An-Najah National University
Faculty of Graduate Studies

Interpolation of Radial Basis Functions Using Trapezoidal Fuzzy Numbers

By

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II

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This Thesis was defended successfully on 27/9/2021 and approved by:

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Dedication

This thesis is dedicated to my wonderful family for their endless love, support and encouragement.

Acknowledgment

The second phase of my dreams comes true. Firstly, I would like to thank Allah for giving me guidance, strength, power of mind, and the moment to see my master's thesis.

I wish to express my sincere thanks to my advisors, Dr. Mohammad Borini Yasin and Dr. Hadi Hamad for their advice and assistance in every stage of my thesis, including wise supervision and suggestions which were so helpful to me.

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V

الإقرار

انا الموقعة أدناه مقدمة الرسالة التي تحمل العنوان:

Interpolation of Radial Basis Functions Using Trapezoidal Fuzzy Numbers

اقر بأن ما اشتملت عليه هذه الرسالة إنما في نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد، وأن هذه الرسالة ككل، أو أي جزء منها لم يقدم من قبل لنيل أية درجة علمية أو بحث علمي أو بحثي لدى أي مؤسسة تعليمية أو بحثيه اخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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List of Abbreviations

RBFs	Radial Basis Functions
FNs	Fuzzy Numbers
TFNs	Trapezoidal Fuzzy Numbers
$\mathbf{E_c}$(**RMFE**)	Root mean forecast error
$\mathbf{E}_{\boldsymbol{\ell}}$(**RMSE**)	Root mean square error

**Interpolation of Radial Basis Functions
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Abstract

Interpolation is one of the important and widespread problems in different scientific technical fields such as image processing, visualization of computational graphics, geometric modeling, design and many others. It is used to create a new determined or estimated data points between known data points on a graph (or the discussed issue).

In this thesis, a new radial basis function's (RBF's) methodology is interpolating a function by using trapezoidal fuzzy numbers (TFNs) through the Gaussian RBFs. The methodology uses two approximation errors: E_c (root mean forecast error) and E_ℓ (root mean square error) to measure the performance of the methodology in approximating the interpolation function to the original function.

Finally, using two numerical examples, results made it possible to conclude that as the number of RBF's centers is increased, the accuracy of the interpolation is increased which is measured through two common metrics: the E_c and E_ℓ errors by using MATLAB software.

Preface

One of the most interesting and important problems in mathematical field of numerical analysis is interpolation; which is a type of estimation. It is widely used in processing images, neural networks, surface reconstruction, numerical solution of partial differential equations … etc.

The interpolation problem of fuzzy data was first introduced by Zadeh in 1965 [39] and can be formulated as:

" Suppose we are given $n+1$ points $x_0, \dots, x_n \in \mathbb{R}$ and for each of these points a Fuzzy Value in $\mathbb{R}$; then, it is possible to construct some function on $\mathbb{R}$, rather than a crisp one to define some kind of smooth function in $\mathbb{R}$ with the given $n+1$ points?". Lowen investigated a fuzzy Lagrange interpolation in [20]. Later, Kaleva [14] proposed some properties of Lagrange interpolation by using cubic spline approximation. Wang and Li gave the definition of simple fuzzy numbers and the expressions of their membership functions in [33]. Different types of fuzzy sets [4] are defined in order to clear the vagueness of the existing problems.

The concept of fuzzy numbers is the generalization of the concept of real numbers. A fuzzy number [17], is a quantity whose values are imprecise, rather than exact as in the case with single-valued function. Recently in 1978, additional important works on the concepts of fuzzy numbers have been written by Nahmias [24]. Dubois and Prade has defined fuzzy number as a fuzzy subset of the real line [6]. So far, fuzzy numbers like triangular fuzzy numbers, trapezoidal fuzzy numbers, Pentagonal fuzzy numbers, Hexagonal, Octagonal and pyramidu fuzzy numbers have been introduced with their membership functions [26]. These numbers have got many applications [19] particularly in nonlinear equations, risk analysis and reliability. A defined set of operations [3,8,11] are implemented on fuzzy numbers.

Radial basis functions (RBFs) are a special class of functions. Their characteristic feature is that their response decreases (or increases) monotonically with distance from a central point; that is a function defined on an Euclidean space $\mathbb{R}^d$ whose value at each point depends only on the distance between that point and the origin, (to view and expand more see [9,27,32,36]).

This thesis is organized as follows: Chapter one is divided into two parts; Frist, review some basic concepts of fuzzy sets and related concepts including fuzzy points, fuzzy functions and their properties. Second, study a special type of fuzzy sets, namely, fuzzy numbers and Operations on Fuzzy numbers. While chapter two will describe RBFs method for fuzzy interpolation. Chapter three, in this thesis, MATLAB software will be used for RBFs techniques for solving trapezoidal fuzzy numbers. Also a comparison of these numerical schemes will be presented in chapter four.

Chapter One

Introduction to Fuzzy Sets

Chapter One

Introduction to Fuzzy Sets

Introduction

The word fuzzy was first introduced by zadeh in his famous paper "Fuzzy Sets" [39]. He used this word to generalize the mathematical concept of set to fuzzy set or fuzzy subset, where in a fuzzy set, a membership function is defined for each element of the referential set (in Boolean algebra[1] this concept is usually called the Characteristic function). The membership function takes its values in the interval $[0,1] \subset \mathbb{R}$ instead of $\{0,1\}$ as in Boolean algebra. The applications of this theory can be found in almost every scientific field as artificial intelligence, expert systems, logic, management science, operations research, robotics, and others.

This chapter contains some definitions and basic operations about fuzzy sets, fuzzy functions and fuzzy numbers.

1.1 Crisp Sets

Definition (1.1.1) [28]: A **Classical**, or a **Crisp set**

The set which assigns grades of membership of either 0 or 1 to objects within their universe of discourse. In other words, objects either belong to or do not belong to the set; or objects either posses a certain property, or they do not;

[1] A Boolean algebra is an algebraic structure defined on a set of elements (0 and 1), two binary operators: + (OR), * (AND).

there is no middle ground; which leads to a new definition of a characteristic function,

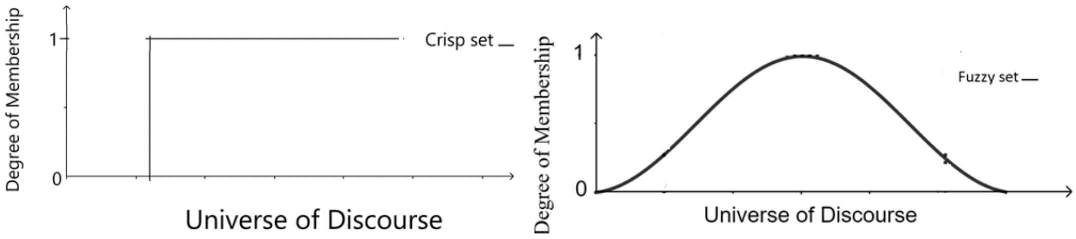
$$\text{i.e} \quad \mu_{\mathcal{A}}(x) = \begin{cases} 1 & \text{if } x \in \mathcal{A}, \\ 0 & \text{if } x \notin \mathcal{A}. \end{cases}$$

1.2 Fuzzy Sets

Zadeh in [39] extended the definition of the characteristic functions by replacing the set $\{0,1\}$ by the closed interval $[0,1]$ which is the bases to the new definition of fuzzy sets. The classical and fuzzy sets are shown in Figure 1.1.

Definition (1.2.1)[4]: **Fuzzy set**

A fuzzy set $\mu_{\mathcal{A}}(x)$ of X is a function from X to $[0,1]$, where $\mu_{\mathcal{A}}(x)$ is called the degree of membership of X in $\mathcal{A}$, it is obvious that a regular (crisp) subset is a special case of fuzzy subset.



(a) (b)

Figure 1.1: Illustration of a Crisp and a Fuzzy set.

Example (1.1):

Consider the regular set $X = \{a, b, c, d, e\}$ and let $\mu_{\mathcal{A}}$ be the fuzzy subset of X that maps X to $[0,1]$ by the following mapping as shown in Figure 1.2:

$$a \to 1, b \to 0.3, c \to 0.2, d \to 0.8, e \to 0.5$$

or $\mu_{\mathcal{A}}$ can be written as the set of ordered pairs:

$$\mu_{\mathcal{A}} = \{(a, 1), (b, 0.3), (c, 0.2), (d, 0.8), (e, 0.5)\},$$

also can be represented as

$$\mu_{\mathcal{A}} = \{a_1, b_{0.3}, c_{0.2}, d_{0.8}, e_{0.5}\}.$$

which is presented graphically as in Figure 1.2:

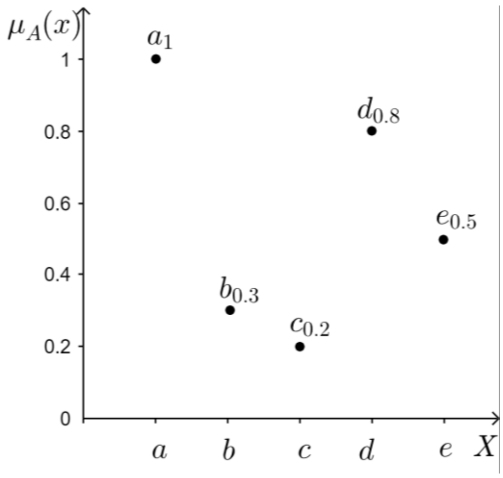


Figure 1.2: Graphical representation of the fuzzy membership of Example (1.1).

Remark:

If $X = \{a\}$, then the set of all subsets of X is denoted as 2^n, n means, the number of all elements in X are either $\{a\}$ or ϕ, while fuzzy subsets of X are $\{a_{0.2}\}, \{a_{0.19}\}, \{a_{0.4}\}, \dots \dots$ (infinite number of fuzzy subsets).

Definition (1.2.2) [18]: **Normal fuzzy sets**

A fuzzy set $\mathcal{A}$ on $\mathbb{R}$ is normal if its maximal degree of membership is unity (i.e. there must exist at least one $x \in X$ for which $\mathcal{A}(x) = 1$). Of course, non-normal fuzzy sets have maximum degree of membership less than one (see Figure 1.3).

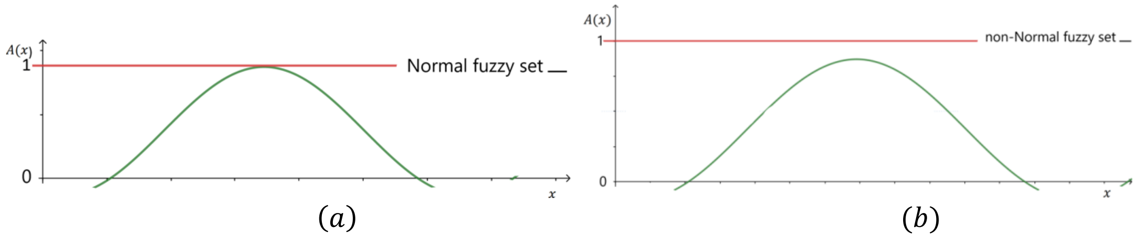


Figure 1.3: Normal and non–Normal fuzzy sets.

Definition (1.2.3) [25]: **Convex fuzzy sets**

A fuzzy subset $\mathcal{A} \subset \mathbb{R}$ is convex if and only if every ordinary subset

$$\mathcal{A}^{\alpha} = \{x \in \mathbb{R}: \mathcal{A}(x) \geq \alpha, \alpha \in (0,1]\}$$

is convex; that is, if it is a closed interval of $\mathbb{R}$, (see Figure 1.4).

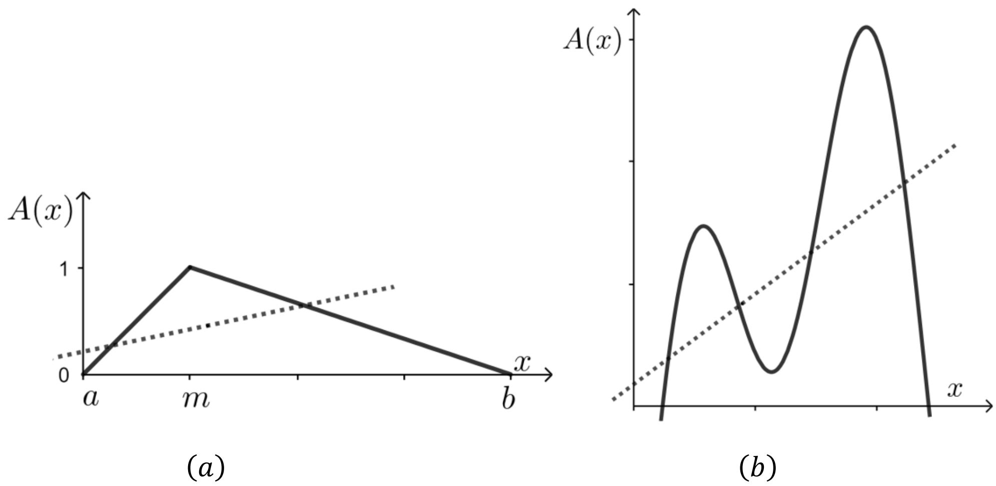


Figure 1.4: Convex and non – Convex fuzzy sets.

1.3 Operations on Fuzzy Sets

Definitions of operations on sets extend from ordinary set theory to fuzzy sets. In most cases, there are various ways to extend these

operations. This section presents the basic definitions of fuzzy intersection, union and complement, in addition to definitions of some other operations used in this thesis [41].

Definition (1.3.1): Intersection of fuzzy sets

The membership function of the intersection of two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ is defined as the minimum so:

$$(\mathcal{A} \cap \mathcal{B})(x) = min\{\mathcal{A}(x), \mathcal{B}(x)\}, \forall x \in X.$$

Or, in notation form $(\mathcal{A} \cap \mathcal{B})(x) = \mathcal{A}(x) \wedge \mathcal{B}(x)$.

Definition (1.3.2): Union of fuzzy sets

The membership function of the union of two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ is defined as the maximum so:

$$(\mathcal{A} \cup \mathcal{B})(x) = max\{\mathcal{A}(x), \mathcal{B}(x)\}, \forall x \in X.$$

Or, in notation form $(\mathcal{A} \cup \mathcal{B})(x) = \mathcal{A}(x) \vee \mathcal{B}(x)$.

Definition (1.3.3): Complement of fuzzy sets

The membership function of the complement is defined as:

$$\mathcal{A}^c(x) = 1 - \mathcal{A}(x), \forall x \in X.$$

Definition (1.3.4) [19]: **The containment**

A fuzzy set $\mathcal{A}$ is contained in a fuzzy set $\mathcal{B}$. If

$$\mathcal{A}(x) \leq \mathcal{B}(x), \forall x \in X.$$

In symbols, $(\mathcal{A} \subseteq \mathcal{B}) \Leftrightarrow \mathcal{A}(x) \leq \mathcal{B}(x), \forall x \in X$.

The previous definitions are illustrated by the following examples.

