutions using the Galerkin method with Laguerre polynomials for system (3.5)

x	**Exact solution $f_1(x)$**	**Exact solution $f_2(x)$**	**approximate solution $f_1(x)$**	**approximate solution $f_2(x)$**	**Absolute error $e_1(x)$**	**Absolute error $e_2(x)$**
0	0	1	0.0005	1.0022	0.0005	0.0022
0.2	0.1987	1.2214	0.1984	1.2206	0.0003	0.0008
0.4	0.3894	1.4918	0.3893	1.4917	0.0001	0.0002
0.6	0.5646	1.8221	0.5647	1.8229	0.0001	0.0008
0.8	0.7174	2.2255	0.7174	2.2263	0.0000	0.0007
1	0.8415	2.7183	0.8414	2.7182	0.0001	0.0001
1.2	0.9320	3.3201	0.9319	3.3193	0.0002	0.0008
1.4	0.9854	4.0552	0.9854	4.0547	0.0001	0.0005
1.6	0.9996	4.9530	0.9996	4.9539	0.0001	0.0008
1.8	0.9738	6.0496	0.9737	6.0506	0.0001	0.0009
2	0.9093	7.3891	0.9079	7.3830	0.0014	0.0061

Figure (3.8.a) compares the exact solution $\boldsymbol{f_1(x)}$ and the approximate solution with $m = 4$.
Figure (3.8.b) compares the exact solution $\boldsymbol{f_2(x)}$ and the approximate solution with $m = 4$.

As shown in appendix A page 58

3.3.2 Chebyshev collocation method:

Since the integrals are bounded in the range [0,2], then solutions can be got through the shifted Chebyshev polynomials.$s_r = 2(x_r + 1)/2$ where x_r, $r = 0,1,2,3,4$ are Chebyshev Gauss points of order m in interval $[-1,1]$.

Take $m = 2, n = 2$ then we have

$$f_i(x) = \sum_{r=0}^{2} a_{ir} T_r(x) \quad , \quad i = 1,2$$

$$f_i(x) = T(x)A_i$$

where

$$T(x) = [1 \quad x \quad 2x^2 - 1]$$

$$A_i^T = [a_{i0} \quad a_{i1} \quad a_{i2}]$$

then

$$F(x) = \mathcal{T}(x)A$$

$$\mathcal{T}(x) = \begin{bmatrix} T(x) & 0 \\ 0 & T(x) \end{bmatrix} \qquad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

and

$$\bar{K} = \begin{bmatrix} K(x_0) & 0 & 0 \\ 0 & K(x_1) & 0 \\ 0 & 0 & K(x_2) \end{bmatrix}, \quad Y = \begin{bmatrix} Y(x_0) & 0 \\ 0 & Y(x_0) \\ Y(x_1) & 0 \\ 0 & Y(x_1) \\ \vdots & \vdots \\ 0 & Y(x_2) \end{bmatrix}, G = \begin{bmatrix} g_{11}(x_0) \\ g_{21}(x_0) \\ \vdots \\ g_{11}(x_2) \\ g_{21}(x_2) \end{bmatrix}$$

Hence, solving the liner algebraic system $WA = G$ we have

$$F \cong \begin{bmatrix} -0.4306x^2 + 1.2742x - 0.0000 \\ 1.3501x^2 + 0.3759x + 1.0000 \end{bmatrix}$$

and if we take $m = 5$, we will get a more accurate answer

$$F = \begin{bmatrix} 0.0025x^5 + 0.0184x^4 - 0.1869x^3 + 0.0072x^2 + x - 0 \\ 0.0191x^5 + 0.0199x^4 + 0.1864x^3 + 0.4901x^2 + 1.0027x + 1 \end{bmatrix}$$

Table (3.9) contains both the exact and the numerical solutions of system (3. 5).
$with\ m = 2$

Table 3.9

A comparison between the exact and the numerical solutions using the Chebyshev collocation method with $m = 2$ for system (3.5).

x	Exact solution $f_1(x)$	Exact solution $f_2(x)$	approximate solution $f_1(x)$	approximate solution $f_2(x)$	Absolute error $e_1(x)$	Absolute error $e_2(x)$
0	0	1	0	1.0000	0.0000	0.0000
0.2	0.1987	1.2214	0.2376	1.1292	0.0389	0.0922
0.4	0.3894	1.4918	0.4408	1.3664	0.0514	0.1254
0.6	0.5646	1.8221	0.6095	1.7116	0.0449	0.1105
0.8	0.7174	2.2255	0.7438	2.1648	0.0264	0.0608
1	0.8415	2.7183	0.8436	2.7260	0.0021	0.0077
1.2	0.9320	3.3201	0.9090	3.3952	0.0231	0.0751
1.4	0.9854	4.0552	0.9399	4.1725	0.0455	0.1173
1.6	0.9996	4.9530	0.9364	5.0577	0.0632	0.1047
1.8	0.9738	6.0496	0.8984	6.0509	0.0754	0.0013
2	0.9093	7.3891	0.8260	7.1522	0.0833	0.2369

Figure (3.9.a) compares the exact solution $f_1(x)$ and the approximate solution with $m = 2$.
Figure (3.9.b) compares the exact solution $f_2(x)$ and the approximate solution with $m = 2$.

Figure 3.9.a

A comparison between the exact solutions f_1 and the numerical solutions using Chebyshev collocation method with $m = 2$ for system (3.5)

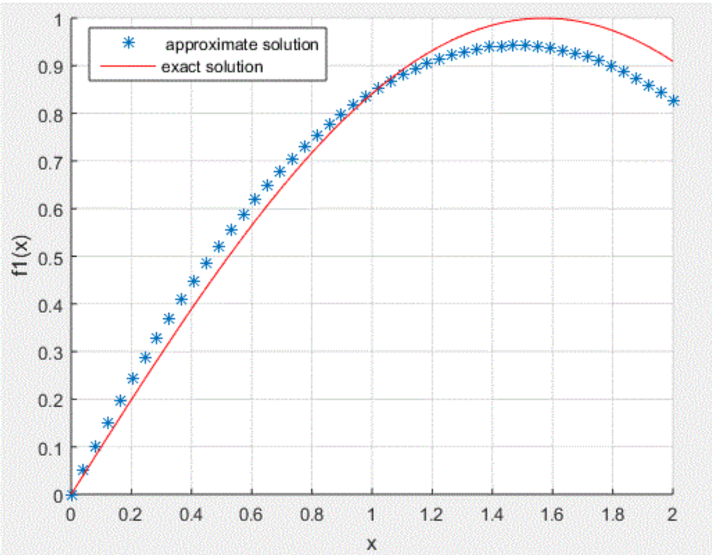


Figure 3.9.b

A comparison between the exact solutions f_2 and the numerical solutions using Chebyshev collocation method with $m = 2$ for system (3.5)

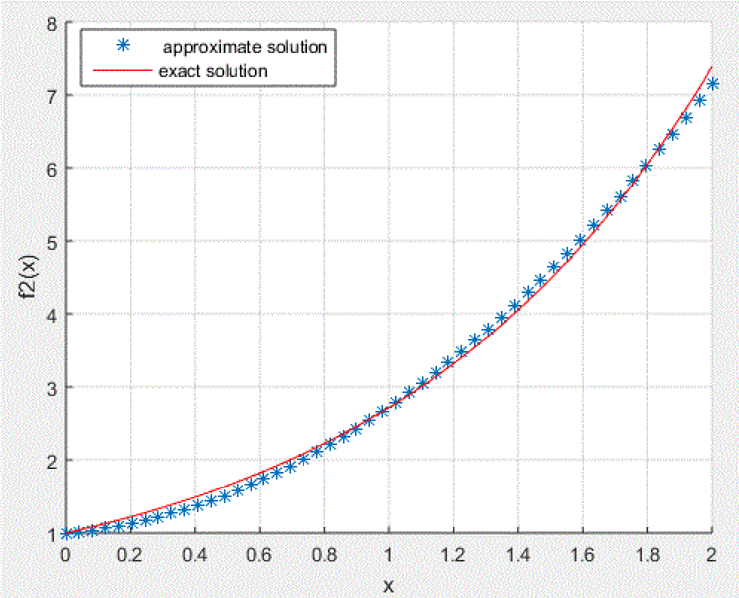


Table (3.10) contains both the exact and the numerical solutions of system (3. 5). *with* $m = 5$

Table 3.10

A comparison between the exact and the numerical solutions using the Chebyshev collocation method with $m = 5$ for system (3.5).

x	Exact solution $f_1(x)$	Exact solution $f_2(x)$	approximate solution $f_1(x)$	approximate solution $f_2(x)$	Absolute error $e_1(x)$	Absolute error $e_2(x)$
0	0	1	0	1.0000	0.0000	0.0000
0.2	0.1987	1.2214	0.1988	1.2217	0.0001	0.0003
0.4	0.3894	1.4918	0.3897	1.4921	0.0003	0.0003
0.6	0.5646	1.8221	0.5648	1.8224	0.0002	0.0003
0.8	0.7174	2.2255	0.7173	2.2257	0.0001	0.0002
1	0.8415	2.7183	0.8412	2.7182	0.0003	0.0001
1.2	0.9320	3.3201	0.9318	3.3199	0.0002	0.0002
1.4	0.9854	4.0552	0.9854	4.0550	0.0000	0.0002
1.6	0.9996	4.9530	0.9997	4.9532	0.0001	0.0002
1.8	0.9738	6.0496	0.9737	6.0497	0.0001	0.0001
2	0.9093	7.3891	0.9080	7.3866	0.0013	0.0025

Figure (3.10.a) compares the exact solution $\boldsymbol{f_1(x)}$ and the approximate solution with $m = 5$.

Figure (3.10.b) compares the exact solution $\boldsymbol{f_2(x)}$ and the approximate solution with $m = 5$.

As shown in appendix A page 59

3.3.3 Bernstein's approximation method:

take $m = 4, n = 2$ then we have

$$f_{i4}(x) = \sum_{r=0}^{4} a_{ir} P_{4r}(x)\,, where\ i = 1,2$$

where

$$T_{qr}^{ij} = \int_{a}^{x_q} k_{ij}(x_q, t)\ p_{rm}(t)\ dt \qquad Y_{qr}^{ij} = c_{ij}(x_q)\ p_{rm}(x_q)$$

where

$$A = [a_{ir}] = [a_{10}, a_{11}, \dots, a_{14} \quad \dots\dots\dots \quad a_{20}, a_{21}, \dots, a_{24}\ ]^T$$

$$G = [g_i(x_r)] = [g_1(x_0), g_1(x_1), \dots, g_1(x_4), g_2(x_0), g_2(x_1), \dots, g_2(x_4)\ ]^T$$

$$Z = \begin{bmatrix} z^{11} & z^{12} \\ z^{21} & z^{22} \end{bmatrix} \text{ and } S = \begin{bmatrix} s^{11} & s^{12} \\ s^{21} & s^{22} \end{bmatrix}$$

and the matrix z^{ij} , s^{ij} for $i,j = 1,2$ are defined with following elements:

$$z^{ij} = \begin{bmatrix} z_{00}^{ij} & \dots & z_{04}^{ij} \\ \vdots & \ddots & \vdots \\ z_{40}^{ij} & \dots & z_{44}^{ij} \end{bmatrix} \quad , \quad where \quad z_{qr}^{ij} = Y_{qr}^{ij}$$

$$s^{ij} = \begin{bmatrix} s_{00}^{ij} & \dots & s_{04}^{ij} \\ \vdots & \ddots & \vdots \\ s_{40}^{ij} & \dots & s_{44}^{ij} \end{bmatrix} \quad , \quad where \quad s_{qr}^{ij} = T_{qr}^{ij}$$

Hence, solving the liner algebraic system $ZA = G + SA$ we have

$$F = \begin{bmatrix} 0.0311x^4 - 0.2112x^3 + 0.0276x^2 + 0.9940x \\ 0.1117x^4 + 0.0296x^3 + 0.6018x^2 + 0.9752x + 1 \end{bmatrix}$$

Table (3.11) contains both the exact and the numerical solutions of system (3. 5).

Table 3.11

A comparison between the exact and the numerical solutions using Bernstein's approximation method for system (3.5)

x	Exact solution $f_1(x)$	Exact solution $f_2(x)$	approximate solution $f_1(x)$	approximate solution $f_2(x)$	Absolute error $e_1(x)$	Absolute error $e_2(x)$
0	0	1	0	1	0	0
0.2	0.1987	1.2214	0.1983	1.2195	0.0004	0.0019
0.4	0.3894	1.4918	0.3893	1.4911	0.0001	0.0007
0.6	0.5646	1.8221	0.5647	1.8226	0.0001	0.0005
0.8	0.7174	2.2255	0.7175	2.2262	0.0001	0.0007
1	0.8415	2.7183	0.8415	2.7183	0.0000	0.0000
1.2	0.9320	3.3201	0.9321	3.3196	0.0000	0.0005
1.4	0.9854	4.0552	0.9856	4.0551	0.0002	0.0001
1.6	0.9996	4.9530	0.9998	4.9542	0.0002	0.0012
1.8	0.9738	6.0496	0.9734	6.0504	0.0005	0.0008
2	0.9093	7.3891	0.9064	7.3816	0.0029	0.0075

Figure (3.11.a) compares the exact solution $\boldsymbol{f_1(x)}$ and the approximate solution with $m = 2$.

Figure (3.11.b) compares the exact solution $\boldsymbol{f_2(x)}$ and the approximate solution with $m = 2$.

As shown in appendix A page 60

3.4 Conclusion

In this thesis several numerical methods have been implemented namely, Galerkin method with Laguerre polynomials, Chebyshev collection method, and Bernstein's approximation method to solve a system of Volterra Integral equations. Numerical results show that the convergence and accuracy of these methods are in good agreement with the analytical solution. However, according to the comparison of these methods, the conclusions were:

1- The results for example (3.1) show clearly that Galerkin method with Laguerre polynomials is more efficient in comparison with Chebyshev collection method, and Bernstein's approximation method. since the kernel is polynomial.

2- The results for examples (3.2) and (3.3) show clearly that Bernstein's approximation method is more efficient in comparison with the other methods since the kernel is not polynomial.

