An-Najah National University

Faculty of Graduate Studies

OPTIMAL CONTROL FOR LINEAR DYNAMICAL SYSTEM WITH ZERO INITIAL CONDITIONS

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Dedication

I dedicate this thesis to the memory of my mother. She was my inspiration, although she was unable to see my graduation. This is for her.

To my father, who helped me in all things great and small, who has been my source of inspiration and continually provides his moral, spiritual, and emotional support.

To my beloved brothers, who have never left my side and given me the extra strength and motivation to get things done.

A special thanks to my lovely husband, who supported me emotionally and encouraged me to pursue my dreams and finish my thesis. I am truly blessed to have you in my life.

And most of all to the Almighty God who gives me strength and good health while doing this.

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Most importantly, I would also like to thank all my family members, my husband and my friends for encouraging and supporting me whenever I needed them.

Declaration

I, the undersigned, declare that I submitted the thesis entitled:

OPTIMAL CONTROL FOR LINEAR DYNAMICAL SYSTEM WITH ZERO INITIAL CONDITIONS

I declare that the work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's Name: ______________________________

Signature: ______________________________

Date: ______________________________

List of Contents

List of Tables

List of Figures

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ABSTRACT

The main focus of this study is on the Linear Dynamic Systems With Zero Initial Conditions. Also the following topics are presented: Optimal control, state space representation, Lyapunov equations, controllability, observability and their Gramians. Some model order reduction methods are also proposed, specifically balanced truncation and singular perturbation approximation.

For the optimal control problems, the study used Feedback control strategies, numerical solution methods and their implementation with various illustrative applications.

A linear quadratic regulator (LQR) is introduced to develop an optimal control that minimizes the quadratic cost function by employing the formal asymptotic solution for the underlying algebraic Riccati equation. This final optimal control was delivered by implementing three types of singular perturbation approximation to test the best performance in closed-loop conditions.

A few numerical examples were given to illustrate one type of singular perturbation approximation and show how the reduced-order may be used to approach the optimal control of the original system.

Numerical results for the clamped beam model and the RLC circuit have shown that singular perturbation approximation method is one of the most efficient methods for model order reduction.

Keywords: Dynamic systems, Model Order Reduction, Singular Perturbation Approximation, Optimal Control.

Chapter One

Preliminaries

1.1 Introduction

Optimal control theory is an outcome of the calculus of variations, with a history stretching back over 360 years, but interest in it mushroomed only with the advent of the computer, launched by the stunning successes of optimal trajectory prediction in aerospace applications in the early 1960s [1].

The modeling of many physical, chemical, or biological phenomena resulting from discretized partial differential equations leads to the famous representation of a linear time-invariant (LTI) dynamic system [2],[3]:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t) \tag{1-1}$$

$$x(t_0) = x_0$$

Where,

$x(t) \in R^n$ is called the system state

$u(t) \in R^r$ is called the system input

$y(t) \in R^m$ is called the system output

$x(t_0)$ represents the initial condition of the system

$A \in R^{n\times n}, B \in R^{n\times r}, C \in R^{m\times n}, D \in R^{m\times r}$ are constant matrices

Where,

A is the state matrix

B is the input matrix

C is the output matrix

D is the feed through matrix

n is the order of the system

Model Order Reduction (MOR) is used in the proposed approach to reduce the number of system states and the computing effort with a minor loss of accuracy in the numerical solutions.

Model order reduction is an essential feature for analyzing and designing modern control systems. The reduced order system should have the properties of the original system. Moreover, the approximation error between the reduced order and the original system is required to be small for the entire frequency range.

Model order reduction for linear systems offers an adequate foundation for many approximation methods, such as easily calculable error bounds. Model reduction's overall goal is to limit the system to the subspace of easily observable and controlled states, which may be identified by the system's Hankel singular values [4], [5].

In this thesis, the focus is laid on the description of finite-dimensional continuous linear and time-invariant systems (FDLTI), described by linear differential equations with zero initial conditions, and the necessary conditions of optimality derived respectively [2].

Two methods of reducing the order in linear time-invariant systems are as follows: balance truncation method (BT) and singular approximation technique (SPA). These methods lead to a stable reduced system and guarantee the upper limit of error reduction.

Balanced truncation (for standard state space system) is a popular scheme for this purpose since it preserves stability and has frequency response error bounds. The fundamental principle of balanced truncation (BT) is determining the states that are simultaneously hard to reach and hard to observe. In other words, these states require a lot of energy to steer from zero as well as generate very little output energy. Furthermore, it preserves stability and provides guaranteed error bounds [6], [7].

In [8] a generalized reciprocal transformation method to reduce the infinite-dimensional system is presented. This fact motivates the generalization (SPA) method of finite dimensional systems to obtain reduced-order models that perform well at low frequencies.

In [9], It has been found that, At high frequencies, reduced systems by balanced truncation have smaller errors and a larger error than at low frequencies. In contrast, reducing the systems using singular perturbation approximation method is showing a reversal of behavior, i.e., the error goes to zero at low frequencies and tends to be large at high frequencies. It also has been shown that the continuous-time results on the singular perturbation approximation of balanced systems are easily extended to the discrete-time case.

For finite time-horizon optimal control problems, among the most actively investigated singularly perturbed optimal control problems is the linear quadratic regulator problems. Most of these approaches are based on the singularly perturbed differential Riccati equations [10].

The linear quadratic regulator (LQR), one of the most significant techniques for solving control problems will be discussed in this thesis for a closed-loop to develop an optimal control that minimizes the quadratic cost function by employing the formal asymptotic solution for the underlying algebraic Riccati equation. A formal calculation shall be supported by numerical experiments, and the theoretical results presented with a test case in order to demonstrate their validity, after showing how it is possible to approximate original system's optimal control using reduced order.

This thesis is organized as follows:

In Chapter 1, state space representation is given for dynamic systems, stability, controllability, observability matrices and Gramians, Lyapunov equations as well as vector and matrix standards.

The topic of model order reduction is covered in Chapter 2, the balancing of linear systems using balanced truncation, the singular perturbation approximation, and a discussion of the transfer function.

Chapter 3 proposes the error bounds using ‘balanced truncation method’ (BT), ‘singular perturbation approximation’ (SPA), and ‘reciprocal system’ (RS) methods. All these methods give stable reduced systems and guarantee the upper bound of the error reduction, i.e., that is $x_0 = 0$.

In Chapter4, the Linear Quadratic Regulator (LQR), one of the most crucial techniques for solving control problems, is presented. The quadratic cost function is minimized using feedback optimal control, and balanced truncation using the so-called Hamiltonian Function and the underlying algebraic Riccati equation is also achieved.

Chapter 5 illustrates the performance of these techniques using a numerical experiment.

1.2 Dynamic System

Linear differential equations that can simulate entrainment and synchronization behavior are among the most useful tools for dynamic systems modelers and the term "dynamic" refers to something active or changing over time.

SISO (single input single output) systems are dynamic systems that have a single input $(r = 1)$ and a single output $(m = 1)$; otherwise, they are known as MIMO (multiple input multiple outputs) systems [5].

1.3 State Space Representation of Dynamic Systems

The state-space representation of a linear dynamic system is given in the following form [5]:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1-2}$$

$$y(t) = Cx(t) + Du(t) \tag{1-3}$$

$$x(t_0) = x_0$$

Where,

$x(t) \in R^n$ is called the system state

$u(t) \in R^r$ is called the system input

$y(t) \in R^m$ is called the system output

$x(t_0)$ represents the initial condition of the system

$A \in R^{n\times n}, B \in R^{n\times r}, C \in R^{m\times n}, D \in R^{m\times r}$ are constant matrices

Where,

A is the state matrix

B is the input matrix

C is the output matrix

D is the feed through matrix

n is defined to be the order of the system

Equation (1-2) is a set of n-coupled first-order linear differential equations and called **the state equation**, where

$$\dot{x} = \frac{dx}{dt}$$

denotes the time derivative of each state variable and is expressed in terms of the state variables $x_1(t), \ldots, x_n(t)$ and the system inputs $u_1(t), \ldots, u_r(t)$.

Equation (1-3) is **the output equation** for the linear continuous dynamic system

The physical quantities that can be measured are called the outputs of the system and are denoted by $y_1(t), \ldots, y_m(t)$. These represent the response of the system. In most dynamic systems, $D \in R^{m\times r}$ is the null matrix, so the form of state space representation transforms into:

$$\dot{x} = Ax + Bu$$

$$y = Cx \tag{1-4}$$

$$x(t_0) = x_0$$

Definition (1.1) [4]: $\Sigma = \left(\begin{array}{c|c} A & B \\ C & \end{array}\right)$ is a linear system in state space representation additionally, it has the same dimension as the system described in the state space description, i.e.

$$Dim(\Sigma) = n$$

Rewriting the system (1-4) in compact matrix form as:

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

Where

$$\begin{pmatrix} A & B \\ C & \end{pmatrix}$$

Is a block matrix.

1.4 Stability, Controllability, and Observability

This subsection examines the foundational notions of stability, controllability, and observability which are commonly dealt with in continuous FDLTI systems and control theory.

1.4.1 Stability

Definition (1.2) [3]: A matrix A is a stable matrix if and only if the eigenvalues of A have strictly negative real parts ($Re(\lambda_i) < 0$), where λ_i is the eigenvalue of A and $i = 1, \ldots, n$.

Theorem 1 [3]: The system is asymptotically stable if and only if all eigenvalues of A have negative real parts ($Re(\lambda_i) < 0$), Where λ_i is the eigenvalue of A and ($i = 1, \ldots, n$).

Controllability and observability are two dual concepts and are closely related to the cancellation of a pole and a zero in the system transfer function.

1.4.2 Controllability

One of the key ideas in modern mathematical control theory is the reduced dynamic system, which is closely related to the concept of controllability.

Definition (1.3) [13]**:** A state x_0 is controllable at time t_0 if, for some finite time $t_1 > 0$, there exists an input $u(t)$ that transfers the state $x(t)$ from x_0 to a desired last state x_1 at time t_1 such that the solution of (1-3) satisfies $x(t_1) = x_1$.

Definition (1.4) [14]**:** A system is called controllable at time t_0 if every state x_0 in the state space is controllable.

Theorem 2 [14]: A dynamic system (1-4) is controllable if and only if

$Rank\ C(A,B) = n$ (i.e., $C(A,B)$ has full row rank)

where,

$$C(A,B) = [B\ \ AB\ \ A^2B\ \ \ldots\ \ A^{n-1}B]$$

is a controllable matrix.

Theorem 3 [5], [4]: The following are equivalent:

1. The pair (A,B), $A \in R^{n\times n}, B \in R^{n\times r}$ is controllable.
2. The rank of the controllable matrix is full, i.e., $Rank\ C(A,B) = n$.
3. The controllability Gramian is positive definite i.e., $W_c(t) > 0$, for some $t > 0$

1.4.3 Observability (Output Controllability)

Observability generally means that we can steer the output of a dynamic system independently of its state vector.

Definition (1.5) [14]**:**

A system with an initial state x_0 is observable if and only if the system output $y(t)$ can be used to estimate the initial state's value on $[0, t_1]$ that has been observed through finite time.

Theorem 4 [14]**:**

Dynamic system (1-4) is observable if and only if

$$rank[CB\ \ CAB\ \ CA^2B\ \ \dots\ \ CA^{n-1}B] = n$$

Definition (1.6) [14]**:**

The observability matrix of the system (1-3) is defined as:

$$O(C,A) = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

Where n is a positive integer.

Theorem 5 [5], [4]**:** The following are equivalent:

1. The pair (C, A), $A \in R^{n\times n},\ C \in R^{m\times n}$ is observable.
2. The rank of the observable matrix is full, i.e., $Rank\ O(C, A) = n$.
3. The observability Gramian is positive definite, i.e., $W_o(t) > 0$, for some $t > 0$

1.5 Lyapunov Equations

This section aims to give a brief of Lyapunov equations. The solutions of them and their connections with system stability, controllability, observability, and so on are discussed.

Definition (1.7) [3]**:** the continuous Lyapunov equation has the form:

$$AX + XA^T = -M$$

Where $A, M \in R^{n\times n}$, and $X \in R^n$is a symmetric matrix.

Theorem 6 [5]**:** Assume that A is stable, then the following statements are equivalent:

1) $X = \int_0^\infty e^{A^T t} M e^{At}\, dt.$

2) $X > 0\ if\ M > 0\ and\ X \geq 0\ if\ M \geq 0.$

3) If $M > 0$, then (M, A) is observable if and only if $X > 0$.

Definition (1.8) [3]**:** Given a stable matrix A, a pair (A, B) is controllable if and only if the solution to the following Lyapunov equation:

$$AW_c + W_c A^T + BB^T = 0$$

is positive definite where,

$$W_c = \int_0^\infty e^{\mathrm{A}t} BB^T e^{A^T t}\, dt$$

is The Controllability Gramian.

Similarly, a pair (C, A) is observable if and only if the solution to

$$W_o A + A^T W_o + C^T C = 0$$

is positive definite and

$$W_o = \int_0^\infty e^{A^T t} C^T C e^{At}\, dt$$

is The Observability Gramian.

Theorem 7 [15], [16]**:** The observability Gramian is a unique solution to the Lyapunov equation

$$W_o A + A^T W_o = -C^T C$$

Proof [17]**:**

$$\begin{aligned} W_o A + A^T W_o &= \left(\int_0^\infty e^{A^T t} C^T C e^{At}\, dt \right) A + A^T \left(\int_0^\infty e^{A^T t} C^T C e^{At}\, dt \right) \\ &= \int_0^\infty e^{A^T t} C^T C e^{At} A\, dt + \int_0^\infty A^T e^{A^T t} C^T C e^{At}\, dt \end{aligned}$$

$$= \int_0^\infty (e^{A^T t} C^T C e^{At} A + A^T e^{A^T t} C^T C e^{At})\, dt$$

$$= \int_0^\infty \frac{d}{dt} (e^{A^T t} C^T C e^{At})\, dt$$

$$= -C^T C$$

$$W_o A + A^T W_o + C^T C = 0$$

This proves that the observability Gramian is an actual solution of the Lyapunov equation.

