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Faculty of Graduate Studies

Model Order Reduction by Balanced Truncation Method

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Dedication

I dedicate this thesis to the best mother and father in this universe, Hanan and Muhammad, who have been the source of motivation and strength during the moments of despair and frustration. Without your everlasting love, endless support, sacrifices, your prayers, your faith in me and your understanding of my goals, I wouldn't have been where I am today and what I am on today.

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الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

Model Order Reduction by Balanced Truncation Method

أقر بأن ما اشتملت عليه هذه الرسالة إنما هو من نتاج جهدي الخاص باستثناء ما تمت الإشارة إليه حيثما ورد، وان هذه الرسالة ككل، أو أي جزء منها لم يقدم لنيل أي درجة أو لقب علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's Name: **اسم الطالب:**

Signature: **التوقيع:**

Date: **التاريخ:**

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Model Order Reduction by Balanced Truncation Method

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Abstract

In this thesis, we focus our attention on one of the most important method for the model order reduction called balanced truncation method for continuous linear time invariant systems.

Balancing system is equivalent to finding the controllability and observability Gramian of the system in diagonal form which is equal to the Hankel singular values.

As a result of the balanced truncation method, we find an approximation error which is H_∞ norm of transfer functions (in frequency domain) and L_2 norm of the outputs.

Numerical results for the mass spring damper system have shown that balanced truncation method is one of the most efficient methods for model order reduction, since the reduced system maintains the properties and simulates the behavior of the original system.

Introduction

Control theory is a branch of applied mathematics specially in physical systems. Also control theory is relevant to the theory of differential equations, it did not become a branch in its own right until the late 1950s and early 1960s, because of problems growing in engineering and economics and were realized as problems in differential equations, therefore, it was realized that these seemingly various problems all had the same mathematical structure, and control theory emerged [24].

Control theory is The mathematical study of how to manipulate the parameters affecting the behavior of a system to produce the desired or optimal outcome [38].

There are many examples of control systems such as Distributed parameter systems, Fractional-order control, Model predictive control, Process control, Vector control and State space (controls).

In control engineering, a state space representation is a mathematical model of a physical system as a set of input, output and state variables related by differential equations. The state of the system can be represented as a vector within that space [21,22].

The most general state space representation of a continuous linear time invariant system (LTI) with p inputs, q outputs and n state variables is written in the following form [7,12,13]:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du$$

where:
n is the dimension of the system,

$x(t) \in \mathbb{R}^n$ is called the state vector,

$x(t_0)$ is called the initial condition of the system,

$y(t) \in \mathbb{R}^q$ is called the output vector,

$u(t) \in \mathbb{R}^p$ is called the input (or control) vector,

A is called the state (or system) matrix,

B is called the input matrix,

C is called the output matrix,

D is called the feed through (or feed forward) matrix,

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{q\times p}$ are constant matrices.

The proposed approach aims to reducing the number of states of the system and lowering the computational effort. To do this we apply model order reduction.

Model Order Reduction (MOR) is a technique for reducing the computational complexity of mathematical models in numerical simulations, for example, in simulations of large scale dynamical system and control system by a reduction of the model's associated state space, an approximation to the original model is computed.

Methods of Model Order Reduction: Proper orthogonal decomposition, Proper generalized decomposition, Approximate balancing, Reduced Basis Method, Matrix Interpolation, Transfer Function Interpolation, Piecewise Tangential Interpolation, Loewner framework, (Empirical) cross Gramian, Krylov subspace methods and Balanced truncation [2]. In this thesis we take a balanced truncation as a method of model order reduction.

Balanced truncation is a model reduction technique for producing simple approximate models of complex linear systems. This technique may have significant applications in physics contexts, and the results suggest it will prove a useful tool for treating large system [29].

To use balanced truncation method the systems have to be stable, controllable and observable. The system is stable if the eigenvalues of the system matrix have a negative real parts, but it is said to be controllable, if the control vector could drive the state from an initial state to desired final state in finite time intervals, and it is observable if the state of the system can be determined by measuring the system output over a finite time interval [1].

Our goal is to apply the balanced truncation method in order to obtain a low dimensional system that has similar response and characteristics as the original system with far lower storage requirements [23].

The general idea of balanced truncation method is to neglect the original states x that are difficult to control, i.e., require a large amount of control energy , and states that are difficult to observe, i.e., produce a small observation energy. Hopefully, this would lead to a system that is of lower order and retains the important dynamic behavior of the original system [11,14,19].

Balanced truncation follows particular procedure which begins with transforming the system to a balanced representation and then ignoring (truncating) some of the state variables, stability is preserved and there is a priori computable error bound for the error system [23].

This thesis is organized as follows:

Chapter (1) introduces some important properties of Laplace transform in particular the derivatives and the integral of matrices.

State space representation, the definition of state transition matrix and its properties, solution of state space system, definitions and theorems of very important concepts such as (stable controllable and observable system, controllability and observability Gramians) and discussion about transfer function are presented in chapter (2).

In chapter (3) we investigate the model order reduction of linear time invariant system by balanced truncation method.

Numerical results for the mass spring damper system are illustrated in chapter (4) followed by conclusion.

Chapter One

Preliminaries

In this chapter we introduce the Laplace transforms and its properties, since it is a very important tool for modeling dynamical systems. Moreover, we present some important properties regarding derivatives and integration of matrices since the state space system is written in matrix form and extends the characteristics of the matrix exponential function.

1.1 Laplace Transforms

A function $f(t)$ defined for $0 \leq t < \infty$ is called exponentially bounded if there exist $\sigma \in \mathbb{R}$ and a positive constant $\mathrm{k} < \infty$ such that $|f(t)| < ke^{\sigma t}, \forall t \geq 0$. The Laplace transform is an integral transform. It takes a function of a real variable$t$ (often time) to a function of a complex variable s (frequency) [36].

Definition (1.1) [36]: The Laplace transform of continuous exponentially bounded function $f(t)$, $t \in [0, \infty)$ is denoted by $\mathcal{L}[f(t)]$ and written as:

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\, dt = F(s)$$

where $s = \sigma + i\omega$, σ and ω are real variables.

Remark (1.1) [8]: If the continuous function $f(t)$ is defined for all real numbers $t \in [0, \infty)$ then it is called inverse Laplace transform and it's written as:

$$\mathcal{L}^{-1}[F(s)] = f(t).$$

Some Properties of Laplace Transforms

The following theorems include some properties of Laplace transforms [8,17].

Theorem (1.1) (Linearity):

If a is a constant or is independent of s and t, and if $\mathcal{L}[f(t)] = F(s)$, then

$$\mathcal{L}[af(t)] = a\mathcal{L}[f(t)] = aF(s).$$

Theorem (1.2) (Superposition):

If $\mathcal{L}[f_1(t)] = F_1(s)$ and $\mathcal{L}[f_2(t)] = F_2(s)$, then

$$\mathcal{L}[f_1(t) + f_2(t)] = \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)] = F_1(s) + F_2(s).$$

Theorem (1.3) (Real Differentiation):

If $\mathcal{L}[f(t)] = F(s)$ and let $\dot{f}(t) = \frac{d}{dt}f(t)$, then

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0).$$

In general, the transform of the n^{th} derivative $\frac{d^n f}{dt^n}$ is

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0).$$

Theorem (1.4) (Real Integration):

If $\mathcal{L}[f(t)] = F(s)$, then the Laplace transform of $\int_0^t f(\tau)d\tau$ given by:

$$\mathcal{L}\left[\int_0^t f(x)dx\right] = \frac{F(s)}{s}.$$

1.2 The Derivative and Integral of Matrices

State space system brings the differential equations that describe the time domain of the system and analyzes them in vector form utilizing state variables. This makes it possible to perform the system through simple matrix, which also allows multiple input, multiple output systems to be evaluated. In this section, we introduce the derivative and integral of a matrix and we discuss its properties.

Definition (1.2) [8]: Let $A(t) = [a_{ij}(t)]$ be an $n \times n$ matrix where the entries of $A(t)$ are a function of time t, then :

1. The derivative of $A(t)$ denoted by $\frac{d}{dt}A(t)$ is:

$$\frac{d}{dt}A(t) = \dot{A}(t) = \frac{d}{dt}(a_{ij}(t))$$

2. The integral of $A(t)$ is :

$$\int A(t)dt = \int a_{ij}(t)dt.$$

Let (α,β) be two constants and (A,B) two matrices, then [13]:

- $\frac{d}{dt}(\alpha A) = \alpha\frac{d}{dt}A = \alpha\dot{A}$
- $\frac{d}{dt}(\alpha A + \beta B) = \alpha\frac{d}{dt}A + \beta\frac{d}{dt}B = \alpha\dot{A} + \beta\dot{B}$
- $\int_a^b \alpha A dt = \alpha\int_a^b A dt$,where a and b are real numbers
- $\int_a^b (\alpha A + \beta B)dt = \alpha\int_a^b A dt + \beta\int_a^b B dt$
- $A^0 = I$
- $\frac{d}{dt}A^n \neq nA^{n-1}\frac{dA}{dt}$.

1.3 The Matrix Exponential Function (e^{At})

This section contains the definition and properties of the matrix exponential function, since it plays very important role in dynamical system.

Definition (1.3) [5]: Let $A \in \mathbb{R}^{n\times n}$ be a constant matrix. The matrix exponential function, denoted by e^{At}, is defined as

$$e^{At} = \sum_{n=0}^{\infty}\frac{t^n A^n}{n!}.$$

Characteristics of The Matrix Exponential Function [17]:

1. The powers A^n make sense, since A is a square matrix. This series converges for all t and every matrix A.

2. Differentiating the series term-by-term,

$$\frac{d}{dt}(e^{At}) = \sum_{n=0}^{\infty} \frac{nt^{n-1}A^n}{n!} = \sum_{n=1}^{\infty} \frac{t^{n-1}A^n}{(n-1)!} = A\sum_{n=1}^{\infty} \frac{t^{n-1}A^{n-1}}{(n-1)!}$$
$$= \sum_{m=0}^{\infty} \frac{t^m A^m}{m!} = Ae^{At}.$$

3. $e^{A0} = I$ (by setting $t = 0$ in the power series).
4. If $AB = BA$, then

$$e^A e^B = e^{A+B}$$

(e^A is just e^{At} with $t = 1$).

Chapter Tow

State Space Representation

In this chapter we give an overview for the state space representation which is a mathematical model of a physical system as a set of input, output and state variables related by differential equations. The major importance of state space representation is its relevancy to dealing with various systems: linear and nonlinear, time varying and time invariant, single input single output (SISO) and multiple input multiple output (MIMO).

Moreover, we will consider the transition matrix and its properties together with the initial condition of state variables for finding the solution of state space system.

Some important properties and concepts of the dynamical system like stability, controllability, observability, Gramians and transfer function are presented in this chapter.

2.1 State Space Representation

The state-space representation of a linear system is written in the following form [7,13]:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ x(t_0) &= x_0 \end{aligned} \qquad (2.1)$$

where:
n is the dimension of the system,

$x(t) \in \mathbb{R}^n$ is called the state vector,

$x(t_0)$ is called the initial condition of the system,

$y(t) \in \mathbb{R}^q$ is called the output vector,

$u(t) \in \mathbb{R}^p$ is called the input (or control) vector,

A is called the state (or system) matrix,

B is called the input matrix,

C is called the output matrix,

D is called the feed through (or feed forward) matrix,

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{q\times p}$ are constant matrices.

The first equation of (2.1) is called the state equation, and the second equation of (2.1) is called the output equation.

A dynamical system can be single input ($p = 1$) and single output ($q = 1$) in this case it is called a SISO (single input and single output) system, otherwise it is called MIMO (multiple input and multiple output) system [39].

If the matrices of the system (A, B, C, D) depend on the time then the system is called linear time varying (LTV) system, otherwise (when the matrices of the system are constant) the system is called linear time invariant (LTI) system. The time variable t can be continuous (i.e. $t \in \mathbb{R}$) or discrete (i.e. $t \in \mathbb{Z}$) [30].

In this thesis, we consider the finite dimensional continuous linear time invariant system.

In most dynamical systems the matrix D is the null matrix.

Now, we rewrite equation (2.1) in a new form as [30,37]:

$$\dot{x} = Ax + Bu$$
$$y = Cx \qquad (2.2)$$

2.2 State Transition Matrix

The state transition matrix is used to find the solution to a general state space equation of a linear system. We use it to find a solution of the state space equations:

$$\dot{x} = Ax + Bu$$
$$y = Cx.$$

Definition (2.1) [15]: The state transition matrix of a dynamical system is a matrix function denoted by $\phi(t, t_0)$ and a acts as a transformation from one state to another

$$\phi(t, t_0) = e^{A(t-t_0)} \qquad (2.3)$$

where A is a constant matrix.

Properties of state transition matrix [6]:

- $\phi(t_2, t_1)\phi(t_1, t_0) = \phi(t_2, t_0)$
- $\phi^{-1}(t, \tau) = \phi(\tau, t)$
- $\phi^{-1}(t, \tau)\phi(t, \tau) = I$
- $\frac{d}{dt}(\phi(t, t_0)) = A\phi(t, t_0)$.

2.3 Solution of State Space System [8]

In this section, our goal is to find the solution of the state space system in equation (2.2). So we solve the state space equation and the output equation, respectively.

