An-Najah National University

Faculty of Graduate Studies

COMPUTATIONAL METHODS FOR SOLVING VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

By

Masa Husam Saeed Desuqi

Supervisor

Prof. Naji Qatanani

This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics, Faculty of Graduate Studies, An-Najah National University, Nablus - Palestine.

2022

COMPUTATIONAL METHODS FOR SOLVING VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

By

Masa Husam Saeed Desuqi

This Thesis was Defended Successfully on 20/10/2022 and approved by

Dr. Naji Qatanani Supervisor	Signature
Dr. Rania Wannan External Examiner	Signature
Dr. Fatima Aqel Internal Examiner	Signature

Dedication

I dedicate this thesis to my beloved Palestine, my parents, my husband and my brother. Without their patience, understanding, support and most of all love, this work would not have been possible.

Acknowledgement

At the beginning, thanks God, I am grateful to have completed this thesis. I am heartily thankful to my Supervisor, Prof. Dr. Naji Qatanani, whose encouragement, guidance and support from the beginning to end level enabled me to develop and understand the topic of my research.

My thanks and appreciation goes to my thesis committee members Dr. Fatima Alzahra Aqel and Dr. Rania Wannan for their support and valuable remarks.

Declaration

I, the undersigned, declare that I submitted the thesis entitled:

COMPUTATIONAL METHODS FOR SOLVING VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

I declare that the work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's Name: ______________________________

Signature: ______________________________

Date: ______________________________

List of Contents

List of Tables

List of Figures

List of Appendices

COMPUTATIONAL METHODS FOR SOLVING VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

By
Masa Husam Saeed Desuqi
Supervisor
Prof. Naji Qatanani

ABSTRACT

Background: Integro-Differential Equations (IDEs), one of the most important mathematical tools in both pure and applied mathematics, arise in many physical problems such as wind ripple in the desert, nano-hydrodynamics, population growth model, glass-forming process..

They have motivated a huge amount of research in recent years. Many researchers have developed numerical schemes for solving these IDEs.

Aim: In this work, the researcher proposed three numerical schemes, namely, the Taylor collocation method, Legendre polynomials method and the Haar wavelets method, to approximate the solution of Volterra Integro-Differential Equations (VIDEs).

Material and method: These three numerical methods have been applied in the form of algorithms, and Maple software has been developed/used to solve some numerical examples.

Results: The numerical results showed that the convergence and accuracy of the aforementioned methods were in good agreement with the analytical solution. Comparison of numerical results, mentioned in tables and figures, showed clearly that the Haar wavelets method provides more accurate results and is, therefore, more effective than other methods.

Keywords: Volterra integro-differential equations; Taylor collocation; Legendre polynomials; Haar wavelets.

Introduction

Importance of Integral Equations:

the subject of IDEs have attracted the attention of many scientists and researchers over the past few years due to their wide range of applications like wind ripple, hydro-dynamics, glass-formation, model of population growth and various models in physics, engineering and medicine, namely the mathematical modelling of epidemics, particular when the models contains age-structure or describe spatial epidemics [5, 6].

Literature Review

Many numerical schemes for solving VIDEs have been constructed and implemented by many researches. For example, Karamete et al. [19] used collocation method based on Taylor expansion to solve VIDEs. Khater et al. [20] implemented the Legendre polynomials method to approximate the solution of VIDEs subject to initial conditions. Ali [2] has applied the Haar wavelets approach to obtain an approximate solution to VIDEs. Draidi et al. [12] used the product Nystrom in parallel with the sinc-collocation scheme to solve integral equations with Carleman kernel. Hamaydi et al. [16] solved fuzzy integral equations using variational iteration and Taylor expansion techniques. Moreover, Fazeli et al. [14] suggested several numerical schemes to approximate the solution of VIDEs. Other numerical techniques for solving VIDEs are: variational iteration [30], Walsh expansion series [26], Chebyshev collocation [1], Nyström method [22], differential transform [11], homotopy [25], power series [3] and finite difference [10]. Burton [7, 8] has investigated in the 1980s some stability results for the VIDEs. Zhang [31] has also presented some stability results for VIDEs. Tunc [28] and Staffans [27] proposed a new stability results based on Lyapunov functional for VIDEs.

In this work, we suggest three numerical techniques to solve VIDEs, namely, Taylor collocation method, Legendre polynomials and Haar wavelets method. The VIDE under consideration has the form:

$$Y^{(n)}(x) = r(x) + \int_{a}^{x} F(x,t)Y(t)dt$$

where

$$Y^{(n)}(x) = \frac{d^n Y}{dx^n}\ , \qquad n \in N$$

subject to the initial conditions:

$$Y^{(s)} = a_s\ ,\ \ s = 0, 1, 2, \dots, (n-1).$$

The kernel $F(x,t)$ and the function $r(x)$ are given. The unknown function $Y(x)$ is to be determined.

A major objective is to compare between these numerical techniques in approximating the solution of the VIDE by solving some numerical examples.

We organize this work as follows: chapter one deals with some general aspects of VIDEs together with their solvability. In chapter two, we address all the aforementioned methods, namely, Taylor collocation, Legendre polynomials and Haar wavelets methods. We conclude chapter three by solving some VIDEs with known exact solution by the aforementioned algorithms.

Chapter One

Mathematical Preliminaries

In this chapter we introduce some basic concepts concerning integral equations and integro-differential equations.

Definition 1.1 [23]

An integral equation is any equation in which the unknown function $Y(x)$ appear inside an integral.

A standard integral equation can be considered in general as follows:

$$(x) = r(x) + \lambda \int_{g(x)}^{h(x)} F(x,t)Y(t)dt \qquad (1.1)$$

where $Y(x)$ is the unknown function, $g(x)$ and $h(x)$ are limits of integration that may both be variables, constants, or mixed, λ is a constant parameter, $F(x,t)$ is a known function of two variables x and t called the kernel of the integral equation and $r(x)$ is a given function.

Definition 1.2 [29]

An IDE is an equation in which the unknown function $Y(x)$ appears inside an integral and is given in the form of derivatives.

A standard IDE can be considered in general as follows:

$$Y^{(n)}(x) = r(x) + \lambda \int_{g(x)}^{h(x)} F(x,t)Y(t)dt \qquad (1.2)$$

where

$$Y^{(n)}(x) = \frac{d^nY}{dx^n}\,, \qquad n = 1,2,\ldots$$

subject to the initial conditions:

$$Y^{(s)}(0) = a_s\,, \qquad s = 0, 1, 2, \dots, (n-1),$$

1.1 Classification of Integro-Differential Equations [29]

1. Volterra integro-differential equation

The VIDE appears in the form:

$$Y^{(n)}(x) = r(x) + \lambda \int_a^x F(x,t)Y(t)dt \tag{1.3}$$

where the upper limit of integration is variable.

2. Fredholm integro-differential equation

The Fredholm integro-differential equation appears in the form:

$$Y^{(n)}(x) = r(x) + \lambda \int_a^b F(x,t)Y(t)dt \tag{1.4}$$

where a, b are constants.

3. Volterra-Fredholm integro-differential equation

The Volterra-Fredholm integro-differential equation appear in two forms, namely:

$$Y^{(n)}(x) = r(x) + \lambda_1 \int_a^x F_1(x,t)Y(t)dt + \lambda_2 \int_a^b F_2(x,t)Y(t)dt \tag{1.5}$$

and

$$Y^{(n)}(x,t) = r(x,t) + \lambda \int_0^t \int_\Omega H\left(x,t,\xi,\tau,\mathrm{Y}(\xi,\tau)\right) d\xi d\tau\,, x,t \in \Omega \times [0,T] \tag{1.6}$$

where $r(x,t)$ and $H\ (x,t,\xi,\tau,y(\xi,\tau))$ are analytic functions on $D=\Omega\times[0,T]$ and Ω is a closed subset of $R^n, n=1,2,3$.

1.2 Linearity

The IDE

$$Y^{(n)}(x)=r(x)+\lambda\int_{g(x)}^{h(x)}F(x,t)Y(t)dt,$$

is said to be linear if the exponent of the unknown function $Y(x)$ inside the integral sign is one and the equation does not contain nonlinear functions of $Y(x)$, otherwise, the equation is called nonlinear.

1.3 Homogeneity

The IDE

$$Y^{(n)}(x)=r(x)+\lambda\int_{g(x)}^{h(x)}F(x,t)Y(t)dt,$$

is said to be homogeneous if the function $r(x)$ is identically zero, otherwise, it is called nonhomogeneous.

1.4 Singularity

When one or both limits of integration become infinite or when the kernel $F(x,t)$ becomes infinite at one or more points within the range of integration, then the equation is said to be singular.

For example,

$$Y^{(n)}(x)=r(x)+\lambda\int_{-\infty}^{\infty}F(x,t)Y(t)dt$$

is a singular IDE

Moreover, if the kernel is of the form

$$F(x,t) = \frac{L(x,t)}{x-t} \tag{1.7}$$

where $L(x,t)$ is differentiable function $a \leq x \leq b,\ a \leq t \leq b$ with $L(x\ ,t\ ) \neq 0$, then the IDE is said to be singular with Cauchy kernel where

$$F(x,t) = \int_a^b \frac{L(x,t)}{x-t}\, r(t)d(t)$$

is understood in the sence of Cauchy Principal Value (CPV) and the notation P.V$\int_a^b \frac{L(x,t)}{x-t} d(t)$ is usually used. Thus

$$\mathrm{P.V.}\int_a^b \frac{L(x,t)}{x-t} d(t) = \lim_{x\to\varepsilon}\left\{\int_a^{x-\varepsilon} \frac{L(x,t)}{x-t} d(t) + \int_{x+\varepsilon}^b \frac{L(x,t)}{x-t} dt\right\}.$$

Weakly singular integro-differential equation

If the kernel is of the form

$$F(x,t) = \frac{L(x,t)}{|x-t|^{\alpha}} \tag{1.8}$$

where $L(x,t)$ is bounded $i.e.$ several times continuously differentiable $a \leq x \leq b$ and $a \leq t \leq b$ with $L(x\ ,t) \neq 0$ and a is a constant such that $0 < \alpha < 1$.

For example,

$$Y^{(n)} = \lambda \int_0^x \frac{1}{|x-t|^{\alpha}} Y(t)dt\,, \qquad 0 < a < 1$$

is a singular IDE with a weakly singular kernel.

1.5 System of VIDEs [18]

A system of VIDE is one of the form:

$$Y_i^{(n)}(x) = r_i(x) + \int_a^x \left(\sum_{j=1}^{N} F_{ij}(x,t) Y_j(t) \right) dt \,, a \le x \le b, 1 \le i \le N \qquad (1.9)$$

subject to the initial conditions

$$Y_i^{(s)} = a_{is} \,, \;\; i = 1,2,3,\dots,N \,, \;\; s = 0,1,2,\dots,(n-1) \qquad (1.10)$$

The kernels $F_{ij}(x,t)$ and the functions $r_i(x)$ are given real valued functions and the unknown functions $Y_i(x)$ are to be determined.

1.6 System of Fredholm Integro-Differential Equations

A system of Fredholm integro-differential equation is one of the form:

$$Y_i^{(n)}(x) = r_i(x) + \int_a^b \left(\sum_{j=1}^{N} F_{ij}(x,t) Y_j(t) \right) dt \,, a \le x \le b, 1 \le i \le N \qquad (1.11)$$

subject to the initial conditions

$$Y_i^{(s)} = a_{is} \,, \;\; i = 1,2,3,\dots,N \,, \;\; s = 0,1,2,\dots,(n-1) \qquad (1.12)$$

The kernels $F_{ij}(x,t)$ and the functions $r_i(x)$ are given real valued functions and the unknown functions $Y_i(x)$ are to be determined.

1.7 Kinds of Kernels

1. Separable kernel

A kernel $F(x,t)$ is said to be separable or (degenerate) if it can be expressed in the form

$$F(x,t) = \sum_{i=1}^{n} g_i\,(x) h_i(t)$$

where the functions $g_i(x)$ and the functions $h_i(t)$ are linearly independent.

2. Symmetric (or Hermitian) kernel

A complex-valued function $F(x,t)$is called symmetric (or Hermitian) if

$$F(x\,,t\,)=F^{*}(x\,,t\,)$$

where the asterisk denotes the complex conjugate. For a real kernel, we have

$$F(x,t)=F(t,x).$$

3. Cauchy kernel

If the Kernel $F(x,t)$ is of the form

$$F(x,t)=\frac{L(x,t)}{x-t}$$

where $L(x,t)$ is a differentiable function of (x,t) with $L(x,t)\neq 0$, then the integral equation is said to be a singular equation with Cauchy kernel.

4. Hilbert-Schmidt Kernel

The kernel $F(x,t)$, for each set of values x and t in the square $a\leq x\leq b$ and $a\leq t\leq b$**,** has the form:

$$\int_a^b\int_a^b |F(x,t)|^2\,dx\,dt\ <\infty$$

also for each value of x in $a\leq x\leq b$**,** is

$$\int_a^b |F(x,t)|^2\ \ dt\ <\infty$$

and for each value of t in $a\leq t\leq b$, is

$$\int_a^b |F(x,t)|^2\ \ dx\ <\infty$$

has a finite value, then we call the kernel $F(x,t)$ a regular kernel.

5. Abel's Kernel

If the kernel $F(x,t)$ is of the form

$$F(x,t) = \frac{L(x,t)}{|x-t|^{\alpha}}$$

where $0 < \alpha < 1$ and $L(x,t)$ is assumed to be several times continuously differentiable function, then the integral equation is called Abel's equation.

6. Skew-symmetric kernel

The kernel is of the form

$$F(x,t) = -F(t,x)$$

Chapter Two

Numerical Methods for Solving Volterra Integro-Differential Equations

There are many numerical methods available for solving VIDEs s. In this Chapter, we will address the following methods: Taylor collocation method, Legendre polynomials and Haar Wavelets method.

2.1 Taylor Collocation Method [19]

Consider the m^{th}-order linear VIDE

$$\sum_{l=0}^{m} P_l(x)Y^{(l)}(x) = r(x) + \lambda \int_a^x F(x,t)Y(t)dt \tag{2.1}$$

where the known functions $P_l(x)$, $r(x)$, $F(x,t)$ are defined on $a \leq x,t \leq b$ and λ is a real parameter and $Y(x)$ is the unknown function subject to the initial conditions:

$$\sum_{j=0}^{m-1} [a_{ij}Y^{(j)}(a) + b_{ij}Y^{(j)}(b) + c_{ij}Y^{(j)}(c)] = \lambda_i \, , i = 0,1, \dots m-1 \tag{2.2}$$

where $a \leq c \leq b,$ and a_{ij}, b_{ij}, c_{ij} and λ_i are some appropriate constants.

The approximate solution is expressed in the truncated Taylor series as:

$$Y(x) = \sum_{n=0}^{N} \frac{Y^{(n)}(c)}{n!}(x-c)^n \, ; \quad a \leq x \leq b \tag{2.3}$$

where $Y^{(n)}(c)$ are the Taylor coefficients to be determined and N is chosen to be any positive integer such that $N \geq m.$ Then the solution (2.3) of equation (2.1) can be expressed in the matrix form

$$[Y(x)] = \mathrm{XM_0A}$$

where

$$X = [1 \quad x-c \quad (x-c)^2 \ \dots \ (x-c)^N]$$
$$A = [Y^{(0)}(c) \quad Y^{(1)}(c) \quad Y^{(2)}(c) \dots Y^{(N)}(c)]^t$$

and

$$M_0 = \begin{bmatrix} \frac{1}{0!} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{1!} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{N!} \end{bmatrix}$$

To obtain such a solution, we can use the following matrix method, known as Taylor Collocation method.

Firstly, we substitute the Taylor collocation points defined by

$$x_i = a + i\frac{b-a}{N} \ ; \quad i = 0, 1, \dots, N \ ; \quad x_0 = a \ , \ \ x_N = b \tag{2.4}$$

into equation (2.1) to obtain

$$\sum_{l=0}^{m} P_l(x_i)Y^{(l)}(x_i) = r(x_i) + \lambda \mathrm{G}(x_i) \tag{2.5}$$

such that

$$\mathrm{G}(x_i) = \int_a^{x_i} F(x_i, t)Y(t)dt$$

Then we can write system (2.5) in the matrix form as

$$\mathrm{P_0Y^{(0)}} + \mathrm{P_1Y^{(1)}} + \cdots + \mathrm{P_mY}^{(m)} = \sum_{s=0}^{m} \mathrm{P_sY}^{(s)} = \mathrm{R} + \lambda\mathrm{G} \tag{2.6}$$

where

$$P_s = \begin{bmatrix} P_s(x_0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_s(x_N) \end{bmatrix}$$

$$R = \begin{bmatrix} r(x_0) \\ r(x_1) \\ \vdots \\ r(x_N) \end{bmatrix}$$

$$Y^{(s)} = \begin{bmatrix} Y^{(s)}(x_0) \\ Y^{(s)}(x_1) \\ \vdots \\ Y^{(s)}(x_N) \end{bmatrix}$$

and

$$G = \begin{bmatrix} G(x_0) \\ G(x_1) \\ \vdots \\ G(x_N) \end{bmatrix}$$

we assume that the s^{th} derivative of the function $Y(x)$ in (2.3) with respect to x has the truncated Taylor series expansion defined by equation (2.3), so

$$Y^{(s)}(x_i) = \sum_{n=0}^{N} \frac{Y^{(n)}(c)}{(n-s)!}(x_i - c)^{n-s} \quad , \quad a \leq x \leq b$$

where $Y^{(s)}(x)$ $(s = 0,1, \ldots\ldots N)$ are Taylor coefficients; clearly $Y^{(0)}(x) = Y(x)$. Then substituting the Taylor collocation points into this expression, we get the matrix form

$$\left[Y^{(s)}(x_i)\,\right] = \mathrm{X}_{x_i}\mathrm{M}_\mathrm{s}\mathrm{A} \qquad (\mathrm{s} = 0,1,\ldots,\mathrm{N}) \tag{2.7}$$

or the matrix equation

$$\mathrm{Y}^{(s)} = \mathrm{X}_{x_i}\mathrm{M}_\mathrm{s}\mathrm{A} \tag{2.8}$$

where

$$X_{x_i}=\begin{bmatrix}X_{x_0}\\ X_{x_1}\\ \vdots\\ X_{x_N}\end{bmatrix}=\begin{bmatrix}(x_0-c)^0 & (x_0-c)^1 & \dots & (x_0-c)^N\\ (x_1-c)^0 & (x_1-c)^1 & \dots & (x_1-c)^N\\ \vdots & \vdots & \ddots & \vdots\\ (x_N-c)^0 & (x_N-c)^1 & \dots & (x_N-c)^N\end{bmatrix}$$

and

$$M_s=\begin{bmatrix}0 & 0 & \dots & \frac{1}{0!} & 0 & \dots & 0\\ 0 & 0 & \dots & 0 & \frac{1}{1!} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0 & 0 & \dots & \frac{1}{(N-s)!}\\ 0 & 0 & \dots & 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0 & 0 & \dots & 0\end{bmatrix}$$

Then we can write the matrix equation (2.6) as

$$\left(\sum_{s=0}^{m} P_s \mathrm{X}_{x_i} M_s\right)A=\mathrm{R}+\lambda\mathrm{G} \qquad (2.9)$$

The kernel $F(x,t)$ is expanded in Taylor series in the form

$$F(x,t)=\sum_{n=0}^{N}\sum_{m=0}^{N} f_{nm}(x-c)^n(t-c)^m$$

$$f_{nm}=\frac{1}{n!\,m!}\frac{\partial^{n+m}F(0,0)}{\partial x^n\partial t^m}\bigg|_{(x=c,t=c)}$$

The matrix representation of $F(x,t)$ can be given as

$$[F(x,t)]=\mathrm{XFT}^{\mathrm{T}} \qquad (2.10)$$

where

$$\mathrm{X} = [1 \;\; (x-c)(x-c)^2 \ldots\ldots \;\; (x-c)^N]$$

$$\mathrm{T} = [1 \;\; (t-c)(t-c)^2 \ldots\ldots \;\; (t-c)^N]$$

$$F = \begin{bmatrix} f_{00} & f_{01} & \cdots & f_{0N} \\ f_{10} & f_{11} & \cdots & f_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{N0} & f_{N1} & \cdots & f_{NN} \end{bmatrix}$$

Moreover, the matrix representation of $Y(x)$ and $Y(t)$ are

$$[Y(x)] = \mathrm{XM_0A}\,, \;\; [Y(t)] = \mathrm{TM_0A} \tag{2.11}$$

Substituting the equations (2.10) and (2.11) into $G(x_i)$, we get

$$[G(x_i)] = X_{x_i} F J_{x_i} M_0 A$$

where

$$X_{x_i} = [1 \quad x_i - c \quad (x_i - c)^2 \;\ldots\; (x_i - c)^N]$$

and

$$J_{x_i} = [j_{nm}] = \int_a^{x_i} T^T T dt$$

$$j_{nm} = \frac{(x_i - c)^{m+n+1} - (a-c)^{m+n+1}}{m+n+1}\Bigg|_{\mathrm{n,m=0,1,\ldots,N}}$$

$$J_{x_i} = \begin{bmatrix} j_{00}(x_i) & j_{01}(x_i) & \cdots & j_{0N}(x_i) \\ j_{10}(x_i) & j_{11}(x_i) & \cdots & j_{1N}(x_i) \\ \vdots & \vdots & \ddots & \vdots \\ j_{N0}(x_i) & j_{N1}(x_i) & \cdots & j_{NN}(x_i) \end{bmatrix}$$

Hence we obtain the matrix as

$$\mathrm{G} = \overline{\mathrm{X}}\,\overline{\mathrm{FJ}}\,\overline{\mathrm{M_0}}\,\mathrm{A} \tag{2.12}$$

where

$$\bar{X} = \begin{bmatrix} X_{x_0} & 0 & \cdots & 0 \\ 0 & X_{x_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{x_N} \end{bmatrix}$$

$$\bar{F} = \begin{bmatrix} F & 0 & \cdots & 0 \\ 0 & F & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F \end{bmatrix}$$

$$\bar{J} = \begin{bmatrix} J_{x_0} & 0 & \cdots & 0 \\ 0 & J_{x_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{x_N} \end{bmatrix}$$

$$\overline{M_0} = \begin{bmatrix} M_0 \\ M_0 \\ \vdots \\ M_0 \end{bmatrix}$$

Finally, substituting equation (2.12) into equation (2.9), we get

$$\left(\sum_{s=0}^{m} P_s X_{x_i} M_s - \lambda \bar{X}\bar{F}\bar{J}\,\overline{M_0}\right) A = R \qquad (2.13)$$

which is the fundamental relation for solving the VIDE.

Therefore, we can write equation (2.13) into the matrix form

$$WA = R \qquad (2.14)$$

where

$$W = [w_{ij}] = \sum_{k=0}^{m} P_s X_{x_i} M_s - \lambda \bar{X}\bar{F}\bar{J}\overline{M_0}$$

Note that if $P_s = 0$, $s = 1, 2, \dots, N$ and $|W| \neq 0$, then the Volterra integral equation has one solution only, that is

$$\mathrm{A} = \mathrm{W}^{-1}\mathrm{R}$$

Now let us form the matrix representation for the initial conditions. For the initial conditions (2.2) in the interval $a \leq x \leq b$ we have:

$$\sum_{j=0}^{m-1} [\, a_{ij} Y^{(j)}(a) + b_{ij} Y^{(j)}(b) + c_{ij} Y^{(j)}(c)] = \lambda_i$$

where

$$i = 0,1, \dots, m-1 \,, \qquad a \leq c \leq b$$

Using the relation (2.7), we find the matrix representations of the functions at the points a, b and c in the forms

$$\left[Y^{(j)}(a)\right] = P M_j A \tag{2.15}$$

$$\left[Y^{(j)}(b)\right] = Q M_j A \tag{2.16}$$

$$\left[Y^{(j)}(c)\right] = S M_j A \tag{2.17}$$

where

$$\mathrm{P} = [1 \;\; (a-c)(a-c)^2 \;\; \dots \;\; (a-c)^N]$$

$$\mathrm{Q} = [1 \;\; (b-c)(b-c)^2 \;\; \dots \;\; (b-c)^N]$$

$$S = [1 \quad 0 \quad 0 \quad \dots \quad 0]$$

Substituting the matrix representations in equations (2.15), (2.16) and (2.17) into equation (2.2) we obtain

$$\sum_{j=0}^{m-1} [a_{ij} P + b_{ij} Q + c_{ij} S] \mathrm{M_j A} = [\lambda_i]$$

Define Y_i as

$$Y_i = \sum_{j=0}^{m-1} [a_{ij} P + b_{ij} Q + c_{ij} S] \mathrm{M_j} = [Y_{i0} \quad Y_{i1} \;\; \dots \;\; Y_{iN}]$$

$$i = 0,1, \dots . m-1$$

So, the matrix form conditions (2.2) is

$$\mathrm{Y_i A} = [\lambda_i]$$

and the augmented matrices are

$$[\mathrm{Y_i}\colon \lambda_i] = [Y_{i0}\ Y_{i1}\ \ \dots\ \ Y_{iN} : \lambda_i] \tag{2.18}$$

Consequently, replacing the m^{th} row matrices (2.18) by the last m rows of the augmented matrix (2.14), we obtain the required augmented matrix

$$[\widetilde{\mathrm{W}}_\mathrm{i}\colon \tilde{F}]$$

where

$$\widetilde{\mathrm{W}} = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} \\ w_{10} & w_{11} & \cdots & w_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} \\ Y_{00} & Y_{01} & \cdots & Y_{0N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{m-1,0} & Y_{m-1,1} & \cdots & Y_{m-1,N} \end{bmatrix}$$

$$\tilde{R} = \begin{bmatrix} r(x_0) \\ r(x_1) \\ \vdots \\ r(x_{N-m}) \\ \lambda_0 \\ \vdots \\ \lambda_{m-1} \end{bmatrix}$$

If $|\widetilde{\mathrm{W}}| \neq 0$ then we have

$$A = \widetilde{\mathrm{W}}^{-1}\tilde{R}$$

where the matrix A is uniquely determined. Consequently the IDE (2.1) with conditions (2.2) has a unique solution in the form (2.3).

2.2 Legendre Polynomials Method

The following second order linear differential equation with variable coefficients is known as Legendre's differential equation:

$$(1 - x^2)Y''(x) - 2xY'(x) + n(n+1)Y(x) = 0 \tag{2.19}$$

where n is a non-negative integer.

Equation (2.19) has bounded solutions $Y_n(x)$ on [-1, 1], these are

$$Y_n = L_n(x) \ , \qquad n \geq 0$$

Generating function for Legendre's polynomials

Theorem (2.1) [4]

The function $(1 - 2xh + h^2)^{-\frac{1}{2}}$ is the generating function for the Legendre polynomials $L_n(x)$ such that

$$(1 - 2xh + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n L_n(x)$$

and this series converges for $|h| < 1$ when $|x| \leq 1$.

Orthogonality relations [13]

One of the most common set of orthogonal polynomials are Legendre polynomials. This set of Legendre polynomials $\{L_0(x), L_1(x), \dots, L_n(x)\}$ is orthogonal on $[-1, 1]$ with respect to the weight function $w(x) \ = \ 1$, with the following properties:

$$(I) \quad \int_{-1}^{+1} L_m(x) L_n(x) dx = 0 \quad if \quad m \neq n$$

$$(II) \quad \int_{-1}^{+1} \left(L_n(x) \right)^2 dx = \frac{2}{2n+1} \quad if \quad m = n$$

Solving VIDE by Legendre polynomials method [15]

Consider the following linear VIDE

$$Y^{(n)}(x) = r(x) + \int_0^x F(x,t) Y(t) dt \qquad (2.20)$$

Subject to the conditions:

$$Y^{(s)}(0) = a_s\,, \qquad s = 0,1,2,\dots,(n-1)$$

The Legendre polynomials $L_n(x)$, for $-1 \le x \le 1$ and $n \ge 0$, are given by the forms

$$L_n(x) = \frac{1}{2^n}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\binom{n}{l}\binom{2n-2l}{n}x^{n-2l}, n = 0, 1, \dots \tag{2.21}$$

where $\left[\frac{n}{2}\right] = \frac{n}{2}$ if n is even, otherwise $\frac{n-1}{2}$.

To use the Legendre polynomials for our purposes, it is preferable to map this to the interval [0,1]. Then we can also define them by the following recursive formula: $L_0(x) = 1$ and $L_1(x) = 2x - 1$ and for $n = 1, 2, \dots$

$$(2n+1)xL_n(x) = (n+1)L_{n+1}(x) + nL_{n-1}(x) \tag{2.22}$$

then the solution is

$$Y(x) = \sum_{n=0}^{\infty} a_n L_n(x) \tag{2.23}$$

if we define

$$L(x) = [L_0(x)L_1(x) \dots \dots . L_M(x)]$$

and

$$\mathrm{A} = [a_0 a_1 \dots \dots\ a_M]^T$$

Then

$$Y(x) = \sum_{n=0}^{M} a_n L_n(x) = \ \mathrm{L(x)A} \tag{2.24}$$

Similarly if we define

$$\mathrm{L'(x)} = [L'_0(x)L'_1(x) \dots \dots . L'_M(x)]$$

we have

$$Y'(x) = \sum_{n=0}^{M} a_n L'_n(x) = \; L'(x)A \tag{2.25}$$

Using Legendre recursive formula (2.21) for $n = 0,1,2,..M$, we obtain the matrix form as follows

$$L'(x) = L(x)\Omega^T \tag{2.26}$$

where Ω has two different forms, namely; for odd and even values of M, that is, for odd values of M we have

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 3 & 0 & 7 & \dots & 2M-3 & 0 & 0 \\ 1 & 0 & 5 & 0 & \dots & 0 & 2M-1 & 0 \end{bmatrix}$$

and for even values of M we have

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 5 & 0 & \dots & 2M-3 & 0 & 0 \\ 0 & 3 & 0 & 7 & \dots & 0 & 2M-1 & 0 \end{bmatrix}$$

From equations (2.25) and (2.26) we get

$$Y'(x) = L(x)\Omega^T A$$

$$Y''(x) = L(x)(\Omega^T)^2 A$$

$$\vdots$$

$$Y^{(n)}(x) = L(x)(\Omega^T)^n A$$

We use this method to approximate the left hand side of the VIDE (2.20) as follows

$$Y^{(n)}(x) \approx L(x)(\Omega^{T})^{n}A$$

thus

$$Y^{(n)}(x) \approx \mathrm{S(x)A} \tag{2.27}$$

where

$$S(x) = [s_0(x)\ \ s_1(x)\ \ \ldots\ \ s_M(x)]$$

and for $i = 0,1,2,..M$, we define

$$s_i(x) = L_i(x)((\Omega^{T})^{n})^{i}$$

To obtain a solution of problem (2.20), for each $x, t \in [0,1]$ we define

$$F(x,t)Y(t) \approx L(x)F^{*}L^{T}(t)$$

where $F^{*} = [f_{nm}]$, and

$$f_{nm} = \frac{< L_n(x), < F(x,t), Y(t), L_m(t) >>}{\|L_n\|^2 \|L_m\|^2}$$

We use this method to approximate the right hand side of the VIDE (2.20) as follows

$$r(x) + \int_0^x \mathrm{L(x)F^{*}L^{T}}(t)dt = r(x) + \mathrm{L(x)F^{*}} \int_0^x \mathrm{L^{T}}(t)dt$$

Using Legendre formulas

$$\int_0^x L(t)dt = L^{*}$$

we have

$$r(x) + \int_0^x F(x,t)Y(t)dt = r(x) + L(x)F^{*}L^{*}$$

Let

$$g(x) = r(x) + L(x)F^{*}L^{*} \tag{2.28}$$

then from equations (2.27) and (2.28) we have

$$S(x)A = g(x) \tag{2.29}$$

Using the matrix method based on Legendre collocation points defined by

$$x_i = \frac{i}{M} \qquad i = 0,1,2,\dots,M$$

then by substituting the collocation points into equation (2.29) we have the following system

$$S(x_i)A = g(x_i) \quad i = 0,1,2,\dots,M \tag{2.30}$$

Equation (2.30) in matrix form is given as

$$\mathrm{S\ A} = \mathrm{G}$$

where for $i = 0,1,2,\dots,M$

$$\mathrm{S} = [\mathrm{S}(x_0)\mathrm{S}(x_1) \quad \dots \quad \mathrm{S}(x_M)]^T$$

and

$$G = [g(x_1)g(x_2)g(x_M)]^T$$

2.3 Haar Wavelet Method

Haar Wavelet Preliminaries [17]

The scaling function for the family of Haar wavelets is defined on the interval $[0,1)$ as:

$$h_1(\mathrm{x}) = \begin{cases} 1 & x \in [0,1) \\ 0 & otherwise \end{cases}$$

The mother wavelet for Haar wavelets family is also defined on the interval [0, 1) as:

$$h_2(\mathrm{x}) = \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ -1 & x \in [\frac{1}{2}, 1) \\ 0 & otherwise \end{cases}$$

Haar wavelet family for $x \in [0,1]$ is defined as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\alpha, \beta) \\ -1 & \text{for } x \in [\beta, \gamma) \\ 0 & otherwise \end{cases} \tag{2.31}$$

where

$\alpha = \frac{s}{m}, \beta = \frac{(s+0.5)}{m},\ \gamma = \frac{(s+1)}{m}, i = 3,4, \dots ,2M$

the integer $m = 2^j$ where $j = 0, 1, \dots, J$, $M = 2^J$ and integer $s = 0, 1, \dots, m - 1$. Integer j indicates the level of the wavelet and s is the translation parameter. The maximal level of resolution is the integer J. The relation between i, m and s is given by $i = m + s + 1$.

Introducing the following notations:

$$p_{i,1}(x) = \int_0^x h_i(x)dx \tag{2.32}$$

$$p_{i,v}(x) = \int_0^x p_{i,v-1}(x)dx \qquad v = 2,3, \dots \tag{2.33}$$

These integrals can be evaluated using equation (2.31) and are given as follows:

$$p_{i,1}(x) = \begin{cases} x - \alpha & \text{for } x \in [\alpha, \beta) \\ \gamma - x & \text{for } x \in [\beta, \gamma) \\ 0 & otherwise \end{cases}$$

$$p_{i,2}(x)=\begin{cases}\frac{1}{2}(x-\alpha)^2 & \text{for } x\in[\alpha,\beta)\\ \frac{1}{4m^2}-\frac{1}{2}(\gamma-x)^2 & \text{for } x\in[\beta,\gamma)\\ \frac{1}{4m^2} & \text{for } x\in[\gamma,1)\\ 0 & otherwise\end{cases}$$

The v^{th} order integrals of the Haar function (2.31) can be computed as:

$$p_{n,v}(x)=\begin{cases}\frac{1}{v!}(x-\alpha)^v & \text{for } x\in[\alpha,\beta)\\ \frac{(x-\alpha)^v-2(x-\beta)^v}{v!} & \text{for } x\in[\beta,\gamma)\\ \frac{(x-\alpha)^v-2(x-\beta)^v+2(x-\gamma)^v}{v!} & \text{for } x\in[\gamma,1)\\ 0 & otherwise\end{cases}$$

Chen and Hsiao [9] showed that the following matrix equation $P_{(\mu\times\mu)}$ is given as

$$p_{(2\mu\times2\mu)}=\frac{1}{4\mu}\begin{bmatrix}4\mu p_{(\mu\times\mu)} & -H_{(\mu\times\mu)}\\ H^{-1}{}_{(\mu\times\mu)} & 0_{(\mu\times\mu)}\end{bmatrix} \tag{2.34}$$

where $0_{(\mu\times\mu)}$ is a null matrix and $p_{(1\times1)}=0.5$

and they introduced the row vector $h_{(\mu)}$ as

$$h_{(\mu)}(t)=\left[h_1(t)h_2(t)\ldots\ldots\ldots\ldots\ldots h_\mu(t)\right] \tag{2.35}$$

where

$$\mu=2M=2^{J+1}$$

Now we have

$$\int_0^x h_\mu(t)dt\ \approx p_{(\mu\times\mu)}h_{(\mu)}(t) \tag{2.36}$$

Equations (2.34) and (2.35) can be used to solve first-order differential equations.

Higher-order equations have to be reduced to a system of first-order equations, but this increases the order of the Haar matrices and may cause complications of computational character.

Orthogonality [17]

Haar wavelets are orthogonal on $[0, 1)$ such that

$$\int_0^1 h_n(t), h_m(t)\, dt = 0, \ \ whenever \quad n \neq m$$

Function Approximation [24]:

Any function $r(t)$ defined over the interval $[0, 1)$, can be expanded in (h_i) functions as

$$r(t) = \sum_{i=1}^{\infty} \alpha_i h_i(t) \tag{2.37}$$

where the function coefficients α_i are given as follows

$$\alpha_i = 2^j \int_0^1 r(t) h_i(t) dt \tag{2.38}$$

Usually, the series in equation (2.37) contains an infinite number of terms. For approximation purposes we consider J is the maximum level of resolution and also define the integer $M = 2^J$. With these notations the function $r(t)$ can be approximated as follows:

$$r(t) \approx \sum_{i=1}^{2M} \alpha_i h_i(t) \tag{2.39}$$

We shall divide the interval $t \in [0, 1]$ into 2M parts of equal length $\Delta t = 1/(2M)$; the grid points are

$$\tau_l = (l - 1)\Delta t \qquad l = 1,2, \dots ,2M + 1 \tag{2.40}$$

Since in the subsequent sections the collocation method is used then we define also the collocation points

$$t_l = (l - 0.5)\Delta t \qquad l = 1,2,\dots,2M \tag{2.41}$$

Solving VIDE by the Haar wavelets method [21]**:**

Consider the VIDE

$$Y^{(n)}(x) = r(x) + \int_0^x F(x,t)Y(t)dt \tag{2.42}$$

for $0 \leq x \leq 1$

subject to the conditions:

$$Y^{(s)}(0) = a_s\,, \qquad s = 0,1,2,\dots,(n-1)$$

the functions F and r are prescribed.

Using the collocation points in equation (2.41) we have

$$Y^{(n)}(x_l) = r(x_l) + \int_0^{x_l} F(x_l,t)Y(t)dt \tag{2.43}$$

where $l = 1,\dots,2M$.