To prove the uniqueness, suppose that there are two different solutions for

$$W_o A + A^T W_o = -C^T C$$

Given by W_{o1} andW_{o2}.

So,

$$(W_{o1} - W_{o2})A + A^T (W_{o1} - W_{o2}) = 0$$

Thus, multiplying the equation by $e^{A^T t}$ from the left and by e^{At} from the right gives

$$e^{A^T t}[(W_{o1} - W_{o2})A + A^T (W_{o1} - W_{o2})]\, e^{At} = 0$$

Taking the integral from 0 to ∞ for both sides to have

$$\int_0^\infty (e^{A^T t}[(W_{o1} - W_{o2})A + A^T (W_{o1} - W_{o2})]\, e^{At})\, dt = 0$$

$$\int_0^\infty (\frac{d}{dx} e^{A^T t} (W_{o1} - W_{o2})\, e^{At}) dt = 0$$

$$e^{A^T t} (W_{o1} - W_{o2})\, e^{At} |_0^\infty = 0$$

$$0 - (W_{o1} - W_{o2}) = 0$$

So,

$$W_{o1} = W_{o2}$$

This means that W_o is unique.

1.6 Vector and Matrix Norms

ℓ -norm is a method for calculating the difference between the system's outputs for arbitrary input signals to measure how far two linear systems are from each other.

1.6.1 Vector Norm

Let X be a vector space, $||x||$ is a norm of X if it satisfies the following properties [5]:

1. $||x|| \geq 0$ for every x and $||x|| = 0$ if and only if $x = 0$.

2. $\left||\alpha x|\right| = |\alpha| \, ||x||$, for any real α.

3. $||x_1 + x_2|| \leq ||x_1|| + ||x_2||$ for every x_1 and x_2.

4. $||x_1 \bullet x_2|| \leq ||x_1|| \, ||x_2||$

Let $x \in C^n$. Then the ℓ_p norm of x is

$$\left||x|\right|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} , \ for \ 1 \leq p \leq \infty$$

A norm of a vector is a measure of a vector length, for example, $||x||_2$ (ℓ_2 norm) is the Euclidean distance of the vector x from the origin [5].

And the ℓ_∞ norm (the infinity or uniform norm) given by:

$$\left||x|\right|_\infty = \max_i |x_i|$$

1.6.2 Matrix Norm

We denote the norm of a matrix by $||A||$ which satisfies the same properties as the vector norm. Consider the ℓ_1 matrix norm given by [5]:

$$||A||_1 = \max_j \sum_{i=1}^{n} \left|a_{ij}\right|$$

Then ℓ_2 matrix norm is defined as:

$$||A||_{\,2} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \left| a_{ij} \right|^2 \right)^{\frac{1}{2}}$$

Finally, the ℓ_∞ matrix norm stated as:

$$||A||_{\,\infty} = \max_{i} \sum_{j=1}^{n} \left| a_{ij} \right|$$

Chapter Two

Model Order Reduction

2.1 System Energy

Reducing the order of a dynamic system by model order reduction makes the system use a small amount of controlled energy and produce a large amount of observed energy.

Consider a dynamic system

$$\dot{x} = Ax + Bu$$

$$y = Cx \qquad (2\text{-}1)$$

$$x(t_0) = x_0$$

Which is assumed to be asymptotically stable, controllable, and observable.

The energy of the system is calculated and estimated using the observability and controllability gramians.

Lemma (2.1) [18]: A linear dynamic system's controllability and observability functions are given as follows:

$$E_c(x_0) = \min_{\substack{u\in(-\infty,0) \\ x(-\infty)=x_0, x(0)=0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 \; dt$$

$$E_o(x_0) = \frac{1}{2}\int_{0}^{\infty} \|y(t)\|^2 \; dt$$

$$x(0) = x_0, u(t) = 0, 0 \le t \le \infty$$

The following quadratic forms can be used to approximate the controllability and observability functions [20]:

$$E_c = \tfrac{1}{2}x^T {W_c}^{-1} x \quad \text{and} \quad E_o = \tfrac{1}{2}x^T W_o x$$

The controllability and observability Gramians, which are the only positive definite solutions to the Lyapunov equations, are represented by W_c and W_o, respectively.

$$AW_c + W_cA^T + BB^T = 0$$

$$W_oA + A^TW_o + C^TC = 0$$

2.2 Transfer Function

Given the state space matrix(A, B, C), the transfer function of the system can be written as:

$$G(s) = C(sI - A)^{-1}B \qquad (2\text{-}2)$$

Definition (2.1) [21]**:**

The transfer function G(s) is defined as:

$$G(s) = \frac{Y(s)}{U(s)} \qquad (2\text{-}3)$$

In this case, $Y(s)$ and $U(s)$ are the Laplace transforms of the system's output $y(t)$ and input $u(t)$, respectively.

The transfer function $G(s)$ in (2-2) can be written as:

$$\Sigma = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right) = G(s) = \frac{Y(s)}{U(s)}$$

2.3 Balanced Truncation method

Consider a linear system

$$\dot{x} = Ax + Bu$$

$$y = Cx \qquad (2\text{-}4)$$

$$x(t_0) = x_0$$

Where $x \in R^n$, $u \in R^r and\ y \in R^m$, we assume that (2-4) is stable, controllable, and observable.

Theorem 1 [18], [22]**:** There exists a state space transformation $x = T\tilde{x}$, (T is a non-singular matrix) for the system (2-4) such that the transformed system

$$\dot{\tilde{x}} = \hat{A}\tilde{x} + \hat{B}u$$

$$\tilde{y} = \hat{C}\tilde{x} \qquad (2\text{-}5)$$

Is in balanced form, i.e.,

$$\widetilde{W_c} = \widetilde{W_o} = \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_n \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, where $\widetilde{W_c}$ *and* $\widetilde{W_o}$ are the controllability and observability Gramians of the transformed system, with

$$\widetilde{W_c} = \widetilde{W_o} = T^{-1}W_cT^{-T} = T^TW_oT = \Sigma$$

Here, the singular values of the system's Hankel operator, are given by $\sigma_i{}'s$, $i = 1,2,\dots,n$.

The Hankel singular value (HSV) for linear systems is defined as the square root of the eigenvalues of the product of the controllability and observability Gramians [22]:

$$\sigma_i(\Sigma) = \sqrt{\lambda_i(W_oW_c)}$$

Proof. [23] [24]

Since $x = T\tilde{x}$, then $\dot{x} = T\dot{\tilde{x}}$ and $\dot{\tilde{x}} = T^{-1}\dot{x}$

Let's start with the state equation:

$$\dot{x} = Ax + Bu \qquad (2\text{-}6)$$

Equation (2-6) from the left is multiplied by T^{-1} to produce

$$T^{-1}\dot{x} = T^{-1}Ax + T^{-1}Bu$$

$$T^{-1}\dot{x} = T^{-1}AT\tilde{x} + T^{-1}Bu$$

$$\dot{\tilde{x}} = \hat{A}\tilde{x} + \hat{B}u$$

And

$$\tilde{y} = Cx$$

$$\tilde{y} = CT\tilde{x}$$

$$\tilde{y} = \hat{C}\tilde{x}$$

Where,

$\hat{A} = T^{-1}AT$, $\hat{B} = T^{-1}B$, and $\hat{C} = CT$

In addition, the two Lyapunov equations are satisfied by the controllability and observability Gramians.

$$T\hat{A}T^{-1}W_c + W_cT^{-T}\hat{A}^TT^T + T\hat{B}\hat{B}^TT^T = 0$$

$$W_oT\hat{A}T^{-1} + T^{-T}\hat{A}^TT^TW_o + T^{-T}\hat{C}^T\hat{C}T^{-1} = 0$$

Notice that the transformed Gramians satisfy the Lyapunov equations

$$\widetilde{W_c} = T^{-1}W_cT^{-T}$$

$$\widetilde{W_o} = T^TW_oT.$$

Lemma (2.2) [4], [23]**:** The balancing transformation T and its inverse T^{-1} are expressed in terms of the Singular Value Decomposition (SVD) as follows:

$$T = UY\Sigma^{-\frac{1}{2}}$$

$$T^{-1} = \Sigma^{-\frac{1}{2}}X^TL^T \qquad (2\text{-}7)$$

Where

$$\Sigma^{-\frac{1}{2}} = diag(\frac{1}{\sqrt{\sigma_1}}, \frac{1}{\sqrt{\sigma_2}}, \dots, \frac{1}{\sqrt{\sigma_n}})$$

Using Cheloskey Decomposition to decompose the two Gramians according to

$$W_c = UU^T, W_o = LL^T$$

such that

$$L^T\, U = X\Sigma Y^T$$

Where X and Y are orthogonal transformations such that:

$$X^T X = I \text{ and } Y^T Y = I$$

Then the state space model of the balanced system is ($\hat{A}, \hat{B}, \hat{C}$) with

$$\hat{A} = T^{-1}AT$$

$$\hat{B} = T^{-1}B$$

$$\hat{C} = CT$$

2.4 The reciprocal system of a linear dynamic system

The linear continuous dynamic system is represented by:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du \tag{2-8}$$

The balanced reciprocal system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is indicated as [25], [26]:

$$\bar{A} = A^{-1}$$

$$\bar{B} = A^{-1}B \tag{2-9}$$

$$\bar{C} = -CA^{-1}$$

$$\bar{D} = D - CA^{-1}B$$

And, the system's initial condition is given as follows:

$$\bar{x}(t_0) = A^{-1}x(t_0)$$

The system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is also asymptotically stable and the Lyapunov equations are given by:

$$\bar{A}\Sigma + \Sigma\bar{A}^T + \bar{B}\bar{B}^T = 0$$

$$\Sigma\bar{A} + \bar{A}^T\Sigma + \bar{C}^T\bar{C} = 0$$

The reciprocal system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is also controllable and observable and has the same diagonal Gramian matrix Σ [27].

Theorem 2 [25]**:** Let $\bar{G}$ be the transfer function of the reciprocal system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$, that is,

$$\bar{G} = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$

In the case of zero initial condition, the relation with two transfer functions G and $\bar{G}$ is then indicated as follows::

$$G(s) = \bar{G}(\frac{1}{s})$$

Proof:

$$\begin{aligned}
G(s) &= C(sI - A)^{-1}B + D \\
&= C(sI - A)^{-1}AA^{-1}B + D \\
&= C(sIA^{-1} - I)^{-1}A^{-1}B + D \\
&= -C\frac{I}{s}\left(\frac{I}{s} - A^{-1}\right)^{-1} A^{-1}B + D \\
&= -C(\frac{I}{s} - A^{-1} + A^{-1})\left(\frac{I}{s} - A^{-1}\right)^{-1} A^{-1}B + D \\
&= -CA^{-1}B - CA^{-1}\left(\frac{I}{s} - A^{-1}\right)^{-1} A^{-1}B + D \\
&= -CA^{-1}\left(\frac{I}{s} - A^{-1}\right)^{-1} A^{-1}B + D - CA^{-1}B \\
&= \bar{C}(\frac{I}{s} - \bar{A})^{-1}\bar{B} + \bar{D} \\
&= \bar{G}\left(\frac{1}{s}\right) \qquad (2\text{-}10)
\end{aligned}$$

From this, it follows that if $G(s)$ has a dominant high-frequency behavior, then $\bar{G}(s)$ will have a dominant low-frequency behavior and vice versa

It's noteworthy to observe that the equation (2-9) changes in our case if the high-order balanced system is strictly proper such that $D = 0$.

$$\bar{A} = A^{-1}$$
$$\bar{B} = A^{-1}B$$
$$\bar{C} = -CA^{-1}$$
$$\bar{D} = -CA^{-1}B$$

Lemma (2.3) [28]: " Let the system (A, B, C) be the minimal and balanced realization with Gramian Σ of a linear, time-invariant, and stable system, then the reciprocal system $(\bar{A}, \bar{B}, \bar{C})$ is also stable and balanced with the same Gramian Σ "

Proof:

Since Σ satisfies the Lyapunov equation

$$\mathrm{A}\Sigma + \Sigma\mathrm{A}^{\mathrm{T}} + \mathrm{B}\mathrm{B}^{\mathrm{T}} = 0$$

$$\mathrm{A}^{\mathrm{T}}\Sigma + \Sigma\mathrm{A} + \mathrm{C}^{\mathrm{T}}\mathrm{C} = 0$$

Multiplying the first equation from the right by A^{-1} and from the left by A^{-T} gives

$$A^{-1}(\mathrm{A}\Sigma + \Sigma\mathrm{A}^{\mathrm{T}} + \mathrm{B}\mathrm{B}^{\mathrm{T}})A^{-T} = 0$$

$$A^{-1}\mathrm{A}\Sigma A^{-T} + A^{-1}\Sigma\mathrm{A}^{\mathrm{T}}A^{-T} + A^{-1}\mathrm{B}\mathrm{B}^{\mathrm{T}}A^{-T} = 0$$

$$\mathrm{A}\Sigma + A^{-1}\Sigma + (A^{-1}\mathrm{B})(A^{-1}B)^{T} = 0$$

Equation (2-9) values are substituted, resulting in

$$\bar{A}\Sigma + \Sigma\bar{A}^{T} + \bar{B}\bar{B}^{T} = 0$$

Similar to equation (2-9), the second Lyapunov equation is multiplied by A^{-T} from the right and by A^{-1} from the left to get

$$\Sigma\bar{A} + \bar{A}^{T}\Sigma + \bar{C}^{T}\bar{C} = 0$$

The nth-order reciprocal system $(\bar{A}, \bar{B}, \bar{C})$ and the Gramian Σ are separated as:

$$\bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix}, \bar{B} = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}, \bar{C} = (\bar{C}_1 \quad \bar{C}_2) \ and \ \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$$

Pick an integer r that is positive and such that $\sigma_r > \sigma_{r+1}$ with

$W_c = W_o = \Sigma = diag(\Sigma_1, \Sigma_2)$

Where,

$$\Sigma_1 = diag(\sigma_1, \dots \ , \sigma_r) \ and \ \Sigma_2 = diag(\sigma_{r+1}, \dots \ , \sigma_n)$$

According to lemma (2.3), we have the following [7]:

The reciprocal system $(\bar{A}, \bar{B}, \bar{C})$ with W_c and W_o as the controllability and observability Gramians of the system, respectively, can be subjected to the balanced truncation approach.