Assume that $t_0 = 0$ so $x(0) = x_0$,

- Solution of state space equation:

$$\dot{x} = Ax + Bu$$

$$\dot{x} - Ax = Bu$$

$$e^{-At}(\dot{x} - Ax) = e^{-At}Bu$$

$$\frac{d}{dt}[e^{-At}x] = e^{-At}Bu$$

$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \qquad (2.4)$$

- Solution of output equation:

$$y(t) = Cx(t)$$

by substituting $x(t)$ in equation (2.4) we obtain:

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\,. \qquad (2.5)$$

2.4 Stability, Controllability and Observability

Here we present some important concepts describing the dynamical system. These are: stability, controllability and observability.

2.4.1 Stability

Here, we introduce the stability of the dynamical system. Eigenvalues of a system are robustly associated to the stability of the system. Stability is the property of a system that verifies that the output of a system is bounded in the interval time [33].

Definition (2.2) [1]: A matrix A is called stable matrix if and only if the eigenvalues of A have a strictly negative real parts (i.e. $Re(\lambda_i) < 0,$ where λ_i is the eigenvalue of A and $i = 1, \ldots, n$).

The following theorem explain the stability of the system.

Theorem (2.1) [1]: The system is asymptotically stable if and only if all eigenvalues of A have negative real parts (i.e. $Re(\lambda_i) < 0,$ where λ_i is the eigenvalue of A and $i = 1, \ldots, n$).

2.4.2 Controllability

Controllability is one of the most important features of dynamical systems. It describes the capability of an exterior input to move the interior state of a system from any initial state to desired state in a finite time interval [27].

Definition (2.3) [27]: A state x_0 is controllable at time t_0 if for some finite time $t_1 > 0$, there exists an input $u(t)$ that transfers the state $x(t)$ from x_0 to a desired final state x_1 at time t_1.

Definition (2.4) [8]: A system is called controllable at time t_0 if every state x_0 in the state space is controllable.

Definition (2.5) [8]: The controllability matrix of the system is defined as

$$C(A,B) = (B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B) \tag{2.6}$$

where n is a positive integer.

2.4.3 Observability

Observability is one of the most important features of dynamical systems. Whereas it is a measurement of how we can deduce internal states of a system from its outputs [25].

Definition (2.6) [34]: A system with an initial state x_0 is observable if and only if the value of the initial state can be determined from the system output $y(t)$ that has been observed through the finite time.

Definition (2.7) [8]: The observability matrix of the system in equation (2.2) is defined as

$$O(C,A) = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{2.7}$$

where n is a positive integer.

2.5 Controllability Gramian and Observability Gramian

Controllability and observability Gramians are matrices that play a very important role in a dynamical system. Now we introduce these two Gramians and clarify some theorems about them.

Consider the linear time invariant system:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

If the system is stable, we define the controllability and observability Gramians denoted by W_c and W_o respectively as:

Definition (2.8) [9]: The Controllability Gramian of the system in equation (2.2) is

$$W_c = \int_0^{\infty} e^{At} B B^T e^{A^T t} dt \tag{2.8}$$

Definition (2.9) [9]: The Observability Gramian of the system in equation (2.2) is

$$W_o = \int_0^{\infty} e^{A^T t} C^T C e^{At} dt \tag{2.9}$$

Theorem (2.2) [9]: The two matrices W_c and W_o are the solutions of the Lyapunov equation, so we have:

$$\begin{aligned} AW_c + W_c A^T + BB^T &= 0 \\ W_o A + A^T W_o + C^T C &= 0. \end{aligned} \tag{1.10}$$

Definition (2.10) [15]: The continuous Lyapunov equation has the form:

$$AX + XA^T = -M$$

where $X \in \mathbb{R}^n$ is a symmetric matrix.

Theorem (2.3) [6,9,39]: The following statements are equivalent:

1. The pair $(A, B)(A, B)$ is controllable.

2. The $n \times n$ matrix

$$W_c(t) = \int_0^{t} e^{A\tau} BB^T e^{A^T \tau} d\tau$$

is positive definite matrix ($W_c(t)$ is nonsingular for any $t > 0$).

3. The $n \times np$ controllability matrix

$$C(A,B) = (B \quad AB \quad A^2B \quad \dots A^{n-1}B)$$

has rank n.

Proof [6,9,39],

(1 → 2):

Let the pair (A,B) is controllable but $W_c(t_1)$ is singular for some $t_1 > 0$.

Since $\int_0^{\infty} e^{At}BB^T e^{A^T t} \geq 0$ for all t, there exists a real vector $v \neq 0$,

$(v \in \mathbb{R}^n)$ such that $v^T e^{At}B = 0,\ t \in [0, t_1]$.

Now let $x(\mathrm{t}_1) = x_1 = 0$, and then from the solution in equation (2.4), we have

$$0 = e^{A\mathrm{t}_1}x_0 + \int_0^{\mathrm{t}_1} e^{A(\mathrm{t}_1-\tau)}Bu(\tau)d\tau.$$

Pre-multiply the above equation by v^{T} to have

$$v^{\mathrm{T}}e^{A\mathrm{t}_1}x_0 = 0.$$

If we chose the initial state $x_0 = e^{-A\mathrm{t}_1}v$, then $v = 0$, and this is a contradiction.

Hence, $W_c\,(t)$ cannot be singular for any $t > 0$.

(2 → 3):

Suppose $W_c\,(t) > 0$ for all $t > 0$ but the controllability matrix $C(A,B)$ does not have full row rank. Then there exists a real vector $v \neq 0$,

$(v \in \mathbb{R}^n)$ such that

$$v^T A^k B = 0, \qquad \text{for all } 0 \leq k \leq n-1.$$

hence,

$$v^T e^{At} B = 0, \qquad \text{for all } t$$

or, equivalently, $v^T W_c(t) = 0$ for all t; this is a contradiction, and hence, the controllability matrix $C(A, B)$ must be full row rank.

(3 → 1):

Let $rank\, C(A, B) = rank\,(B \quad AB \quad A^2B \quad \dots A^{n-1}B) = n$

Assume that the system is not controllable, i.e., *rank* $(W_c(t)) < n$.

Then there exists a real vector $v \neq 0, (v \in \mathbb{R}^n)$ such that $W_c(t)v = 0$, which implies that $v^T e^{At} B = 0$.

Expanding the matrix exponential we get $v^T A^k B = 0$ for all $k \geq 0$.

However, this yields that

$$rank\, C(A, B) = rank\,(B \quad AB \quad A^2B \quad \dots A^{n-1}B) < n$$

which contradicts the assumption, and hence, the system is controllable.

Theorem (2.4) [6]: The controllability Gramian is a unique solution of the Lyapunov equation

$$AW_c + W_c A^T = -BB^T.$$

Proof [6]:

$$\begin{aligned} AW_c + W_c A^T &= \int_0^\infty A e^{At} BB^T e^{A^T t} dt + \int_0^\infty e^{At} BB^T e^{A^T t} A^T dt \\ &= \int_0^\infty \frac{d}{dt} (e^{At} BB^T e^{A^T t}) dt \\ &= e^{At} BB^T e^{A^T t} |_{t=0}^{\infty} \\ &= -BB^T. \end{aligned}$$

This proves that W_c is actually a solution for the Lyapunov equation.

Now, we prove that W_c is unique. Suppose we have two different solutions for

$$AW_c + W_cA^T = -BB^T.$$

and they are given by W_{c1} and W_{c2} . Then we have:

$$A(W_{c1} - W_{c2}) + (W_{c1} - W_{c2})A^T = 0$$

Multiplying by e^{At} by the left and by $e^{A^T t}$ by the right, to have

$$e^{At}[A(W_{c1} - W_{c2}) + (W_{c1} - W_{c2})A^T]e^{A^T t} = \frac{d}{dt}[e^{At}[(W_{c1} - W_{c2})e^{A^T t}] = 0$$

Integrating from 0 to ∞:

$$[e^{At}[(W_{c1} - W_{c2})e^{A^T t}]|_{t=0}^{\infty} = 0$$

$$0 - (W_{c1} - W_{c2}) = 0$$

so,

$$W_{c1} = W_{c2}$$

In other words, W_c is unique.

Theorem (2.5) [9,39]: The following statements are equivalent:

1. The pair (A, C) is observable.

2. The $n \times n$ matrix

$$W_o(t) = \int_0^t e^{A^T \tau} C^T C e^{A\tau} d\tau$$

is positive definite matrix ($W_o(t)$ is nonsingular for any $t > 0$).

3. The $nq \times n$ controllability matrix

$$\begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

has *rank* n.

Theorem (2.6) [6]: The Observability Gramian is a unique solution of the Lyapunov equation

$$W_o A + A^T W_o + C^T C = 0.$$

2.6 Transfer Function

The transfer function of a control system yields the output of the system for every possible input, so it performs the direct relation between input and output in the system [33].

For a linear dynamical system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

with the constraint that all initial conditions are zero, the transfer Function denoted by $G(s)$ and defined as:

$$G(S) = \frac{Y(s)}{U(s)} \tag{2.11}$$

Where, $Y(s)$ and $U(s)$ are the Laplace transforms of $y(t)$ and $u(t)$ respectively.

Equation (2.11) can be written as

$$Y(s) = G(s)U(s) \tag{2.12}$$

Definition (2.11) [10]: The (A, B, C)matrices is called a realization of the transfer function $G(s)$ if

$$G(s) = C(sI - A)^{-1}B \qquad (2.13).$$

Consider the system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= 0 \end{aligned} \qquad (2.14)$$

If we take the Laplace transform of system (2.14), we obtain

$$sX(s) = AX(s) + BU(s)$$

$$X(s) = (sI - A)^{-1}BU(s) \qquad (2.15)$$

$$Y(s) = CX(s)$$

$$Y(s) = C(sI - A)^{-1}BU(s) \qquad (2.16)$$

so that,

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B \qquad (2.17).$$

Theorem (2.7) [8,10]: A state space realization of a system that has a transfer function $G(s)$ is minimal if and only if it is controllable and observable.

One main feature employing transfer functions is they permit for taking a system in a time domain and perform it in frequency domain [4].

Definition (2.12) [4]: For the continuous time state space representation given by (2.14), the matrix

$$G(i\omega) = C(i\omega I - A)^{-1}B$$

is called the frequency response matrix, where $\omega \in \mathbb{R}$ is the frequency.

Mainly a frequency response function is a mathematical description of the connection between the input and the output of a system.

$$G(iw) = \frac{Y(i\omega)}{U(i\omega)}$$

where $Y(i\omega)$ is the output of the system in the frequency domain, and $U(i\omega)$is the input to the system in the frequency domain.

The transfer function analyzes the LTI system and model the output signal of possible inputs. But the frequency response is a function that relates the output response to a sinusoidal input at frequency ω. Clearly the frequency response of a system is merely its transfer function by substituting $s = i\omega$ [16].

Chapter Three

Model Order Reduction of Linear Time Invariant System by Balanced Truncation Method

Model order reduction simplify and simulate the behavior of dynamical system within limit time interval and store space [33].

Balanced truncation is very important method of model order reduction and talking about it having two parts, first one is balancing the dynamical system and the other is truncation some state variables for finding a reduced system has the same properties like the original one, such as, stability and conserve most of the original system energy. Then we have to compute the norm of the error system, such that there is an a priori computable error bound for the error system [18], the details are clarified in this chapter.

3.1 System Energy

When we reduce the order of dynamical system, we try to make the system use a small amount of control energy and produce a large amount of observe energy [19]. This section includes some theorems about system energy.

Consider a linear time invariant system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

and assume it is asymptotically stable, controllable and observable.

The following lemma is shown that the controllability and observability Gramians can be used to measure the energy of the system.

Lemma (3.1) [1]: Let W_c and W_o denote the infinite Gramians of a stable linear system in equation (2.2)

a. For a controllable system, the inner product based on ${W_c}^{-1}$ describes the minimal energy required to steer the state from x_0 to x

$$E_c = \frac{1}{2} x^{\mathrm{T}} {W_c}^{-1} x.$$

b. The inner product based on W_o indicates the maximal energy generated by observing the output of the system corresponding to an initial state x when no input is applied

$$E_o = \frac{1}{2} x^{\mathrm{T}} W_o x.$$

Also W_c , W_o are the unique positive definite solutions of the Lyapunov equations

$$AW_c + W_c A^T + BB^T = 0$$

$$W_o A + A^T W_o + C^T C = 0.$$

From Lemma (3.1) we conclude that the states that are difficult to control, i.e., those that need a large amount of energy to control, are in the span of the eigenvectors of the controllability Gramian W_c corresponding to small eigenvalues. likewise, the states that are difficult to observe, i.e., those that give small amounts of observation energy, are those that be in the span of the eigenvectors of the observability Gramian W_o corresponding to small eigenvalues as well [1].

Theorem (3.1) [32,35]:

(a) The controllability function of a linear system is

$$E_c(x_0) = \min_{\substack{u\in L_2(-\infty,0) \\ x(-\infty)=0, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 dt$$

(b) The observability function of a linear system is

$$E_o(x_0) = \frac{1}{2}\int_{0}^{\infty} \|y(t)\|^2 dt, \qquad x(0) = x_0, u(t) = 0, \qquad 0 \le t < \infty.$$

The value of controllability function at x is the minimum amount of control energy required to control the state x_0 and the value of observability function at x is the amount of output energy generated by the state x.

Proof [32,35]:

By lemma (3.1)

$$E_c = \frac{1}{2} x^T W_c^{-1} x$$

This is a measure of the smallest energy needed in the input to obtain a certain state x.