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Appendices

Appendix A

Figure 3.1.a

A comparison between the exact solution $\boldsymbol{f_1}$ *and the numerical solution using Galerkin method with Laguerre polynomials for system (3. 1)*

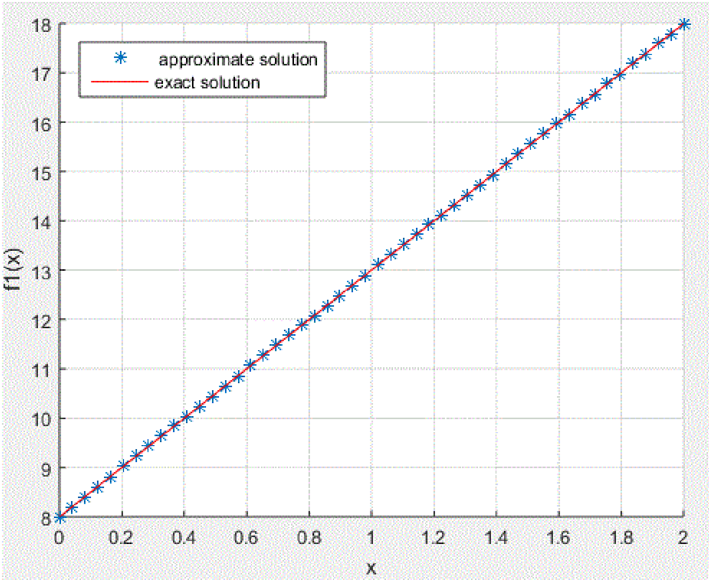


Figure 3.1.b

A comparison between the exact solution $\boldsymbol{f_2}$ *and the numerical solution using Galerkin method with Laguerre polynomials for system (3. 1)*

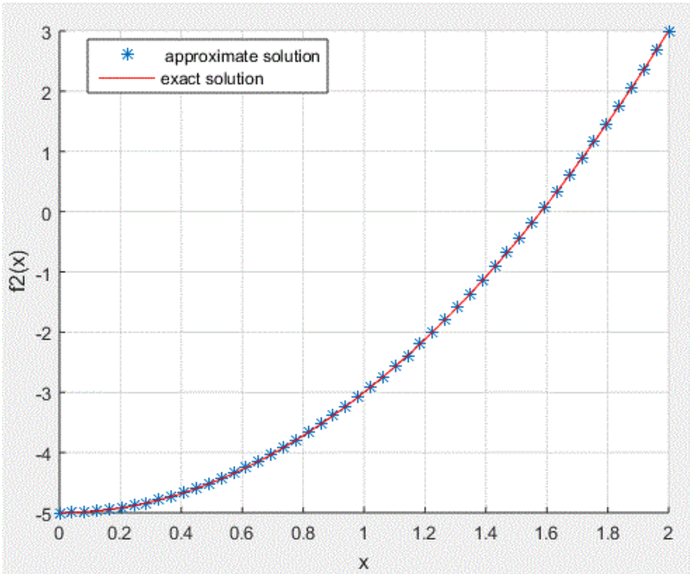


Figure 3.1.c

A comparison between the exact solution f_3 and the numerical solution using Galerkin method with Laguerre polynomials for system (3. 1)

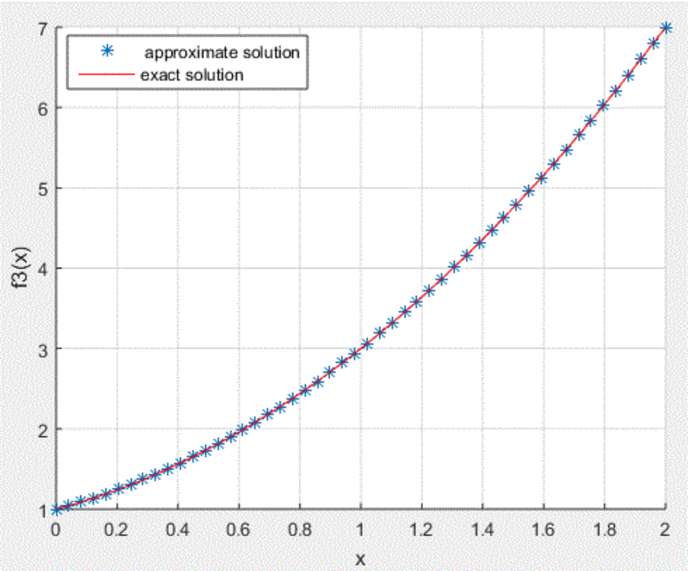


Figure 3.2.a

A comparison between the exact solutions f_1 and the numerical solutions using Chebyshev collocation method for system (3.1)

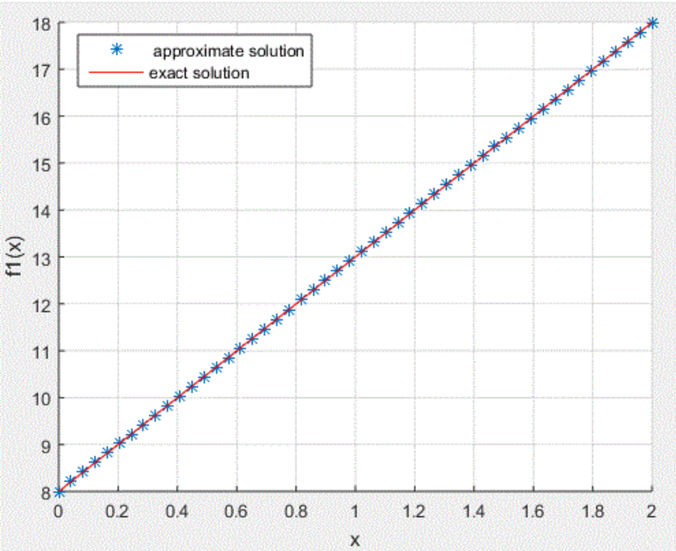


Figure 3.2.b

A comparison between the exact solutions f_2 *and the numerical solutions using Chebyshev collocation method for system (3.1)*

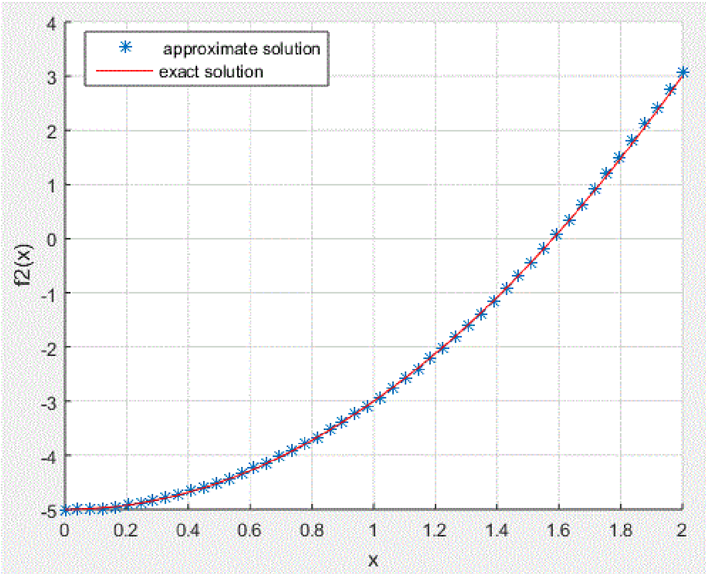


Figure 3.2.c

A comparison between the exact solutions f_3 *and the numerical solutions using Chebyshev collocation method for system (3.1)*

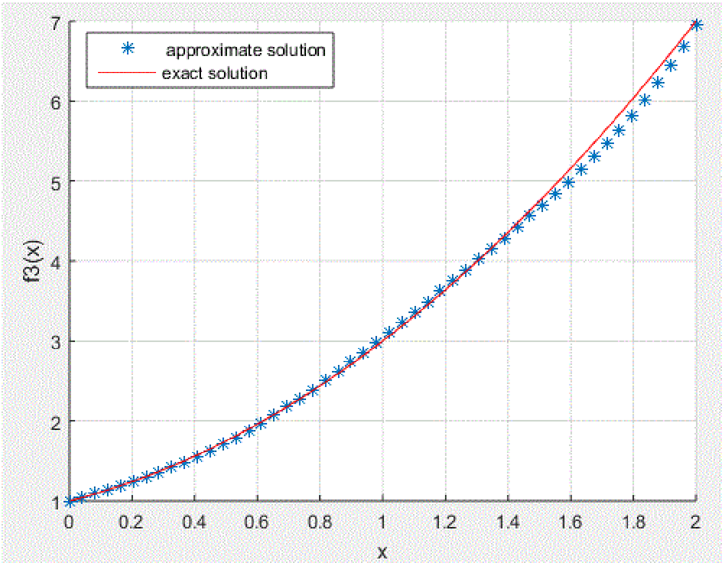


Figure 3.4.a

A comparison between the exact solutions f_1 and the numerical solutions using Galerkin method with Laguerre polynomials for system (3.4)

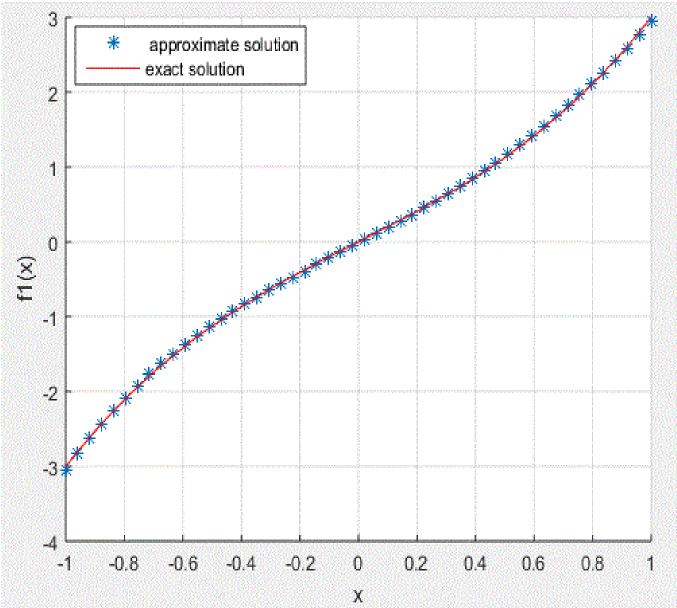


Figure 3.4.b

A comparison between the exact solutions f_2 and the numerical solutions using Galerkin method with Laguerre polynomials for system (3. 4)

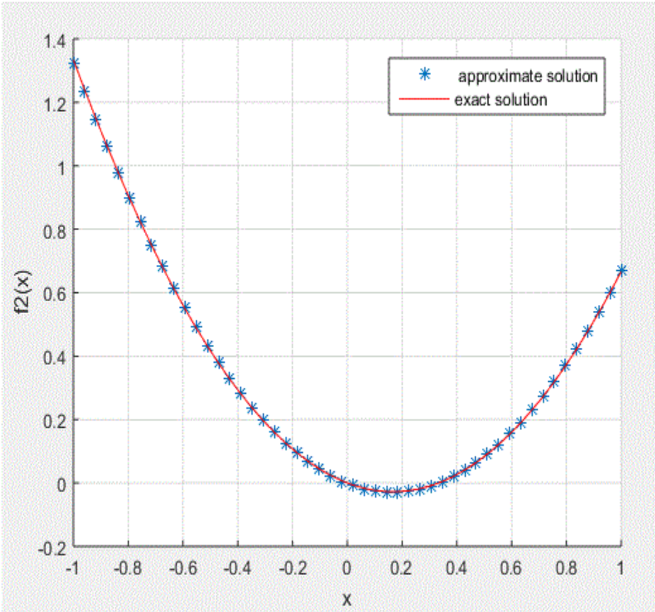


Figure 3.7.a

A comparison between the exact solutions f_1 and the numerical solutions using Bernstein's approximation method for system (3.4)

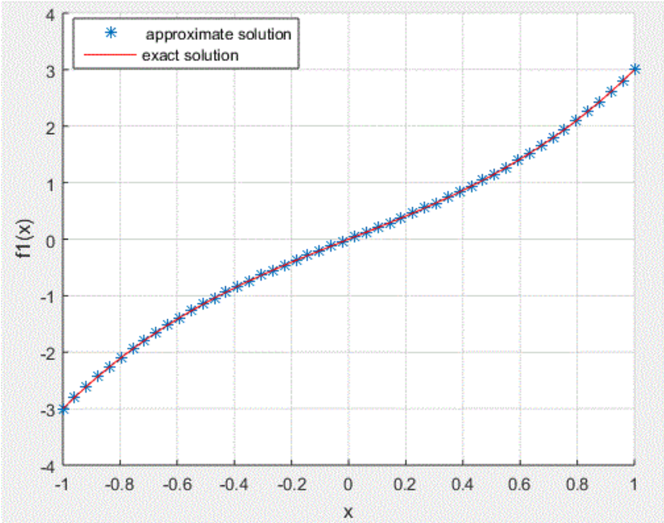


Figure 3.7.b

A comparison between the exact solutions f_2 and the numerical solutions using Bernstein's approximation method for system (3.4)

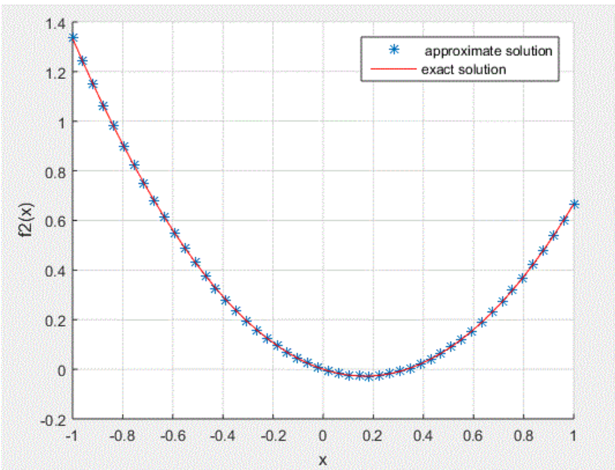


Figure 3.8.a

A comparison between the exact solutions f_1 *and the numerical solutions using Galerkin method with Laguerre polynomials for system (3.5)*