Example (1.2):

Take $X = \{a, b, c, d, e\}$ and the fuzzy subsets

$$\mathcal{A} = \{a_{0.4}, b_{0.5}, c_{0.9}, d_{0.7}, e_1\}, \mathcal{B} = \{a_{0.2}, b_{0.1}, c_{0.8}, d_{0.6}, e_{0.3}\}$$

Then

$$\mathcal{A} \cap \mathcal{B} = \{a_{0.2}, b_{0.1}, c_{0.8}, d_{0.6}, e_{0.3}\}.$$

$$\mathcal{A} \cup \mathcal{B} = \{a_{0.4}, b_{0.5}, c_{0.9}, d_{0.7}, e_1\}.$$

$$\mathcal{A}^c = \{a_{0.6}, b_{0.5}, c_{0.1}, d_{0.3}, e_0\}.$$

This shows that $\mathcal{B} \subseteq \mathcal{A}$, also notice that it is possible to have $\mathcal{A} \cup \mathcal{A}^c \neq X$ and $\mathcal{A} \cap \mathcal{A}^c \neq \phi$.

The following example is given for the continuous graph case:

Example (1.3):

Take $X = [0,10], \mathcal{A}(x)$ and $\mathcal{B}(x)$ are defined as follows in Figure 1.5.a. Then their union and intersection is interpreted in Figure 1.5.b,c, respectively.

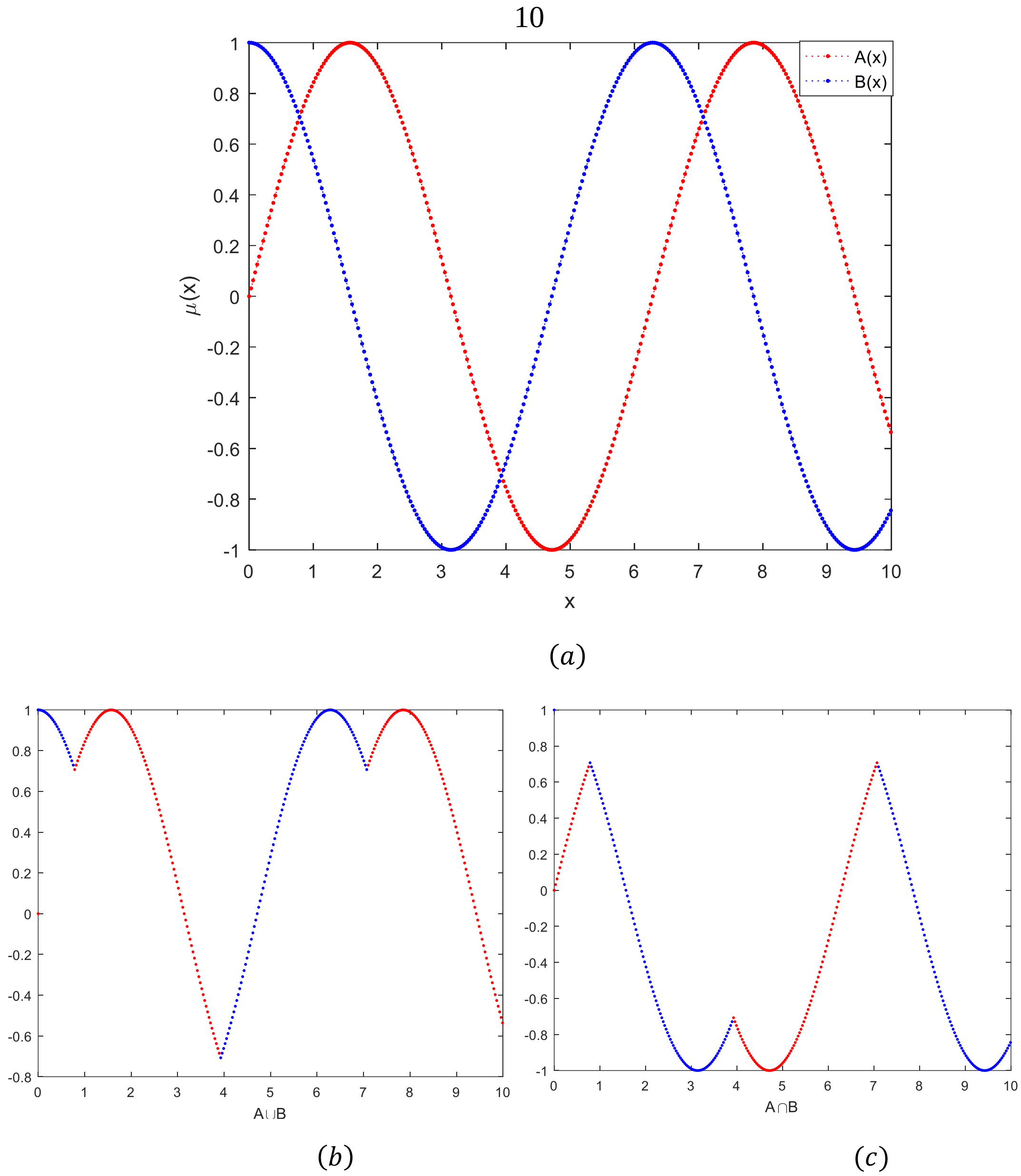


Figure 1.5: (a) Two fuzzy sets, (b) Union and (c) Intersection of fuzzy sets, respectively.

Some important properties of fuzzy sets (1.3.5)[34]:

Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy subsets of X, then:

1. $\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$ and $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$. (Commutativity)

2. $\begin{cases} (\mathcal{A} \cup \mathcal{B})^c = \mathcal{A}^c \cap \mathcal{B}^c, \\ (\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c \cup \mathcal{B}^c. \end{cases}$ (De Morgan's laws).

Proof:

1. $(\mathcal{A} \cup \mathcal{B})(x) = max\{\mathcal{A}(x), \mathcal{B}(x)\}$

$$= max\{\mathcal{B}(x), \mathcal{A}(x)\}$$

$$= (\mathcal{B} \cup \mathcal{A})(x).$$

$$\therefore\ \mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}.$$

And similarly can prove $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$.

2. $(\mathcal{A} \cup \mathcal{B})^c(x) = 1 - (\mathcal{A} \cup \mathcal{B})(x)$

$$= 1 - max\{\mathcal{A}(x), \mathcal{B}(x)\}$$

$$= \begin{cases} 1 - \mathcal{A}(x) & \text{if } \mathcal{A}(x) \geq \mathcal{B}(x), \\ 1 - \mathcal{B}(x) & \text{if } \mathcal{B}(x) \geq \mathcal{A}(x). \end{cases}$$

$$= \begin{cases} 1 - \mathcal{A}(x) & \text{if } 1 - \mathcal{A}(x) < 1 - \mathcal{B}(x), \\ 1 - \mathcal{B}(x) & \text{if } 1 - \mathcal{B}(x) < 1 - \mathcal{A}(x). \end{cases}$$

$$= min\{1 - \mathcal{A}(x), 1 - \mathcal{B}(x)\}$$

$$= min\{\mathcal{A}^c(x), \mathcal{B}^c(x)\}$$

$$= (\mathcal{A}^c \cap \mathcal{B}^c)(x), \forall x \in X.$$

$$\therefore (\mathcal{A} \cup \mathcal{B})^c = \mathcal{A}^c \cap \mathcal{B}^c.$$

And similarly can prove $(\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c \cup \mathcal{B}^c$.

Definition (1.3.6) [8]: **The Support of a fuzzy set**

The support of a fuzzy set $\mathcal{A}$ defined on X is a crisp set defined as $Supp(\mathcal{A}) = \{x \in X: \mathcal{A}(x) > 0\}$.

The support of $\mathcal{A}$ is the open interval $\left(a_1(0), a_2(0)\right)$,which is shown in Figure 1.6.

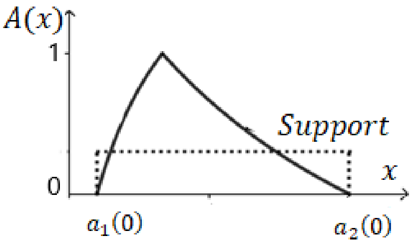


Figure 1.6: Support of a fuzzy set.

Definition (1.3.7)[40]: $\boldsymbol{\alpha}-$ **Cut**

An $\alpha -$ Level set of a fuzzy set $\mathcal{A}$ of X is a non-fuzzy set denoted by $\mathcal{A}^{\alpha}$ and is defined as follows:

$$\mathcal{A}^{\alpha} = \begin{cases} x \in X : \mathcal{A}(x) \geq \alpha & \text{if } \alpha \,\epsilon (0,1], \\ \mathrm{Cl}\left(\mathrm{Supp}(\mathcal{A})\right) & \text{if } \alpha = 0. \end{cases}$$

where $\mathrm{Cl}\left(\mathrm{Supp}(\mathcal{A})\right) = \text{Closure of the Support of } \mathcal{A}$.

Using $\alpha -$ cut one can get the following crisp range according to Figure1.7:

Minimum value of a crisp range x_1 is $= a^{\alpha}$.

Maximum value of a crisp range x_2 is $= b^{\alpha}$.

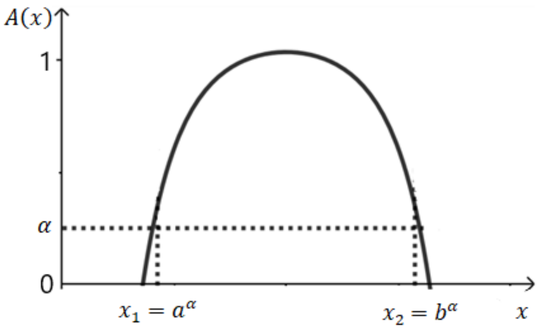


Figure 1.7: Illustration of an $\boldsymbol{\alpha} -$ cut.

Example (1.4):

$\mathcal{A}$ is a fuzzy subset of $\mathbb{R}$ defined by

$$\mathcal{A}(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ \frac{4-x}{3} & \text{if } 1 \leq x \leq 4 \\ 0 & \text{elsewhere} \end{cases},$$

let $\alpha = 0.1$. Find $\mathcal{A}^\alpha$ and in general $\alpha -$ cut values for $0 < \alpha \leq 1$.

Solution:

The 0.1 level of this fuzzy set is $\mathcal{A}^{0.1} = \{x \in X : \mathcal{A}(x) \geq 0.1\}$ is shown in the following, Figure 1.8.

In the first interval, the function is defined as:

$y = x^2$

Hence,

$0.1 = x^2 \to x = \pm\sqrt{0.1} \to x_1 \cong 0.31623$

while in the second interval of the function

$y = \frac{4-x}{3}$

$$0.1 = \frac{4-x}{3} \to x_2 = 3.7$$

$\therefore\ \mathcal{A}^{0.1} = [x_1, x_2] = [0.31623\,, 3.7]$

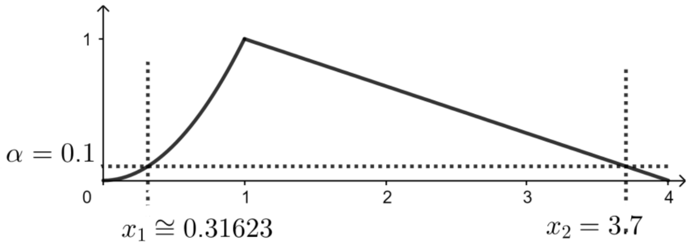


Figure 1.8: $0.1 -$ cut of the semi triangular fuzzy number (Example 1.4).

In general, for any $0 < \alpha \leq 1$ the $\alpha -$ cut can be found for this example as follows:

In the first side

$$y = x^2$$

$$\alpha = x^2 \rightarrow x = \pm\sqrt{\alpha} \rightarrow x_1 = \sqrt{\alpha}$$

In the second side

$$y = \frac{4 - x}{3}$$

$$\alpha = \frac{4 - x}{3} \rightarrow x_2 = 4 - 3\alpha$$

$$\therefore \mathcal{A}^\alpha = [x_1, x_2] = \left[\sqrt{\alpha}, 4 - 3\alpha\right]$$

1.4 Fuzzy Functions

A fuzzy function is a generalization of the concept of a classical function, which was defined between two families of fuzzy subsets corresponding to a function between two regular sets (i.e. we also extend any function f between any two regular sets; $f: X \rightarrow Y$; to a fuzzy function f^* between two families of fuzzy subsets; $f^*: \mathbb{F}(X) \rightarrow \mathbb{F}(Y)$).

Definition (1.4.1) [19]:

Let X and Y be two sets, $\mathbb{F}(X), \mathbb{F}(Y)$, denote the family of all fuzzy subsets of X and Y, respectively. For any crisp function $f: X \to Y$ there exist a function (fuzzy function) such that:

$$f^*: \mathbb{F}(X) \to \mathbb{F}(Y)$$

defined by: for any $\mathcal{A} \in \mathbb{F}(X)$, $f^*(\mathcal{A})$ is a fuzzy subset of Y where:

$$[f^*(\mathcal{A})](y) = \begin{cases} max\{\mathcal{A}(x): x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi, \\ 0 & \text{if } f^{-1}(y) = \phi. \end{cases}$$

and, if L is a fuzzy subset of Y then $(f^*)^{-1}(L)$ is defined by:

$$(f^*)^{-1}(L)(x) = L\big(f(x)\big).$$

Example (1.5):

Let $X = \{a, b, c\}, Y = \{u, v, w\}$,

$f: X \to Y$,	$\mathcal{A}: X \to [0,1]$	and	$L: Y \to [0,1]$
$a \to u$	$a \to 0.4$		$u \to 0.3$
$b \to v$	$b \to 0.9$		$v \to 0.8$
$c \to v$	$c \to 0.2$		$w \to 0.4$

Find $f^*(\mathcal{A}), (f^*)^{-1}(L)$.

Solution:

Then $f^*(\mathcal{A})$ be fuzzy subset of Y defined as:

$$f^*(\mathcal{A}): Y \to [0,1]$$

$$u \to 0.4$$

$$v \to max\{0.9,0.2\} = 0.9$$

$$w \to 0$$

And $(f^*)^{-1}(L)$ be fuzzy subset of X defined as:

$$(f^*)^{-1}(L): X \to [0,1]$$

$$a \to 0.3$$

$$b \to 0.8$$

$$c \to 0.8$$

Theorem (1.4.2) [1]:

Let $f: X \to Y$ be a function

1. If L and M are fuzzy subsets of Y, then:

a. $(f^*)^{-1}(L \cup M) = (f^*)^{-1}(L) \cup (f^*)^{-1}(M)$

b. $(f^*)^{-1}(L \cap M) = (f^*)^{-1}(L) \cap (f^*)^{-1}(M)$

2. If $\mathcal{A}$ and $\mathcal{B}$ are fuzzy subsets of X, then:

a. $(f^*)(\mathcal{A} \cup \mathcal{B}) = (f^*)(\mathcal{A}) \cup (f^*)(\mathcal{B})$

b. $(f^*)(\mathcal{A} \cap \mathcal{B}) \subseteq (f^*)(\mathcal{A}) \cap (f^*)(\mathcal{B})$

Proof:

1.

a. $(f^*)^{-1}(L \cup M) = (f^*)^{-1}(L) \cup (f^*)^{-1}(M)$

$(f^*)^{-1}(L \cup M)(x) = (L \cup M)f(x)$ for any $x \in X$

$$= max\{L(f(x)), M(f(x))\}$$

$$= max\{(f^*)^{-1}(L)(x), (f^*)^{-1}(M)(x)\}$$

$$= ((f^*)^{-1}(L) \cup (f^*)^{-1}(M))(x).$$

Hence, the equation is true for all x

$(f^*)^{-1}(L \cup M) = (f^*)^{-1}(L) \cup (f^*)^{-1}(M)$.

b. And similarly can prove $(f^*)^{-1}(L \cap M) = (f^*)^{-1}(L) \cap (f^*)^{-1}(M)$.