We take the derivative of E_c with $\dot{x} = Ax + Bu$

$$\dot{E}_c = \frac{1}{2}(x^T W_c^{-1}\dot{x} + \dot{x}^T W_c^{-1} x)$$

$$\dot{E}_c = \frac{1}{2}(x^T W_c^{-1}(Ax + Bu) + (Ax + Bu)^T W_c^{-1} x)$$

$$= \frac{1}{2}(x^T W_c^{-1} Bu + u^T B^T W_c^{-1} x + x^T W_c^{-1} Ax + x^T A^T W_c^{-1} x)$$

$$= \frac{1}{2}(x^T W_c^{-1} Bu + u^T B^T W_c^{-1} x - x^T W_c^{-1} B B^T W_c^{-1} x)$$

$$= \frac{1}{2}(-\left(B^T W_c^{-1} x - u\right)^T \left(B^T W_c^{-1} x - u\right) + u^T u) \le u^T u$$

The control vector $u_{optimal} = B^T W_c^{-1} x$, gives the smallest control energy.

The energy required for control a certain state is given by

$$\frac{1}{2}\int_{-\infty}^{0} \|u\|^2 = E_c(x) + \frac{1}{2}\int_{-\infty}^{0} \left\|B^T W_c^{-1} x - u\right\|^2$$

Thus

$$\frac{1}{2}\|u\|^2 \geq E_c(x) = \frac{1}{2} x^T W_c^{-1} x$$

with equality when $u = u_{optimal}$.

(b) By assuming $x(0) = x_0, u(t) = 0,\ 0 \leq t < \infty$

$$\dot{x} = Ax$$

and by lemma (3.1)

$$E_o = \frac{1}{2} x^T W_o x.$$

Now, we take its time derivative

$$\dot{E}_o = \frac{1}{2}(x^T W_o \dot{x} + \dot{x}^T W_o x)$$

$$\dot{E}_o = \frac{1}{2}(x^T W_o A x + x^T A^T W_o x)$$

$$= \frac{1}{2}(x^T (W_o A + A^T W_o) x)$$

$$= -\frac{1}{2} x^T C^T C x$$

$$= -\frac{1}{2} y^T y$$

$$= -\frac{1}{2} \|y\|_2{}^2$$

Then, we integrate $\dot{E}_o$:

$$\int_0^t \dot{E}_o d\tau = E_o(x(t)) - E_o(x(0))$$

$$= -\frac{1}{2}\int_0^t \|y\|_2{}^2 d\tau$$

If we assume that $t \to \infty$ then $y(t) \to 0$.

Thus

$$E_o(x(0)) = \frac{1}{2}x^T(0)W_o x(0) = \frac{1}{2}\int_0^t \|y\|_2{}^2 d\tau$$

The Gramian W_o, includes information about how much a certain state becomes "visible" (as energy) in the output signal.

3.2 Balanced Truncation System

Here, we explain the procedure of balancing and truncation by theorems and lemmas then introduce the properties of the balanced truncation system.

2.2.1 Balanced System

For balancing, we start with a linear time invariant continuous asymptotically stable system with zero initial condition written as [31]:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$x(0) = x_0 = 0.$$

Balancing system depends on the controllability and observability Gramians. When the system is balanced, both Gramians are diagonal and equal.

The following theorem discuss the main idea of balancing system.

Theorem (3.2) [10,28]: There exists a state space transformation $x = T\tilde{x}$ (T is a nonsingular matrix) for system in equation (2.14) such that the transformed system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$

$$\tilde{y} = \tilde{C}\tilde{x}$$

is in balanced form, i.e.:

$$\widetilde{W_c} = \widetilde{W_o} = \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_n \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, are the controllability and observability Gramians of the transformed system, where

$$\widetilde{W_c} = \widetilde{W_o} = T^{-1} W_c T^{-T} = T^T W_o T.$$

Here the $\sigma_i' s$, $i = 1,2, \dots, n$, are the Hankel singular values.

Proof:

Let a state space transformation be given as

$$x = T\tilde{x} \tag{3.1}$$

so,

$$\tilde{x} = T^{-1} x \tag{3.2}$$

Substituting the equations (3.1) and (3.2) into equation (2.14), we obtain:

$$T\dot{\tilde{x}} = AT\tilde{x} + Bu$$

$$\tilde{y} = CT\tilde{x}$$

then,

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$
$$\tilde{y} = \tilde{C}\tilde{x}$$

where,

$$\tilde{A} = T^{-1}AT, \qquad \tilde{B} = T^{-1}B \qquad and\ \tilde{C} = CT,$$

$$A = T\tilde{A}T^{-1}, \qquad B = T\tilde{B} \qquad and\ C = \tilde{C}T^{-1}$$

But the controllability and observability Gramians satisfy the two Lyapunov equations:

$$T\tilde{A}T^{-1}W_c + W_c T^{-T}\tilde{A}^T T^T + T\tilde{B}\tilde{B}^T T^T = 0$$

$$W_o T\tilde{A}T^{-1} + T^{-T}\tilde{A}^T T^T W_o + T^{-T}\tilde{C}^T \tilde{C}T^{-1} = 0$$

we have:

$$\tilde{A}T^{-1}W_cT^{-T} + T^{-1}W_cT^{-T}\tilde{A}^T + \tilde{B}\tilde{B}^T = 0$$

$$T^TW_oT\tilde{A} + \tilde{A}^TT^TW_oT + \tilde{C}^T\tilde{C} = 0$$

Notice that we have a transformed Gramians which satisfy the two Lyapunov equations,

$$\widetilde{W_c} = T^{-1}W_cT^{-T}$$

$$\widetilde{W_o} = T^TW_oT.$$

Definition (3.1) [28]: The Hankel singular values of a system (2.14) denoted by (HSVs) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are defined as the square roots of the eigenvalues of the product of W_cW_o and given by:

$$\sigma_i(\Sigma) = \sqrt{\lambda_i(W_cW_o)}$$

Usually the σ_i's are placed in a matrix as:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_n \end{bmatrix}.$$

The controllability and observability Gramians are respective bases for the controllable and observable subspaces. From theorem (3.1) theorem we can suppose a basis where both concepts are equivalent, i.e., where the system is transformed (balanced). So the Hankel singular values are the basis for controllable and observable states system and are a measure tool of energy for each state in a system [26].

Definition (3.2) [1] (Balanced System): The controllable, observable and asymptotically stable system with

$$\widetilde{W_c} = \widetilde{W_o} = diag(\sigma_1, \sigma_2, \dots, \sigma_n) = \Sigma$$

such that , $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ is called a balanced system.

Balancing requires modification the basis for the state employing a transformation T. So T be a transformation matrix. Then the correlation between the original and the transformed state vector is [8,36]:

$$x = T\tilde{x}$$

$$\tilde{x} = T^{-1}x$$

When we substitute this into equation (2.14), we obtain:

$$T\dot{\tilde{x}} = AT\tilde{x} + Bu$$

$$y = CT\tilde{x}$$

then,

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$

$$y = \tilde{C}\tilde{x}$$

where,

$$\tilde{A} = T^{-1}AT, \qquad \tilde{B} = T^{-1}B \qquad and\ \tilde{C} = CT \tag{3.3}$$

Lemma (3.2) [1,36]:

a. For a balanced system, the inner product based on $\widetilde{W_c}^{-1}$ describes the minimal energy required to steer the state from $\tilde{x}_0$ to $\tilde{x}$

$$\widetilde{E_c} = \frac{1}{2}\tilde{x}^{\mathrm{T}}\widetilde{W_c}^{-1}\tilde{x}.$$

b. The inner product based on $\widetilde{W_o}$ indicates the maximal energy generated by observing the output of the balanced system corresponding to an initial state $\tilde{x}$ when no input is applied

$$\widetilde{\mathrm{E_o}} = \frac{1}{2}\tilde{x}^{\mathrm{T}}\widetilde{W_o}\tilde{x}.$$

Also, in a balanced system the Hankel singular matrix is a unique solution of the Lyapunov equations

$$\tilde{A}\Sigma + \Sigma\tilde{A}^T + \tilde{B}\tilde{B}^T = 0$$

$$\Sigma\tilde{A} + \tilde{A}^T\Sigma + \tilde{C}^T\tilde{C} = 0.$$

To find the transformation matrix T let's turn on this lemma,

Lemma (3.3) [3,4,12]: Given the controllable, observable, and stable system in (2.14) and the corresponding Gramians W_c , W_o, balancing transformation is given as follows:

$$T = L_o{}^{-T} U \Sigma^{\frac{1}{2}}$$

and

$$T^{-1} = \Sigma^{-\frac{1}{2}} U^T L_o{}^T$$

where $\Sigma^{-\frac{1}{2}} = diag\left(\frac{1}{\sqrt{\sigma_1}}, \frac{1}{\sqrt{\sigma_2}}, \dots, \frac{1}{\sqrt{\sigma_n}}\right)$.

Proof [3,10,26]:

From definition of Hankel singular value, we have the correspondence

$$(T^{-1} W_c T^{-T})(T^T W_o T) = \Sigma^2$$

$$T^{-1} W_c W_o T = \Sigma^2 \qquad (3.4)$$

Since $W_o = W_o{}^T$ and is positive definite, we can make factorization for W_o based on Cholesky decompositions

$$W_o = L_o L_o{}^T \qquad (3.5)$$

Where L_o is the lower triangular matrix with non-zero diagonal entries. we can rewrite equation (3.4) as

$$T^{-1}W_cL_oL_o{}^TT = \Sigma^2$$

which is equivalent to,

$$\left(L_o{}^TT\right)^{-1}L_o{}^TW_cL_o(L_o{}^TT) = \Sigma^2$$

This Equation means that $L_o{}^TW_cL_o$ is similar to Σ^2 and is positive definite. Therefore, there exists an orthogonal transformation U ,$U^TU = I$, such that

$$L_o{}^TW_cL_o = U\Sigma^2U^T$$

By setting $\left(L_o{}^TT\right)^{-1}U\Sigma^{\frac{1}{2}} = I$, we reach at a definition for T and T^{-1} as

$$T = L_o{}^{-T}U\Sigma^{\frac{1}{2}} \tag{3.6}$$

and

$$T^{-1} = \Sigma^{-\frac{1}{2}}U^TL_o{}^T \tag{3.7}$$

By this transformation in equations (3,6) and (3.7), we have that,

$$\widetilde{W_c} = \Sigma^{-\frac{1}{2}}U^TL_o{}^TW_cL_oU\Sigma^{-\frac{1}{2}}$$
$$= \left(\Sigma^{-\frac{1}{2}}U^T\right)(U\Sigma^2U)\left(U\Sigma^{-\frac{1}{2}}\right)$$
$$= \Sigma$$

and

$$\widetilde{W_o} = \left(\Sigma^{\frac{1}{2}}U^TL_o{}^{-1}\right)\left(L_oL_o{}^T\right)\left(L_o{}^{-T}U\Sigma^{\frac{1}{2}}\right)$$
$$= \left(\Sigma^{\frac{1}{2}}V^T\right)\left(L_o{}^{-1}L_oL_o{}^TL_o{}^{-T}\right)(V\Sigma^{\frac{1}{2}})$$
$$= \Sigma.$$

Remark (3.1): If we make a factorization based on Cholesky decompositions of the controllability Gramian

$$W_c = L_cL_c{}^T \tag{3.8}$$

Where L_c is the lower triangular matrices with non-zero diagonal entries, then we have

$$T = L_c V \Sigma^{-\frac{1}{2}}$$

and

$$T^{-1} = \Sigma^{-\frac{1}{2}} V L_c{}^T.$$

Proof: like the proof of Lemma (3.3).

Now, if we compute the singular value decomposition (SVD) of the matrix $L_o{}^T L_c$,we obtain:

$$L_o{}^T L_c = U \Sigma V^T = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1{}^T \\ V_2{}^T \end{pmatrix}$$

such that,

$$\Sigma_1 = \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$$

and

$$\Sigma_2 = \sigma_{r+1} \geq \sigma_{r+2} \geq \cdots \geq \sigma_n$$

The other matrices satisfy

$$U_1{}^T U_1 = V_1{}^T V_1 = I_{r \times r}$$

$$U_2{}^T U_2 = V_2{}^T V_2 = I_{n-r \times n-r}.$$

Lemma (3.4) [1,26]: Balancing transformation is given as follows:

$$T = L_c V \Sigma^{-\frac{1}{2}} \quad ,$$

$$T^{-1} = \Sigma^{-\frac{1}{2}} U^T L_o{}^T.$$

where

$$\Sigma^{-\frac{1}{2}} = diag\left(\frac{1}{\sqrt{\sigma_1}}, \frac{1}{\sqrt{\sigma_2}}, \dots, \frac{1}{\sqrt{\sigma_n}}\right).$$

Then find the balanced system $(\tilde{A}, \tilde{B}, \tilde{C})$ with

$$\tilde{A} = T^{-1} A T, \qquad \tilde{B} = T^{-1} B \quad and\ \tilde{C} = CT.$$

3.2.2 Balanced Truncation System

In balanced system we try to conserve the high energy states and eliminate the low energy states which are least observable and controllable states.