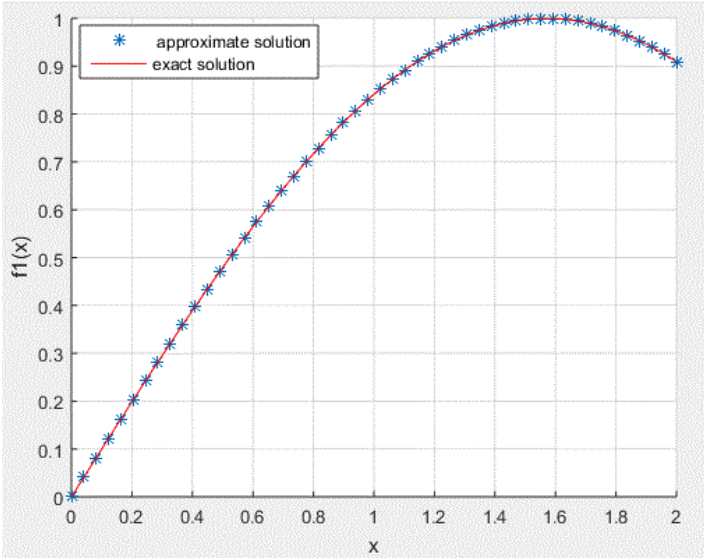


Figure 3.8.b

A comparison between the exact solutions f_2 *and the numerical solutions using Galerkin method with Laguerre polynomials for system (3.5)*

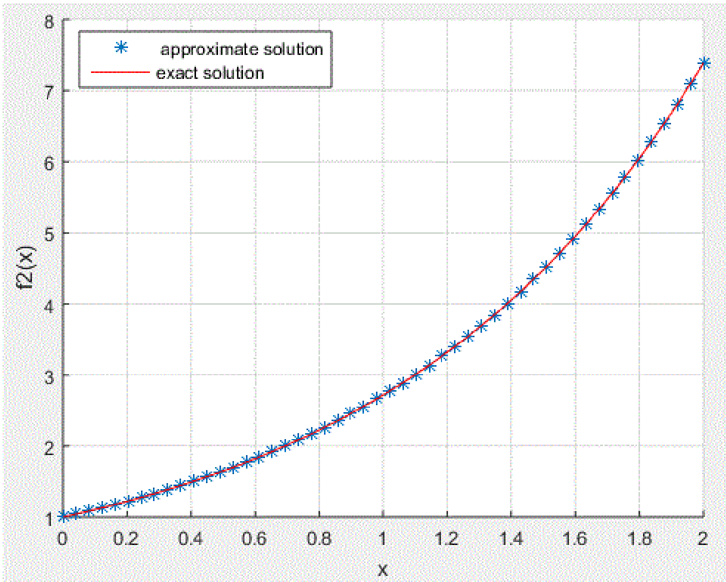


Figure 3.10.a

A comparison between the exact solutions f_1 *and the numerical solutions using Chebyshev collocation method with* $m = 5$ *for system (3.5)*

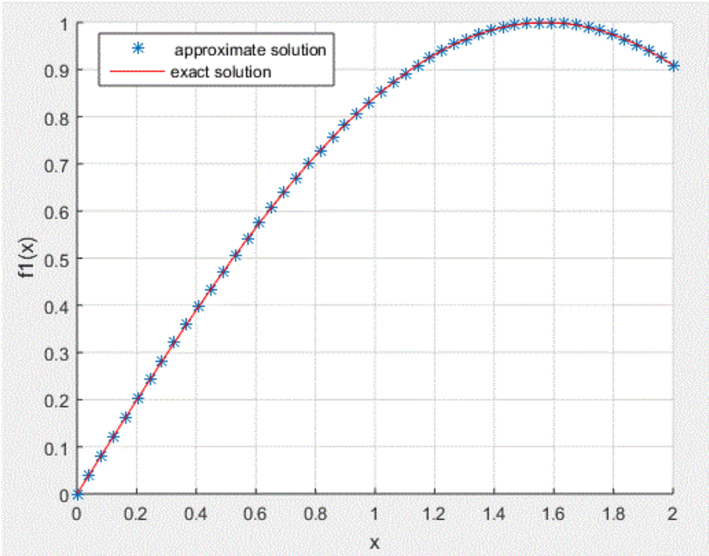


Figure (3.10.b

A comparison between the exact solutions f_2 *and the numerical solutions using Chebyshev collocation method with* $m = 5$ *for system (3.5)*

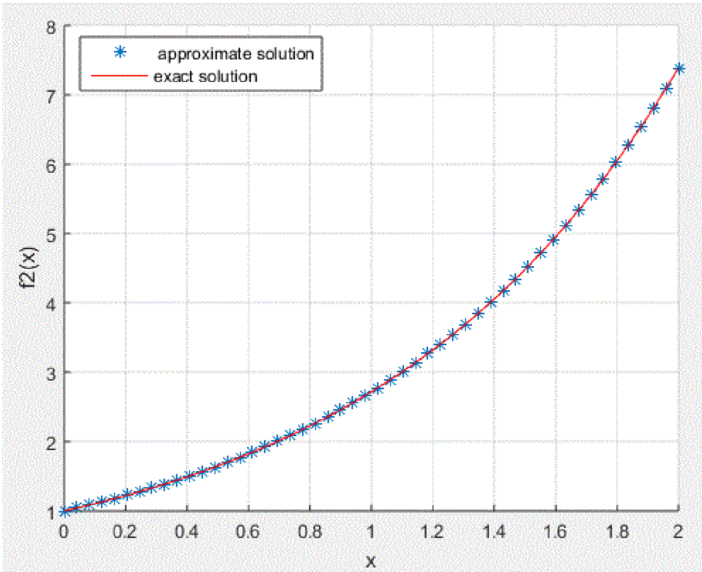


Figure 3.11.a

A comparison between the exact solutions $\boldsymbol{f_1}$ *and the numerical solutions using Bernstein's approximation method for system (3.5)*

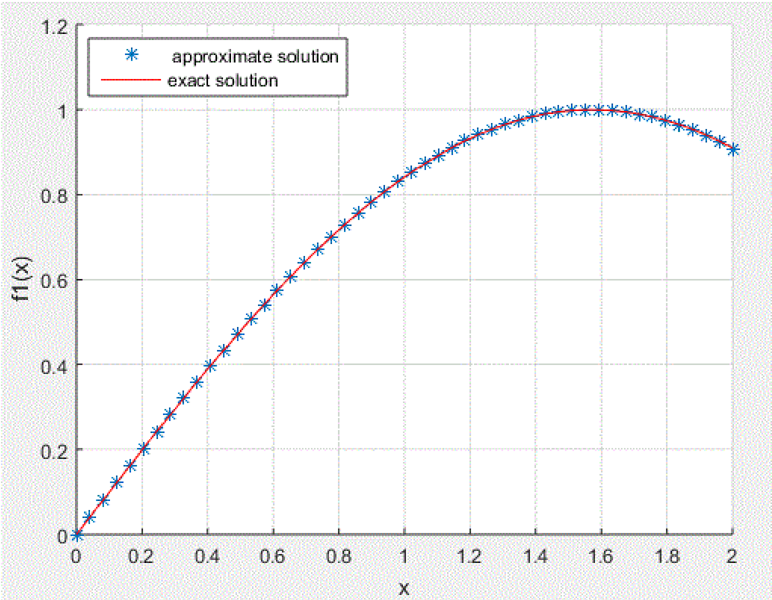
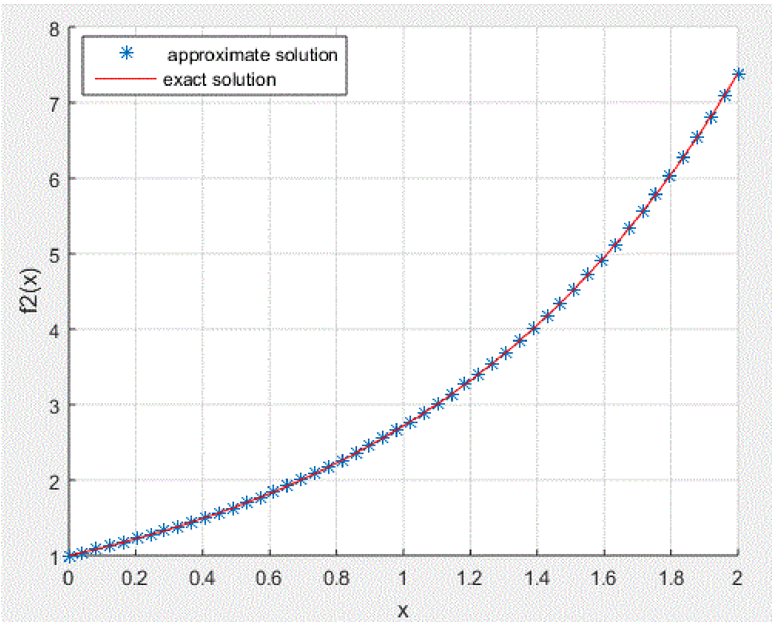


Figure 3.11.b

A comparison between the exact solutions $\boldsymbol{f_2}$ *and the numerical solutions using Bernstein's approximation method for system (3.5)*



Appendix B

Table 3.5

A comparison between the exact and the numerical solutions using Chebyshev collocation method with $\boldsymbol{m = 4}$ *for system (3.4)*

x	Exact solution $f_1(x)$	Exact solution $f_2(x)$	approximate solution $f_1(x)$	approximate solution $f_2(x)$	Absolute error $e_1(x)$	Absolute error $e_2(x)$
1	3	0.6667	3.0001	0.1670	0.0001	0.4997
0.75	1.9219	0.3125	1.9222	0.0212	0.0003	0.2812
0.5	1.1250	0.0833	1.1254	-0.0416	0.0004	0.1250
0.25	0.5156	-0.0208	0.5159	-0.0520	0.0003	0.0312
0	0	0	0.0002	0.0001	0.0002	0.0001
-0.25	-0.5156	0.1458	-0.5155	0.1146	0.0001	0.0312
-0.5	-1.1250	0.4167	-1.1250	0.2916	0.0000	0.1250
-0.75	-1.9219	0.8125	-1.9218	0.5311	0.0001	0.2814
-1	-3	1.3333	-2.9999	0.8332	0.0001	0.5001

Appendix C

MATLAB code for Galerkin method with Laguerre polynomials

system (3.1)

```
syms x t ;
L0=1;
L1=1-x;
L2=0.5*(2-4*x+x^2);
L3=(1/6)*(6-18*x+9*x^2-x^3);
L0t=1;
L1t=1-t;
L2t=0.5*(2-4*t+t^2);
L3t=(1/6)*(6-18*t+9*t^2-t^3);
Lt=[L0t L1t L2t L3t];
L=[L0 L1 L2 L3];
p=zeros(1,4);
LLt=[Lt p p;p Lt p;p p Lt];
LLx=[L p p;p L p;p p L];
k11= x^2 - 2*t;
k12 = t^2 - x;
k13= 2*t;
k21=t*(x+1);
k22=t*x*(x^2 +1);
k23=2*t^2 +x^3;
k31=t-x;
k32=t^2 - x^3;
k33=2*t*x+t^2;
K=[k11 k12 k13; k21 k22 k23; k31 k32 k33];
Kr=int(K*LLt,t,0,x);
krs=Kr'*LLx;
Krs=int(krs,0,2);
c11=2*x^2 +3;
c12=0;
c13=0;
c21=0;
c22=1-3*x^2;
c23=0;
c31=0;
c32=0;
c33=3*x^2 +6;
C=[c11 c12 c13;c21 c22 c23;c31 c32 c33];
Cr=C*LLx;
crs=Cr'*LLx;
Crs=int(crs,0,2);
w=Crs-Krs;
v=inv(w);
g11=(-1/15)*(6*x^5 + 35*x^4 - 95*x^3 - 270*x^2 - 225*x - 360);
g21=(-1/30)*(15*x^7 + 10*x^6 -33*x^5 +275*x^4 +115*x^3 -390*x^2 +150);
g31=(1/60)*(40*x^6 -66*x^5 -175*x^4 +250*x^3 +780*x^2 +360*x +360);
G=[g11;g21;g31];
gs=G'*LLx;
Gs=int(gs,0,2);
A=Gs*v;
Lx=    [    1,    0,0;
    1 - (x),    0,0;
  (x)^2/2 - 2*(x) + 1,   0,0;
 (3*(x)^2)/2 - 3*(x) - (x)^3/6 + 1,   0,0;
```

```
    0,     1,0;
     0,     1 - (x),0;
   0,  (x)^2/2 - 2*(x)  + 1,0;
     0,  (3*(x)^2)/2 - 3*(x)  - (x)^3/6 + 1,0;
     0,  0,     1;
    0,  0,     1 - (x);
  0,  0,  (x)^2/2 - 2*(x)  + 1;
    0,  0,  (3*(x)^2)/2 - 3*(x)  - (x)^3/6 + 1];
 F=A*Lx;
 x=linspace(0 , 2 ,50);
 fa1=5*x + 8;
fa2=2*x.^2 - 5;
fa3=x.^2 + x + 1;
fe1=5*x + 8;
fe2=2*x.^2 - 5;
fe3=x.^2 + x + 1;
e1=fe1-fa1;
e2=fe2-fa2;
e3=fe3-fa3;
figure
hold on
plot(x,fa1,'*');
plot(x,fe1,'r');
xlabel ('x');
ylabel ('f1(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa2,'*');
plot(x,fe2,'r');
xlabel ('x');
ylabel ('f2(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa3,'*');
plot(x,fe3,'r');
xlabel ('x');
ylabel ('f3(x)');
legend(' approximate solution','exact solution');
grid on
hold off
```