2.

a. $(f^*)(\mathcal{A} \cup \mathcal{B}) = (f^*)(\mathcal{A}) \cup (f^*)(\mathcal{B})$

for any $y \in Y$ if $f^{-1}(y) = \phi$ then $(f^*(\mathcal{A} \cup \mathcal{B})(y)) = 0$

Now $(f^*(\mathcal{A}) \cup f^*(\mathcal{B}))(y) = max\{(f^*(\mathcal{A}))(y), (f^*(\mathcal{B}))(y)\}$

$$= max\{0,0\} = 0$$

$\Longrightarrow (f^*(\mathcal{A} \cup \mathcal{B})(y)) = (f^*(\mathcal{A}) \cup f^*(\mathcal{B}))(y)$

$\therefore$ $(f^*)(\mathcal{A} \cup \mathcal{B}) = (f^*)(\mathcal{A}) \cup (f^*)(\mathcal{B})$.

If $f^{-1}(y) \neq \phi$ and let $f^{-1}(y) = \{x_1, x_2, \dots \dots \dots, x_n\}$

$(f^*(\mathcal{A} \cup \mathcal{B})(y)) = max\{(\mathcal{A} \cup \mathcal{B})(x_1), (\mathcal{A} \cup \mathcal{B})(x_2), \dots \dots, (\mathcal{A} \cup \mathcal{B})(x_n)\}$

$$= max\{max\{\mathcal{A}(x_1), \mathcal{B}(x_1)\}, max\{\mathcal{A}(x_2), \mathcal{B}(x_2)\}, \dots, max\{\mathcal{A}(x_n), \mathcal{B}(x_n)\}\}$$

Where $\mathcal{A}(x_1) = \lambda_1, \mathcal{B}(x_1) = r_1,\ \mathcal{A}(x_2) = \lambda_2, \mathcal{B}(x_2) = r_2, \dots \dots,\ \mathcal{A}(x_n) = \lambda_n, \mathcal{B}(x_n) = r_n$.

$$= max\{max\{\lambda_1, r_1\}, max\{\lambda_2, r_2\}, \dots \dots, max\{\lambda_n, r_n\}\}$$

But $\big(f^*(\mathcal{A}) \cup f^*(\mathcal{B})\big)(y) = max\{\big(f^*(\mathcal{A})\big)(y), \big(f^*(\mathcal{B})\big)(y)\}$

$$= max\{max\{\mathcal{A}(x_1), \mathcal{A}(x_2), \dots \mathcal{A}(x_n)\}, max\{\mathcal{B}(x_1), \mathcal{B}(x_2), \dots, \mathcal{B}(x_n)\}\}$$

$$= max\{max\{\lambda_1, \lambda_2, \dots, \lambda_n\}, max\{r_1, r_2, \dots, r_n\}\}$$

$$= max\{max\{\lambda_1, r_1\}, max\{\lambda_2, r_2\}, \dots \dots, max\{\lambda_n, r_n\}\}$$

$\Longrightarrow\ \big(f^*(\mathcal{A} \cup \mathcal{B})(y)\big) = \big(f^*(\mathcal{A}) \cup f^*(\mathcal{B})\big)(y)$

$\therefore\ (f^*)(\mathcal{A} \cup \mathcal{B}) = (f^*)(\mathcal{A}) \cup (f^*)(\mathcal{B})$.

b. To show $(f^*)(\mathcal{A} \cap \mathcal{B}) \subseteq (f^*)(\mathcal{A}) \cap (f^*)(\mathcal{B})$

for any $y \in Y$ if $f^{-1}(y) = \phi \Longrightarrow \big(f^*(\mathcal{A} \cap \mathcal{B})(y)\big) = 0$

Now $\big(f^*(\mathcal{A}) \cap f^*(\mathcal{B})\big)(y) = min\{\big(f^*(\mathcal{A})\big)(y), \big(f^*(\mathcal{B})\big)(y)\}$

$$= min\{0,0\} = 0$$

i.e $\big(f^*(\mathcal{A} \cap \mathcal{B})(y)\big) = \big(f^*(\mathcal{A}) \cap f^*(\mathcal{B})\big)(y)$

If $f^{-1}(y) \neq \phi$ and let $f^{-1}(y) = \{x_1, x_2, \dots \dots \dots, x_n\}$

$\big(f^*(\mathcal{A} \cap \mathcal{B})(y)\big) = max\{(\mathcal{A} \cap \mathcal{B})(x_1), (\mathcal{A} \cap \mathcal{B})(x_2), \dots \dots, (\mathcal{A} \cap \mathcal{B})(x_n)\}$

$$= max\{min\{\mathcal{A}(x_1), \mathcal{B}(x_1)\}, min\{\mathcal{A}(x_2), \mathcal{B}(x_2)\}, \dots, min\{\mathcal{A}(x_n), \mathcal{B}(x_n)\}\}$$

Where $\mathcal{A}(x_1) = \lambda_1, \mathcal{B}(x_1) = r_1,\ \mathcal{A}(x_2) = \lambda_2, \mathcal{B}(x_2) = r_2, \dots \dots,\ \mathcal{A}(x_n) = \lambda_n, \mathcal{B}(x_n) = r_n$.

$$= max\{\{\lambda_1 \wedge r_1\}, \{\lambda_2 \wedge r_2\}, \dots \dots, \{\lambda_n \wedge r_n\}\}$$

$$= (\lambda_1 \wedge r_1) \vee (\lambda_2 \wedge r_2) \vee \dots \dots (\lambda_n \wedge r_n)$$

let $\lambda_0 = max\{\lambda_1, \lambda_2, \dots, \lambda_n\}$

$r_0 = max\{r_1, r_2, \dots, r_n\}$

$(\lambda_1 \wedge r_1) \leq (\lambda_0 \wedge r_0)$

$(\lambda_2 \wedge r_2) \leq (\lambda_0 \wedge r_0)$

$\vdots$

$\vdots$

$(\lambda_n \wedge r_n) \leq (\lambda_0 \wedge r_0)$

$\therefore \quad (\lambda_1 \wedge r_1) \vee (\lambda_2 \wedge r_2) \vee \dots \dots \dots (\lambda_n \wedge r_n) \leq (\lambda_0 \wedge r_0)$

But $\left(f^*(\mathcal{A}) \cap f^*(\mathcal{B})\right)(y) = min\{\left(f^*(\mathcal{A})\right)(y), \left(f^*(\mathcal{B})\right)(y)\}$

$$= min\{max\{\mathcal{A}(x_1), \mathcal{A}(x_2), \dots, \mathcal{A}(x_n)\}, max\{\mathcal{B}(x_1), \mathcal{B}(x_2), \dots, \mathcal{B}(x_n)\}\}$$

$$= min\{max\{\lambda_1, \lambda_2, \dots, \lambda_n\}, max\{r_1, r_2, \dots, r_n\}\}$$

$$= min\{\{\lambda_1 \vee \lambda_2 \vee \dots \lambda_n\}, \{r_1 \vee r_2 \vee \dots r_n\}\}$$

$$= min\{\lambda_0, r_0\} = \lambda_0 \wedge r_0$$

$\Longrightarrow \ \left(f^*(\mathcal{A} \cup \mathcal{B})(y)\right) \leq \lambda_0 \wedge r_0 = \left(f^*(\mathcal{A}) \cup f^*(\mathcal{B})\right)(y)$

In general $(f^*)(\mathcal{A} \cap \mathcal{B}) \subseteq (f^*)(\mathcal{A}) \cap (f^*)(\mathcal{B})$.

1.5 Fuzzy Points

As a special case of fuzzy subsets of X are the fuzzy points. They were defined by Wong [35], later on, other definitions were presented by Pu Ming and Liu Ming [23] as well as and made their contribution.

Definition (1.5.1) [35]:

A fuzzy point in X call it P, $P = a_\lambda$ where $a \in X$, $\lambda \in (0,1]$

such that a goes to λ, $X - \{a\}$ goes to 0,

i.e $P = a_\lambda : X \to [0,1]$

$a \to \lambda$

$b \to 0$

$c \to 0$

Definition (1.5.2) [35]:

Let $P = a_\lambda$ be a fuzzy point and $\mathcal{A}$ a fuzzy subset of X, then we may say P in $\mathcal{A}$ or $\mathcal{A}$ contains P denoted by: $P = a_\lambda \in \mathcal{A} \iff \lambda \leq \mathcal{A}(a)$.

Example (1.6):

Let $X = \{a, b, c\}$, $\mathcal{A} = \{a_{0.4}, b_{0.3}, c_{0.1}\}$

$P = a_{0.15} \in \mathcal{A}$?

$P: X \to [0,1]$ $\qquad$ $\mathcal{A}: X \to [0,1]$

$a \to 0.15$ $\qquad$ $a \to 0.4$

$b \to 0$ $\qquad$ $b \to 0.3$

$c \to 0$ $\qquad$ $c \to 0.1$

$\therefore$ $a_{0.15} \in \mathcal{A}$.

Theorem (1.5.3) [38]:

1. $P = a_\lambda \in \mathcal{A} \cup \mathcal{B}$ if and only if $P \in \mathcal{A}$ or $P \in \mathcal{B}$.

Proof:

$$P = a_\lambda \in \mathcal{A} \cup \mathcal{B} \Leftrightarrow \lambda \le (\mathcal{A} \cup \mathcal{B})(a)$$

$$\Leftrightarrow \lambda \le max\{\mathcal{A}(a), \mathcal{B}(a)\}$$

$$\Leftrightarrow \lambda \le \mathcal{A}(a) \text{ or } \lambda \le \mathcal{B}(a)$$

$$\Leftrightarrow P = a_\lambda \in \mathcal{A} \text{ or } P = a_\lambda \in \mathcal{B}$$

$\therefore$ $P = a_\lambda \in \mathcal{A} \cup \mathcal{B} \Leftrightarrow P \in \mathcal{A}$ or $P \in \mathcal{B}$.

2. In a simillar way, one can prove

$P = a_\lambda \in \mathcal{A} \cap \mathcal{B} \Leftrightarrow P \in \mathcal{A}$ and $P \in \mathcal{B}$.

Definition (1.5.4) [23]:

Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy subsets of X, then $\mathcal{A}$ and $\mathcal{B}$ are quasi-coincident (we write $\mathcal{A}\mathbb{Q}\mathcal{B}$) to mean $\exists a \in X$ such that $\mathcal{A}(a) + \mathcal{B}(a) > 1$ and if $P = a_\lambda$ is a fuzzy point then write $P\mathbb{Q}\mathcal{A}$ to mean $\lambda + \mathcal{A}(a) > 1$.

Example (1.7):

Let $X = \{a, b, c, d\}, \mathcal{A} = \{a_{0.4}, b_{0.7}, c_{0.5}, d_{0.1}\}$ and

$\mathcal{B} = \{a_{0.5}, b_{0.4}, c_{0.2}, d_{0.9}\}$.

Is $\mathcal{A}\mathbb{Q}\mathcal{B}$?

the answer is yes, because $\mathcal{A}(b) + \mathcal{B}(b) = 0.7 + 0.4 = 1.1 > 1$.

Is $a_{0.55}\mathbb{Q}\mathcal{B}$?

the answer is yes, because $\lambda + \mathcal{B}(a) = 0.55 + 0.5 = 1.05 > 1$.

Remark:

It is clear that: $a_\lambda\mathbb{Q}\mathcal{A} \Leftrightarrow a_\lambda \notin \mathcal{A}^c$.

Proof:

$$a_\lambda\mathbb{Q}\mathcal{A} \Leftrightarrow \lambda + \mathcal{A}(a) > 1$$

$$\Leftrightarrow \lambda > 1 - \mathcal{A}(a) = \mathcal{A}^c(a)$$

$$\Leftrightarrow \lambda > \mathcal{A}^c(a)$$

$$\therefore a_\lambda \notin \mathcal{A}^c.$$

This means that a_λ is said to be quasi–coincident with $\mathcal{A}$ if and only if

$\lambda > \mathcal{A}^c(a)$, or $\lambda + \mathcal{A}(a) > 1$.

1.6 Fuzzy Numbers (FNs)

A Fuzzy Number is a special form of fuzzy sets on the real line $\mathbb{R}$. FNs play a fundamental role in fuzzy mathematics, analogous to the role played by the ordinary numbers in classical mathematics. The concept of fuzzy numbers was introduced by Zadeh [39] in 1965 and by several other authors who have investigated properties and proposed applications of fuzzy number.

Definition (1.6.1) [37]: **Fuzzy Numbers**

A fuzzy number is a mapping $u: \mathbb{R} \rightarrow [0,1]$ such that it is normal and convex; an example is shown in Figure 1.9, satisfying the following properties:

- $u(x)$ is an upper semi-continuous function on $\mathbb{R}$.
- $u(x) = 0$ outside some interval $[a_1, a_4] \subset \mathbb{R}$.
- There exist real numbers a_2 and a_3 such that $a_1 \leq a_2 \leq a_3 \leq a_4$ with

1. $u(x)$ a monotonic increasing function on $[a_1, a_2]$.

2. $u(x)$ a monotonic decreasing function on $[a_3, a_4]$.

3. $u(x) = 1$, for all $x \in [a_2, a_3]$.

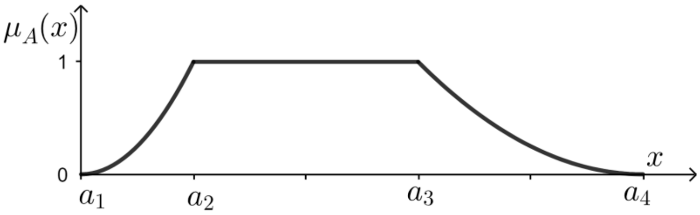


Figure 1.9: A fuzzy number.

1.7 Types of Fuzzy Numbers

Most common types of fuzzy numbers are triangular and trapezoidal ones. Other types of fuzzy numbers are possible, such as bell-shaped or gaussian fuzzy numbers. Here, the most popular types of fuzzy numbers are discussed namely:

1. Trapezoidal Fuzzy Numbers (TFNs)

A trapezoidal fuzzy number, $\mathcal{A} = (a_1, a_2, a_3, a_4)$ is described as any fuzzy subset of the real line $\mathbb{R}$, with the membership function $\boldsymbol{\mu_{\mathcal{A}}}$ is expressed as follows [2]: (this is shown in Figure 1.10).

$$\mu_{\mathcal{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ 1\ , & a_2 \leq x \leq a_3 \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \leq x \leq a_4 \\ 0\ , & \text{otherwise} \end{cases}$$

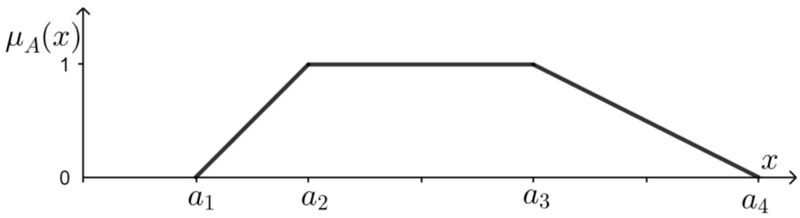


Figure 1.10: A Trapezoidal fuzzy number.