The function $Y^{(n)}(t)$ is developed into the Haar series as:

$$Y^{(n)}(t) = \sum_{i=1}^{2M} a_i\, h_i(t) \tag{2.44}$$

Integrating equation (2.44) n-times we obtain

$$Y^{(n-1)}(t) = \sum_{i=1}^{2M} a_i\, p_{1,i}(t) + Y^{(n-1)}(0)$$

$$Y^{(n-2)}(t)=\sum_{i=1}^{2M}a_i\,p_{2,i}(t)+Y^{(n-1)}(0)t+\ Y^{(n-2)}(0)$$

$$\vdots$$

$$Y(t)=\sum_{i=1}^{2M}a_i\,p_{n,i}(t)+Y^{(n-1)}(0)\frac{t^{(n-1)}}{(n-1)!}+\ Y^{(n-2)}(0)\frac{t^{(n-2)}}{(n-2)!}+\cdots+Y(0)\quad(2.45)$$

Let

$$F^{*}=F(x,t)Y(t)$$

Then we solve the system

$$-Y^{(n)}(x_l)+r(x_l)+\int_0^{x_l}F^{*}dt=0\,,l=1,2,..2M\qquad(2.46)$$

Let

$$\varphi(t)=\int_0^{x_l}F^{*}\,dt\qquad(2.47)$$

and

$$q_p(t)=p_{1,i}(t),\ \ k_p(t)=p_{2,i}(t),\ \ O_p(t)=p_{n,i}(t)$$

and consider the subinterval $t\in[\tau_s,\tau_{s+1}]$, $s=1,2,\dots 2M$ where τ_s is the s^{th} grid point defined in equation (2.40). In each subinterval

$$h_p(t)=h_p(t_s)=const$$

$$q_p(t)=q_p(t_s)+(t-\tau_s)h_p(t_s)$$

Here t_s denotes the s^{th} collocation point defined in equation (2.41) and

$$t_s=\tau_s+0.5\Delta t\qquad(2.48)$$

Now we have

$$Y^{(n)}(t)=Y^{(n)}(\tau_s)+(t-\tau_s)Y^{(n+1)}(t_s)$$

Integrating n-times we obtain

$$Y^{(n-1)}(t) = Y^{(n)}(\tau_s)t + \left(\frac{t^2}{2} - \tau_s t\right) Y^{(n+1)}(t_s)$$

$$Y^{(n-2)}(t) = Y^{(n)}(\tau_s)\frac{t^2}{2} + \left(\frac{t^3}{3!} - \tau_s \frac{t^2}{2}\right) Y^{(n+1)}(t_s)$$

$$Y^{(n-3)}(t) = Y^{(n)}(\tau_s)\frac{t^3}{3!} + \left(\frac{t^4}{4!} - \tau_s \frac{t^3}{3!}\right) Y^{(n+1)}(t_s)$$

$$\vdots$$

$$Y(t) = Y^{(n)}(\tau_s)\frac{t^n}{n!} + \left(\frac{t^{(n+1)}}{(n+1)!} - \tau_s \frac{t^n}{n!}\right) Y^{(n+1)}(t_s)$$

Introducing the following notations:

$$\Delta G\big(x_l, t, Y(t)\big) = \int_{\tau_s}^{\tau_{s+1}} F(x_l, t) Y(t) dt \tag{2.49}$$

$$\Delta \tilde{G}\big(x_l, t_l, Y(t_l)\big) = \int_{\tau_l}^{t_l} F(x_l, t) Y(t) d \tag{2.50}$$

Evaluating the integrals (2.47) for each subinterval $t \in [\tau_s, \tau_{s+1}]$, and summing up the results we obtain

$$\varphi(l) = \sum_{s=1}^{l-1} \Delta G + \Delta \tilde{G} \tag{2.51}$$

It is convenient to put our results into the matrix form. For this purpose we introduce the row vectors

$$\mathrm{R} = [\mathrm{r}(l)]\,, \qquad \varphi = [\varphi(l)]\,, \qquad Y = [Y(t_l)]\ \ , \qquad Y^{(n)} = \left[Y^{(n)}(t_l)\right]$$

We solve the system of algebraic equations

$$-Y^{(n)} + R + \varphi = 0$$

to get the solution $Y(x)$ of the VIDE

Chapter Three

Numerical Examples and Results

In this chapter, we will implement the aforementioned methods in chapter two to solve some numerical examples

Example 3.1

We assume the following VIDE

$$\mathrm{Y}''(x) = 4\mathrm{e}^{x} - 2x - 2 + \int_{0}^{x} (x - \mathrm{t})\mathrm{Y}(\mathrm{t})\mathrm{dt} \tag{3.1}$$

subject to the initial conditions

$\mathrm{Y}(0)=0$, $\mathrm{Y}'(0)=1$

Equation (3.1) has the exact solution

$$Y(x) = xe^{x}$$

3.1 The numerical treatment of equation (3.1) using Taylor collocation method

To solve equation (3.1) by the Taylor collocation method we implement algorithm 3.1.

Thus for $N = 3$ we have

$$F(x,t) = x - t$$

$$\mathrm{Y}(x) = \mathrm{a}_0 + \mathrm{a}_1 x + \mathrm{a}_2 x^2 + \mathrm{a}_3 x^3$$

$$x_0 = 0\ , x_1 = \frac{1}{3}\ , x_2 = \frac{2}{3}\ , x_3 = 1$$

Algorithm 3.1

Input $\bar{M}_0, M_2, X_{x_i}, \bar{X}, \bar{J}, P_2$

Input $F, \bar{F}$

Calculate $W_1 = Multiply(\bar{X}, \bar{F}, \bar{J}, \bar{M}_0)$

```
Calculate $W_2 = Multiply\left(P_2, X_{x_i}, M_2\right)$

Calculate $W = W_1 - W_2$

Input $R$

Calculate $A = W^{-1}R$

Input initial condition $Y_1(0), Y_2(0)$

Input $\widetilde{W}, \tilde{R}$

Calculate $A = \widetilde{W}^{-1}R$

Input $Y_{exact}(x), Y_{approximate}(x)$

Define error $= \left|Y_{exact}(x) - Y_{approximate}(x)\right|$

Plot $Y_{exact}(x), Y_{approximate}(x)$
```

$$J_0 = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$J_{\frac{1}{3}} = \begin{bmatrix} 0.333333 & 0.055556 & 0.012346 & 0.003086 \\ 0.055556 & 0.012346 & 0.003086 & 0.000823 \\ 0.012346 & 0.003086 & 0.000823 & 0.000229 \\ 0.003086 & 0.000823 & 0.000229 & 0.000065 \end{bmatrix}$$

$$J_{\frac{2}{3}} = \begin{bmatrix} 0.666667 & 0.222222 & 0.098765 & 0.049383 \\ 0.222222 & 0.098765 & 0.049383 & 0.026337 \\ 0.098765 & 0.049383 & 0.026337 & 0.014632 \\ 0.049383 & 0.026337 & 0.014632 & 0.084824 \end{bmatrix}$$

$$J_1 = \begin{bmatrix} 1.0 & 0.5 & 0.333333 & 0.25 \\ 0.5 & 0.333333 & 0.25 & 0.2 \\ 0.333333 & 0.25 & 0.2 & 0.166667 \\ 0.25 & 0.2 & 0.166667 & 0.142857 \end{bmatrix}$$

$$M_0 = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.16666667 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$X_{x_i} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.33333333 & 0.11111111 & 0.03703704 \\ 1.0 & 0.66666667 & 0.44444444 & 0.2962963 \\ 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$$W = \begin{bmatrix} 0.0 & 0.0 & -1.0 & 0.0 \\ 0.055556 & 0.006173 & -0.999486 & -0.333299 \\ 0.222222 & 0.049383 & -0.991770 & -0.665569 \\ 0.5 & 0.166667 & -0.958333 & -0.991667 \end{bmatrix}$$

$$R = \begin{bmatrix} 2.0 \\ 2.915783033 \\ 4.457602831 \\ 6.87312731 \end{bmatrix}$$

From initial condition:

$$Y_1 = (1 \quad 0 \quad 0 \quad 0) \ , \ \lambda_1 = 0$$

$$Y_2 = (0 \quad 1 \quad 0 \quad 0) \ , \ \lambda_2 = 1$$

Hence

$$\widetilde{W} = \begin{bmatrix} 0.0 & 0.0 & -1.0 & 0.0 \\ 0.055556 & 0.006173 & -0.999486 & -0.333299 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\tilde{R} = \begin{bmatrix} 2.0 \\ 2.915783033 \\ 0.0 \\ 1.0 \end{bmatrix}$$

The solution for this system is:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \widetilde{W}^{-1}\tilde{R} = \begin{bmatrix} 0.0 \\ 1.0 \\ -2.0 \\ -2.73219809057310 \end{bmatrix}$$

$$Y(x) = x - 2x^2 + 2.73219809057310x^3$$

Table 3.1 contains both the exact and approximate solutions of equation (3.1) and the absolute error $\left|\mathrm{Y_{app}} - \mathrm{Y_{ex}}\right|$ using Taylor collocation method.

Table 3.1

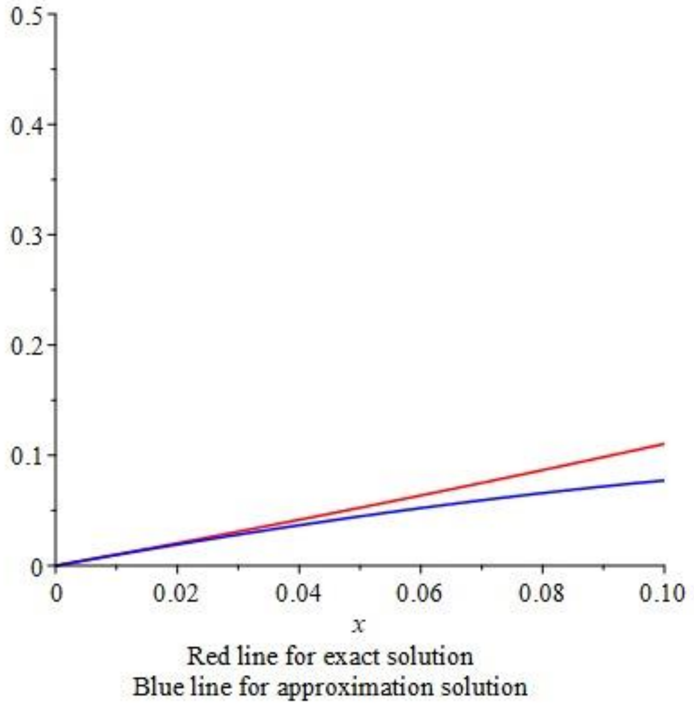
The approximate and the exact solutions of applying Taylor collocation method for equation (3.1)

x	$\mathrm{Y_{app}}$	$\mathrm{Y_{ex}}$	$\left\|\mathrm{Y_{app}} - \mathrm{Y_{ex}}\right\|$
0.0	0.0	0.0	0.0
0.001	$9.979972678\ 10^{-4}$	$1.001000500\ 10^{-3}$	$3.0032322\ 10^{-6}$
0.002	$1.991978142\ 10^{-3}$	$2.004004002\ 10^{-3}$	$1.2025860\ 10^{-5}$
0.003	$2.981926231\ 10^{-3}$	$3.009013515\ 10^{-3}$	$2.7087284\ 10^{-5}$
0.004	$3.967825139\ 10^{-3}$	$4.016032044\ 10^{-3}$	$4.8206905\ 10^{-5}$
0.005	$4.949658475\ 10^{-3}$	$5.025062605\ 10^{-3}$	$7.5404130\ 10^{-5}$
0.006	$5.927409845\ 10^{-3}$	$6.036108216\ 10^{-3}$	$1.08698371\ 10^{-4}$
0.007	$6.901062856\ 10^{-3}$	$7.049171899\ 10^{-3}$	$1.48109043\ 10^{-4}$
0.008	$7.870601115\ 10^{-3}$	$8.064256688\ 10^{-3}$	$1.93655573\ 10^{-4}$
0.009	$8.836008228\ 10^{-3}$	$9.081365598\ 10^{-3}$	$2.45357370\ 10^{-4}$
0.01	$9.797267802\ 10^{-3}$	$1.010050167\ 10^{-2}$	$3.03233868\ 10^{-4}$

Figure 3.1 compares the exact solution with the approximate solution using Taylor collocation method.

Figure 3.1

The approximate and the exact solutions of applying Taylor collocation method for equation (3.1)



The maximum error corresponding to y_{app} is $E \approx 3.03233868 \times 10^{-4}$

3.2 The numerical treatment of equation (3.1) using Legendre polynomials method

To solve equation (3.1) by the Legendre polynomials method we implement algorithm 3.2.

Thus for $N = 3$ we have

We write the solution $\mathrm{Y}(x)$ into the form

$$\mathrm{Y}(x) = \mathrm{aL}_0(x) + \mathrm{bL}_1(x) + \mathrm{cL}_2(x) + d\mathrm{L}_3(x)$$

$$\mathrm{L}_0(x) = 1\,,\ \ \mathrm{L}_1(x) = x\,,\ \ \mathrm{L}_2(x) = \frac{3}{2}x^2 - \frac{3}{2}\,,\ \ \mathrm{L}_3(x) = \frac{1}{3}(5x^3 - 3x)$$

Algorithm 3.2

```
Define $L_0(x)$, $L_1(x)$, $L_2(x)$, $L_3(x)$

Input $L, A, \Omega$

Calculate $S = multiply\ ( L, \Omega^{T}, \Omega^{T} )$

Calculate $g_1 = multiply\ ( S, A)$

Input $F$

Input $L^{*}$= integral $L$

Calculate $g_2 = multiply(L, F,\ L^{*})$

Evaluate $g_3 = R + g_2$

$For\ i\ from\ 0\ to\ 3$
    $x[i] = \frac{i}{3}$
$end\ do$:

Evaluate $g_1(x[i]) = g_3(x[i])$

Input initial condition $Y_1(0), Y_2(0)$

Solving the system of equations to get $a, b, c, d$

Input $Y_{exact}(x), Y_{approximate}(x)$

Define error $= \left|Y_{exact}(x) - Y_{approximate}(x)\right|$

Plot $Y_{exact}(x), Y_{approximate}(x)$
```

$$\mathrm{A} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\Omega = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 3.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 5.0 & 0.0 \end{pmatrix}$$

We find

$$\mathrm{S} = \mathrm{L}(x)\left(\Omega^{\mathrm{T}}\right)^{2}$$

$$S = [0 \quad 0 \quad 3 \quad 15\,x]$$

and we compute the integral:

$$L^{*} = \int_{0}^{x} \mathrm{L(t)dt}$$

$$L^{*} := \begin{bmatrix} x \\ \frac{1}{2}x^{2} \\ \frac{1}{2}x^{3} - \frac{3}{2}x \\ \frac{5}{12}x^{4} - \frac{1}{2}x^{2} \end{bmatrix}$$

Finally we obtain the exact solution as

$$\left\{ \begin{matrix} a = 0.9999999999, b = 1.120463527, c = 0.6666666666, \\ d = 0.12046352266 \end{matrix} \right\}$$

Hence the solution of the Volterra integro-differential equation is:

$$Y := x \rightarrow 1.0000000004\,x + 0.9999999999\,x^{2} + 0.20077254433\,x^{3}$$

Table 3.2 contains both the exact and approximate solutions of equation (3.1) and the absolute error $\left|\mathrm{Y}_{\mathrm{app}} - \mathrm{Y}_{\mathrm{ex}}\right|$ using Legendre polynomials method.

Table 3.2

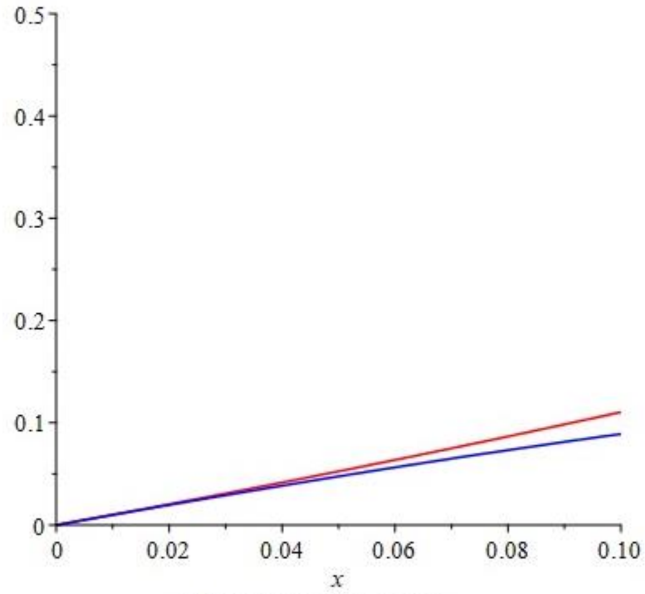
The approxim`ate and the exact solutions of applying Legendre polynomials method for equation (3.1)

x	$\mathrm{Y}_{\mathrm{app}}$	Y_{ex}	$\left\|\mathrm{Y}_{\mathrm{app}} - \mathrm{Y}_{\mathrm{ex}}\right\|$
0.0	0.0	0.0	0.0
0.001	$1.001000201\ 10^{-3}$	$1.001000500\ 10^{-3}$	$2.99\ 10^{-10}$
0.002	$2.004001606\ 10^{-3}$	$2.004004002\ 10^{-3}$	$2.396\ 10^{-9}$
0.003	$3.009005421\ 10^{-3}$	$3.009013515\ 10^{-3}$	$8.094\ 10^{-9}$
0.004	$4.016012849\ 10^{-3}$	$4.016032044\ 10^{-3}$	$1.9195\ 10^{-8}$
0.005	$5.025025097\ 10^{-3}$	$5.025062605\ 10^{-3}$	$3.7508\ 10^{-8}$
0.006	$6.036043367\ 10^{-3}$	$6.036108216\ 10^{-3}$	$6.4849\ 10^{-8}$
0.007	$7.049068865\ 10^{-3}$	$7.049171899\ 10^{-3}$	1.0303410^{-7}
0.008	$8.064102796\ 10^{-3}$	$8.064256688\ 10^{-3}$	1.5389210^{-7}
0.009	$9.081146363\ 10^{-3}$	$9.081365598\ 10^{-3}$	2.1923510^{-7}
0.01	$1.010020077\ 10^{-2}$	$1.010050167\ 10^{-2}$	3.009010^{-7}

Figure 3.2 compares the exact solution with the approximate solution using Legendre polynomials method.

Figure 3.2

The approximate and the exact solutions of applying Legendre polynomials method for equation (3.1)



Red line for exact solution
Blue line for approximation solution

The maximum error corresponding to $\mathrm{Y}_{\mathrm{app}}$ is $E \approx 3.0090 \times 10^{-7}$

3.3 The numerical treatment of equation (3.1) using Haar wavelets method

To solve equation (3.1) by the Haar wavelets method we implement algorithm 3.3.

Thus for M=4 we have

Algorithm 3.3

```
Define $h_i, p_i, k_i \; for \; i = 1,2,\dots,2M$
Define $\tau[l], t[l]$
Input $Y''(l)$
Integral $Y''(l)$ two times
Define $\Delta G\big(x_l, t, Y(t)\big), \Delta\tilde{G}\big(x_l, t, Y(t)\big)$
Calculate $\varphi(l) = \sum_{s=1}^{l-1} \Delta G + \Delta\tilde{G}$
Input F
Solve the equation $-Y'' + \varphi + R = 0$
Input $Y_{exact}(x), Y_{approximate}(x)$
Define error $= \left|Y_{exact}(x) - Y_{approximate}(x)\right|$
Plot $Y_{exact}(x), Y_{approximate}(x)$
```

$$r(x) = 4\mathrm{e}^x - 2x - 2$$

$$R1 = [r(t[1]), r(t[2]), r(t[3]), r(t[4]), r(t[5]), r(t[6]), r(t[7]), r(t[8])]^T$$

$$Y'' = [Y''(t[1]), Y''(t[2]), Y''(t[3]), Y''(t[4]), Y''(t[5]), Y''(t[6]), Y''(t[7]), Y''(t[8])]^T$$

Then we solve the system*:*

$-Y'' + \varphi + F1 = 0$

$$E[1] := -Y11[1,1] + F1[1,1] + \varphi[1,1] = 0:$$

$$E[2] := -Y11[2,1] + F1[2,1] + \varphi[2,1] = 0:$$

$$E[3] := -Y11[3,1] + F1[3,1] + \varphi[3,1] = 0:$$

$$E[4] := -Y11[4,1] + F1[4,1] + \varphi[4,1] = 0:$$

$$E[5] := -Y11[5,1] + F1[5,1] + \varphi[5,1] = 0:$$

$$E[6] := -Y11[6,1] + F1[6,1] + \varphi[6,1] = 0:$$

$$E[7] := -Y11[7,1] + F1[7,1] + \varphi[7,1] = 0:$$

$$E[8] := -Y11[8,1] + F1[8,1] + \varphi[8,1] = 0:$$

solve($\{E[1], E[2], E[3], E[4], E[5], E[6], E[7], E[8]\}$, {a, b, c, d, e, f, g, k});

{a=3.925071313, b=-1.232524531, c=-0.4004527336, d=-0.8646885785, e=-0.1590730266, f=-0.2446142819, g=-0.3594240400, k=-0.5101371752}

The solution $Y(x)$ is:

$$\begin{aligned} \mathrm{Y} := \mathrm{x} \to\ & 3.925071313 \cdot k_1(x) - 1.232524531 \cdot k_2(x) - 0.4004527336 \cdot k_3(x) \\ & - 0.8646885785 \cdot k_4(x) - 0.1590730266 \cdot k_5(x) - 0.2446142819 \\ & \cdot k_6(x) - 0.3594240400 \cdot k_7(x) - 0.5101371752 \cdot k_8(x) + x \end{aligned}$$

Table 3.3 contains both the exact and approximate solutions of equation (3.1) and the absolute error $\left|\mathrm{Y}_{\mathrm{app}} - \mathrm{Y}_{\mathrm{ex}}\right|$ using Haar wavelets method.

Table 3.3

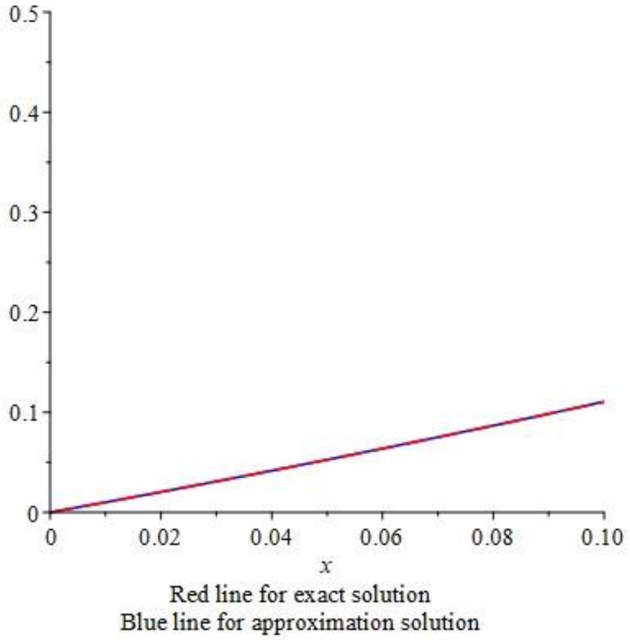
The approximate and the exact solutions of applying Haar wavelets method for equation (3.1)

x	$\mathrm{Y}_{\mathrm{app}}$	Y_{ex}	$\left\|\mathrm{Y}_{\mathrm{app}} - \mathrm{Y}_{\mathrm{ex}}\right\|$
0.0	0.0	0.0	0.0
0.001	$1.001066511\ 10^{-3}$	$1.001000500\ 10^{-3}$	$6.6011\ 10^{-8}$
0.002	$2.004266042\ 10^{-3}$	$2.004004002\ 10^{-3}$	$2.62040\ 10^{-7}$
0.003	$3.009598595\ 10^{-3}$	$3.009013515\ 10^{-3}$	$5.85080\ 10^{-7}$
0.004	$4.017064168\ 10^{-3}$	$4.016032044\ 10^{-3}$	$1.032124\ 10^{-6}$
0.005	$5.026662763\ 10^{-3}$	$5.025062605\ 10^{-3}$	$1.600158\ 10^{-6}$
0.006	$6.038394378\ 10^{-3}$	$6.036108216\ 10^{-3}$	$2.286162\ 10^{-6}$
0.007	$7.052259015\ 10^{-3}$	$7.049171899\ 10^{-3}$	$3.087116\ 10^{-6}$
0.008	$8.068256673\ 10^{-3}$	$8.064256688\ 10^{-3}$	$3.999985\ 10^{-6}$
0.009	$9.086387351\ 10^{-3}$	$9.081365598\ 10^{-3}$	$5.021753\ 10^{-6}$
0.01	$1.010665105\ 10^{-3}$	$1.010050167\ 10^{-3}$	$6.14938\ 10^{-6}$

Figure 3.3 compares the exact solution with the approximate solution using Haar wavelets method.

Figure 3.3

The approximate and the exact solutions of applying Haar wavelets method for equation(3.1)

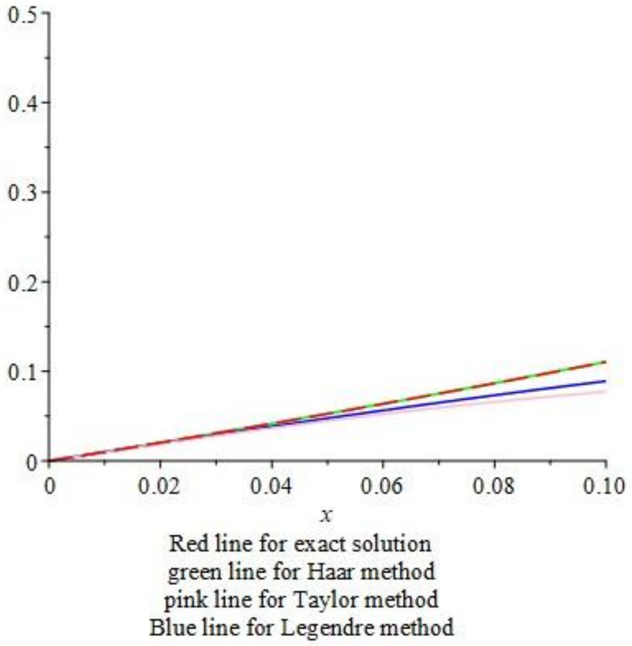


The maximum error corresponding to $\mathrm{Y}_{\mathrm{app}}$ is $E \approx 6.14938 \times 10^{-6}$

Figure 3.4 compares the exact solution with the solution of three methods.

Figure 3.4

The solution of three methods and exact solution for equation (3.1)



Example 3.2

We assume the following VIDE

$$\mathrm{Y}''(x) = \mathrm{e}^x - \frac{1}{2}\cos x - \frac{1}{2}\sin x - \frac{1}{2}(\cos x)^2\mathrm{e}^x + (\cos x)^2 - \sin x - \frac{1}{2}(\sin x)^2\mathrm{e}^x + (\sin x)^2 + \int_0^x cos(x - \mathrm{t})\,\mathrm{Y}(\mathrm{t})\mathrm{dt} \qquad (3.2)$$

subject to the initial conditions:

$$Y(0) = 1\,, Y'(0) = 0$$

Equation (3.2) has the exact solution

$$Y(x) = e^x - x$$

3.4 The numerical treatment of equation (3.2) using Taylor collocation method

To solve equation (3.2) by the Taylor collocation method we implement algorithm 3.1.

Thus for N=3 we have

$$\mathrm{F}(x, \mathrm{t}) = cos(x - \mathrm{t})$$

$$\mathrm{Y}(x) = \mathrm{a}_0 + \mathrm{a}_1 x + \mathrm{a}_2 x^2 + \mathrm{a}_3 x^3$$

$$x_0 = 0\,,\ \ x_1 = \frac{1}{3}\,,\ \ x_2 = \frac{2}{3}\,,\ \ x_3 = 1$$

$$F = \begin{bmatrix} 1.0 & 0.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 & -0.1666666667 \\ -0.5 & 0.0 & 0.25 & 0.0 \\ 0.0 & -0.1666666667 & 0.0 & 0.02777777778 \end{bmatrix}$$

$$J_0 = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$J_{\frac{1}{3}} = \begin{bmatrix} 0.333333 & 0.055556 & 0.012346 & 0.003086 \\ 0.055556 & 0.012346 & 0.003086 & 0.000823 \\ 0.012346 & 0.003086 & 0.000823 & 0.000229 \\ 0.003086 & 0.000823 & 0.000229 & 0.000065 \end{bmatrix}$$

$$J_{\frac{2}{3}} = \begin{bmatrix} 0.666667 & 0.222222 & 0.098765 & 0.049383 \\ 0.222222 & 0.098765 & 0.049383 & 0.026337 \\ 0.098765 & 0.049383 & 0.026337 & 0.014632 \\ 0.049383 & 0.026337 & 0.014632 & 0.084824 \end{bmatrix}$$

$$J_1 = \begin{bmatrix} 1.0 & 0.5 & 0.333333 & 0.25 \\ 0.5 & 0.333333 & 0.25 & 0.2 \\ 0.333333 & 0.25 & 0.2 & 0.166667 \\ 0.25 & 0.2 & 0.166667 & 0.142857 \end{bmatrix}$$

$$M_0 = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.16666667 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$X_{x_i} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.333333 & 0.111111 & 0.037037 \\ 1.0 & 0.666667 & 0.444444 & 0.296296 \\ 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$$W = \begin{bmatrix} 0.0 & 0.0 & -1.0 & 0.0 \\ 0.326992201391962 & 0.0550058002675501 & -0.993865781902416 & -0.332821217174043 \\ 0.61220342471495 & 0.211891818657536 & -0.952223482945827 & -0.658647320165794 \\ 0.798611111086667 & 0.437499999982667 & -0.849074074080833 & -0.961640211642063 \end{bmatrix}$$

$$R = \begin{bmatrix} 1.0 \\ 1.061730391 \\ 1.271738488 \\ 1.668254269 \end{bmatrix}$$

From initial condition:

$Y_1 = (1 \quad 0 \quad 0 \quad 0) \ , \ \lambda_1 = 1$

$$Y_2 = (0 \quad 1 \quad 0 \quad 0) \ , \ \lambda_2 = 0$$

Hence

$$\widetilde{W} = \begin{bmatrix} 0.0 & 0.0 & -1.0 & 0.0 \\ 0.326992201391962 & 0.0550058002675501 & -0.993865781902416 & -0.332821217174043 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\tilde{R} = \begin{bmatrix} 1.0 \\ 1.061730391 \\ 1.0 \\ 0.0 \end{bmatrix}$$

The solution for this system is:

$$\widetilde{W}^{-1}\tilde{R} = \begin{bmatrix} 1.0 \\ 0.0 \\ -1.0 \\ 0.778578945460895 \end{bmatrix}$$

$$\mathrm{Y}(x) = 1 - x^2 + 0.778578945460895x^3$$

Table 3.4 contains both the exact and approximate solutions of equation (3.2) and the absolute error $\left|\mathrm{Y}_{\mathrm{app}} - \mathrm{Y}_{\mathrm{ex}}\right|$ using Taylor collocation method.

Table 3.4

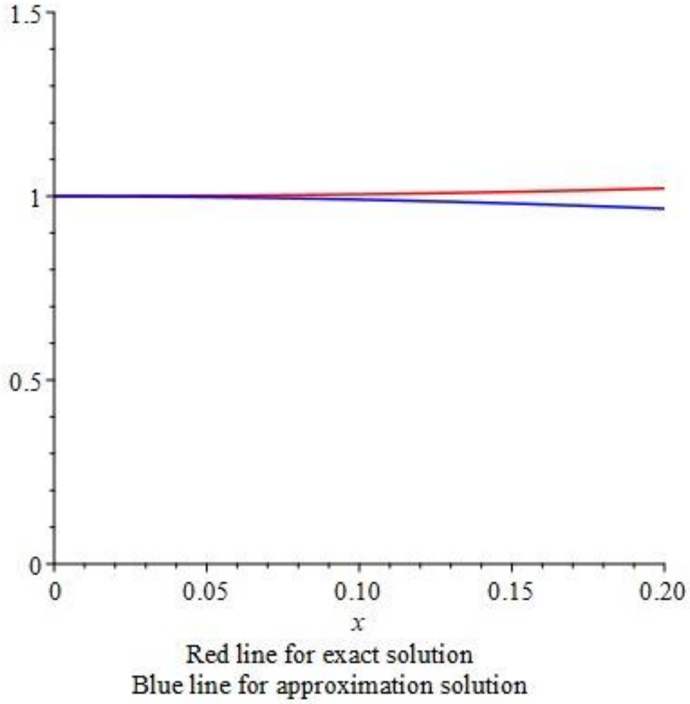
The approximate and the exact solutions of applying Taylor collocation method for equation (3.2)

x	$\mathrm{Y_{app}}$	$\mathrm{Y_{ex}}$	$\lvert\mathrm{Y_{app}} - \mathrm{Y_{ex}}\rvert$
0.0	1.0	1.0	0.0
0.001	0.9999990008	1.000000500	$1.499\ 10^{-6}$
0.002	0.9999960062	1.000002001	$5.995\ 10^{-6}$
0.003	0.9999910210	1.000004505	$1.3484\ 10^{-5}$
0.004	0.9999840498	1.000008011	$2.3961\ 10^{-5}$
0.005	0.9999750973	1.000012521	$3.7424\ 10^{-5}$
0.006	0.9999641682	1.000018036	$5.3868\ 10^{-5}$
0.007	0.9999512671	1.000024557	$7.3290\ 10^{-5}$
0.008	0.9999363986	1.000032086	$9.5687\ 10^{-5}$
0.009	0.9999195676	1.000040622	$1.21054\ 10^{-4}$
0.01	0.9999007786	1.000050167	$1.49388\ 10^{-4}$

Figure 3.5 compares the exact solution with the approximate solution using Taylor collocation method.

Figure 3.5

The approximate and the exact solutions of applying Taylor collocation method for equation(3.2)



The maximum error corresponding to $\mathrm{Y_{app}}$ is $E \approx 1.49388 \times 10^{-4}$

3.5 The numerical treatment of equation (3.2) using Legendre polynomials method

To solve equation (3.2) by the Legendre polynomials method we implement algorithm 3.2.

Thus for N=3 we have

We write the solution $\mathrm{Y}(x)$ into the form

$$\mathrm{Y}(x) = \mathrm{aL}_0(x) + \mathrm{bL}_1(x) + \mathrm{cL}_2(x) + d\mathrm{L}_3(x)$$

$$\mathrm{L}_0(x) = 1,\ \ \mathrm{L}_1(x) = \mathrm{x}\,,\ \ \mathrm{L}_2(x) = \frac{3}{2}x^2 - \frac{3}{2},\ \ \mathrm{L}_3(x) = \frac{1}{3}(5x^3 - 3x)$$

In matrix form:

$$\mathrm{L}(x) = [\mathrm{L}_0(x) \quad \mathrm{L}_1(x) \quad \mathrm{L}_2(x) \quad \mathrm{L}_3(x)]$$

$$\mathrm{A} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\Omega^{\mathrm{T}} = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 3.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 5.0 & 0.0 \end{pmatrix}$$

we find

$$\mathrm{S} = \mathrm{L}(x)\left(\Omega^{\mathrm{T}}\right)^2$$

$$S = [0 \quad 0 \quad 3 \quad 15\,x]$$

And we compute the integral

$$L^* = \int_0^x \mathrm{L}(\mathrm{t})\mathrm{dt}$$

$$L^* := \begin{bmatrix} x \\ \frac{1}{2}x^2 \\ \frac{1}{2}x^3 - \frac{3}{2}x \\ \frac{5}{12}x^4 - \frac{1}{2}x^2 \end{bmatrix}$$

Substituting the following values

$$x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1$$

Finally we obtain the exact solution as

$$\{a = 1.5000000000, b = 0.5199945646, c = 0.3333333333,$$
$$d = 0.5199945646\}$$

Hence the solution of the VIDEs is:

$$Y := x \rightarrow 1.00000000005 + 1.0399891292\, x + 0.49999999995\, x^2$$
$$+ 0.86665760766\, x^3$$

Table 3.5 contains both the exact and approximate solutions of equation (3.1) and the absolute error $\left|\mathrm{Y_{app}} - \mathrm{Y_{ex}}\right|$ using Legendre polynomials method.

Table 3.5

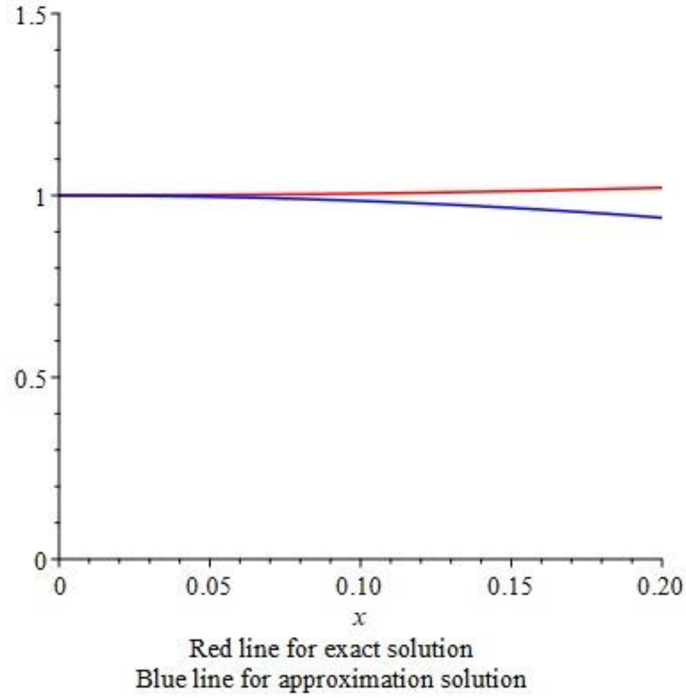
The approximate and the exact solutions of applying Legendre polynomials method for equation (3.2)

x	$\mathbf{Y_{app}}$	$\mathbf{Y_{ex}}$	$\left\|\mathbf{Y_{app}} - \mathbf{Y_{ex}}\right\|$
0.0	1.0	1.0	0.0
0.001	1.001040490	1.000000500	1.039990 $\mathbf{10^{-3}}$
0.002	1.002081985	1.000002001	2.079984 $\mathbf{10^{-3}}$
0.003	1.003124490	1.000004505	3.119985 $\mathbf{10^{-3}}$
0.004	1.004168012	1.000008011	4.160001 $\mathbf{10^{-3}}$
0.005	1.005212554	1.000012521	5.200033 $\mathbf{10^{-3}}$
0.006	1.006258122	1.000018036	6.240086 $\mathbf{10^{-3}}$
0.007	1.007304721	1.000024557	7.280164 $\mathbf{10^{-3}}$
0.008	1.008352357	1.000032086	8.320271 $\mathbf{10^{-3}}$
0.009	1.009401034	1.000040622	9.360412 $\mathbf{10^{-3}}$
0.01	1.010450758	1.000050167	1.0400591 $\mathbf{10^{-2}}$

Figure 3.6 compares the exact solution with the approximate solution using Legendre polynomials method.