system (3.2)

```
syms x t ;
L0=1;
L1=1-x;
L2=0.5*(2-4*x+x^2);
L3=(1/6)*(6-18*x+9*x^2-x^3);
L4=(1/24)*(24-29*x+72*x^2-16*x^3+x^4);
L5=(1/120)*(120-600*x+600*x^2-200*x^3+25*x^4-x^5);
L0t=1;
L1t=1-t;
L2t=0.5*(2-4*t+t^2);
```

```
L3t=(1/6)*(6-18*t+9*t^2-t^3);
L4t=(1/24)*(24-29*t+72*t^2-16*t^3+t^4);
L5t=(1/120)*(120-600*t+600*t^2-200*t^3+25*t^4-t^5);
Lt=[L0t L1t L2t L3t L4t L5t];
L=[L0 L1 L2 L3 L4 L5];
p=zeros(1,6);
LLt=[Lt p;p Lt];
LLx=[L p;p L];
k11=sin(x-t);
k12 =exp(x-t);
k21=cos(x-t);
k22=3*t.*x.^2;
K=[k11 k12 ; k21 k22];
Kr=int(K*LLt,t,-1,x);
krs=Kr'*LLx;
Krs=int(krs,-1,1);
c11=1;
c12=0;
c21=0;
c22=1;
C=[c11 c12 ; c21 c22];
Cr=C*LLx;
crs=Cr'*LLx;
Crs=int(crs,-1,1);
w=Crs-Krs;
W=[0.9093     1.4161     2.0522     2.8671     2.5017     5.2640    -1.4161
-1.0907   -0.9665   -1.0116   -2.1120   -1.5277;
    0.4024     1.7416     3.1379     4.7227     2.8913     9.0004    -1.7416
-1.5976   -1.6518   -1.8903   -2.7090   -2.9029;
    0.0248     2.2683     4.5644     7.1331     3.9348    13.9936    -2.2683
-2.3085   -2.5728   -3.0637   -3.6794   -4.7758;
   -0.2733     3.0280     6.4380    10.2815     5.6757    20.6232    -3.0280
-3.2733   -3.7979   -4.6219   -5.0898   -7.2949;
    0.8794     3.1537     5.8513     9.2004     7.1073    18.9092    -3.1537
-3.1373   -3.4957   -4.2192   -5.3594   -6.8000;
   -0.8014     5.4223    12.0412    19.6784    11.6135    40.6782    -5.4223
-6.2181   -7.5036   -9.3433   -9.5954  -15.0221;
   -4.3891    -2.3891    -1.2503    -0.7690    -6.3709    -1.1867     2.4000
2.4000     2.8190     3.6571     4.9349     6.6937;
   -6.3891    -3.7224    -2.2503    -1.7023    -9.3153    -2.6009     3.0667
3.4476     4.4476     6.0963     7.3923    11.6218;
   -9.2503    -5.5836    -3.6087    -2.9535   -13.5698    -4.5395     4.2095
4.9714     6.6857     9.4116    11.1335    18.3973;
  -13.1763    -8.1096    -5.4311    -4.6249   -19.4432    -7.1656     5.8952
7.0825     9.7089    13.8681    16.4196    27.5643;
  -12.8334    -7.7445    -5.0524    -4.2335   -19.0957    -6.7281     6.3553
6.9934     9.1804    12.9718    16.7274    26.0199;
  -25.2743   -15.8553   -10.9996    -9.7424   -37.6496   -15.3086    11.2939
13.6449    18.9900    27.5268    33.0600    55.8231];
v=inv(W);
g11=5/3 -exp(x+1) + 23/3*x + x.^2 + 3*cos(x+1)-sin(x+1);
g21=1/12*(48  -  4*x  -11*x.^2  +  4*x.^5  -  9*x.^6  -  12*cos(x+1)  -
36*sin(x+1));
G=[g11;g21];
gs=G'*LLx;
Gs=int(gs,-1,1);
a=Gs*v;
A=[-0.8970    0.5941   -2.6211    2.7524    1.6829   -1.5311    2.7697
-3.2473   -1.1524    2.1902   -0.5622   -0.0001];
Lx=[1,0;
```

```
1-x,0;
0.5*(2-4*x+x^2),0;
(1/6)*(6-18*x+9*x^2-x^3),0;
(1/24)*(24-29*x+72*x^2-16*x^3+x^4),0;
(1/120)*(120-600*x+600*x^2-200*x^3+25*x^4-x^5),0
0,1;
0,1-x;
0,0.5*(2-4*x+x^2);
0,(1/6)*(6-18*x+9*x^2-x^3);
0,(1/24)*(24-29*x+72*x^2-16*x^3+x^4);
0,(1/120)*(120-600*x+600*x^2-200*x^3+25*x^4-x^5)];
F=A*Lx;
x=linspace(-1 , 1 , 50);
x=[1 0.75 0.5 0.25 0 -0.25 -0.5 -0.75 -1];
fe1=x.^3 + 2*x;
fe2=x.^2 -x/3;
fa1=0.0128*x.^5 - 0.2489*x.^4 + 0.9712*x.^3 + 0.2112*x.^2+2.0129*x -
0.0198;
fa2=0.0000*x.^5 - 0.0234*x.^4 + 0.0099*x.^3 + 1.0220*x.^2 - 0.3387*x-
0.0021;
 e1=fe1-fa1
e2=fe2-fa2
figure
hold on
plot(x,fa1,'*');
plot(x,fe1,'r');
xlabel ('x');
ylabel ('f1(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa2,'*');
plot(x,fe2,'r');
xlabel ('x');
ylabel ('f2(x)');
legend(' approximate solution','exact solution');
grid on
hold off
```

system (3.3)

```
syms x t ;
L0=1;
L1=1-x;
L2=0.5*(2-4*x+x^2);
L3=(1/6)*(6-18*x+9*x^2-x^3);
L4=(1/24)*(24-29*x+72*x^2-16*x^3+x^4);
L0t=1;
L1t=1-t;
L2t=0.5*(2-4*t+t^2);
L3t=(1/6)*(6-18*t+9*t^2-t^3);
L4t=(1/24)*(24-29*t+72*t^2-16*t^3+t^4);
Lt=[L0t L1t L2t L3t L4t];
L=[L0 L1 L2 L3 L4];
p=zeros(1,5);
LLt=[Lt p;p Lt];
LLx=[L p;p L];
```

```
k11=x+t;
k12=x*t;
k21=2*x-t;
k22=t+t*x;
K=[k11 k12 ; k21 k22];
Kr=int(K*LLt,t,0,x);
krs=Kr'*LLx;
Krs=int(krs,0,2);
c11=3*x-8;
c12=-2*x +5;
c21=4*x;
c22=x-5;
C=[c11 c12 ; c21 c22];
Cr=C*LLx;
crs=Cr'*LLx;
Crs=int(crs,0,2);
w=Crs-Krs;
v=inv(w);
g11=5*exp(x)-2*x -9*sin(x)-x.^2*exp(x)+2*x*cos(x)-x*exp(x)+3*x*sin(x);
g21=-3*x-4*exp(x)+sin(x)-x.^2*exp(x)+x*cos(x)+x*exp(x)+4*x*sin(x)-1;
G=[g11;g21];
gs=G'*LLx;
Gs=int(gs,0,2);
a=Gs*v;
A=[ -1.6467     1.8246     0.8774    -1.8396     0.7848    -1.5692    -
7.1092   18.0059  -11.0571    2.7318];
Lx=[1,0;
1-x,0;
0.5*(2-4*x+x^2),0;
(1/6)*(6-18*x+9*x^2-x^3),0;
(1/24)*(24-29*x+72*x^2-16*x^3+x^4),0;
0,1;
0,1-x;
0,0.5*(2-4*x+x^2);
0,(1/6)*(6-18*x+9*x^2-x^3);
0,(1/24)*(24-29*x+72*x^2-16*x^3+x^4)];
F=A*Lx;
x=linspace(0 , 2 , 50);
fe1=sin(x);
fe2=exp(x);
fa1=0.0327*x.^4 -0.2166*x.^3 +0.0337*x.^2+ 0.9911*x + 0.0005;
fa2=0.1138*x.^4+ 0.0217*x.^3 +0.6127*x.^2 +0.9678*x +1.0022;
e1=fe1-fa1;
e2=fe2-fa2;
figure
hold on
plot(x,fa1,'*');
plot(x,fe1,'r');
xlabel ('x');
ylabel ('f1(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa2,'*');
plot(x,fe2,'r');
xlabel ('x');
ylabel ('f2(x)');
legend(' approximate solution','exact solution');
```

```
grid on
```

MATLAB code for Chebyshev collocation method

System (3.1)

```
syms x xs tr t;
tr =[2.0000    1.8090    1.3090    0.6910    0.1910         0];
k11= xs.^2 - 2*tr;
k12 = tr.^2 - xs;
k13= 2*tr;
k21=tr*(xs+1);
k22=tr*xs*(xs.^2 +1);
k23=2*tr.^2 +xs.^3;
k31=tr-xs;
k32=tr.^2 - xs.^3;
k33=2*tr*xs+tr.^2;
K=[k11 k12 k13; k21 k22 k23; k31 k32 k33];
T0 = [1 1 1 1 1 1];
T1= tr-1 ;
T2=2*(tr-1).^2 -1;
T3=4*(tr-1).^3 -3*(tr-1);
T4=8*(tr-1).^4 -8*(tr-1).^2 +1;
T5=16*(tr-1).^5-20*(tr-1).^3 +5*(tr-1);
Tt=[1,t-1,2*(t-1)^2-1,4*(t-1).^3-3*(t-1),8*(t-1).^4-8*(t-
1).^2+1,16*(t-1).^5-20*(t-1).^3+5*(t-1)];
t=[T0' T1' T2' T3' T4' T5'];
z=zeros(6,6);
tt=[t z z; z t z;z z t];
n=zeros(1,6);
T=[T0 n n;
    n T0 n;
    n n T0;
    T1 n n;
    n T1 n;
    n n T1;
    T2 n n;
    n T2 n;
    n n T2;
    T3 n n;
    n T3 n;
    n n T3;
    T4 n n;
    n T4 n;
    n n T4;
    T5 n n;
    n T5 n;
    n n T5];
Kxs=K *inv(tt);
K1=[1.9999   -1.9999    0.0001   -0.0001    0.0000   -0.0000   -0.4999
1.9999    0.4999    0.0000    0.0000    0.0000    2.0001   1.9999    -
0.0001    0.0001   -0.0000    0.0000
    3.0001    2.9999   -0.0001    0.0001   -0.0000    0.0000   10.0004
9.9996   -0.0003    0.0003   -0.0001    0.0000   11.0001   3.9999
0.9999    0.0001    0.0000    0.0000
   -1.0000    1.0000   -0.0000    0.0000   -0.0000    0.0000   -6.4999
1.9999    0.4999    0.0000    0.0000    0.0000    5.5002   5.9998
0.4998    0.0002   -0.0000    0.0000];
```

```
K2=[ 1.2724    -1.9999     0.0001    -0.0001     0.0000    -0.0000    -
0.3089     1.9999     0.4999     0.0000     0.0000     0.0000     2.0001
1.9999   -0.0001    0.0001   -0.0000    0.0000
    2.8091    2.8089   -0.0001    0.0001   -0.0000    0.0000    7.7293
7.7286    -0.0003     0.0002    -0.0001     0.0000     8.9201    3.9999
0.9999    0.0001    0.0000    0.0000
   -0.8090    1.0000   -0.0000    0.0000   -0.0000    0.0000   -4.4199
1.9999     0.4999     0.0000     0.0000     0.0000     5.1182    3.9999
0.9999    0.0001    0.0000    0.0000];
K3=[ -0.2866     -1.9999      0.0001     -0.0001      0.0000     -0.0000
0.1911     1.9999     0.4999     0.0000     0.0000     0.0000     2.0001
1.9999   -0.0001    0.0001   -0.0000    0.0000
    2.3091    2.3089   -0.0001    0.0001   -0.0000    0.0000    3.5521
3.5518    -0.0001     0.0001    -0.0000     0.0000     5.2431    3.9999
0.9999    0.0001    0.0000    0.0000
   -0.3090    1.0000   -0.0000    0.0000   -0.0000    0.0000   -0.7429
1.9999     0.4999     0.0000     0.0000     0.0000     4.1182    4.6179
0.4998    0.0001   -0.0000    0.0000];
K4=[-1.5226     -1.9999      0.0001     -0.0001      0.0000     -0.0000
0.8091     1.9999     0.4999     0.0000     0.0000     0.0000     2.0001
1.9999   -0.0001    0.0001   -0.0000    0.0000
    1.6911    1.6909   -0.0001    0.0001   -0.0000    0.0000    1.0210
1.0209    -0.0000     0.0000    -0.0000     0.0000     3.3301   3.9999
0.9999    0.0001    0.0000    0.0000
    0.3090    1.0000   -0.0000    0.0000   -0.0000    0.0000    1.1701
1.9999     0.4999     0.0000     0.0000     0.0000     2.8821   3.3819
0.4999    0.0001   -0.0000    0.0000];
K5=[ -1.9636     -1.9999      0.0001     -0.0001      0.0000     -0.0000
1.3091     1.9999     0.4999     0.0000     0.0000     0.0000     2.0001
1.9999   -0.0001    0.0001   -0.0000    0.0000
    1.1911    1.1910   -0.0000    0.0000   -0.0000    0.0000    0.1980
0.1980    -0.0000     0.0000    -0.0000     0.0000     3.0071   3.9999
0.9999    0.0001    0.0000    0.0000
    0.8090    1.0000   -0.0000    0.0000   -0.0000    0.0000    1.4931
1.9999     0.4999     0.0000     0.0000     0.0000     1.8821   2.3819
0.4999    0.0001   -0.0000    0.0000];
K6=[-2.0001     -1.9999      0.0001     -0.0001      0.0000     -0.0000
1.5001     1.9999     0.4999     0.0000     0.0000     0.0000     2.0001
1.9999   -0.0001    0.0001   -0.0000    0.0000
    1.0000    1.0000   -0.0000    0.0000   -0.0000    0.0000         0
0         0         0         0         0    3.0001    3.9999    0.9999
0.0001    0.0000    0.0000
    1.0000    1.0000   -0.0000    0.0000   -0.0000    0.0000    1.5001
1.9999     0.4999     0.0000     0.0000     0.0000     1.5001    1.9999
0.4999    0.0000    0.0000    0.0000];
v=zeros(3,18);
KK=[K1 v v v v v;
    v K2 v v v v;
    v v K3 v v v;
    v v v K4 v v;
    v v v v K5 v;
    v v v v v K6];
yt=Tt'*Tt;
y=int(yt,0,xs);
Y1=[    2,      0,    -2/3,       0,    -2/15,       0;
      0,    2/3,        0,    -2/5,        0,   -2/21;
   -2/3,      0,    14/15,       0,  -38/105,       0;
      0,   -2/5,        0,   34/35,        0,  -22/63;
  -2/15,      0,  -38/105,       0,    62/63,       0;
      0,  -2/21,        0,  -22/63,        0,   98/99];
```

```
Y2=[1.8090   -0.1728   -0.7893   -0.0534   -0.1152    0.0754;
   -0.1728    0.5098   -0.1131   -0.4522    0.0110   -0.0329;
   -0.7893   -0.1131    0.8469   -0.0487   -0.3700    0.0271;
   -0.0534   -0.4522   -0.0487    0.9292   -0.0326   -0.3691;
   -0.1152    0.0110   -0.3700   -0.0326    0.9301   -0.0648;
    0.0754   -0.0329    0.0271   -0.3691   -0.0648    0.8954];
Y3=[1.3090   -0.4523   -0.6227    0.3659    0.1682    0.0288;
   -0.4523    0.3432   -0.0432   -0.2273    0.1973   -0.0091;
   -0.6227   -0.0432    0.7386   -0.2117   -0.4045    0.1552;
    0.3659   -0.2273   -0.2117    0.5613   -0.2539   -0.2818;
    0.1682    0.1973   -0.4045   -0.2539    0.6840   -0.1696;
    0.0288   -0.0091    0.1552   -0.2818   -0.1696    0.6479];
Y4=[0.6910   -0.4523   -0.0440    0.3659   -0.3015    0.0288;
   -0.4523    0.3235   -0.0432   -0.1727    0.1973   -0.0861;
   -0.0440   -0.0432    0.1948   -0.2117    0.0426    0.1552;
    0.3659   -0.1727   -0.2117    0.4101   -0.2539   -0.0674;
   -0.3015    0.1973    0.0426   -0.2539    0.3001   -0.1696;
    0.0288   -0.0861    0.1552   -0.0674   -0.1696    0.3420];
Y5=[ 0.1910   -0.1728    0.1227   -0.0534   -0.0182    0.0754;
   -0.1728    0.1568   -0.1131    0.0522    0.0110   -0.0623;
    0.1227   -0.1131    0.0864   -0.0487    0.0081    0.0271;
   -0.0534    0.0522   -0.0487    0.0422   -0.0326    0.0199;
   -0.0182    0.0110    0.0081   -0.0326    0.0541   -0.0648;
    0.0754   -0.0623    0.0271    0.0199   -0.0648    0.0945];
Y6=zeros(6,6);
Y=[Y1 z z;
    z Y1 z;
    z z Y1;
    Y2 z z;
    z Y2 z;
    z z Y2;
    Y3 z z;
    z Y3 z;
    z z Y3;
    Y4 z z;
    z Y4 z;
    z z Y4;
    Y5 z z;
    z Y5 z;
    z z Y5;
    Y6 z z;
    z Y6 z;
    z z Y6];
g11=(-1/15)*(6*xs^5 + 35*xs^4 - 95*xs^3 - 270*xs^2 - 225*xs - 360);
g21=(-1/30)*(15*xs^7 + 10*xs^6 -33*xs^5 +275*xs^4 +115*xs^3 -390*xs^2
+150);
g31=(1/60)*(40*xs^6  -66*xs^5  -175*xs^4  +250*xs^3  +780*xs^2  +360*xs
+360);
G=[126.5333;
  -180.4667;
   64.1333;
  114.7954;
  -105.3881;
  54.8811;
  80.2950;
  -18.9782;
   36.0378;
   44.4543;
   -2.0480;
    16.9623;
```

```
   27.5626;
   -4.5644;
   7.6452;
   24;
   -5;
    6];
c11=2*xs^2 +3;
c12=0;
c13=0;
c21=0;
c22=1-3*xs^2;
c23=0;
c31=0;
c32=0;
c33=3*xs^2 +6;
c=[c11 c12 c13;c21 c22 c23;c31 c32 c33];
h = zeros(3,3);
c1=[11     0     0
     0   -11     0
     0     0    18];
 c2=[ 9.5450         0         0
         0   -8.8174         0
         0         0   15.8174];
c3=[ 6.4270         0         0
         0   -4.1404         0
         0         0   11.1404];
c4=[ 3.9550         0         0
         0   -0.4324         0
         0         0    7.4324];
c5=[3.0730         0         0
         0    0.8906         0
         0         0    6.1094];
c6=[ 3     0     0
     0     1     0
     0     0     6];
C=[c1 h h h h h;
    h c2 h h h h;
    h h c3 h h h;
    h h h c4 h h;
    h h h h c5 h;
    h h h h h c6];
w=C*T-KK*Y;
s=inv(w);
A=s*G;
Tx=[1,x-1,2*(x-1)^2-1,4*(x-1).^3-3*(x-1),8*(x-1).^4-8*(x-
1).^2+1,16*(x-1).^5-20*(x-1).^3+5*(x-1)];
Txx=[Tx n n;n Tx n;n n Tx];
F=Txx*A;
x=linspace(0 , 2 ,50);
fa1 =0.0352*x.^5 - 0.1696*x.^4 + 0.2650*x.^3 - 0.1504*x.^2+ 5.0229*x +
8.0000;
fa2  =-0.1463*x.^5  +  0.7065*x.^4  -  1.1203*x.^3  +  2.6520  *x.^2  -
0.1003*x - 5.0000;
fa3 =0.5769*x.^5 - 2.4425*x.^4 + 3.3321*x.^3 - 0.6230*x.^2 + 1.2037*x
+ 1.0000;
fe1=5*x + 8;
fe2=2*x.^2 - 5;
fe3=x.^2 + x + 1;
e1=fe1-fa1;
e2=fe2-fa2;
```

```
e3=fe3-fa3;
figure
hold on
plot(x,fa1,'*');
plot(x,fe1,'r');
xlabel ('x');
ylabel ('f1(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa2,'*');
plot(x,fe2,'r');
xlabel ('x');
ylabel ('f2(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa3,'*');
plot(x,fe3,'r');
xlabel ('x');
ylabel ('f3(x)');
legend(' approximate solution','exact solution');
grid on
hold off
```