Where, $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1)$ and $(\bar{A}_{22}, \bar{B}_2, \bar{C}_2)$ are rth-order and $(n-r)$th-order subsystems, respectively. And the diagonal Gramians $\Sigma_i, i = 1,2$, can be associated with the balanced subsystems $(\bar{A}_{ii}, \bar{B}_i, \bar{C}_i), i = 1,2$.

The subsystem $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1)$ is called the 'strong' subsystem, and the subsystem $(\bar{A}_{22}, \bar{B}_2, \bar{C}_2)$ is called the 'weak' subsystem [27], [6].

The reciprocal system $(\bar{A}, \bar{B}, \bar{C})$ should be balanced truncated, assume that the HSVs $\sigma_i' s, \ i = 1,2, \dots, r$ are distinct and $\sigma_1 > \sigma_2 > \cdots > \sigma_r > \sigma_n > 0$

with $\Sigma_1 > 0$, then the rth-order truncated system of $(\bar{A}, \bar{B}, \bar{C})$ is given by $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1)$ with the subspace equation:

$$\dot{\bar{x}} = \bar{A}_{11}\bar{x} + \bar{B}_1 u$$

$$\bar{y} = \bar{C}_1 \bar{x} \qquad (2\text{-}11)$$

Where $\bar{A}_{11}$, $\bar{B}_1$, and $\bar{C}_1$ can be computed from equation (2-9), and are defined as:

$$\bar{A}_{11} = \left(A_{11} - A_{12}{A_{22}}^{-1}A_{21}\right)^{-1}$$
$$\bar{B}_1 = \left(A_{11} - A_{12}{A_{22}}^{-1}A_{21}\right)^{-1}\left(B_1 - A_{12}{A_{22}}^{-1}B_2\right)$$
$$\bar{C}_1 = \left(C_1 - C_2{A_{22}}^{-1}A_{21}\right)\left(A_{11} - A_{12}{A_{22}}^{-1}A_{21}\right)^{-1} \tag{2-12}$$

The transfer function $\bar{G}_r$ for the reciprocal reduced system is defined as:

$$\bar{G}_r(s) = \bar{C}_1(sI - \bar{A}_{11})^{-1}\bar{B}_1$$

2.5 Singular perturbation approximation method (SPA)

The linear dynamic system has the following form:

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$

Here, once again $A \in R^{n\times n}, B \in R^{n\times r}, C \in R^{m\times n}$ and

$$x(t_0) = \begin{pmatrix} x_0 \\ w_0 \end{pmatrix}$$

is the initial condition.

This system's output equation is:

$$y = (C_1 \quad C_2)\begin{pmatrix} x \\ w \end{pmatrix}$$

We select $r < n$ in such a way that the reduced system is provided as:

$$\dot{\breve{x}} = \breve{A}\breve{x} + \breve{B}u$$

$$\breve{y} = \breve{C}\breve{x} \tag{2-13}$$

Where,

$$\breve{A} = A_{11} - A_{12}{A_{22}}^{-1}A_{21}$$
$$\breve{B} = \left(B_1 - {A_{22}}^{-1}B_2\right)$$
$$\breve{C} = \left(C_1 - C_2{A_{22}}^{-1}A_{21}\right) \tag{2-14}$$

The equations (2-11), (2-12) and (2-14) lead to the following results:

$$\bar{A}_{11} = \left(A_{11} - A_{12}{A_{22}}^{-1}A_{21}\right)^{-1}$$

$$= \left(\breve{A}\right)^{-1}$$

$$\bar{B}_1 = \left(A_{11} - A_{12}{A_{22}}^{-1}A_{21}\right)^{-1}\left(B_1 - A_{12}{A_{22}}^{-1}B_2\right) \tag{2-15}$$

$$= (\breve{A})^{-1}\breve{B}$$

$$\bar{C}_1 = \left(C_1 - C_2{A_{22}}^{-1}A_{21}\right)\left(A_{11} - A_{12}{A_{22}}^{-1}A_{21}\right)^{-1}$$

$$= -\breve{C}(\breve{A})^{-1}$$

And also through singular perturbation approximation ($\breve{A},\breve{B}$,$\breve{C}$), $\breve{G}_r$ is the transfer function of the $rth-order$ model.

$$\breve{G}_r = \breve{C}(sI - \breve{A})^{-1}\breve{B}$$

And consider that $\bar{G}_r$ is the reduced reciprocal system's transfer function ($\bar{A}_{11}, \bar{B}_1$,$\bar{C}_1$).

Using equation (2-15) connects the transfer functions $\breve{G}_r$ of the reduced SPA system and $\bar{G}_r$ of the reduced reciprocal system.

$$\breve{G}_r(s) = \breve{C}[\left(sI - \breve{A}\right)^{-1} + \breve{A}^{-1}]\breve{B}$$

$$= -\breve{C}\breve{A}^{-1}\left(\frac{1}{s}I - \breve{A}^{-1}\right)^{-1}\breve{A}^{-1}\breve{B}$$

$$= \bar{G}_r\left(\frac{1}{s}\right) \tag{2-16}$$

Chapter Three

Error Bounds of Model Order Reduction Methods

3.1 An error bound using balanced truncation model reduction (BT)

Take into consideration the following linear, time-invariant, finite-dimensional state space system:

$$\dot{x} = Ax + Bu$$

$$y = Cx \qquad (3\text{-}1)$$

$$x(0) = x_0$$

The associated transfer function is given by:

$$G(s) = C(sI - A)^{-1}B$$

State space model (A, B, C) presented by

$$G(s) = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right)$$

Is a realization of $G(s)$ which is minimal if and only if (C, A) is observable and (A, B) is controllable [5]:

Definition (3.1) [7]**:**

Let $1 \leq r \leq n$ the system (3-1) with the balanced system (A, B, C), and the matrices A, B, C are partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \; and \; C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$$

With $\Sigma_1 = diag(\sigma_1, \dots, \sigma_r) \; and \; \Sigma_2 = diag(\sigma_{r+1}, \dots, \sigma_n)$

The order of the A_{11} is $r \times r$ as well as the order of Σ_1 and $(n-r) \times (n-r)$ is the order of A_{22}, where the original system's order matches the order of the other block matrices.

Let $\sigma_r > \sigma_{r+1}$ a balanced truncation (BT) produces a system of lower order r defined by:

$$\dot{x}_r = A_{11}x_r + B_1 u$$

$$y_r = C_1 x_r$$

Applying balanced truncation to the balanced system yields a reduced rth order system with a transfer function:

$$G_r(s) = C_1(sI - A_{11})^{-1}B_1$$

Lemma (3.1) [10]**:** The matrices $A_{ii}, i = 1{,}2$ are asymptotically stable, i.e.,

$$Re\big(\lambda_j\{A_{ii}\} < 0\big), i = 1{,}2, \forall j$$

if there are no diagonal elements in common between Σ_1 and Σ_2 then the subsystem (A_{ii}, B_i, C_i) is also observable and controllable.

The $H_\infty -$ norm between the original and reduced order systems can be used as an error criterion.

Definition (3.2) [9]**:**

The $H_\infty -$ norm of the transfer function is defined as

$$\|G\|_{H_\infty} = \sup_{\omega\in\mathcal{R}} \sigma_{max}\big(G(i\omega)\big) = \sup_{U} \frac{\|Y(i\omega)\|_{H_\infty}}{\|U(i\omega)\|_{H_\infty}} = \sup_{u} \frac{\|y\|_2}{\|u\|_2}$$

Where $\sup\limits_{\omega\in\mathcal{R}}$ is the least upper bound over all real-valued frequencies and, $\|y\|_2$ and $\|u\|_2$ are the $\ell_2 - norm$ of $y(t), u(t)$ respectively.

The $H_\infty -$ norm (the maximum of the frequency response) of the error system is bounded above by twice the sum of the neglacted singular values $2(\sigma_{r+1}, \dots, \sigma_n)$.

Theorem 1 [4], [7]**:**

For the zero-initial condition, G_r is stable, and satisfies

$$\|G - G_r\|_{H_\infty} \leq 2 \sum_{i=r+1}^{n} \sigma_i$$

Where σ_{r+1} is the first deleted Hankel singular value of $G(s)$.

And $\|G\|_{H_\infty} = max\{\|G(i\omega\|_2 : \omega \in \mathcal{R}\}$ denotes the $H_\infty -$ norm.

Remark:

if

$$\|G - G_r\|_{H_\infty} \leq 2 \sum_{i=r+1}^{n} \sigma_i$$

then

$$\|y - y_r\|_2 \leq (2 \sum_{i=r+1}^{n} \sigma_i) \|u\|_2$$

proof:

since

$$\|G\|_{H_\infty} = \sup_{\omega \in \mathcal{R}} \sigma_{max}\big(G(i\omega)\big) = \sup_{u} \frac{\|y\|_2}{\|u\|_2}$$

Then $\|y\|_2 \leq \|G\|_{H_\infty} \|u\|_2$

So,

$$\|y - y_r\|_2 \leq \|G - G_r\|_{H_\infty} \|u\|_2 \leq (2 \sum_{i=r+1}^{n} \sigma_i) \|u\|_2$$

3.2 An error bound of the Reciprocal system

The balanced system's reciprocal system is represented by

$$\dot{\bar{x}} = \bar{A}x + \bar{B}u$$

$$y = \bar{C}x + \bar{D}u$$

With

$\bar{A} = A^{-1}$, $\bar{B} = A^{-1}B$, $\bar{C} = -CA^{-1}$, and $\bar{D} = -CA^{-1}B$

And the transfer function of the reciprocal system$(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is defined as:

$$\bar{G} = \bar{C}(sI - \bar{A})^{-1}\bar{B}$$

Lemma (3.2) [28]**:** The $H_\infty -$ norm is invariant under the reciprocal transformation, i.e.,

$$\|G(s)\|_{H_\infty} = \|\bar{G}(s)\|_{H_\infty}$$

Lemma (3.3) [7], [29]**:** The reduced reciprocal system's error bound is presented as follows for zero initial conditions:

$$\|G - \bar{G}_r\|_{H_\infty} \leq 2 \sum_{i=r+1}^{n} \sigma_i$$

Thus, a direct duality exists between the balanced and reciprocal elimination of subsystems through the reciprocal transformation.

3.3 Error Bound for the Singular Perturbation Approximation with Zero Initial Condition

The singular perturbation approximation approach $(\breve{A}, \breve{B}, \breve{C})$ reduces the system to:

$$\dot{\breve{x}} = \breve{A}\breve{x} + \breve{B}u$$

$$\breve{y} = \breve{C}\breve{x}$$

where,

$$\breve{A} = A_{11} - A_{12}{A_{22}}^{-1}A_{21}$$
$$\breve{B} = \left(B_1 - {A_{22}}^{-1}B_2\right)$$
$$\breve{C} = \left(C_1 - C_2{A_{22}}^{-1}A_{21}\right)$$

And also through singular perturbation approximation, $\breve{G}_r$ is the transfer function of the $rth - order$ model.

$$\breve{G}_r = \breve{C}(sI - \breve{A})^{-1}\breve{B}$$

Utilize the reciprocal system technique and expand it using the singular perturbation approximation to obtain such an error bound.

Theorem 2 [4], [7]**:**

The error bound shall be given as follows in the form of the $H_\infty -$ norm:

$$\left\|G - \breve{G}_r\right\|_{H_\infty} \leq 2\sum_{i=r+1}^{n} \sigma_i$$

In particular, $\left\|G(s) - \breve{G}(s)\right\|_{H_\infty} \to 0 \;\; as \;\; r \to \infty$

Proof [16], [25] :

From equation (2-10), equation (2-16), and lemma (3.3)

$$\left\|G(s) - \breve{G}(s)\right\|_{H_\infty}$$

$$= \left\|G(s) - \bar{G}\left(\frac{1}{s}\right) + \bar{G}\left(\frac{1}{s}\right) - \bar{G}_r\left(\frac{1}{s}\right) + \bar{G}_r\left(\frac{1}{s}\right) - \breve{G}(s)\right\|_{H_\infty}$$

$$\leq \left\|G(s) - \bar{G}\left(\frac{1}{s}\right)\right\|_{H_\infty} + \left\|\bar{G}\left(\frac{1}{s}\right) - \bar{G}_r\left(\frac{1}{s}\right)\right\|_{H_\infty} + \left\|\bar{G}_r\left(\frac{1}{s}\right) - \breve{G}(s)\right\|_{H_\infty}$$

$$\leq \left\|\bar{G}\left(\frac{1}{s}\right) - \bar{G}_r\left(\frac{1}{s}\right)\right\|_{H_\infty} \quad \leq 2\sum_{i=r+1}^{n} \sigma_i$$

Chapter Four

Optimal Control

One of the key techniques for handling control issues is covered in this chapter, using the fundamental Algebraic Riccati Equation and the Hamiltonian Function, the Linear Quadratic Regulator (LQR) is developed to discover an optimal control that minimizes "The quadratic cost function," with regard to the restrictions on the states and the inputs.

4.1 Linear Quadratic Regulator (LQR)

Consider the following continuous linear dynamic system:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x(0) = x_0 \tag{4-1}$$

Where, x and u are the state and the input of the system, respectively.

Assume that the system is controllable and observable, as one of the most useful applications of dynamic programming, consider the general quadratic cost linear control problem.