Figure 3.6

The approximate and the exact solutions of applying Legendre polynomials method for equation (3.2)



The maximum error corresponding to $\mathrm{Y}_{\mathrm{app}}$ is $E \approx 1.0400591 \times 10^{-2}$

3.6 The numerical treatment of equation (3.2) using Haar wavelets method

To solve equation (3.2) by the Haar wavelets method we implement algorithm 3.3.

Thus for M=4 we have

Then we calculate the values of

$$\tau[l]\,,\ \ t[l]\,,\ \ h_s(t[l])\,,\ \ h_s\big(\tau[l]\big)\,,\ \ p_s(t[l]),\ \ p_s\big(\tau[l]\big)\,,\ \ k_s\big(\tau[l]\big)\,,\ \ k_s(t[l]):$$

$$r(x) = \mathrm{e}^x - \frac{1}{2}\cos x - \frac{1}{2}\sin x - \frac{1}{2}(\cos x)^2\mathrm{e}^x + (\cos x)^2 - \sin x - \frac{1}{2}(\sin x)^2\mathrm{e}^x + (\sin x)^2$$

$$R = [r(t[1]), r(t[2]), r(t[3]), r(t[4]), r(t[5]), r(t[6]), r(t[7]), r(t[8])]^T$$

$$Y'' = [Y''(t[1]), Y''(t[2]), Y''(t[3]), Y''(t[4]), Y''(t[5]), Y''(t[6]), Y''(t[7]), Y''(t[8])]^T$$

Then we solve the system*:*

$$-Y'' + \varphi + R = 0$$

$$E[1] := -Y11[1,1] + R[1,1] + \varphi[1,1] = 0:$$
$$E[2] := -Y11[2,1] + R[2,1] + \varphi[2,1] = 0:$$
$$E[3] := -Y11[3,1] + R[3,1] + \varphi[3,1] = 0:$$
$$E[4] := -Y11[4,1] + R[4,1] + \varphi[4,1] = 0:$$
$$E[5] := -Y11[5,1] + R[5,1] + \varphi[5,1] = 0:$$
$$E[6] := -Y11[6,1] + R[6,1] + \varphi[6,1] = 0:$$
$$E[7] := -Y11[7,1] + R[7,1] + \varphi[7,1] = 0:$$
$$E[8] := -Y11[8,1] + R[8,1] + \varphi[8,1] = 0:$$

solve($\{E[1], E[2], E[3], E[4], E[5], E[6], E[7], E[8]\}$, {a, b, c, d, e, f, g, k});

{a=1.247171251, b=-0.1956079221, c=-0.04055846470, d=-0.1610515520, e=-0.008960311422, f=-0.03253769161, g=-0.06303626163, k=-0.09845368482}

The solution $Y(x)$ is:

$$\begin{aligned}\mathrm{Y} := \mathrm{x} \rightarrow 1.247171251 \cdot k_1(x) - 0.1956079221 \cdot k_2(x) - 0.04055846470 \cdot k_3(x) \\ - 0.1610515520 \cdot k_4(x) - 0.008960311422 \cdot k_5(x) \\ - 0.03253769161 \cdot k_6(x) - 0.06303626163 \cdot k_7(x) \\ - 0.09845368482 \cdot k_8(x) + 1\end{aligned}$$

Table 3.6 contains both the exact and approximate solutions of equation (3.2) and the absolute error $|\mathrm{Y}_{\mathrm{app}} - \mathrm{Y}_{\mathrm{ex}}|$ using Haar wavelets method.

Table 3.6

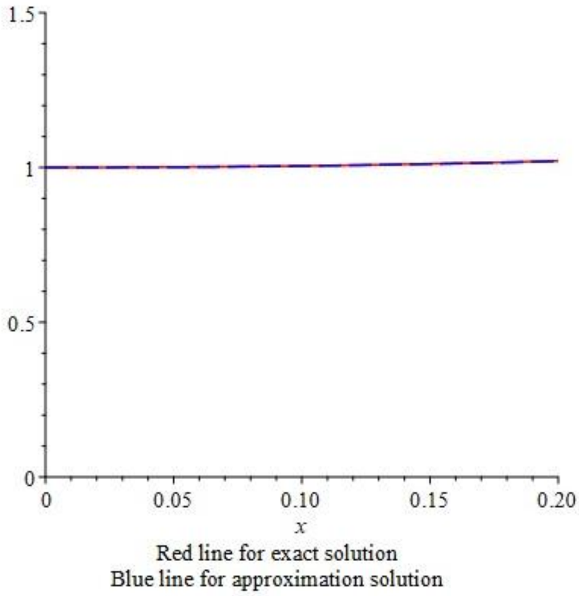
The approximate and the exact solutions of applying Haar wavelets method for equation (3.2)

x	Y_{app}	Y_{ex}	$\lvert Y_{app} - Y_{ex} \rvert$
0.0	1.0	1.0	0.0
0.001	1.000000501	1.000000500	1.10^{-9}
0.002	1.000002004	1.000002001	3.10^{-9}
0.003	1.000004509	1.000004505	4.10^{-9}
0.004	1.000008016	1.000008011	5.10^{-9}
0.005	1.000012526	1.000012521	5.10^{-9}
0.006	1.000018037	1.000018036	1.10^{-9}
0.007	1.000024550	1.000024557	7.10^{-9}
0.008	1.000032065	1.000032086	$2.1\ 10^{-8}$
0.009	1.000040583	1.000040622	$3.9\ 10^{-8}$
0.01	1.000050102	1.000050167	$6.5\ 10^{-8}$

Figure 3.7 compares the exact solution with the approximate solution using Haar wavelets method.

Figure 3.7

The approximate and the exact solutions of applying Haar Wavelets method for equation (3.2)

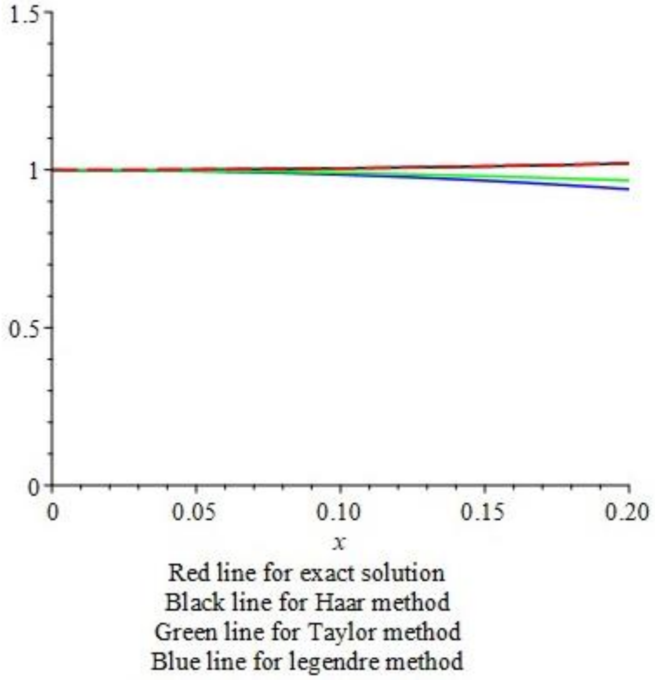


The maximum error corresponding to Y_{app} is $E \approx 6.5\ 10^{-8}$

Figure 3.8 compares the exact solution with the solution of three methods.

Figure 3.8

The solution of three methods and exact solution for equation (3.2)



Example 3.3

We assume the following VIDE

$$Y'(x) + Y(x) = \int_0^x e^{(t-x)} Y(t) dt \tag{3.3}$$

subject to the initial condition

$$Y(0) = 1$$

Equation (3.3) has the exact solution

$$Y(x) = e^{(-x)} \cosh(x)$$

3.7 The numerical treatment of equation (3.3) using Taylor collocation method

To solve equation (3.3) by the Taylor collocation method we implement algorithm 3.4. Thus for N=2 we have

$$F(x,t) = e^{(t-x)}$$

$$\mathrm{Y}(x) = \mathrm{a}_0 + \mathrm{a}_1 x + \mathrm{a}_2 x^2$$

$$x_0 = 0 \,, x_1 = 0.5 \,, x_2 = 1$$

Algorithm 3.4

```
Input $\bar{M}_0, M_1, X_{x_i}, \bar{X}, \bar{J}, P_1, P_2$
Input $F, \bar{F}$
Calculate $W_2 = Multiply(\bar{X}, \bar{F}, \bar{J}, \bar{M}_0)$
Calculate $W_1 = Multiply(P_1, X_{x_i}, M_1)$
Calculate $W_0 = Multiply(P_0, X_{x_i}, M_0)$
Calculate $W_3 = W_0 + W_1$
Calculate $W = W_3 - W_2$
Input $R$
Calculate $A = W^{-1}R$
Input initial condition $Y(0)$
Input $\widetilde{W}, \tilde{R}$
Calculate $A = \widetilde{W}^{-1}\tilde{R}$
Input $Y_{exact}(x), Y_{approximate}(x)$
Define error $= |Y_{exact}(x) - Y_{approximate}(x)|$
```

Plot $Y_{exact}(x), Y_{approximate}(x)$

$$F = \begin{bmatrix} 1.0 & 1.0 & 0.5 \\ -1.0 & -1.0 & -0.5 \\ 0.5 & 0.5 & 0.25 \end{bmatrix}$$

$$X_{x_i} = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 1.0 & 0.5 & 0.25 \\ 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$$J = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.125 & 0.04166667 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.125 & 0.04166667 & 0.015625 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.04166667 & 0.015625 & 0.00625 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1 & 0.5 & 0.33333333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.33333333 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.33333333 & 0.25 & 0.2 \end{bmatrix}$$

$$M = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.5 \\ 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.5 \\ 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$\bar{X} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.5 & 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$$W = \begin{bmatrix} 1.0 & 1.0 & 0.0 \\ 0.596354166666667 & 1.39095052083333 & 0.606119791666667 \\ 0.166666666666667 & 1.52083333333333 & 1.32916666666667 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}$$

From initial condition:

$$Y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \ , \ \lambda_1 = 1$$

Hence

$$\widetilde{W} = \begin{bmatrix} 1.0 & 1.0 & 0.0 \\ 0.596354166666667 & 1.39095052083333 & 0.606119791666667 \\ 1.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\tilde{R} = \begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$$

The solution for this system is:

$$\widetilde{W}^{-1}\tilde{R} = \begin{bmatrix} 1.0 \\ -1.0 \\ 1.31095596133189 \end{bmatrix}$$

$$Y(x) = 1 - x + 1.31095596133189 \cdot x^2$$

Table 3.7 contains both the exact and approximate solutions of equation (3.3) and the absolute error $\left|\mathrm{Y_{app}} - \mathrm{Y_{ex}}\right|$ using Taylor collocation method.

Table 3.7

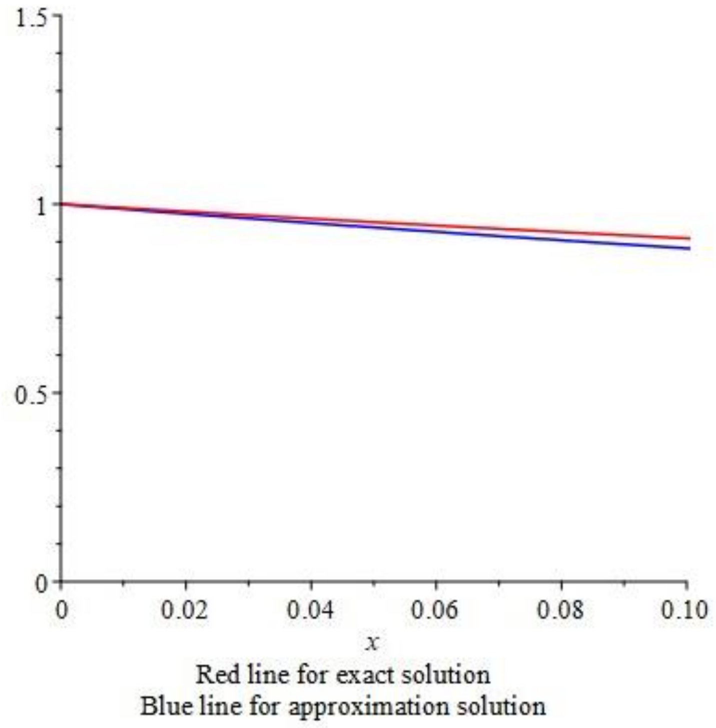
The approximate and the exact solution of applying Taylor collocation method for equation (3.3)

$\boldsymbol{x}$	$\mathbf{Y_{app}}$	$\mathbf{Y_{ex}}$	$\left\|\mathbf{Y_{app} - Y_{ex}}\right\|$
0.0	1.0	1.0	0.0
0.001	0.9990013110	0.9990009993	3.117 $\mathbf{10^{-7}}$
0.002	0.9980052438	0.9980039947	1.2491 $\mathbf{10^{-6}}$
0.003	0.9970117986	0.9970089820	2.8166 $\mathbf{10^{-6}}$
0.004	0.9960209753	0.9960159574	5.0179 $\mathbf{10^{-6}}$
0.005	0.9950327739	0.9950249169	7.8570 $\mathbf{10^{-6}}$
0.006	0.9940471944	0.9940358564	1.13380 $\mathbf{10^{-5}}$
0.007	0.9930642368	0.9930487720	1.54648 $\mathbf{10^{-5}}$
0.008	0.9920839012	0.9920636598	2.02414 $\mathbf{10^{-5}}$
0.009	0.9911061874	0.9910805159	2.56715 $\mathbf{10^{-5}}$
0.01	0.9901310956	0.9900993362	3.17594 $\mathbf{10^{-5}}$

Figure 3.9 compares the exact solution with the approximate solution using Taylor collocation method.

Figure 3.9

The approximate and the exact solution of applying Taylor collocation method for equation (3.3)



The maximum error corresponding to $\mathrm{Y_{app}}$ is $E \approx 3.17594 \times 10^{-5}$

3.8 The numerical treatment of equation (3.3) using Legendre polynomials method

To solve equation (3.3) by the Legendre polynomials method we implement algorithm 3.5.

Thus for N=2 we have

We write the solution $\mathrm{Y}(x)$ into the form

$$\mathrm{Y}(x) = \mathrm{aL_0}(x) + \mathrm{bL_1}(x) + \mathrm{cL_2}(x)$$

$$\mathrm{L_0}(x) = 1, \mathrm{L_1}(x) = x \,, \mathrm{L_2}(x) = \frac{3}{2}x^2 - \frac{3}{2}$$

Algorithm 3.5

```
Define $\mathrm{L}_0(x)$, $\mathrm{L}_1(x)$, $\mathrm{L}_2(x)$
Input $L, A, \Omega$
Calculate $S = multiply(\mathrm{L}, \Omega^T)$
Calculate $h_1 = multiply(\mathrm{S}+\mathrm{L}, \mathrm{A})$
Input $H$
Input $L^*$= integral $L$
Calculate $j_2 = multiply(\mathrm{L}, \mathrm{F}, L^*)$
Evaluate $j_3 = R + h_2$
$For\ i\ from\ 0\ to\ 2$
    $x[i] = \frac{i}{2}$
$end\ do$:
Evaluate $j_1(x[i]) = j_3(x[i])$
Input initial condition $Y(0)$
Solving the system of equations to get $a, b, c$
Input $Y_{exact}(x), Y_{approximate}(x)$
Define error $= \left|Y_{exact}(x) - Y_{approximate}(x)\right|$
Plot $Y_{exact}(x), Y_{approximate}(x)$
```

$$\mathrm{L}(x) = [\mathrm{L}_0(x) \quad \mathrm{L}_1(x) \quad \mathrm{L}_2(x)]$$

$$\mathrm{A} = \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix}$$

$$\Omega^{T} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

we find S

$$S = L(x)\Omega^{T} = [x \quad 4.5x^2 - 4.5 \quad 0]^{T}$$

and the matrix F:

$$\begin{bmatrix} 1.08616127\,a + 0.6321205588\,b - 0.9481808382\,c & 0.6321205588\,a + 0.4540407108\,b - 0.4139412942\,c & -0.9481808382\,a - 0.4139412942\,b + 1.061376508\,c \\ 0.4540407108\,a + 0.2642411177\,b - 0.3963616765\,c & 0.2642411177\,a + 0.1897995931\,b - 0.1730371029\,c & -0.3963616765\,a - 0.1730371029\,b + 0.443680103\,c \\ -1.21530061\,a - 0.707276647\,b + 1.060914971\,c & -0.707276647\,a - 0.5080239633\,b + 0.4631569192\,c & 1.060914971\,a + 0.4631569192\,b - 1.187569060\,c \end{bmatrix}$$

$$\int_0^x L(t)dt = \begin{pmatrix} x \\ 0.5x^2 \\ 0.5x^3 - 1.5x \end{pmatrix}$$

Hence

$$g(x) = L(x)F\int_0^x L^{T}(t)dt$$

Finally we obtain the exact solution as

{a = -7.051555864, b = 0.222222222, c = -5.367703907}

Hence the solution of the VIDE is:

$$x \rightarrow -7.051555864 + 0.222222222\,x - 5.367703907 \cdot (1.5)(x^2 - 1)$$

$$Y := x \rightarrow 0.9999999965 + 0.222222222\,x - 8.0515558605\,x^2$$

Table 3.8 contains both the exact and approximate solutions of equation (3.3) and the absolute error $|Y_{app} - Y_{ex}|$ using Legendre polynomials method.

Table 3.8

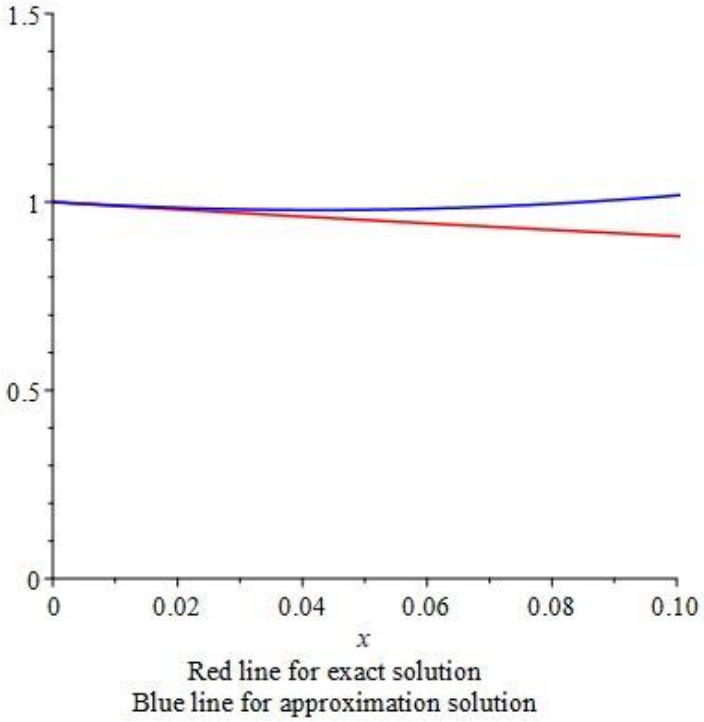
The approximate and the exact solution of applying Legendre polynomials method for equation (3.3)

x	$\mathbf{Y_{app}}$	$\mathbf{Y_{ex}}$	$\lvert\mathbf{Y_{app}} - \mathbf{Y_{ex}}\rvert$
0.0	0.9999999965	1.0	$\mathbf{3.5\ 10^{-9}}$
0.001	1.000214167	0.9990009993	$1.2131677\ \mathbf{10^{-3}}$
0.002	1.000412235	0.9980039947	$2.4082403\ \mathbf{10^{-3}}$
0.003	1.000594199	0.9970089820	$3.5852170\ \mathbf{10^{-3}}$
0.004	1.000760060	0.9960159574	$4.7441026\ \mathbf{10^{-3}}$
0.005	1.000909819	0.9950249169	$5.8849021\ \mathbf{10^{-3}}$
0.006	1.001043474	0.9940358564	$7.0076176\ \mathbf{10^{-3}}$
0.007	1.001161026	0.9930487720	$8.1122540\ \mathbf{10^{-3}}$
0.008	1.001262474	0.9920636598	$9.1988142\ \mathbf{10^{-3}}$
0.009	1.001347820	0.9910805159	$1.02673041\ \mathbf{10^{-2}}$
0.01	1.001417063	0.9900993362	$1.13177268\ \mathbf{10^{-2}}$

Figure 3.10 compares the exact solution with the approximate solution using Legendre polynomials method.

Figure 3.10

The approximate and the exact solution of applying Legendre polynomials method for equation (3.3)



The maximum error corresponding to $\mathrm{Y_{app}}$ is $E \approx 1.13177268 \times 10^{-2}$

3.9 The numerical treatment of equation (3.3) using Haar wavelets method

To solve equation (3.3) by the Haar wavelets method we implement algorithm 3.6. Thus for M=4 we have

Algorithm 3.6

Define $h_i, p_i, for\ i = 1,2,...$

Define $\tau[l], t[l]$

Input $Y'(l)$

Integral $Y'(l)$ once

Define $\Delta G\big(x_l, t, Y(t)\big), \Delta\tilde{G}\big(x_l, t, Y(t)\big)$

Calculate $\varphi(l) = \sum_{s=1}^{l-1} \Delta G + \Delta\tilde{G}$

Input F

Solve the equation $-Y' - Y + \varphi + R = 0$

Input $Y_{exact}(x), Y_{approximate}(x)$

Define error $= \left|Y_{exact}(x) - Y_{approximate}(x)\right|$

Plot $Y_{exact}(x), Y_{approximate}(x)$

$$R(x) \ = 0$$

$$R = [r(t[1]), r(t[2]), r(t[3]), r(t[4]), r(t[5]), r(t[6]), r(t[7]), r(t[8])]^T$$

$$Y' = [Y'(t[1]), Y'(t[2]), Y'(t[3]), Y'(t[4]), Y'(t[5]), Y'(t[6]), Y'(t[7]), Y'(t[8])]^T$$

$$Y = [Y(t[1]), Y(t[2]), Y(t[3]), Y(t[4]), Y(t[5]), Y(t[6]), Y(t[7]), Y(t[8])]^T$$

$$\varphi \coloneqq array\Big(\big[[\varphi[1]], [\varphi[2]], [\varphi[3]], [\varphi[4]], [\varphi[5]], [\varphi[6]], [\varphi[7]], [\varphi[8]]\big]\Big):$$

Then we solve the system*:*

$$-Y' - Y + \varphi + R = 0$$

$$E[1] := -Y1[1,1] - Y[1,1] + R[1,1] + \varphi[1,1] = 0:$$

$$E[2] := -Y1[2,1] - Y[2,1] + R[2,1] + \varphi[2,1] = 0:$$

$$E[3] := -Y1[3,1] - Y[3,1] + R[3,1] + \varphi[3,1] = 0:$$

$$E[4] := -Y1[4,1] - Y[4,1] + R[4,1] + \varphi[4,1] = 0:$$

$$E[5] := -Y1[5,1] - Y[5,1] + R[5,1] + \varphi[5,1] = 0:$$

$$E[6] := -Y1[6,1] - Y[6,1] + R[6,1] + \varphi[6,1] = 0:$$

$$E[7] := -Y1[7,1] - Y[7,1] + R[7,1] + \varphi[7,1] = 0:$$

$$E[8] := -Y1[8,1] - Y[8,1] + R[8,1] + \varphi[8,1] = 0:$$

solve($\{E[1], E[2], E[3], E[4], E[5], E[6], E[7], E[8]\}$, {a, b, c, d, e, f, g, k});
{a=-0.4294829821, b=-0.1994100106, c=-0.1564973539, d=-0.05550281296, e=-0.09953642127, f=-0.05949646172, g=-0.03543716762, k=-0.02098129873}

The solution $Y(x)$ is:

$$\begin{aligned} \mathrm{Y} := \mathrm{x} \rightarrow & -0.4294829821 \cdot P_1(x) - 0.1994100106 \cdot P_2(x) - 0.1564973539 \cdot P_3(x) \\ & - 0.05550281296 \cdot P_4(x) - 0.09953642127 \cdot P_5(x) \\ & - 0.05949646172 \cdot P_6(x) - 0.03543716762 \cdot P_7(x) \\ & - 0.02098129873 \cdot P_8(x) + 1 \end{aligned}$$

Table 3.9 contains both the exact and approximate solutions of equation (3.3) and the absolute error $\left|\mathrm{Y}_{\mathrm{app}} - \mathrm{Y}_{\mathrm{ex}}\right|$ using Haar wavelets method.

Table 3.9

The approximate and the exact solution of applying Haar wavelets methods for equation (3.3)

x	$\mathbf{Y_{app}}$	$\mathbf{Y_{ex}}$	$\lvert \mathbf{Y_{app}} - \mathbf{Y_{ex}} \rvert$
0.0	1.0	1.0	0.0
0.001	0.9991150732	0.9990009993	1.140739 $\mathbf{10^{-4}}$
0.002	0.9982301465	0.9980039947	2.261518 $\mathbf{10^{-4}}$
0.003	0.9973452197	0.9970089820	3.362377 $\mathbf{10^{-4}}$
0.004	0.9964602929	0.9960159574	4.443355 $\mathbf{10^{-4}}$
0.005	0.9955753662	0.9950249169	5.504493 $\mathbf{10^{-4}}$
0.006	0.9946904394	0.9940358564	6.545830 $\mathbf{10^{-4}}$
0.007	0.9938055126	0.9930487720	7.567406 $\mathbf{10^{-4}}$
0.008	0.9929205859	0.9920636598	8.569261 $\mathbf{10^{-4}}$
0.009	0.9920356591	0.9910805159	9.551432 $\mathbf{10^{-4}}$
0.01	0.9911507323	0.9900993362	1.0513961 $\mathbf{10^{-3}}$

Figure A1 in Appendix (A) shows the comparison of the exact solution with the approximate solution using Haar wavelets method.

The maximum error corresponding to $\mathrm{Y_{app}}$ is $E \approx 1.0513961 \times 10^{-3}$

Figure A2 in Appendix (A) shows the comparison of the exact solution with the solution of three methods.

Example 3.4

We assume the following VIDE

$$\mathrm{Y}'(x) = x + \frac{1}{6}x^3 + \int_0^{\mathrm{x}} (x - \mathrm{t})\mathrm{Y}(\mathrm{t})\mathrm{dt} \tag{3.4}$$

subject to the initial condition

$$Y(0) = 1$$

Equation (3.4) has the exact solution

$$Y(x) = \mathrm{e}^{x} - x$$

3.10 The numerical treatment of equation (3.4) using Taylor collocation method

To solve equation (3.3) by the Taylor collocation method we implement algorithm 3.4.

Thus for N=2 we have

$$\mathrm{F}(x,\mathrm{t}) = (x - \mathrm{t})$$

$$\mathrm{Y}(x) = \mathrm{a}_0 + \mathrm{a}_1 x + \mathrm{a}_2 x^2$$

$$x_0 = 0\,, x_1 = 0.5\,, x_2 = 1$$

First, we find the matrixes

$$K, \bar{M}_0, \bar{J}, \bar{X}\,, \bar{F}, M_1, C, W, Y:$$

$$J = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.125 & 0.04166667 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.125 & 0.04166667 & 0.015625 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.04166667 & 0.015625 & 0.00625 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1 & 0.5 & 0.33333333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.33333333 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.33333333 & 0.25 & 0.2 \end{bmatrix}$$

$$M = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.5 \\ 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.5 \\ 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$X_{x_i} = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 1.0 & 0.5 & 0.25 \\ 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$$\bar{X} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.5 & 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$$W = \begin{bmatrix} 0.0 & 1.0 & 0.0 \\ -0.125 & 0.979167 & 0.497396 \\ -0.5 & 0.833333 & 0.958333 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.0 \\ 0.520833 \\ 1.166667 \end{bmatrix}$$

From initial condition:

$$Y = (1 \quad 0 \quad 0) \ , \ \lambda_1 = 1$$

Hence

$$\widetilde{W} = \begin{bmatrix} 0.0 & 1.0 & 0.0 \\ -0.125 & 0.979167 & 0.497396 \\ 1.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\tilde{R} = \begin{bmatrix} 0.0 \\ 0.520833 \\ 1.0 \end{bmatrix}$$

The solution for this system is:

$$\widetilde{W}^{-1}\tilde{R} = \begin{bmatrix} 1.0 \\ 0.0 \\ 1.298429 \end{bmatrix}$$

$$Y(x) = 1 + 1.298429 \cdot x^2$$

Table 3.10 contains both the exact and approximate solutions of equation (3.4) and the absolute error $|Y_{app} - Y_{ex}|$ using Taylor collocation method.

Table 3.10

The approximate and the exact solution of applying Taylor collocation method for equation (3.4)

x	$\mathbf{Y_{app}}$	$\mathbf{Y_{ex}}$	$\lvert\mathbf{Y_{app}-Y_{ex}}\rvert$
0.0	1.0	1.0	0.0
0.001	1.000001298	1.000000500	$\mathbf{7.98\ 10^{-7}}$
0.002	1.000005194	1.000002001	3.193 $\mathbf{10^{-6}}$
0.003	1.000011686	1.000004505	7.181 $\mathbf{10^{-6}}$
0.004	1.000020775	1.000008011	1.2764 $\mathbf{10^{-5}}$
0.005	1.000032461	1.000012521	1.9940 $\mathbf{10^{-5}}$
0.006	1.000046743	1.000018036	2.8707 $\mathbf{10^{-5}}$
0.007	1.000063623	1.000024557	3.9066 $\mathbf{10^{-5}}$
0.008	1.000083099	1.000032086	5.1013 $\mathbf{10^{-5}}$
0.009	1.000105173	1.000040622	6.4551 $\mathbf{10^{-5}}$
0.01	1.000129843	1.000050167	7.9676 $\mathbf{10^{-5}}$

Figure A3 in Appendix (A) shows the comparison of the exact solution with the approximate solution using Taylor collocation method.

The maximum error corresponding to $\mathrm{Y_{app}}$ is $E \approx 7.9676 \times 10^{-5}$

3.11 The numerical treatment of equation (3.4) using Legendre polynomials method

To solve equation (3.4) by the Legendre polynomials method we implement algorithm 3.5.

Thus for N=2 we have

We write the solution $\mathrm{Y}(x)$ into the form

$$\mathrm{Y}(x) = \mathrm{aL_0}(x) + \mathrm{bL_1}(x) + \mathrm{cL_2}(x)$$

$$\mathrm{L_0}(x) = 1, \mathrm{L_1}(x) = x\,, \mathrm{L_2}(x) = \frac{3}{2}x^2 - \frac{3}{2}$$

In matrix form:

$$\mathrm{L}(x) = [\mathrm{L_0}(x) \quad \mathrm{L_1}(x) \quad \mathrm{L_2}(x)]$$

$$\mathrm{A} = \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix}$$

$$\Omega^{\mathrm{T}} = \begin{pmatrix} 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 \\ 0.0 & 3.0 & 0.0 \end{pmatrix}$$

we find S

$$S = L(x)\Omega^{T} = [x \quad 4.5x^2 - 4.5 \quad 0]^{T}$$

and the matrix F:

$$\begin{matrix} -0.125\,c - 0.0833333333\,b & 0.0125\,c - 0.08333333335\,b - 0.0833333333\,a & 0.225\,c + 0.0125\,b - 0.125\,a \\ -0.1458333333\,c + 1.666666667\;10^{-11}\,b + 0.08333333333\,a & -0.025\,c - 0.0138888889\,b + 1.666666667\;10^{-11}\,a & 0.2125\,c - 0.025\,b - 0.1458333333\,a \\ 0.1458333333\,b + 0.125\,a & -0.059375\,c + 0.125\,b + 0.1458333333\,a & -0.075\,c - 0.059375\,b \end{matrix}$$

And we compute the integral $\int_0^x \mathrm{L(t)dt}$

$$\int_0^x \mathrm{L(t)dt} = \begin{pmatrix} x \\ 0.5x^2 \\ 0.5x^3 - 1.5x \end{pmatrix}$$

Finally we obtain the exact solution as

{a = 1.110030746, b = -0, c = 0.0733538307}

Hence the solution of the VIDE is:

$$x \to 1.110030746 + 0.0733538307 \cdot (1.5)(x^2 - 1)$$

$$Y := x \to 0.99999999995 + 0.11003074605\, x^2$$

Table B1 in Appendix (B) contains both the exact and approximate solutions of equation (3.4) and the absolute error $|Y_{app} - Y_{ex}|$ using Legendre polynomials method.

Figure A4 in Appendix (A) shows the comparison of the exact solution with the approximate solution using Legendre polynomials method.

The maximum error corresponding to Y_{app} is $E \approx 3.9164 \times 10^{-5}$

3.12 The numerical treatment of equation (3.4) using Haar wavelets method

To solve equation (3.4) by the Haar wavelets method we implement algorithm 3.6.

Thus for M=4 we have

Then we calculate the values of

$$\tau[l]\,,\ \ t[l]\,,\ \ h_s(t[l])\,,\ \ h_s\big(\tau[l]\big)\,,\ \ p_s(t[l]),\ \ p_s\big(\tau[l]\big)$$

Then we solve the system*:*

$$-Y' + \varphi + R = 0$$

$$R \coloneqq array\Big(\Big[[r\,[1]],[r\,[2]],[r\,[3]],[r\,[4]],[r\,[5]],[r\,[6]],[r\,[7]],[r\,[8]]\Big]\Big):$$

$$\varphi \coloneqq array\Big(\Big[[\varphi[1]],[\varphi[2]],[\varphi[3]],[\varphi[4]],[\varphi[5]],[\varphi[6]],[\varphi[7]],[\varphi[8]]\Big]\Big):$$

$$Y1 \coloneqq array\Big(\Big[[\mathrm{Y}1[1]],[\mathrm{Y}1[2]],[\mathrm{Y}1[3]],[\mathrm{Y}1[4]],[\mathrm{Y}1[5]],[\mathrm{Y}1[6]],[\mathrm{Y}1[7]],[\mathrm{Y}1[8]]\Big]\Big):$$

$$E[1] \coloneqq -Y1[1,1] + R1[1,1] + \varphi[1,1] = 0:$$
$$E[2] \coloneqq -Y1[2,1] + R[2,1] + \varphi[2,1] = 0:$$
$$E[3] \coloneqq -Y1[3,1] + R[3,1] + \varphi[3,1] = 0:$$
$$E[4] \coloneqq -Y1[4,1] + R[4,1] + \varphi[4,1] = 0:$$
$$E[5] \coloneqq -Y1[5,1] + R[5,1] + \varphi[5,1] = 0:$$
$$E[6] \coloneqq -Y1[6,1] + R[6,1] + \varphi[6,1] = 0:$$
$$E[7] \coloneqq -Y1[7,1] + R[7,1] + \varphi[7,1] = 0:$$
$$E[8] \coloneqq -Y1[8,1] + R[8,1] + \varphi[8,1] = 0:$$

solve($\{E[1], E[2], E[3], E[4], E[5], E[6], E[7], E[8]\}$, {a, b, c, d, e, f, g, k});

{a=0.7176923574, b = -0.4208165660, c = -0.1613974385, d = -0.2659168783, e = -0.07095533476, f = -0.09106966578, g = -0.1168956508, k = -0.1500560606}

The solution $Y(x)$ is:

$$\mathrm{Y} \coloneqq \mathrm{x} \rightarrow 0.7176923574 \cdot P_1(x) - 0.4208165660 \cdot P_2(x) - 0.1613974385 \cdot P_3(x) - 0.2659168783 \cdot P_4(x) - 0.07095533476 \cdot P_5(x) - 0.09106966578 \cdot P_6(x) - 0.1168956508 \cdot P_7(x) - 0.1500560606 \cdot P_8(x) + 1$$

Table B2 in Appendix (B) contains both the exact and approximate solutions of equation (3.4) and the absolute error $\left|\mathrm{Y}_{\mathrm{app}} - \mathrm{Y}_{\mathrm{ex}}\right|$ using Haar wavelets method.

Figure A5 in Appendix (A) shows the comparison of the exact solution with the approximate solution using Haar wavelets method.

The maximum error corresponding to $\mathrm{Y}_{\mathrm{app}}$ is $E \approx 5.95063 \times 10^{-4}$

Figure A6 in Appendix (A) shows the comparison of the exact solution with the solution of three methods.

Chapter Four

Discussions and Conclusions

Conclusion

In this thesis we have solved VIDEs using three numerical methods. These include the Taylor collocation method, Legendre polynomial and Haar wavelets methods.

These numerical methods are implemented in the form of algorithms to solve some numerical examples using Maple software.

The numerical results show clearly that the Haar wavelet method is more efficient for solving the proposed numerical examples in comparison with the Taylor collocation method and Legendre polynomials method.

References

[1] A. Aakyüz and M. sezer, **A Chebyshev collocation method for the solution of linear integro-differential equations**, International Journal of Computer Mathematics, Vol.72, 2007, 491-507.

[2] I. S. Ali, **Haar Wavelet Collocation Technique for Solving Linear Volterra Integro Differential Equations**, Neuro Quantology, Vol. 18, Issue 7, 2020, 39-44.

[3] M. Alshammari, M. H. Al-Smadi, I. Hashim and M. A. Alias, **Numerical Investigation to Fuzzy Volterra Integro-Differential Equations via Residual Power Series Method**, ASM Science Journal, Vol. 13, 2020.