system (3.2) for m = 4

```
syms x xs tr t;
tr = [1 0.7071 0 -0.7071 -1];
k11=sin(xs-tr);
k12 =exp(xs-tr);
k21=cos(xs-tr);
k22=3*tr.*xs.^2;
K = [ k11 k12;k21 k22];
T0 = [1 1 1 1 1];
T1= tr ;
T2=2*tr.^2 -1;
T3=4*tr.^3-3*tr;
T4=8*tr.^4-8*tr.^2+1;
Tt=[1 t 2*t^2-1 4*t^3-3*t 8*t^4-8*t^2+1];
t=[T0' T1' T2' T3' T4'];
z=zeros(5,5);
tt=[t z ; z t];
T=[T0 0 0 0 0 0;
    0 0 0 0 0 T0;
    T1 0 0 0 0 0;
    0 0 0 0 0 T1;
    T2 0 0 0 0 0 ;
    0 0 0 0 0 T2;
    T3 0 0 0 0 0;
    0 0 0 0 0 T3;
    T4 0 0 0 0 0;
    0 0 0 0 0 T4];
Kxs=K *inv(tt);
K1=[ 0.6439    -0.4755    -0.1934     0.0209     0.0042     3.4415    -
3.0725    0.7382   -0.1220    0.0149
```

```
    0.4134    0.7406   -0.1242   -0.0325    0.0027    0.0000    3.0000
-0.0000    0.0000   -0.0000];
K2=  [ 0.4971   -0.6691   -0.1493    0.0294    0.0032    2.5677   -
2.2924    0.5507   -0.0910    0.0111
    0.5817    0.5717   -0.1747   -0.0251    0.0038    0.0000    1.5000
-0.0000    0.0000   -0.0000];
K3=[ -0.0000   -0.8801    0.0000    0.0386    0.0000    1.2661   -
1.1303    0.2716   -0.0449    0.0055
    0.7652    0.0000   -0.2298    0.0000    0.0050         0         0
0         0         0];
K4=[-0.4971   -0.6691    0.1493    0.0294   -0.0032    0.6243   -
0.5573    0.1339   -0.0221    0.0027
    0.5817   -0.5717   -0.1747    0.0251    0.0038    0.0000    1.5000
-0.0000    0.0000   -0.0000];
K5=[ -0.6439   -0.4755    0.1934    0.0209   -0.0042    0.4658   -
0.4158    0.0999   -0.0165    0.0020
    0.4134   -0.7406   -0.1242    0.0325    0.0027    0.0000    3.0000
-0.0000    0.0000   -0.0000];
v=zeros(2,10);
KK=[K1 v v v v;
    v K2 v v v;
    v v K3 v v ;
    v v v K4 v;
    v v v v K5];
yt=Tt'*Tt;
y=int(yt,-1,xs);
Y1=[  2.0000         0   -0.6667         0   -0.1333
         0    0.6667         0   -0.4000         0
   -0.6667         0    0.9333         0   -0.3619
         0   -0.4000         0    0.9714         0
   -0.1333         0   -0.3619         0    0.9841];
Y2=[1.7071   -0.2500   -0.8047    0.0000   -0.0195
   -0.2500    0.4512   -0.1250   -0.4121    0.0833
   -0.8047   -0.1250    0.8438   -0.0417   -0.3560
    0.0000   -0.4121   -0.0417    0.8999   -0.0833
   -0.0195    0.0833   -0.3560   -0.0833    0.8400];

Y3=[ 1.0000   -0.5000   -0.3333    0.5000   -0.0667
   -0.5000    0.3333         0   -0.2000    0.1667
   -0.3333         0    0.4667   -0.3333   -0.1810
    0.5000   -0.2000   -0.3333    0.4857   -0.1667
   -0.0667    0.1667   -0.1810   -0.1667    0.4921];
Y4=[0.2929   -0.2500    0.1381    0.0000   -0.1138
   -0.2500    0.2155   -0.1250    0.0121    0.0833
    0.1381   -0.1250    0.0895   -0.0417   -0.0059
    0.0000    0.0121   -0.0417    0.0716   -0.0833
   -0.1138    0.0833   -0.0059   -0.0833    0.1441];
Y5=zeros(5,5);
YY=[Y1 z ; z Y1 ; Y2 z ; z Y2 ; Y3 z ; z Y3 ; Y4 z ; z Y4 ; Y5 z ; z
Y5];
g11=5/3 -exp(xs+1) + 23/3*xs + xs.^2 + 3*cos(xs+1)-sin(xs+1);
g21=1/12*(48 - 4*xs -11*xs.^2 + 4*xs.^5 - 9*xs.^6 - 12*cos(xs+1) -
36*sin(xs+1));
G=[0.7865;0.0216;0.6764;        0.4349;-0.2722;0.9353;-2.0112;1.8011;-
3.0000;1.3333];
w=T-KK*YY;
s=inv(w);
A=s*G;
tx=[1 x x^2-1 4*x.^3-3*x 8*x.^4-8*x.^2+1];
r=zeros(1,5);
```

```
Tx=[tx r ; r tx];
F=Tx*A;
x=linspace(-1 , 1 , 50);
fe1=x.^3 + 2*x;
fe2=x.^2 -x/3;
fa1= - 0.0002*x.^4 + 0.9995*x.^3 + 0.0001*x.^2 + 2.0005*x + 0.0002;
fa2= 0.0005*x.^4 + 0.0002*x.^3+ 0.4995*x.^2 - 0.3333*x +0.0001;
e1=fe1-fa1;
e2=fe2-fa2;
figure
hold on
plot(x,fa1,'*');
plot(x,fe1,'r');
xlabel ('x');
ylabel ('f1(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa2,'*');
plot(x,fe2,'r');
xlabel ('x');
ylabel ('f2(x)');
legend(' approximate solution','exact solution');
grid on
hold off
```