The following equation defines the quadratic cost function J:

$$J = \frac{1}{2}\int_0^\infty (y^T y + u^T R u)dt$$

or, equivalently

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)dt$$

While the positive definite matrix $R > 0$ is used to represent the cost penalty of the input u, the positive semi-definite matrix $Q = C^T C \geq 0$ is used to represent the cost penalty of the states x.

selecting the optimal control u that satisfies the requirement and minimizes the quadratic cost function J.

$$\dot{x} = Ax + Bu.$$

The optimal control can be indicated by u°such that:

$$J(u^{\circ}) \leq J(u), \qquad \forall u \in L^2$$

Substitute u° in the constraint equation and the optimal solution of this equation x°, gives:

$$\dot{x} = Ax + Bu^{\circ}$$

To find the optimal control, we apply the maximum principle theorem by computing the Hamiltonian H:

$$H = \frac{1}{2}(x^T Q x + u^T R u) + \lambda^T (Ax + Bu).$$

where $\lambda \in \mathcal{R}^n$ are functions of time and are often referred as the costate variables.

Theorem 1 [30] (Maximum Principle):

If (x°, u°) is optimal, then there exists $\lambda^{\circ} \in \mathcal{R}^n$ such that

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \quad and \quad -\dot{\lambda}_i = \frac{\partial H}{\partial x_i}$$

Let H be a differentiable function, consider $\frac{\partial H}{\partial u} = 0$ which is a necessary condition for the optimal input. A little computation shows that the minimum occurs at:

$$u = -R^{-1} B^T \lambda \tag{4-2}$$

Applying the maximum principle and equation (2) to obtain the necessary conditions:

$$\dot{x} = \frac{\partial H}{\partial \lambda} = Ax - BR^{-1}B^T\lambda\,, \qquad x(0) = x_0$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -Qx - A^T\lambda \tag{4-3}$$

This is a coupled system, linear in x and λ of order $2n \times 2n$.

The optimal control is found by solving a two-point boundary value problem using the initial condition $x(0)$ and the final condition $\lambda(T)$. Unfortunately, it is not simple to figure out how to solve systems like this or how to establish the relationship between x and λ in the form:

$$\lambda(t) = M(t)x(t) \text{ where } M(t) \in \mathcal{R}^{n\times n} \tag{4-4}$$

In the linear quadratic regulator issue, there is a crucial differential equation known as the Matrix Riccati Equation (MRE), which begins with (4-4) and applies the essential conditions as follows:

$$\begin{aligned}
&\lambda = Mx \\
&\dot{\lambda} = \dot{M}x + M\dot{x} \\
&-Qx - A^T\lambda = \dot{M}x + M(Ax - BR^{-1}B^T\lambda) \\
&-Qx - A^TMx = \dot{M}x + MAx - M\,BR^{-1}B^TMx \\
&\dot{M}x + MAx + A^TMx - M\,BR^{-1}B^TMx + Qx = 0 \\
&\quad \dot{M} = -\,MA - A^TM + M\,BR^{-1}B^TM - Q
\end{aligned} \tag{4-5}$$

Since $u \in L^2$ over an infinite time horizon, it follows that [30]:

$$\lim_{t\to\infty} \dot{M} = 0$$

It suggests that the unique positive definite solution to the algebraic Riccati equation (ARE) might be used to replace M in equation (4-5).

$$MA + A^TM - M\,BR^{-1}B^TM + Q = 0$$

Once the function M has been computed and stored, the best optimal control u by reducing J is represented by:

$$u = -R^{-1}B^TMx \tag{4-6}$$

This represents a constant gain $K = R^{-1}B^TM$ where M is the solution of the (ARE) is completely determined in the feedback form.

Finally, the optimum trajectory is the solution of:

$$\dot{x} = (A - R^{-1}B^TM)x \tag{4-7}$$

4.2 Optimal Control for Reduced Order Model

4.2.1 Optimal Control for a Reduced System type 1

Think about the dynamic system that is linear and time-invariant that has the following definition [10]

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\alpha}A_{21} & \frac{1}{\alpha}A_{22} \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$

$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \tag{4-8}$$

In another form:

$$\dot{x} = A_{11}x + A_{12}w + B_1 u$$

$$\dot{w} = \frac{1}{\alpha}A_{21}x + \frac{1}{\alpha}A_{22}w + B_2 u$$

$$\alpha\dot{w} = A_{21}x + A_{22}w + \alpha B_2 u$$

$$y = C_1 x + C_2 w$$

Thus,

$$\dot{x} = A_{11}x + A_{12}w + B_1 u$$

$$\alpha\dot{w} = A_{21}x + A_{22}w + \alpha B_2 u \tag{4-9}$$

$$y = C_1 x + C_2 w$$

Assuming that A_{22} is stable and that ${A_{22}}^{-1}$ exists, set $\alpha = 0$ to produce the equation shown below:

$$\dot{x} = A_{11}x + A_{12}w + B_1 u$$

$$\alpha\dot{w} = A_{21}x + A_{22}w + \alpha B_2 u$$

$$0 = A_{21}x + A_{22}w$$

$$\breve{w} = -{A_{22}}^{-1}A_{21}\breve{x} \tag{4-10}$$

Here is the reduced system which was got by substituting equation (4-10) in the system described by (4-9):

$$\dot{\breve{x}} = \breve{A}\breve{x} + \breve{B}u$$

$$\breve{y} = \breve{C}\breve{x} \tag{4-11}$$

Where,

$$\breve{A} = A_{11} - A_{12}{A_{22}}^{-1}A_{21}$$

$$\breve{B} = B_1$$

$$\breve{C} = C_1 - C_2{A_{22}}^{-1}A_{21}$$

Let's start by defining the quadratic cost function J for the original system (4-8) as:

$$J \ = \frac{1}{2}\int_0^{\infty}(x^TQx + u^TRu)dt$$

Where $Q = C^TC \geq 0$ and $R > 0$

$$u = -R^{-1}B^TMx \ = -R^{-1}({B_1}^T \quad {B_2}^T)M\begin{pmatrix} x \\ w \end{pmatrix}$$

The matrix M is the solution of the following (ARE):

$$MA + A^TM - M\,BR^{-1}B^TM + Q = 0$$

Substituting the matrices A, B and C gives:

$$M\begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\alpha}A_{21} & \frac{1}{\alpha}A_{22} \end{pmatrix} + \begin{pmatrix} {A_{11}}^T & \frac{1}{\alpha}{A_{21}}^T \\ {A_{12}}^T & \frac{1}{\alpha}{A_{22}}^T \end{pmatrix}M - M\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}R^{-1}({B_1}^T \quad {B_2}^T)M +$$
$$\begin{pmatrix} {C_1}^T \\ {C_2}^T \end{pmatrix}(C_1 \quad C_2) = 0 \tag{4-12}$$

To avoid unboundedness, $\alpha \rightarrow 0$ the solution is requested as follows:

$$M = \begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha {M_{12}}^T & \alpha M_{22} \end{pmatrix}$$

So equation (4-12) becomes

$$\begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha M_{12}{}^T & \alpha M_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\alpha} A_{21} & \frac{1}{\alpha} A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}{}^T & \frac{1}{\alpha} A_{21}{}^T \\ A_{12}{}^T & \frac{1}{\alpha} A_{22}{}^T \end{pmatrix} \begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha M_{12}{}^T & \alpha M_{22} \end{pmatrix}$$

$$- \begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha M_{12}{}^T & \alpha M_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} R^{-1} \begin{pmatrix} B_1{}^T & B_2{}^T \end{pmatrix} \begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha M_{12}{}^T & \alpha M_{22} \end{pmatrix}$$

$$+ \begin{pmatrix} C_1{}^T \\ C_2{}^T \end{pmatrix} \begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} M_{11}A_{11} + M_{12}A_{21} & M_{11}A_{12} + M_{12}A_{22} \\ \alpha M_{12}{}^T A_{11} + M_{22}A_{21} & \alpha M_{12}{}^T A_{12} + M_{22}A_{22} \end{pmatrix}$$

$$+ \begin{pmatrix} A_{11}{}^T M_{11} + A_{21}{}^T M_{12}{}^T & \alpha A_{11}{}^T M_{12} + A_{21}{}^T M_{22} \\ A_{12}{}^T M_{11} + A_{22}{}^T M_{12}{}^T & \alpha A_{12}{}^T M_{12} + A_{22}{}^T M_{22} \end{pmatrix}$$

$$- \begin{pmatrix} M_{11}B_1 + \alpha M_{12}B_2 \\ \alpha M_{12}{}^T B_1 + \alpha M_{22}B_2 \end{pmatrix} R^{-1} \begin{pmatrix} B_1{}^T M_{11} + \alpha B_2{}^T M_{12}{}^T & \alpha B_1{}^T M_{12} + \alpha B_2{}^T M_{22} \end{pmatrix}$$

$$+ \begin{pmatrix} C_1{}^T C_1 & C_1{}^T C_2 \\ C_2{}^T C_1 & C_2{}^T C_2 \end{pmatrix} = 0$$

Obtain the following set of equations:

$$M_{11}A_{11} + M_{12}A_{21} + A_{11}{}^T M_{11} + A_{21}{}^T M_{12}{}^T$$

$$-(M_{11}B_1 + \alpha M_{12}B_2)R^{-1}(B_1{}^T M_{11} + \alpha B_2{}^T M_{12}{}^T) + C_1{}^T C_1 = 0$$

$$M_{11}A_{12} + M_{12}A_{22} + \alpha A_{11}{}^T M_{12} + A_{21}{}^T M_{22}$$

$$-(M_{11}B_1 + \alpha M_{12}B_2)R^{-1}(\alpha B_1{}^T M_{12} + \alpha B_2{}^T M_{22}) + C_1{}^T C_2 = 0$$

$$\alpha M_{12}{}^T A_{11} + M_{22}A_{21} + A_{12}{}^T M_{11} + A_{22}{}^T M_{12}{}^T$$

$$-(\alpha M_{12}{}^T B_1 + \alpha M_{22}B_2)R^{-1}(B_1{}^T M_{11} + \alpha B_2{}^T M_{12}{}^T) + C_2{}^T C_1 = 0$$

$$\alpha M_{12}{}^T A_{12} + M_{22}A_{22} + \alpha A_{12}{}^T M_{12} + A_{22}{}^T M_{22}$$

$$-(\alpha M_{12}{}^T B_1 + \alpha M_{22}B_2)R^{-1}\left(\alpha B_1{}^T M_{12} + \alpha B_2{}^T M_{22}\right) + C_2{}^T C_2 = 0$$

Let $\alpha = 0$ in the previous equations, the following reduced system Riccati Equations are obtained:

$$M_{11}A_{11} + M_{12}A_{21} + {A_{11}}^T M_{11} + {A_{21}}^T {M_{12}}^T - M_{11}B_1R^{-1}{B_1}^T M_{11} + {C_1}^T C_1 = 0$$

(4-13)

$$M_{11}A_{12} + M_{12}A_{22} + {A_{21}}^T M_{22} + {C_1}^T C_2 = 0$$

(4-14)

$$M_{22}A_{21} + {A_{12}}^T M_{11} + {A_{22}}^T\, {M_{12}}^T + {C_2}^T C_1 = 0$$

(4-15)

$$M_{22}A_{22} + {A_{22}}^T M_{22} + {C_2}^T C_2 = 0$$

(4-16)

writing $\bar{\bar{M}}_{12}$ and ${\bar{\bar{M}}_{12}}^{\,T}$ in equations (4-14) and (4-15) in terms of $\bar{\bar{M}}_{11}$ and $\bar{\bar{M}}_{22}$ as follows:

$$\bar{\bar{M}}_{12} = -(\bar{\bar{M}}_{11}A_{12} + {A_{21}}^T\bar{\bar{M}}_{22} + {C_1}^T C_2){A_{22}}^{-1} \tag{4-17}$$

$${\bar{\bar{M}}_{12}}^{\,T} = -({A_{22}}^T)^{-1}(\bar{\bar{M}}_{22}A_{21} + {A_{12}}^T\bar{\bar{M}}_{11} + {C_2}^T C_1) \tag{4-18}$$

Equation (4-16) can be expressed in another form:

$${A_{21}}^T({A_{22}}^T)^{-1}\bar{\bar{M}}_{22}A_{21} + {A_{21}}^T\bar{\bar{M}}_{22}{A_{22}}^{-1}A_{21} = -{A_{21}}^T({A_{22}}^T)^{-1}{C_2}^T C_2{A_{22}}^{-1}A_{21} \tag{4-19}$$

Using equations (4-17), (4-18), and (4-19) to obtain:

$$\bar{\bar{M}}_{11}\breve{A} + \breve{A}^T\bar{\bar{M}}_{11} - \bar{\bar{M}}_{11}\,\breve{B}\bar{\bar{R}}^{-1}\breve{B}^T\bar{\bar{M}}_{11} + \breve{C}^T\breve{C} = 0 \tag{4-20}$$

where

$$\begin{aligned}\breve{A} &= A_{11} - A_{12}{A_{22}}^{-1}A_{21}\\ \breve{B} &= B_1\\ \breve{C} &= C_1 - C_2{A_{22}}^{-1}A_{21}\end{aligned} \tag{4-21}$$

Theorem 2 [31], [32]**:**

Applying the implicit function theorem, if $(\breve{A}, \breve{B})$ is controllable and $(\breve{A}, \breve{C})$ is observable, then

$$M_{ij} = \bar{\bar{M}}_{ij} + O(\alpha), i, j = 1,2.$$

Assuming the pair $(\breve{A}, \breve{B})$ is controllable, then the values of M_{ij} and $\bar{\bar{M}}_{ij}, i, j = 1,2$ satisfy theorem 2.

substitute $\bar{\bar{M}}_{ij}$ for M_{ij} rewriting the feedback optimal control u:

$$u = -R^{-1}\begin{pmatrix} {B_1}^T & {B_2}^T \end{pmatrix}\begin{pmatrix} \bar{\bar{M}}_{11} & \alpha\bar{\bar{M}}_{12} \\ \alpha{\bar{\bar{M}}_{12}}^T & \alpha\bar{\bar{M}}_{22} \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}$$

$$u = -R^{-1}\left({B_1}^T\bar{\bar{M}}_{11} + \alpha{B_2}^T\bar{\bar{M}}_{12}\right)x - R^{-1}\left(\alpha{B_1}^T\bar{\bar{M}}_{12} + \alpha{B_2}^T\bar{\bar{M}}_{22}\right)w \tag{4-22}$$

Therefore, the modified version of the original system defined by equation (4-9) is:

$$\begin{aligned} \dot{x} &= (A_{11} - B_1R^{-1}\left({B_1}^T\bar{\bar{M}}_{11} + \alpha{B_2}^T\bar{\bar{M}}_{12}\right))x + (A_{12} - B_1R^{-1}\left(\alpha{B_1}^T\bar{\bar{M}}_{12} + \alpha{B_2}^T\bar{\bar{M}}_{22}\right))w \\ \alpha\dot{w} &= (A_{21} - \alpha B_2R^{-1}\left({B_1}^T\bar{\bar{M}}_{11} + \alpha{B_2}^T\bar{\bar{M}}_{12}\right))x + (A_{22} - \alpha B_2R^{-1}\left(\alpha{B_1}^T\bar{\bar{M}}_{12} + \alpha{B_2}^T\bar{\bar{M}}_{22}\right))w \end{aligned} \tag{4-23}$$

If the new dynamic system is asymptotically stable and theorem 2 is fulfilled, then there is a solution $x(t)$ and $w(t)$ within the $O(\alpha)$ for this system.