Consider the balanced system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$

$$\tilde{y} = \tilde{C}\tilde{x}$$

Let $1 \leq r \leq n$; the balanced system can be partitioned in blocks as [20]:

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}, \quad \tilde{C} = (\tilde{C}_1 \quad \tilde{C}_2)$$

We partition the balanced Gramian as

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$$

Which satisfy the following Lyapunov equations [18,27]:

$$\tilde{A}\Sigma + \Sigma\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 \tag{3.9}$$

$$\Sigma\tilde{A} + \tilde{A}^T\Sigma + \tilde{C}^T\tilde{C} = 0 \tag{3.10}$$

Which can be developed as:

$$\begin{bmatrix} \tilde{A}_{11}\Sigma_1 & \tilde{A}_{12}\Sigma_2 \\ \tilde{A}_{21}\Sigma_1 & \tilde{A}_{22}\Sigma_2 \end{bmatrix} + \begin{bmatrix} \Sigma_1\tilde{A}_{11}{}^T & \Sigma_1\tilde{A}_{21}{}^T \\ \Sigma_2\tilde{A}_{12}{}^T & \Sigma_2\tilde{A}_{22}{}^T \end{bmatrix} + \begin{bmatrix} \tilde{B}_1\tilde{B}_1{}^T & \tilde{B}_1\tilde{B}_2{}^T \\ \tilde{B}_2\tilde{B}_1{}^T & \tilde{B}_2\tilde{B}_2{}^T \end{bmatrix} = 0 \tag{3.11}$$

$$\begin{bmatrix} \Sigma_1\tilde{A}_{11} & \Sigma_1\tilde{A}_{12} \\ \Sigma_2\tilde{A}_{21} & \Sigma_2\tilde{A}_{22} \end{bmatrix} + \begin{bmatrix} \tilde{A}_{11}{}^T\Sigma_1 & \tilde{A}_{21}{}^T\Sigma_2 \\ \tilde{A}_{12}{}^T\Sigma_1 & \tilde{A}_{22}{}^T\Sigma_2 \end{bmatrix} + \begin{bmatrix} \tilde{C}_1{}^T\tilde{C}_1 & \tilde{C}_1{}^T\tilde{C}_2 \\ \tilde{C}_2{}^T\tilde{C}_1 & \tilde{C}_2{}^T\tilde{C}_2 \end{bmatrix} = 0 \tag{3.12}$$

From the two equations (3.11) and (3.12) we have these following equations:

$$A_{11}\Sigma_1 + \Sigma_1 A_{11}{}^T + B_1 B_1{}^T = 0 \tag{3.13}$$

$$A_{21}\Sigma_1 + \Sigma_2 A_{12}{}^T + B_2 B_1{}^T = 0 \tag{3.14}$$

$$A_{12}\Sigma_2 + \Sigma_1 A_{21}{}^T + B_1 B_2{}^T = 0 \tag{3.15}$$

$$A_{22}\Sigma_2 + \Sigma_2 A_{22}{}^T + B_2 B_2{}^T = 0 \tag{3.16}$$

$$\Sigma_1 A_{11} + A_{11}{}^T \Sigma_1 + C_1{}^T C_1 = 0 \quad (3.17)$$

$$\Sigma_2 A_{21} + A_{12}{}^T \Sigma_1 + C_2{}^T C_1 = 0 \quad (3.18)$$

$$\Sigma_2 A_{22} + A_{22}{}^T \Sigma_2 + C_2{}^T C_2 = 0 \quad (3.19)$$

The subscripts 1 and 2 denote the dimensions r and $n-r$ respectively.

Theorem (3.3) [10,39]: Assume that Σ_1 and Σ_2 have no diagonal entries in common. Then both subsystems (A_{ii}, B_i, C_i); i = 1, 2 are asymptotically stable.

The model order reduction of size r obtained by Balanced Truncation as in the following form [38]:

$(A_{trnc}, B_{trnc}, C_{trnc}) = \left(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1\right) \in \mathbb{R}^{r\times r},\ \mathbb{R}^{p\times r},\ \mathbb{R}^{r\times q}$ respectively.

If $G(s)$ is the transfer function of original and balanced system, we will introduce the transfer function of truncated system as

$$G_{trnc}(s) = \tilde{C}_1 (sI - \tilde{A}_{11})^{-1} \tilde{B}_1$$

and

$$W_{c_{trnc}} = W_{o_{trnc}} = \Sigma_1$$

So the Lyapunov equations are [9]:

$$\tilde{A}_{11}\Sigma_1 + \Sigma_1 \tilde{A}_{11}{}^T + \tilde{B}_1 \tilde{B}_1{}^T = 0$$

$$\Sigma_1 \tilde{A}_{11} + \tilde{A}_{11}{}^T \Sigma_1 + \tilde{C}_1{}^T \tilde{C}_1 = 0.$$

In common applications, to reduce the original system into an r^{th} order system there should be a large gap between σ_r and σ_{r+1}, i.e. $\sigma_r >> \sigma_{r+1}$. So we ignore the state components x_{r+1} to x_n.

$\tilde{x}_1$ is the most controllable and observable state, and $\tilde{x}_n$ is the least controllable and observable state.

The new system we obtained of reduced original system by balanced truncation method is [20]:

$$\dot{x}_{trnc} = \tilde{A}_{11}x_{trnc} + \tilde{B}_1 u$$
$$y_{trnc} = \tilde{C}_1 x_{trnc}.$$

3.2.3 Properties of Balanced Truncation System

The balanced truncation system has to preserve the properties of original system. Now we discuss the properties that inherited from the original system.

Theorem (3.4) [39]: Let the original system be an asymptotically stable system. If Σ_1 and Σ_2 do not have any common diagonal elements then the balanced truncation system is asymptotically stable.

For more explanation, we can insert this lemma

Lemma (3.5) [1]:

(a) Transfer matrices

$$G(S) = C(sI - A)^{-1}B \ and \ G_{trnc}(s) = \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1$$

are asymptotically stable.

(b) The Lyapunov equalities

$$\tilde{A}_{11}\Sigma_1 + \Sigma_1\tilde{A}_{11}{}^T + \tilde{B}_1\tilde{B}_1{}^T = 0$$
$$\Sigma_1\tilde{A}_{11} + \tilde{A}_{11}{}^T\Sigma_1 + \tilde{C}_1{}^T\tilde{C}_1 = 0.$$

are satisfied.

(c) $\|G - G_{trnc}\|_\infty \leq 2(\sigma_{r+1} + \cdots + \sigma_n)$ (error bound).

3.4 Error Bound Using Balanced Truncation

This section introduces H_∞ norm of LTI system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

that is essential for system estimation error. Estimation by balanced truncation maintains stability, and the H_∞ norm of the error system is exists and bounded as it clear in this section. The H_∞ norm between the original and reduced order systems can be used as an error criterion [4].

Definition (3.3) [4]: The H_∞ norm of the transfer function $G(s)$, denoted as $\|G\|_\infty$ is defined as

$$\|G\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{max}\big(G(i\omega)\big) = \sup_{U} \frac{\|Y(i\omega)\|_\infty}{\|U(i\omega)\|_\infty}$$

where $\sup_{\omega\in\mathbb{R}}$ denotes the supremum or least upper bound over all real valued frequencies.

Note:

$$\|G\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{max}\big(G(i\omega)\big) = \sup_{u} \frac{\|y\|_2}{\|u\|_2}$$

where, $\|y\|_2$, $\|u\|_2$ are the L_2 norm of $y(t)$, $u(t)$ respectively.

Theorem (3.5) [39]: Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ be the ordered set of different Hankel singular numbers of a stable LTI system with transfer function G. Let G_{trnc} be the reduced model obtained by removing the states corresponding to singular numbers not larger than σ_r from a balanced realization of G. Then G_{trnc} is stable, and satisfies

$$\|G - G_{trnc}\|_\infty \leq 2(\sigma_{r+1} + \cdots + \sigma_n) \qquad (3.20).$$

Proof [39]:

The stability of G_{trunc} follows from Theorem (3.3)

Let

$$\gamma(s) = (sI - A_{11})^{-1}$$

$$\varphi(s) = sI - A_{22} - A_{21}\gamma(s)A_{12}$$

$$B_* = A_{21}\gamma(s)B_1 + B_2$$

$$C_* = C_1\gamma(s)A_{12} + C_2$$

then take the partitioned matrix,

$$G(s) - G_{trnc}(s) = C(sI - A)^{-1}B - C_1\gamma(s)B_1$$
$$= [C_1 \quad C_2]\begin{bmatrix} sI - A_{11} & -A_{12} \\ -A_{21} & sI - A_{22} \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - C_1\gamma(s)B_1$$

computing this value on the imaginary axis to have

$$\sigma_{max}\big(G(i\omega) - G_{trnc}(i\omega)\big)$$
$$= \lambda_{max}{}^{1/2}\left(\varphi^{-1}(i\omega)B_*(i\omega)B_*{}^T(i\omega)\varphi^{-T}(i\omega)C_*{}^T(i\omega)C_*(i\omega)\right) \qquad (3.21)$$

by the equations (3.13) and (3.19).

$$B_*(i\omega)B_*{}^T(i\omega) = \varphi(i\omega)\Sigma_2 + \Sigma_2\varphi^T(i\omega) \qquad (3.22)$$

$$C_*{}^T(i\omega)C_*(i\omega) = \Sigma_2\varphi(i\omega) + \varphi^T(i\omega)\Sigma_2 \qquad (3.23)$$

then by substituted the equations (3.22) and (3.23) into (3.21) we get

$$\sigma_{max}\big(G(i\omega) - G_{trnc}(i\omega)\big)$$

$$= \lambda_{max}{}^{1/2}((\Sigma_2 + \varphi^{-1}(i\omega)\Sigma_2\varphi^T(i\omega))(\Sigma_2 + \varphi^{-T}(i\omega)\Sigma_2\varphi(i\omega)))$$

Now consider the reduced order is, $r = n - 1$, then $\Sigma_2 = \sigma_n$ and

$$\sigma_{max}\big(G(i\omega) - G_{trnc}(i\omega)\big) = \sigma_n\lambda_{max}{}^{1/2}\left(\left(1 + \rho^{-1}(i\omega)\right)\left(1 + \rho(i\omega)\right)\right)$$

where $\rho(i\omega) = \varphi^{-T}(i\omega)\varphi(i\omega) = \rho^{-T}(i\omega)$. But $|\rho(i\omega)| = 1$.

Using triangle inequality, we get

$$\sigma_{max}\big(G(i\omega) - G_{trnc}(i\omega)\big) \leq \sigma_n|1 + \rho(i\omega)| = 2\sigma_n \qquad (3.24)$$

This completes the bound for $r = n - 1$.

The rest of the proof is completed by applying the order reduction by one step results and by considering that $G_k(s)$, $(A_k, B_k, C_k) = (A_{11}, B_1, C_1)$ gained by the k^{th} order, balanced Gramian given by

$$\Sigma_1 = diag(\sigma_1, \sigma_2, \dots, \sigma_k)$$

Let

$$E_k(s) = G_{k+1}(s) - G_k(s) \text{ for } k = 1,2,\dots,n-1,$$

and let

$$G_n(s) = G(s).$$

then

$$\sigma_{max}(E_k(i\omega) \leq 2\sigma_{k+1}$$

Since $G_k(s)$ is a reduced order model gained from the balanced realization of $G_{k+1}(s)$ and the bound for one step order reduction, (3.24) holds.

Notice that

$$G(s) - G_{trnc}(s) = \sum_{k=r}^{n-1} E_k(s)$$

by the assumption of $E_k(s)$, we get

$$\sigma_{max}\big(G(i\omega) - G_{trnc}(i\omega)\big) \leq \sum_{k=r}^{n-1} \sigma_{max}\big(E_k(s)\big) \leq 2\sum_{k=r}^{n-1} \sigma_{k+1}$$

This is the desired.

Remark (3.2): If $\|G - G_{trnc}\|_\infty \leq 2(\sigma_{r+1} + \cdots + \sigma_n)$, then

$$\|y - y_{trnc}\|_2 \leq 2(\sigma_{r+1} + \cdots + \sigma_n)\|u\|_2 \qquad (3.25)$$

Proof:

$$\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{max}\big(G(i\omega)\big) = \sup_u \frac{\|y\|_2}{\|u\|_2}$$

$$\|y\|_2 \leq \|G\|_\infty \|u\|_2$$

$$\|y - y_{trnc}\|_2 \leq \|G - G_{trnc}\|_\infty \|u\|_2 \leq 2(\sigma_{r+1} + \cdots + \sigma_n)\|u\|_2.$$

Chapter Four

Numerical Results

In this chapter we insert some numerical examples and MATLAB software is implemented to solve the examples and we construct a model order reduction by balanced truncation method for mass spring damper system.

4.1 Numerical examples

Here, we clarify the state space representation and check the properties of the dynamical system then we apply balanced truncation method.

Example (4.1): Consider the equations of state space representation of the system defined by

$$\dddot{v} - 2\ddot{v} + \dot{v} - v = u$$

our objective is to convert this differential equation into state space equations. We start by defining variables x_1, x_2 andx_3 as follow:

$$x_1 = v$$

$$x_2 = \dot{x}_1 = \dot{v}$$

$$x_3 = \dot{x}_2 = \ddot{v}$$

but

$$\dot{x}_3 = \dddot{v}$$

so

$$\dddot{v} = \dot{x}_3 = 2x_3 - x_2 + x_1 + u$$

Now, rewrite the above equations as first order equations:

$$\dot{x}_1 = x_2 = \dot{v}$$

$$\dot{x}_2 = x_3 = \ddot{v}$$

$$\dot{x}_3 = x_1 - x_2 + 2x_3 + u$$

This may represent in state space form as:

$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

and

$$y = Cx$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Example (4.2): Let the LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -7.5 & 5 & -0.75 & 0.5 \\ 2.5 & -2.5 & 0.25 & -0.25 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.25 \end{bmatrix} \text{ and } C = [1 \quad 1 \quad 0 \quad 0],$$

We check if the system is stable, controllable or observable.