system (3.2) for m = 7

```
syms x xs tr t;
tr = [1.0000    0.9010    0.6235    0.2225   -0.2225   -0.6235   -
0.9010   -1.0000];
k11=sin(xs-tr);
k12 =exp(xs-tr);
k21=cos(xs-tr);
k22=3*tr.*xs.^2;
K = [ k11 k12;k21 k22];
T0 = [1 1 1 1 1 1 1 1];
T1= tr ;
T2=2*tr.^2 -1;
T3=4*tr.^3-3*tr;
T4=8*tr.^4-8*tr.^2+1;
T5=16*tr.^5-20*tr.^3+5*tr;
T6=32*tr.^6-48*tr.^4+18*tr.^2-1;
T7=64*tr.^7-112*tr.^5+56*tr.^3-7*tr;
Tt=[1  t  2*t^2-1  4*t^3-3*t  8*t^4-8*t^2+1  16*t.^5-20*t.^3+5*t  32*t.^6-
48*t.^4+18*t.^2-1 64*t.^7-112*t.^5+56*t.^3-7*t];
t=[T0' T1' T2' T3' T4' T5' T6' T7'];
z=zeros(8,8);
tt=[t z ; z t];
h=zeros(1,8);
T=[T0 h;
    h T0;
    T1 h;
    h T1;
    T2 h;
    h T2;
    T3 h;
```

```
    h T3;
    T4 h;
    h T4;
    T5 h;
    h T5;
    T6 h;
    h T6;
    T7 h;
    h T7];
Kxs=K *inv(tt);
K1=[ 0.6440    -0.4756    -0.1934     0.0212     0.0042    -0.0003    -
0.0000     0.0000     3.4420    -3.0728     0.7377    -0.1204     0.0148  -
0.0013    0.0000    0.0000
    0.4133    0.7406    -0.1241    -0.0330     0.0027     0.0004    -0.0000
-0.0000    -0.0005     3.0003     0.0003    -0.0001     0.0001    -0.0001
0.0001   -0.0000];
K2=[ 0.6000    -0.5464    -0.1802     0.0243     0.0039    -0.0003    -
0.0000     0.0000     3.1176    -2.7832     0.6682    -0.1090     0.0134  -
0.0012    0.0000    0.0000
    0.4749    0.6900    -0.1426    -0.0307     0.0031     0.0004    -0.0000
-0.0000    -0.0004     2.4356     0.0003    -0.0001     0.0001    -0.0001
0.0000   -0.0000];
K3=[0.4469   -0.7146   -0.1343    0.0318    0.0029   -0.0004   -0.0000
0.0000     2.3621    -2.1088     0.5062    -0.0826     0.0101    -0.0009
0.0000    0.0000
    0.6211    0.5139   -0.1865   -0.0229    0.0040    0.0003   -0.0000
-0.0000    -0.0002     1.1664     0.0001    -0.0000     0.0000   -0.0001
0.0000   -0.0000];
K4=[0.1690   -0.8585   -0.0508    0.0382    0.0011   -0.0005   -0.0000
0.0000     1.5818    -1.4121     0.3390    -0.0553     0.0068    -0.0006
0.0000    0.0000
    0.7463    0.1942   -0.2241   -0.0086    0.0048    0.0001   -0.0000
-0.0000    -0.0000     0.1485     0.0000    -0.0000     0.0000   -0.0000
0.0000   -0.0000];
K5=[-0.1687    -0.8585     0.0506     0.0382    -0.0011    -0.0005    -
0.0000     0.0000     1.0137    -0.9049     0.2172    -0.0355     0.0043  -
0.0004    0.0000    0.0000
    0.7464   -0.1942   -0.2242    0.0086    0.0048   -0.0001   -0.0000
0.0000    -0.0000     0.1485     0.0000    -0.0000     0.0000    -0.0000
0.0000   -0.0000];
K6=[ -0.4467     -0.7146      0.1341      0.0318     -0.0029     -0.0004
0.0000     0.0000     0.6788    -0.6060     0.1455    -0.0237     0.0029    -
0.0003    0.0000    0.0000
    0.6213   -0.5139   -0.1866    0.0229    0.0040   -0.0003   -0.0000
0.0000    -0.0002     1.1664     0.0001    -0.0000     0.0000    -0.0001
0.0000   -0.0000];
K7=[ -0.5998     -0.5464      0.1801      0.0243     -0.0039     -0.0003
0.0000     0.0000     0.5143    -0.4591     0.1102    -0.0180     0.0022  -
0.0002    0.0000    0.0000
    0.4752   -0.6900   -0.1428    0.0307    0.0031   -0.0004   -0.0000
0.0000    -0.0004     2.4356     0.0003    -0.0001     0.0001    -0.0001
0.0000   -0.0000];
K8=[-0.6438     -0.4756      0.1933      0.0212     -0.0042     -0.0002
0.0000     0.0000     0.4658    -0.4159     0.0998    -0.0163     0.0020  -
0.0002    0.0000    0.0000
    0.4135   -0.7406   -0.1243    0.0330    0.0027   -0.0004   -0.0000
0.0000    -0.0005     3.0003     0.0003    -0.0001     0.0001    -0.0001
0.0001   -0.0000];
v=zeros(2,16);
KK=[K1 v v v v v v v;
```

```
     v K2 v v v v v v;
     v v K3 v v v v v;
     v v v K4 v v v v;
     v v v v K5 v v v;
     v v v v v K6 v v;
     v v v v v v K7 v;
     v v v v v v v K8];
yt=Tt'*Tt;
y=int(yt,-1,xs);
Y1=[ 2.0000         0    -0.6667         0    -0.1333         0    -
0.0571         0
         0    0.6667         0    -0.4000         0    -0.0952         0
-0.0444
   -0.6667         0     0.9333         0    -0.3619         0    -0.0825
0
         0   -0.4000         0     0.9714         0    -0.3492         0
-0.0768
   -0.1333         0    -0.3619         0     0.9841         0    -0.3434
0
         0   -0.0952         0    -0.3492         0     0.9899         0
-0.3403
   -0.0571         0    -0.0825         0    -0.3434         0     0.9930
0
         0   -0.0444         0    -0.0768         0    -0.3403         0
0.9949];
 Y2=[1.9010    -0.0941    -0.7467    -0.0587    -0.1661    -0.0056    -
0.0377    0.0396
   -0.0941    0.5771   -0.0764   -0.4564   -0.0321   -0.1019    0.0170
-0.0084
   -0.7467   -0.0764    0.8674   -0.0499   -0.3922   -0.0096   -0.0726
0.0260
   -0.0587   -0.4564   -0.0499    0.9317   -0.0273   -0.3629   -0.0005
-0.0657
   -0.1661   -0.0321   -0.3922   -0.0273    0.9609   -0.0182   -0.3560
-0.0066
   -0.0056   -0.1019   -0.0096   -0.3629   -0.0182    0.9678   -0.0243
-0.3646
   -0.0377    0.0170   -0.0726   -0.0005   -0.3560   -0.0243    0.9593
-0.0392
    0.0396   -0.0084    0.0260   -0.0657   -0.0066   -0.3646   -0.0392
0.9456];
Y3=[ 1.6235    -0.3056    -0.7952      0.0680      0.0612      0.2062
0.0651    0.0079
   -0.3056    0.4141   -0.1188   -0.3670    0.1371    0.0632    0.1070
-0.0173
   -0.7952   -0.1188    0.8424   -0.0497   -0.3651    0.0379   -0.0192
0.0674
    0.0680   -0.3670   -0.0497    0.8443   -0.1489   -0.4474   -0.0018
0.0113
    0.0612    0.1371   -0.3651   -0.1489    0.7619   -0.1886   -0.4170
0.0561
    0.2062    0.0632    0.0379   -0.4474   -0.1886    0.7924   -0.1308
-0.3687
    0.0651    0.1070   -0.0192   -0.0018   -0.4170   -0.1308    0.8407
-0.1273
    0.0079   -0.0173    0.0674    0.0113    0.0561   -0.3687   -0.1273
0.8076];
Y4=[1.2225   -0.4752   -0.5485    0.4282    0.1273   -0.0548   -0.1901
0.0255
```

```
   -0.4752    0.3370   -0.0235   -0.2106    0.1867   -0.0314   -0.0147
-0.0422
   -0.5485   -0.0235    0.6749   -0.2650   -0.3693    0.2268    0.1165
0.0014
    0.4282   -0.2106   -0.2650    0.5162   -0.2249   -0.2214    0.2429
0.0194
    0.1273    0.1867   -0.3693   -0.2249    0.6641   -0.2088   -0.3185
0.1839
   -0.0548   -0.0314    0.2268   -0.2214   -0.2088    0.5670   -0.2678
-0.2593
   -0.1901   -0.0147    0.1165    0.2429   -0.3185   -0.2678    0.6262
-0.1980
    0.0255   -0.0422    0.0014    0.0194    0.1839   -0.2593   -0.1980
0.6081];
Y5=[0.7775   -0.4752   -0.1182    0.4282   -0.2607   -0.0548    0.1329
0.0255
   -0.4752    0.3297   -0.0235   -0.1894    0.1867   -0.0639   -0.0147
-0.0022
   -0.1182   -0.0235    0.2584   -0.2650    0.0074    0.2268   -0.1990
0.0014
    0.4282   -0.1894   -0.2650    0.4552   -0.2249   -0.1278    0.2429
-0.0962
   -0.2607    0.1867    0.0074   -0.2249    0.3201   -0.2088   -0.0249
0.1839
   -0.0548   -0.0639    0.2268   -0.1278   -0.2088    0.4229   -0.2678
-0.0810
    0.1329   -0.0147   -0.1990    0.2429   -0.0249   -0.2678    0.3668
-0.1980
    0.0255   -0.0022    0.0014   -0.0962    0.1839   -0.0810   -0.1980
0.3868];
Y6=[0.3765   -0.3056    0.1286    0.0680   -0.1946    0.2062   -0.1223
0.0079
   -0.3056    0.2525   -0.1188   -0.0330    0.1371   -0.1584    0.1070
-0.0272
    0.1286   -0.1188    0.0910   -0.0497    0.0032    0.0379   -0.0633
0.0674
    0.0680   -0.0330   -0.0497    0.1271   -0.1489    0.0982   -0.0018
-0.0880
   -0.1946    0.1371    0.0032   -0.1489    0.2222   -0.1886    0.0735
0.0561
    0.2062   -0.1584    0.0379    0.0982   -0.1886    0.1975   -0.1308
0.0283
   -0.1223    0.1070   -0.0633   -0.0018    0.0735   -0.1308    0.1523
-0.1273
    0.0079   -0.0272    0.0674   -0.0880    0.0561    0.0283   -0.1273
0.1873];
Y7=[0.0990   -0.0941    0.0800   -0.0587    0.0328   -0.0056   -0.0195
0.0396
   -0.0941    0.0895   -0.0764    0.0564   -0.0321    0.0067    0.0170
-0.0360
    0.0800   -0.0764    0.0659   -0.0499    0.0303   -0.0096   -0.0099
0.0260
   -0.0587    0.0564   -0.0499    0.0398   -0.0273    0.0137   -0.0005
-0.0110
    0.0328   -0.0321    0.0303   -0.0273    0.0232   -0.0182    0.0126
-0.0066
   -0.0056    0.0067   -0.0096    0.0137   -0.0182    0.0221   -0.0243
0.0242
   -0.0195    0.0170   -0.0099   -0.0005    0.0126   -0.0243    0.0337
-0.0392
```

```
    0.0396   -0.0360    0.0260   -0.0110   -0.0066    0.0242   -0.0392
0.0492];
Y8=zeros(8,8);
Y=[Y1 z;
    z Y1;
    Y2 z;
    z Y2;
    Y3 z;
    z Y3;
    Y4 z;
    z Y4;
    Y5 z;
    z Y5;
    Y6 z;
    z Y6;
    Y7 z;
    z Y7;
    Y8 z;
    z Y8];
g11=5/3 -exp(xs+1) + 23/3*xs + xs.^2 + 3*cos(xs+1)-sin(xs+1);
g21=1/12*(48 - 4*xs -11*xs.^2 + 4*xs.^5 - 9*xs.^6 - 12*cos(xs+1) -
36*sin(xs+1));
G=[0.7865;
   0.0216;
  0.7749;
  0.2385;
  0.6081;
  0.4800;
  0.1103;
  0.7194;
  -0.7292;
  1.2113;
  -1.7597;
  1.7430;
 -2.6468;
 1.6654;
 -3;
1.3333];
w=T-KK*Y;
s=inv(w);
A=s*G;
Tx=[1 x 2*x^2-1 4*x^3-3*x 8*x^4-8*x^2+1 16*x.^5-20*x.^3+5*x 32*x.^6-
48*x.^4+18*x.^2-1 64*x.^7-112*x.^5+56*x.^3-7*x];
TTx=[Tx h;
    h Tx];
F=TTx*A;
x=linspace(-1 , 1 , 50);
x=[1 0.75 0.5 0.25 0 -0.25 -0.5 -0.75 -1];
fe1=x.^3 + 2*x;
fe2=x.^2 -x/3;
fa1=-0.0072*x.^7 - 0.0006*x.^6 + 0.0095*x.^5 - 0.0005*x.^4 +
0.9956*x.^3 - 0.0004*x.^2 + 1.9987*x - 0.0018;
fa2= -0.0013*x.^7 -0.0051*x.^6 - 0.0008*x.^5 + 0.0038*x.^4 -
0.0007*x.^3 + 0.9982*x.^2 - 0.3344*x - 0.0008;
e1=fe1-fa1;
e2=fe2-fa2;
figure
hold on
plot(x,fa1,'*');
plot(x,fe1,'r');
```

```
xlabel ('x');
ylabel ('f1(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa2,'*');
plot(x,fe2,'r');
xlabel ('x');
ylabel ('f2(x)');
legend(' approximate solution','exact solution');
grid on
hold off
```

system (3.3) for m = 2

```
syms x xs tr t;
tr =[2 1 0];
k11=(xs)+(tr);
k12=(xs)*(tr);
k21=2*(xs)-(tr);
k22=(tr)+(tr)*(xs);
K = [ k11 k12;k21 k22];
T0 = [1 1 1];
T1= tr-1 ;
T2=2*(tr-1).^2 -1;
Tt=[1, t-1 ,2*(t-1)^2-1];
t=[T0' T1' T2'];
z=zeros(3,3);
tt=[t z ; z t];
n=zeros(1,3);
T=[T0 n;
    n T0;
    T1 n;
    n T1;
    T2 n;
    n T2];
Kxs=K *inv(tt);
K1=[  3     1     0     2     2     0
     3    -1     0     3     3     0];
K2=[ 2     1     0     1     1     0
     1    -1     0     2     2     0];
K3=[  1     1     0     0     0     0
    -1    -1     0     1     1     0];
v=zeros(2,6);
KK=[K1 v v ;
    v K2 v ;
    v v K3 ];

yt=Tt'*Tt;
y=int(yt,0,xs);
Y1=[ 2,    0,  -2/3;
    0, 2/3,     0;
 -2/3,    0, 14/15];
Y2=[  1, -1/2, -1/3;
 -1/2,  1/3,     0;
 -1/3,     0, 7/15];
Y3=zeros(3,3);
```

```
Y=[Y1 z ; z Y1 ; Y2 z ; z Y2 ; Y3 z ; z Y3 ];
g11=5*exp(xs)-2*(xs)         -9*sin(xs)-(xs).^2*exp(xs)+2*(xs)*cos(xs)-
(xs)*exp(xs)+3*(xs)*sin(xs);
g21=-3*(xs)-4*exp(xs)+sin(xs)-
(xs).^2*exp(xs)+(xs)*cos(xs)+(xs)*exp(xs)+4*(xs)*sin(xs)-1;
G=[-15.7815; -43.9830; 2.1866; -10.1255;5;-5];
c11=3*(xs)-8;
c12=-2*(xs) +5;
c21=4*(xs);
c22=(xs)-5;
c=[c11 c12;c21 c22];
h = zeros(2,2);
c1=[-2     1
     8    -3];
c2=[-5     3
     4    -4];
c3=[-8     5
     0    -5];
C=[c1 h h;h c2 h ; h h c3];
w=C*T-KK*Y;
s=inv(w);
A=s*G;
Tx=[1 x-1 2*(x-1)^2-1 0 0 0 ; 0 0 0 1 x-1 2*(x-1)^2-1];
F=Tx*A;
x=linspace(0 , 2 , 50);
fe1=sin(x);
fe2=exp(x);
fa1=1.2742*x - 0.4306*x.^2 - 0.0000;
fa2=1.3501*x.^2 + 0.3759*x + 1.0000;
e1=fe1-fa1;
e2=fe2-fa2;
figure
hold on
plot(x,fa1,'*');
plot(x,fe1,'r');
xlabel ('x');
ylabel ('f1(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa2,'*');
plot(x,fe2,'r');
xlabel ('x');
ylabel ('f2(x)');
legend(' approximate solution','exact solution');
grid on
```