2. Triangular Fuzzy Numbers

For a triangular fuzzy number $\mathcal{A} = (a_1, a_2, a_3)$, it can be represented with the membership function $\mu_{\mathcal{A}}(x)$ given as [5], (this is shown in Figure 1.11).

$$\mu_{\mathcal{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3-x}{a_3-a_2}, & a_2 \le x \le a_3 \\ 0, & \text{otherwise} \end{cases}$$

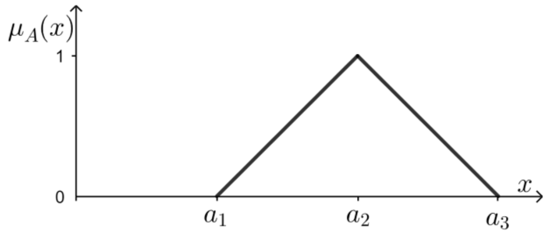


Figure 1.11: A Triangular fuzzy number.

1.8 Operations on Fuzzy Numbers

1.8.1 Addition of Fuzzy Numbers:

In this section [31] it is showed how to add two fuzzy numbers with intervals of confidence. Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy numbers and their intervals of confidence for the level of presumption α , $\alpha \in (0,1]$, then:

$$\mathcal{A}^\alpha\ (+)\ \mathcal{B}^\alpha = [a_1^\alpha, a_2^\alpha](+)[b_1^\alpha, b_2^\alpha]$$

$$= [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha].$$

Example (1.8):

Let $\mathcal{A} = [1,2,4]$ and $\mathcal{B} = [3,5,6]$ be two fuzzy numbers whose membership functions are:

$$\mathcal{A}(x) = \begin{cases} x-1 & \text{if } x \in [1,2] \\ \dfrac{4-x}{2} & \text{if } x \in [2,4] \\ 0 & \text{elsewhere} \end{cases} \qquad \mathcal{B}(x) = \begin{cases} \dfrac{x-3}{2} & \text{if } x \in [3,5] \\ 6-x & \text{if } x \in [5,6] \\ 0 & \text{elsewhere} \end{cases}$$

To determined $\mathcal{A}+\mathcal{B}$ use $\mathcal{A}(x)$ and $\mathcal{B}(x)$ respectively

$x-1=\alpha \Longrightarrow x=\alpha+1 \Rightarrow a_1^\alpha=\alpha+1$

And

$\frac{4-x}{2}=\alpha \Longrightarrow x=4-2\alpha \ \Rightarrow a_2^\alpha=4-2\alpha$

Then

$\mathcal{A}^\alpha=[a_1^\alpha,a_2^\alpha]=[\alpha+1,4-2\alpha]$

Also for $\mathcal{B}^\alpha$ as

$\frac{x-3}{2}=\alpha \Longrightarrow x=3+2\alpha \Rightarrow b_1^\alpha=3+2\alpha$

$6-x=\alpha \Longrightarrow x=6-\alpha \Rightarrow b_2^\alpha=6-\alpha$

Then

$\mathcal{B}^\alpha=[b_1^\alpha,b_2^\alpha]=[3+2\alpha,6-\alpha]$

To calculate addition of the fuzzy numbers $\mathcal{A}$ and $\mathcal{B}$ by adding the

$\alpha-$cuts of $\mathcal{A}$ and $\mathcal{B}$ using the interval of confidence.

$(\mathcal{A}+\mathcal{B})^\alpha=[\alpha+1+3+2\alpha,4-2\alpha+6-\alpha]=[4+3\alpha,10-3\alpha]$

To find $(\mathcal{A}+\mathcal{B})(x)=C(x)$ then c_1^α and c_2^α respectively

$c_1^\alpha=4+3\alpha,\alpha\in[0,1]$,

when $\alpha=0$ then $c_1^0=4\rightarrow x_1=4$

when $\alpha=1$ then $c_1^1=4+3(1)=7\rightarrow x_2=7$

replace c_1^α by x and α by y then $c_1^\alpha=4+3\alpha$ becomes

$x = 4 + 3y \rightarrow y = \dfrac{x-4}{3}, x \in [4,7]$

$c_2^{\alpha} = 10 - 3\alpha, \alpha \in [0,1],$

when $\alpha = 0$ then $c_2^0 = 10 \rightarrow x_1 = 10$

when $\alpha = 1$ then $c_2^1 = 10 - 3(1) = 7 \rightarrow x_2 = 7$

replace c_2^{α} by x and α by y then $c_2^{\alpha} = 10 - 3\alpha$ becomes

$x = 10 - 3y \rightarrow y = \dfrac{10-x}{3}, x \in [7,10]$

$$C(x) = (\mathcal{A} + \mathcal{B})(x) = \begin{cases} \dfrac{x-4}{3} & \text{if } x \in [4,7] \\ \dfrac{10-x}{3} & \text{if } x \in [7,10] \\ 0 & \text{elsewhere} \end{cases}$$

The summation of the two fuzzy numbers is shown in Figure 1.12.

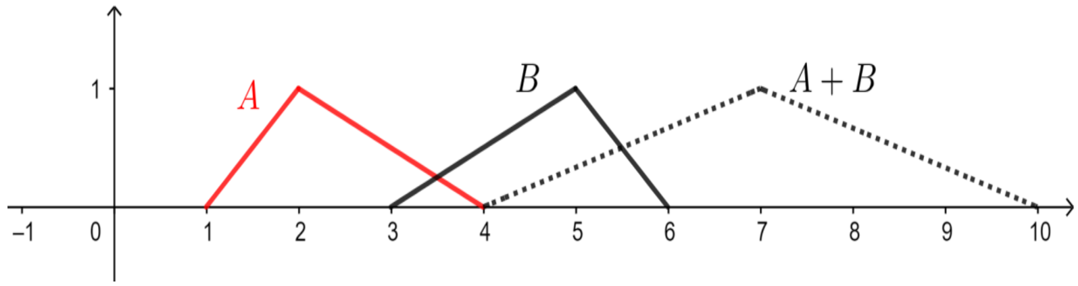


Figure 1.12: Addition of two linear triangular fuzzy numbers (Example 1.8).

Example (1.9):

Let $\mathcal{A} = [1,2,4,6]$ and $\mathcal{B} = [3,5,7]$ be two fuzzy numbers whose membership functions are defined as follows (Figure 1.13)

$$\mathcal{A}(x) = \begin{cases} x-1 & \text{if } x \in [1,2] \\ 1 & \text{if } x \in [2,4] \\ \dfrac{6-x}{2} & \text{if } x \in [4,6] \\ 0 & \text{elsewhere} \end{cases} \qquad \mathcal{B}(x) = \begin{cases} \dfrac{x-3}{2} & \text{if } x \in [3,5] \\ \dfrac{7-x}{2} & \text{if } x \in [5,7] \\ 0 & \text{elsewhere} \end{cases}$$

To find $\mathcal{A} + \mathcal{B}$, go through the following steps.

In case $\mathcal{A}(x)$, let $a_1^\alpha = \alpha + 1$ and $a_2^\alpha = 6 - 2\alpha$, from which

$$\mathcal{A}^\alpha = [a_1^\alpha, a_2^\alpha] = [\alpha + 1, 6 - 2\alpha] \qquad (1.1)$$

In case $\mathcal{B}(x)$, let $b_1^\alpha = 3 + 2\alpha$ and $b_2^\alpha = 7 - 2\alpha$, from which

$$\mathcal{B}^\alpha = [b_1^\alpha, b_2^\alpha] = [3 + 2\alpha, 7 - 2\alpha] \qquad (1.2)$$

Adding (1.1) and (1.2) gives

$$\begin{aligned}\mathcal{A}^\alpha\ (+)\ \mathcal{B}^\alpha &= [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha] \\ &= [\alpha + 1, 6 - 2\alpha](+)[3 + 2\alpha, 7 - 2\alpha] \\ &= [\alpha + 1 + 3 + 2\alpha, 6 - 2\alpha + 7 - 2\alpha] = [4 + 3\alpha, 13 - 4\alpha]\end{aligned}$$

From

$$a_1^\alpha + b_1^\alpha = 4 + 3\alpha$$

and

$$a_2^\alpha + b_2^\alpha = 13 - 4\alpha$$

Thus obtain

$$(\mathcal{A} + \mathcal{B})(x) = \begin{cases} \dfrac{x - 4}{3} & \text{if } x \in [4,7] \\ 1 & \text{if } x \in [7,9] \\ \dfrac{13 - x}{4} & \text{if } x \in [9,13] \\ 0 & \text{elsewhere} \end{cases}$$

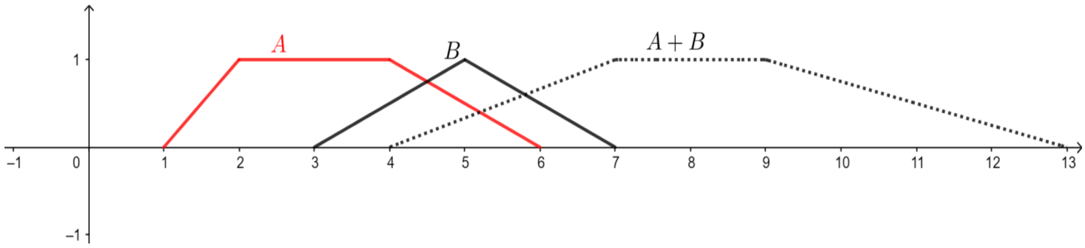


Figure 1.13: Addition of linear triangular and trapezoidal fuzzy numbers (Example 1.9).

1.8.2 Subtraction of Fuzzy Numbers:

The definition of addition can also be extended to the definition of subtraction, which is expressed as [17];

$$\mathcal{A}^{\alpha}(-)\mathcal{B}^{\alpha} = [a_1^{\alpha}, a_2^{\alpha}](-)[b_1^{\alpha}, b_2^{\alpha}], \forall\alpha \in [0,1]$$

$$= [a_1^{\alpha} - b_2^{\alpha}, a_2^{\alpha} - b_1^{\alpha}].$$

Now an example is given involving subtraction (see Figure 1.14).

Example (1.10):

Let $\mathcal{A} = [1,2,4]$ and $\mathcal{B} = [2,5,6]$ be two fuzzy numbers whose membership functions are

$$\mathcal{A}(x) = \begin{cases} x-1 & \text{if } x \in [1,2] \\ \dfrac{4-x}{2} & \text{if } x \in [2,4] \\ 0 & \text{elsewhere} \end{cases} \qquad \mathcal{B}(x) = \begin{cases} \dfrac{x-2}{3} & \text{if } x \in [2,5] \\ 6-x & \text{if } x \in [5,6] \\ 0 & \text{elsewhere} \end{cases}$$

Then $\mathcal{A} - \mathcal{B}$ can be calculated as follows:

Redefine the components of and $\mathcal{B}$.

Now using $\mathcal{A}(x)$ and $\mathcal{B}(x)$, then

$$a_1^{\alpha} = \alpha + 1 \;\;,\;\; a_2^{\alpha} = 4 - 2\alpha \;\; \text{and} \;\; b_1^{\alpha} = 2 + 3\alpha \;\;,\;\; b_2^{\alpha} = 6 - \alpha.$$

Hence,

$\mathcal{A}^{\alpha} = [a_1^{\alpha}, a_2^{\alpha}] = [\alpha + 1, 4 - 2\alpha]$ and $\mathcal{B}^{\alpha} = [b_1^{\alpha}, b_2^{\alpha}] = [2 + 3\alpha, 6 - \alpha]$

Now

$(-\mathcal{B})^{\alpha} = [-b_2^{\alpha}, -b_1^{\alpha}] = [-6 + \alpha, -3\alpha - 2]$

Using $-b_2^{\alpha}$ and $-b_1^{\alpha}$ to define $(-\mathcal{B})$

$-b_2^{\alpha} = -6 + \alpha, \alpha \in [0,1]$, when $\alpha = 0$ then $-b_2^{0} = -6 \rightarrow x_1 = -6$

when $\alpha = 1$ then $-b_2^{1} = -6 + (1) = -5 \rightarrow x_2 = -5$

replace $-b_2^{\alpha}$ by x and α by y then $-b_2^{\alpha} = -6 + \alpha$ becomes

$$x = -6 + y \rightarrow y = x + 6, x \in [-6, -5]$$

$-b_1^{\alpha} = -2 - 3\alpha, \alpha \in [0,1]$, when $\alpha = 0$ then $-b_1^{0} = -2 \rightarrow x_1 = -2$

when $\alpha = 1$ then $-b_2^{1} = -2 - 3(1) = -5 \rightarrow x_2 = -5$

replace $-b_1^{\alpha}$ by x and α by y then $-b_1^{\alpha} = -2 - 3\alpha$ becomes:

$$x = -2 - 3y \rightarrow y = \frac{-2 - x}{3}, x \in [-5, -2]$$

$$\therefore (-\mathcal{B}) = [-6, -5, -2]$$

To calculate addition of the fuzzy numbers $\mathcal{A}$ and $(-\mathcal{B})$ by adding the $\alpha -$ cuts of $\mathcal{A}$ and $(-\mathcal{B})$ using the interval of confidence.

$$(\mathcal{A} - \mathcal{B})^{\alpha} = [a_1^{\alpha} - b_2^{\alpha}, a_2^{\alpha} - b_1^{\alpha}]$$

$$= [\alpha + 1 - 6 + \alpha, 4 - 2\alpha - 2 - 3\alpha]$$

$$= [-5 + 2\alpha, 2 - 5\alpha]$$

Finally,

$$(\mathcal{A}-\mathcal{B})(x)=\begin{cases}\dfrac{x+5}{2} & \text{if } x\in[-5,-3]\\ \dfrac{2-x}{5} & \text{if } x\in[-3,\ \ 2]\\ 0 & \text{elsewhere}\end{cases}$$

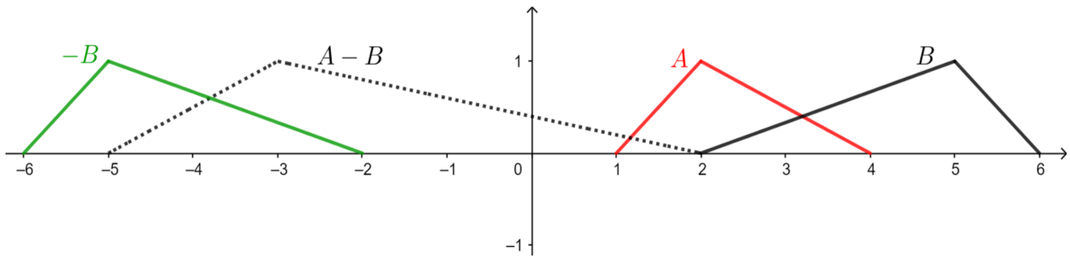


Figure 1.14: Subtraction of two fuzzy numbers (Example 1.10).

Remark:

Notice that $\mathcal{B}+(-\mathcal{B})\neq 0$.

Proof:

$$(\mathcal{B})^{\alpha}=[b_1^{\alpha},b_2^{\alpha}],(-\mathcal{B})^{\alpha}=[-b_2^{\alpha},-b_1^{\alpha}]$$

$$[\mathcal{B}+(-\mathcal{B})]^{\alpha}=[b_1^{\alpha}-b_2^{\alpha},b_2^{\alpha}-b_1^{\alpha}]\neq 0$$

1.8.3 Multiplication of Fuzzy Numbers:

Now consider two fuzzy numbers $\mathcal{A}$ and $\mathcal{B}$ where

$$(\mathcal{A})^{\alpha}=[a_1^{\alpha},a_2^{\alpha}],(\mathcal{B})^{\alpha}=[b_1^{\alpha},b_2^{\alpha}];$$

define $\mathcal{A}\cdot\mathcal{B}$ by using the definition $(\mathcal{A}\cdot\mathcal{B})^{\alpha}=[a_1^{\alpha}\cdot b_1^{\alpha},a_2^{\alpha}\cdot b_2^{\alpha}]$, [19].