The order of the system is 4. We insert A, B and C in MATLAB, then we check the properties of the system:

- Stability

We find the eigenvalues of matrix A to check the stability;

$$\lambda_i(A) = \{-0.4665 + 3.0187i, \quad -0.4665 - 3.0187i, \quad -0.0335 + 0.8178i, \quad -0.0335 - 0.8178i\}$$

where λ_i is the eigenvalue of A and $i = 1, \ldots, n$

By theorem (2.1) the system is asymptotically stable, since every eigenvalue of A has a negative real part ($Re(\lambda_i) < 0$).

- Controllability

We find the controllability matrix $C(A,B)$and its rank;

$$C(A,B) = \begin{bmatrix} 0 & 0 & 0.125 & 1.125 \\ 0 & 0.25 & -0.0625 & -0.5781 \\ 0 & 0.125 & 1.125 & -2.3828 \\ 0.25 & -0.0625 & -0.5781 & 0.8945 \end{bmatrix}$$

$$rank(C(A,B)) = 4$$

By theorem (2.3) the system is controllable, since $C(A,B)$ has rank

equal the order of the system.

- Observability

We find the observability matrix $O(A,C)$and its rank;

$$O(A,C) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -5 & 2.5 & -.5 & 0.25 \\ 4.375 & -3.125 & -4.5625 & 2.1875 \end{bmatrix}$$

$$rank(O(A,C)) = 4$$

By theorem (2.5) the system is observable, since $O(A,C)$ has rank

equal the order of the system.

So the given LTI system is asymptotically stable, controllable and observable.

Now, we reduce the order of LTI system by balanced truncation method using MATLAB.

First we compute the controllability and observability Gramians of the original system then factor it using Cholesky factorization:

$$W_c = \begin{bmatrix} 0.2328 & 0.3164 & -0.000 & -0.0041 \\ 0.3164 & 0.4336 & 0.0041 & -0.000 \\ -0.000 & 0.0041 & 0.1660 & 0.2080 \\ -0.0041 & -0.000 & 0.2080 & 0.29220 \end{bmatrix}$$

$$W_o = \begin{bmatrix} 2.0828 & 5.0828 & 0.2033 & 0.4098 \\ 5.0828 & 13.9844 & 0.5967 & 1.3934 \\ 0.2033 & 0.5967 & 2.8066 & 7.6066 \\ 0.4098 & 1.3934 & 7.6066 & 20.7869 \end{bmatrix}$$

By equations ((3.8), (3.5)) we have:

$$L_c = \begin{bmatrix} 0.4825 & 0.000 & 0.000 & 0.000 \\ 0.6558 & 0.0598 & 0.000 & 0.000 \\ -0.000 & 0.0685 & 0.4016 & 0.000 \\ -0.0085 & 0.0931 & 0.5020 & 0.1768 \end{bmatrix}$$

$$L_o = \begin{bmatrix} 1.4432 & 0.0000 & 0.0000 & 0.0000 \\ 3.5219 & 1.2572 & 0.000 & 0.0000 \\ 0.1409 & 0.0801 & 1.6674 & 0.0000 \\ 0.2840 & 0.3128 & 4.5228 & 0.3902 \end{bmatrix}$$

Then we find the singular value decomposition for ${L_o}^T L_c$ to have

$$\Sigma = \begin{bmatrix} 3.2445 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 2.9898 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0183 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0136 \end{bmatrix}$$

Now, we can find T as it being in lemma (3.3).

$$T = \begin{bmatrix} -0.1904 & 0.1952 & -0.1877 & 0.1926 \\ -0.2635 & 0.2627 & 0.0737 & -0.0671 \\ -0.1533 & -0.1622 & -0.5088 & -0.6856 \\ -0.2065 & -0.2256 & 0.1853 & 0.2519 \end{bmatrix}$$

By equation (3.3), we get:

$$\tilde{A} = \begin{bmatrix} -0.0319 & 0.8180 & -0.0161 & 0.0175 \\ -0.8180 & -0.0351 & 0.0173 & -0.0193 \\ 0.0161 & 0.0173 & -0.3541 & 3.0207 \\ 0.0175 & 0.0193 & -3.0207 & -0.5789 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} -0.4550 \\ -0.4580 \\ 0.1139 \\ 0.1255 \end{bmatrix},$$

$$\tilde{C} = [-0.4550 \quad 0.4580 \quad -0.1139 \quad 0.1255]$$

The Hankel singular values $= \{3.2445, 2.9898, 0.0183, 0.0136\}$, and we have the following figure:

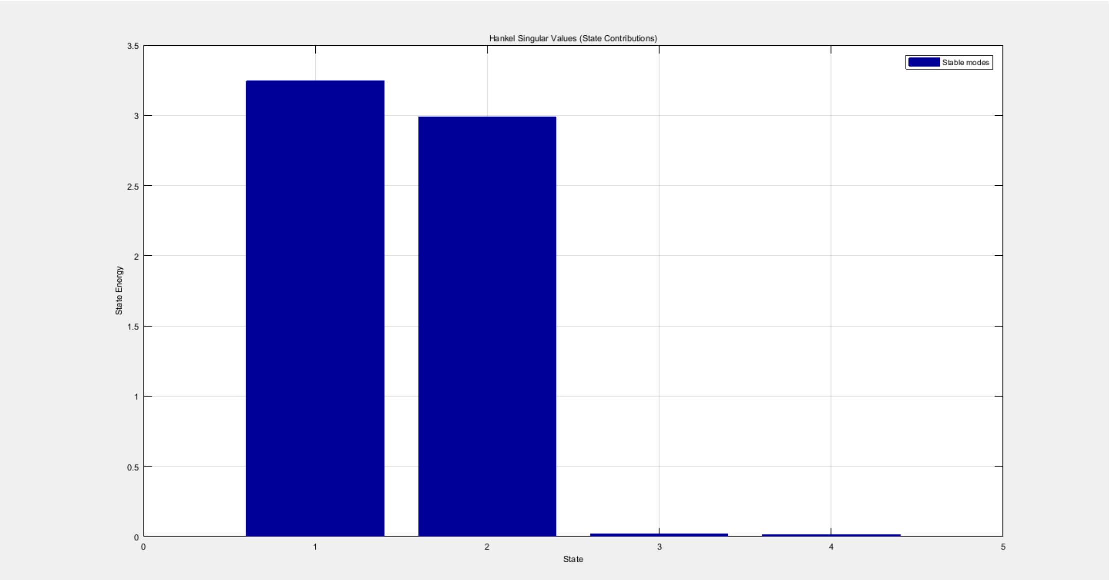

Fig (4.1): Hankel singular values of a system in example (4.2)

We notice there is a big gap between the value of the 2^{nd} and 3^{rd} Hankel singular values, choose the reduced order to be 2, then truncate the 3^{rd} and 4^{th} Hankel singular values to have:

$$A_{trnc} = \begin{bmatrix} -0.0319 & 0.8180 \\ -0.8180 & -0.0351 \end{bmatrix},$$

$$B_{trnc} = \begin{bmatrix} -0.4550 \\ -0.4580 \end{bmatrix},$$

$$C_{trnc} = [-0.4550 \quad 0.4580]$$

With balanced truncation Gramian:

$$\Sigma_1 = \begin{bmatrix} 3.2445 & 0 \\ 0 & 2.9898 \end{bmatrix}$$

Now, we check the properties of balanced truncation system:

$$\lambda_i(A_{trnc}) = \{-0.0335 + 0.8180i, -0.0335 - 0.8180i\}$$

where λ_i be the eigenvalues of A_{trnc}

$rank(C(A_{trnc}, B_{trnc}) = 2$

$rank(O(A_{trnc}, C_{trnc}) = 2$

Therefore, the system is asymptotically stable, controllable and observable.

To compute the error bound, find the transfer function of the original system $G(s)$ and the transfer function of balanced truncation system:

$$G(s) = \frac{0.25\,s^2 + 0.3125\,s + 3.125}{s^4 + s^3 + 10.06s^2 + 1.25\,s + 6.25},$$

$$G_{trnc}(s) = \frac{-0.002762\,s + 0.3415}{s^2 + 0.06698\,s + 0.6702}$$

then,

$$\|G - G_{trnc}\|_\infty = 0.0315$$

and

$$2(\sigma_3 + \sigma_4) = 0.0639$$

so we achieve the desired inequality which is

$$\|G - G_{trnc}\|_\infty \leq 2(\sigma_{r+1} + \cdots + \sigma_n) \text{ (error bound).}$$

Now, we choose various reduced order (r) then calculate the error bound for each r and check of achieving theorem (3.5).

If we alter the reduced order r, such that $r = 1,2,3$ and create the balanced truncation system for each r then the $\|G - G_{trunc}\|_\infty$ and the error bounds are listed in the following table:

Table (4.1): The $\boldsymbol{H}_{\infty}$ norm of $(\boldsymbol{G}-\boldsymbol{G}_{\boldsymbol{trunc}})$ and the error bound of the system in example (4.2)

r	$\|G-G_{trunc}\|_{\infty}$	$2\sum_{i=r+1}^{10}\sigma_i$
1	5.9891	6.0435
2	0.0315	0.0639
3	0.0272	0.0272

Notice that all the reduced order chosen achieve the desired inequality

$$\|G-G_{trnc}\|_{\infty} \leq 2(\sigma_{r+1}+\cdots+\sigma_n).$$

In this chapter we construct a model order reduction by balanced truncation method for mass spring damper system.

The mass spring damper system is utilized in many areas like engineering to have a mathematical model such that, it can be modeled by a linear differential equation.

The mass spring damper system be composed of multi masses dispensed during a connection with springs and dampers.

The spring is capable of save energy when it is elongated or squeezed from its original length and then the spring liberates that energy back onto the mass until the energy is wasted.

The energy squandered out of the dynamic system through the damper in the mass spring damper system. Damper is a device that be resistant to the springs activity.

4.2 Derivative The State Space Equation of Mass Spring Damper System

Here, we introduce a numerical example of mechanical system that is mass spring damper system as a model of a real engineering system and represent it in state space equations form.

First, we consider the simple example where one mass is connected by a spring and damper. The control force u is applied to a mass m, which directly effects the position, x of this mass.

System analysis is based on Newton's second law: the sum of all forces applied to a body equals the product of the vector acceleration of the body times its mass. The equation for Newton's second law is

$$\sum F = m\ddot{x} \tag{4.1}$$

where $\sum F$ are the sum of forces that are act on the body in Newton's (N), m is the mass, and $\ddot{x} = \frac{d^2x}{dt^2}$.

We suppose the mass at time $t = 0$ is in a rest condition, so that $x_0 = 0$ and $\dot{x}_0 = 0$.

The mass spring damper system with single mass (m) described in figure (4.2).

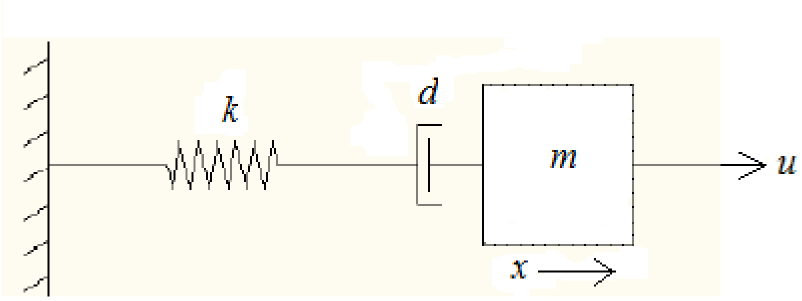

Figure (4.2): Mass spring damper system with single mass so, the forces will always act on the object are shown in figure (4.3)

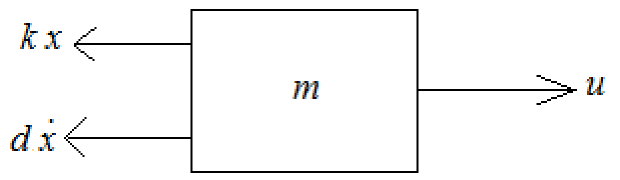

Figure (4.3): Analysis forces on the single mass

1) Spring force, denoted by (F_s):

   Which is a linear force and given by:

$$F_s = kx$$

   where, k is a positive real number (stiffness coefficient) and x is a position. This force will be in the opposite direction.

2) Damping force, denoted by (F_d):

$$F_d = d\dot{x}$$

   where, d is a positive real number (damping coefficient) and $\dot{x} = \frac{dx}{dt}$. This force will be in the opposite direction.

3) External or control force, denoted by $u(t)$.

   so

$$\sum F = -kx - d\dot{x} + u(t)$$

By Newton's second law

$$m\ddot{x} = -kx - d\dot{x} + u(t)$$

By rearranging this equation, we have:

$$m\ddot{x} + kx + d\dot{x} = u(t) \tag{4.2}$$

So we have a linear differential equation that we can represent it in a state space equation by we take:

$$\dot{x} = z \tag{4.3}$$

and

$$\ddot{x} = \dot{z} \tag{4.4}$$

Then, by substituting the equations (4.3) and (4.4) into (4.2) equation we have:

$$\dot{x} = z \tag{4.5}$$

$$\dot{z} = -\frac{k}{m}x - \frac{d}{m}z + \frac{u}{m} \tag{4.6}$$

But in matrix form we have:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u \tag{4.7}$$

Let $A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{pmatrix}$ of size (2×2) and $B = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$ of size (2×1), then the state space equation for this system is:

$$\dot{X} = AX + Bu \tag{4.8}$$

where, $\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix}$ and $X = \begin{pmatrix} x \\ z \end{pmatrix}$ of size (2×1) for the both.

and the output equation is:

$$Y = CX \tag{4.9}$$

where $C = (1 \quad 0)$.