[4] R. Belinsky**, Integrals of Legendre Polynomials and Solution of Some Partial Differential Equations**, Journal of Applied Analysis, Vol. 6, No. 2, 2000, 259-282.

[5] F. Bloom, **Concavity arguments and growth estimates for damped linear integro-differential equations with applications to a class of holohedral isotropic diectrics,** Zeitschrift für Angewandte Mathematik und Physik (ZAMP), Vol. 29, Issue 4, 1978, 644-663.

[6] T. L. Bo, L. Xie and X. J. Zheng, **Numerical approach to wind ripple in desert**, International Journal of Nonlinear Science and Numerical Simulation, Vol. 8, 2007, 223-228.

[7] T. A. Burton, **Stability and Periodic Solutions of Ordinary and Functional Differential Equations**, Mathematics in Science and Engineering, Vol. 178, 1985.

[8] T. A. Burton, **Volterra Integral and Differential Equations**, Mathematics in Science and Engineering, Vol. 167, 1983.

[9] C. F. Chen and C. H. Hsiao, **Haar wavelet method for solving lumped and distributed-parameter systems**, IEEE, Vol.144, 1997, 87-94.

[10] E. Cimen, **A computational Method for volterra Integro-Differential Equation**, Journal of Science and Technology, Vol. 11(3), 2018, 347-352.

[11] P. Darania and A. Ebadian, **A method for the numerical solution of the integro-differential equations**, Applied Mathematics and Computation, Vol. 188, 2007, 657-668.

[12] W. Draidi and N. Qatanani, **Numerical schemes for solving Volterra integral equations with Carleman kernel**, International Journal of Applied Mathematics, Vol. 31, no. 5, 2018, 647-669.

[13] M. E. A. El-Mikkawy and G.S. Cheon, **Combinatorial and hypergeometric identities via the Legendre polynomials a computational approach**, Applied Mathematics and Computation, Vol. 166, 2005, 181–195.

[14] S. Fazeli and G. Hojjati, **Numerical solution of Volterra integro-differential equations by superimplicit multistep collocation methods**, Numerical Algorithms, Vol. 68, 2015, 741 – 768.

[15] M. Gachpazan, M. Erfanian and H. Beiglo, **Solving nonlinear Volterra integro-differential equation by using Legendre polynomial approximations**, Iranian Journal of Numerical Analysis and Optimization, Vol. 4, 2014, 73-83.

[16] J. Hamaydi and N. Qatanani, **Computational methods for solving linear fuzzy Volterra integral equation**, Journal of Applied Mathematics, Vol. 2017, 2017.

[17] S. Islam, I. Aziz and M. Fayyaz, **A new approach for numerical solution of integro-differential equations via Haar wavelets**, International Journal of Computer Mathematics, Vol. 90, 2013, 1971-1989.

[18] A. Issa, N. Qatanani and A. Daraghmeh, **Approximation Techniques for Solving Linear Systems of Volterra Integro-Differential Equations**, Journal of Applied Mathematics, Vol. 2020, 2020.

[19] A. Karamete and M. Sezer, **A Taylor Collocation Method for the Solution of Linear Integro Differential Equations**, International Journal of Computer Mathematics, Vol.79, 2002.

[20] A. H. Khater, A. B. Shamardan, D. K. Callebaut and M. R. A. Sakran, **Numerical solutions of integral and integro-differential equation using Legendre polynomials**, Numerical Algorithms, Vol. 46, 2007, 195-218.

[21] U. Lepik, **Haar wavelet method for nonlinear integro-differential equations,** Institute of Applied Mathematics, University of Tartu, Vol.176, 2006, 324-333.

[22] T. Okayama, **Theoretical analysis of a Sinc-Nysröm method for Volterra integro-differential equations and its improvement**, Applied Mathematics and Computation, Vol. 84(293), 2015, 1189-1215.

[23] M. Rahman, **Integral Equations and their Applications**, Dalhousie University, Canada, 2007.

[24] M. H. Reihani and Z. Abadi, **Rationalized Haar functions method for solving Fredholm and Volterra integral equations**, Journal of Computational and Applied Mathematics, Vol. 200, 2007, 12-20.

[25] K. Sayevand, M. Fardi, E. Moradi and F. H. Boroujeni, **Convergence analysis of homotopy perturbation method for Volterra integro-differential equations of fractional order**, Alexandria Engineering Journal, Vol. 52, 2013.

[26] B. Sepehrian and M. Razzaghi, **Single-term Walsh series method for the Volterra integro-differential equations**, Engineering Analysis with Boundary Elements, Vol. 28, 2004, 1315-1319.

[27] O. J. Staffans, **A Direct Lyapunov Approach to Volterra integro-differential Equations**, SIAM Journal on Mathematical Analysis, Vol. 19, 1988, 879-901.

[28] C. Tunç, **New stability and boundedness results to Volterra integro-differential equations with delay**, Journal of the Egyptian Mathematical society, Vol. 24, 2016, 210-213.

[29] A. Wazwaz, **Linear and Nonlinear integral equations: Methods and Application**, Springer Heidelberg, Dordrechi London, 2011.

[30] A. Wazwaz , **The variational iteration method for solving linear and nonlinear Volterra integral and integro-differential equations**, International Journal of Computer Mathematics, Vol. 87, 2010, 1131-1141.

[31] B. Zhang, **Construction of Liapunov functionals for linear Volterra integro-differential equations and stability of delay systems**, Electronic Journal of Qualitative Theory of Differential Equations, Vol. 30, 2000, 1-17.

Appendices

Appendix A

Figures of Study

Figure A1

The approximate and the exact solution of applying Haar wavelets methods for equation (3.3).

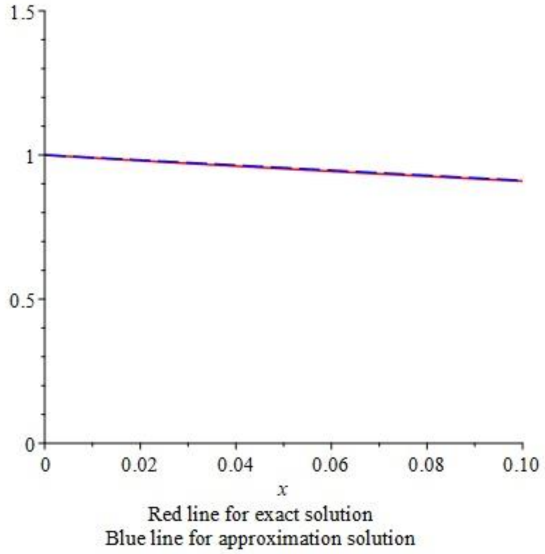


Figure A2

The solution of three methods and exact solution for equation (3.3).

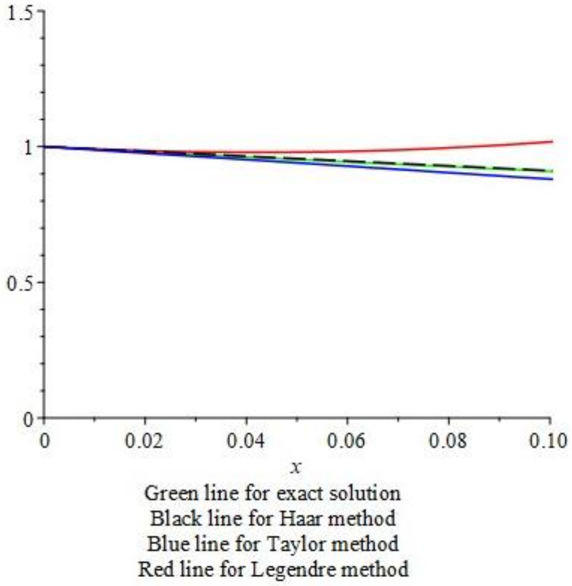


Figure A3

The approximate and the exact solution of applying Taylor collocation method for equation (3.4)

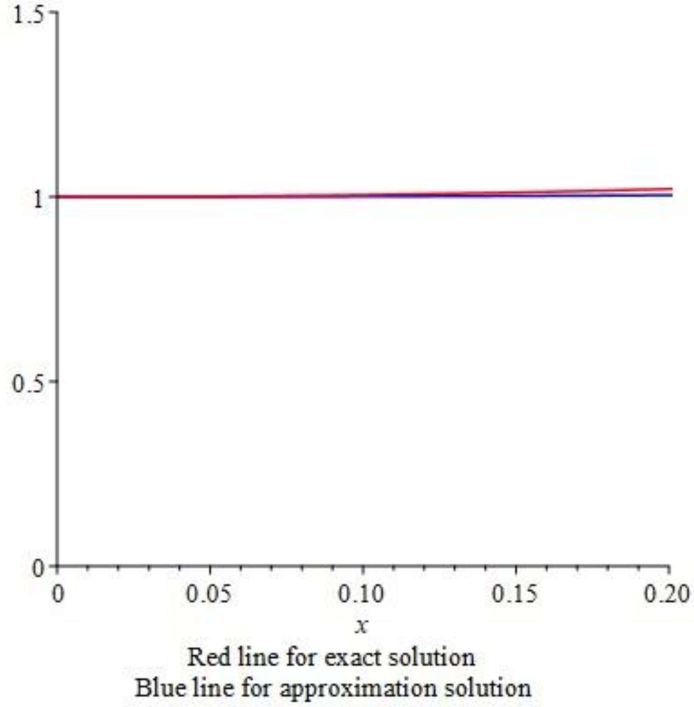


Figure A4

The approximate and the exact solution of applying Legendre polynomials method for equation (3.4)

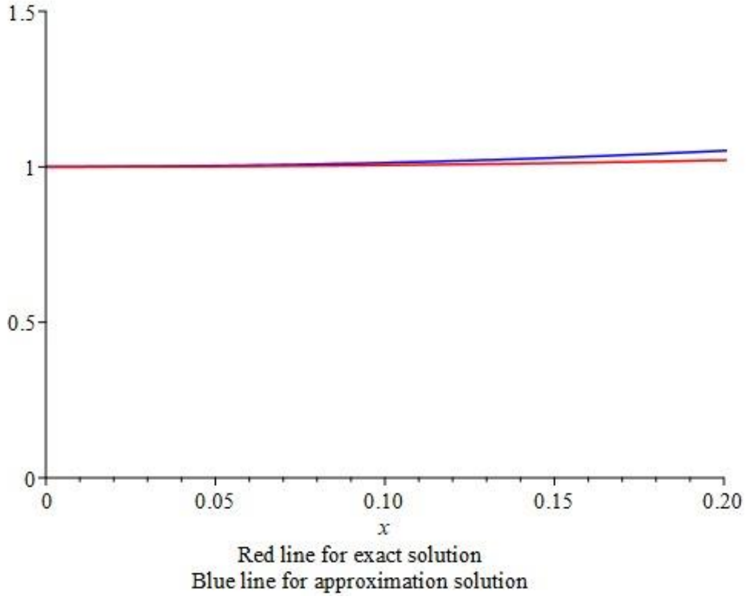


Figure A5

The approximate and the exact solutions of applying Haar wavelets method for equation (3.4).

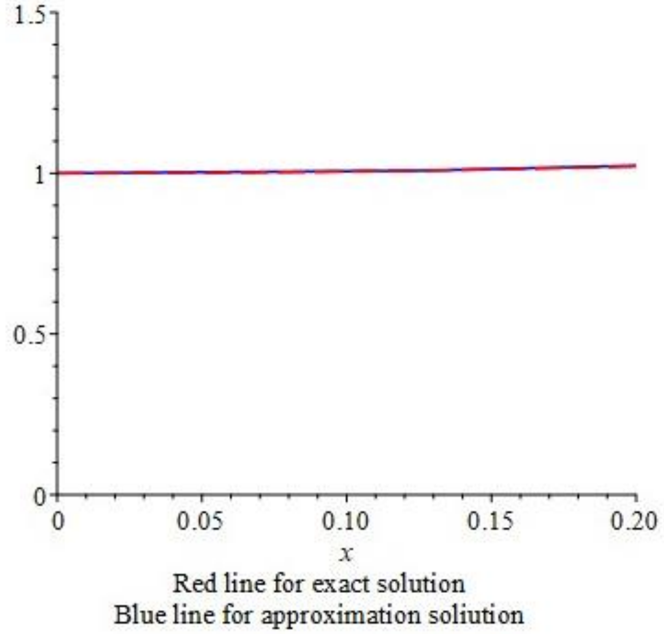
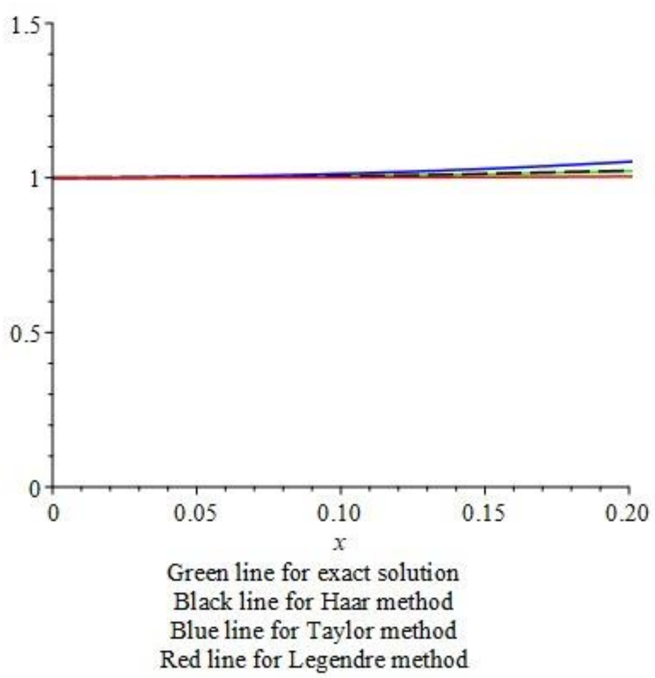


Figure A6

The solution of three methods and exact solution for equation (3.1).



Appendix B

Tables of Study

Table B1

The approximate and the exact solution of applying Legendre polynomials method for equation (3.4)

x	$\mathbf{Y_{app}}$	$\mathbf{Y_{ex}}$	$\lvert\mathbf{Y_{app}-Y_{ex}}\rvert$
0.0	1.0	1.0	0.0
0.001	1.000000110	1.000000500	3.90 $\mathbf{10^{-7}}$
0.002	1.000000440	1.000002001	1.561$\mathbf{10^{-6}}$
0.003	1.000000990	1.000004505	3.515 $\mathbf{10^{-6}}$
0.004	1.000001760	1.000008011	6.251 $\mathbf{10^{-6}}$
0.005	1.000002751	1.000012521	9.770 $\mathbf{10^{-6}}$
0.006	1.000003961	1.000018036	1.4075 $\mathbf{10^{-5}}$
0.007	1.000005392	1.000024557	1.9165 $\mathbf{10^{-5}}$
0.008	1.000007042	1.000032086	2.5044 $\mathbf{10^{-5}}$
0.009	1.000008912	1.000040622	3.1710 $\mathbf{10^{-5}}$
0.01	1.000011003	1.000050167	3.9164 $\mathbf{10^{-5}}$

Table B2

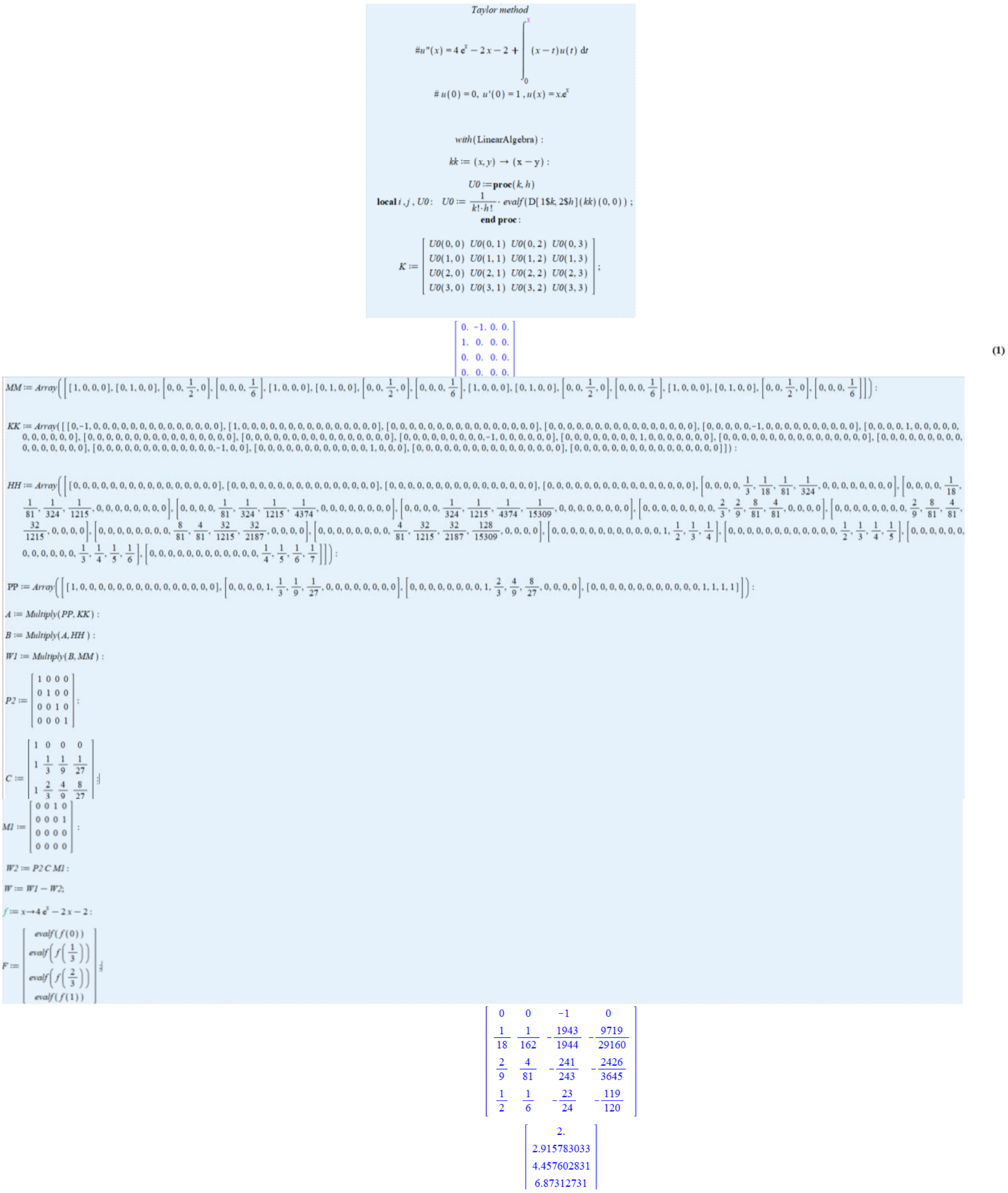
The approximate and the exact solution of applying Haar wavelets method for equation (3.4).

x	$\mathbf{Y_{app}}$	$\mathbf{Y_{ex}}$	$\lvert\mathbf{Y_{app}-Y_{ex}}\rvert$
0.0	1.0	1.0	0.0
0.001	1.000064523	1.000000500	6.4023 $\mathbf{10^{-5}}$
0.002	1.000129046	1.000002001	1.27045 $\mathbf{10^{-4}}$
0.003	1.000193569	1.000004505	1.89064 $\mathbf{10^{-4}}$
0.004	1.000258092	1.000008011	2.50081 $\mathbf{10^{-4}}$
0.005	1.000322615	1.000012521	3.10094 $\mathbf{10^{-4}}$
0.006	1.000387138	1.000018036	3.69102 $\mathbf{10^{-4}}$
0.007	1.000451661	1.000024557	4.27104 $\mathbf{10^{-4}}$
0.008	1.000516184	1.000032086	4.84098 $\mathbf{10^{-4}}$
0.009	1.000580707	1.000040622	5.40085 $\mathbf{10^{-4}}$
0.01	1.000645230	1.000050167	5.95063 $\mathbf{10^{-4}}$

Appendix C

Maple Code

Example 1:

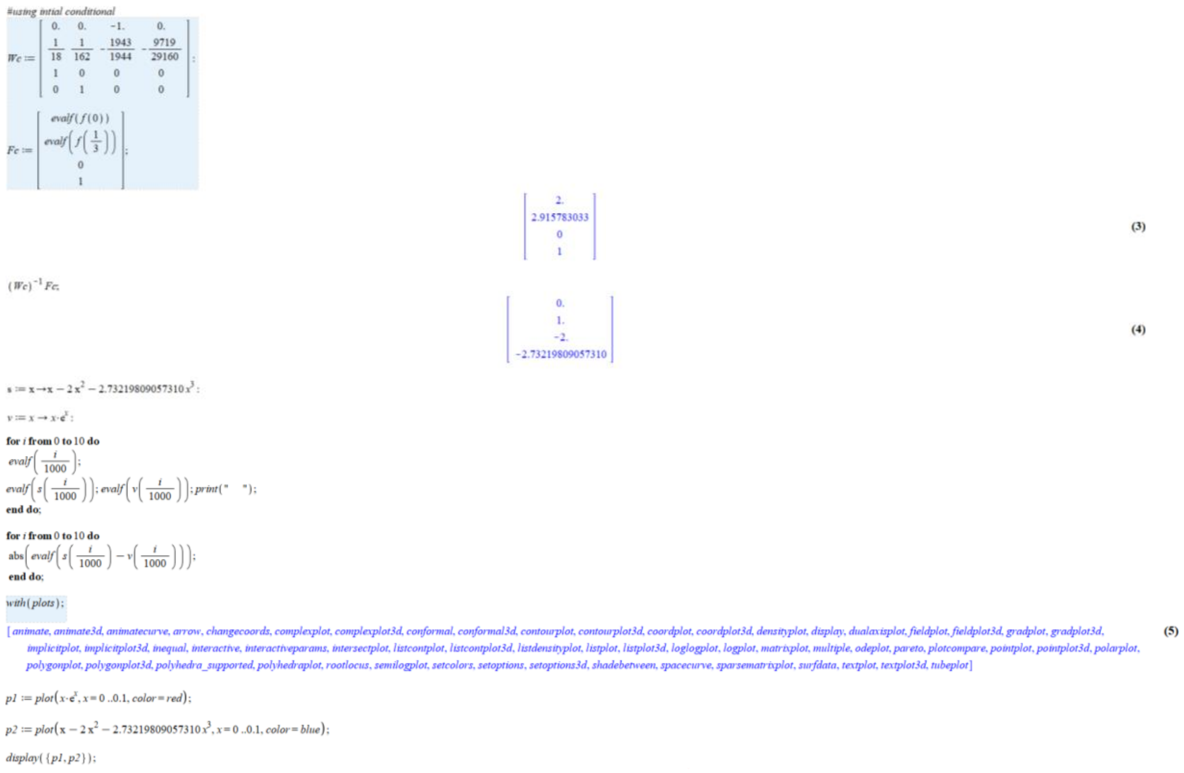


```
Taylor method

#u"(x) = 4 e^x − 2 x − 2 + ∫_0^x (x − t) u(t) dt

# u(0) = 0, u'(0) = 1 , u(x) = x.e^x


with(LinearAlgebra) :

kk := (x, y) → (x − y) :

U0 :=proc(k, h)
local i , j , U0:   U0 := 1/(k!·h!) · evalf(D[1$k, 2$h](kk)(0, 0)) ;
end proc :

      [ U0(0, 0)  U0(0, 1)  U0(0, 2)  U0(0, 3) ]
K :=  [ U0(1, 0)  U0(1, 1)  U0(1, 2)  U0(1, 3) ] ;
      [ U0(2, 0)  U0(2, 1)  U0(2, 2)  U0(2, 3) ]
      [ U0(3, 0)  U0(3, 1)  U0(3, 2)  U0(3, 3) ]
```

```
[ 0.  -1.  0.  0. ]
[ 1.   0.  0.  0. ]                                   (1)
[ 0.   0.  0.  0. ]
[ 0.   0.  0.  0. ]
```

```
MM := Array([[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1/2, 0], [0, 0, 0, 1/6], [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1/2, 0], [0, 0, 0, 1/6], [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1/2, 0], [0, 0, 0, 1/6], [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1/2, 0], [0, 0, 0, 1/6]]) :


KK := Array([[0,−1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0,−1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0,−1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,−1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]]) :


HH := Array([[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1/3, 1/18, 1/81, 1/324, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1/18,
    1/81, 1/324, 1/1215, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1/81, 1/324, 1/1215, 1/4374, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1/324, 1/1215, 1/4374, 1/15309, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 2/3, 2/9, 8/81, 4/81, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 2/9, 8/81, 4/81,
    32/1215, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 8/81, 4/81, 32/1215, 32/2187, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 4/81, 32/1215, 32/2187, 128/15309, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1/2, 1/3, 1/4], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 1/3, 1/4, 1/5], [0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 1/3, 1/4, 1/5, 1/6], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/4, 1/5, 1/6, 1/7]]) :

PP := Array([[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 1/3, 1/9, 1/27, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 2/3, 4/9, 8/27, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1]]) :

A := Multiply(PP, KK) :

B := Multiply(A, HH ) :

W1 := Multiply(B, MM ) :

       [ 1 0 0 0 ]
P2 :=  [ 0 1 0 0 ] :
       [ 0 0 1 0 ]
       [ 0 0 0 1 ]

      [ 1   0    0    0   ]
C :=  [ 1  1/3  1/9  1/27 ] :
      [ 1  2/3  4/9  8/27 ]

       [ 0 0 1 0 ]
M1 :=  [ 0 0 0 1 ] :
       [ 0 0 0 0 ]
       [ 0 0 0 0 ]

W2 := P2 C M1 :

W := W1 − W2;

f := x→4 e^x − 2 x − 2 :

      [ evalf(f(0))   ]
F :=  [ evalf(f(1/3)) ] :
      [ evalf(f(2/3)) ]
      [ evalf(f(1))   ]
```

```
[  0      0        −1          0       ]
[ 1/18   1/162   −1943/1944  −9719/29160 ]
[ 2/9    4/81    −241/243    −2426/3645  ]
[ 1/2    1/6     −23/24      −119/120    ]

[      2.      ]
[ 2.915783033 ]
[ 4.457602831 ]
[ 6.87312731  ]
```



#using intial conditional

$$Wc := \begin{bmatrix} 0. & 0. & -1. & 0. \\ \frac{1}{18} & \frac{1}{162} & -\frac{1943}{1944} & -\frac{9719}{29160} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}:$$

$$Fc := \begin{bmatrix} evalf(f(0)) \\ evalf\left(f\left(\frac{1}{3}\right)\right) \\ 0 \\ 1 \end{bmatrix};$$

$$\begin{bmatrix} 2. \\ 2.915783033 \\ 0 \\ 1 \end{bmatrix} \quad (3)$$

$$(Wc)^{-1}\, Fc;$$

$$\begin{bmatrix} 0. \\ 1. \\ -2. \\ -2.73219809057310 \end{bmatrix} \quad (4)$$

$$s := x \rightarrow x - 2x^2 - 2.73219809057310\, x^3 :$$

$$v := x \rightarrow x \cdot e^x :$$

for i **from** 0 **to** 10 **do**

$$evalf\left(\frac{i}{1000}\right);$$

$$evalf\left(s\left(\frac{i}{1000}\right)\right); evalf\left(v\left(\frac{i}{1000}\right)\right); print(\text{"\quad "});$$

end do;

for i **from** 0 **to** 10 **do**

$$\mathrm{abs}\left(evalf\left(s\left(\frac{i}{1000}\right) - v\left(\frac{i}{1000}\right)\right)\right);$$

end do;

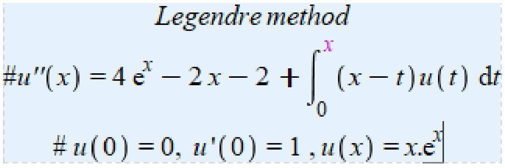
with(plots);

[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot] (5)

$$p1 := plot(x \cdot e^x, x = 0 ..0.1, color = red);$$

$$p2 := plot(x - 2x^2 - 2.73219809057310\, x^3, x = 0 ..0.1, color = blue);$$

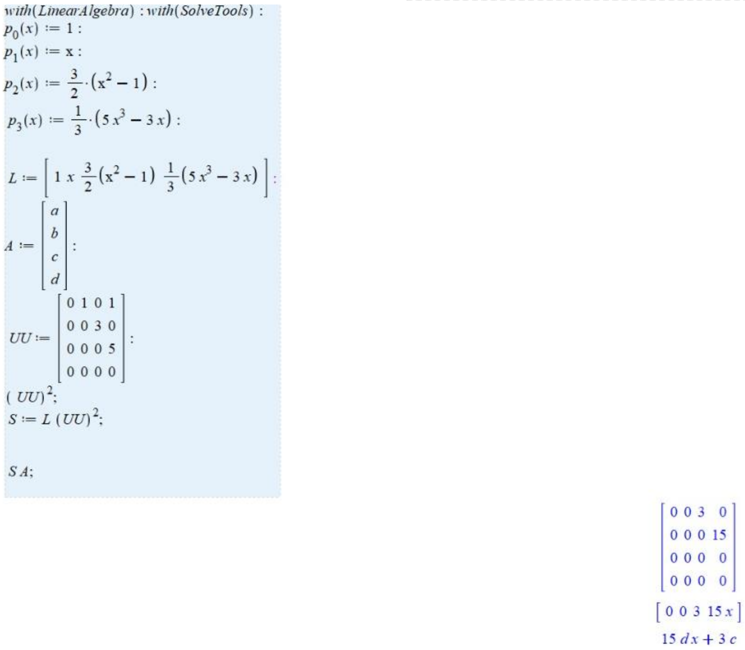
$$display(\{p1, p2\});$$



Legendre method

$$\#u''(x) = 4\, e^x - 2x - 2 + \int_0^x (x - t)\, u(t)\, dt$$

$$\#\, u(0) = 0,\ u'(0) = 1,\ u(x) = x.e^x$$



$$with(LinearAlgebra) : with(SolveTools) :$$

$$p_0(x) := 1 :$$

$$p_1(x) := x :$$

$$p_2(x) := \frac{3}{2} \cdot (x^2 - 1) :$$

$$p_3(x) := \frac{1}{3} \cdot (5x^3 - 3x) :$$

$$L := \begin{bmatrix} 1 & x & \frac{3}{2}(x^2 - 1) & \frac{1}{3}(5x^3 - 3x) \end{bmatrix} :$$

$$A := \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} :$$

$$UU := \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} :$$

$$(UU)^2;$$

$$S := L\,(UU)^2;$$

$$S\,A;$$

$$\begin{bmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 3 & 15x \end{bmatrix}$$

$$15\,d\,x + 3\,c$$

$hh := x \rightarrow 3\,c + 15\,x\;d;$

$$x \rightarrow 3\,c + 15\,x\,d$$

for n **from** 0 **to** 3 **do**
 for m **from** 0 **to** 3 **do**

$$e := evalf\left(\int_0^1 \left((x-t)\cdot\left(a + b\cdot t + c\cdot\left(\frac{3}{2}\cdot t^2 - \frac{3}{2}\right) + d\cdot\left(\frac{1}{3}\left(5\,t^3 - 3\,t\right)\right)\right)\cdot\left(p_m(t)\right)\right)\mathrm{d}t\right):$$

$$k[n, m] := \int_0^1 \left(\left(e\right)\cdot p_n(x)\right)\mathrm{d}x :$$

 end do:
 end do :

$$K := \begin{bmatrix} k[0,0] & k[0,1] & k[0,2] & k[0,3] \\ k[1,0] & k[1,1] & k[1,2] & k[1,3] \\ k[2,0] & k[2,1] & k[2,2] & k[2,3] \\ k[3,0] & k[3,1] & k[3,2] & k[3,3] \end{bmatrix};$$

$$IL := \begin{bmatrix} x \\ \frac{1}{2}x^2 \\ \frac{1}{2}x^3 - \frac{3}{2}x \\ \frac{5}{12}x^4 - \frac{1}{2}x^2 \end{bmatrix} :/$$

$$[[-0.04166666665\,d - 0.1250000000\,c - 0.08333333330\,b, 0.01250000000\,c - 0.08333333335\,b - 0.08333333330\,a - 0.02777777778\,d, 0.02619047621\,d + 0.2250000000\,c + 0.01250000000\,b - 0.1250000000\,a, -0.009920634925\,d + 0.02619047619\,c - 0.04166666666\,a - 0.02777777778\,b], [-0.02777777777\,d - 0.1458333333\,c + 1.666666667\,10^{-11}\,b + 0.08333333333\,a, -0.02500000000\,c - 0.01388888890\,b + 1.666666667\,10^{-11}\,a - 0.01388888889\,d, 0.02698412700\,d + 0.2125000000\,c - 0.02500000000\,b - 0.1458333333\,a, 0.0003306878283\,d + 0.02698412698\,c - 0.02777777778\,a - 0.01388888889\,b], [0.03124999999\,d + 0.1458333333\,b + 0.1250000000\,a, -0.05937500000\,c + 0.1250000000\,b + 0.1458333333\,a + 0.02777777778\,d, -0.005357142872\,d - 0.07500000000\,c - 0.05937500000\,b, 0.01785714286\,d - 0.005357142859\,c + 0.03125000000\,a + 0.02777777778\,b], [-0.03125000000\,c + 0.02777777778\,b + 0.04166666667\,a, -0.01666666667\,c + 0.02083333333\,b + 0.02777777778\,a + 0.002314814815\,d, 0.004761904762\,d + 0.03125000000\,c - 0.01666666667\,b - 0.03125000000\,a, 0.003472222222\,d + 0.004761904762\,c + 0.002314814815\,b]] \tag{3}$$

$Y := K\,IL:$
$N := L\,Y;$

$$(-0.04166666665\,d - 0.1250000000\,c - 0.08333333330\,b)\,x + \frac{1}{2}\,(0.01250000000\,c - 0.08333333335\,b - 0.08333333330\,a - 0.02777777778\,d)\,x^2 + (0.02619047621\,d + 0.2250000000\,c + 0.01250000000\,b - 0.1250000000\,a)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) + (-0.009920634925\,d + 0.02619047619\,c - 0.04166666666\,a - 0.02777777778\,b)\left(\frac{5}{12}x^4 - \frac{1}{2}x^2\right) + x\left((-0.02777777777\,d - 0.1458333333\,c + 1.666666667\,10^{-11}\,b + 0.08333333333\,a)\,x + \frac{1}{2}\,(-0.02500000000\,c - 0.01388888890\,b + 1.666666667\,10^{-11}\,a - 0.01388888889\,d)\,x^2 + (0.02698412700\,d + 0.2125000000\,c - 0.02500000000\,b - 0.1458333333\,a)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) + (0.0003306878283\,d + 0.02698412698\,c - 0.02777777778\,a - 0.01388888889\,b)\left(\frac{5}{12}x^4 - \frac{1}{2}x^2\right)\right) + \left(\frac{3}{2}x^2 - \frac{3}{2}\right)\left((0.03124999999\,d + 0.1458333333\,b + 0.1250000000\,a)\,x + \frac{1}{2}\,(-0.05937500000\,c + 0.1250000000\,b + 0.1458333333\,a + 0.02777777778\,d)\,x^2 + (-0.005357142872\,d - 0.07500000000\,c - 0.05937500000\,b)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) + (0.01785714286\,d - 0.005357142859\,c + 0.03125000000\,a + 0.02777777778\,b)\left(\frac{5}{12}x^4 - \frac{1}{2}x^2\right)\right) + \left(\frac{5}{3}x^3 - x\right)\left((-0.03125000000\,c + 0.02777777778\,b + 0.04166666667\,a)\,x + \frac{1}{2}\,(-0.01666666667\,c + 0.02083333333\,b + 0.02777777778\,a + 0.002314814815\,d)\,x^2 + (0.004761904762\,d + 0.03125000000\,c - 0.01666666667\,b - 0.03125000000\,a)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) + (0.003472222222\,d + 0.004761904762\,c + 0.002314814815\,b)\left(\frac{5}{12}x^4 - \frac{1}{2}x^2\right)\right) \tag{4}$$

$4\,e^x - 2\,x - 2 + N;$

$$4\,e^x - 2\,x - 2 + (-0.04166666665\,d - 0.1250000000\,c - 0.08333333330\,b)\,x + \frac{1}{2}\,(0.01250000000\,c - 0.08333333335\,b - 0.08333333330\,a - 0.02777777778\,d)\,x^2 + (0.02619047621\,d + 0.2250000000\,c + 0.01250000000\,b - 0.1250000000\,a)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) + (-0.009920634925\,d + 0.02619047619\,c - 0.04166666666\,a - 0.02777777778\,b)\left(\frac{5}{12}x^4 - \frac{1}{2}x^2\right) + x\left((-0.02777777777\,d - 0.1458333333\,c + 1.666666667\,10^{-11}\,b + 0.08333333333\,a)\,x + \frac{1}{2}\,(-0.02500000000\,c - 0.01388888890\,b + 1.666666667\,10^{-11}\,a - 0.01388888889\,d)\,x^2 + (0.02698412700\,d + 0.2125000000\,c - 0.02500000000\,b - 0.1458333333\,a)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) + (0.0003306878283\,d + 0.02698412698\,c - 0.02777777778\,a - 0.01388888889\,b)\left(\frac{5}{12}x^4 - \frac{1}{2}x^2\right)\right) + \left(\frac{3}{2}x^2 - \frac{3}{2}\right)\left((0.03124999999\,d + 0.1458333333\,b + 0.1250000000\,a)\,x + \frac{1}{2}\,(-0.05937500000\,c + 0.1250000000\,b + 0.1458333333\,a + 0.02777777778\,d)\,x^2 + (-0.005357142872\,d - 0.07500000000\,c - 0.05937500000\,b)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) + (0.01785714286\,d - 0.005357142859\,c + 0.03125000000\,a + 0.02777777778\,b)\left(\frac{5}{12}x^4 - \frac{1}{2}x^2\right)\right) + \left(\frac{5}{3}x^3 - x\right)\left((-0.03125000000\,c + 0.02777777778\,b + 0.04166666667\,a)\,x + \frac{1}{2}\,(-0.01666666667\,c + 0.02083333333\,b + 0.02777777778\,a + 0.002314814815\,d)\,x^2 + (0.004761904762\,d + 0.03125000000\,c - 0.01666666667\,b - 0.03125000000\,a)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) + (0.003472222222\,d + 0.004761904762\,c + 0.002314814815\,b)\left(\frac{5}{12}x^4 - \frac{1}{2}x^2\right)\right) \tag{5}$$