Let

$$\bar{\bar{J}} = \frac{1}{2}\int_0^\infty (\bar{\bar{y}}^T\bar{\bar{y}} + \bar{\bar{u}}^T\bar{\bar{R}}\bar{\bar{u}})dt$$

or, equivalently

$$\bar{\bar{J}} = \frac{1}{2}\int_0^\infty (\bar{\bar{x}}^T\bar{\bar{Q}}\bar{\bar{x}} + \bar{\bar{u}}^T\bar{\bar{R}}\bar{\bar{u}})dt$$

Where $\bar{\bar{Q}} = \bar{\bar{C}}^T\bar{\bar{C}} \geq 0$ and $\bar{\bar{R}} = R > 0$.

be the reduced system's quadratic cost function in equation (4-11).

Additionally, the following is the definition of the optimal feedback control for the linear dynamic system's reduced order model:

$$\breve{u} = -\bar{\bar{R}}^{-1}\breve{B}^{T}\bar{\bar{M}}\breve{x}. \tag{4-24}$$

Where $\bar{\bar{M}}$ is the (ARE) solution for the reduced order linear dynamic system, which is represented by:

$$\bar{\bar{M}}A_{11} + {A_{11}}^{T}\bar{\bar{M}} - \bar{\bar{M}}\,B_{1}\bar{\bar{R}}^{-1}{B_{1}}^{T}\bar{\bar{M}} + {C_{1}}^{T}C_{1} = 0 \tag{4-25}$$

From [9], [32], $\bar{\bar{M}}$ and $\bar{\bar{M}}_{11}$ are both identical.

Thus, the reduced system (4-11) is given by substituting u in (4-24) and $\bar{\bar{M}}$ in (4-25)

$$\dot{\breve{x}} = \breve{A}\breve{x} - \breve{B}\bar{\bar{R}}^{-1}\breve{B}^{T}\bar{\bar{M}}\breve{x}$$

$$\dot{\breve{x}} = (\breve{A} - \breve{B}\bar{\bar{R}}^{-1}\breve{B}^{T}\bar{\bar{M}})\breve{x} \tag{4-26}$$

$$\breve{y} = \breve{C}\breve{x}$$

Where,

$$\breve{A} = A_{11} - A_{12}{A_{22}}^{-1}A_{21}$$

$$\breve{B} = B_{1}$$

$$\breve{C} = C_{1} - C_{2}{A_{22}}^{-1}A_{21}$$

By finding a solution to the new reduced system described by equation (4-26),

With the assistance of $\breve{x}(t)$, the feedback optimal control $\breve{x}(t)$, that must be determined in order to identify the quadratic cost function's least value $\bar{\bar{J}}$, is discovered.

4.2.2 Optimal Control for a Reduced System type 2

Consider the full linear time-invariant dynamic system:

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & \frac{1}{\alpha}A_{22} \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{1}{\alpha}B_2 \end{pmatrix} u$$

$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \tag{4-27}$$

A different way to express the original system (4-27) is as follows:

$$\dot{x} = A_{11}x + A_{12}w + B_1u$$

$$\alpha\dot{w} = \alpha A_{21}x + A_{22}w + B_2u \tag{4-28}$$

$$y = C_1x + C_2w$$

Then, set $\alpha = 0$ to obtain the following equation:

$$\dot{x} = A_{11}x + A_{12}w + B_1u$$

$$\alpha\dot{w} = \alpha A_{21}x + A_{22}w + B_2u$$

$$0 = A_{22}w + B_2u$$

$$w = -{A_{22}}^{-1}B_2u$$

$$\dot{x} = A_{11}x - A_{12}{A_{22}}^{-1}B_2u + B_1u$$

$$\dot{x} = A_{11}x + (B_1 - A_{12}{A_{22}}^{-1}B_2)u$$

$$y = C_1x - C_2{A_{22}}^{-1}B_2u \tag{4-29}$$

Here is the reduced system which was got by substituting equation (4-29) in the system described by (4-28):

$$\dot{\breve{x}} = \breve{A}\breve{x} + \breve{B}\breve{u}$$

$$\breve{y} = \breve{C}\breve{x} + \breve{D}\breve{u} \tag{4-30}$$

where,

$$\breve{A} = A_{11}$$

$$\breve{B} = B_1 - A_{12}{A_{22}}^{-1}B_2$$

$$\breve{C} = C_1$$

$$\breve{D} = -C_2{A_{22}}^{-1}B_2$$

For the initial system (4-27), the quadratic cost function J is defined as:

$$J = \frac{1}{2}\int_0^\infty (y^T y + u^T R u)dt$$

or, equivalently

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)dt$$

For the optimal system, we define the optimal control u as follows:

$$u = -R^{-1}B^T M x = -R^{-1}\begin{pmatrix} {B_1}^T & \frac{1}{\alpha}{B_2}^T \end{pmatrix} M \begin{pmatrix} x \\ w \end{pmatrix}$$

The matrix M is the solution of the following (ARE):

$$MA + A^T M - M\,BR^{-1}B^T M + Q = 0$$

Substituting the matrices A, B and C gives:

$$M\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & \frac{1}{\alpha}A_{22} \end{pmatrix} + \begin{pmatrix} {A_{11}}^T & {A_{21}}^T \\ {A_{12}}^T & \frac{1}{\alpha}{A_{22}}^T \end{pmatrix} M - M\begin{pmatrix} B_1 \\ \frac{1}{\alpha}B_2 \end{pmatrix} R^{-1}\begin{pmatrix} {B_1}^T & \frac{1}{\alpha}{B_2}^T \end{pmatrix} M + \begin{pmatrix} {C_1}^T \\ {C_2}^T \end{pmatrix}\begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0 \qquad (4\text{-}31)$$

To avoid unboundedness, $\alpha \to 0$ the solution is requested as follows:

$$M = \begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha {M_{12}}^T & \alpha M_{22} \end{pmatrix}$$

So equation (4-31) becomes

$$\begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha M_{12}{}^T & \alpha M_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & \frac{1}{\alpha} A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}{}^T & A_{21}{}^T \\ A_{12}{}^T & \frac{1}{\alpha} A_{22}{}^T \end{pmatrix} \begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha M_{12}{}^T & \alpha M_{22} \end{pmatrix}$$

$$- \begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha M_{12}{}^T & \alpha M_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ \frac{1}{\alpha} B_2 \end{pmatrix} R^{-1} \begin{pmatrix} B_1{}^T & \frac{1}{\alpha} B_2{}^T \end{pmatrix} \begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha M_{12}{}^T & \alpha M_{22} \end{pmatrix}$$

$$+ \begin{pmatrix} C_1{}^T \\ C_2{}^T \end{pmatrix} \begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0$$

Thus, the following set of equations is obtained:

$$M_{11}A_{11} + \alpha M_{12}A_{21} + A_{11}{}^T M_{11} + \alpha A_{21}{}^T M_{12}{}^T$$

$$-(M_{11}B_1 + M_{12}B_2)R^{-1}(B_1{}^T M_{11} + B_2{}^T M_{12}{}^T) + C_1{}^T C_1 = 0$$

$$M_{11}A_{12} + M_{12}A_{22} + \alpha A_{11}{}^T M_{12} + \alpha A_{21}{}^T M_{22}$$

$$-(M_{11}B_1 + M_{12}B_2)R^{-1}(\alpha B_1{}^T M_{12} + B_2{}^T M_{22}) + C_1{}^T C_2 = 0$$

$$\alpha M_{12}{}^T A_{11} + \alpha M_{22}A_{21} + A_{12}{}^T M_{11} + A_{22}{}^T M_{12}{}^T$$

$$-(\alpha M_{12}{}^T B_1 + M_{22}B_2)R^{-1}(B_1{}^T M_{11} + B_2{}^T M_{12}{}^T) + C_2{}^T C_1 = 0$$

$$\alpha M_{12}{}^T A_{12} + M_{22}A_{22} + \alpha A_{12}{}^T M_{12} + A_{22}{}^T M_{22}$$

$$-(\alpha M_{12}{}^T B_1 + M_{22}B_2)R^{-1}\left(\alpha B_1{}^T M_{12} + B_2{}^T M_{22}\right) + C_2{}^T C_2 = 0$$

Let $\alpha = 0$ in the previous equations, the reduced system Riccati equations is given as:

$$M_{11}A_{11} + A_{11}{}^T M_{11} - (M_{11}B_1 + M_{12}B_2)R^{-1}(B_1{}^T M_{11} + B_2{}^T M_{12}{}^T) + C_1{}^T C_1 = 0$$

$$M_{11}A_{12} + M_{12}A_{22} - (M_{11}B_1 + M_{12}B_2)R^{-1}(B_2{}^T M_{22}) + C_1{}^T C_2 = 0$$

$$A_{12}{}^T M_{11} + A_{22}{}^T M_{12}{}^T - (M_{22}B_2)R^{-1}(B_1{}^T M_{11} + B_2{}^T M_{12}{}^T) + C_2{}^T C_1 = 0$$

$$M_{22}A_{22} + A_{22}{}^T M_{22} - M_{22}B_2R^{-1}B_2{}^T M_{22} + C_2{}^T C_2 = 0$$

Then, the following $m \times m$ reduced equation for $\bar{\bar{M}}_{22}$:

$$M_{22}A_{22} + {A_{22}}^T M_{22} - M_{22}B_2R^{-1}{B_2}^T M_{22} + {C_2}^T C_2 = 0$$

Another $n \times n$ equation for $\bar{\bar{M}}_{11}$ is obtained when we express $\bar{\bar{M}}_{12}$ and ${M_{12}}^T$ in terms of $\bar{\bar{M}}_{11}$ and $\bar{\bar{M}}_{22}$, this equation takes the form:

$$\bar{\bar{M}}_{11}\breve{A} + \breve{A}^T\bar{\bar{M}}_{11} - \bar{\bar{M}}_{11}\,\breve{B}\bar{\bar{R}}^{-1}\breve{B}^T\bar{\bar{M}}_{11} + \breve{C}^T\breve{C} = 0 \qquad (4\text{-}32)$$

If $(\breve{A}, \breve{B})$ is controllable and $(\breve{A}, \breve{C})$ is observable, then theorem 2 is satisfied by the values of M_{ij} and $\bar{\bar{M}}_{ij}, i, j = 1{,}2$.

Substituting $\bar{\bar{M}}_{ij}$ for M_{ij} to rewrite the feedback optimal control u:

$$u = -R^{-1}\begin{pmatrix} {B_1}^T & \frac{1}{\alpha}{B_2}^T \end{pmatrix}\begin{pmatrix} \bar{\bar{M}}_{11} & \alpha\bar{\bar{M}}_{12} \\ \alpha{\bar{\bar{M}}_{12}}^{\;T} & \alpha\bar{\bar{M}}_{22} \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}$$

$$u = -R^{-1}\left({B_1}^T\bar{\bar{M}}_{11} + {B_2}^T\bar{\bar{M}}_{12}\right)x - R^{-1}\left(\alpha{B_1}^T\bar{\bar{M}}_{12} + {B_2}^T\bar{\bar{M}}_{22}\right)w. \qquad (4\text{-}33)$$

The original system's modified form, as given by equation (4-28), is then:

$$\dot{x} = (A_{11} - B_1R^{-1}\left({B_1}^T\bar{\bar{M}}_{11} + {B_2}^T\bar{\bar{M}}_{12}\right))x + (A_{12} - B_1R^{-1}\left(\alpha{B_1}^T\bar{\bar{M}}_{12} + {B_2}^T\bar{\bar{M}}_{22}\right))w.$$

$$\alpha\dot{w} = (\alpha A_{21} - B_2R^{-1}\left({B_1}^T\bar{\bar{M}}_{11} + {B_2}^T\bar{\bar{M}}_{12}\right))x + (A_{22} - B_2R^{-1}\left(\alpha{B_1}^T\bar{\bar{M}}_{12} + {B_2}^T\bar{\bar{M}}_{22}\right))w \qquad (4\text{-}34)$$

If the new dynamic system is asymptotically stable and it fulfilled theorem 2, then there is a solution $x(t)$ and $w(t)$ within the $O(\alpha)$ for this system.