Example (1.11):

Let $\mathcal{A} = [2,3,5]$ and $\mathcal{B} = [3,5,6]$ be two fuzzy numbers, which are defined by

$$\mathcal{A}(x) = \begin{cases} x-2 & \text{if } x \in [2,3] \\ \dfrac{5-x}{2} & \text{if } x \in [3,5] \\ 0 & \text{elsewhere} \end{cases} \qquad \mathcal{B}(x) = \begin{cases} \dfrac{x-3}{2} & \text{if } x \in [3,5] \\ 6-x & \text{if } x \in [5,6] \\ 0 & \text{elsewhere} \end{cases}$$

to apply the multiplication operation using their membership functions, for the level α in Figure 1.15 and using $\mathcal{A}(x), \mathcal{B}(x)$ we have

$a_1^\alpha = \alpha + 2$ and $a_2^\alpha = 5 - 2\alpha$. Hence $\mathcal{A}^\alpha = [a_1^\alpha, a_2^\alpha] = [\alpha + 2,\ 5 - 2\alpha]$.

Also,

$b_1^\alpha = 3 + 2\alpha$ and $b_2^\alpha = 6 - \alpha$. Hence $\mathcal{B}^\alpha = [b_1^\alpha, b_2^\alpha] = [3 + 2\alpha,\ 6 - \alpha]$.

Thus the multiplication result gives

$$(\mathcal{A} \cdot \mathcal{B})^\alpha = [(\alpha + 2) \cdot (3 + 2\alpha), (5 - 2\alpha) \cdot (6 - \alpha)]$$

$$= [2\alpha^2 + 7\alpha + 6, 2\alpha^2 - 17\alpha + 30]$$

Now, two equations to solve, namely,

$x = 2\alpha^2 + 7\alpha + 6$ replace α by y then becomes $x = 2y^2 + 7y + 6$

by Quadratic Formula solve the equation $2y^2 + 7y + 6 - x = 0$

$$y = \frac{-7 \pm \sqrt{49 - 4 \cdot 2 \cdot (6 - x)}}{4} = \frac{-7 \pm \sqrt{1 + 8x}}{4}$$

then the root $y = \frac{-7+\sqrt{1+8x}}{4}$ such that $\alpha \in [0,1]$

and the other root

$x = 2\alpha^2 - 17\alpha + 30$ replace α by y then become $x = 2y^2 - 17y + 30$

by Quadratic Formula $2y^2 - 17y + 30 - x = 0$

$$y = \frac{17 \pm \sqrt{289 - 4 \cdot 2 \cdot (30 - x)}}{4} = \frac{17 \pm \sqrt{49 + 8x}}{4}$$

then the root $y = \frac{17 - \sqrt{49+8x}}{4}$ such that $\alpha \in [0,1]$

Finally,

$$(\mathcal{A} \cdot \mathcal{B})(x) = \begin{cases} \dfrac{-7 + \sqrt{1 + 8x}}{4} & \text{where } x \in [6,15] \\ \dfrac{17 - \sqrt{49 + 8x}}{4} & \text{where } x \in [15,30] \\ 0 & \text{elsewhere} \end{cases}$$

The resulting multiplication curve is shown in Figure 1.15. Note that $\mathcal{A}(\cdot)\mathcal{B}$ does not yield a triangular shape.

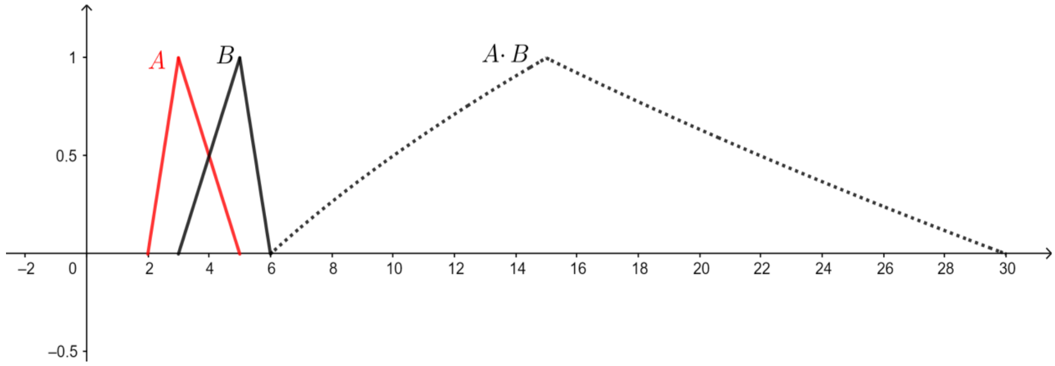


Figure 1.15: Multiplication of two fuzzy numbers (Example 1.11).

1.8.4 Division of Fuzzy Numbers:

Division of two fuzzy numbers is defined by [17];

$$\mathcal{A}^{\alpha}(:)\mathcal{B}^{\alpha} = \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right](:)\left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] = \left[a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}\right]$$

where support of $\mathcal{B} > 0,\ \forall\alpha \in [0,1]$.

Example (1.12):

Let us now look at the example illustrated in Figure 1.16, where $\mathcal{A} = [1,2,4]$ and $\mathcal{B} = [2,3,5]$ be two fuzzy numbers whose membership functions are:

$$\mathcal{A}(x) = \begin{cases} x-1 & \text{if } x \in [1,2] \\ \dfrac{4-x}{2} & \text{if } x \in [2,4] \\ 0 & \text{elsewhere} \end{cases} \qquad \mathcal{B}(x) = \begin{cases} x-2 & \text{if } x \in [2,3] \\ \dfrac{5-x}{2} & \text{if } x \in [3,5] \\ 0 & \text{elsewhere} \end{cases}$$

Find $\frac{\mathcal{A}}{\mathcal{B}}$.

To compute the intervals of confidence for each level α in Figure 1.16, which will be described by a functions of α, is illustrated in the following manner:

From $\mathcal{A}(x)$:

$a_1^{(\alpha)} = 1 + \alpha$ and $a_2^{(\alpha)} = 4 - 2\alpha$

Hence, the interval of confidence at the level α is given by

$\mathcal{A}^{\alpha} = \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] = [1 + \alpha, 4 - 2\alpha]$

From $\mathcal{B}(x)$:

$b_1^{(\alpha)} = 2 + \alpha$ and $b_2^{(\alpha)} = 5 - 2\alpha$

Therefore

$$\mathcal{B}^{\alpha} = \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] = [2+\alpha, 5-2\alpha]$$

Thus

$$(\mathcal{A}(:)\mathcal{B})^{\alpha} = [1+\alpha, 4-2\alpha](:)[2+\alpha, 5-2\alpha]$$
$$= \left[\frac{1+\alpha}{5-2\alpha}, \frac{4-2\alpha}{2+\alpha}\right]$$

I obtain

$$\left(\frac{\mathcal{A}}{\mathcal{B}}\right)(x) = \begin{cases} \dfrac{5x-1}{2x+1} & \text{if } x \in \left[\frac{1}{5}, \frac{2}{3}\right] \\ \dfrac{4-2x}{2+x} & \text{if } x \in \left[\frac{2}{3}, 2\right] \\ 0 & \text{elsewhere} \end{cases}$$

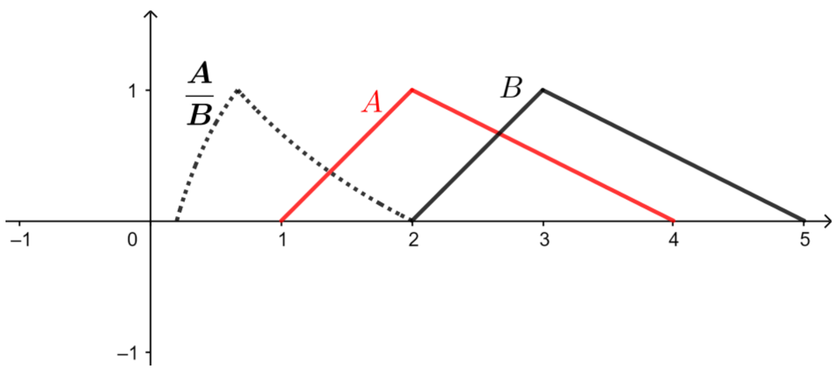


Figure 1.16: Division of two fuzzy numbers (Example 1.12).

Remark:

Notice that: dividing a fuzzy number by itself, it is not always equal to 1. That is $\frac{\mathcal{A}}{\mathcal{A}} \neq 1$.

Proof:

$$\mathcal{A}^{(\alpha)} = \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right],\ \left(\frac{1}{\mathcal{A}}\right)^{\alpha} = \left[\frac{1}{a_2^{(\alpha)}}, \frac{1}{a_1^{(\alpha)}}\right]$$

$$\therefore \quad \left(\frac{\mathcal{A}}{\mathcal{A}}\right)^{\alpha} = \left[\frac{a_1^{(\alpha)}}{a_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{a_1^{(\alpha)}}\right] \neq 1.$$

Chapter Two

Radial Basis Functions

Chapter Two

Radial Basis Functions

Introduction

Radial basis functions (RBFs) interpolation, which is the topic of this thesis, can be used in [13] cartography, Geodesy and digital terrain models applications in order to reduce error in data interpolation, it has been successfully used in a variety of applications. Some examples of RBFs are: measurements of the earth’s temperature from meteorological stations situated at scattered sites and learning application areas including pattern recognition, medical imaging, image warping and specially neural networks. While interpolating multidimensional scattered data, the RBF has been developed in the literature [27,32].

The history of RBFs goes back to 1970s by Rolland Hardy [12]. It uses multiquadric method to interpolate as a linear combination of translations of a radially symmetric basis function. The multiquadric method was generalized to other “radial functions”, such as the thin plate spline [7], the gaussian, the cubic, ... etc, and the method was called the RBFs method. Franke in 1982 [10] was concerned in the evaluation of methods for scattered data interpolation. In the 1990s, multiquadric is used as the spatial to approximate scheme for parabolic, hyperbolic and the elliptic Poisson's equation [15,16]. Skala in 2016, described novel approaches based on RBFs for data interpolation and approximation generally in d-dimensional space [30]. The authors in [21] introduced a comparison of RBFs approximation methods which are made with respect to the stability and accuracy of computation. A method to approximate a fuzzy function by using RBFs interpolation was presented in [9]. This chapter will give an introduction to the interpolation method using RBF's.

2.1 Radial Basis Function Formulation

The RBFs are based on input-output relation on linear combinations of terms, which include a single bivariate function[2]. Applying RBFs to interpolate a fuzzy function, a linear system will be obtained by defining coefficient vector, target function will be interpolated by the formula:

$$s(\boldsymbol{x}) = \sum_{i=1}^{M} \alpha_i \phi(\|\boldsymbol{x} - t_i\|) \qquad (2.1)$$

where $\boldsymbol{x} = [x_1, \dots, x_d]^T$ is the input, α_i are weights, and the radial function ϕ on $\mathbb{R}^d$ is defined through a univariate function ϕ: $[0, \infty) \to \mathbb{R}$ in such a way that $\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{\phi}(\|\boldsymbol{x}\|)^2$, where $\|.\|$ is the usual Euclidean norm in $\mathbb{R}^d$, and t_i are called the centers. A set of commonly used types of RBFs are given in Table (2.1).

Suppose given a set of M points $\{(x_1, f_1), \dots, (x_M, f_M)\}$, the centers are placed at the observed points, $t_i = x_i$, for $i = 1, \dots, M$. Our goal is to find an interpolant $s(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$, satisfying:

$$s(x_i) = f_i,\ \ i = 1, \dots, M. \qquad (2.2)$$

From equation (2.1), the problem follows that

---

[2] A function of two variables.

$$\sum_{i=1}^{M} \alpha_i \phi(\|x_j - x_i\|) = f_i, \quad i,j = 1, \dots, M. \tag{2.3}$$

Also, the formula can be rewrite in matrix form

$$\begin{pmatrix} \phi(\|x_1 - x_1\|) & \phi(\|x_2 - x_1\|) & \cdots & \phi(\|x_M - x_1\|) \\ \phi(\|x_1 - x_2\|) & \phi(\|x_2 - x_2\|) & \cdots & \phi(\|x_M - x_2\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\|x_1 - x_M\|) & \phi(\|x_2 - x_M\|) & \vdots & \phi(\|x_M - x_M\|) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_M \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_M \end{pmatrix} \tag{2.4}$$

Or the linear system of equation (2.4) can be expressed in the form:

$$\boldsymbol{\phi\alpha} = \boldsymbol{f}. \tag{2.5}$$

The matrix ϕ is of order $M \times M$, symmetric[3], nonsingular ($det(\phi) \neq 0$)[4] and our system is well-posed (i.e. ∃ a unique solution).

If matrix ϕ is said to be Strictly Diagonally Dominant (SDD) when $|a_{ii}| > \sum_{j=1,i\neq j}^{n} |a_{ij}|$ for each $i = 1,2,\dots,n$, then ϕ is nonsingular, hence the system $\boldsymbol{\phi\alpha} = \boldsymbol{f}$[5] , for any α, has a unique solution.

Some classical choices of smooth basis functions for RBFs can be found in (Table 2.1 and Figure 2.1). It is important to notice that the basis functions depend on parameter ε that determines the width of the basis functions and it is also used to scale the input of the radial kernel.

---

[3] A matrix $\phi = [a_{ij}]$ is Symmetric (i.e. $a_{ji} = a_{ij}$ for all possible values of i and j.

[4] Nonsingular matrix – a matrix whose determinant is not zero.

[5] The Problem will be Well-Posed (i.e. the interpolation matrix ϕ is non- singular).

Table 2.1: A List of commonly used RBFs.

Type of basis function	$\phi(r), (r \geq 0)$
Gaussian (GA)	$\phi(r) = e^{-(\varepsilon r)^2}$
Inverse quadratic (IQ)	$\phi(r) = \dfrac{1}{1+(\varepsilon r)^2}$
Inverse multiquadric (IMQ)	$\phi(r) = \dfrac{1}{\sqrt{1+(\varepsilon r)^2}}$
Multiquadric (MQ)	$\phi(r) = \sqrt{1+(\varepsilon r)^2}$
Polyharmonic spline (PS)	$\phi(r) = r^k, k = 1,3,5,..$ $\phi(r) = r^k ln(k), k = 2,4,6,..$

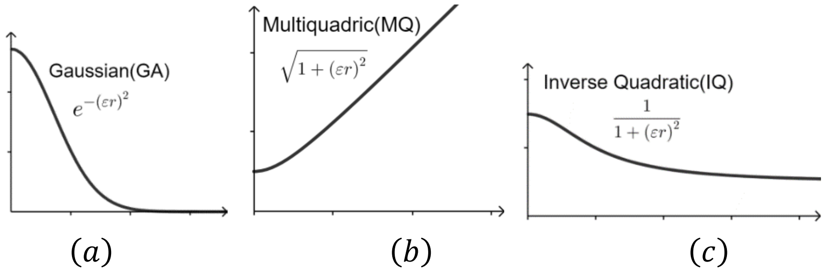


Figure 2.1: Infinitely smooth RBFs for interpolation.