Now, we take a mass spring damper system composed of two masses (m_1, m_2) that are connected by two springs and two dampers as it shown in figure (4.4).

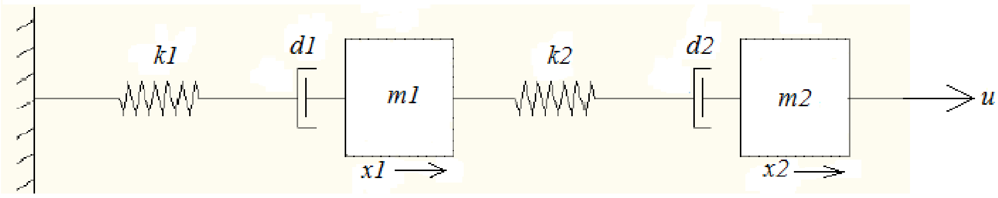

Figure (4.4): Double mass spring damper system

The control force u is applied to a mass m_2, which directly effects the position (x_1, x_2) of this masses (m_1, m_2) respectively.

where (k_1, k_2) are the stiffness coefficient of springs, (d_1, d_2) are the damping coefficient.

The analysis of the forces that acts on m_1 is described in figure (4.5)

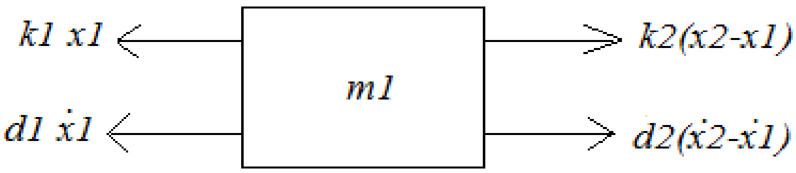

Figure (4.5): Analysis forces on $\boldsymbol{m_1}$

By applying Newton's second law we have:

$$m_1\ddot{x}_1 = -k_1x_1 - d_1\dot{x}_1 + k_2(x_2 - x_1) + d_2(\dot{x}_2 - \dot{x}_1)$$

$$m_1\ddot{x}_1 + (d_1 + d_2)\dot{x}_1 - d_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = 0 \qquad (4.10)$$

Similarity for the analysis forces on mass m_2 as shown in figure (4.6)

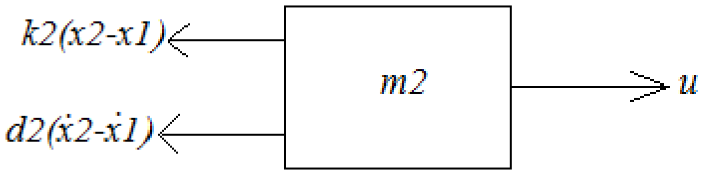

Figure (4.6): Analysis forces on $\boldsymbol{m_2}$

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - d_2(\dot{x}_2 - \dot{x}_1) + u$$

$$m_2\ddot{x}_2 - d_2\dot{x}_1 + d_2\dot{x}_2 - k_2x_1 + k_2x_2 = u \qquad (4.11).$$

The set of differential equations in (4.10) and (4.11) can be written in the matrix form as:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} d_1 + d_2 & -d_2 \\ -d_2 & d_2 \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \qquad (4.12)$$

and the differential equation that represents this system has the form:

$$M\ddot{x} + D\dot{x} + Kx = Lu \qquad (4.13)$$

where M is the mass matrix, D is the damping matrix and K is the stiffness matrix, and $L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an (2×1) column vector.

To find the state space equation for the linear continuous system in equation (4.13) we take:

$$\dot{x} = z \qquad (4.14)$$

and

$$\ddot{x} = \dot{z} \qquad (4.15)$$

Consider that M^{-1} exists, then by substituting the equations (4,14) and (4.15) into equation (4.13), we get the following system:

$$\dot{x} = z \qquad (4.16)$$

$$\dot{z} = -M^{-1}Kx - M^{-1}Dz + M^{-1}Lu \qquad (4.17)$$

and in matrix form we have:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}L \end{pmatrix} u$$

Consider $A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}$ be of size (4×4) and $B = \begin{pmatrix} 0 \\ M^{-1}L \end{pmatrix}$ be of size (4×1),then the state space equation for this system is:

$$\dot{X} = AX + Bu$$

where, $\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix}$ and $X = \begin{pmatrix} x \\ z \end{pmatrix}$ of size (4×1) for the both.

and the output equation is:

$$Y = CX$$

where $C = (1 \quad 1 \quad 0 \quad 0)$.

In general, we can represent the state space equations for mass spring damper system which has n masses by applying Newton's second law for each mass m_i $(i = 1 \cdots n)$ to obtain n differential equations,

- For $i = 1, \cdots, n-1$:

$$m_i\ddot{x}_i - d_i\dot{x}_{i-1} + (d_i + d_{i+1})\dot{x}_i - d_{i+1}\dot{x}_{i+1} - k_ix_{i-1} + (k_i + k_{i+1})x_i - k_{i+1}x_{i+1} = 0 \qquad (4.18)$$

- But for $i = n$, that is in the form:

$$m_n\ddot{x}_n - d_n\dot{x}_{n-1} + d_n\dot{x}_n - k_nx_{n-1} + k_nx_n = u \qquad (4.19).$$

This equations (4.18) and (4.19) can be summarized in this representation:

$$M\ddot{x} + D\dot{x} + Kx = Lu \qquad (4.20)$$

where:

$$M = \begin{pmatrix} m_1 & 0 & 0 & \dots & \dots & 0 \\ 0 & m_2 & 0 & \dots & \dots & 0 \\ 0 & 0 & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & 0 & m_i & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & m_n \end{pmatrix}_{n\times n}$$

is called the mass matrix of the system,

$$D = \begin{pmatrix} d_1 + d_2 & -d_2 & 0 & 0 & \dots & & 0 \\ -d_2 & d_2 + d_3 & -d_3 & 0 & \dots & & 0 \\ 0 & -d_3 & d_3 + d_4 & -d_4 & \dots & & 0 \\ 0 & 0 & -d_4 & \ddots & & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \ddots & -d_n \\ 0 & 0 & 0 & 0 & & -d_n & d_n \end{pmatrix}_{n\times n}$$

is called the damper matrix of the system,

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & 0 & 0 & \dots & & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 & \dots & & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 & \dots & & 0 \\ 0 & 0 & -k_4 & \ddots & & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \ddots & -k_n \\ 0 & 0 & 0 & 0 & & -k_n & k_n \end{pmatrix}_{n\times n}$$

is called the stiffness matrix of the system,

and

$$L = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_{n\times 1}$$

To find the state space equation for the linear continuous system in equation (4.20), we take:

$$\dot{x} = z \tag{4.21}$$

and

$$\ddot{x} = \dot{z} \tag{4.22}$$

Consider that M^{-1} exists, then by substituting the equations (4.21) and (4.22) into equation (4.20), we get the following system:

$$\dot{x} = z$$

$$\dot{z} = -M^{-1}Kx - M^{-1}Dz + M^{-1}Lu$$

and in matrix form we have:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}L \end{pmatrix} u$$

Consider $A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}_{2n\times 2n}$ and $B = \begin{pmatrix} 0 \\ M^{-1}L \end{pmatrix}_{2n\times 1}$, then the state space equation for this system is:

$$\dot{X} = AX + Bu$$

where, $\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix}_{2n\times 1}$ and $X = \begin{pmatrix} x \\ z \end{pmatrix}_{2n\times 1}$.

and the output equation is:

$$Y = CX$$

where $C = \begin{pmatrix} 1 & 1 & \dots & 1_{(1\times n)} & 0 & 0 & \dots & 0 \end{pmatrix}_{1\times 2n}$.

4.3 Using MATLAB for Applying Balanced Truncation Method on Mass Spring Damper System.

This section contains two numerical examples to reduce the order of mass spring damper system which have the state space representation by balanced truncation method.

Example (4.3): We take five masses such that they are connected with five springs and five dampers. So the order of the system is 10.

We check the properties of the system:

- Stability

$\lambda_i(A) = \{-0.0614 + 1.1062i, -0.0614 - 1.1062i, -0.0472 + 0.9702i, -0.0472 - 0.9702i, -0.0286 + 0.7556i, -0.0286 - 0.7556i, -0.0115 + 0.4795i, -0.0115 - 0.4795i, -0.0014 + 0.1643i, -0.0014 - 0.1643i\}$

By definition (2.2) the system is stable. Since every eigenvalue of the system matrix A has negative real part ($Re(\lambda_i) < 0$).

- Controllability and observability

$rank\big(C(A,B)\big) = 10$

$rank\big(O(A,C)\big) = 10$

By theorems (2.3) and (2.5) the system is controllable and observable. Since $rank\big(C(A,B)\big) = rank\big(O(A,C)\big) =$ The order of the system $(2n)$.

Now, we compute the Hankel singular values and plot it by two ways as it shown in figures (4.7) and (4.8) then we find the balancing system,

HSVs = σ = {473.9815, 466.2567, 5.6058, 5.3461, 0.6304, 0.5904, 0.1060, 0.1012, 0.0104, 0.0101}

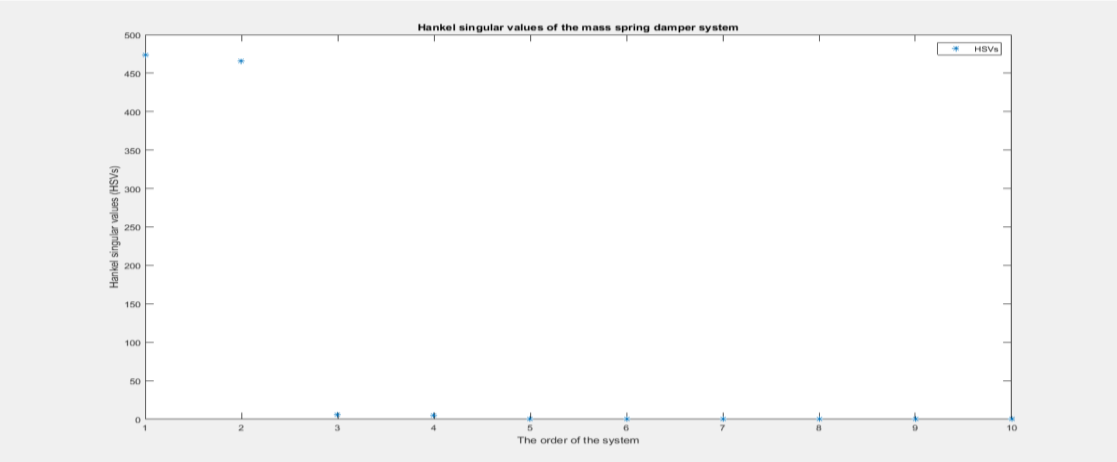

Figure (4.7): Hankel singular values of the mass spring damper system in example (4.3)

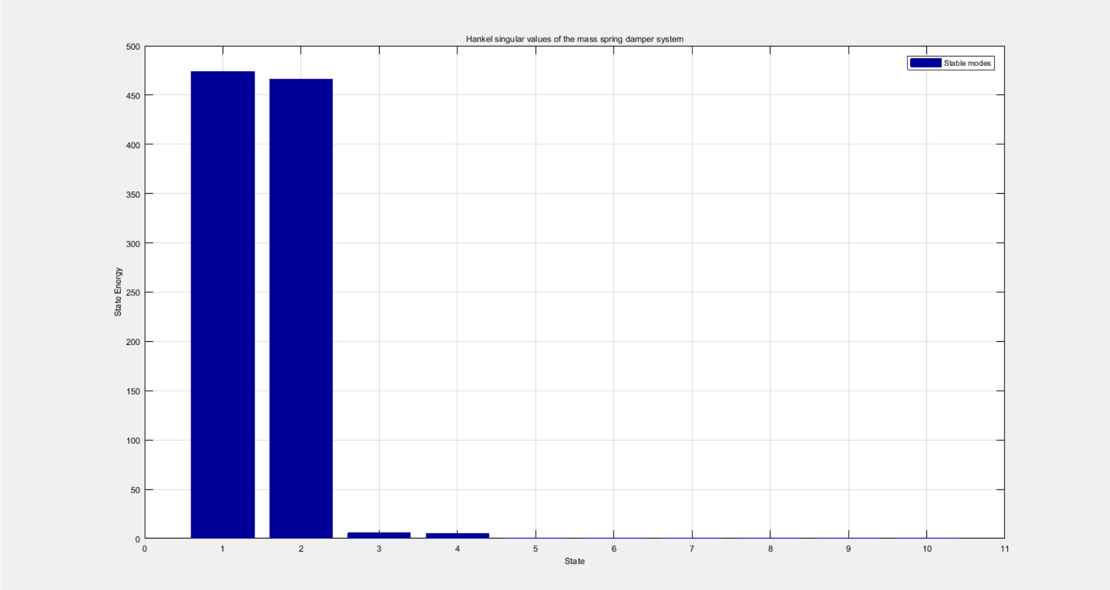

Figure (4.8): Hankel singular values of the mass spring damper system in example (4.3) by "hsvd" command

From two figures, there is a large gap between the second and the third Hankel singular values, so we eliminate from third to tenth Hankel singular values to have a balanced truncation system has order $= 2$.

We check the properties of new system (balanced truncation system):

- Stability

$$\lambda_i(A_{trnc}) = \{-0.0014 + 0.1643i,\ -0.0014 - 0.1643i\}$$

By definition (2.2) the balanced truncation system is stable. Since the eigenvalues of the system matrix A_{trnc} have negative real parts.