Let

$$\bar{\bar{J}} = \frac{1}{2}\int_0^\infty (\bar{\bar{y}}^T\bar{\bar{y}} + \bar{\bar{u}}^T\bar{\bar{R}}\bar{\bar{u}})dt$$

or, equivalently

$$\bar{\bar{J}} = \frac{1}{2}\int_0^\infty (\bar{\bar{x}}^T\bar{\bar{Q}}\bar{\bar{x}} + 2\bar{\bar{x}}^T\bar{\bar{C}}\bar{\bar{D}}\bar{\bar{u}} + \bar{\bar{u}}^T\bar{\bar{R}}\bar{\bar{u}})dt$$

where $\bar{\bar{Q}} = \bar{\bar{C}}^T\bar{\bar{C}} \geq 0$ and $\bar{\bar{R}} = R + \bar{\bar{D}}^T\bar{\bar{D}} > 0$.

be the reduced system's quadratic cost function in equation (4-30).

Additionally, the following is the definition of the optimal feedback control for the linear dynamic system's reduced order model:

$$\breve{u} = -\bar{\bar{R}}^{-1}\breve{B}^T\bar{\bar{M}}\breve{x}. \tag{4-35}$$

Where $\bar{\bar{M}}$ is the reduced order linear dynamic system's constant (ARE) solution, which is defined as:

$$\bar{\bar{M}}A_{11} + {A_{11}}^T\bar{\bar{M}} - \bar{\bar{M}}\,B_1\bar{\bar{R}}^{-1}{B_1}^T\bar{\bar{M}} + {C_1}^TC_1 = 0 \tag{4-36}$$

From [9], [32], $\bar{\bar{M}}$ and $\bar{\bar{M}}_{11}$ are both identical.

The Feedback Optimal Control $\breve{u}$ in equations (4-35) and $\bar{\bar{M}}$ in equations (4-36) are substituted into the reduced system (4-30), and the result is:

$$\dot{\breve{x}} = (\breve{A} - \breve{B}\bar{\bar{R}}^{-1}\breve{B}^T\bar{\bar{M}})\breve{x}$$

$$\breve{y} = \breve{C}\breve{x} + \breve{D}\breve{u} \tag{4-37}$$

where,

$$\breve{A} = A_{11}$$

$$\breve{B} = B_1 - A_{12}{A_{22}}^{-1}B_2$$

$$\breve{C} = C_1$$

$$\breve{D} = -C_2{A_{22}}^{-1}B_2$$

Assume that the pairs $(\breve{A}, \breve{C})$ and $(\breve{A}, \breve{B})$, are observable and controllable respectively as well as that the matrix $\breve{A} - \breve{B}\bar{\bar{R}}^{-1}\breve{B}^T\bar{\bar{M}}$ is stable.

Finding the minimal value of the quadratic cost function $\bar{\bar{J}}$ requires the optimal control $\breve{u}$, which can be obtained by finding a solution $\breve{x}(t)$ to the new reduced system denoted by equation (4-37).

4.2.3 Optimal Control for a Reduced System type 3

Consider the full linear time-invariant dynamic system:

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A_{11} & \frac{1}{\alpha} A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$

$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \tag{4-38}$$

The original system (4-38) is also represented by the following alternative form:

$$\dot{x} = A_{11}x + \frac{1}{\alpha} A_{12}w + B_1 u$$

$$\dot{w} = A_{21}x + A_{22}w + B_2 u \tag{4-39}$$

$$y = C_1 x + C_2 w$$

Then, set $\alpha = 0$ to obtain the following equation:

$$\alpha\dot{x} = \alpha A_{11}x + A_{12}w + \alpha B_1 u$$
$$0 = A_{12}w$$
$$w = 0$$
$$\dot{w} = A_{21}x + B_2 u$$
$$y = C_1 x$$

Here is the reduced system:

$$\dot{\breve{w}} = \breve{A}\breve{x} + \breve{B}\breve{u}$$

$$\breve{y} = \breve{C}\breve{x} \tag{4-40}$$

Where,

$$\breve{A} = A_{21}$$
$$\breve{B} = B_2$$
$$\breve{C} = C_1$$

The original system's (4-38) quadratic cost function J is as follows:

$$J \;=\; \frac{1}{2}\int_0^{\infty}(y^T y + u^T R u)dt$$

or, equivalently

$$J \;=\; \frac{1}{2}\int_0^{\infty}(x^T Q x + u^T R u)dt$$

Our optimal control u is specified as the following for the original system:

$$u = -R^{-1}B^T M x \;= -R^{-1}({B_1}^T \quad {B_2}^T)M\begin{pmatrix} x \\ w \end{pmatrix}$$

The matrix M is the solution of the following (ARE):

$$MA + A^T M - M\, B R^{-1} B^T M + Q = 0$$

Substituting the matrices A, B and C gives:

$$M\begin{pmatrix} A_{11} & \frac{1}{\alpha}A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} {A_{11}}^T & {A_{21}}^T \\ \frac{1}{\alpha}{A_{12}}^T & {A_{22}}^T \end{pmatrix}M - M\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}R^{-1}({B_1}^T \quad {B_2}^T)M + \begin{pmatrix} {C_1}^T \\ {C_2}^T \end{pmatrix}(C_1 \quad C_2) = 0 \tag{4-41}$$

To avoid unboundedness, $\alpha \to 0$ it requested the solution as follows:

$$M = \begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha {M_{12}}^T & \alpha M_{22} \end{pmatrix}.$$

So equation (41) becomes

$$\begin{aligned}
&\begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha {M_{12}}^T & \alpha M_{22} \end{pmatrix}\begin{pmatrix} A_{11} & \frac{1}{\alpha}A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} {A_{11}}^T & {A_{21}}^T \\ \frac{1}{\alpha}{A_{12}}^T & {A_{22}}^T \end{pmatrix}\begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha {M_{12}}^T & \alpha M_{22} \end{pmatrix} \\
&\qquad - \begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha {M_{12}}^T & \alpha M_{22} \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}R^{-1}({B_1}^T \quad {B_2}^T)\begin{pmatrix} M_{11} & \alpha M_{12} \\ \alpha {M_{12}}^T & \alpha M_{22} \end{pmatrix} \\
&\qquad + \begin{pmatrix} {C_1}^T \\ {C_2}^T \end{pmatrix}(C_1 \quad C_2) = 0
\end{aligned}$$

Thus, the following set of equations is obtained:

$$M_{11}A_{11} + \alpha M_{12}A_{21} + A_{11}{}^{T}M_{11} + \alpha A_{21}{}^{T}M_{12}{}^{T}$$

$$-(M_{11}B_1 + \alpha M_{12}B_2)R^{-1}(B_1{}^{T}M_{11} + \alpha B_2{}^{T}\, M_{12}{}^{T}) + C_1{}^{T}C_1 = 0$$

$$\frac{1}{\alpha}M_{11}A_{12} + \alpha M_{12}A_{22} + \alpha A_{11}{}^{T}M_{12} + \alpha A_{21}{}^{T}M_{22}$$

$$-(M_{11}B_1 + \alpha M_{12}B_2)R^{-1}(\alpha B_1{}^{T}M_{12} + \alpha B_2{}^{T}M_{22}) + C_1{}^{T}C_2 = 0$$

$$\alpha M_{12}{}^{T}A_{11} + \alpha M_{22}A_{21} + \frac{1}{\alpha}A_{12}{}^{T}M_{11} + \alpha A_{22}{}^{T}\, M_{12}{}^{T}$$

$$-(\alpha M_{12}{}^{T}B_1 + \alpha M_{22}B_2)R^{-1}(\alpha B_1{}^{T}M_{11} + \alpha B_2{}^{T}\, M_{12}{}^{T}) + C_2{}^{T}C_1 = 0$$

$$M_{12}{}^{T}A_{12} + \alpha M_{22}A_{22} + A_{12}{}^{T}M_{12} + \alpha A_{22}{}^{T}M_{22}$$

$$-(\,\alpha M_{12}{}^{T}B_1 + \alpha M_{22}B_2)R^{-1}\left(\alpha B_1{}^{T}M_{12} + \alpha B_2{}^{T}M_{22}\right) + C_2{}^{T}C_2 = 0$$

Let $\alpha = 0$ in the previous equations, the reduced system Riccati equations is given as:

$$M_{11}A_{11} + A_{11}{}^{T}M_{11} - M_{11}B_1R^{-1}B_1{}^{T}M_{11} + C_1{}^{T}C_1 = 0 \qquad (4\text{-}42)$$

$$M_{11}A_{12} = 0$$

$$A_{12}{}^{T}M_{11} = 0$$

$$M_{12}{}^{T}A_{12} + A_{12}{}^{T}M_{12} + C_2{}^{T}C_2 = 0$$

Assumption 1 [6] : $A_{11} - B_1R^{-1}B_1{}^{T}\bar{\bar{M}}_{11}$ is stable since the pair (A_{11}, B_1) is controllable and $\bar{\bar{M}}_{11}$ is the only positive semi-definite solution for equation (4-42).

If $(\breve{A}, \breve{B})$ is controllable and $(\breve{A}, \breve{C})$ is observable, then theorem 2 is satisfied by the values of M_{ij} and $\bar{\bar{M}}_{ij}, i, j = 1,2$.

Substituting $\bar{\bar{M}}_{ij}$ for M_{ij} to rewrite the feedback optimal control u:

$$u = -R^{-1}({B_1}^T \quad {B_2}^T)\begin{pmatrix} \bar{\bar{M}}_{11} & \alpha\bar{\bar{M}}_{12} \\ \alpha{\bar{\bar{M}}_{12}}^T & \alpha\bar{\bar{M}}_{22} \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}$$

$$u = -R^{-1}\left({B_1}^T\bar{\bar{M}}_{11} + \alpha{B_2}^T\bar{\bar{M}}_{12}\right)x - R^{-1}\left(\alpha{B_1}^T\bar{\bar{M}}_{12} + \alpha{B_2}^T\bar{\bar{M}}_{22}\right)w \tag{4-43}$$

The original system given by equation (4-39) has the following updated form:

$$\alpha\dot{x} = (\alpha A_{11} - \alpha B_1 R^{-1}\left({B_1}^T\bar{\bar{M}}_{11} + \alpha{B_2}^T\bar{\bar{M}}_{12}\right))x + (A_{12} - \alpha B_1 R^{-1}\left(\alpha{B_1}^T\bar{\bar{M}}_{12} + \alpha{B_2}^T\bar{\bar{M}}_{22}\right))w$$

$$\dot{w} = (A_{21} - B_2 R^{-1}\left({B_1}^T\bar{\bar{M}}_{11} + \alpha{B_2}^T\bar{\bar{M}}_{12}\right))x + (A_{22} - B_2 R^{-1}\left(\alpha{B_1}^T\bar{\bar{M}}_{12} + \alpha{B_2}^T\bar{\bar{M}}_{22}\right))w \tag{4-44}$$

If the new dynamic system is asymptotically stable and theorem 2 is fulfilled, then there is a solution $x(t)$ and $w(t)$ within the $O(\alpha)$ for this system.

Let

$$\bar{\bar{J}} = \frac{1}{2}\int_0^\infty (\bar{\bar{y}}^T\bar{\bar{y}} + \bar{\bar{u}}^T\bar{\bar{R}}\bar{\bar{u}})dt$$

or, equivalently

$$\bar{\bar{J}} = \frac{1}{2}\int_0^\infty (\bar{\bar{x}}^T\bar{\bar{Q}}\bar{\bar{x}} + 2\bar{\bar{x}}^T\bar{\bar{C}}\bar{\bar{D}}\bar{\bar{u}} + \bar{\bar{u}}^T\bar{\bar{R}}\bar{\bar{u}})dt$$

Where $\bar{\bar{Q}} = \bar{\bar{C}}^T\bar{\bar{C}} \geq 0$ and $\bar{\bar{R}} = R + \bar{\bar{D}}^T\bar{\bar{D}} > 0$.

be the reduced system's quadratic cost function in equation (4-40).

Additionally, the following is the definition of the optimal feedback control for the linear dynamic system's reduced order model:

$$\breve{u} = -\bar{\bar{R}}^{-1}\breve{B}^T\bar{\bar{M}}\breve{x} \tag{4-45}$$

where $\bar{\bar{M}}$ is the constant solution of the (ARE) for the reduced order linear dynamic system and defined as:

$$\bar{\bar{M}} A_{11} + {A_{11}}^T \bar{\bar{M}} - \bar{\bar{M}}\, B_1 \bar{\bar{R}}^{-1} {B_1}^T \bar{\bar{M}} + {C_1}^T C_1 = 0 \tag{4-46}$$

From [9], [32], $\bar{\bar{M}}$ and $\bar{\bar{M}}_{11}$ are both identical.

Then

$$\dot{\breve{w}} = (\breve{A} - \breve{B} \bar{\bar{R}}^{-1} \breve{B}^T \bar{\bar{M}}) \breve{x}$$

$$\breve{y} = \breve{C} \breve{x} \tag{4-47}$$

where,

$$\breve{A} = A_{21}$$
$$\breve{B} = B_2$$
$$\breve{C} = C_1$$

Assume that the pairs $(\breve{A}, \breve{C})$ and $(\breve{A}, \breve{B})$, are observable and controllable respectively as well as that the matrix $\breve{A} - \breve{B} \bar{\bar{R}}^{-1} \breve{B}^T \bar{\bar{M}}$ is stable.

Finding the minimal value of the quadratic cost function $\bar{\bar{J}}$ requires the optimal control $\breve{u}$, which can be obtained by finding a solution $\breve{x}(t)$ to the new reduced system denoted by equation (4-47).

Chapter Five

Numerical Examples

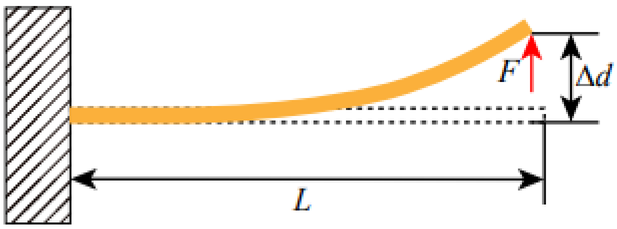
This chapter describes how to build a low order model using the singular perturbation approximation approach.