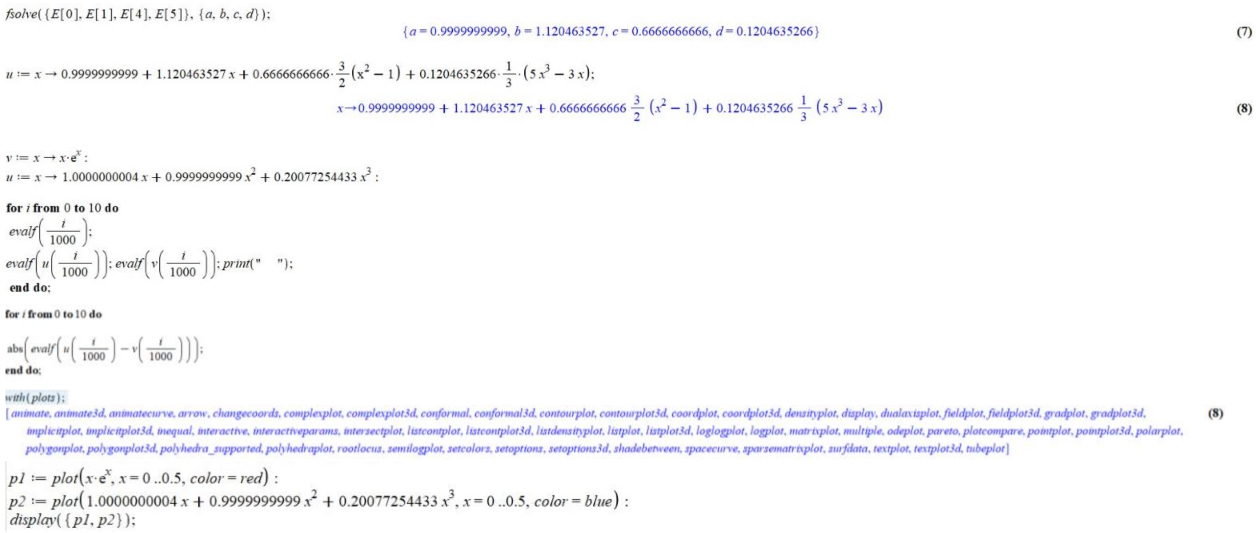
for i **from** 0 **to** 3 **do**
$x[i] := \frac{i}{3};$
end do:

$$h := x \rightarrow 4\,e^x - 2\,x - 2 + (-0.04166666665\cdot d - 0.1250000000\cdot c - 0.08333333330\cdot b)\cdot x + \frac{1}{2}\,(0.01250000000\cdot c - 0.08333333335\cdot b - 0.08333333330\cdot a - 0.02777777778\cdot d)\cdot x^2 + (0.02619047621\cdot d + 0.2250000000\cdot c + 0.01250000000\cdot b - 0.1250000000\cdot a)\cdot\left(\frac{1}{2}\cdot x^3 - \frac{3}{2}\cdot x\right) + (-0.009920634925\cdot d + 0.02619047619\cdot c - 0.04166666666\cdot a - 0.02777777778\cdot b)\cdot\left(\frac{5}{12}\cdot x^4 - \frac{1}{2}\cdot x^2\right) + x\cdot\left((-0.02777777777\cdot d - 0.1458333333\cdot c + 1.666666667\,10^{-11}\cdot b + 0.08333333333\cdot a)\cdot x + \frac{1}{2}\,(-0.02500000000\cdot c - 0.01388888890\cdot b + 1.666666667\,10^{-11}\cdot a - 0.01388888889\cdot d)\cdot x^2 + (0.02698412700\cdot d + 0.2125000000\cdot c - 0.02500000000\cdot b - 0.1458333333\cdot a)\left(\frac{1}{2}\cdot x^3 - \frac{3}{2}\cdot x\right) + (0.0003306878283\cdot d + 0.02698412698\cdot c - 0.02777777778\cdot a - 0.01388888889\cdot b)\left(\frac{5}{12}\cdot x^4 - \frac{1}{2}\cdot x^2\right)\right) + \left(\frac{3}{2}\cdot x^2 - \frac{3}{2}\right)\cdot\left((0.03124999999\cdot d + 0.1458333333\cdot b + 0.1250000000\cdot a)\cdot x + \frac{1}{2}\,(-0.05937500000\cdot c + 0.1250000000\cdot b + 0.1458333333\cdot a + 0.02777777778\cdot d)\cdot x^2 + (-0.005357142872\cdot d - 0.07500000000\cdot c - 0.05937500000\cdot b)\cdot\left(\frac{1}{2}\cdot x^3 - \frac{3}{2}\cdot x\right) + (0.01785714286\cdot d - 0.005357142859\cdot c + 0.03125000000\cdot a + 0.02777777778\cdot b)\left(\frac{5}{12}\cdot x^4 - \frac{1}{2}\cdot x^2\right)\right) + \left(\frac{5}{3}\cdot x^3 - x\right)\cdot\left((-0.03125000000\cdot c + 0.02777777778\cdot b + 0.04166666667\cdot a)\cdot x + \frac{1}{2}\,(-0.01666666667\cdot c + 0.02083333333\cdot b + 0.02777777778\cdot a + 0.002314814815\cdot d)\cdot x^2 + (0.004761904762\cdot d + 0.03125000000\cdot c - 0.01666666667\cdot b - 0.03125000000\cdot a)\cdot\left(\frac{1}{2}x^3 - \frac{3}{2}\cdot x\right) + (0.003472222222\cdot d + 0.004761904762\cdot c + 0.002314814815\cdot b)\left(\frac{5}{12}\cdot x^4 - \frac{1}{2}\cdot x^2\right)\right):$$

for i **from** 0 **to** 3 **do**
$h(x[i]);$
$hh(x[i]);$
$E[i] := h(x[i]) = hh(x[i]);$
end do:

$E[4] := a - \frac{3}{2}c = 0:$

$E[5] := b - d = 1:$



$$fsolve(\{E[0], E[1], E[4], E[5]\}, \{a, b, c, d\});$$

$$\{a = 0.9999999999,\ b = 1.120463527,\ c = 0.6666666666,\ d = 0.1204635266\} \qquad (7)$$

$$u := x \rightarrow 0.9999999999 + 1.120463527\,x + 0.6666666666 \cdot \frac{3}{2} \cdot (x^2 - 1) + 0.1204635266 \cdot \frac{1}{3} \cdot (5\,x^3 - 3\,x);$$

$$x \rightarrow 0.9999999999 + 1.120463527\,x + 0.6666666666\,\frac{3}{2}\,(x^2 - 1) + 0.1204635266\,\frac{1}{3}\,(5\,x^3 - 3\,x) \qquad (8)$$

$$v := x \rightarrow x \cdot e^x :$$
$$u := x \rightarrow 1.0000000004\,x + 0.9999999999\,x^2 + 0.20077254433\,x^3 :$$

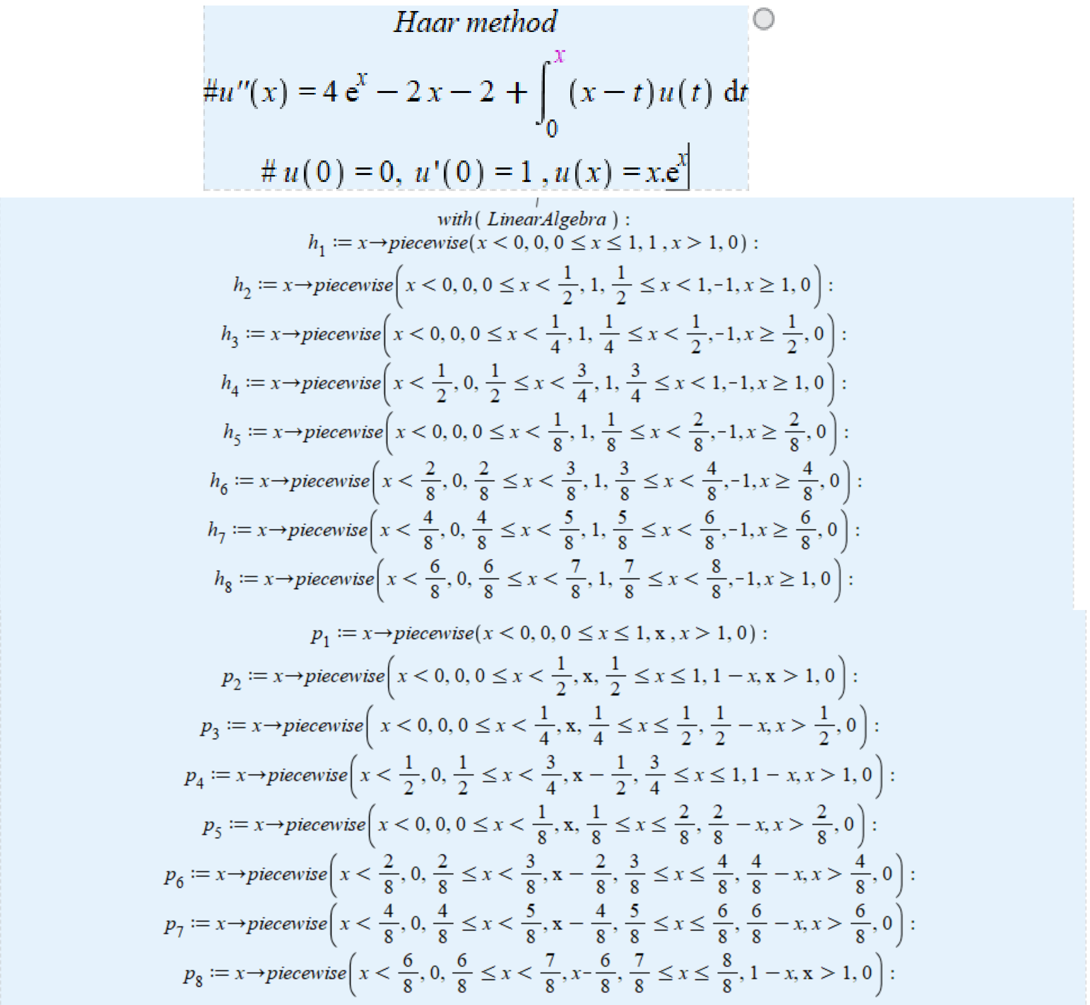
for i **from** 0 **to** 10 **do**
$evalf\left(\frac{i}{1000}\right);$
$evalf\left(u\left(\frac{i}{1000}\right)\right); evalf\left(v\left(\frac{i}{1000}\right)\right); print("\quad");$
end do;

for i **from** 0 **to** 10 **do**

$abs\left(evalf\left(u\left(\frac{i}{1000}\right) - v\left(\frac{i}{1000}\right)\right)\right);$
end do;

with(plots);
[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot] (8)

$$p1 := plot(x \cdot e^x, x = 0 ..0.5, color = red) :$$
$$p2 := plot(1.0000000004\,x + 0.9999999999\,x^2 + 0.20077254433\,x^3, x = 0 ..0.5, color = blue) :$$
$$display(\{p1, p2\});$$



Haar method

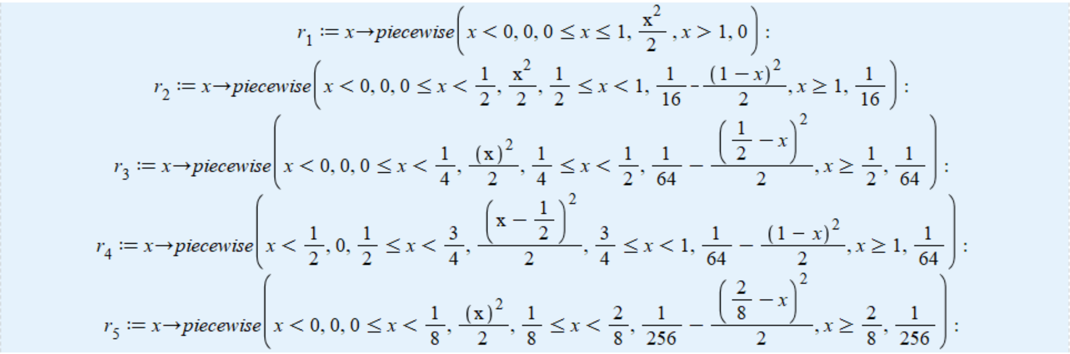
$$\#u''(x) = 4\,e^x - 2\,x - 2 + \int_0^x (x - t)\,u(t)\,dt$$

$$\#\,u(0) = 0,\ u'(0) = 1\,, u(x) = x.e^x$$

$$with(\,LinearAlgebra\,) :$$

$$h_1 := x \rightarrow piecewise(x < 0, 0, 0 \le x \le 1, 1\,, x > 1, 0) :$$
$$h_2 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{2}, 1, \frac{1}{2} \le x < 1, -1, x \ge 1, 0\right) :$$
$$h_3 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{4}, 1, \frac{1}{4} \le x < \frac{1}{2}, -1, x \ge \frac{1}{2}, 0\right) :$$
$$h_4 := x \rightarrow piecewise\left(x < \frac{1}{2}, 0, \frac{1}{2} \le x < \frac{3}{4}, 1, \frac{3}{4} \le x < 1, -1, x \ge 1, 0\right) :$$
$$h_5 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{8}, 1, \frac{1}{8} \le x < \frac{2}{8}, -1, x \ge \frac{2}{8}, 0\right) :$$
$$h_6 := x \rightarrow piecewise\left(x < \frac{2}{8}, 0, \frac{2}{8} \le x < \frac{3}{8}, 1, \frac{3}{8} \le x < \frac{4}{8}, -1, x \ge \frac{4}{8}, 0\right) :$$
$$h_7 := x \rightarrow piecewise\left(x < \frac{4}{8}, 0, \frac{4}{8} \le x < \frac{5}{8}, 1, \frac{5}{8} \le x < \frac{6}{8}, -1, x \ge \frac{6}{8}, 0\right) :$$
$$h_8 := x \rightarrow piecewise\left(x < \frac{6}{8}, 0, \frac{6}{8} \le x < \frac{7}{8}, 1, \frac{7}{8} \le x < \frac{8}{8}, -1, x \ge 1, 0\right) :$$

$$p_1 := x \rightarrow piecewise(x < 0, 0, 0 \le x \le 1, \mathrm{x}\,, x > 1, 0) :$$
$$p_2 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{2}, \mathrm{x}, \frac{1}{2} \le x \le 1, 1 - x, \mathrm{x} > 1, 0\right) :$$
$$p_3 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{4}, \mathrm{x}, \frac{1}{4} \le x \le \frac{1}{2}, \frac{1}{2} - x, x > \frac{1}{2}, 0\right) :$$
$$p_4 := x \rightarrow piecewise\left(x < \frac{1}{2}, 0, \frac{1}{2} \le x < \frac{3}{4}, \mathrm{x} - \frac{1}{2}, \frac{3}{4} \le x \le 1, 1 - x, x > 1, 0\right) :$$
$$p_5 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{8}, \mathrm{x}, \frac{1}{8} \le x \le \frac{2}{8}, \frac{2}{8} - x, x > \frac{2}{8}, 0\right) :$$
$$p_6 := x \rightarrow piecewise\left(x < \frac{2}{8}, 0, \frac{2}{8} \le x < \frac{3}{8}, \mathrm{x} - \frac{2}{8}, \frac{3}{8} \le x \le \frac{4}{8}, \frac{4}{8} - x, x > \frac{4}{8}, 0\right) :$$
$$p_7 := x \rightarrow piecewise\left(x < \frac{4}{8}, 0, \frac{4}{8} \le x < \frac{5}{8}, \mathrm{x} - \frac{4}{8}, \frac{5}{8} \le x \le \frac{6}{8}, \frac{6}{8} - x, x > \frac{6}{8}, 0\right) :$$
$$p_8 := x \rightarrow piecewise\left(x < \frac{6}{8}, 0, \frac{6}{8} \le x < \frac{7}{8}, x - \frac{6}{8}, \frac{7}{8} \le x \le \frac{8}{8}, 1 - x, \mathrm{x} > 1, 0\right) :$$



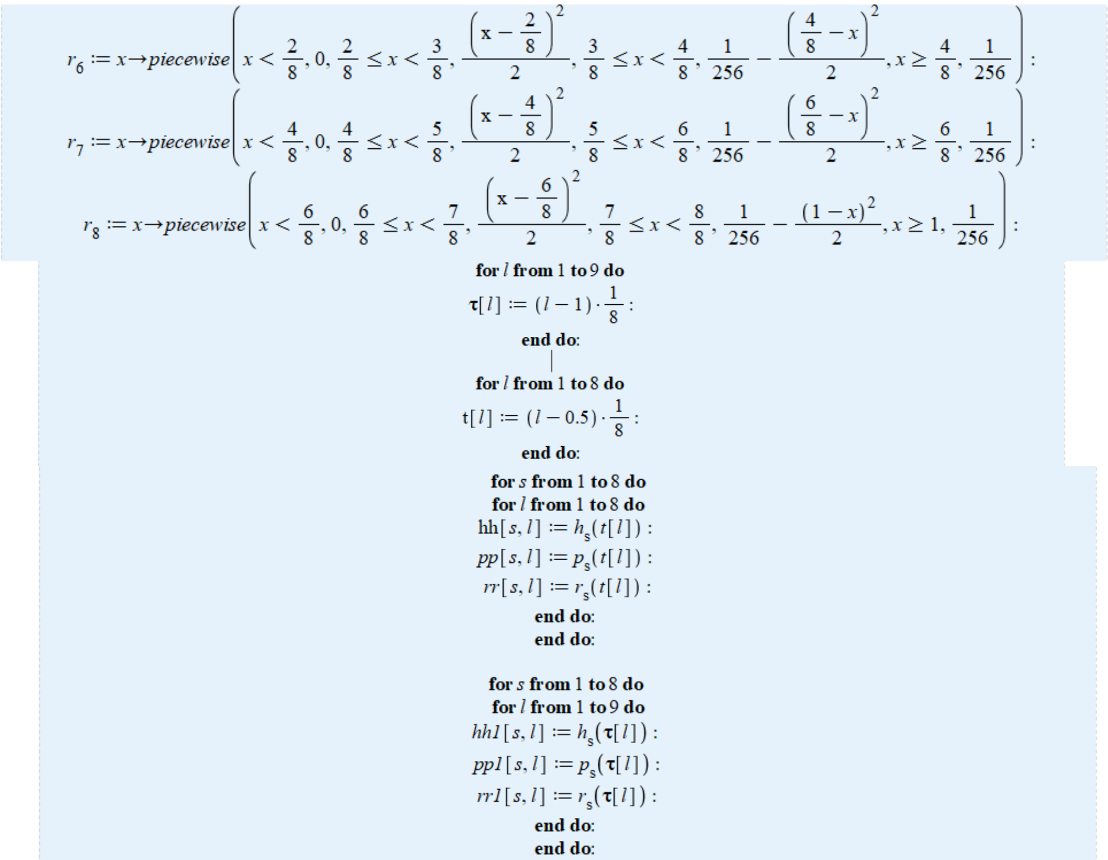
$$r_1 := x \to piecewise\left(x < 0, 0, 0 \le x \le 1, \frac{x^2}{2}, x > 1, 0\right):$$

$$r_2 := x \to piecewise\left(x < 0, 0, 0 \le x < \frac{1}{2}, \frac{x^2}{2}, \frac{1}{2} \le x < 1, \frac{1}{16} - \frac{(1-x)^2}{2}, x \ge 1, \frac{1}{16}\right):$$

$$r_3 := x \to piecewise\left(x < 0, 0, 0 \le x < \frac{1}{4}, \frac{(x)^2}{2}, \frac{1}{4} \le x < \frac{1}{2}, \frac{1}{64} - \frac{\left(\frac{1}{2} - x\right)^2}{2}, x \ge \frac{1}{2}, \frac{1}{64}\right):$$

$$r_4 := x \to piecewise\left(x < \frac{1}{2}, 0, \frac{1}{2} \le x < \frac{3}{4}, \frac{\left(x - \frac{1}{2}\right)^2}{2}, \frac{3}{4} \le x < 1, \frac{1}{64} - \frac{(1-x)^2}{2}, x \ge 1, \frac{1}{64}\right):$$

$$r_5 := x \to piecewise\left(x < 0, 0, 0 \le x < \frac{1}{8}, \frac{(x)^2}{2}, \frac{1}{8} \le x < \frac{2}{8}, \frac{1}{256} - \frac{\left(\frac{2}{8} - x\right)^2}{2}, x \ge \frac{2}{8}, \frac{1}{256}\right):$$

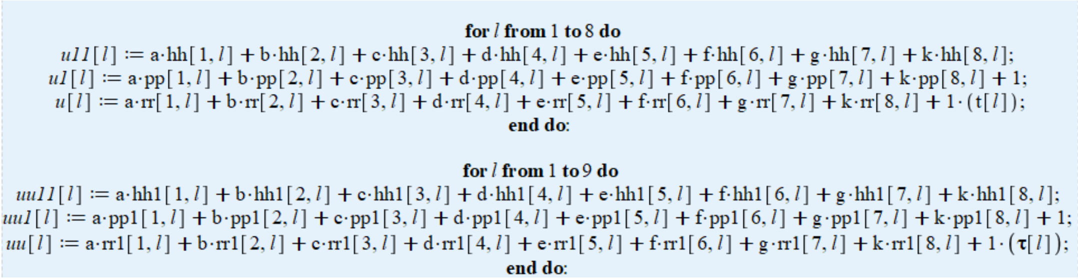


$$r_6 := x \to piecewise\left(x < \frac{2}{8}, 0, \frac{2}{8} \le x < \frac{3}{8}, \frac{\left(x - \frac{2}{8}\right)^2}{2}, \frac{3}{8} \le x < \frac{4}{8}, \frac{1}{256} - \frac{\left(\frac{4}{8} - x\right)^2}{2}, x \ge \frac{4}{8}, \frac{1}{256}\right):$$

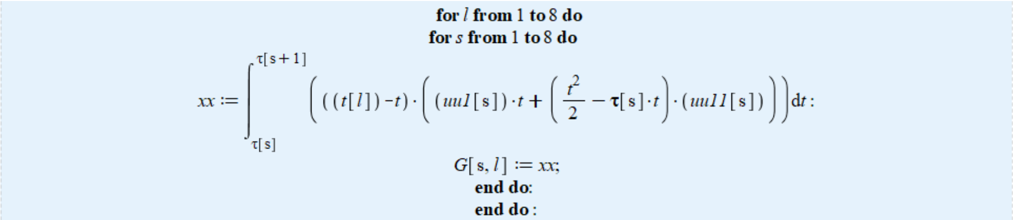
$$r_7 := x \to piecewise\left(x < \frac{4}{8}, 0, \frac{4}{8} \le x < \frac{5}{8}, \frac{\left(x - \frac{4}{8}\right)^2}{2}, \frac{5}{8} \le x < \frac{6}{8}, \frac{1}{256} - \frac{\left(\frac{6}{8} - x\right)^2}{2}, x \ge \frac{6}{8}, \frac{1}{256}\right):$$

$$r_8 := x \to piecewise\left(x < \frac{6}{8}, 0, \frac{6}{8} \le x < \frac{7}{8}, \frac{\left(x - \frac{6}{8}\right)^2}{2}, \frac{7}{8} \le x < \frac{8}{8}, \frac{1}{256} - \frac{(1-x)^2}{2}, x \ge 1, \frac{1}{256}\right):$$

```
for l from 1 to 9 do
τ[l] := (l − 1)·1/8 :
end do:

for l from 1 to 8 do
t[l] := (l − 0.5)·1/8 :
end do:

for s from 1 to 8 do
for l from 1 to 8 do
hh[s, l] := h_s(t[l]) :
pp[s, l] := p_s(t[l]) :
rr[s, l] := r_s(t[l]) :
end do:
end do:

for s from 1 to 8 do
for l from 1 to 9 do
hh1[s, l] := h_s(τ[l]) :
pp1[s, l] := p_s(τ[l]) :
rr1[s, l] := r_s(τ[l]) :
end do:
end do:
```



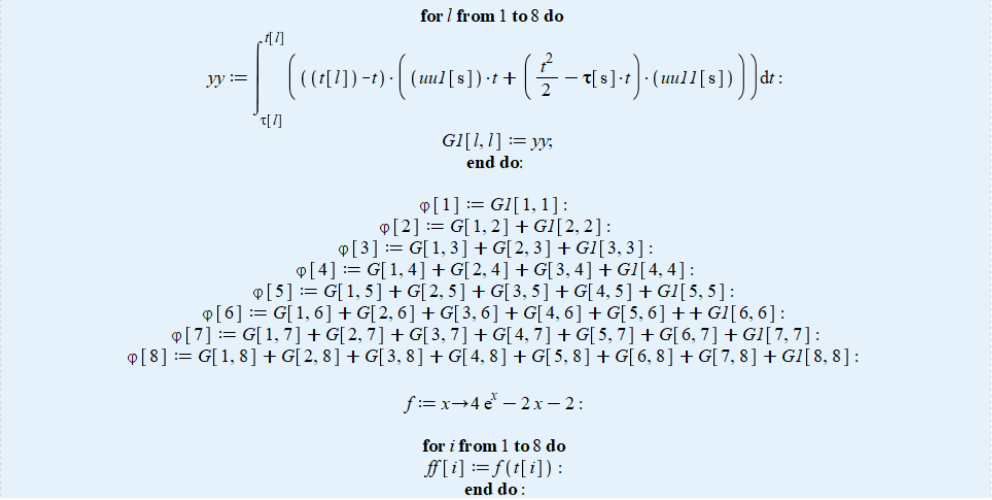
```
for l from 1 to 8 do
u11[l] := a·hh[1, l] + b·hh[2, l] + c·hh[3, l] + d·hh[4, l] + e·hh[5, l] + f·hh[6, l] + g·hh[7, l] + k·hh[8, l];
u1[l] := a·pp[1, l] + b·pp[2, l] + c·pp[3, l] + d·pp[4, l] + e·pp[5, l] + f·pp[6, l] + g·pp[7, l] + k·pp[8, l] + 1;
u[l] := a·rr[1, l] + b·rr[2, l] + c·rr[3, l] + d·rr[4, l] + e·rr[5, l] + f·rr[6, l] + g·rr[7, l] + k·rr[8, l] + 1·(t[l]);
end do:

for l from 1 to 9 do
uu11[l] := a·hh1[1, l] + b·hh1[2, l] + c·hh1[3, l] + d·hh1[4, l] + e·hh1[5, l] + f·hh1[6, l] + g·hh1[7, l] + k·hh1[8, l];
uu1[l] := a·pp1[1, l] + b·pp1[2, l] + c·pp1[3, l] + d·pp1[4, l] + e·pp1[5, l] + f·pp1[6, l] + g·pp1[7, l] + k·pp1[8, l] + 1;
uu[l] := a·rr1[1, l] + b·rr1[2, l] + c·rr1[3, l] + d·rr1[4, l] + e·rr1[5, l] + f·rr1[6, l] + g·rr1[7, l] + k·rr1[8, l] + 1·(τ[l]);
end do:
```



```
for l from 1 to 8 do
for s from 1 to 8 do
```

$$xx := \int_{\tau[s]}^{\tau[s+1]} \left(((t[l]) - t)\cdot\left((uu1[s])\cdot t + \left(\frac{t^2}{2} - \tau[s]\cdot t\right)\cdot(uu11[s])\right)\right) dt :$$

```
G[s, l] := xx;
end do:
end do :
```



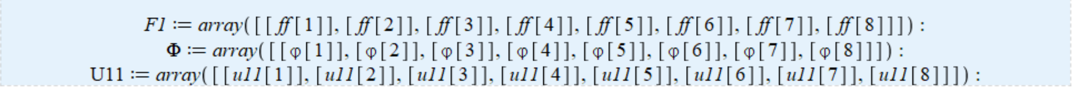
for *l* **from** 1 **to** 8 **do**

$$yy := \int_{\tau[l]}^{t[l]} \left(\left((t[l]) - t \right) \cdot \left((uu1[s]) \cdot t + \left(\frac{t^2}{2} - \tau[s] \cdot t \right) \cdot (uu11[s]) \right) \right) \mathrm{d}t :$$

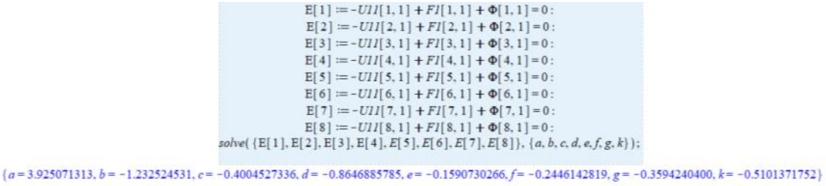
```
G1[l, l] := yy;
end do:

φ[1] := G1[1, 1] :
φ[2] := G[1, 2] + G1[2, 2] :
φ[3] := G[1, 3] + G[2, 3] + G1[3, 3] :
φ[4] := G[1, 4] + G[2, 4] + G[3, 4] + G1[4, 4] :
φ[5] := G[1, 5] + G[2, 5] + G[3, 5] + G[4, 5] + G1[5, 5] :
φ[6] := G[1, 6] + G[2, 6] + G[3, 6] + G[4, 6] + G[5, 6] + + G1[6, 6] :
φ[7] := G[1, 7] + G[2, 7] + G[3, 7] + G[4, 7] + G[5, 7] + G[6, 7] + G1[7, 7] :
φ[8] := G[1, 8] + G[2, 8] + G[3, 8] + G[4, 8] + G[5, 8] + G[6, 8] + G[7, 8] + G1[8, 8] :
```

$$f := x \rightarrow 4\,e^{x} - 2\,x - 2 :$$

```
for i from 1 to 8 do
ff[i] := f(t[i]) :
end do :
```

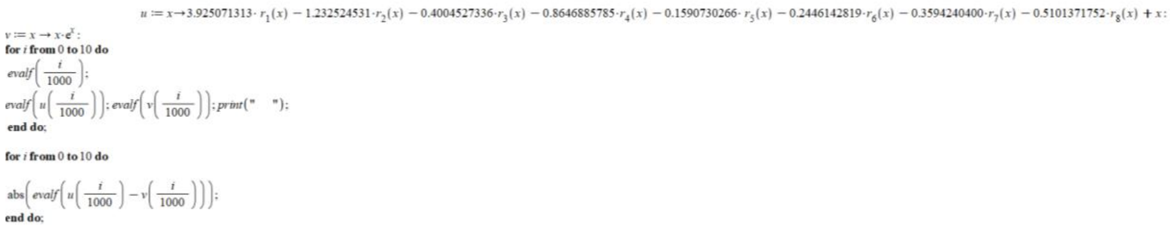


```
F1 := array([[ff[1]], [ff[2]], [ff[3]], [ff[4]], [ff[5]], [ff[6]], [ff[7]], [ff[8]]]) :
Φ := array([[φ[1]], [φ[2]], [φ[3]], [φ[4]], [φ[5]], [φ[6]], [φ[7]], [φ[8]]]) :
U11 := array([[u11[1]], [u11[2]], [u11[3]], [u11[4]], [u11[5]], [u11[6]], [u11[7]], [u11[8]]]) :
```



```
E[1] := -U11[1, 1] + F1[1, 1] + Φ[1, 1] = 0 :
E[2] := -U11[2, 1] + F1[2, 1] + Φ[2, 1] = 0 :
E[3] := -U11[3, 1] + F1[3, 1] + Φ[3, 1] = 0 :
E[4] := -U11[4, 1] + F1[4, 1] + Φ[4, 1] = 0 :
E[5] := -U11[5, 1] + F1[5, 1] + Φ[5, 1] = 0 :
E[6] := -U11[6, 1] + F1[6, 1] + Φ[6, 1] = 0 :
E[7] := -U11[7, 1] + F1[7, 1] + Φ[7, 1] = 0 :
E[8] := -U11[8, 1] + F1[8, 1] + Φ[8, 1] = 0 :
solve({E[1], E[2], E[3], E[4], E[5], E[6], E[7], E[8]}, {a, b, c, d, e, f, g, k});
```

```
{a = 3.925071313, b = -1.232524531, c = -0.4004527336, d = -0.8646885785, e = -0.1590730266, f = -0.2446142819, g = -0.3594240400, k = -0.5101371752}
```



$$u := x \rightarrow 3.925071313 \cdot r_1(x) - 1.232524531 \cdot r_2(x) - 0.4004527336 \cdot r_3(x) - 0.8646885785 \cdot r_4(x) - 0.1590730266 \cdot r_5(x) - 0.2446142819 \cdot r_6(x) - 0.3594240400 \cdot r_7(x) - 0.5101371752 \cdot r_8(x) + x :$$

$$v := x \rightarrow x \cdot e^{x} :$$

for *i* **from** 0 **to** 10 **do**

$evalf\left(\frac{i}{1000}\right);$

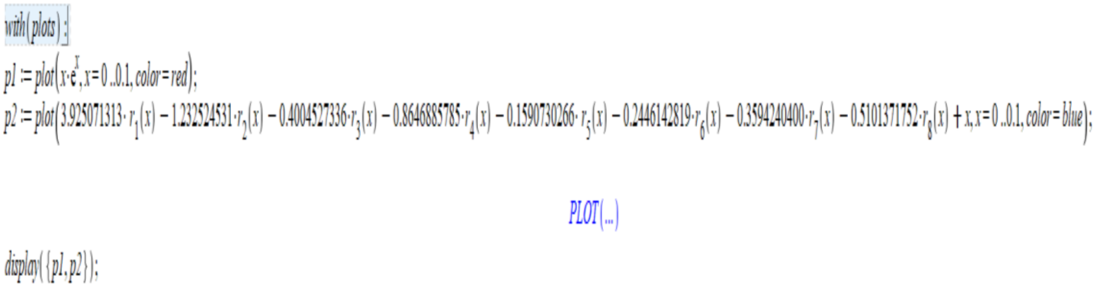
$evalf\left(u\left(\frac{i}{1000}\right)\right); evalf\left(v\left(\frac{i}{1000}\right)\right); print("\quad");$

end do;

for *i* **from** 0 **to** 10 **do**

$abs\left(evalf\left(u\left(\frac{i}{1000}\right) - v\left(\frac{i}{1000}\right)\right)\right);$

end do;



```
with(plots) :
```

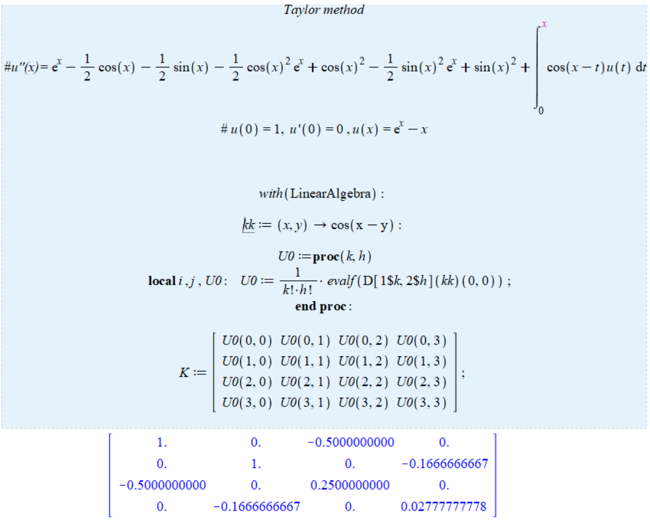
$$p1 := plot\left(x \cdot e^{x}, x = 0..0.1, color = red\right);$$

$$p2 := plot\left(3.925071313 \cdot r_1(x) - 1.232524531 \cdot r_2(x) - 0.4004527336 \cdot r_3(x) - 0.8646885785 \cdot r_4(x) - 0.1590730266 \cdot r_5(x) - 0.2446142819 \cdot r_6(x) - 0.3594240400 \cdot r_7(x) - 0.5101371752 \cdot r_8(x) + x, x = 0..0.1, color = blue\right);$$

```
PLOT(...)
```

```
display({p1, p2});
```

Example 2:



Taylor method

$$\#u''(x)= e^{x} - \frac{1}{2}\cos(x) - \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)^{2} e^{x} + \cos(x)^{2} - \frac{1}{2}\sin(x)^{2} e^{x} + \sin(x)^{2} + \int_{0}^{x} \cos(x-t)u(t)\, dt$$

$$\# u(0) = 1,\ u'(0) = 0 ,\ u(x) = e^{x} - x$$

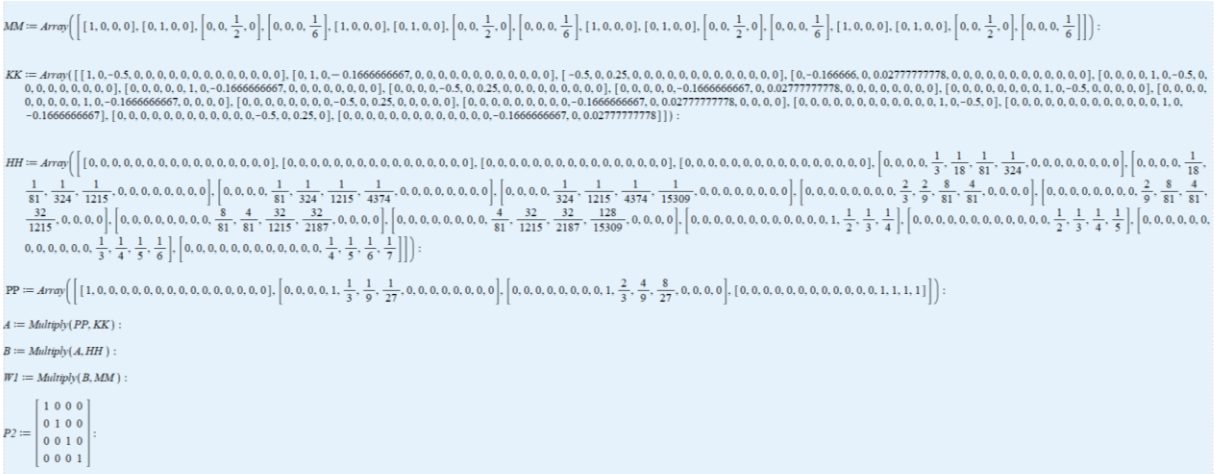
$$with(\mathrm{LinearAlgebra}) :$$

$$kk := (x, y) \rightarrow \cos(x - y) :$$

$$U0 := \mathbf{proc}(k, h)$$
$$\mathbf{local}\ i, j, U0:\quad U0 := \frac{1}{k!\cdot h!}\cdot evalf(\mathrm{D}[1\$k, 2\$h](kk)(0, 0)) ;$$
$$\mathbf{end\ proc}:$$

$$K := \begin{bmatrix} U0(0,0) & U0(0,1) & U0(0,2) & U0(0,3) \\ U0(1,0) & U0(1,1) & U0(1,2) & U0(1,3) \\ U0(2,0) & U0(2,1) & U0(2,2) & U0(2,3) \\ U0(3,0) & U0(3,1) & U0(3,2) & U0(3,3) \end{bmatrix};$$

$$\begin{bmatrix} 1. & 0. & -0.5000000000 & 0. \\ 0. & 1. & 0. & -0.1666666667 \\ -0.5000000000 & 0. & 0.2500000000 & 0. \\ 0. & -0.1666666667 & 0. & 0.02777777778 \end{bmatrix}$$



$$MM := Array\left(\left[[1,0,0,0],[0,1,0,0],\left[0,0,\frac{1}{2},0\right],\left[0,0,0,\frac{1}{6}\right],[1,0,0,0],[0,1,0,0],\left[0,0,\frac{1}{2},0\right],\left[0,0,0,\frac{1}{6}\right],[1,0,0,0],[0,1,0,0],\left[0,0,\frac{1}{2},0\right],\left[0,0,0,\frac{1}{6}\right],[1,0,0,0],[0,1,0,0],\left[0,0,\frac{1}{2},0\right],\left[0,0,0,\frac{1}{6}\right]\right]\right):$$

$$KK := Array([[1, 0, -0.5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, - 0.1666666667, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [-0.5, 0, 0.25, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, -0.166666, 0, 0.02777777778, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, -0.5, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, -0.1666666667, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, -0.5, 0, 0.25, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, -0.1666666667, 0, 0.02777777778, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -0.5, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -0.1666666667, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, -0.5, 0, 0.25, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, -0.1666666667, 0, 0.02777777778, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -0.5, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -0.1666666667], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -0.5, 0, 0.25, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -0.1666666667, 0, 0.02777777778]]) :$$

$$HH := Array\left(\left[[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],\left[0,0,0,0,\frac{1}{3},\frac{1}{18},\frac{1}{81},\frac{1}{324},0,0,0,0,0,0,0,0\right],\left[0,0,0,0,\frac{1}{18},\frac{1}{81},\frac{1}{324},\frac{1}{1215},0,0,0,0,0,0,0,0\right],\left[0,0,0,0,\frac{1}{81},\frac{1}{324},\frac{1}{1215},\frac{1}{4374},0,0,0,0,0,0,0,0\right],\left[0,0,0,0,\frac{1}{324},\frac{1}{1215},\frac{1}{4374},\frac{1}{15309},0,0,0,0,0,0,0,0\right],\left[0,0,0,0,0,0,0,0,\frac{2}{3},\frac{2}{9},\frac{8}{81},\frac{4}{81},0,0,0,0\right],\left[0,0,0,0,0,0,0,0,\frac{2}{9},\frac{8}{81},\frac{4}{81},\frac{32}{1215},0,0,0,0\right],\left[0,0,0,0,0,0,0,0,\frac{8}{81},\frac{4}{81},\frac{32}{1215},\frac{32}{2187},0,0,0,0\right],\left[0,0,0,0,0,0,0,0,\frac{4}{81},\frac{32}{1215},\frac{32}{2187},\frac{128}{15309},0,0,0,0\right],\left[0,0,0,0,0,0,0,0,0,0,0,0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4}\right],\left[0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}\right],\left[0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6}\right],\left[0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7}\right]\right]\right):$$

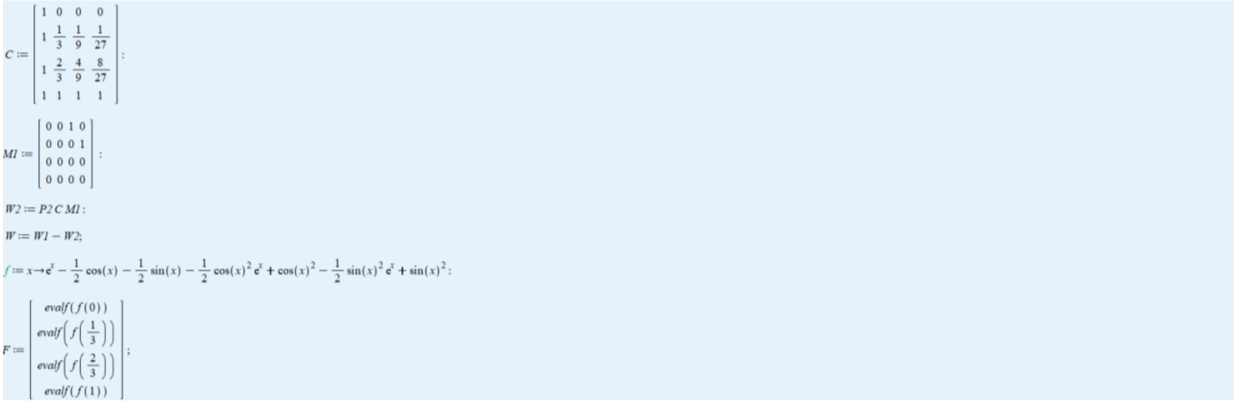
$$PP := Array\left(\left[[1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],\left[0,0,0,0,1,\frac{1}{3},\frac{1}{9},\frac{1}{27},0,0,0,0,0,0,0,0\right],\left[0,0,0,0,0,0,0,0,1,\frac{2}{3},\frac{4}{9},\frac{8}{27},0,0,0,0\right],[0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1]\right]\right):$$

$$A := Multiply(PP, KK) :$$

$$B := Multiply(A, HH) :$$

$$W1 := Multiply(B, MM) :$$

$$P2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}:$$



$$C := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} \\ 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} \\ 1 & 1 & 1 & 1 \end{bmatrix}:$$

$$M1 := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}:$$

$$W2 := P2\,C\,M1 :$$

$$W := W1 - W2;$$

$$f := x \rightarrow e^{x} - \frac{1}{2}\cos(x) - \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)^{2} e^{x} + \cos(x)^{2} - \frac{1}{2}\sin(x)^{2} e^{x} + \sin(x)^{2} :$$

$$F := \begin{bmatrix} evalf(f(0)) \\ evalf\left(f\left(\frac{1}{3}\right)\right) \\ evalf\left(f\left(\frac{2}{3}\right)\right) \\ evalf(f(1)) \end{bmatrix};$$

$$\begin{bmatrix} 0. & 0. & -1. & 0. \\ 0.326992201391962 & 0.0550058002675501 & -0.993865781902416 & -0.332821217174043 \\ 0.612203424271495 & 0.211891818657536 & -0.952223482945827 & -0.658647320165794 \\ 0.798611111086667 & 0.437499999982667 & -0.849074074080833 & -0.961640211642063 \end{bmatrix}$$

$$\begin{bmatrix} 1. \\ 1.061730391 \\ 1.271738488 \\ 1.668254269 \end{bmatrix}$$

#using intial conditional

$$Wc := \begin{bmatrix} 0. & 0. & -1. & 0. \\ 0.326992201391962 & 0.0550058002675501 & -0.993865781902416 & -0.332821217174043 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix};$$

$$Fc := \begin{bmatrix} evalf(f(0)) \\ evalf\left(f\left(\frac{1}{3}\right)\right) \\ 1 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} 0. & 0. & -1. & 0. \\ 0.326992201391962 & 0.0550058002675501 & -0.993865781902416 & -0.332821217174043 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1. \\ 1.061730391 \\ 1 \\ 0 \end{bmatrix}$$

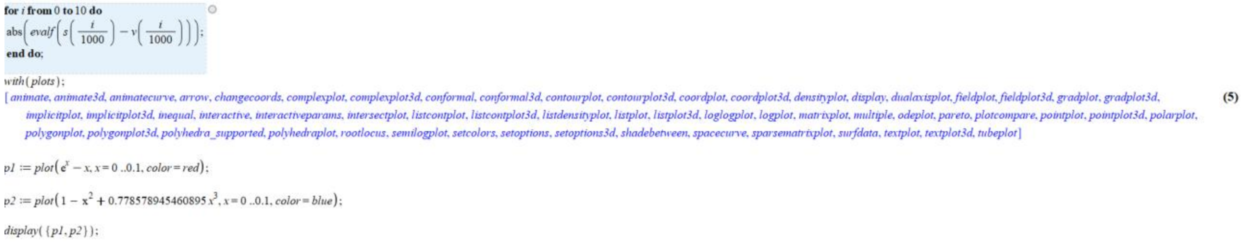
$(Wc)^{-1}\,Fc;$

$$\begin{bmatrix} 1. \\ 0. \\ -1. \\ 0.778578945460895 \end{bmatrix}$$

$s := x \rightarrow 1 - x^2 + 0.778578945460895\,x^3 :$

$v := x \rightarrow e^x - x :$

for *i* **from** 0 **to** 10 **do**
$evalf\left(\frac{i}{1000}\right);$
$evalf\left(s\left(\frac{i}{1000}\right)\right); evalf\left(v\left(\frac{i}{1000}\right)\right); print("\quad");$
end do;



for *i* **from** 0 **to** 10 **do**
$abs\left(evalf\left(s\left(\frac{i}{1000}\right) - v\left(\frac{i}{1000}\right)\right)\right);$
end do;

with(plots);

[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot*] **(5)**

$p1 := plot(e^x - x, x = 0..0.1, color = red);$

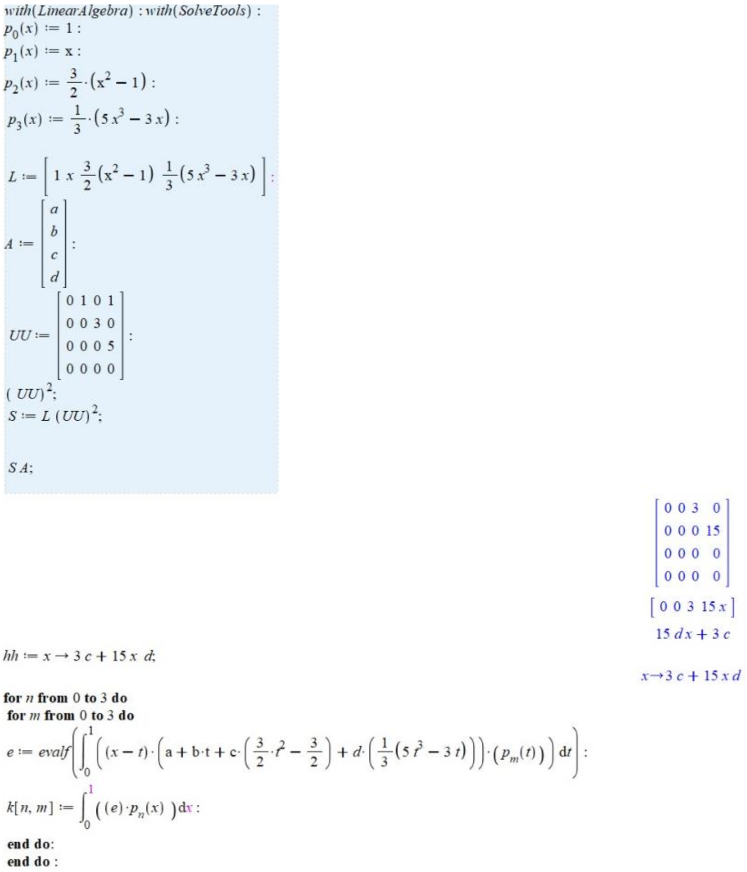
$p2 := plot(1 - x^2 + 0.778578945460895\,x^3, x = 0..0.1, color = blue);$

$display(\{p1, p2\});$

Legendre method

$$\#u''(x) = e^x - \frac{1}{2}\cos(x) - \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)^2 e^x + \cos(x)^2 - \frac{1}{2}\sin(x)^2 e^x + \sin(x)^2 + \int_0^x \cos(x-t)\,u(t)\,dt$$

$$\#\,u(0) = 1,\ u'(0) = 0\,, u(x) = e^x - x$$



$with(LinearAlgebra) : with(SolveTools) :$

$p_0(x) := 1 :$

$p_1(x) := \mathrm{x} :$

$p_2(x) := \frac{3}{2}\cdot(\mathrm{x}^2 - 1) :$

$p_3(x) := \frac{1}{3}\cdot(5\,x^3 - 3\,x) :$

$$L := \begin{bmatrix} 1 & x & \frac{3}{2}(\mathrm{x}^2-1) & \frac{1}{3}(5\,x^3-3\,x) \end{bmatrix} :$$

$$A := \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} :$$

$$UU := \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} :$$

$(\ UU)^2;$

$S := L\ (UU)^2;$

$S\,A;$

$$\begin{bmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 3 & 15\,x \end{bmatrix}$$

$$15\,d\,x + 3\,c$$

$hh := x \rightarrow 3\,c + 15\,x\ d;$

$$x \rightarrow 3\,c + 15\,x\,d$$

for n **from** 0 **to** 3 **do**

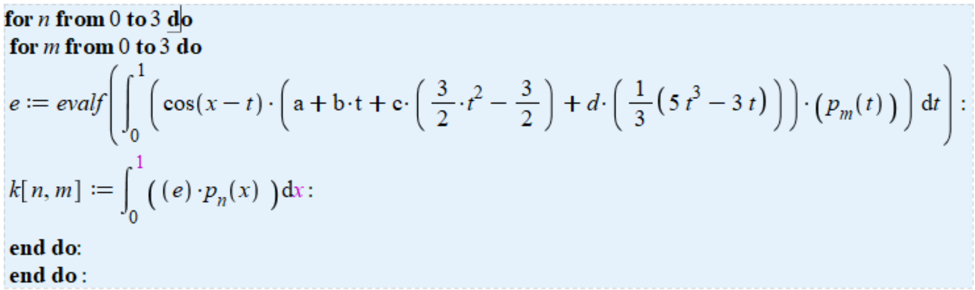
for m **from** 0 **to** 3 **do**

$$e := evalf\left(\int_0^1 \left((x-t)\cdot\left(\mathrm{a}+\mathrm{b}\cdot\mathrm{t}+\mathrm{c}\cdot\left(\frac{3}{2}\cdot t^2-\frac{3}{2}\right)+d\cdot\left(\frac{1}{3}\left(5\,t^3-3\,t\right)\right)\right)\cdot\left(p_m(t)\right)\right)\mathrm{d}t\right) :$$

$$k[n, m] := \int_0^1 \left(\,(e)\cdot p_n(x)\ \right)\mathrm{d}x :$$

end do:

end do :



for n **from** 0 **to** 3 **do**

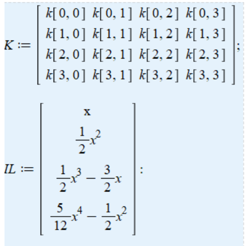
for m **from** 0 **to** 3 **do**

$$e := evalf\left(\int_0^1 \left(\cos(x-t)\cdot\left(\mathrm{a}+\mathrm{b}\cdot\mathrm{t}+\mathrm{c}\cdot\left(\frac{3}{2}\cdot t^2-\frac{3}{2}\right)+d\cdot\left(\frac{1}{3}\left(5\,t^3-3\,t\right)\right)\right)\cdot\left(p_m(t)\right)\right)\mathrm{d}t\right) :$$

$$k[n, m] := \int_0^1 \left(\,(e)\cdot p_n(x)\ \right)\mathrm{d}x :$$

end do:

end do :



$$K := \begin{bmatrix} k[0,0] & k[0,1] & k[0,2] & k[0,3] \\ k[1,0] & k[1,1] & k[1,2] & k[1,3] \\ k[2,0] & k[2,1] & k[2,2] & k[2,3] \\ k[3,0] & k[3,1] & k[3,2] & k[3,3] \end{bmatrix};$$

$$IL := \begin{bmatrix} \mathrm{x} \\ \frac{1}{2}x^2 \\ \frac{1}{2}x^3 - \frac{3}{2}x \\ \frac{5}{12}x^4 - \frac{1}{2}x^2 \end{bmatrix} :$$

$$\begin{gathered}[[0.9193953883\,a - 0.9233197516\,c - 0.08315688770\,d + 0.4596976941\,b, -0.00485723834\,d + 0.3038488872\,b + 0.4596976941\,a - 0.3506598156\,c, -0.9233197516\,a + 1.104958351\,c + 0.1572559633\,d - 0.3506598156\,b, -0.00485723834\,b \\ + 0.057737772\,d - 0.08315688770\,a + 0.1572559633\,c], \\ [0.2364534186\,b - 0.4517530184\,c - 0.03828410910\,d + 0.4596976941\,a, 0.2364534186\,a - 0.1763277494\,c + 0.1585290152\,b - 0.000242997445\,d, -0.4517530184\,a + 0.0765382148\,d - 0.1763277494\,b + 0.5346253646\,c, \\ -0.03828410909\,a + 0.0765382148\,c - 0.00024299745\,b + 0.0296427927\,d], \\ [-0.4517530184\,b + 0.9421211517\,c + 0.08845333815\,d - 0.9233197516\,a, -0.4517530184\,a + 0.0081564036\,d - 0.2952389837\,b + 0.3506598156\,c, 0.9421211517\,a - 1.136455491\,c + 0.3506598156\,b - 0.1610618477\,d, -0.056823370\,d \\ + 0.00815640361\,b + 0.08845333815\,a - 0.1610618477\,c], \\ [0.00916452261\,d - 0.03828410915\,b + 0.08845333834\,c - 0.08315688786\,a, -0.03828410915\,a - 0.02418799563\,b + 0.001529504071\,d + 0.03121853582\,c, -0.015265755\,d + 0.03121853577\,b + 0.08845333834\,a - 0.1088459803\,c, \\ 0.00916452260\,a - 0.015265755\,c + 0.001529504071\,b - 0.00483620785\,d]] \qquad (3)\end{gathered}$$

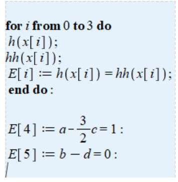
$Y := K\,IL:$
$N := L\,Y:$

$e^x - \frac{1}{2}\cos(x) - \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)^2 e^x + \cos(x)^2 - \frac{1}{2}\sin(x)^2 e^x + \sin(x)^2 + N;$

$$\begin{aligned}&e^x - \frac{1}{2}\cos(x) - \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)^2 e^x + \cos(x)^2 - \frac{1}{2}\sin(x)^2 e^x + \sin(x)^2 + (0.9193953883\,a - 0.9233197516\,c - 0.08315688770\,d + 0.4596976941\,b)\,x + \frac{1}{2}\,(-0.00485723834\,d + 0.3038488872\,b + 0.4596976941\,a \\ &- 0.3506598156\,c)\,x^2 + (-0.9233197516\,a + 1.104958351\,c + 0.1572559633\,d - 0.3506598156\,b)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) + (-0.00485723834\,b + 0.057737772\,d - 0.08315688770\,a + 0.1572559633\,c)\left(\frac{5}{12}x^4 - \frac{1}{2}x^2\right) + x\Big((0.2364534186\,b \\ &- 0.4517530184\,c - 0.03828410910\,d + 0.4596976941\,a)\,x + \frac{1}{2}\,(0.2364534186\,a - 0.1763277494\,c + 0.1585290152\,b - 0.000242997445\,d)\,x^2 + (-0.4517530184\,a + 0.0765382148\,d - 0.1763277494\,b + 0.5346253646\,c)\Big(\frac{1}{2}x^3 \\ &- \frac{3}{2}x\Big) + (-0.03828410909\,a + 0.0765382148\,c - 0.00024299745\,b + 0.0296427927\,d)\left(\frac{5}{12}x^4 - \frac{1}{2}x^2\right)\Big) + \left(\frac{3}{2}x^2 - \frac{3}{2}\right)\Big((-0.4517530184\,b + 0.9421211517\,c + 0.08845333815\,d - 0.9233197516\,a)\,x + \frac{1}{2}\,(-0.4517530184\,a \\ &+ 0.0081564036\,d - 0.2952389837\,b + 0.3506598156\,c)\,x^2 + (0.9421211517\,a - 1.136455491\,c + 0.3506598156\,b - 0.1610618477\,d)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) + (-0.056823370\,d + 0.00815640361\,b + 0.08845333815\,a \\ &- 0.1610618477\,c)\left(\frac{5}{12}x^4 - \frac{1}{2}x^2\right)\Big) + \left(\frac{5}{3}x^3 - x\right)\Big((0.00916452261\,d - 0.03828410915\,b + 0.08845333834\,c - 0.08315688786\,a)\,x + \frac{1}{2}\,(-0.03828410915\,a - 0.02418799563\,b + 0.001529504071\,d + 0.03121853582\,c)\,x^2 + (\\ &-0.015265755\,d + 0.03121853577\,b + 0.08845333834\,a - 0.1088459803\,c)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) + (0.00916452260\,a - 0.015265755\,c + 0.001529504071\,b - 0.00483620785\,d)\left(\frac{5}{12}x^4 - \frac{1}{2}x^2\right)\Big) \qquad (4)\end{aligned}$$

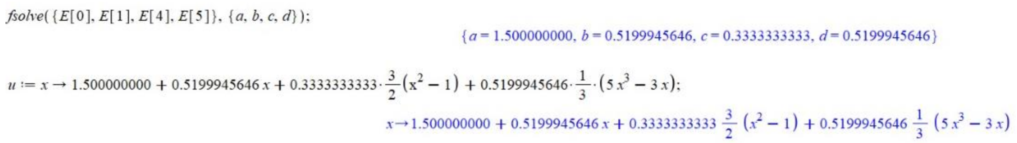
for *i* **from** 0 **to** 3 **do**
$x[i] := \frac{i}{3};$
end do:

$$\begin{aligned}h &:= x \rightarrow e^x - \frac{1}{2}\cos(x) - \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)^2 e^x + \cos(x)^2 - \frac{1}{2}\sin(x)^2 e^x + \sin(x)^2 + (0.9193953883\cdot a - 0.9233197516\cdot c - 0.08315688770\cdot d + 0.4596976941\cdot b)\cdot x + \frac{1}{2}\,(-0.00485723834\cdot d + 0.3038488872\cdot b + 0.4596976941\cdot a \\ &- 0.3506598156\cdot c)\cdot x^2 + (-0.9233197516\cdot a + 1.104958351\cdot c + 0.1572559633\cdot d - 0.3506598156\cdot b)\cdot\left(\frac{1}{2}\cdot x^3 - \frac{3}{2}\cdot x\right) + (0.057737772\cdot d - 0.00485723834\cdot b + 0.1572559633\cdot c - 0.08315688770\cdot a)\cdot\left(\frac{5}{12}\cdot x^4 - \frac{1}{2}\cdot x^2\right) + x\cdot\Big((\\ &-0.03828410910\cdot d + 0.2364534186\cdot b - 0.4517530184\cdot c + 0.4596976941\cdot a)\cdot x + \frac{1}{2}\,(0.2364534186\cdot a - 0.1763277494\cdot c - 0.000242997445\cdot d + 0.1585290152\cdot b)\cdot x^2 + (0.0765382148\cdot d - 0.1763277494\cdot b - 0.4517530184\cdot a \\ &+ 0.5346253646\cdot c)\left(\frac{1}{2}\cdot x^3 - \frac{3}{2}\cdot x\right) + (-0.03828410909\cdot a + 0.0765382148\cdot c + 0.0296427927\cdot d - 0.000242997445\cdot b)\left(\frac{5}{12}\cdot x^4 - \frac{1}{2}\cdot x^2\right)\Big) + \left(\frac{3}{2}\cdot x^2 - \frac{3}{2}\right)\cdot\Big((0.08845333815\cdot d + 0.9421211517\cdot c - 0.4517530184\cdot b - 0.9233197516 \\ &\cdot a)\cdot x + \frac{1}{2}\,(0.0081564036\cdot d - 0.2952389837\cdot b + 0.3506598156\cdot c - 0.4517530184\cdot a)\cdot x^2 + (-0.1610618477\cdot d - 1.136455491\cdot c + 0.3506598156\cdot b + 0.9421211518\cdot a)\cdot\left(\frac{1}{2}\cdot x^3 - \frac{3}{2}\cdot x\right) + (-0.056823367\cdot d + 0.0081564036\cdot b \\ &- 0.1610618477\cdot c + 0.08845333815\cdot a)\left(\frac{5}{12}\cdot x^4 - \frac{1}{2}\cdot x^2\right)\Big) + \left(\frac{5}{3}\cdot x^3 - x\right)\cdot\Big((-0.03828410915\cdot b - 0.08315688786\cdot a + 0.08845333834\cdot c + 0.00916452261\cdot d)\cdot x + \frac{1}{2}\,(-0.03828410915\cdot a + 0.03121853582\cdot c + 0.001529504071\cdot d \\ &- 0.02418799563\cdot b)\cdot x^2 + (0.03121853582\cdot b + 0.08845333834\cdot a - 0.1088459803\cdot c - 0.015265755\cdot d)\cdot\left(\frac{1}{2}x^3 - \frac{3}{2}\cdot x\right) + (0.0091645226\cdot a - 0.015265755\cdot c - 0.00483620785\cdot d + 0.001529504071\cdot b)\left(\frac{5}{12}\cdot x^4 - \frac{1}{2}\cdot x^2\right)\Big):\end{aligned}$$



for *i* **from** 0 **to** 3 **do**
$h(x[i]);$
$hh(x[i]);$
$E[i] := h(x[i]) = hh(x[i]);$
end do:

$E[4] := a - \frac{3}{2}c = 1:$
$E[5] := b - d = 0:$

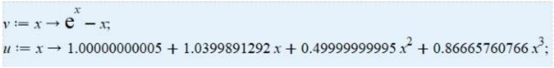


$fsolve(\{E[0], E[1], E[4], E[5]\}, \{a, b, c, d\});$

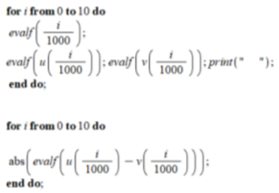
$$\{a = 1.500000000,\ b = 0.5199945646,\ c = 0.3333333333,\ d = 0.5199945646\}$$

$u := x \rightarrow 1.500000000 + 0.5199945646\,x + 0.3333333333\cdot\frac{3}{2}(x^2 - 1) + 0.5199945646\cdot\frac{1}{3}\cdot(5x^3 - 3x);$

$$x \rightarrow 1.500000000 + 0.5199945646\,x + 0.3333333333\,\frac{3}{2}\,(x^2 - 1) + 0.5199945646\,\frac{1}{3}\,(5x^3 - 3x)$$

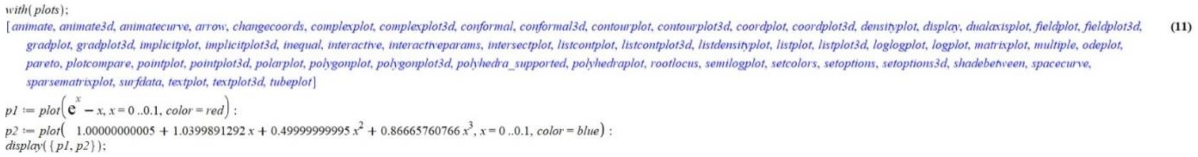


$v := x \rightarrow e^x - x;$
$u := x \rightarrow 1.00000000005 + 1.0399891292\,x + 0.49999999995\,x^2 + 0.86665760766\,x^3;$



for *i* **from** 0 **to** 10 **do**
$evalf\left(\frac{i}{1000}\right);$
$evalf\left(u\left(\frac{i}{1000}\right)\right); evalf\left(v\left(\frac{i}{1000}\right)\right); print("\quad");$
end do;

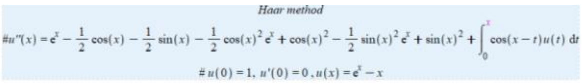
for *i* **from** 0 **to** 10 **do**
$abs\left(evalf\left(u\left(\frac{i}{1000}\right) - v\left(\frac{i}{1000}\right)\right)\right);$
end do;



with(*plots*);

[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot*] (11)

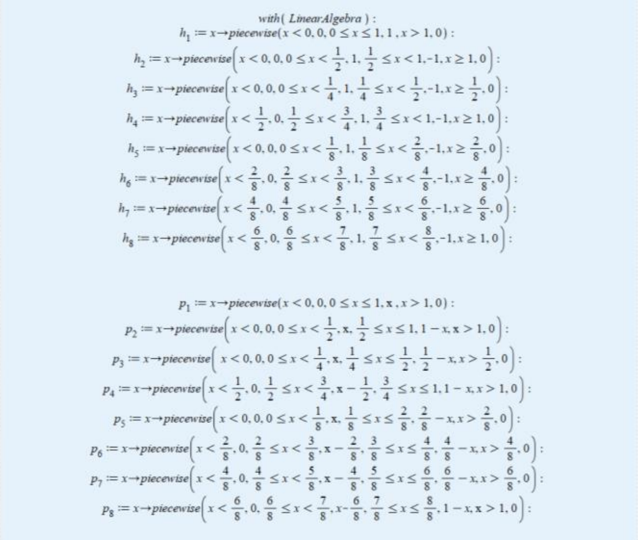
$p1 := plot\left(e^x - x, x = 0..0.1, color = red\right):$
$p2 := plot\left(1.00000000005 + 1.0399891292\,x + 0.49999999995\,x^2 + 0.86665760766\,x^3, x = 0..0.1, color = blue\right):$
$display(\{p1, p2\});$



Haar method

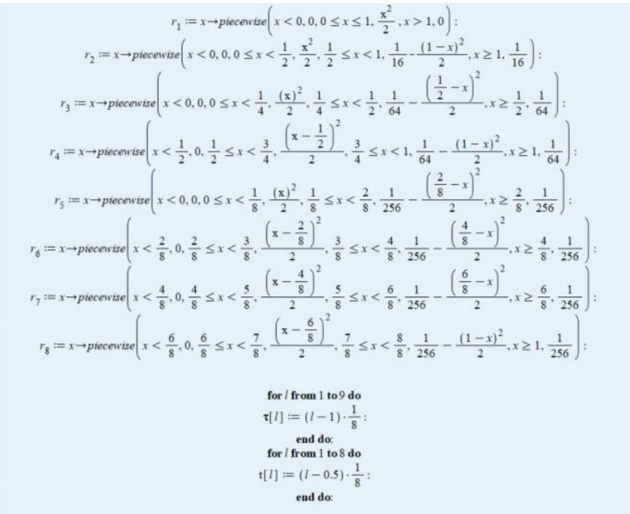
$\#u''(x) = e^x - \frac{1}{2}\cos(x) - \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)^2 e^x + \cos(x)^2 - \frac{1}{2}\sin(x)^2 e^x + \sin(x)^2 + \int_0^x \cos(x - t)u(t)\,dt$

$\# u(0) = 1,\ u'(0) = 0,\ u(x) = e^x - x$



$$with(\,LinearAlgebra\,):$$
$$h_1 := x \rightarrow piecewise(x < 0, 0, 0 \le x \le 1, 1\,, x > 1, 0):$$
$$h_2 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{2}, 1, \frac{1}{2} \le x < 1, -1, x \ge 1, 0\right):$$
$$h_3 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{4}, 1, \frac{1}{4} \le x < \frac{1}{2}, -1, x \ge \frac{1}{2}, 0\right):$$
$$h_4 := x \rightarrow piecewise\left(x < \frac{1}{2}, 0, \frac{1}{2} \le x < \frac{3}{4}, 1, \frac{3}{4} \le x < 1, -1, x \ge 1, 0\right):$$
$$h_5 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{8}, 1, \frac{1}{8} \le x < \frac{2}{8}, -1, x \ge \frac{2}{8}, 0\right):$$
$$h_6 := x \rightarrow piecewise\left(x < \frac{2}{8}, 0, \frac{2}{8} \le x < \frac{3}{8}, 1, \frac{3}{8} \le x < \frac{4}{8}, -1, x \ge \frac{4}{8}, 0\right):$$
$$h_7 := x \rightarrow piecewise\left(x < \frac{4}{8}, 0, \frac{4}{8} \le x < \frac{5}{8}, 1, \frac{5}{8} \le x < \frac{6}{8}, -1, x \ge \frac{6}{8}, 0\right):$$
$$h_8 := x \rightarrow piecewise\left(x < \frac{6}{8}, 0, \frac{6}{8} \le x < \frac{7}{8}, 1, \frac{7}{8} \le x < \frac{8}{8}, -1, x \ge 1, 0\right):$$

$$p_1 := x \rightarrow piecewise(x < 0, 0, 0 \le x \le 1, \mathbf{x}\,, x > 1, 0):$$
$$p_2 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{2}, \mathbf{x}, \frac{1}{2} \le x \le 1, 1 - x, \mathbf{x} > 1, 0\right):$$
$$p_3 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{4}, \mathbf{x}, \frac{1}{4} \le x \le \frac{1}{2}, \frac{1}{2} - x, x > \frac{1}{2}, 0\right):$$
$$p_4 := x \rightarrow piecewise\left(x < \frac{1}{2}, 0, \frac{1}{2} \le x < \frac{3}{4}, \mathbf{x} - \frac{1}{2}, \frac{3}{4} \le x \le 1, 1 - x, x > 1, 0\right):$$
$$p_5 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{8}, \mathbf{x}, \frac{1}{8} \le x \le \frac{2}{8}, \frac{2}{8} - x, x > \frac{2}{8}, 0\right):$$
$$p_6 := x \rightarrow piecewise\left(x < \frac{2}{8}, 0, \frac{2}{8} \le x < \frac{3}{8}, \mathbf{x} - \frac{2}{8}, \frac{3}{8} \le x \le \frac{4}{8}, \frac{4}{8} - x, x > \frac{4}{8}, 0\right):$$
$$p_7 := x \rightarrow piecewise\left(x < \frac{4}{8}, 0, \frac{4}{8} \le x < \frac{5}{8}, \mathbf{x} - \frac{4}{8}, \frac{5}{8} \le x \le \frac{6}{8}, \frac{6}{8} - x, x > \frac{6}{8}, 0\right):$$
$$p_8 := x \rightarrow piecewise\left(x < \frac{6}{8}, 0, \frac{6}{8} \le x < \frac{7}{8}, x - \frac{6}{8}, \frac{7}{8} \le x \le \frac{8}{8}, 1 - x, \mathbf{x} > 1, 0\right):$$



$$r_1 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x \le 1, \frac{\mathbf{x}^2}{2}\,, x > 1, 0\right):$$
$$r_2 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{2}, \frac{\mathbf{x}^2}{2}, \frac{1}{2} \le x < 1, \frac{1}{16} - \frac{(1-x)^2}{2}, x \ge 1, \frac{1}{16}\right):$$
$$r_3 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{4}, \frac{(\mathbf{x})^2}{2}, \frac{1}{4} \le x < \frac{1}{2}, \frac{1}{64} - \frac{\left(\frac{1}{2} - x\right)^2}{2}, x \ge \frac{1}{2}, \frac{1}{64}\right):$$
$$r_4 := x \rightarrow piecewise\left(x < \frac{1}{2}, 0, \frac{1}{2} \le x < \frac{3}{4}, \frac{\left(\mathbf{x} - \frac{1}{2}\right)^2}{2}, \frac{3}{4} \le x < 1, \frac{1}{64} - \frac{(1-x)^2}{2}, x \ge 1, \frac{1}{64}\right):$$
$$r_5 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{8}, \frac{(\mathbf{x})^2}{2}, \frac{1}{8} \le x < \frac{2}{8}, \frac{1}{256} - \frac{\left(\frac{2}{8} - x\right)^2}{2}, x \ge \frac{2}{8}, \frac{1}{256}\right):$$
$$r_6 := x \rightarrow piecewise\left(x < \frac{2}{8}, 0, \frac{2}{8} \le x < \frac{3}{8}, \frac{\left(\mathbf{x} - \frac{2}{8}\right)^2}{2}, \frac{3}{8} \le x < \frac{4}{8}, \frac{1}{256} - \frac{\left(\frac{4}{8} - x\right)^2}{2}, x \ge \frac{4}{8}, \frac{1}{256}\right):$$
$$r_7 := x \rightarrow piecewise\left(x < \frac{4}{8}, 0, \frac{4}{8} \le x < \frac{5}{8}, \frac{\left(\mathbf{x} - \frac{4}{8}\right)^2}{2}, \frac{5}{8} \le x < \frac{6}{8}, \frac{1}{256} - \frac{\left(\frac{6}{8} - x\right)^2}{2}, x \ge \frac{6}{8}, \frac{1}{256}\right):$$
$$r_8 := x \rightarrow piecewise\left(x < \frac{6}{8}, 0, \frac{6}{8} \le x < \frac{7}{8}, \frac{\left(\mathbf{x} - \frac{6}{8}\right)^2}{2}, \frac{7}{8} \le x < \frac{8}{8}, \frac{1}{256} - \frac{(1-x)^2}{2}, x \ge 1, \frac{1}{256}\right):$$

for *l* **from** 1 **to** 9 **do**

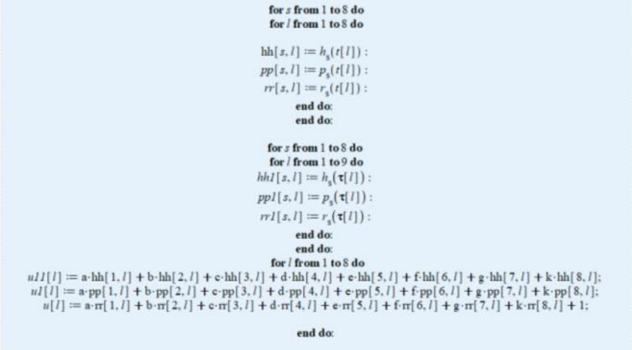
$$\tau[l] := (l-1)\cdot\frac{1}{8}:$$

end do:

for *l* **from** 1 **to** 8 **do**

$$t[l] := (l-0.5)\cdot\frac{1}{8}:$$

end do:



for *s* **from** 1 **to** 8 **do**
for *l* **from** 1 **to** 8 **do**

$$\mathrm{hh}[s, l] := h_s(t[l]):$$
$$pp[s, l] := p_s(t[l]):$$
$$rr[s, l] := r_s(t[l]):$$

end do:
end do:

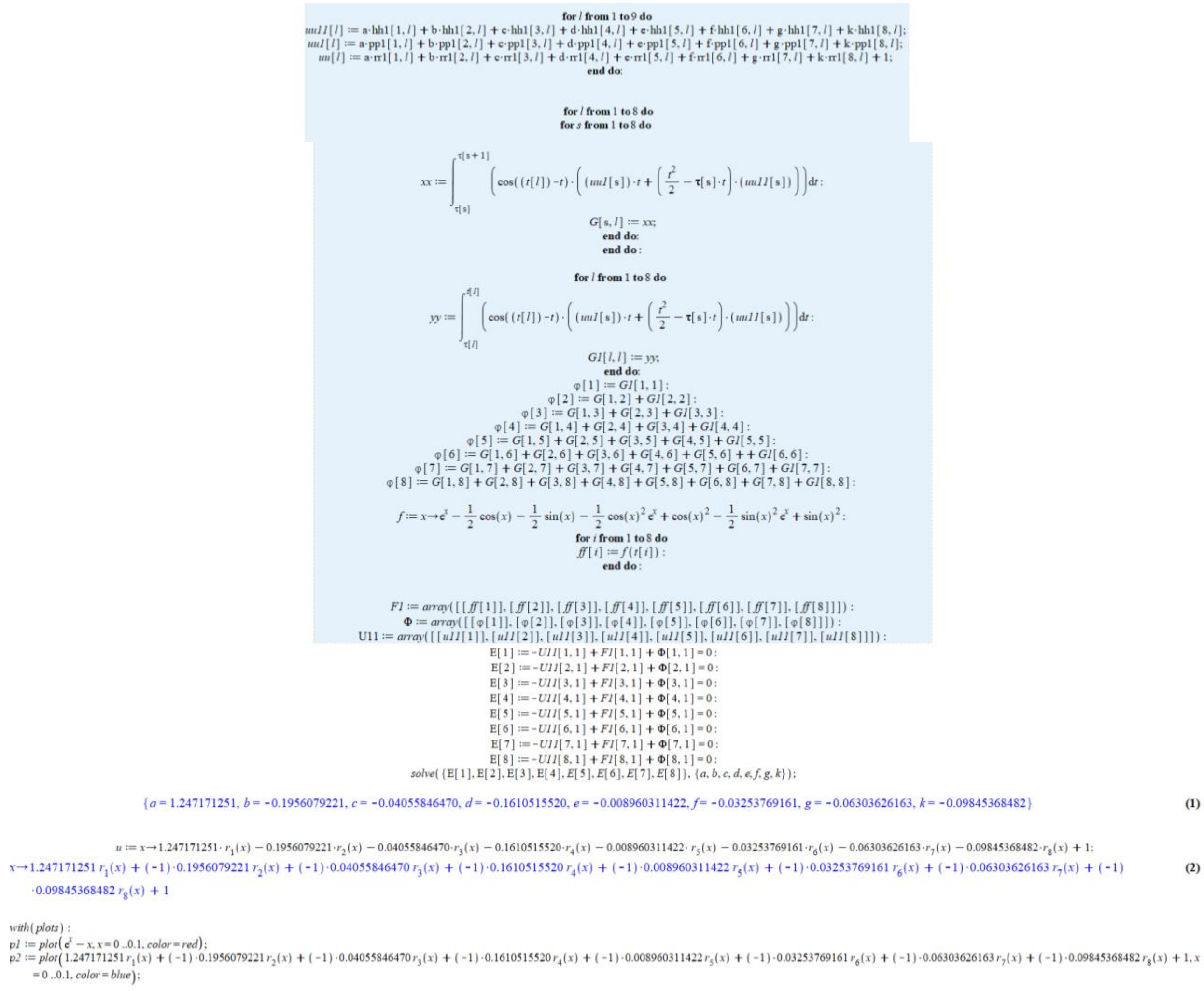
for *s* **from** 1 **to** 8 **do**
for *l* **from** 1 **to** 9 **do**

$$hh1[s, l] := h_s(\tau[l]):$$
$$pp1[s, l] := p_s(\tau[l]):$$
$$rr1[s, l] := r_s(\tau[l]):$$

end do:
end do:
for *l* **from** 1 **to** 8 **do**

$$u11[l] := \mathrm{a}\cdot\mathrm{hh}[1, l] + \mathrm{b}\cdot\mathrm{hh}[2, l] + \mathrm{c}\cdot\mathrm{hh}[3, l] + \mathrm{d}\cdot\mathrm{hh}[4, l] + \mathrm{e}\cdot\mathrm{hh}[5, l] + \mathrm{f}\cdot\mathrm{hh}[6, l] + \mathrm{g}\cdot\mathrm{hh}[7, l] + \mathrm{k}\cdot\mathrm{hh}[8, l];$$
$$u1[l] := \mathrm{a}\cdot\mathrm{pp}[1, l] + \mathrm{b}\cdot\mathrm{pp}[2, l] + \mathrm{c}\cdot\mathrm{pp}[3, l] + \mathrm{d}\cdot\mathrm{pp}[4, l] + \mathrm{e}\cdot\mathrm{pp}[5, l] + \mathrm{f}\cdot\mathrm{pp}[6, l] + \mathrm{g}\cdot\mathrm{pp}[7, l] + \mathrm{k}\cdot\mathrm{pp}[8, l];$$
$$u[l] := \mathrm{a}\cdot\mathrm{rr}[1, l] + \mathrm{b}\cdot\mathrm{rr}[2, l] + \mathrm{c}\cdot\mathrm{rr}[3, l] + \mathrm{d}\cdot\mathrm{rr}[4, l] + \mathrm{e}\cdot\mathrm{rr}[5, l] + \mathrm{f}\cdot\mathrm{rr}[6, l] + \mathrm{g}\cdot\mathrm{rr}[7, l] + \mathrm{k}\cdot\mathrm{rr}[8, l] + 1;$$

end do:



for l **from** 1 **to** 9 **do**
$uu11[l] := a\cdot hh1[1,l] + b\cdot hh1[2,l] + c\cdot hh1[3,l] + d\cdot hh1[4,l] + e\cdot hh1[5,l] + f\cdot hh1[6,l] + g\cdot hh1[7,l] + k\cdot hh1[8,l];$
$uu1[l] := a\cdot pp1[1,l] + b\cdot pp1[2,l] + c\cdot pp1[3,l] + d\cdot pp1[4,l] + e\cdot pp1[5,l] + f\cdot pp1[6,l] + g\cdot pp1[7,l] + k\cdot pp1[8,l];$
$uu[l] := a\cdot rr1[1,l] + b\cdot rr1[2,l] + c\cdot rr1[3,l] + d\cdot rr1[4,l] + e\cdot rr1[5,l] + f\cdot rr1[6,l] + g\cdot rr1[7,l] + k\cdot rr1[8,l] + 1;$
end do:

for l **from** 1 **to** 8 **do**
for s **from** 1 **to** 8 **do**

$$xx := \int_{\tau[s]}^{\tau[s+1]} \left(\cos((t[l]) - t)\cdot\left((uu1[s])\cdot t + \left(\frac{t^2}{2} - \tau[s]\cdot t\right)\cdot(uu11[s])\right)\right)dt:$$

$G[s,l] := xx;$
end do:
end do:

for l **from** 1 **to** 8 **do**

$$yy := \int_{\tau[l]}^{t[l]} \left(\cos((t[l]) - t)\cdot\left((uu1[s])\cdot t + \left(\frac{t^2}{2} - \tau[s]\cdot t\right)\cdot(uu11[s])\right)\right)dt:$$

$G1[l,l] := yy;$
end do:
$\varphi[1] := G1[1,1]:$
$\varphi[2] := G[1,2] + G1[2,2]:$
$\varphi[3] := G[1,3] + G[2,3] + G1[3,3]:$
$\varphi[4] := G[1,4] + G[2,4] + G[3,4] + G1[4,4]:$
$\varphi[5] := G[1,5] + G[2,5] + G[3,5] + G[4,5] + G1[5,5]:$
$\varphi[6] := G[1,6] + G[2,6] + G[3,6] + G[4,6] + G[5,6] + + G1[6,6]:$
$\varphi[7] := G[1,7] + G[2,7] + G[3,7] + G[4,7] + G[5,7] + G[6,7] + G1[7,7]:$
$\varphi[8] := G[1,8] + G[2,8] + G[3,8] + G[4,8] + G[5,8] + G[6,8] + G[7,8] + G1[8,8]:$

$$f := x \rightarrow e^x - \frac{1}{2}\cos(x) - \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)^2 e^x + \cos(x)^2 - \frac{1}{2}\sin(x)^2 e^x + \sin(x)^2:$$

for i **from** 1 **to** 8 **do**
$ff[i] := f(t[i]):$
end do:

$F1 := array([[ff[1]], [ff[2]], [ff[3]], [ff[4]], [ff[5]], [ff[6]], [ff[7]], [ff[8]]]):$
$\Phi := array([[\varphi[1]], [\varphi[2]], [\varphi[3]], [\varphi[4]], [\varphi[5]], [\varphi[6]], [\varphi[7]], [\varphi[8]]]):$
$U11 := array([[u11[1]], [u11[2]], [u11[3]], [u11[4]], [u11[5]], [u11[6]], [u11[7]], [u11[8]]]):$
$E[1] := -U11[1,1] + F1[1,1] + \Phi[1,1] = 0:$
$E[2] := -U11[2,1] + F1[2,1] + \Phi[2,1] = 0:$
$E[3] := -U11[3,1] + F1[3,1] + \Phi[3,1] = 0:$
$E[4] := -U11[4,1] + F1[4,1] + \Phi[4,1] = 0:$
$E[5] := -U11[5,1] + F1[5,1] + \Phi[5,1] = 0:$
$E[6] := -U11[6,1] + F1[6,1] + \Phi[6,1] = 0:$
$E[7] := -U11[7,1] + F1[7,1] + \Phi[7,1] = 0:$
$E[8] := -U11[8,1] + F1[8,1] + \Phi[8,1] = 0:$
$solve(\{E[1], E[2], E[3], E[4], E[5], E[6], E[7], E[8]\}, \{a, b, c, d, e, f, g, k\});$

$$\{a = 1.247171251,\ b = -0.1956079221,\ c = -0.04055846470,\ d = -0.1610515520,\ e = -0.008960311422,\ f = -0.03253769161,\ g = -0.06303626163,\ k = -0.09845368482\} \tag{1}$$

$u := x \rightarrow 1.247171251\cdot r_1(x) - 0.1956079221\cdot r_2(x) - 0.04055846470\cdot r_3(x) - 0.1610515520\cdot r_4(x) - 0.008960311422\cdot r_5(x) - 0.03253769161\cdot r_6(x) - 0.06303626163\cdot r_7(x) - 0.09845368482\cdot r_8(x) + 1;$

$$x \rightarrow 1.247171251\, r_1(x) + (-1)\cdot 0.1956079221\, r_2(x) + (-1)\cdot 0.04055846470\, r_3(x) + (-1)\cdot 0.1610515520\, r_4(x) + (-1)\cdot 0.008960311422\, r_5(x) + (-1)\cdot 0.03253769161\, r_6(x) + (-1)\cdot 0.06303626163\, r_7(x) + (-1)\cdot 0.09845368482\, r_8(x) + 1 \tag{2}$$

$with(plots):$
$p1 := plot(e^x - x, x = 0\,..0.1, color = red);$
$p2 := plot(1.247171251\, r_1(x) + (-1)\cdot 0.1956079221\, r_2(x) + (-1)\cdot 0.04055846470\, r_3(x) + (-1)\cdot 0.1610515520\, r_4(x) + (-1)\cdot 0.008960311422\, r_5(x) + (-1)\cdot 0.03253769161\, r_6(x) + (-1)\cdot 0.06303626163\, r_7(x) + (-1)\cdot 0.09845368482\, r_8(x) + 1, x = 0\,..0.1, color = blue);$

Example 3:

$$\#Taylor\ method$$

$$\#u'(x)+u(x)=\int_0^x \exp(t-x)u(t)\,dt$$

$$\#\,u(0)=1\,,\,u(x)=\exp(-x)\cdot\cosh(x)$$

$$kk := (x,y) \rightarrow \exp(y-x):$$

$$U0 := \mathbf{proc}(k,h)$$
$$\mathbf{local}\ i\,,j\,,U0:\quad U0 := \frac{1}{k!\cdot h!}\cdot evalf(\mathrm{D}[1\$k,2\$h](kk)(0,0));$$
$$\mathbf{end\ proc}:$$

$$K := \begin{bmatrix} U0(0,0) & U0(0,1) & U0(0,2) \\ U0(1,0) & U0(1,1) & U0(1,2) \\ U0(2,0) & U0(2,1) & U0(2,2) \end{bmatrix};$$

$$\begin{bmatrix} 1. & 1. & 0.5000000000 \\ -1. & -1. & -0.5000000000 \\ 0.5000000000 & 0.5000000000 & 0.2500000000 \end{bmatrix}$$

$$MM := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}:$$

$$\mathrm{KK} := \begin{bmatrix} 1 & 1 & 0.5000000000 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -0.5000000000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5000000000 & 0.5000000000 & 0.2500000000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0.5000000000 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -0.5000000000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5000000000 & 0.5000000000 & 0.2500000000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0.5000000000 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -0.5000000000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5000000000 & 0.5000000000 & 0.2500000000 \end{bmatrix}:$$

$$HH := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} & \frac{1}{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{24} & \frac{1}{64} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{24} & \frac{1}{64} & \frac{1}{160} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}:$$

$$\mathrm{PP} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}:$$

$$W2 := PP\ KK\ HH\ MM;$$

$$\begin{bmatrix} 0. & 0. & 0. \\ 0.403645833333333 & 0.109049479166667 & 0.0188802083333333 \\ 0.833333333333333 & 0.479166666666667 & 0.170833333333333 \end{bmatrix}$$

$$P1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} :$$

$$C := \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix} :$$

$$M0 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} :$$

$$P0 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} :$$

$$M1 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} :$$

$W0 := P0\, C\, M0;$

$$\begin{bmatrix} 1.0 & .0 & .0 \\ 1.0 & .5 & .1 \\ 1.0 & 1.0 & .5 \end{bmatrix}$$

$W1 := P1\, C\, M1;$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

$W3 := W0 + W1;$

$$\begin{bmatrix} 1.0 & 1.0 & .0 \\ 1.0 & 1.5 & .6 \\ 1.0 & 2.0 & 1.5 \end{bmatrix}$$

$W := W3 - W2;$

$$\begin{bmatrix} 1. & 1. & 0. \\ 0.596354166666667 & 1.39095052083333 & 0.606119791666667 \\ 0.166666666666667 & 1.52083333333333 & 1.32916666666667 \end{bmatrix}$$

$$F := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} :$$

$$W := \begin{bmatrix} 1. & 1. & 0. \\ 0.596354166666667 & 1.39095052083333 & 0.606119791666667 \\ 0.166666666666667 & 1.52083333333333 & 1.32916666666667 \end{bmatrix} :$$

$W^{-1} F;$

$$\begin{bmatrix} 0. \\ 0. \\ 0. \end{bmatrix}$$

$$Wc := \begin{bmatrix} 1. & 1. & 0. \\ 0.596354166666667 & 1.39095052083333 & 0.606119791666667 \\ 1 & 0 & 0 \end{bmatrix} :$$

$$Fc := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} :$$

$(Wc)^{-1} Fc;$

$$\begin{bmatrix} 1. \\ -1. \\ 1.31095596133189 \end{bmatrix}$$

$s := x \rightarrow 1 - x + 1.31095596133189 \cdot x^2 :$

$v := x \rightarrow e^{-x} \cdot \cosh(x) :$

for i **from** 0 **to** 10 **do**

$evalf\left(\frac{i}{1000}\right);$

$; evalf\left(s\left(\frac{i}{1000}\right)\right); evalf\left(v\left(\frac{i}{1000}\right)\right); print(\text{"   "});$

end do;

for i **from** 0 **to** 10 **do**

$\operatorname{abs}\left(evalf\left(s\left(\frac{i}{1000}\right)-v\left(\frac{i}{1000}\right)\right)\right);$

end do;

$with(plots);$

[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot*] **(9)**

$p1 := plot\left(\mathrm{e}^{-x}\cdot\cosh(x), x=0..0.2, color=red\right);$

$p2 := plot\left(1-\mathrm{x}+1.31095596133189\cdot \mathrm{x}^2, x=0..0.2, color=blue\right);$

$display(\{p1, p2\});$

Legendre method

$\#u'(x)+u(x)=\int_0^x \exp(t-x)u(t)\,\mathrm{d}t$

$\#\, u(0)=1\,, u(x)=\exp(-x)\cdot\cosh(x)$

$with(LinearAlgebra):$

$with(SolveTools):$

$p_0(x) := 1:$

$p_1(x) := \mathrm{x}:$

$p_2(x) := \frac{3}{2}(\mathrm{x}^2-1):$

$L := \begin{bmatrix} 1 & \mathrm{x} & \frac{3}{2}(\mathrm{x}^2-1) \end{bmatrix}:$

$A := \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix}:$

$UU := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}:$

$S := L\,UU;$

$$\begin{bmatrix} 0 & 1 & 3\,x \end{bmatrix}$$

$(S+L)\,A;$

$$a+(1+x)\,b+\left(3\,x+\frac{3}{2}\,x^2-\frac{3}{2}\right)c$$

$hh := x \rightarrow a+(1+x)\,b+\left(3\,x+\frac{3}{2}\,x^2-\frac{3}{2}\right)c:$

for n **from** 0 **to** 2 **do**

for m **from** 0 **to** 2 **do**

$e := evalf\left(\int_0^1 \left((\exp(t-x))\cdot\left(\mathrm{a}+\mathrm{b}\cdot\mathrm{t}+\mathrm{c}\cdot\left(\frac{3}{2}\cdot t^2-\frac{3}{2}\right)\right)\cdot(p_m(t))\right)\mathrm{d}t\right):$

$k[n, m] := \int_0^1 ((e)\cdot p_n(x)\,)\mathrm{d}x:$

end do:

end do :

$$K := \begin{bmatrix} k[0,0] & k[0,1] & k[0,2] \\ k[1,0] & k[1,1] & k[1,2] \\ k[2,0] & k[2,1] & k[2,2] \end{bmatrix};$$

$$\begin{bmatrix} 1.086161270\,a + 0.6321205588\,b - 0.9481808382\,c & 0.6321205588\,a + 0.4540407108\,b - 0.4139412942\,c & -0.9481808382\,a - 0.4139412942\,b + 1.061376508\,c \\ 0.4540407108\,a + 0.2642411177\,b - 0.3963616765\,c & 0.2642411177\,a + 0.1897995931\,b - 0.1730371029\,c & -0.3963616765\,a - 0.1730371029\,b + 0.4436801030\,c \\ -1.215300610\,a - 0.7072766470\,b + 1.060914971\,c & -0.7072766470\,a - 0.5080239633\,b + 0.4631569192\,c & 1.060914971\,a + 0.4631569192\,b - 1.187569060\,c \end{bmatrix} \tag{3}$$

$$IL := \begin{bmatrix} \mathrm{x} \\ \frac{1}{2}x^2 \\ \frac{1}{2}x^3 - \frac{3}{2}x \end{bmatrix}:$$

$Y := K\,IL;$

$$\Big[\Big[(1.086161270\,a + 0.6321205588\,b - 0.9481808382\,c)\,x + \frac{1}{2}(0.6321205588\,a + 0.4540407108\,b - 0.4139412942\,c)\,x^2 + (-0.9481808382\,a - 0.4139412942\,b + 1.061376508\,c)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right)\Big], \Big[(0.4540407108\,a + 0.2642411177\,b - 0.3963616765\,c)\,x + \frac{1}{2}(0.2642411177\,a + 0.1897995931\,b - 0.1730371029\,c)\,x^2 + (-0.3963616765\,a - 0.1730371029\,b + 0.4436801030\,c)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right)\Big], \Big[(-1.215300610\,a - 0.7072766470\,b + 1.060914971\,c)\,x + \frac{1}{2}(-0.7072766470\,a - 0.5080239633\,b + 0.4631569192\,c)\,x^2 + (1.060914971\,a + 0.4631569192\,b - 1.187569060\,c)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right)\Big]\Big] \tag{4}$$

$N := L\,Y;$

$$(1.086161270\,a + 0.6321205588\,b - 0.9481808382\,c)\,x + \frac{1}{2}(0.6321205588\,a + 0.4540407108\,b - 0.4139412942\,c)\,x^2 + (-0.9481808382\,a - 0.4139412942\,b + 1.061376508\,c)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) + x\left((0.4540407108\,a + 0.2642411177\,b - 0.3963616765\,c)\,x + \frac{1}{2}(0.2642411177\,a + 0.1897995931\,b - 0.1730371029\,c)\,x^2 + (-0.3963616765\,a - 0.1730371029\,b + 0.4436801030\,c)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right)\right) + \left(\frac{3}{2}x^2 - \frac{3}{2}\right)\left((-1.215300610\,a - 0.7072766470\,b + 1.060914971\,c)\,x + \frac{1}{2}(-0.7072766470\,a - 0.5080239633\,b + 0.4631569192\,c)\,x^2 + (1.060914971\,a + 0.4631569192\,b - 1.187569060\,c)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right)\right) \tag{5}$$

for *i* **from** 0 **to** 2 **do**

$x[i] := \frac{i}{2};$

end do :

$$h := x \rightarrow (1.086161270 \cdot a + 0.6321205588 \cdot b - 0.9481808382 \cdot c) \cdot x + \frac{1}{2} \cdot (0.6321205588 \cdot a + 0.4540407108 \cdot b - 0.4139412942 \cdot c) \cdot x^2 + (-0.9481808382 \cdot a - 0.4139412942 \cdot b + 1.061376508 \cdot c) \cdot \left(\frac{1}{2} \cdot x^3 - \frac{3}{2} \cdot x\right) + x \cdot \left((0.4540407108 \cdot a + 0.2642411177 \cdot b - 0.3963616765 \cdot c) \cdot x + \frac{1}{2} \cdot (0.2642411177 \cdot a + 0.1897995931 \cdot b - 0.1730371029 \cdot c) \cdot x^2 + (-0.3963616765 \cdot a - 0.1730371029 \cdot b + 0.4436801030 \cdot c)\left(\frac{1}{2} \cdot x^3 - \frac{3}{2} \cdot x\right)\right) + \left(\frac{3}{2} \cdot x^2 - \frac{3}{2}\right) \cdot \left((-1.215300610 \cdot a - 0.7072766470 \cdot b + 1.060914971 \cdot c) \cdot x + \frac{1}{2} \cdot (-0.7072766470 \cdot a - 0.5080239633 \cdot b + 0.4631569192 \cdot c) \cdot x^2 + (1.060914971 \cdot a + 0.4631569192 \cdot b - 1.187569060 \cdot c) \cdot \left(\frac{1}{2} \cdot x^3 - \frac{3}{2} \cdot x\right)\right):$$

for *i* **from** 0 **to** 2 **do**
$h(x[i]);$
$hh(x[i]);$
$E[i] := h(x[i]) = hh(x[i]);$
end do ;

$E[3] := a - \frac{3}{2}c = 1;$

$$0$$

$$a + b - \frac{3}{2}c$$

$$0 = a + b - \frac{3}{2}c$$

$$3.143863400\,a + 1.622311180\,b - 3.098356261\,c$$

$$a + \frac{3}{2}b + \frac{3}{8}c$$

$$3.143863400\,a + 1.622311180\,b - 3.098356261\,c = a + \frac{3}{2}b + \frac{3}{8}c$$

$$3.332925333\,a + 1.805260225\,b - 3.143088324\,c$$

$$a + 2\,b + 3\,c$$

$$3.332925333\,a + 1.805260225\,b - 3.143088324\,c = a + 2\,b + 3\,c$$

$$a - \frac{3}{2}c = 1$$

$fsolve(\{E[0], E[1], E[3]\}, \{a, b, c\});$

$$\{a = 12.77323599,\ b = -0.9999999998,\ c = 7.848823993\}$$

$$x \rightarrow 12.77323599 - 0.9999999998\, x + 7.848823993 \cdot \frac{3}{2}\left(\mathrm{x}^2 - 1\right) :$$

$$\mathrm{u} := \mathrm{x} \rightarrow 1.0000000005 - 0.9999999998\,\mathrm{x} + 11.7732359895\,\mathrm{x}^2 :$$

$$v := x \rightarrow \exp(-x)\cdot\cosh(x) :$$

for *i* **from** 0 **to** 10 **do**

$$evalf\left(\frac{i}{1000}\right);$$

$$;\ evalf\left(u\left(\frac{i}{1000}\right)\right);\ evalf\left(v\left(\frac{i}{1000}\right)\right);\ print("\quad\ ");$$

end do;

for *i* **from** 0 **to** 10 **do**

$$\mathrm{abs}\left(evalf\left(u\left(\frac{i}{1000}\right) - v\left(\frac{i}{1000}\right)\right)\right);$$

end do;

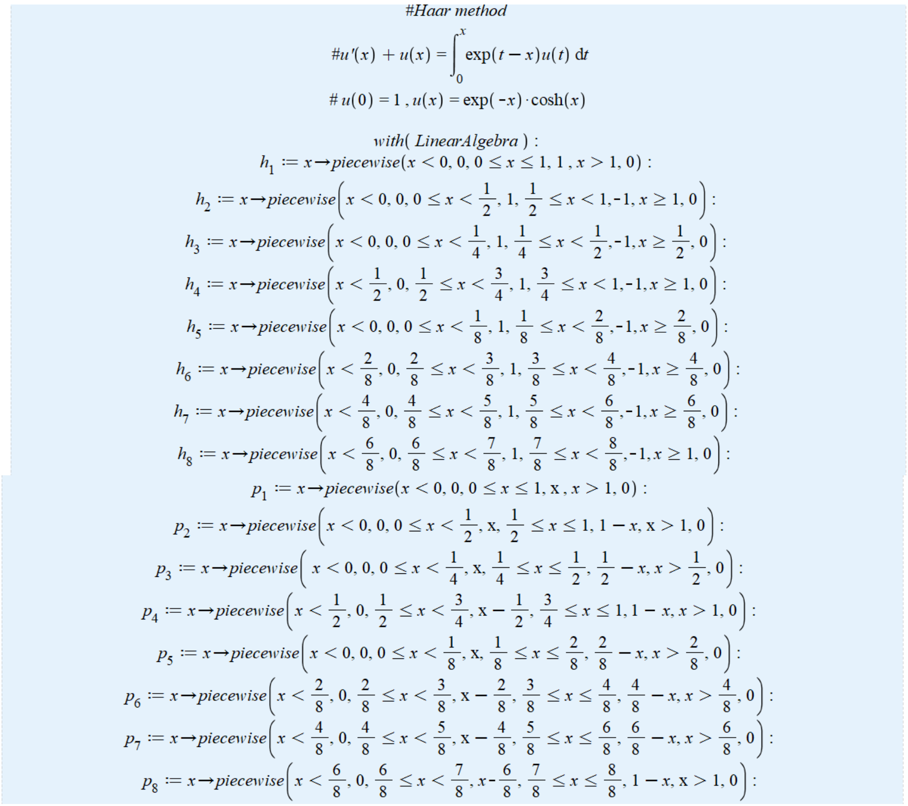
with(*plots*);

[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot*] (8)

$$p1 := plot(\exp(-x)\cdot\cosh(x), x = 0..0.2, color = red) :$$

$$p2 := plot\left(1.0000000005 - 0.9999999998\,\mathrm{x} + 11.7732359895\,\mathrm{x}^2, x = 0..0.2, color = blue\right) :$$

$$display(\{p1, p2\});$$



#Haar method

$$\#u'(x) + u(x) = \int_0^x \exp(t - x)u(t)\,\mathrm{d}t$$

$$\#\,u(0) = 1\,,\, u(x) = \exp(-x)\cdot\cosh(x)$$

$$with(\,LinearAlgebra\,) :$$

$$h_1 := x \rightarrow piecewise(x < 0, 0, 0 \le x \le 1, 1, x > 1, 0) :$$

$$h_2 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{2}, 1, \frac{1}{2} \le x < 1, -1, x \ge 1, 0\right) :$$

$$h_3 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{4}, 1, \frac{1}{4} \le x < \frac{1}{2}, -1, x \ge \frac{1}{2}, 0\right) :$$

$$h_4 := x \rightarrow piecewise\left(x < \frac{1}{2}, 0, \frac{1}{2} \le x < \frac{3}{4}, 1, \frac{3}{4} \le x < 1, -1, x \ge 1, 0\right) :$$

$$h_5 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{8}, 1, \frac{1}{8} \le x < \frac{2}{8}, -1, x \ge \frac{2}{8}, 0\right) :$$

$$h_6 := x \rightarrow piecewise\left(x < \frac{2}{8}, 0, \frac{2}{8} \le x < \frac{3}{8}, 1, \frac{3}{8} \le x < \frac{4}{8}, -1, x \ge \frac{4}{8}, 0\right) :$$

$$h_7 := x \rightarrow piecewise\left(x < \frac{4}{8}, 0, \frac{4}{8} \le x < \frac{5}{8}, 1, \frac{5}{8} \le x < \frac{6}{8}, -1, x \ge \frac{6}{8}, 0\right) :$$

$$h_8 := x \rightarrow piecewise\left(x < \frac{6}{8}, 0, \frac{6}{8} \le x < \frac{7}{8}, 1, \frac{7}{8} \le x < \frac{8}{8}, -1, x \ge 1, 0\right) :$$

$$p_1 := x \rightarrow piecewise(x < 0, 0, 0 \le x \le 1, \mathrm{x}, x > 1, 0) :$$

$$p_2 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{2}, \mathrm{x}, \frac{1}{2} \le x \le 1, 1 - x, \mathrm{x} > 1, 0\right) :$$

$$p_3 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{4}, \mathrm{x}, \frac{1}{4} \le x \le \frac{1}{2}, \frac{1}{2} - x, x > \frac{1}{2}, 0\right) :$$

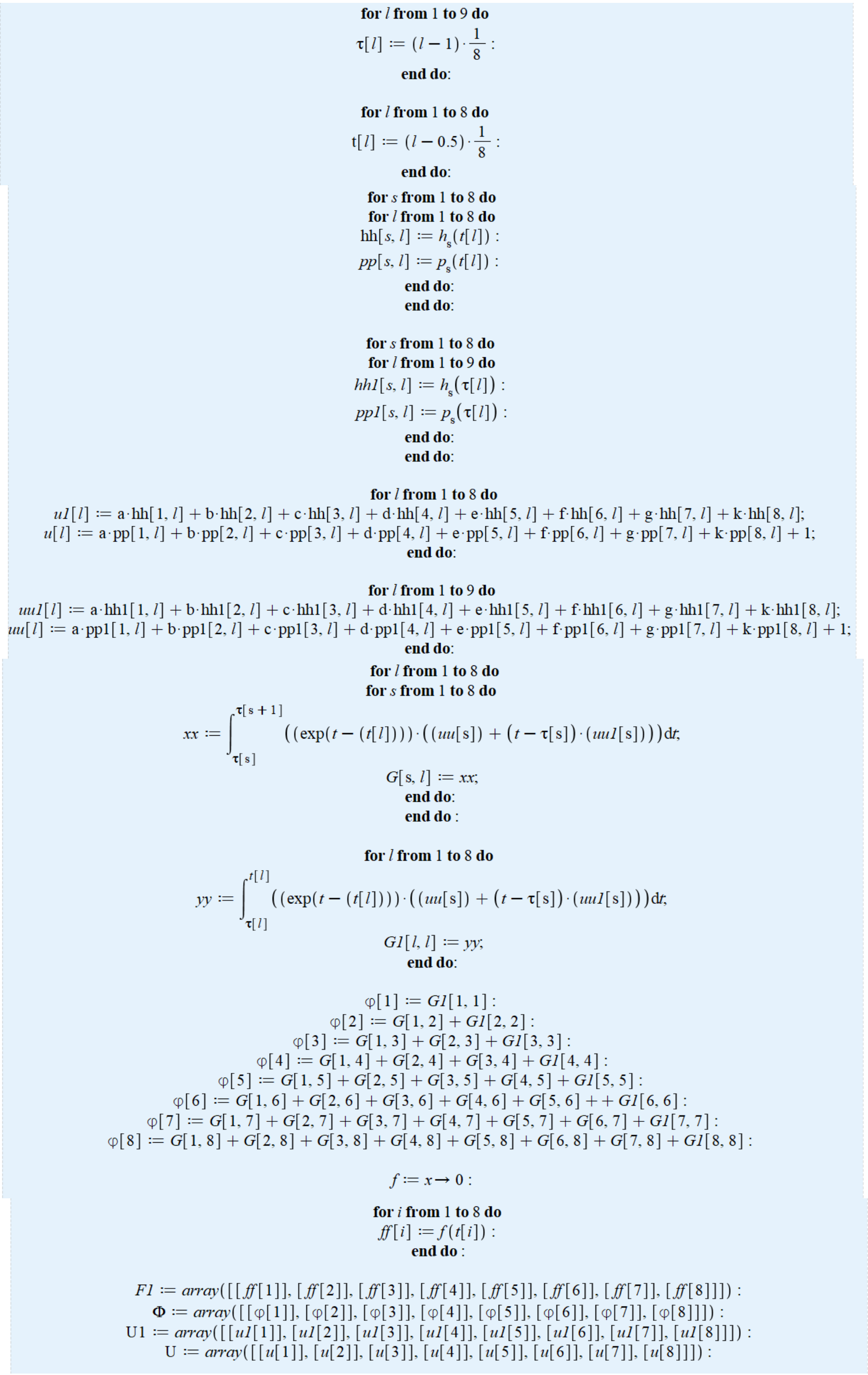
$$p_4 := x \rightarrow piecewise\left(x < \frac{1}{2}, 0, \frac{1}{2} \le x < \frac{3}{4}, \mathrm{x} - \frac{1}{2}, \frac{3}{4} \le x \le 1, 1 - x, x > 1, 0\right) :$$

$$p_5 := x \rightarrow piecewise\left(x < 0, 0, 0 \le x < \frac{1}{8}, \mathrm{x}, \frac{1}{8} \le x \le \frac{2}{8}, \frac{2}{8} - x, x > \frac{2}{8}, 0\right) :$$

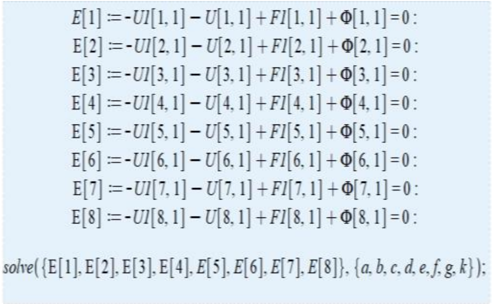
$$p_6 := x \rightarrow piecewise\left(x < \frac{2}{8}, 0, \frac{2}{8} \le x < \frac{3}{8}, \mathrm{x} - \frac{2}{8}, \frac{3}{8} \le x \le \frac{4}{8}, \frac{4}{8} - x, x > \frac{4}{8}, 0\right) :$$

$$p_7 := x \rightarrow piecewise\left(x < \frac{4}{8}, 0, \frac{4}{8} \le x < \frac{5}{8}, \mathrm{x} - \frac{4}{8}, \frac{5}{8} \le x \le \frac{6}{8}, \frac{6}{8} - x, x > \frac{6}{8}, 0\right) :$$

$$p_8 := x \rightarrow piecewise\left(x < \frac{6}{8}, 0, \frac{6}{8} \le x < \frac{7}{8}, x - \frac{6}{8}, \frac{7}{8} \le x \le \frac{8}{8}, 1 - x, \mathrm{x} > 1, 0\right) :$$



```
for l from 1 to 9 do
  τ[l] := (l − 1)·$\frac{1}{8}$ :
end do:

for l from 1 to 8 do
  t[l] := (l − 0.5)·$\frac{1}{8}$ :
end do:

for s from 1 to 8 do
  for l from 1 to 8 do
    hh[s, l] := $h_s$(t[l]) :
    pp[s, l] := $p_s$(t[l]) :
  end do:
end do:

for s from 1 to 8 do
  for l from 1 to 9 do
    hh1[s, l] := $h_s$(τ[l]) :
    pp1[s, l] := $p_s$(τ[l]) :
  end do:
end do:

for l from 1 to 8 do
  u1[l] := a·hh[1, l] + b·hh[2, l] + c·hh[3, l] + d·hh[4, l] + e·hh[5, l] + f·hh[6, l] + g·hh[7, l] + k·hh[8, l];
  u[l] := a·pp[1, l] + b·pp[2, l] + c·pp[3, l] + d·pp[4, l] + e·pp[5, l] + f·pp[6, l] + g·pp[7, l] + k·pp[8, l] + 1;
end do:

for l from 1 to 9 do
  uu1[l] := a·hh1[1, l] + b·hh1[2, l] + c·hh1[3, l] + d·hh1[4, l] + e·hh1[5, l] + f·hh1[6, l] + g·hh1[7, l] + k·hh1[8, l];
  uu[l] := a·pp1[1, l] + b·pp1[2, l] + c·pp1[3, l] + d·pp1[4, l] + e·pp1[5, l] + f·pp1[6, l] + g·pp1[7, l] + k·pp1[8, l] + 1;
end do:

for l from 1 to 8 do
  for s from 1 to 8 do
    xx := $\int_{\tau[s]}^{\tau[s+1]} ((\exp(t - (t[l]))) \cdot ((uu[s]) + (t - \tau[s]) \cdot (uu1[s])))\,dt$;
    G[s, l] := xx;
  end do:
end do :

for l from 1 to 8 do
  yy := $\int_{\tau[l]}^{t[l]} ((\exp(t - (t[l]))) \cdot ((uu[s]) + (t - \tau[s]) \cdot (uu1[s])))\,dt$;
  G1[l, l] := yy;
end do:

φ[1] := G1[1, 1] :
φ[2] := G[1, 2] + G1[2, 2] :
φ[3] := G[1, 3] + G[2, 3] + G1[3, 3] :
φ[4] := G[1, 4] + G[2, 4] + G[3, 4] + G1[4, 4] :
φ[5] := G[1, 5] + G[2, 5] + G[3, 5] + G[4, 5] + G1[5, 5] :
φ[6] := G[1, 6] + G[2, 6] + G[3, 6] + G[4, 6] + G[5, 6] + + G1[6, 6] :
φ[7] := G[1, 7] + G[2, 7] + G[3, 7] + G[4, 7] + G[5, 7] + G[6, 7] + G1[7, 7] :
φ[8] := G[1, 8] + G[2, 8] + G[3, 8] + G[4, 8] + G[5, 8] + G[6, 8] + G[7, 8] + G1[8, 8] :

f := x → 0 :

for i from 1 to 8 do
  ff[i] := f(t[i]) :
end do :

F1 := array([[ff[1]], [ff[2]], [ff[3]], [ff[4]], [ff[5]], [ff[6]], [ff[7]], [ff[8]]]) :
Φ := array([[φ[1]], [φ[2]], [φ[3]], [φ[4]], [φ[5]], [φ[6]], [φ[7]], [φ[8]]]) :
U1 := array([[u1[1]], [u1[2]], [u1[3]], [u1[4]], [u1[5]], [u1[6]], [u1[7]], [u1[8]]]) :
U := array([[u[1]], [u[2]], [u[3]], [u[4]], [u[5]], [u[6]], [u[7]], [u[8]]]) :
```



$$E[1] := -UI[1,1] - U[1,1] + FI[1,1] + \Phi[1,1] = 0 :$$
$$E[2] := -UI[2,1] - U[2,1] + FI[2,1] + \Phi[2,1] = 0 :$$
$$E[3] := -UI[3,1] - U[3,1] + FI[3,1] + \Phi[3,1] = 0 :$$
$$E[4] := -UI[4,1] - U[4,1] + FI[4,1] + \Phi[4,1] = 0 :$$
$$E[5] := -UI[5,1] - U[5,1] + FI[5,1] + \Phi[5,1] = 0 :$$
$$E[6] := -UI[6,1] - U[6,1] + FI[6,1] + \Phi[6,1] = 0 :$$
$$E[7] := -UI[7,1] - U[7,1] + FI[7,1] + \Phi[7,1] = 0 :$$
$$E[8] := -UI[8,1] - U[8,1] + FI[8,1] + \Phi[8,1] = 0 :$$

$$solve(\{E[1], E[2], E[3], E[4], E[5], E[6], E[7], E[8]\}, \{a, b, c, d, e, f, g, k\});$$

$$\{a = -0.4294829821, b = -0.1994100106, c = -0.1564973539, d = -0.05550281296, e = -0.09953642127, f = -0.05949646172, g = -0.03543716762, k = -0.02098129873\} \quad (1)$$

$$u := x \rightarrow -0.4294829821 \cdot p_1(x) - 0.1994100106 \cdot p_2(x) - 0.1564973539 \cdot p_3(x) - 0.05550281296 \cdot p_4(x) - 0.09953642127 \cdot p_5(x) - 0.05949646172 \cdot p_6(x) - 0.03543716762 \cdot p_7(x) - 0.02098129873 \cdot p_8(x) + 1 :$$