- Controllability and observability

$rank\big(C(A_{trnc}, B_{trnc})\big) = 2$

$rank\big(O(A_{trnc}, C_{trnc})\big) = 2$

By theorems (2.3) and (2.5) the balanced truncation system is controllable and observable.

Since $rank\big(C(A_{trnc}, B_{trnc})\big) = rank\big(O(A_{trnc}, C_{trnc})\big) =$ The order of balanced truncation system.

We conclude that the balanced truncation system inherited the main characteristics of the original mass spring damper system in the current example.

Now we need to find the approximation error. We compute $\|G - G_{trnc}\|_\infty$ and compare it with error bound to check if the inequality (3.20) achieved and draw it to have figure (4.9)

$$\|G - G_{trnc}\|_\infty = 10.9141$$

$$2\sum_{i=3}^{10} \sigma_i = 24.8008$$

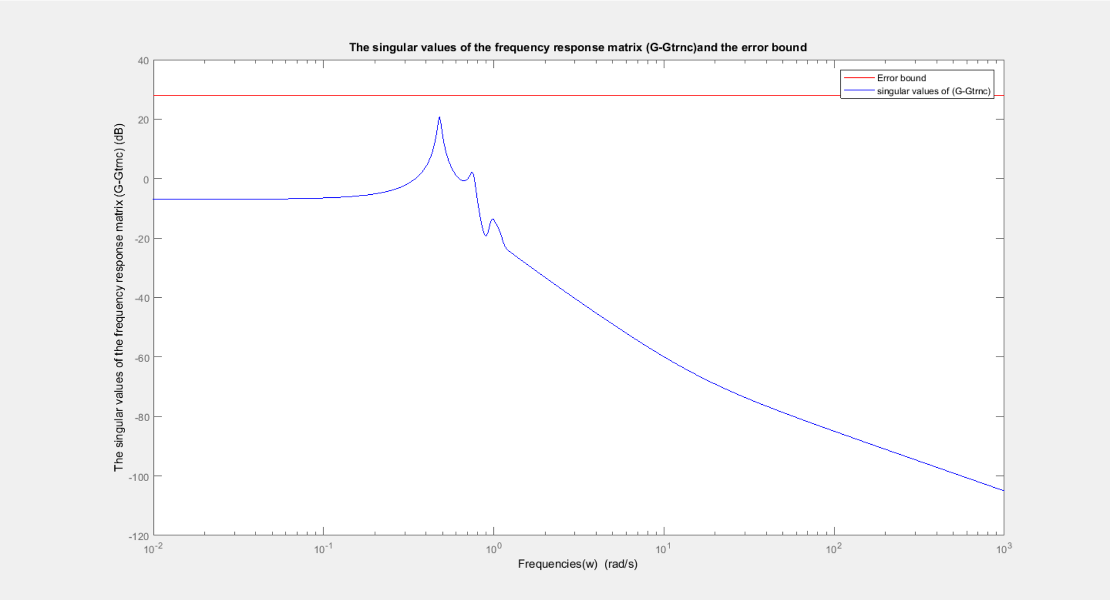

Figure (4.9): The singular values of the frequency response matrix$(\boldsymbol{G} - \boldsymbol{G_{trnc}})$ and the error bound in example (4.3).

We find the outputs of the original system, the balanced truncation system and the difference between them $(y,\ y_{trnc}$ and $(y - y_{trnc}))$, respectively, on the interval [0,5000]. Then draw both of them to have almost the same behavior as it clarified in the following figure:

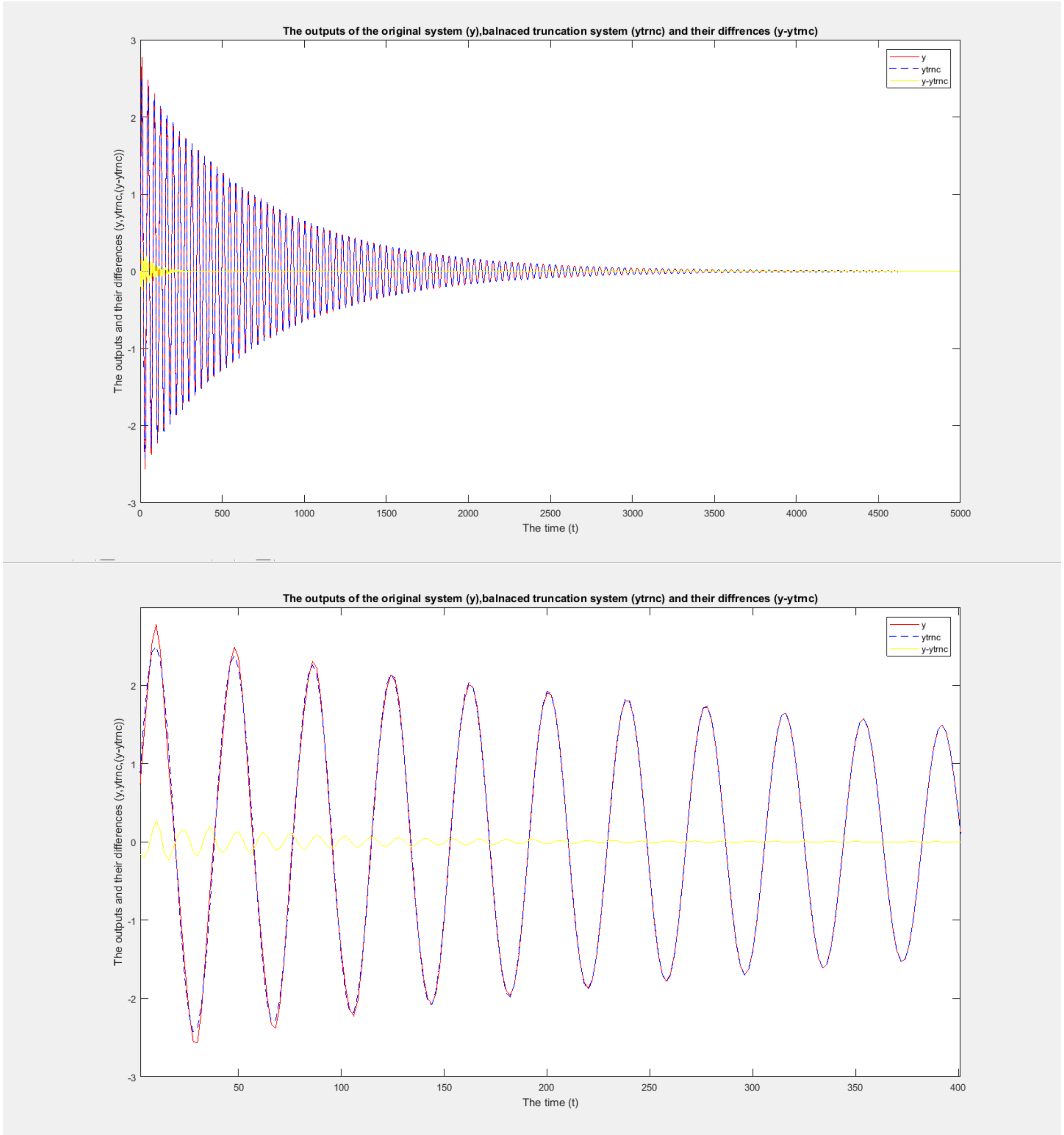

Figure (4.10): The outputs of the mass spring damper system in example (4.3) and a part of its zoom.

Now, computing H_∞ norm of $(G - G_{trunc})$ and comparing it with the error bound as it described in theorem (3.5) for various reduced order r, such as in the following table:

Table (4.2): The $\boldsymbol{H_\infty}$ norm of $(\boldsymbol{G - G_{trunc}})$ and the error bound of mass spring damper system in example (4.3).

r	$\|G - G_{trunc}\|_\infty$	$2\sum_{i=r+1}^{10} \sigma_i$
2	10.9141	24.8008
4	1.1849	2.8970
6	0.1939	0.4554
8	0.0208	0.0409

The following table includes the L_2 norm of difference of the outputs of original system and output of reduced system comparing with $2\sum_{i=r+1}^{10} \sigma_i \|u\|_2$:

Table (4.3): The $\boldsymbol{L_2}$ norm of $(\boldsymbol{y - y_{trunc}})$ and the error bound of the mass spring damper system in example (4.3).

r	$\|y - y_{trunc}\|_2$	$2\sum_{i=r+1}^{10} \sigma_i \|u\|_2$
2	0.8394	17.5368
4	0.1449	2.0485
6	0.0320	0.3220
8	0.0046	0.0290

Example (4.4): We take 20 masses such that they are connected with 20 springs and 20 dampers. So the order of the system is 40.

Check the properties of the system:

- Stability

$\lambda_i(A) = \{-0.0663 + 1.1494i,\ -0.0663 - 1.1494i,\ -0.0651 + 1.1393i,\ -0.0651 - 1.1393i,\ -0.0632 + 1.1225i,\ -0.0632 - 1.1225i,\ -0.0606 + 1.0992i,\ -0.0606 - 1.0992i,\ -0.0574 + 1.0695i,\ -0.0574 - 1.0695i,$

$-0.0535 + 1.0334i$, $-0.0535 - 1.0334i$, $-0.0493 + 0.9913i$, $-0.0493 - 0.9913i$, $-0.0446 + 0.9434i$, $-0.0446 - 0.9434i$, $-0.0397 + 0.8900i$, $-0.0397 - 0.8900i$, $-0.0346 + 0.8313i$, $-0.0346 - 0.8313i$, $-0.0295 + 0.7677i$, $-0.0295 - 0.7677i$, $-0.0245 + 0.6996i$, $-0.0245 - 0.6996i$, $-0.0197 + 0.6273i$, $-0.0197 - 0.6273i$, $-0.0152 + 0.5514i$, $-0.0152 - 0.5514i$, $-0.0112 + 0.4722i$, $-0.0112 - 0.4722i$, $-0.0076 + 0.3902i$, $-0.0076 - 0.3902i$, $-0.0047 + 0.3059i$, $-0.0047 - 0.3059i$, $-0.0024 + 0.2198i$, $-0.0024 - 0.2198i$, $-0.0009 + 0.1324i$, $-0.0009 - 0.1324i$, $-0.0001 + 0.0442i$, $-0.0001 - 0.0442i$ }

By definition (2.2) the system is stable. Since every eigenvalue of the system matrix A has negative real part.

- Controllability and observability,

$$rank\big(C(A,B)\big) = 40$$

$$rank\big(O(A,C)\big) = 40$$

By theorems (2.3) and (2.5) the system is controllable and observable. Since $rank\big(C(A,B)\big) = rank\big(O(A,C)\big) =$ The order of the system.

Computing the Hankel singular values and plot it by two ways as it shown in figures (4.11) and (4.12). then find the balancing system,

HSVs $= \sigma = \{10^4 \times ($2.4552, 2.4444, 0.0303, 0.0299, 0.0039, 0.0038, 0.0010, 0.0010, 0.0004, 0.0004, 0.0002, 0.0002, 0.0001, 0.0001, 0.0000}

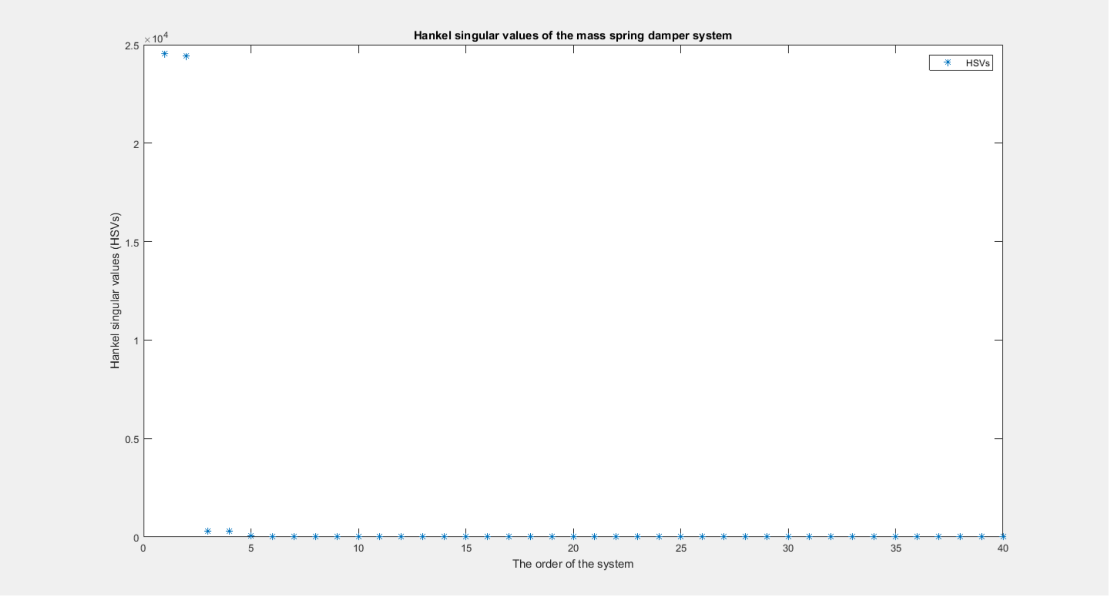

Figure (4.11): Hankel singular values of the mass spring damper system in example (4.4).