Example (2.1): one Dimension

Suppose we have 4 datapoints: $x_1 = 1, x_2 = 3, x_3 = 3.5, x_4 = 4$ and also given the 4 function values: $f_1 = 1, f_2 = 0.2, f_3 = 0.1, f_4 = 2$, using the RBF $\phi(r) = e^{-(r)^2}$. To calculate our interpolant $s(x)$ we need to calculate α for the matrix as in equation (2.3) which is plotted in Figure 2.2, using our dataset we have:

$$\begin{pmatrix} \phi(\|1-1\|) & \phi(\|3-1\|) & \phi(\|3.5-1\|) & \phi(\|4-1\|) \\ \phi(\|1-3\|) & \phi(\|3-3\|) & \phi(\|3.5-3\|) & \phi(\|4-3\|) \\ \phi(\|1-3.5\|) & \phi(\|3-3.5\|) & \phi(\|3.5-3.5\|) & \phi(\|4-3.5\|) \\ \phi(\|1-4\|) & \phi(\|3-4\|) & \phi(\|3.5-4\|) & \phi(\|4-4\|) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.2 \\ 0.1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.02 & 0.002 & 0.0001 \\ 0.02 & 1 & 0.78 & 0.37 \\ 0.002 & 0.78 & 1 & 0.78 \\ 0.0001 & 0.37 & 0.78 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.2 \\ 0.1 \\ 2 \end{pmatrix}$$

To determine the inverse of matrix ϕ:

$$\phi = \begin{pmatrix} 1 & 0.02 & 0.002 & 0.0001 \\ 0.02 & 1 & 0.78 & 0.37 \\ 0.002 & 0.78 & 1 & 0.78 \\ 0.0001 & 0.37 & 0.78 & 1 \end{pmatrix}$$

First, set up the matrix (ϕ | I): (use the Gauss method for the multiplicative inverse, which has finally the form (I | ϕ^{-1})).

$$\left(\begin{array}{cccc|cccc} 1 & 0.02 & 0.002 & 0.0001 & 1 & 0 & 0 & 0 \\ 0.02 & 1 & 0.78 & 0.37 & 0 & 1 & 0 & 0 \\ 0.002 & 0.78 & 1 & 0.78 & 0 & 0 & 1 & 0 \\ 0.0001 & 0.37 & 0.78 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Next, using elementary row operations on the matrix(ϕ | I) to get (I |ϕ^{-1}).Thus,

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1.0013 & -0.0713 & 0.0846 & -0.0397 \\ 0 & 1 & 0 & 0 & -0.0713 & 4.0624 & -5.0974 & 2.4729 \\ 0 & 0 & 1 & 0 & 0.0846 & -5.0974 & 8.9497 & -5.0947 \\ 0 & 0 & 0 & 1 & -0.0397 & 2.4729 & -5.0947 & 4.0589 \end{array}\right)$$

Solving the system by using the formula $\alpha = \phi^{-1} f$ which is an exact interpolation of the dataset.

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 1.0013 & -0.0713 & 0.0846 & -0.0397 \\ -0.0713 & 4.0624 & -5.0974 & 2.4729 \\ 0.0846 & -5.0974 & 8.9497 & -5.0947 \\ -0.0397 & 2.4729 & -5.0947 & 4.0589 \end{pmatrix} \begin{pmatrix} 1 \\ 0.2 \\ 0.1 \\ 2 \end{pmatrix}$$

gives the solution

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0.916 \\ 5.177 \\ -10.229 \\ 8.063 \end{pmatrix}$$

This means that our interpolating RBF is

$$s(x) = 0.916\,\phi(\|x-1\|) + 5.177\,\phi(\|x-3\|) - 10.229\,\phi(\|x-3.5\|) + 8.063\,\phi(\|x-4\|).$$

This basic RBF interpolation with $\varepsilon = 1$ is plotted in Figure 2.2, it shows how the interpolant s interpolates f by using Gaussian basis functions. It can be seen that s interpolates f exactly at the points $\{1, 3, 3.5, 4\}$.

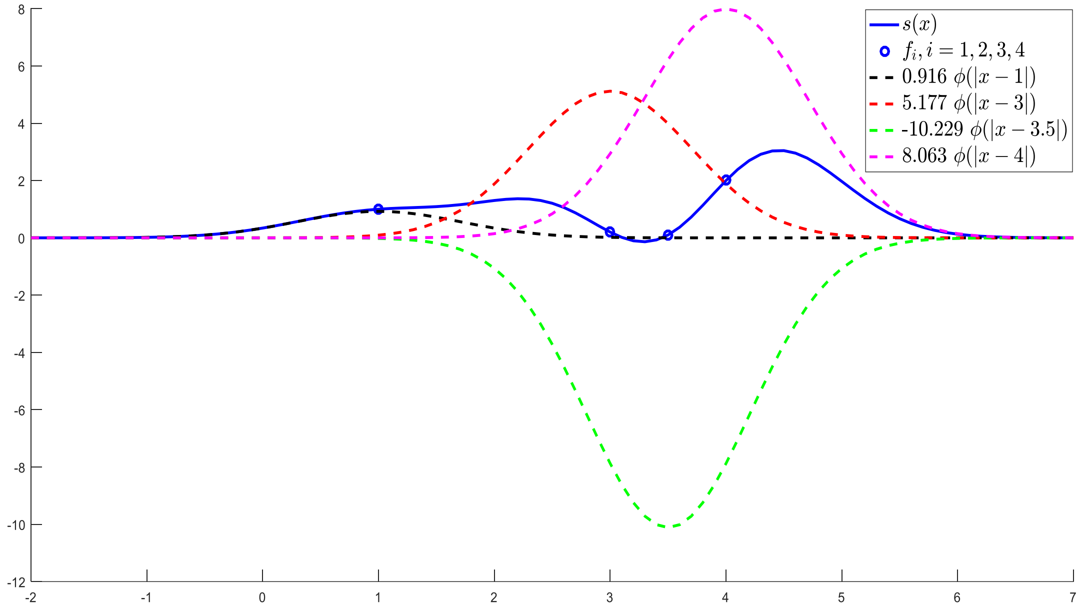


Figure 2.2: Basic RBFs interpolation with Gaussian basis functions where $\boldsymbol{\varepsilon = 1}$, interpolating $\boldsymbol{\phi(r) = e^{-(r)^2}}$ at $\boldsymbol{\{1, 3, 3.5, 4\}}$.

Figure 2.3, shows an example for 2D and 3D data, which uses an interpolation radial function and gives a result similar to the original function, as explained in section 3.4.

Original data 86% data loss RBF interpolation

Stanford bunny RBF interpolation

Figure 2.3: Illustrations of RBFs in 2D and 3D.

2.2 Stability

The condition number of a matrix plays an important role when discussing RBF interpolation, as observed by Schaback [29] there is a trade-off between the accuracy of the interpolation and the condition number of the matrix ϕ in equation (2.5). The condition number of the nonsingular matrix ϕ relative to a norm $\|\cdot\|$ which is defined as:

$$k(\phi) = \|\phi\| \cdot \|\phi^{-1}\|. \tag{2.6}$$

If ϕ is normal (a matrix ϕ is normal, if $\phi^*\phi = \phi\phi^*$ where ϕ^* is the conjugate transpose[6] of ϕ.) and the 2-norm (Euclidean norm) of a matrix is given by

[6] If ϕ is an $m \times n$ matrix with entries from the field $\mathbb{F}$, then the conjugate transpose of ϕ is obtained by taking the complex conjugate of each entry in ϕ and then transposing ϕ.

$\|\phi\|_2 = \sqrt{\rho(\phi^*\phi)}$ (the spectral norm) is chosen, (2.6) becomes

$$k\|\cdot\|_2(\phi) = \left|\frac{\lambda_{\max(\phi)}}{\lambda_{\min(\phi)}}\right|, \tag{2.7}$$

so the condition is the ratio of the biggest to the smallest eigenvalue of ϕ.

In order to improve the accuracy of the interpolant one can change the shape parameter ε to control the condition of the resulting interpolation matrix, which leads to a more ill-conditioned problem since the condition number of the matrix ϕ increases (a matrix ϕ is ill-conditioned when $k(\phi)$ is significantly greater than 1). This trade-off has been called the uncertainty principle, and a matrix ϕ is well-conditioned if $k(\phi)$ is close to 1.

Example (2.2):

Determine the condition number for the matrix

$$\phi = \begin{pmatrix} 2 & 0 & -1 \\ 2 & 3 & 5 \\ 0 & -4 & 3 \end{pmatrix}$$

Solution:

First, calculate $\rho(\phi^*\phi)$ the eigenvalues(λ) of $\phi^*\phi$.

$$\phi^*\phi = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 3 & -4 \\ -1 & 5 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 & -1 \\ 2 & 3 & 5 \\ 0 & -4 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -1 & -3 \\ -1 & 38 & 3 \\ -3 & 3 & 25 \end{pmatrix}$$

Then $\lambda_1 = 4.550,\ \lambda_2 = 24.712,\ \lambda_3 = 38.738$

Hence $\rho(\phi^*\phi) = max\{4.550, 24.712, 38.738\} = 38.738$

$$\|\phi\|_2 = \sqrt{38.738} \approx 6.224$$

Also, $\phi^{-1} = \begin{pmatrix} 0.439 & 0.061 & 0.046 \\ -0.091 & 0.091 & -0.182 \\ -0.121 & 0.121 & 0.091 \end{pmatrix}$

Then, $(\phi^{-1})^* \phi^{-1}$

$$= \begin{pmatrix} 0.439 & -0.091 & -0.121 \\ 0.061 & 0.091 & 0.121 \\ 0.046 & -0.182 & 0.091 \end{pmatrix} \begin{pmatrix} 0.439 & 0.061 & 0.046 \\ -0.091 & 0.091 & -0.182 \\ -0.121 & 0.121 & 0.091 \end{pmatrix}$$

$$= \begin{pmatrix} 0.199 & -0.043 & -0.042 \\ -0.043 & 0.050 & 0.006 \\ -0.042 & 0.006 & 0.038 \end{pmatrix}$$

Eigenvalues of $(\phi^{-1})^*(\phi^{-1})$:

$$\lambda_1 = 0.026,\ \lambda_2 = 0.041,\ \lambda_3 = 0.220$$

Hence $\rho\big((\phi^{-1})^*(\phi^{-1})\big) = max\{0.026, 0.041, 0.220\} = 0.220$

$$\|\phi^{-1}\|_2 = \sqrt{0.220} \approx 0.469$$

and for the Euclidean norm, $k(\phi) = (6.224)(0.469) \approx 2.919$, is ill-conditioned when $k(\phi)$ is significantly greater than 1. This is called uncertainty principle.

From previous example we notice that: for small values of condition number and small values of errors this provide high accuracy interpolation.

Chapter Three

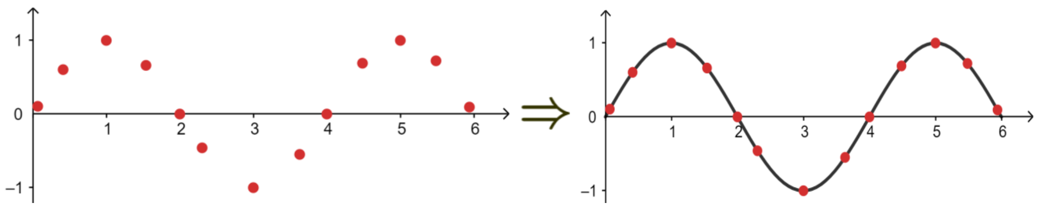
Interpolation of Radial Basis Functions Using Trapezoidal Fuzzy Numbers

Chapter Three

Interpolation of Radial Basis Functions Using Trapezoidal Fuzzy Numbers

3.1 What is Interpolation?

Interpolation is a method of using data points with known values or sample points to estimate unknown values. The interpolation problem is to construct a base function that passes exactly through these points (see Figure 3.1).



(a) Given Points. (b) The interpolated function.

Figure 3.1: Interpolation.

In this chapter, matlab program is used to implement the algorithms which are presented to solve numerically two examples through the Trapezoidal Fuzzy functions interpolation by smoothies method in functional spaces to show the efficiency of the method.

3.2 Basic Concepts

In this section, some definitions are presented about methods for constructing functional spaces generated by RBFs which will be used throughout this thesis.

Denfinition (3.2.1)[22]: **Lebesgue spaces**

The space of functions which are Lebesgue integrable on Ω to the power of $p \in [1,\infty)$ is denoted by:

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p(x)dx\right)^{1/p} < \infty\,.$$

Denfinition (3.2.2)[22]: **Sobolev spaces**

Let $\Omega \subset \mathbb{R}^d (d = 1,2,3)$ be an open set and $1 \le p < \infty$, the Sobolev space $H^{k,p}(\Omega)$ consists of function $u \in L^p(\Omega)$ such that for every multi – index α with $|\alpha| \le k$, the partial derivative $D^\alpha u$ exists and $D^\alpha u \in L^p(\Omega)$ thus:

$$H^{k,p}(\Omega) := \{u \in L^p(\Omega): D^\alpha u \in L^p(\Omega), \qquad \forall \alpha \text{ with } |\alpha| \le k\}$$

Denfinition (3.2.3):

A function f from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is **Lipschitz continuous** at $v \in S$ if there is a constant c such that:

$$\|f(u) - f(v)\| \le c\|u - v\|$$

for all $u \in S$ sufficiently near v.

Denfinition (3.2.4): Positive Definite Matrices

An $M \times M$ matrix ϕ is positive definite, if it is symmetric $(\phi^T = \phi)$ and $x^T \phi\, x > 0$ for any $M -$ dimensional vector $x \neq 0$.

Theorem:

If ϕ is an $M \times M$ positive definite matrix, then:

- ϕ has an inverse (nonsingular).
- $a_{ii} > 0$ for each $i = 1,2,\dots,M$.
- $\max_{1\le k,j\le M} |a_{kj}| \le \max_{1\le i\le M} |a_{ii}|$.
- $(a_{ij})^2 < a_{ii}a_{jj}$ for each $i \neq j$.

3.3 Description of the Method

Given a set of M distinct points $\{x_i\}_{i=1}^{M}$, the RBF interpolant is given as described in the following equation:

$$\sum_{i=1}^{M} \boldsymbol{\alpha_i \phi}\left(\left\|\boldsymbol{x_j} - \boldsymbol{x_i}\right\|\right) = \boldsymbol{f_i}, \quad \boldsymbol{i}, \boldsymbol{j} = \mathbf{1}, \dots, \boldsymbol{M} \tag{3.1}$$

when we interpolate the data values (f_i) at the scattered node locations x_i,

$i = 1,2,\dots M$ in d dimensions $(d = 1,2)$, where $\|.\|$ denote the Euclidean norm and x_i is the center of RBF.

We obtain the expansion coefficients α_i by solving a linear system defined as:

$$\boldsymbol{\phi\alpha} = \boldsymbol{f}, \tag{3.2}$$

based on the interpolation conditions, the estimated function will be

$$\boldsymbol{s}(\boldsymbol{x_i}) = \boldsymbol{f_i}. \tag{3.3}$$

The system takes the form

$$\begin{pmatrix} \phi(\|x_1 - x_1\|) & \phi(\|x_2 - x_1\|) & \cdots & \phi(\|x_M - x_1\|) \\ \phi(\|x_1 - x_2\|) & \phi(\|x_2 - x_2\|) & \cdots & \phi(\|x_M - x_2\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\|x_1 - x_M\|) & \phi(\|x_2 - x_M\|) & \vdots & \phi(\|x_M - x_M\|) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_M \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_M \end{pmatrix}, \quad (3.4)$$

where ϕ is called a distance matrix or interpolation matrix. In conclusion, to ensure a unique solution to the interpolation problem, the RBF requires to be conditionally positive definite of order M on $\mathbb{R}^d (d = 1,2)$.