$v := x \rightarrow \exp(-x) \cdot \cosh(x) :$

for i **from** 0 **to** 10 **do**

$evalf\left(\frac{i}{1000}\right);$

$evalf\left(u\left(\frac{i}{1000}\right)\right); evalf\left(v\left(\frac{i}{1000}\right)\right);$ *print*("   ");

end do;

for i **from** 0 **to** 10 **do**

$\mathrm{abs}\left(evalf\left(u\left(\frac{i}{1000}\right) - v\left(\frac{i}{1000}\right)\right)\right);$

end do;

$with(plots):$

$p1 := plot(\exp(-x) \cdot \cosh(x), x = 0..0.1, color = red):$

$p2 := plot(-0.4294829821 \cdot p_1(x) - 0.1994100106 \cdot p_2(x) - 0.1564973539 \cdot p_3(x) - 0.05550281296 \cdot p_4(x) - 0.09953642127 \cdot p_5(x) - 0.05949646172 \cdot p_6(x) - 0.03543716762 \cdot p_7(x) - 0.02098129873 \cdot p_8(x) + 1, x = 0..0.1, color = blue):$

$display(\{p1, p2\});$

Example 4:

Taylor method

$$kk := (x, y) \rightarrow x - y :$$

$$U0 := \mathbf{proc}(k, h)$$

$$\mathbf{local}\ i\ , j\ , U0 : \quad U0 := \frac{1}{k! \cdot h!} \cdot evalf(\mathrm{D}[1\$k, 2\$h](kk)(0, 0))\ ;$$

$$\mathbf{end\ proc} :$$

$$K := \begin{bmatrix} U0(0, 0) & U0(0, 1) & U0(0, 2) \\ U0(1, 0) & U0(1, 1) & U0(1, 2) \\ U0(2, 0) & U0(2, 1) & U0(2, 2) \end{bmatrix};$$

$$\begin{bmatrix} 0. & -1. & 0. \\ 1. & 0. & 0. \\ 0. & 0. & 0. \end{bmatrix}$$

$$MM := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} :$$

$$\mathrm{KK} := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} :$$

$$HH := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} & \frac{1}{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{24} & \frac{1}{64} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{24} & \frac{1}{64} & \frac{1}{160} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} :$$

$$\mathrm{PP} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} :$$

$$W2 := PP\ KK\ HH\ MM;$$

$$\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{48} & \frac{1}{384} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \end{bmatrix}$$

$$P1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} :$$

$$C := \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix} :$$

$$M1 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} :$$

$W1 := P1\ C\ M1;$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

$W := W1 - W2;$

$$\begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{8} & \frac{47}{48} & \frac{191}{384} \\ -\frac{1}{2} & \frac{5}{6} & \frac{23}{24} \end{bmatrix}$$

$$F := \begin{bmatrix} 0 \\ \frac{1}{2} + \frac{1}{48} \\ \frac{7}{6} \end{bmatrix} :$$

$$W := \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{8} & \frac{47}{48} & \frac{191}{384} \\ -\frac{1}{2} & \frac{5}{6} & \frac{23}{24} \end{bmatrix} :$$

$W^{-1}\ F;$

$$\begin{bmatrix} -\frac{17}{27} \\ 0 \\ \frac{8}{9} \end{bmatrix}$$

$$Wc := \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{8} & \frac{47}{48} & \frac{191}{384} \\ 1 & 0 & 0 \end{bmatrix} :$$

$$Fc := \begin{bmatrix} 0 \\ \frac{25}{48} \\ 1 \end{bmatrix} :$$

$(Wc)^{-1}\ Fc;$

$$\begin{bmatrix} 1 \\ 0 \\ \frac{248}{191} \end{bmatrix}$$

$s := x \rightarrow 1 + \frac{248}{191} \cdot x^2 :$

$v := x \rightarrow e^x - x :$

for i **from** 0 **to** 10 **do**

$evalf\left(\frac{i}{1000}\right);$

$evalf\left(s\left(\frac{i}{1000}\right)\right);\ evalf\left(v\left(\frac{i}{1000}\right)\right);\ print("\quad ");$

end do;

for i **from** 0 **to** 10 **do**

$\mathrm{abs}\left(evalf\left(s\left(\frac{i}{1000}\right) - v\left(\frac{i}{1000}\right)\right)\right);$

end do;

$with(plots);$

$[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra\_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot] \quad (8)$

$p1 := plot\left(e^x - x, x = 0..0.5, color = red\right);$

$p2 := plot\left(1 + \frac{248}{191} \cdot x^2, x = 0..0.5, color = blue\right);$

$display(\{p1, p2\});$

Legendre method

$with(LinearAlgebra) : with(SolveTools) :$

$$p_0(x) := 1 :$$

$$p_1(x) := x :$$

$$p_2(x) := \frac{3}{2}(x^2 - 1) :$$

$$L := \begin{bmatrix} 1 & x & \frac{3}{2}(x^2 - 1) \end{bmatrix} :$$

$$A := \begin{bmatrix} a \\ b \\ c \end{bmatrix} :$$

$$UU := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} :$$

$$S := L\,UU;$$

$$\begin{bmatrix} x & \frac{9}{2}x^2 - \frac{9}{2} & 0 \end{bmatrix}$$

$$S\,A;$$

$$x\,a + \left(\frac{9}{2}x^2 - \frac{9}{2}\right)b$$

$$hh := x \rightarrow x\,a + \left(\frac{9}{2}x^2 - \frac{9}{2}\right)b :$$

```
for n from 0 to 2 do
 for m from 0 to 2 do
 $e := evalf\left(\int_0^1 \left((x - t)\cdot\left(a + b\cdot t + c\cdot\left(\frac{3}{2}\cdot t^2 - \frac{3}{2}\right)\right)\cdot(p_m(t))\right) dt\right) :$
 $k[n, m] := \int_0^1 ((e)\cdot p_n(x))\,dx :$
 end do:
 end do :
```

$$K := \begin{bmatrix} k[0,0] & k[0,1] & k[0,2] \\ k[1,0] & k[1,1] & k[1,2] \\ k[2,0] & k[2,1] & k[2,2] \end{bmatrix};$$

$$\begin{bmatrix} -0.1250000000\,c - 0.08333333330\,b & 0.01250000000\,c - 0.08333333335\,b - 0.08333333330\,a & 0.2250000000\,c + 0.01250000000\,b - 0.1250000000\,a \\ -0.1458333333\,c + 1.666666667\,10^{-11}\,b + 0.08333333333\,a & -0.02500000000\,c - 0.01388888890\,b + 1.666666667\,10^{-11}\,a & 0.2125000000\,c - 0.02500000000\,b - 0.1458333333\,a \\ 0.1458333333\,b + 0.1250000000\,a & -0.05937500000\,c + 0.1250000000\,b + 0.1458333333\,a & -0.07500000000\,c - 0.05937500000\,b \end{bmatrix} \quad (3)$$

$$IL := \begin{bmatrix} x \\ \frac{1}{2}x^2 \\ \frac{1}{2}x^3 - \frac{3}{2}x \end{bmatrix} :$$

$$Y := K\,IL;$$

$$\left[\left[(-0.1250000000\,c - 0.08333333330\,b)\,x + \frac{1}{2}(0.01250000000\,c - 0.08333333335\,b - 0.08333333330\,a)\,x^2 + (0.2250000000\,c + 0.01250000000\,b - 0.1250000000\,a)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right)\right],\right.$$
$$\left[\left(-0.1458333333\,c + 1.666666667\,10^{-11}\,b + 0.08333333333\,a\right)x + \frac{1}{2}\left(-0.02500000000\,c - 0.01388888890\,b + 1.666666667\,10^{-11}\,a\right)x^2 + (0.2125000000\,c - 0.02500000000\,b - 0.1458333333\,a)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right)\right],$$
$$\left.\left[(0.1458333333\,b + 0.1250000000\,a)\,x + \frac{1}{2}(-0.05937500000\,c + 0.1250000000\,b + 0.1458333333\,a)\,x^2 + (-0.07500000000\,c - 0.05937500000\,b)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right)\right]\right] \quad (4)$$

$$N := L\,Y;$$

$$(-0.1250000000\,c - 0.08333333330\,b)\,x + \frac{1}{2}(0.01250000000\,c - 0.08333333335\,b - 0.08333333330\,a)\,x^2 + (0.2250000000\,c + 0.01250000000\,b - 0.1250000000\,a)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right)$$
$$+ x\left(\left(-0.1458333333\,c + 1.666666667\,10^{-11}\,b + 0.08333333333\,a\right)x + \frac{1}{2}\left(-0.02500000000\,c - 0.01388888890\,b + 1.666666667\,10^{-11}\,a\right)x^2 + (0.2125000000\,c - 0.02500000000\,b - 0.1458333333\,a)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right)\right)$$
$$+ \left(\frac{3}{2}x^2 - \frac{3}{2}\right)\left((0.1458333333\,b + 0.1250000000\,a)\,x + \frac{1}{2}(-0.05937500000\,c + 0.1250000000\,b + 0.1458333333\,a)\,x^2 + (-0.07500000000\,c - 0.05937500000\,b)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right)\right) \quad (5)$$

$$x + \frac{1}{6}x^3 + N$$

$$x + \frac{1}{6}x^3 + (-0.1250000000\,c - 0.08333333330\,b)\,x + \frac{1}{2}(0.01250000000\,c - 0.08333333335\,b - 0.08333333330\,a)\,x^2 + (0.2250000000\,c + 0.01250000000\,b - 0.1250000000\,a)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) \quad (6)$$
$$+ x\left(\left(-0.1458333333\,c + 1.666666667\,10^{-11}\,b + 0.08333333333\,a\right)x + \frac{1}{2}\left(-0.02500000000\,c - 0.01388888890\,b + 1.666666667\,10^{-11}\,a\right)x^2 + (0.2125000000\,c - 0.02500000000\,b - 0.1458333333\,a)\left(\frac{1}{2}x^3 - \frac{3}{2}x\right)\right)$$
$$+ \left(\frac{3}{2}x^2 - \frac{3}{2}\right)\left((0.1458333333\,b + 0.1250000000\,a)\,x + \frac{1}{2}(-0.05937500000\,c + 0.1250000000\,b + 0.1458333333\,a)\,x^2 + (\right.$$

$x+\frac{1}{6}x^3+N$

$$x+\frac{1}{6}x^3+(-0.1250000000\,c-0.08333333330\,b)\,x+\frac{1}{2}(0.01250000000\,c-0.08333333335\,b-0.08333333330\,a)\,x^2+(0.2250000000\,c+0.01250000000\,b-0.1250000000\,a)\left(\frac{1}{2}x^3-\frac{3}{2}x\right)+x\left((-0.1458333333\,c+1.666666667\,10^{-11}\,b+0.08333333333\,a)\,x+\frac{1}{2}(-0.02500000000\,c-0.01388888890\,b+1.666666667\,10^{-11}\,a)\,x^2+(0.2125000000\,c-0.02500000000\,b-0.1458333333\,a)\left(\frac{1}{2}x^3-\frac{3}{2}x\right)\right)+\left(\frac{3}{2}x^2-\frac{3}{2}\right)\left((0.1458333333\,b+0.1250000000\,a)\,x+\frac{1}{2}(-0.05937500000\,c+0.1250000000\,b+0.1458333333\,a)\,x^2+(-0.07500000000\,c-0.05937500000\,b)\left(\frac{1}{2}x^3-\frac{3}{2}x\right)\right) \quad (6)$$

for i **from** 0 **to** 2 **do**
$x[i]:=\frac{i}{2};$
end do :

$$h:=x\rightarrow x+\frac{1}{6}\cdot x^3+(-0.1250000000\cdot c-0.08333333330\cdot b)\cdot x+\frac{1}{2}(0.01250000000\cdot c-0.08333333335\cdot b-0.08333333330\cdot a)\cdot x^2+(0.2250000000\cdot c+0.01250000000\cdot b-0.1250000000\cdot a)\left(\frac{1}{2}\cdot x^3-\frac{3}{2}\cdot x\right)+x\cdot\left((-0.1458333333\cdot c+1.666666667\cdot 10^{-11}\cdot b+0.08333333333\cdot a)\cdot x+\frac{1}{2}(-0.02500000000\cdot c-0.01388888890\cdot b+1.666666667\cdot 10^{-11}\cdot a)\cdot x^2+(0.2125000000\cdot c-0.02500000000\cdot b-0.1458333333\cdot a)\left(\frac{1}{2}\cdot x^3-\frac{3}{2}\cdot x\right)\right)+\left(\frac{3}{2}\cdot x^2-\frac{3}{2}\right)\cdot\left((0.1458333333\cdot b+0.1250000000\cdot a)\cdot x+\frac{1}{2}\cdot(-0.05937500000\cdot c+0.1250000000\cdot b+0.1458333333\cdot a)\cdot x^2+(-0.07500000000\cdot c-0.05937500000\cdot b)\left(\frac{1}{2}\cdot x^3-\frac{3}{2}\cdot x\right)\right):$$

for i **from** 0 **to** 2 **do**
$h(x[i]);$
$hh(x[i]);$
$E[i]:=h(x[i])=hh(x[i]);$
end do ;
$E[3]:=a-\frac{3}{2}c=1;$

$$0$$
$$-\frac{9}{2}b$$
$$0=-\frac{9}{2}b$$
$$\frac{25}{48}-0.3763509114\,c-0.1984836155\,b+0.05566406245\,a$$
$$\frac{1}{2}a-\frac{27}{8}b$$
$$\frac{25}{48}-0.3763509114\,c-0.1984836155\,b+0.05566406245\,a=\frac{1}{2}a-\frac{27}{8}b$$
$$\frac{7}{6}-0.7145833333\,c-0.1194444444\,b+0.3125000000\,a$$
$$a$$
$$\frac{7}{6}-0.7145833333\,c-0.1194444444\,b+0.3125000000\,a=a$$
$$a-\frac{3}{2}c=1$$

$fsolve(\{E[0],E[1],E[3]\},\{a,b,c\});$

$$\{a=1.110030746,\ b=-0.,\ c=0.0733538307\} \quad (8)$$

$u:=x\rightarrow 1.110030746+0.0733538307\cdot\frac{3}{2}(x^2-1):$
$v:=x\rightarrow e^x-x:$

for i **from** 0 **to** 10 **do**
$evalf\left(\frac{i}{1000}\right);$
$evalf\left(u\left(\frac{i}{1000}\right)\right);\ evalf\left(v\left(\frac{i}{1000}\right)\right);\ print("\quad");$
end do;

for i **from** 0 **to** 10 **do**
$\mathrm{abs}\left(evalf\left(u\left(\frac{i}{1000}\right)-v\left(\frac{i}{1000}\right)\right)\right);$
end do;

with(*plots*);

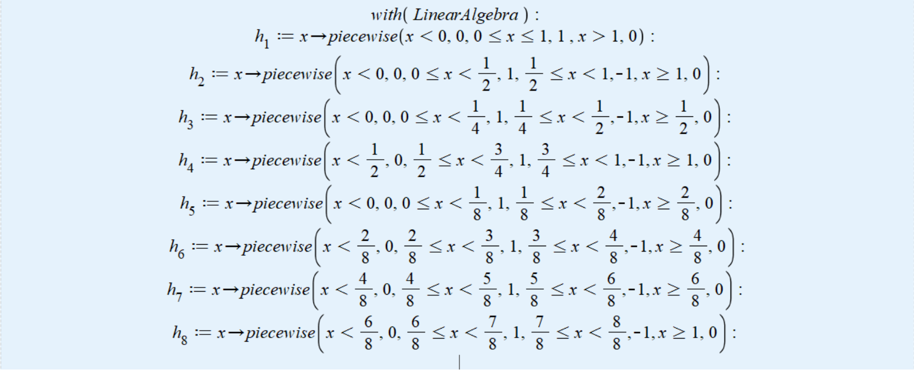
[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot*] (9)

$p1:=plot(e^x-x,\ x=0..0.5,\ color=red);$
$p2:=plot\left(1.110030746+0.0733538307\cdot\frac{3}{2}(x^2-1),\ x=0..0.5,\ color=blue\right);$
$display(\{p1,p2\});$

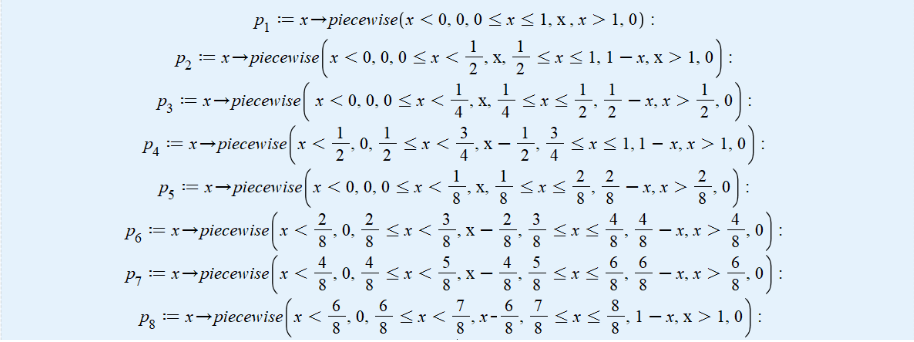
#Haar method

$\#u'(x)=x+\frac{1}{6}x^3+\int_0^x(x-t)u(t)\,dt$

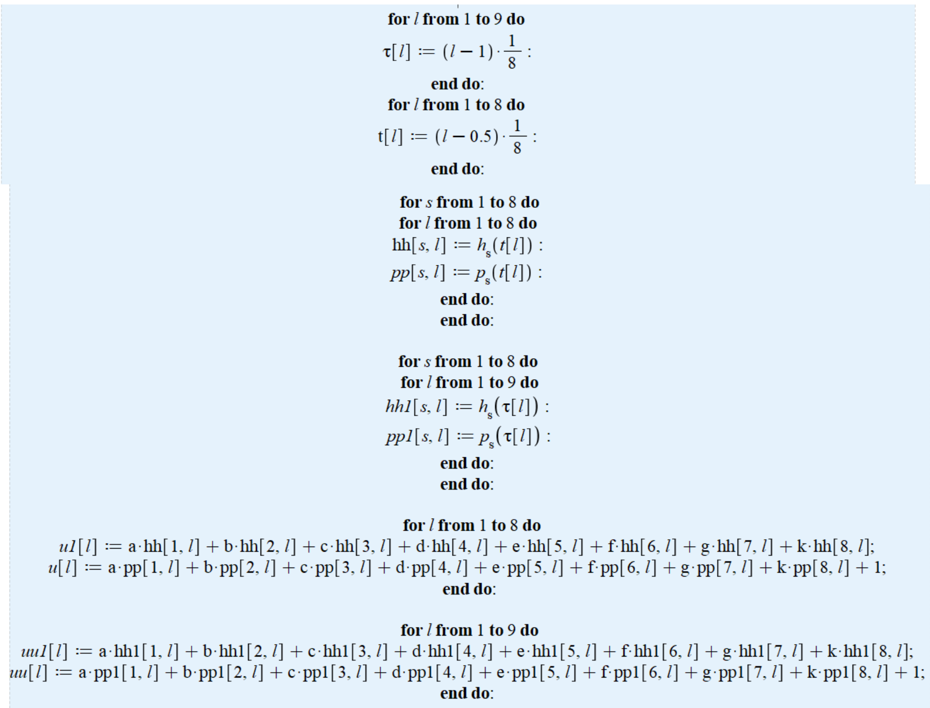
$\#\ u(0)=1,\ u(x)=e^x-x$



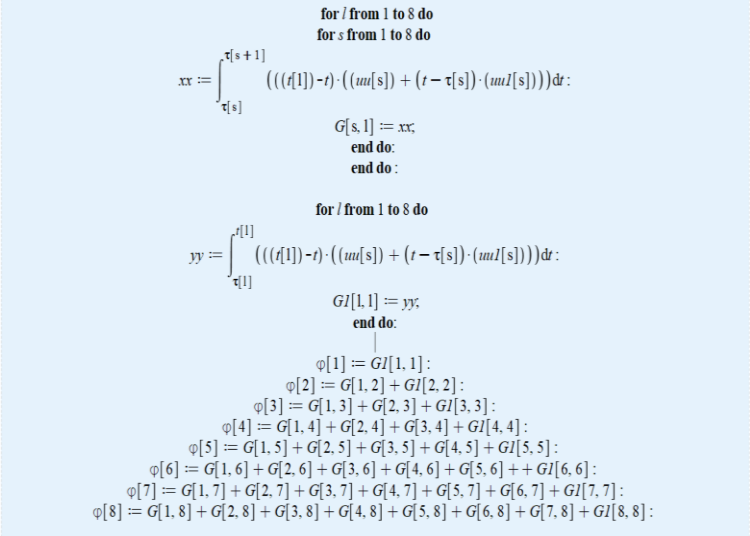
```
with( LinearAlgebra ) :
h_1 ≔ x→piecewise(x < 0, 0, 0 ≤ x ≤ 1, 1 , x > 1, 0) :
h_2 ≔ x→piecewise(x < 0, 0, 0 ≤ x < 1/2, 1, 1/2 ≤ x < 1,-1, x ≥ 1, 0) :
h_3 ≔ x→piecewise(x < 0, 0, 0 ≤ x < 1/4, 1, 1/4 ≤ x < 1/2,-1, x ≥ 1/2, 0) :
h_4 ≔ x→piecewise(x < 1/2, 0, 1/2 ≤ x < 3/4, 1, 3/4 ≤ x < 1,-1, x ≥ 1, 0) :
h_5 ≔ x→piecewise(x < 0, 0, 0 ≤ x < 1/8, 1, 1/8 ≤ x < 2/8,-1, x ≥ 2/8, 0) :
h_6 ≔ x→piecewise(x < 2/8, 0, 2/8 ≤ x < 3/8, 1, 3/8 ≤ x < 4/8,-1, x ≥ 4/8, 0) :
h_7 ≔ x→piecewise(x < 4/8, 0, 4/8 ≤ x < 5/8, 1, 5/8 ≤ x < 6/8,-1, x ≥ 6/8, 0) :
h_8 ≔ x→piecewise(x < 6/8, 0, 6/8 ≤ x < 7/8, 1, 7/8 ≤ x < 8/8,-1, x ≥ 1, 0) :
```



```
p_1 ≔ x→piecewise(x < 0, 0, 0 ≤ x ≤ 1, x , x > 1, 0) :
p_2 ≔ x→piecewise(x < 0, 0, 0 ≤ x < 1/2, x, 1/2 ≤ x ≤ 1, 1 − x, x > 1, 0) :
p_3 ≔ x→piecewise( x < 0, 0, 0 ≤ x < 1/4, x, 1/4 ≤ x ≤ 1/2, 1/2 − x, x > 1/2, 0) :
p_4 ≔ x→piecewise(x < 1/2, 0, 1/2 ≤ x < 3/4, x − 1/2, 3/4 ≤ x ≤ 1, 1 − x, x > 1, 0) :
p_5 ≔ x→piecewise(x < 0, 0, 0 ≤ x < 1/8, x, 1/8 ≤ x ≤ 2/8, 2/8 − x, x > 2/8, 0) :
p_6 ≔ x→piecewise(x < 2/8, 0, 2/8 ≤ x < 3/8, x − 2/8, 3/8 ≤ x ≤ 4/8, 4/8 − x, x > 4/8, 0) :
p_7 ≔ x→piecewise(x < 4/8, 0, 4/8 ≤ x < 5/8, x − 4/8, 5/8 ≤ x ≤ 6/8, 6/8 − x, x > 6/8, 0) :
p_8 ≔ x→piecewise(x < 6/8, 0, 6/8 ≤ x < 7/8, x-6/8, 7/8 ≤ x ≤ 8/8, 1 − x, x > 1, 0) :
```



```
for l from 1 to 9 do
τ[l] ≔ (l − 1)·1/8 :
end do:
for l from 1 to 8 do
t[l] ≔ (l − 0.5)·1/8 :
end do:

for s from 1 to 8 do
for l from 1 to 8 do
hh[s, l] ≔ h_s(t[l]) :
pp[s, l] ≔ p_s(t[l]) :
end do:
end do:

for s from 1 to 8 do
for l from 1 to 9 do
hh1[s, l] ≔ h_s(τ[l]) :
pp1[s, l] ≔ p_s(τ[l]) :
end do:
end do:

for l from 1 to 8 do
u1[l] ≔ a·hh[1, l] + b·hh[2, l] + c·hh[3, l] + d·hh[4, l] + e·hh[5, l] + f·hh[6, l] + g·hh[7, l] + k·hh[8, l];
u[l] ≔ a·pp[1, l] + b·pp[2, l] + c·pp[3, l] + d·pp[4, l] + e·pp[5, l] + f·pp[6, l] + g·pp[7, l] + k·pp[8, l] + 1;
end do:

for l from 1 to 9 do
uu1[l] ≔ a·hh1[1, l] + b·hh1[2, l] + c·hh1[3, l] + d·hh1[4, l] + e·hh1[5, l] + f·hh1[6, l] + g·hh1[7, l] + k·hh1[8, l];
uu[l] ≔ a·pp1[1, l] + b·pp1[2, l] + c·pp1[3, l] + d·pp1[4, l] + e·pp1[5, l] + f·pp1[6, l] + g·pp1[7, l] + k·pp1[8, l] + 1;
end do:
```



for l **from** 1 **to** 8 **do**
for s **from** 1 **to** 8 **do**

$$xx := \int_{\tau[s]}^{\tau[s+1]} \left(\left(\left(t[1]\right)-t\right)\cdot\left(\left(uu[s]\right)+\left(t-\tau[s]\right)\cdot\left(uu1[s]\right)\right)\right)\mathrm{d}t :$$

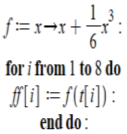
$G[s,1] := xx;$
end do:
end do :

for l **from** 1 **to** 8 **do**

$$yy := \int_{\tau[1]}^{t[1]} \left(\left(\left(t[1]\right)-t\right)\cdot\left(\left(uu[s]\right)+\left(t-\tau[s]\right)\cdot\left(uu1[s]\right)\right)\right)\mathrm{d}t :$$

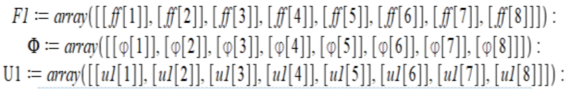
$G1[1,1] := yy;$
end do:

$\varphi[1] := G1[1,1]:$
$\varphi[2] := G[1,2]+G1[2,2]:$
$\varphi[3] := G[1,3]+G[2,3]+G1[3,3]:$
$\varphi[4] := G[1,4]+G[2,4]+G[3,4]+G1[4,4]:$
$\varphi[5] := G[1,5]+G[2,5]+G[3,5]+G[4,5]+G1[5,5]:$
$\varphi[6] := G[1,6]+G[2,6]+G[3,6]+G[4,6]+G[5,6]++G1[6,6]:$
$\varphi[7] := G[1,7]+G[2,7]+G[3,7]+G[4,7]+G[5,7]+G[6,7]+G1[7,7]:$
$\varphi[8] := G[1,8]+G[2,8]+G[3,8]+G[4,8]+G[5,8]+G[6,8]+G[7,8]+G1[8,8]:$

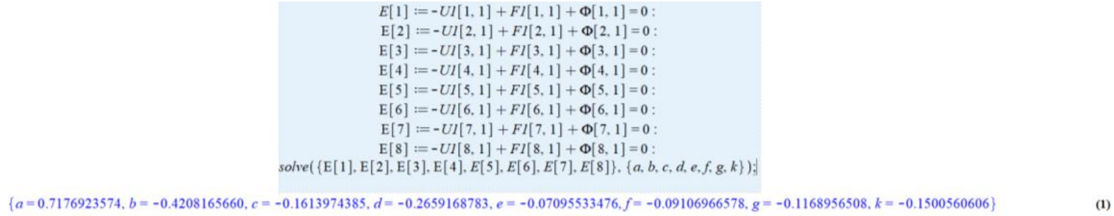


$$f := x \rightarrow x+\frac{1}{6}x^3 :$$

for i **from** 1 **to** 8 **do**
$ff[i] := f(t[i]):$
end do :

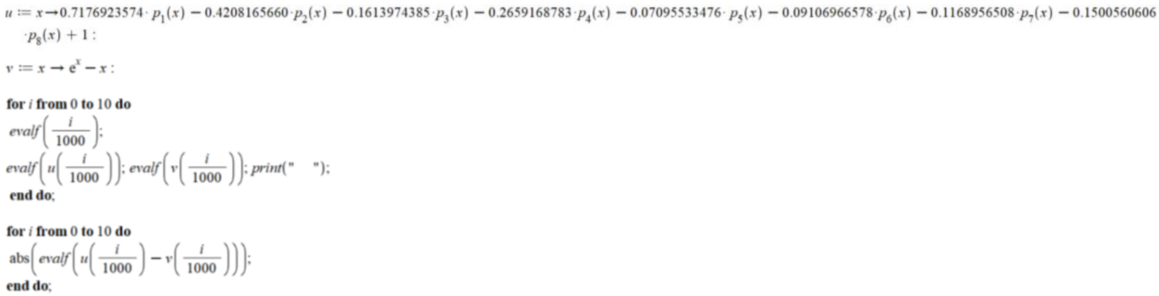


$F1 := array([[ff[1]],[ff[2]],[ff[3]],[ff[4]],[ff[5]],[ff[6]],[ff[7]],[ff[8]]]):$
$\Phi := array([[\varphi[1]],[\varphi[2]],[\varphi[3]],[\varphi[4]],[\varphi[5]],[\varphi[6]],[\varphi[7]],[\varphi[8]]]):$
$U1 := array([[u1[1]],[u1[2]],[u1[3]],[u1[4]],[u1[5]],[u1[6]],[u1[7]],[u1[8]]]):$



$E[1] := -U1[1,1]+F1[1,1]+\Phi[1,1]=0:$
$E[2] := -U1[2,1]+F1[2,1]+\Phi[2,1]=0:$
$E[3] := -U1[3,1]+F1[3,1]+\Phi[3,1]=0:$
$E[4] := -U1[4,1]+F1[4,1]+\Phi[4,1]=0:$
$E[5] := -U1[5,1]+F1[5,1]+\Phi[5,1]=0:$
$E[6] := -U1[6,1]+F1[6,1]+\Phi[6,1]=0:$
$E[7] := -U1[7,1]+F1[7,1]+\Phi[7,1]=0:$
$E[8] := -U1[8,1]+F1[8,1]+\Phi[8,1]=0:$
$solve(\{E[1],E[2],E[3],E[4],E[5],E[6],E[7],E[8]\},\{a,b,c,d,e,f,g,k\});$

$$\{a=0.7176923574, b=-0.4208165660, c=-0.1613974385, d=-0.2659168783, e=-0.07095533476, f=-0.09106966578, g=-0.1168956508, k=-0.1500560606\} \quad (1)$$

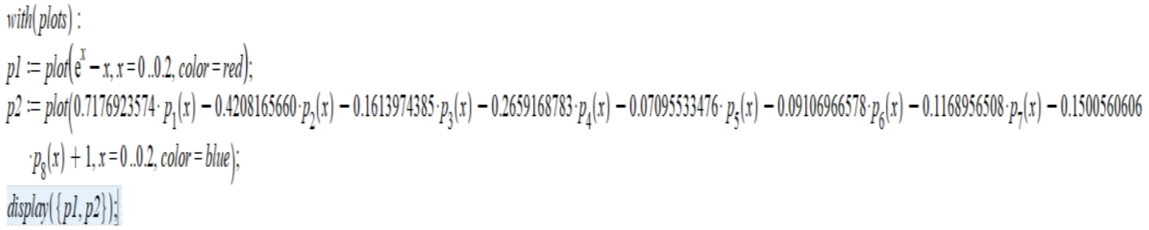


$$u := x \rightarrow 0.7176923574\cdot p_1(x)-0.4208165660\cdot p_2(x)-0.1613974385\cdot p_3(x)-0.2659168783\cdot p_4(x)-0.07095533476\cdot p_5(x)-0.09106966578\cdot p_6(x)-0.1168956508\cdot p_7(x)-0.1500560606\cdot p_8(x)+1:$$

$v := x \rightarrow e^x - x:$

for i **from** 0 **to** 10 **do**
$evalf\left(\frac{i}{1000}\right);$
$evalf\left(u\left(\frac{i}{1000}\right)\right); evalf\left(v\left(\frac{i}{1000}\right)\right); print("\quad");$
end do;

for i **from** 0 **to** 10 **do**
$\mathrm{abs}\left(evalf\left(u\left(\frac{i}{1000}\right)-v\left(\frac{i}{1000}\right)\right)\right);$
end do;



$with(plots):$
$p1 := plot(e^x - x, x=0..0.2, color=red);$
$p2 := plot(0.7176923574\cdot p_1(x)-0.4208165660\cdot p_2(x)-0.1613974385\cdot p_3(x)-0.2659168783\cdot p_4(x)-0.07095533476\cdot p_5(x)-0.09106966578\cdot p_6(x)-0.1168956508\cdot p_7(x)-0.1500560606\cdot p_8(x)+1, x=0..0.2, color=blue);$
$display(\{p1,p2\});$

جامعة النجاح الوطنية

كلية الدراسات العليا

الطرق الحسابية لحل معادلات فولتيرا التكاملية التفاضلية

إعداد

ماسة حسام سعيد الدسوقي

إشراف

أ. د. ناجي قطناني

قدمت هذه الرسالة استكمالا لمتطلبات الحصول علي درجه الماجستير في الرياضيات، من كلية الدراسات العليا، في جامعة النجاح الوطنية، نابلس– فلسطين.

2022

الطرق الحسابية لحل معادلات فولتيرا التكاملية التفاضلية

إعداد

ماسة حسام سعيد الدسوقي

إشراف

أ. د. ناجي قطناني

الملخص

الخلفية: المعادلات التفاضلية التكاملية (IDEs)، إحدى أهم الأدوات الرياضية في كل من الرياضيات البحتة والتطبيقية تنشأ في العديد من المشكلات الفيزيائية مثل تموج الرياح في الصحراء، والديناميكا المائية النانوية، ونموذج النمو السكاني، وعملية تشكيل الزجاج.

لقد حفزوا قدرًا هائلاً من الأبحاث في السنوات الأخيرة. طور العديد من الباحثين مخططات عددية لحل IDEs.

الهدف: في هذا العمل، اقترح الباحث ثلاثة مخططات عددية، وهي طريقة تايلور التجميعية، وطريقة ليجيندر متعددة الحدود وطريقة مويجات هار، لتقريب حل معادلات فولتيرا التكاملية التفاضلية (VIDEs).

الطريقة والإجراءات: تم تطبيق هذه الطرق العددية الثلاثة في شكل خوارزميات، وتم تطوير/ استخدام برنامج Maple لحل بعض الأمثلة العددية.

النتائج: أوضحت النتائج العددية أن تقارب ودقة الطرق المذكورة أعلاه كانا متفقين بشكل جيد مع الحل التحليلي. أظهرت مقارنة النتائج العددية المذكورة في الجداول والأشكال بوضوح أن طريقة موجات هار توفر نتائج أكثر دقة وبالتالي فهي أكثر فاعلية من الطرق الأخرى.

الكلمات المفتاحية: معادلات فولتيرا التكاملية التفاضلية، تايلور التجميعية، ليجيندر متعددة الحدود، موجات هار.