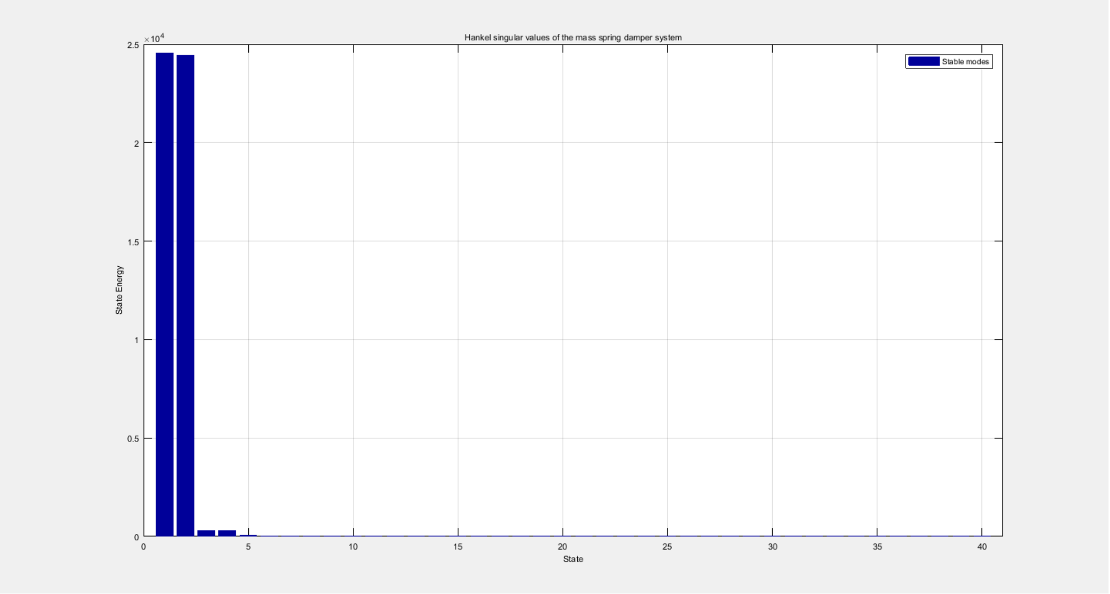

Figure (4.12): Hankel singular values of the mass spring damper system in example (4.4) by "hsvd" command

From two figures, there is a large gap between the second and the third Hankel singular values, so we eliminate from 3^{rd} to 40^{th} Hankel singular values to have a balanced truncation system with reduced order $r = 2$.

Check the properties of the new system:

- Stability

$$\lambda_i(A_{trnc}) = \{-0.0001 + 0.0442i, \quad -0.0001 - 0.0442i\}.$$

- Controllability and observability

$$rank\big(C(A_{trnc}, B_{trnc})\big) = 2$$

$$rank\big(O(A_{trnc}, C_{trnc})\big) = 2$$

By theorems (2.3) and (2.5) the balanced truncation system is controllable and observable.

We conserve the main properties of the original system also in this example.

Trivially, there exists an error system, so we compute $\|G - G_{trnc}\|_\infty$ and compare it with error bound to check if the inequality (3.20) achieved and draw it to have figure (4.13)

$$\|G - G_{trnc}\|_\infty = 602.4162$$

$$2\sum_{i=3}^{10} \sigma_i = 1.4264 \times 10^3$$

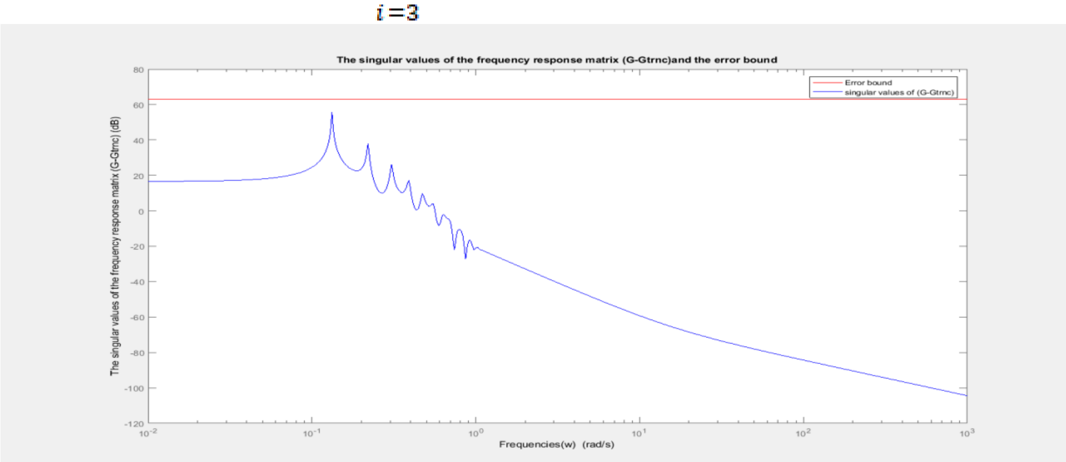

Figure (4.13): The singular values of the frequency response matrix $(G - G_{trnc})$ and the error bound in example (4.4).

Then we have the output of the original system, the output of the balanced truncation system y, y_{trnc} respectively and their difference $(y - y_{trnc})$ and draw them to have almost the same behavior for both systems as it clarified in the following figure:

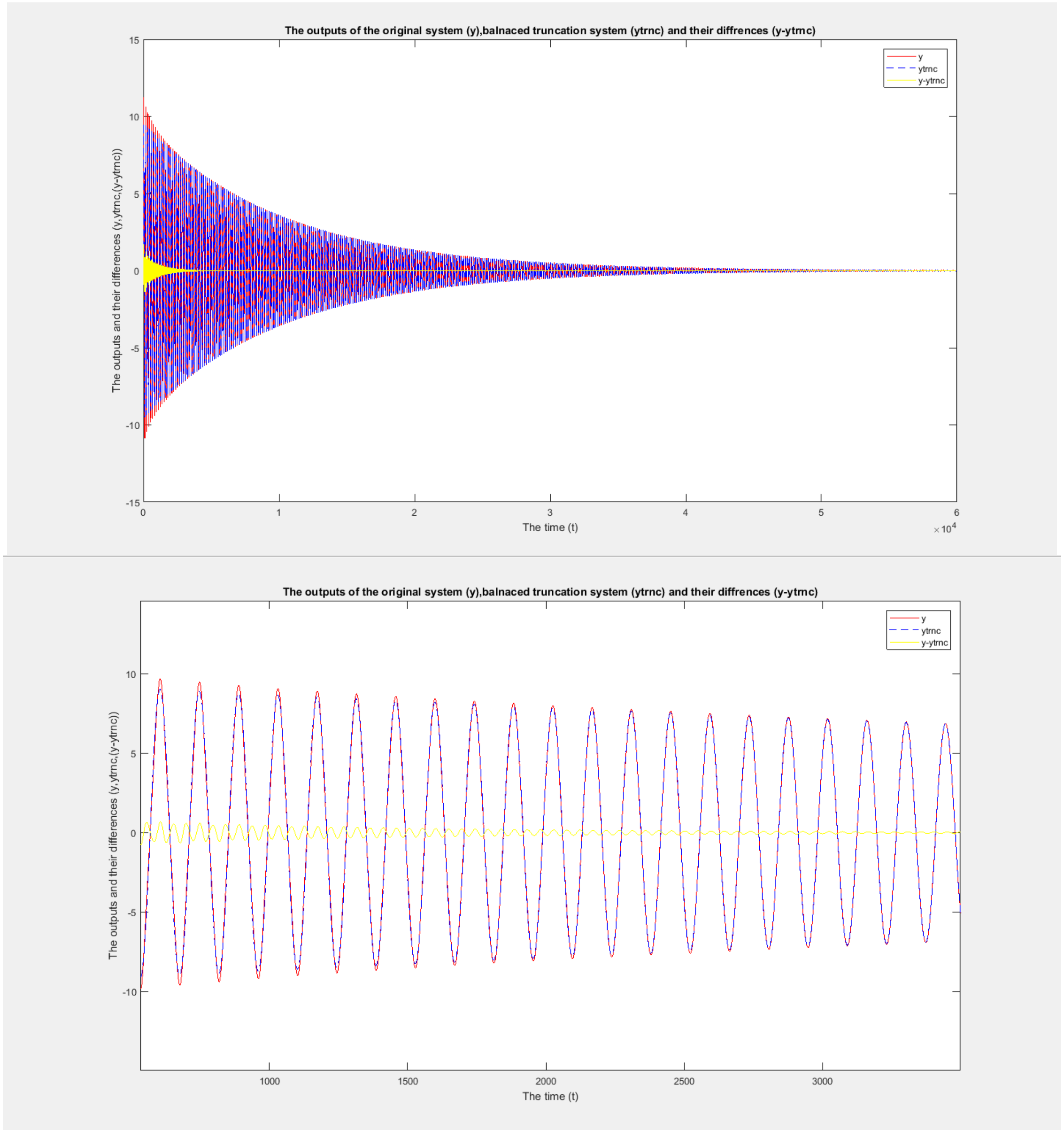

Figure (4.14): The outputs of the mass spring damper system in example (4.4) and a part of its zoom.

Now, computing H_∞ norm of the frequency response matrix $(G - G_{trunc})$ and comparing it with the error bound as it described in theorem (3.5) for various reduced order r, such as in the following table:

Table (4.4): The $\boldsymbol{H_\infty}$ norm of $(\boldsymbol{G} - \boldsymbol{G_{trunc}})$ and the error bound of mass spring damper system in example (4.4).

r	$\|G - G_{trunc}\|_\infty$	$2\sum_{i=r+1}^{10} \sigma_i$
2	602.4162	1.4264×10^3
4	77.2968	221.2623
6	19.6911	66.2461
8	6.8929	26.3655
10	2.8472	12.0434
12	1.2804	5.8417
14	0.7058	2.8651
16	0.3425	1.3723
18	0.1427	0.6251
20	0.0738	0.2651

The following table includes the L_2 norm of difference of the output of original system and output of reduced system comparing with $2\sum_{i=r+1}^{10} \sigma_i \|u\|_2$:

Table (4.5): The $\boldsymbol{L_2}$ norm of $(\boldsymbol{y} - \boldsymbol{y_{trunc}})$ and the error bound of the mass spring damper system in example (4.4).

r	$\|y - y_{trunc}\|_2$	$2\sum_{i=r+1}^{10} \sigma_i \|u\|_2$
2	12.9411	1.0086×10^3
4	2.8951	156.4498
6	1.0809	46.8412
8	0.5114	18.6425
10	0.2786	8.5156
12	0.1647	4.1306
14	0.0980	2.0259
16	0.0542	0.9703
18	0.0282	0.4420
20	0.0155	0.1874

In both examples we notice that, the balanced truncation systems are stable, controllable, observable and there exist an error system achieved the inequality (3.20). Then after we apply the balanced truncation method for various reduce order r, we conclude that the H_∞ norm of $(G - G_{trnc})$ and the error bound $2\sum_{i=r+1}^{2n} \sigma_i$ decreasing when the reduced order increasing as it explained in table (4.2) and (4.4) Since the balanced truncation system of high order include the energy of the original system more than the low order.

Also from table (4.3) and (4.5) we realize that the inequality (3.25) in remark (3.2) are achieved. And as it shown in figures (4.10), (4.14) the outputs of balanced truncation systems have almost the same behavior of the outputs of the original systems. Thereafter when we applied the balanced truncation method for various reduce order r we induce that the L_2 norm of $(y - y_{trunc})$ and the error bound $2\sum_{i=r+1}^{10} \sigma_i \|u\|_2$ decreasing when the reduced order increasing as it explained in table (4.2) and (4.4) Since the outputs of balanced truncation systems almost have the same act of the original systems when we have high order of reduced system, this is logically.

4.4 Conclusion

The simulation and control of large order engineering and physical systems makes mathematical operations complex, therefore the storage space is large and the design of the system and its implementation is costly. Therefore, the model order reduction appears. In this work we have used the balanced truncation as the main method of model order reduction after the system was represented as state space representation. The main concentration is having reduced system simulates the original system and inherit its properties by MATLAB. It has been noticed that the numerical results have coincided the theoretical results so the performance of balanced truncation systems is very close to performance of the original system with an approximation error has achieved the theoretical results. So we have considered the balanced truncation method is a very effective and successfully method of model order reduction.

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جامعة النجاح الوطنية
كلية الدراسات العليا

تقليل رتبة النموذج الرياضي بطريقة الاقتطاع المتوازن

إعداد
آية محمد محمود شرشير

إشراف
أ. د. ناجي قطناني
د. عدنان دراغمة

قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدارسات العليا في جامعة النجاح الوطنية، نابلس –فلسطين.

2019

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الملخص

في هذه الأطروحة ركزنا اهتمامنا على أحد أهم طرق تقليل رتبة النموذج الرياضي ألا وهي طريقة الاقتطاع المتوازن لنظام متصل خطي مستمر.

موازنة النظام تكافئ إيجاد قابلية التحكم والملاحظة لجراميان على شكل مصفوفة قطرية والتي تساوي قيم هانكل المنفردة.

وكنتيجة لطريقة الاقتطاع المتوازن، نجد خطأ تقريبي وهو المعيار اللانهائي H_∞ لدوال التحويل (في فضاء التردد) ومعيار L_2 لمخرجات النظام.

وقد أظهرت النتائج العددية لنظام الكتلة النابض والمخمد أن طريقة الاقتطاع المتوازن من أكثر الطرق فاعلية لتقليل رتبة النموذج الرياضي، لأن النظام ذو الرتبة المخفضة يحافظ على صفات النظام الأصلي ويحاكي سلوكه.