system (3.3) for m = 5

```
syms x xs tr t;
tr =[2.0000    1.8090    1.3090    0.6910    0.1910         0];
k11=xs+tr;
k12=xs*tr;
k21=2*xs-tr;
k22=tr+tr*xs;
K = [ k11 k12;k21 k22];
T0 = [1 1 1 1 1 1];
```

```
T1= tr-1 ;
T2=2*(tr-1).^2 -1;
T3=4*(tr-1).^3 -3*(tr-1);
T4=8*(tr-1).^4 -8*(tr-1).^2 +1;
T5=16*(tr-1).^5-20*(tr-1).^3 +5*(tr-1);
Tt=[1,t-1,2*(t-1)^2-1,4*(t-1).^3-3*(t-1),8*(t-1).^4-8*(t-
1).^2+1,16*(t-1).^5-20*(t-1).^3+5*(t-1)];
t=[T0' T1' T2' T3' T4' T5'];
z=zeros(6,6);
tt=[t z ; z t];
n=zeros(1,6);
T=[T0 n;
    n T0;
    T1 n;
    n T1;
    T2 n;
    n T2;
    T3 n;
    n T3;
    T4 n;
    n T4;
    T5 n;
    n T5];
Kxs=K *inv(tt);
K1=[3.0000    1.0000   -0.0000    0.0000   -0.0000    0.0000    2.0001
1.9999   -0.0001    0.0001   -0.0000    0.0000
    3.0000   -1.0000    0.0000   -0.0000    0.0000   -0.0000    3.0001
2.9999   -0.0001    0.0001   -0.0000    0.0000];
K2=[2.8090    1.0000   -0.0000    0.0000   -0.0000    0.0000    1.8091
1.8089   -0.0001    0.0001   -0.0000    0.0000
    2.6180   -1.0000    0.0000   -0.0000    0.0000   -0.0000    2.8091
2.8089   -0.0001    0.0001   -0.0000    0.0000];
K3=[ 2.3090      1.0000     -0.0000      0.0000     -0.0000      0.0000
1.3091    1.3090   -0.0000    0.0000   -0.0000    0.0000
    1.6180   -1.0000    0.0000   -0.0000    0.0000   -0.0000    2.3091
2.3089   -0.0001    0.0001   -0.0000    0.0000];
K4=[1.6910    1.0000   -0.0000    0.0000   -0.0000    0.0000    0.6910
0.6910   -0.0000    0.0000   -0.0000    0.0000
    0.3820   -1.0000    0.0000   -0.0000    0.0000   -0.0000    1.6911
1.6909   -0.0001    0.0001   -0.0000    0.0000];
K5=[1.1910    1.0000   -0.0000    0.0000   -0.0000    0.0000    0.1910
0.1910   -0.0000    0.0000   -0.0000    0.0000
   -0.6180   -1.0000    0.0000   -0.0000    0.0000   -0.0000    1.1911
1.1910   -0.0000    0.0000   -0.0000    0.0000];
K6=[1.0000    1.0000   -0.0000    0.0000   -0.0000    0.0000         0
0         0         0         0         0
   -1.0000   -1.0000    0.0000   -0.0000    0.0000   -0.0000    1.0000
1.0000   -0.0000    0.0000   -0.0000    0.0000];
v=zeros(2,12);
KK=[K1 v v v v v;
    v K2 v v v v;
    v v K3 v v v;
    v v v K4 v v;
    v v v v K5 v;
    v v v v v K6];
yt=Tt'*Tt;
y=int(yt,0,xs);
Y1=[    2,     0,    -2/3,      0,   -2/15,      0;
      0,   2/3,       0,   -2/5,       0,  -2/21;
  -2/3,     0,   14/15,      0, -38/105,      0;
```

```
     0,   -2/5,        0,   34/35,        0,  -22/63;
 -2/15,      0,  -38/105,       0,    62/63,       0;
     0,  -2/21,        0,  -22/63,        0,   98/99];
Y2=[1.8090   -0.1728   -0.7893   -0.0534   -0.1152    0.0754;
   -0.1728    0.5098   -0.1131   -0.4522    0.0110   -0.0329;
   -0.7893   -0.1131    0.8469   -0.0487   -0.3700    0.0271;
   -0.0534   -0.4522   -0.0487    0.9292   -0.0326   -0.3691;
   -0.1152    0.0110   -0.3700   -0.0326    0.9301   -0.0648;
    0.0754   -0.0329    0.0271   -0.3691   -0.0648    0.8954];
Y3=[1.3090   -0.4523   -0.6227    0.3659    0.1682    0.0288;
   -0.4523    0.3432   -0.0432   -0.2273    0.1973   -0.0091;
   -0.6227   -0.0432    0.7386   -0.2117   -0.4045    0.1552;
    0.3659   -0.2273   -0.2117    0.5613   -0.2539   -0.2818;
    0.1682    0.1973   -0.4045   -0.2539    0.6840   -0.1696;
    0.0288   -0.0091    0.1552   -0.2818   -0.1696    0.6479];
Y4=[0.6910   -0.4523   -0.0440    0.3659   -0.3015    0.0288;
   -0.4523    0.3235   -0.0432   -0.1727    0.1973   -0.0861;
   -0.0440   -0.0432    0.1948   -0.2117    0.0426    0.1552;
    0.3659   -0.1727   -0.2117    0.4101   -0.2539   -0.0674;
   -0.3015    0.1973    0.0426   -0.2539    0.3001   -0.1696;
    0.0288   -0.0861    0.1552   -0.0674   -0.1696    0.3420];
Y5=[ 0.1910   -0.1728    0.1227   -0.0534   -0.0182    0.0754;
   -0.1728    0.1568   -0.1131    0.0522    0.0110   -0.0623;
    0.1227   -0.1131    0.0864   -0.0487    0.0081    0.0271;
   -0.0534    0.0522   -0.0487    0.0422   -0.0326    0.0199;
   -0.0182    0.0110    0.0081   -0.0326    0.0541   -0.0648;
    0.0754   -0.0623    0.0271    0.0199   -0.0648    0.0945];
Y6=zeros(6,6);
Y=[Y1 z ;
    z Y1 ;
    Y2 z ;
    z Y2 ;
    Y3 z ;
    z Y3 ;
    Y4 z;
    z Y4;
    Y5 z;
    z Y5;
    Y6 z;
    z Y6];
g11=5*exp(xs)-2*(xs)         -9*sin(xs)-(xs).^2*exp(xs)+2*(xs)*cos(xs)-
(xs)*exp(xs)+3*(xs)*sin(xs);
g21=-3*(xs)-4*exp(xs)+sin(xs)-
(xs).^2*exp(xs)+(xs)*cos(xs)+(xs)*exp(xs)+4*(xs)*sin(xs)-1;
G=[-15.7815;
   -43.9830;
   -8.4412;
   -32.2013;
   0.4811;
   -14.8721;
    2.9150;
    -7.6984;
    4.1702;
    -5.7054;
    5;
    -5];
c11=3*xs-8;
c12=-2*xs +5;
c21=4*xs;
c22=xs-5;
```

```
c=[c11 c12;c21 c22];
h = zeros(2,2);
c1=[-2     1
     8    -3];
 c2=[ -2.5730    1.3820
    7.2360   -3.1910];
c3=[-4.0730    2.3820
    5.2360   -3.6910];
c4=[-5.9270    3.6180
    2.7640   -4.3090];
c5=[-7.4270    4.6180
    0.7640   -4.8090];
c6=[-8     5
     0    -5];
C=[c1 h h h h h;
    h c2 h h h h;
    h h c3 h h h;
    h h h c4 h h;
    h h h h c5 h;
    h h h h h c6];
w=C*T-KK*Y;
s=inv(w);
A=s*G;
Tx=[1,x-1,2*(x-1)^2-1,4*(x-1).^3-3*(x-1),8*(x-1).^4-8*(x-
1).^2+1,16*(x-1).^5-20*(x-1).^3+5*(x-1)];
Txx=[Tx n;n Tx];
F=Txx*A;
x=linspace(0 , 2 , 50);
x=[0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2];
fe1=sin(x);
fe2=exp(x);
fa1=0.0025*x.^5 + 0.0184*x.^4 - 0.1869*x.^3 + 0.0072*x.^2 + 1.0000*x -
0.0000
 fa2=0.0191*x.^5 + 0.0199*x.^4 + 0.1864*x.^3 + 0.4901*x.^2 + 1.0027*x
+ 1.0000
e1=fe1-fa1
e2=fe2-fa2
figure
hold on
plot(x,fa1,'*');
plot(x,fe1,'r');
xlabel ('x');
ylabel ('f1(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa2,'*');
plot(x,fe2,'r');
xlabel ('x');
ylabel ('f2(x)');
legend(' approximate solution','exact solution');
grid on
```

MATLAB code for Bernstein's approximation method

System (3.1)

```
syms x t xq xr;
%xq= 0    0.5000    1.0000    1.5000    2.0000
p04= nchoosek(4,0)*(1-t).^4;
p14= nchoosek(4,1)*t*(1-t).^3;
p24= nchoosek(4,2)*t^2*(1-t).^2;
p34= nchoosek(4,3)*t^3*(1-t);
p44= nchoosek(4,4)*t^4;
p04xq= nchoosek(4,0)*(1-xq).^4;
p14xq= nchoosek(4,1)*xq*(1-xq).^3;
p24xq= nchoosek(4,2)*xq^2*(1-xq).^2;
p34xq= nchoosek(4,3)*xq^3*(1-xq);
p44xq= nchoosek(4,4)*xq^4;
p04x= nchoosek(4,0)*(1-x).^4;
p14x= nchoosek(4,1)*x*(1-x).^3;
p24x= nchoosek(4,2)*x^2*(1-x).^2;
p34x= nchoosek(4,3)*x^3*(1-x);
p44x= nchoosek(4,4)*x^4;
k11= xq^2 - 2*t;
k12 = t^2 - xq;
k13= 2*t;
k21=t*(xq+1);
k22=t*xq*(xq^2 +1);
k23=2*t^2 +xq^3;
k31=t-xq;
k32=t^2 - xq^3;
k33=2*t*xq+t^2;
K=[k11 k12 k13; k21 k22 k23; k31 k32 k33];
pr4=[p04 p14 p24 p34 p44];
pr4xq=[p04xq p14xq p24xq p34xq p44xq];
pr4x=[p04x; p14x; p24x; p34x; p44x];
T11=int(k11*pr4,0,xq);
s11=[0 0 0 0 0;
    -0.0109   -0.0469   -0.0437   -0.0198   -0.0036;
    0.1333    0.0667         0   -0.0667   -0.1333;
    0.3797    0.3656         0    0.7594   -0.3797;
     0.8000         0    3.2000   -4.2667    4.2667];
T12=int(k12*pr4,0,xq);
s12=[0 0 0 0 0;
    -0.0895   -0.0670   -0.0371   -0.0128   -0.0020;
    -0.1905   -0.1714   -0.1429   -0.1048   -0.0571;
    -0.2873   -0.3134   -0.0362   -0.6509    0.1627;
     -0.1143   -2.7429    7.3143  -11.2762    5.4857];
 T13=int(k13*pr4,0,xq);
 s13=[0 0 0 0 0;
    0.0594    0.0875    0.0688    0.0292    0.0052;
   0.0667    0.1333    0.2000    0.2667    0.3333;
   0.0844   -0.1125    1.5188   -3.0375    3.7969;
    0.8000   -6.4000   22.4000  -34.1333   21.3333];
T21=int(k21*pr4,0,xq);
s21=[0 0 0 0 0;
     0.0445    0.0656    0.0516    0.0219    0.0039;
     0.0667    0.1333    0.2000    0.2667    0.3333;
    0.1055   -0.1406    1.8984   -3.7969    4.7461;
    1.2000   -9.6000   33.6000  -51.2000   32.0000];
T22=int(k22*pr4,0,xq);
s22=[0 0 0 0 0;
```

```
    0.0186    0.0273    0.0215    0.0091    0.0016;
     0.0667    0.1333    0.2000    0.2667    0.3333;
     0.2057   -0.2742    3.7020   -7.4039    9.2549;
     4.0000  -32.0000  112.0000 -170.6667  106.6667];
 T23=int(k23*pr4,0,xq);
 s23=[0 0 0 0 0;
     0.0390    0.0489    0.0384    0.0166    0.0030;
      0.2190    0.2571    0.3143    0.3905    0.4857;
      0.7403    0.0904    4.2308   -7.7565   10.0075;
     4.5714  -24.6857   91.4286 -137.7524   87.7714];
 T31=int(k31*pr4,0,xq);
 s31=[0 0 0 0 0;
    -0.0672   -0.0375   -0.0156   -0.0042   -0.0005;
     -0.1667   -0.1333   -0.1000   -0.0667   -0.0333;
      -0.2672   -0.2250   -0.2531         0   -0.3797;
      -0.4000         0   -1.6000    2.1333   -2.1333];
  T32=int(k32*pr4,0,xq);
  s32=[0 0 0 0 0;
     -0.0169   -0.0060    0.0004    0.0013    0.0003;
     -0.1905   -0.1714   -0.1429   -0.1048   -0.0571;
    -0.6740   -0.5243   -1.3018    1.2475   -2.6849;
    -2.5143    6.8571  -31.0857   46.3238  -32.9143];
T33=int(k33*pr4,0,xq);
s33=[0 0 0 0 0;
     0.0371    0.0580    0.0473    0.0205    0.0037;
    0.0762    0.1619    0.2571    0.3619    0.4762;
   0.1487   -0.3134    3.2545   -6.7259    8.1362;
   2.2857  -18.7429   64.9143  -98.7429   60.9524];
c11=2*xq^2 +3;
c12=0;
c13=0;
c21=0;
c22=1-3*xq^2;
c23=0;
c31=0;
c32=0;
c33=3*xq^2 +6;
C=[c11 c12 c13;c21 c22 c23;c31 c32 c33];
y11=c11*pr4xq;
z11=[3     0     0     0     0;
    0.2188    0.8750    1.3125    0.8750    0.2188;
    0     0     0     0     5;
     0.4688   -5.6250   25.3125  -50.6250   37.9688;
      11   -88   264   -352   176];
y12=c12*pr4xq;
z12=zeros(5,5);
y13=c13*pr4xq;
z13=zeros(5,5);
y21=c21*pr4xq;
z21=zeros(5,5);
 y22=c22*pr4xq;
 z22=[1     0     0     0     0;
    0.0156    0.0625    0.0938    0.0625    0.0156;
    0     0     0     0    -2;
    -0.3594    4.3125  -19.4063   38.8125  -29.1094;
    -11    88  -264   352  -176];
y23=c23*pr4xq;
z23=zeros(5,5);
y31=c31*pr4xq;
z31=zeros(5,5);
```

```
y32=c32*pr4xq;
z32=zeros(5,5);
y33=c33*pr4xq;
z33=[6     0     0     0     0;
     0.4219    1.6875    2.5313    1.6875    0.4219;
     0     0     0     0     9;
     0.7969   -9.5625   43.0313  -86.0625   64.5469;
     18   -144   432   -576   288];
S=[s11 s12 s13;s21 s22 s23;s31 s32 s33];
Z=[z11 z12 z13;z21 z22 z23;z31 z32 z33];
w=Z-S;
v=inv(w);
xq=[0 0.5 1 1.5 2];
g11=(-1/15)*(6*xq.^5 + 35*xq.^4 - 95*xq.^3 - 270*xq.^2 - 225*xq -
360);
g21=(-1/30)*(15*xq.^7 + 10*xq.^6 -33*xq.^5 +275*xq.^4 +115*xq.^3 -
390*xq.^2 +150);
g31=(1/60)*(40*xq.^6  -66*xq.^5  -175*xq.^4  +250*xq.^3  +780*xq.^2
+360*xq +360);
G=[g11';g21';g31'];
A=v*G;
A1=[8.0000 9.2502 10.5001 11.7500 12.9999];
A2=[-5.0000 -4.9959 -4.6648 -3.9998 -3.0003];
A3=[1.0000 1.2500 1.6667 2.2500 2.9999];
f1=A1*pr4x;
f2=A2*pr4x;
f3=A3*pr4x;
 x=linspace(0 , 2 ,50);
 x=[0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2];
fa1 =-0.0003*x.^4 + 0.0012*x.^3 - 0.0018*x.^2 + 5.0008*x + 8
fa2 =- 0.0063*x.^4 + 0.0276*x.^3 +1.9620*x.^2 +0.0164*x - 5
fa3 = 0.0001*x.^4 - 0.0004*x.^3 +1.0002*x.^2 + x + 1
fe1=5*x + 8;
fe2=2*x.^2 - 5;
fe3=x.^2 + x + 1;
e1=fe1-fa1
e2=fe2-fa2
e3=fe3-fa3
figure
hold on
plot(x,fa1,'*');
plot(x,fe1,'r');
xlabel ('x');
ylabel ('f1(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa2,'*');
plot(x,fe2,'r');
xlabel ('x');
ylabel ('f2(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa3,'*');
plot(x,fe3,'r');
```

```
xlabel ('x');
ylabel ('f3(x)');
legend(' approximate solution','exact solution');
grid on
hold off
```