5.1 Clamped Beam Model

To test our results, we take the following example, namely clamped beam model [33], which has 348 states, (i.e., $N = 348$), which is single input single output system. According to Figure 1, the input $u(t)$, is the force F that is applied to the structure at the free end, and the output $y(t)$ is the consequent displacement Δd.

Figure 1

mechanical analysis of the single clamped cantilever beam [34]



First, we approximate the system with zero-initial conditions $(x(0) = 0)$ using the singular perturbation approximation approach.

For testing purposes, we apply the results obtained in equations (4-35) and (4-45) from sections **(**4.2.2) and (4.2.3), and compute the ℓ_2 norm, for the two types, between the optimal control U and $\breve{U}$ of the original system and reduced one respectively. The reduced model has a $r = 3$ size. The value of U is determined via the computation of the full system's solution to the Riccati equation M. Applying the techniques from section (4.2), we first determine $\bar{\bar{M}}$, the reduced system's Riccati equation solution, and then $\breve{U}$, its optimal control.

Figure (2) represents the Hankel singular values (HSVs) for the clamped beam model for type 2 when $r = 3$.

Figure (3) represents the plots of the two optimal controls U, $\breve{U}$ and the difference between them $(U - \breve{U})$ using type 2 in section (4.2.2) when $r = 3$.

Table (1) contains the values of $\|U - \breve{U}\|_{\ell_2}$ by applying the SPA to the clamped beam model using type 2 in section (4.2.2).

Figure 2

HSVs of the clamped beam model for type 2

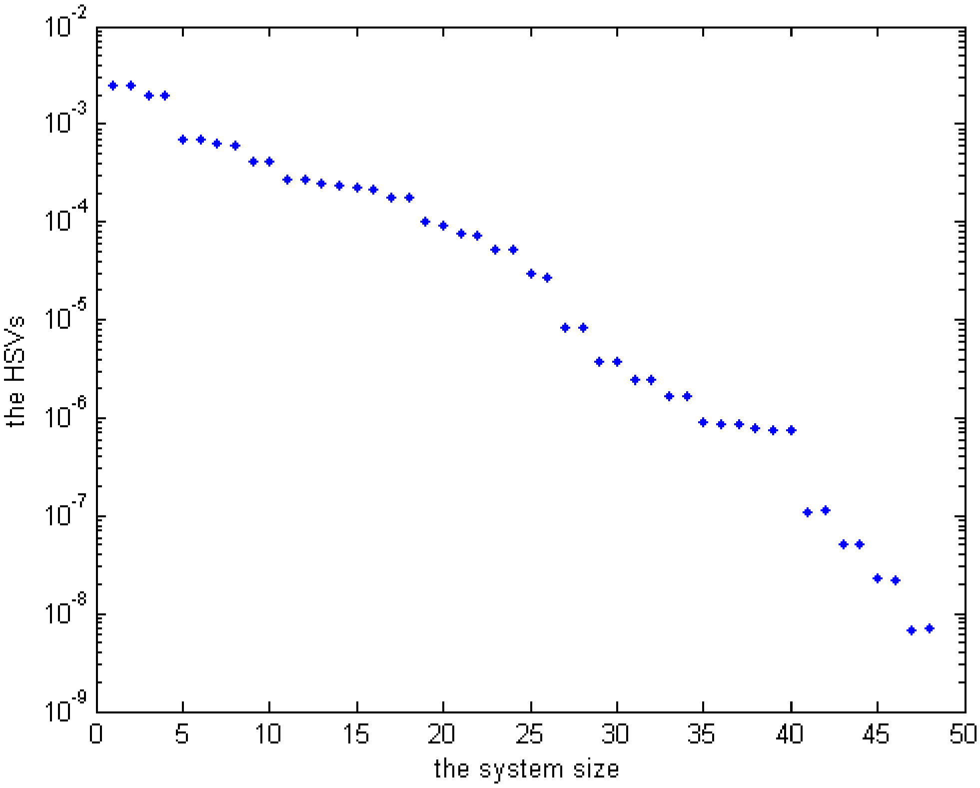


Table 1

The $\ell_2$norm of $U - \breve{U}$ of clamped beam model of type 2

r	$\|U - \breve{U}\|_{\ell_2}$
2	5.9517×10^{-15}
3	6.0476×10^{-15}
4	6.0470×10^{-15}
5	6.0470×10^{-15}
6	8.5802×10^{-15}
7	8.5802×10^{-15}
8	8.1149×10^{-15}
9	8.5005×10^{-15}
10	8.5006×10^{-15}
11	9.3912×10^{-15}

Figure 3

The optimal controls of the clamped beam model (type 2)(r=3)

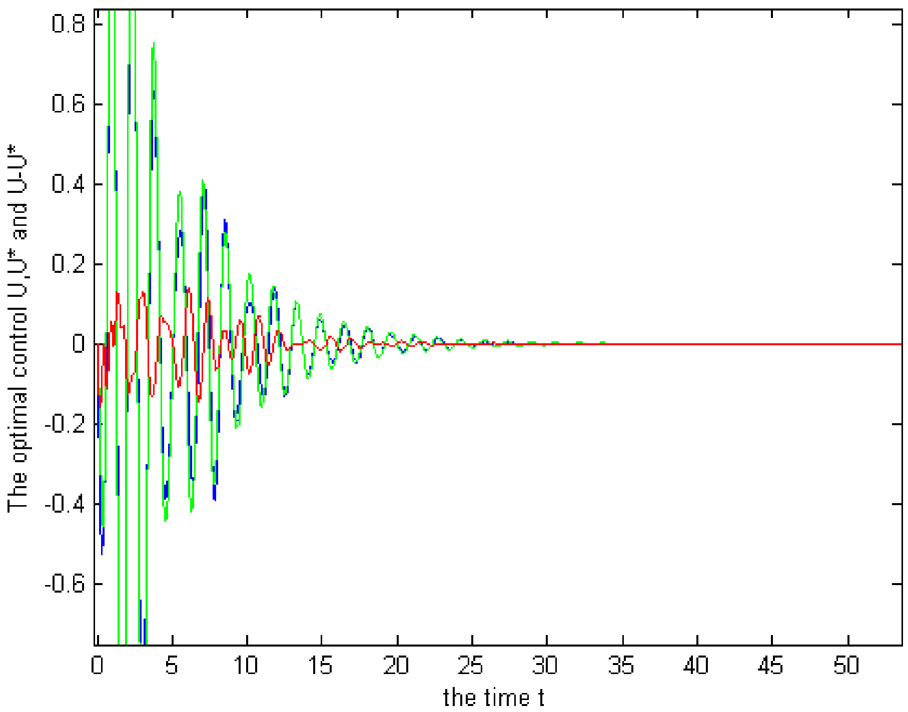


Figure (4) represents the plots of the two optimal controls U, $\breve{U}$ and the difference between them $(U - \breve{U})$ using type 3 in section (4.2.3) when $r = 3$.

Figure (5) represents the Hankel singular values (HSVs) for the clamped beam model for type 3 when $r = 3$.

Table (2) contains the values of $\|U - \breve{U}\|_{\ell_2}$ by applying the SPA to the clamped beam model using type 3 in section (4.2.3).

Table 2

The ℓ_2 norm of $U - \breve{U}$ of clamped beam model of type 3

r	$\|U - \breve{U}\|_{\ell_2}$
2	0.0071
3	0.0620
4	0.0033
5	0.0131
6	0.0011
7	0.0048
8	$7.2999e-005$
9	$5.4983e-004$
10	$7.3210e-005$
11	$2.7392e-004$

Figure 4

The optimal controls of the clamped beam model (type 3)(r=3)

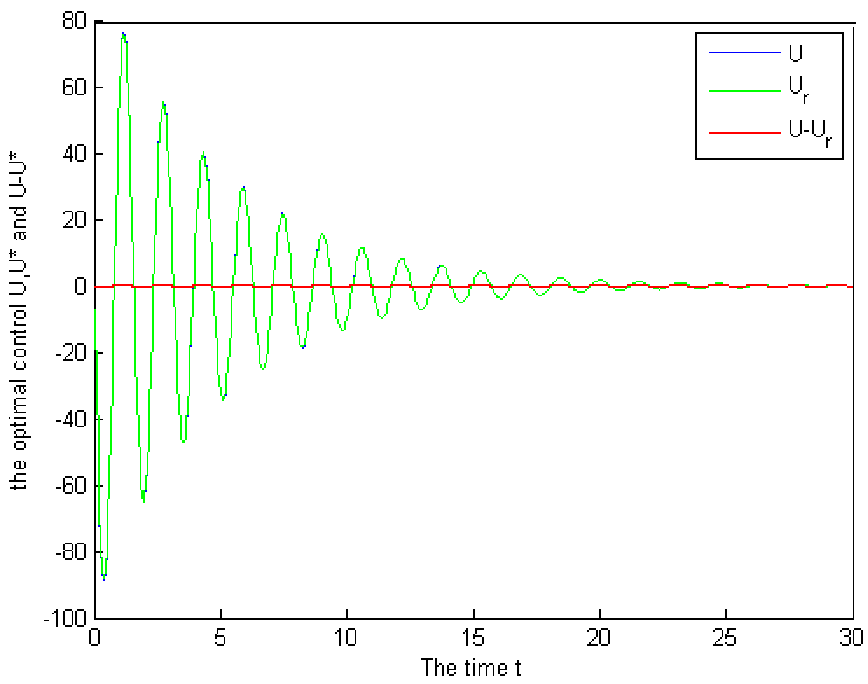
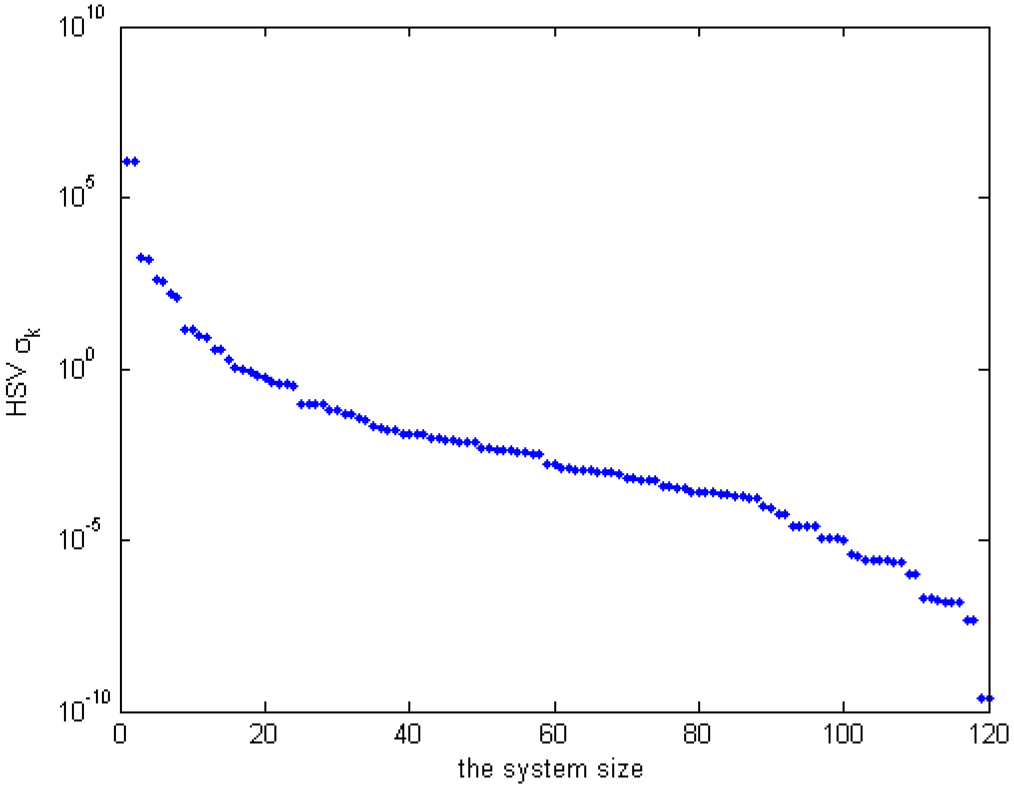


Figure 5

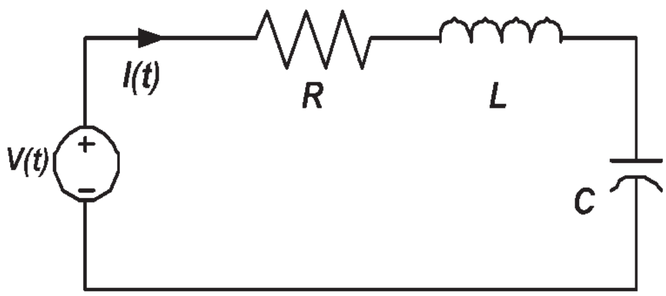
HSVs of the clamped beam model for type 3



5.2 RLC Circuit

Figure 6

RLC series circuit



An RLC circuit is an electrical circuit in which there is a resistor (R), an inductor (L), and a capacitor (C). these may be connected in series or in parallel. The RLC circuit in Figure (6) has a current i which varies with time t when subject to a step input of V and is described by:

$$v = iR + v_L + v_C$$

With $i = C\frac{dv_C}{dt}$ and $v_L = L\frac{di}{dt}$

So

$$v = iR + L\frac{di}{dt} + v_C$$

Then

$$\frac{di}{dt} = \frac{1}{L}v - \frac{R}{L}i - \frac{1}{L}v_C$$

And

$$\frac{dv_C}{dt} = \frac{i}{C}$$

We now that $x_1 = v_C$, $x_2 = i$, $y_1 = x_1 = v_C$, $y_2 = Ri = Rx_2$ and $u_1 = v$ then we get:

$$\dot{x}_1 = \frac{dv_C}{dt} = \frac{i}{C} = \frac{x_2}{C}$$

And

$$\dot{x}_2 = \frac{1}{L}u - \frac{R}{L}x_2 - \frac{1}{L}x_1$$

The last two equations are the differential equations of the RLC circuit and described in the matrix form as:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} u$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Then the state space representation of the RLC circuit is:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Where

$A = \begin{pmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$

5.2.1 The singular perturbation approximation method on the RLC circuit

In this section we show all results obtained by the SPA method to determine the optimal control for RLC circuit.