Let Ω be an open bounded connected nonempty subset of $\mathbb{R}^d$, with Lipschitz - continuous boundary. Also using the classical notation $H^k(\Omega)$ to denote the usual Sobolev space of all distributions $u_i, i = 1, \dots, M,$ whose partial derivatives, up to order $k,$ are the Classial Lebesgue space $L^2(\Omega)$ (see Definitions 3.2.1 - 3.2.3).

Let us give an arbitrary finite set $\{b_1, \dots, b_M\} \subset \mathbb{R}$ of M distinct interpolation points and a set of fuzzy numbers $\mathbb{U} = \{u_1, u_2, \dots, u_M\}$ be defined such as $u_i = \{u_{i1}, u_{i2}, u_{i3}, u_{i4}\}, i = 1, \dots, M$ is a trapezoidal fuzzy number ($\mathbb{TFN}$), i.e. $\mathbb{U} \subset \mathbb{TFN}$. Also require a set of knots (center points) $\{\alpha_1, \dots, \alpha_M\}$ and for each of them a radial function $\phi(\cdot - a_\ell)$ with
$\ell = 1, \dots, M$.

After this introduction, we come to the main goal of this section which is to solve the following interpolation problem: Given the interpolating data set $\{(b_i, u_i), i = 1, \dots, M\} \subset \mathbb{R} * \mathbb{TFN}$. To obtain a fuzzy function
$s \in H^T$, let H^T be the set of functions satisfying the following conditions, $s: \Omega \longrightarrow \mathbb{TFN}$. such that $s(x_i) = u_i,\ i = 1, \dots, M$. expressed as:

$$s(x) = \sum_{\ell=1}^{M} \alpha_\ell \, w_\ell(x)$$

where $w_\ell(x) = \{\phi_\varepsilon(\cdot - a_\ell), \ell = 1, \dots, M\}$ and $[\alpha_1, \dots, \alpha_M]^T \in \mathbb{TFN}$

is the solution of the linear system $\mathcal{A}\alpha = b$ where

$$\mathcal{A} = \left(w_i(b_j)\right), i = 1, \dots, M, \; j = 1, \dots, M.$$

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_M \end{pmatrix} \in (\mathbb{R}^4)^M,$$

that is the usual Gaussian function

$\phi_\varepsilon(r) = e^{-(\varepsilon r)^2}, \varepsilon \in \mathbb{R}^+, r \geq 0.$ $b = \{u_{i1}, u_{i2}, u_{i3}, u_{i4}\}, i = 1, \dots, M.$

The interpolation matrix is nonsingular since it is a positive definite matrix, so we have the unique existence of the coefficients

$$\alpha = (\alpha_1 \dots \alpha_M)^T \in (\mathbb{R}^4)^M.$$

In order to verify and analyze the fuzzy data interpolation using RBFs, consider two approximation error estimaters that are appropriate normalized of the discrete version of the usual norms in $\mathbb{C}(\Omega)$ and $L^2(\Omega)$ respectively, defined by the expressions:

$$E_c = \frac{\max\limits_{i=1,\dots,M} |f(a_i) - s(a_i)|}{\max\limits_{i=1,\dots,M} |f(a_i)|}$$

$$E_\ell = \sqrt{\frac{\sum_{i=1}^{M} \left(f(a_i) - s(a_i)\right)^2}{\sum_{i=1}^{M} \left(f(a_i)\right)^2}}$$

where f is a given function, and s is the interpolating RBF.

Two examples of RBF approximation are presented in the following.

3.4 Numerical Examples and Results

In this section, we want to test our method by choosing two examples of fuzzy functions in order to analyze the presented fuzzy radial basis interpolation method.

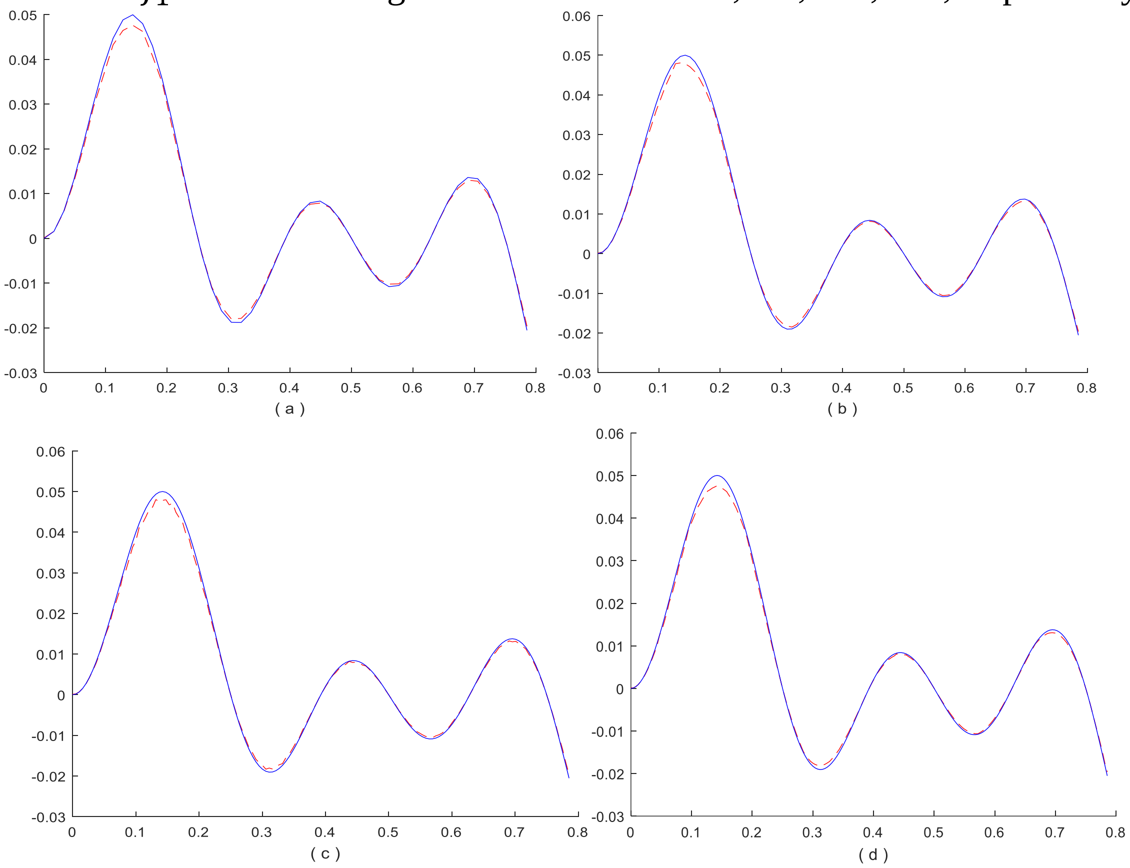
Example (3.1):

In this example we consider $f_1 : \left[0, \frac{\pi}{4}\right] \rightarrow \mathbb{TFN}$

and used $f_1(x) = \left(a_1(x), a_2(x), a_3(x), a_4(x)\right)$

$$= \left(e^{-0.5x}, 0.2\, sin(4\pi x), 0.5\, cos(4x), sin(5x)\right)$$

The next figure (Figure 3.2) shows the radial basis interpolation of the function f_1 with increasing number of centers: 50,100, 150, 200, respectively.



——Original ------Interpolation

Figure 3.2: Radial interpolation of the function f_1 for a) 50, b)100, c)150, d)200 centers (Example 3.1).

On the following Table (Table 3.1), which describes the two approximation errors: E_c and E_ℓ between the original and the interpolated function f_1 which are decreasing when the number of centers increases.

Table 3.1: Error estimates for RBF interpolation of Example 3.1.

$t_i = x_i$	Error	
	E_c	E_ℓ
50	$1.818e^{-05}$	$1.286e^{-04}$
100	$9.091e^{-06}$	$9.091e^{-05}$
150	$6.061e^{-06}$	$7.423e^{-05}$
200	$4.546e^{-06}$	$6.429e^{-05}$

On the following figure (Figure 3.3), which shows the results of the two approximation errors: E_c and E_ℓ for f_1, and it's RBFs interpolation, which are decreasing when the number of centers increases.

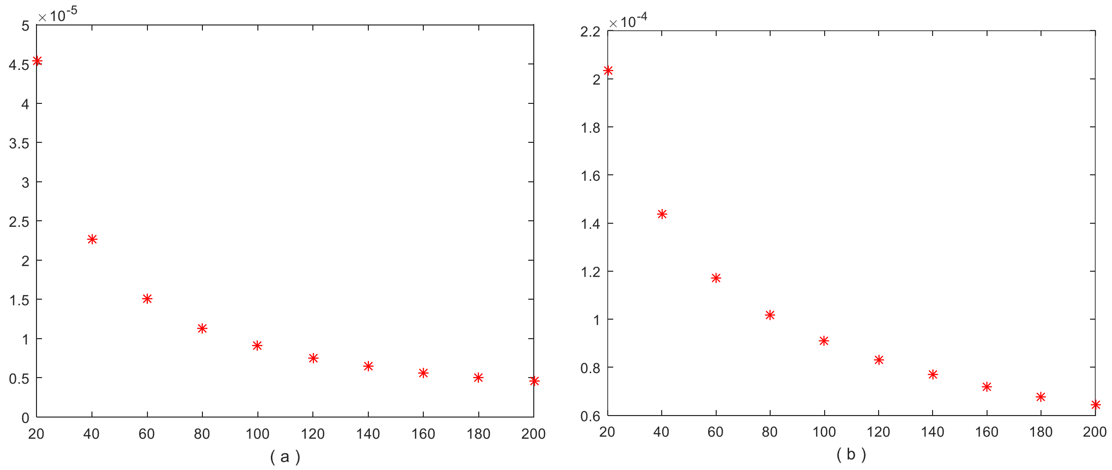


Figure 3.3: Pareto front of E_c, E_ℓ errors vs. number of centers for $f_1(x)$.

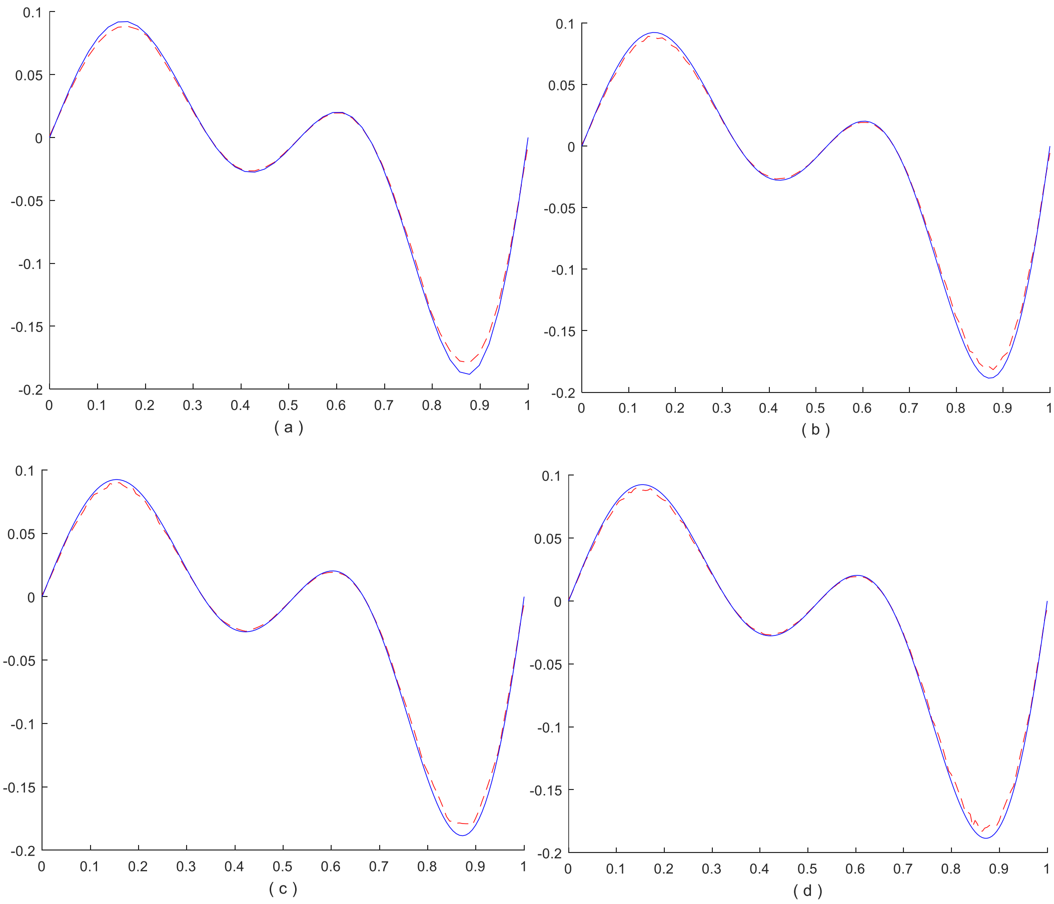
Example (3.2):

In this example we consider $f_2: [0,1] \rightarrow \mathbb{TFN}$

and used $f_2(x) = \left(a_1(x), a_2(x), a_3(x), a_4(x)\right)$

$$= \left(e^{0.1x}, 0.1sin(3\pi x), cos(3x), e^{x^2}\right)$$

This fuzzy function f_2 is interpreted in Figure 3.4 to show the radial basis interpolation when increasing the number of centers: 50, 100, 150, 200, respectively.



—Original ------Interpolation

Figure 3.4: Comparison between original function and interpolation function in Example 3.2 for a)50, b)100, c)150, d)200 centers.

On the following Table (Table 3.2), which describes the two approximation errors: E_c and E_ℓ between the original and the interpolated function f_2 which are decreasing when the number of centers increases.

Table 3.2: Error estimates for RBF interpolation of Example 3.2.

$t_i = x_i$	Error	
	E_c	E_ℓ
50	$1.125e^{-04}$	$7.955e^{-04}$
100	$5.625e^{-05}$	$5.625e^{-04}$
150	$3.750e^{-05}$	$4.593e^{-04}$
200	$2.813e^{-05}$	$3.978e^{-04}$

On the following figure (Figure 3.5), which shows the results of the two approximation errors: $E_c$and E_ℓ for f_2, and it's RBFs approximation, which are decreasing when the number of centers increases.

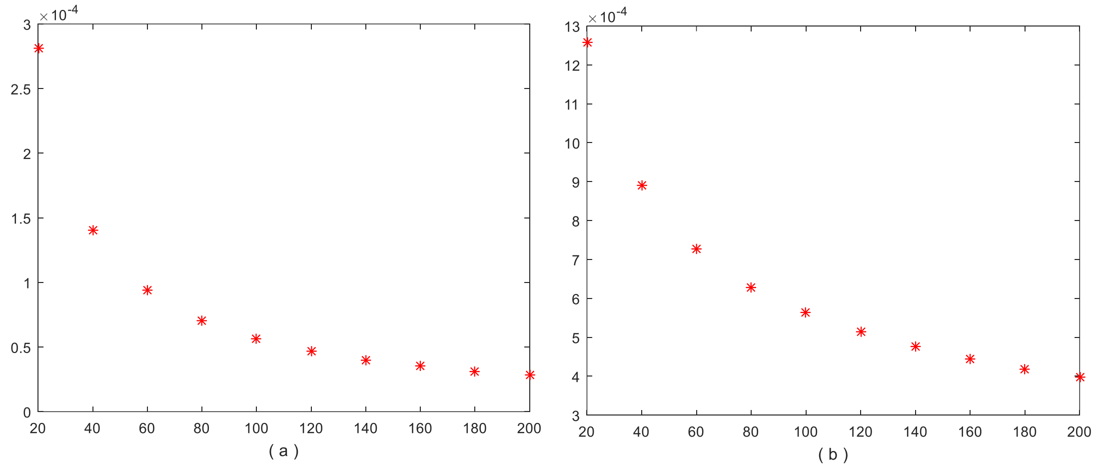


Figure 3.5: Pareto front of E_c, E_ℓ errors vs. number of centers for $f_2(x)$.