System (3.2)

```
syms t xr xq x;
%xq=[-1 -0.5 0 0.5 1]
p40=(1-t).^4;
p41=4*t*(1-t).^3;
p42=6*t^2*(1-t).^2;
p43=4*t^3*(1-t);
p44=t^4;
p04=(1-xq).^4;
p14=4*xq*(1-xq).^3;
p24=6*xq^2*(1-xq).^2;
p34=4*xq^3*(1-xq);
p44x=xq^4;
p4r=[p40 p41 p42 p43 p44];
pr4=[p04 p14 p24 p34 p44x];
k11=sin(xq-t);
k12 =exp(xq-t);
k21=cos(xq-t);
k22=3*t.*xq.^2;
k=[k11 k12;k21 k22];
T11=int(k11*p4r,-1,xq);
s11=[0, 0, 0, 0, 0;
    1.4124   -2.6019    1.8067   -0.5602    0.0654;
    3.8589   -6.5954    4.3446   -1.2950    0.1467;
    5.6685   -9.1022    5.8869   -1.7169    0.1930;
    6.1221   -9.3235    6.0549   -1.6613    0.2240];
T12=int(k12*p4r,-1,xq);
s12=[0, 0, 0, 0, 0;
     6.6273  -11.8193    7.9790   -2.4142    0.2760;
     12.7463  -21.2388   13.8581   -4.1119    0.4645;
     21.2910  -34.8105   22.9658   -6.7373    0.7727;
     35.1124  -57.3373   38.0060  -10.8936    1.5015];
T21=int(k21*p4r,-1,xq);
s21=[0, 0, 0, 0, 0;
    4.6233   -8.1019    5.3812   -1.6040    0.1809;
    4.6234   -6.9899    4.2941   -1.2193    0.1331;
     2.3882   -2.8152    1.7838   -0.4120    0.0527;
     -0.6160    1.9278   -1.1633    0.6170    0.1438];
T22=int(k22*p4r,-1,xq);
s22=[0, 0, 0, 0, 0;
   -2.9152    5.2203   -3.5367    1.0734   -0.1230;
   0, 0, 0, 0, 0;
    -3.2027    5.5828   -3.6492    1.1109   -0.1230;
    -12.8000   22.4000  -14.4000    4.8000         0];
c11=1;
c12=0;
c21=0;
c22=1;
C=[c11 c12;c21 c22];
y11=c11*pr4;
z11=[16   -32    24    -8     1;
    5.0625   -6.7500    3.3750   -0.7500    0.0625;
```

```
   1     0     0     0     0;
   0.0625    0.2500    0.3750    0.2500    0.0625;
   0     0     0     0     1];
y12=c12*pr4;
z12=zeros(5,5);
y21=c21*pr4;
z21=zeros(5,5);
 y22=c22*pr4;
 z22=[16   -32    24    -8     1;
    5.0625   -6.7500    3.3750   -0.7500    0.0625;
   1     0     0     0     0;
   0.0625    0.2500    0.3750    0.2500    0.0625;
   0     0     0     0     1];
S=[s11 s12 ; s21 s22];
Z=[z11 z12 ; z21 z22];
w=Z-S;
v=inv(w);
xq=[-1 -0.5 0 0.5 1];
g11=5/3 -exp(xq+1) + 23/3*xq + xq.^2 + 3*cos(xq+1)-sin(xq+1);
g21=1/12*(48  -  4*xq  -11*xq.^2  +  4*xq.^5  -  9*xq.^6  -  12*cos(xq+1)  -
36*sin(xq+1));
G=[g11';g21'];
A=v*G;
A1=[0.0002 0.5002 1.0000 1.7497 3.0006];
A2=[-0.0000 -0.0834 0.0001 0.2504 0.6672];
pr4x=[(1-x).^4;
4*x*(1-x).^3;
6*x^2*(1-x).^2;
4*x^3*(1-x);
x^4];
f1=A1*pr4x;
f2=A2*pr4x;
x=linspace(-1 , 1 , 50);
fa1=0.0012*x.^4 + 1.0004*x.^3 - 0.0012*x.^2 + 2*x + 0.0002;
fa2=- 0.0002*x.^4 - 0.0004*x.^3 + 1.0014*x.^2 - 0.3336*x;
fe1=x.^3 + 2*x;
fe2=x.^2 -x/3;
e1=fe1-fa1;
e2=fe2-fa2;
figure
hold on
plot(x,fa1,'*');
plot(x,fe1,'r');
xlabel ('x');
ylabel ('f1(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa2,'*');
plot(x,fe2,'r');
xlabel ('x');
ylabel ('f2(x)');
legend(' approximate solution','exact solution');
grid on
hold off
```

System (3.2)

```
syms x t xq xr;
%xq= 0    0.5000    1.0000    1.5000    2.0000
p04= nchoosek(4,0)*(1-t).^4;
p14= nchoosek(4,1)*t*(1-t).^3;
p24= nchoosek(4,2)*t^2*(1-t).^2;
p34= nchoosek(4,3)*t^3*(1-t);
p44= nchoosek(4,4)*t^4;
p04xq= nchoosek(4,0)*(1-xq).^4;
p14xq= nchoosek(4,1)*xq*(1-xq).^3;
p24xq= nchoosek(4,2)*xq^2*(1-xq).^2;
p34xq= nchoosek(4,3)*xq^3*(1-xq);
p44xq= nchoosek(4,4)*xq^4;
p04x= nchoosek(4,0)*(1-x).^4;
p14x= nchoosek(4,1)*x*(1-x).^3;
p24x= nchoosek(4,2)*x^2*(1-x).^2;
p34x= nchoosek(4,3)*x^3*(1-x);
p44x= nchoosek(4,4)*x^4;
k11=xq+t;
k12=xq*t;
k21=2*xq-t;
k22=t+t*xq;
K=[k11 k12 ; k21 k22];
pr4=[p04 p14 p24 p34 p44];
pr4xq=[p04xq p14xq p24xq p34xq p44xq];
pr4x=[p04x; p14x; p24x; p34x; p44x];
T11=int(k11*pr4,0,xq);
s11=[0 0 0 0 0 ;
    81/640, 1/8, 27/320, 1/30, 11/1920;
    7/30, 4/15, 3/10, 1/3, 11/30;
    45/128, 9/80, 567/320, -243/80, 2673/640;
    6/5, -32/5, 24, -544/15, 352/15];
T12=int(k12*pr4,0,xq);
s12=[0 0 0 0 0 ;
    19/1280, 7/320, 11/640, 7/960, 1/768;
    1/30, 1/15, 1/10, 2/15, 1/6;
    81/1280, -27/320, 729/640, -729/320, 729/256;
    4/5, -32/5, 112/5, -512/15, 64/3];
T21=int(k21*pr4,0,xq);
s21=[0 0 0 0 0;
    21/128, 19/160, 21/320, 11/480, 7/1920;
    11/30, 1/3, 3/10, 4/15, 7/30;
    369/640, 63/160, 81/64, -243/160, 1701/640;
    6/5, -16/5, 72/5, -64/3, 224/15];
T22=int(k22*pr4,0,xq);
s22=[0 0 0 0 0 ;
    57/1280, 21/320, 33/640, 7/320, 1/256;
    1/15, 2/15, 1/5, 4/15, 1/3;
    27/256, -9/64, 243/128, -243/64, 1215/256;
    6/5, -48/5, 168/5, -256/5, 32];
c11=3*xq -8;
c12=-2*xq +5;
c21=4*xq;
c22=xq-5;
C=[c11 c12;c21 c22];
y11=c11*pr4xq;
z11=[ -8      0      0      0      0;
     -0.4063   -1.6250   -2.4375   -1.6250   -0.4063;
     0      0      0      0     -5;
```

```
     -0.2188    2.6250  -11.8125   23.6250  -17.7188;
      -2    16    -48     64    -32];
y12=c12*pr4xq;
z12=[ 5     0     0     0     0;
    0.2500    1.0000    1.5000    1.0000    0.2500;
     0     0     0     0     3;
      0.1250   -1.5000    6.7500  -13.5000   10.1250;
       1    -8    24   -32    16];
   y21=c21*pr4xq;
   z21=[0 0 0 0 0;
     0.1250    0.5000    0.7500    0.5000    0.1250;
      0     0     0     0     4;
      0.3750   -4.5000   20.2500  -40.5000   30.3750;
       8   -64   192  -256   128];
   y22=c22*pr4xq;
  z22=[-5     0     0     0     0;
    -0.2813   -1.1250   -1.6875   -1.1250   -0.2813;
    0     0     0     0    -4;
     -0.2188    2.6250  -11.8125   23.6250  -17.7188;
      -3    24    -72    96    -48];
  S=[s11 s12 ; s21 s22];
Z=[z11 z12 ; z21 z22];
w=Z-S;
v=inv(w);
g11=5*exp(xq)-2*xq               -9*sin(xq)-xq.^2*exp(xq)+2*xq*cos(xq)-
xq*exp(xq)+3*xq*sin(xq);
g21=-3*xq-4*exp(xq)+sin(xq)-
xq.^2*exp(xq)+xq*cos(xq)+xq*exp(xq)+4*xq*sin(xq)-1;
G=[5;3.2890; 2.1866;-1.6744;-15.7815; -5;-6.8056; -10.1255;-19.6995;-
43.9830];
A=v*G;
A1=[0 0.2485 0.5016 0.7065 0.8415];
A2=[1.0000 1.2438 1.5879 2.0397 2.7183];
f1=A1*pr4x;
f2=A2*pr4x;
x=linspace(0 , 2 , 50);
fa1=0.0311*x.^4 - 0.2112*x.^3+ 0.0276*x.^2+ 0.9940*x;
fa2=0.1117*x.^4 +0.0296*x.^3 + 0.6018*x.^2 + 0.9752*x + 1;
fe1=sin(x);
fe2=exp(x);
 e1=fe1-fa1;
 e2=fe2-fa2;
figure
hold on
plot(x,fa1,'*');
plot(x,fe1,'r');
xlabel ('x');
ylabel ('f1(x)');
legend(' approximate solution','exact solution');
grid on
hold off
figure
hold on
plot(x,fa2,'*');
plot(x,fe2,'r');
xlabel ('x');
ylabel ('f2(x)');
legend(' approximate solution','exact solution');
grid on
```

جــامعـــة النجاح الوطنية

كليــة الدراســـات العليــا

الحلول العددية لنظام معادلات فولتيرا التكاملية

إعداد

تسنيم عيسى محمد مغربي

إشراف

أ.د. ناجي قطناني

قدمت هذه الرسالة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات، من كلية الدراسات العليا، في جامعة النجاح الوطنية، نابلس – فلسطين.

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الحلول العددية لنظام معادلات فولتيرا التكاملية

اعداد
تسنيم عيسى محمد مغربي
اشراف
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الملخص

في هذه الأطروحة ركزنا على الحلول التقريبية لنظام معادلات فولتيرا التكاملية (Numerical Solutions of System of Volterra Integral Equations) وقمنا باستقصاء بعض الطرق العددية لحل نظام معادلات فولتيرا التكاملية. هذه الطرق هي: طريقة جلاركن مع كثيرات حدود لاجير(Galerkin method with Laguerre polynomials)، طريقة التجميع شيبيشيف (Chebyshev collocation method) ، طريقة التقريب لبرنشتاين (Bernstein's approximation method). إن الأمثلة العددية التي تناولناها نفذت باستخدام هذه الطرق العددية لحل نظام معادلات فولتيرا التكاملية.

تم وضع مقارنة بين هذه الطرق العددية حيث أظهرت لنا النتائج العددية أن طريقة التقريب لبرنشتاين أكثر كفاءة وفاعلية بالمقارنة مع الطرق العددية الأخرى التي تم دراستها وذلك بناء على الأمثلة التي استخدمناها في الرسالة.