For type 2:

Figure (7) represents the plots of the two outputs Y, $\breve{Y}$ and the difference between them $(Y - \breve{Y})$ when $r = 3$.

Figure (8) represents the Hankel singular values (HSVs) for the RLC circuit when $r = 3$.

Table (3) contains the values of $\|Y - \breve{Y}\|_\infty$ by applying the SPA to the RLC circuit.

Figure 7

The outputs of the RLC circuit for SPA

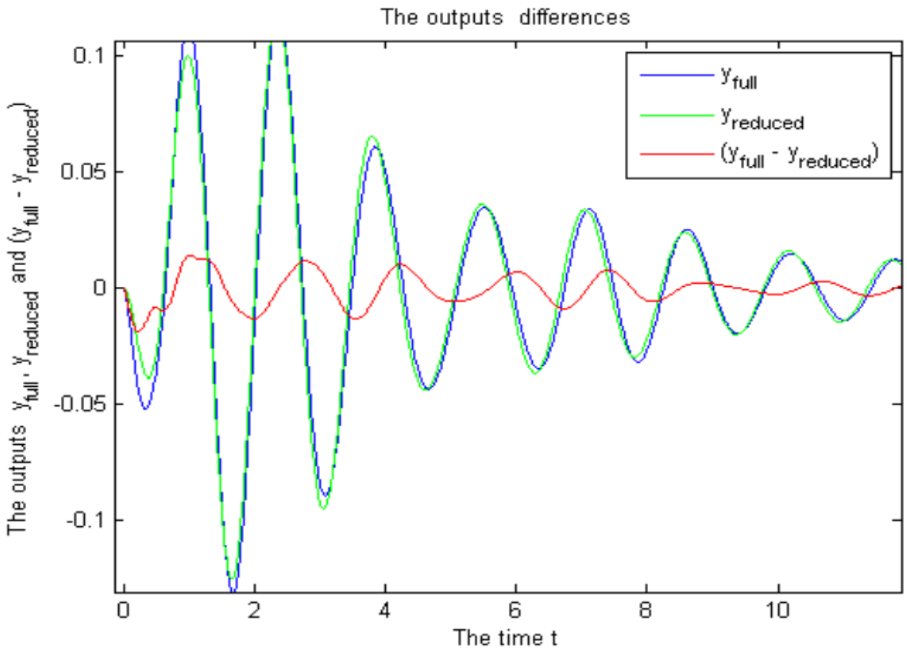


Figure 8

HSVs of the RLC circuit for SPA

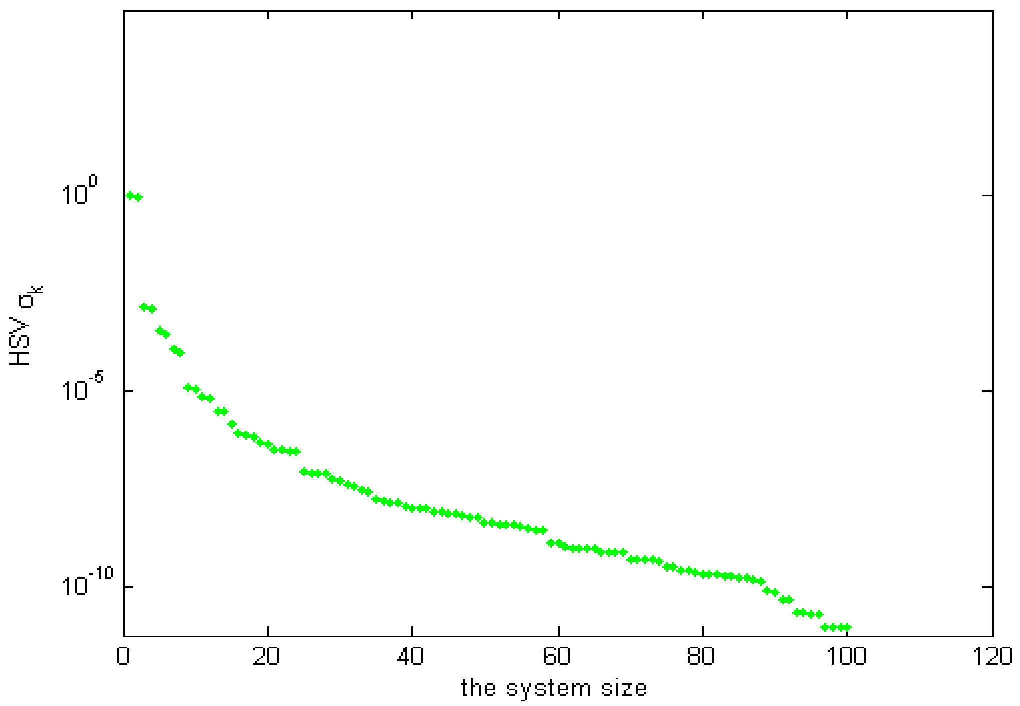


Table 3

The H_∞ norm of $Y - \breve{Y}$ of RLC circuit for SPA

$\boldsymbol{r}$	$\|\boldsymbol{Y} - \breve{\boldsymbol{Y}}\|_\infty$
2	3.7147×10^{-23}
3	1.1278×10^{-20}
4	2.3938×10^{-23}
5	1.7610×10^{-21}
6	2.6064×10^{-23}
7	7.7775×10^{-22}
8	1.3447×10^{-23}
9	3.3785×10^{-22}
10	1.0582×10^{-23}
11	9.6521×10^{-23}

5.3 Conclusion

In this thesis, new techniques for finding the optimal control for a closed loop system were investigated. Singular perturbation approximation and balanced truncation methods were used to obtain the reduced optimal control. These methods have been successfully applied to systems with zero-initial conditions.

The quadratic cost function J is minimized by the linear quadratic regulator LQR. The calculations were validated by numerical experiments, illustrating that the reduced order systems can be used to approximate the optimal control of the original system. Finally, error bounds for two types of SPA were derived, and numerical results for a specific example were presented to support the theory. It has been noticed that although the two types result in smaller errors at low frequencies, type two is more efficient than type three. This is due to the fact that the ℓ_2 norm of type two converges to zero faster than that of type three.

References

[1] R. Sargent, “Optimal control,” Journal of Computational and Applied Mathematics, vol. 124, 1-2, pp. 361–371, 2000.

[2] D. Rowell, Analysis and Design of Feedback Control Systems: State-Space Representation of LTI Systems, Available: http://web.mit.edu/2.14/www/ Handouts/StateSpace.pdf.

[3] B. N. Datta, *Numerical methods for linear control systems: Design and analysis / Biswa Nath Datta*, Amsterdam, London, Elsevier Academic Press, 2004.

[4] A. C. Antoulas, *Approximation of large-scale dynamical systems*, Philadelphia, Society for Industrial and Applied Mathematics, 2005.

[5] K. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal control*, Upper Saddle River, N.J., Prentice Hall; London : Prentice-Hall International, 1996.

[6] B. C. Moore, Ed., *Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction*, 1981.

[7] Y. I. LIU and B. D. O. ANDERSON, “Singular perturbation approximation of balanced systems,” International Journal of Control, vol. 50, no. 4, pp. 1379–1405, 1989.

[8] Fatmawati *et al.*, “Balanced truncation for unstable infinite dimensional systems via reciprocal transformation,” Int. J. Control Autom. Syst., vol. 9, no. 2, pp. 249–257, 2011.

[9] A. Daraghmeh and N. Qatanani, “Archives of Control Sciences,” 2019.

[10] A. Daraghmeh, “Model Order Reduction of Linear Control Systems: Comparison of Balance Truncation and Singular Perturbation Approximation with Application to Optimal Control,” Freie Universität Berlin, Berlin, 2016.

[11] S. P. Sethi, *Optimal Control Theory*, Cham, Springer International Publishing, 2021.

[12] D. E. Kirk, *Optimal control theory: An introduction / Donald E. Kirk*, Mineola, N.Y., Dover Publications, 2004.

[13] P. Benner and A. Schneider, Eds., *Balanced Truncation Model Order Reduction for LTI Systems with many Inputs or Outputs*, 2010.

[14] S. Aniaku, “Review of Controllability Results of Dynamical System,” IOSR Journal of Mathematics (IOSR-JM), vol. 13, 4 Ver. II, 2017.

[15] C.-T. Chen, *Linear system theory and design,* 3rd ed., New York, Oxford, Oxford University Press, 1999.

[16] A. Daraghmeh, N. Qatanani, and C. Hartmann, “Optimal control of linear systems with balanced reduced-order models: Perturbation approximations,” Applied Mathematics and Computation, vol. 337, pp. 119–136, 2018.

[17] D. N. Burghes and A. Graham, *Control and optimal control theories with applications*, Chichester, Horwood, 2004.

[18] J. Scherpen, “Balancing for nonlinear systems,” Systems & Control Letters, vol. 21, no. 2, pp. 143–153, 1993.

[19] P. Benner, P. Goyal, and I. P. Duff, “Gramians, Energy Functionals and Balanced Truncation for Linear Dynamical Systems with Quadratic Outputs,” 2019.

[20] M. Condon and R. Ivanov, “Nonlinear systems – algebraic gramians and model reduction,” COMPEL - The international journal for computation and mathematics in electrical and electronic engineering, vol. 24, no. 1, pp. 202–219, 2005.

[21] J. J. D'Azzo, C. H. Houpis, and S. N. Sheldon, *Linear control system analysis and design with MATLAB,* 5th ed., New York, Marcel Dekker, 2003.

[22] P. Rabiei and M. Pedram, “Model order reduction of large circuits using balanced truncation,” in 237-240 vol.1.

[23] S. R. Desai and R. Prasad, Eds., *Balanced Truncation Method of Model Order Reduction on TIDSP*, 2014.

[24] C. Hartmann, V.-M. Vulcanov, and C. Schütte, “Balanced Truncation of Linear Second-Order Systems: A Hamiltonian Approach,” Multiscale Model. Simul., vol. 8, no. 4, pp. 1348–1367, 2010.

[25] R. Saragih and F. Armawi, “Singular Perturbation Approximation of Balanced Infinite-Dimensional Systems,” IJCA, vol. 6, no. 5, pp. 409–420, 2013.

[26] A. Daraghmeh and N. Qatanani, “Error Bound for Non-Zero Initial Condition Using the Singular Perturbation Approximation Method,” Mathematics, vol. 6, no. 11, p. 232, 2018.

[27] K. Fernando and H. Nicholson, Eds., *Reciprocal transformations in balanced model-order reduction*, 1983.

[28] G. Muscato, G. Nunnari, and L. Fortuna, “Singular perturbation approximation of bounded real balanced and stochastically balanced transfer matrices,” International Journal of Control, vol. 66, no. 2, pp. 253–270, 1997.

[29] A. Daraghmeh, C. Hartmann, and N. Qatanani, “Balanced model reduction of linear systems with nonzero initial conditions: Singular perturbation approximation,” Applied Mathematics and Computation, vol. 353, pp. 295–307, 2019.

[30] R. M. Murray, *Optimization-Based Control: Control and Dynamical Systems*. California Institute of Technology, 2009.

[31] P. Kokotovic and R. Yackel, “Singular perturbation of linear regulators: Basic theorems,” IEEE Trans. Automat. Contr., vol. 17, no. 1, pp. 29–37, 1972.

[32] P. V. Kokotovic, R. E. O'Malley, and P. Sannuti, “Singular perturbations and order reduction in control theory — An overview,” Automatica, vol. 12, no. 2, pp. 123–132, 1976.

[33] A. C. Antoulas, D. C. Sorensen, and S. Gugercin, “A survey of model reduction methods for large-scale systems,” in *Structured Matrices in Mathematics, Computer Science, and Engineering I,* vol. 280, V. Olshevsky, Ed. Providence, Rhode Island: American Mathematical Society, 2001, pp. 193–219.

[34] Y. Gao *et al.,* “Design and Implementation of a Dual-Mass MEMS Gyroscope with High Shock Resistance,” Sensors (Basel, Switzerland), vol. 18, no. 4, 2018.

جـامعـــة النجاح الوطنية

كليـــة الدراســـــات العليـــا

التحكم الأمثل للأنظمة الديناميكية الخطية بشروط أولية صفرية

إعداد

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قدمت هذه الرسالة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات المحوسبة، من كلية الدراسات العليا، في جامعة النجاح الوطنية، نابلس – فلسطين.

2023

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الملخص

تبحث هذه الرسالة في التحكم الأمثل في الأنظمة الديناميكية الخطية بشروط أولية صفرية.

ناقشت الدراسة تمثيل المتجهات الفضائي ومعادلة ليابونوف، بالإضافة إلى قابلية التحكم وقابلية الملاحظة والجراميان الخاص بكل منهما.

تم طرح متحكم تربيعي خطي للوصول إلى تحكم أمثل يقلل دالة التكلفة التربيعية من خلال تطبيق حل مقارب لمعادلة ريكاتي الجبرية ذات الصلة.

وقد أظهرت النتائج العددية أن الحالة الثانية من طريقة تقريب الاضطراب المفرد أعطت تحكم أمثل أفضل منه في الحالة الثالثة عند تطبيقها على نموذج شعاع مثبت وعلى دائرة الرنين التوافقي.

الكلمات المفتاحية: الأنظمة الديناميكية، تقليص درجة النظم، تقريب الاضطراب المفرد، التحكم الأمثل.