Chapter Four

Conclusions and Suggestions

Chapter Four

Conclusions and Suggestions

4.1 Conclusions

Interpolation using RBF appears very attractive in many applications as in image reconstruction, GIS systems, computer graphics, optics, fuzzy systems … etc. The RBF interpolation was originally introduced by [13] and is defined by the function:

$$f_i = \sum_{i=1}^{M} \alpha_i \phi\left(\|x_j - x_i\|\right)$$

where the coefficients α_i are computed such that:

$$s(x_i) = f_i \text{ for } i = 1, \dots, M.$$

M is the number of interpolation points, and ϕ is the RBF.

The interpolation errors: E_c (RMFE) and E_ℓ (RMSE) are computed for the following two functions:

$f_1(x) = \left(e^{-0.5x}, 0.2\, sin(4\pi x), 0.5\, cos(4x), sin(5x)\right)$ and

$f_2(x) = \left(e^{0.1x}, 0.1sin(3\pi x), cos(3x), e^{x^2}\right),$

where f is defined on $\Omega \subset \mathbb{R}$ to the $\mathbb{TFN}s$ set and visualized results in Figures 4.1 and 4.2. which shows the interpolation error is reduced rapidly as the number of centers increases actually implementing the following number of centers: 50, 100, 150, 200, respectively. One can conclude from small error values, that the interpolated function is close to the original data. These results where implemented by MATLAB.

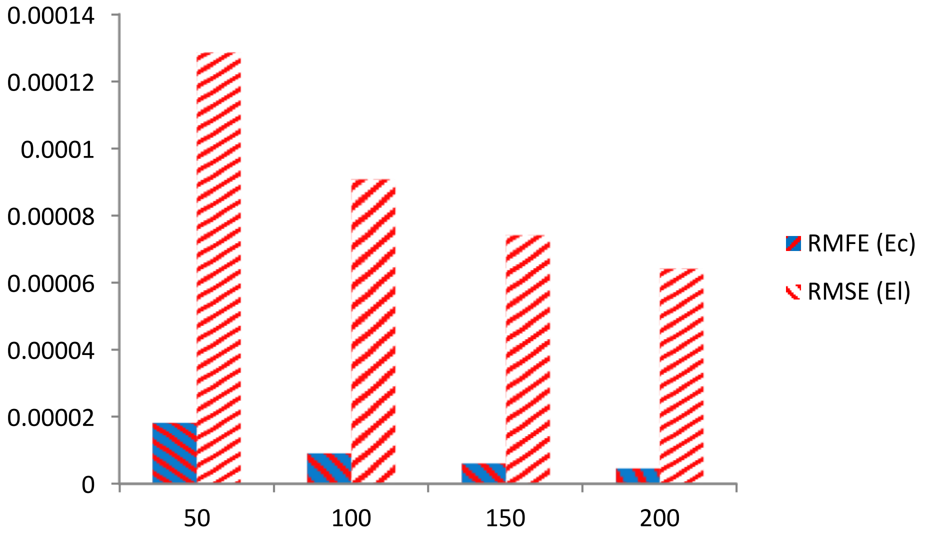


Figure 4.1: Plot of the errors with respect to the number of centers for $f_1(x) = \left(e^{-0.5x}, 0.2\ sin(4\pi x), 0.5\ cos(4x), sin(5x)\right)$.

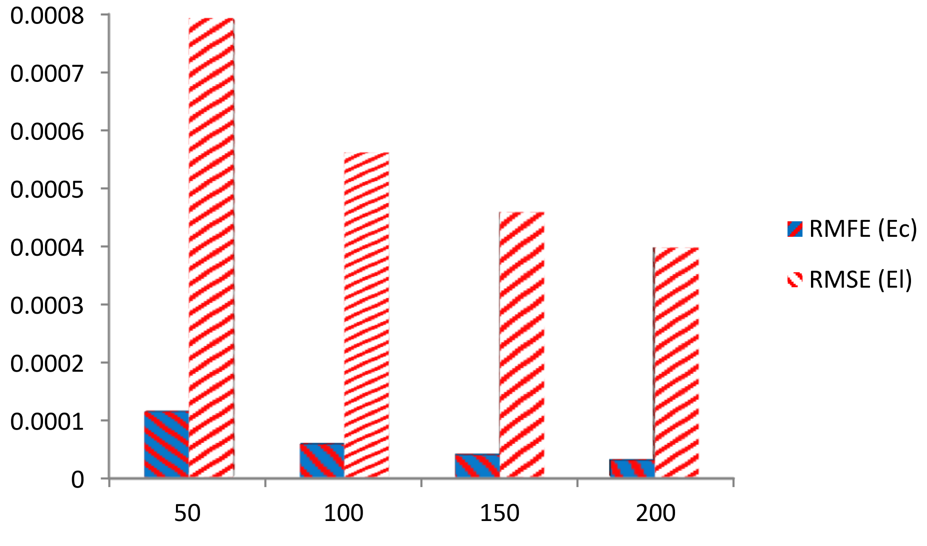


Figure 4.2: Plot of the errors with respect to the number of centers for $f_2(x) = (e^{0.1x}, 0.1sin(3\pi x), cos(3x), e^{x^2})$.

A conclusion can be drawn after the analysis of the results mentioned in section 3.4, by revising the errors: E_c and E_ℓ that the interpolation of radical function of Gaussian type presentes a simillar behavior to the original function.

4.2 Suggestions for Further Work

The work of this thesis can be extended in the following areas:

1- The one dimension problem can be modified and replaced by two or three dimensional problems.

2- This technique can be used for other types of fuzzy numbers rather than $\mathbb{TFN}s$.

3- Investigating the influence of replacing the gaussian with other types of radial function (see table 2.1).

References:

[1] Azad, K. K., 1981. On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity. J. Math. Anal. Appl.: vol. 82,14–32.

[2] Bansal, A., 2011. Trapezoidal fuzzy numbers (a, b, c, d); arithmetic behavior. International Journal of Physical and Mathematical Sciences. (ISSN:2010-1791): vol. 2, iss. 1.

[3] Bansal, A., 2010. Some nonlinear arithmetic operations on triangular fuzzy number (m, α, β). Advances in Fuzzy Mathematics: vol. 5, iss. 2, 147-156.

[4] Bede, B., Mathematics of fuzzy sets and fuzzy logic. Springer Publications, 295, New York.1992, 51-78.

[5] Dinagar, D. S. and Latha, K., 2013. Some types of Type-2 triangular fuzzy matrices. International Journal of Pure and Applied Mathematics: vol. 82, no. 1, 21-32.

[6] Dubois, D. and Prade, H., 1978. Operations on fuzzy numbers. International Journal of Systems Science: vol. 9, no. 6, 613-626.

[7] Duchon, J., Splines minimizing rotation-invariant semi-norms in sobolev spaces. constructive theory of functions of several variables, springer, Berlin.1977. 85-100.

[8] Dutta, P., Boruah, H. and Ali, T., 2011 March. Fuzzy arithmetic with and without using α-cut method: A comparative study. International

Journal of latest trends in Computing. (E-ISSN:2045-5364): vol. 2, iss. 1, 98-107.

[9] Firouzdor, R. and Amirfakhrian, M., 2017. Approximation of a fuzzy function by using radial basis functions interpolation. International Journal of Mathematical Modelling & Computations: vol. 7, no. 3, 299-307.

[10] Franke, R., 1982 January. Scattered data interpolation: tests of some methods. Math. Comp.: vol. 38, no. 157, 181-200.

[11] Hanss, M., Applied fuzzy arithmetic. Springer Berlin Heidelberg, New York; 2005. doi:10.1007/b138914.

[12] Hardy, R. L., 1990. Theory and applications of the multiquadric - biharmonic method 20 years of discovery 1968-1988. Comput. Math. Appl.: vol.19, no. 8/9, 163-208.

[13] Hardy, R. L., 1971 March. Multiquadric equations of topography and other irregular surfaces. J. Geophys. Res.: vol.76, no. 8, 1905–1915.

[14] Kaleva, O., 1994. Interpolation of fuzzy data. Fuzzy Sets Syst.: vol. 61,63–70.

[15] Kansa, E. J., 1990. Multiquadrics - A scattered data approximation scheme with applications to computational fluid-dynamics- I surface approximations and partial derivative estimates. Comput. Math. Appl.: vol. 19, no. 8/9,127-145.

[16] Kansa, E. J., 1990. Multiquadrics - A scattered data approximation

scheme with applications to computational fluid-dynamics-II. Solutions to parabolic, hyperbolic and elliptic partial differential equations. Comput. Math. Appl.: vol. 19, no. 8/9, 147- 161.

[17] Klir, G. J. and Yuan, B., Fuzzy sets and fuzzy Theory and applications. Prentice Hall PTR, Upper Saddle River, New Jersey 07458: Library of Congress Cataloging-in-Publication Data;1995. ISBN:0-13-101171-5.

[18] Klir, G. J., Yuan, B. and Clair, U. S., Fuzzy Set Theory: Foundations and Applications: Upper Saddle River, NJ: Prentice Hall; 1997.

[19] Lee, K. H., First course on Fuzzy Theory and its Applications. Springer Publications, Poland; 2005. ISBN: 3-540-22988-4.

[20] Lowen, R., 1990. A fuzzy lagrange interpolation theorem. Fuzzy Sets and Systems: vol. 34, 33–38.

[21] Majdisovaa, Z. and Skalaa, V., 2017 November. Radial Basis Function Approximations: Comparison and Applications. Applied Mathematical Modelling. vol. 51, 728–743.

[22] Maz'ja, V. G., Sobolev Spaces, Springer Series in Soviet Mathematics, New York;1985. doi:10.1007/978-3-662-09922-3.

[23] Ming, P. P. and Ming, L. Y., 1980. Fuzzy topology. 1. Neighborhood structure of a fuzzy point and moore-smith convergence. J. Math. Anal. Appl.: vol. 76, 571-599.

[24] Nahmias, S., 1978. Fuzzy variables. Fuzzy sets : vol. 1, no. 2, 97-110. doi:10.1016/0165-0114(78)90011-8.

[25] Parvathi, R. and Malathi, C., 2012 May. Arithmetic operations on Symmetric Trapezoidal Intuitionistic Fuzzy Numbers. International Journal of Soft Computing and Engineering: ISSN:2231-2307: vol. 2, iss. 2, 268-273.

[26] Pathinathan, T. and Ponnivalavan, K., 2014. Pentagonal Fuzzy numbers. International Journal of Computing Algorithm, ISSN:2278-2397: vol. 3.

[27] Piret, C., Analytical and numerical advances in radial basis functions. (Ph.D. Thesis). University of Colorado. 2007.

[28] Prodanovic, P., Fuzzy set ranking methods and multiple expert decision making. (Report No. 039). The University of Western Ontario. August 2001.

[29] Schaback, R., 1995. Error estimates and condition numbers for radial basis function interpolation. Adv. Comput. Math.: vol. 3, no. 3, 251–264.

[30] Skala, V., 2016 January. A practical use of radial Basis functions interpolation and approximation. Investigacion Operacional: vol. 37, no. 2, 136-144.

[31] Stefanini, L., Sorini, L. and Guerra, M. L., 2006. Parametric representation of fuzzy numbers and application to fuzzy calculus. Fuzzy Sets and Systems: vol. 157:2423-2455.

[32] Toit, W. d., Radial basis function interpolation. University of

Stellenbosch.(Master's Thesis). Matieland. South Africa. March 2008.

[33] Wang, G. and Li, J., 2017 March. Approximations of fuzzy numbers by step type fuzzy numbers. Fuzzy Sets and Systems: vol. 310, 47–59.

[34] Wong, C. K., Fuzzy topology, fuzzy sets and their application to Cognitive and Decision processes. New York.1975.171-190. https://doi.org/10.1016/B978-0-12-775260-0.50012-8.

[35] Wong, C. K., 1974. Fuzzy points and local properties of fuzzy topology. J. Math. Anal. Appl.: vol. 46, 316–328.

[36] Wright, G. B., Radial basis funtion interpolation: numerical and analytical developments. University of Colorado. (Ph.D. Thesis). 2003.

[37] Wu, C. and Wang, G., 2002. Convergence of sequences of fuzzy numbers and fixed point theorems for inceasing fuzzy mappings and application. Fuzzy Sets and Systems: vol. 130, 383-390. doi:10.1016/s0165-0114(01)00231-7.

[38] Yalvac, T. H.,1987. Fuzzy sets and functions on fuzzy spaces. Journal of Mathematical Analysis and Applications: vol. 126, 409-423.

[39] Zadeh, L. A., 1965. Fuzzy sets. Information and contoral.: vol. 8, 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[40] Zhang, Z., Gao, S. and Cao, C., 2009 June. Multiplication operation on fuzzy numbers. Journal of Software: vol. 4, no. 4, 331-338.

[41] Zimmermann, H. J., 2010 May/June. Fuzzy set theory. John Wiley & Sons, Inc. WIREs Comp Stat. Germany: vol. 2 , 317–332.

جامعة النجاح الوطنية

كلية الدراسات العليا

استيفاء اقترانات الراديال الأساسية باستخدام الأنظمة الضبابية لشبه المنحرف

إعداد

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إشراف

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قدمت هذه الأطروحة استكمالاً لمتطلبات الحصول على درجة الماجستير في الرياضيات من كلية الدراسات العليا في جامعة النجاح الوطنية في نابلس، فلسطين.

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الملخص

الاستيفاء هو أحد المشاكل المهمّة والمنتشرة في المجالات التقنية العلمية المختلفة مثل معالجة الصور، الرسومات الحسابية، النمذجة والتصميم الهندسي، وغيرها حيث يُستخدم في إنشاء نقاط بيانات جديدة محددة أو مقدرة بين نقاط البيانات المعروفة ويمكن أن يمثل الاقتران بيانياً.

في هذه الأطروحة، تم عرض منهجية الراديال من النوع (Gaussian) باستخدام عدد ضبابي من النوع شبه منحرف (TFNs) واسُتخدم الخطأين (E_c: root mean forecast error and E_ℓ: root mean square error) في قياس جودة ودقة الاستيفاء في المنهجية المقترحة بالمقارنة مع الاقتران الأصلي.

وأخيراً، لفحص فعالية الطريقة ونجاعتها، قدمنا مثالين عددين، وتبين أنه عند زيادة عدد المراكز (RBF centers)، فإن ذلك ينعكس على أداء أفضل بتقليل الخطأ و زيادة الدقة للقيم المقدرة مقارنة مع الاقتران الأصلي و الذي تم قياسها باستخدام (E_c and E_ℓ) مستخدمين برنامج MATLAB.