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NUMERICAL COMPARISON OF METHODS FOR SOLVING SECOND ORDER ODES

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Dedication

This thesis is dedicated to my loving family, whose unwavering support and encouragement have been the driving force behind my academic journe. To my dear friends, for their laughter, camaraderie, and moments of respite amidst the academic rigors. And to all my relatives and loved ones who have stood by me with love and encouragement, this work is a testament to your belief in me.

Acknowledgement

All praise and thanks are due to Allah, the Lord of the worlds, and may the peace and blessings of Allah be upon the Seal of the Prophets and Messengers. Muhammad ibn Abdullah (peace be upon him)."

As Allah Almighty says. "And whoever is grateful it is only for [the benefit of] himself." (Quran 31:12) And the Messenger (peace be upon him) said, "Whoever does not thank people has not thanked Allah."

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My appreciation is also extended to the discussion committee for their efforts, time, and constructive observations. May Allah reward them greatly.

To the beacon of knowledge and the destination of its seekers, our pioneering An-Najah National University, represented by its President and the administrative and academic bodies, and to all who have contributed to the completion of this humble study, My heartfelt thanks, gratitude, and appreciation.

Declaration

I, the undersigned, declare that I submitted the thesis entitled:

NUMERICAL COMPARISON OF METHODS FOR SOLVING SECOND ORDER ODES

I declare that the work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's Name: عكرمة سامر فتحي دالي

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Abstract

In this study, two numerical methods for solving second order ODEs were tested, namely finite difference method and Runge _ Kutta method . The study aims to find out which of the two methods is more efficient and accurate. To obtain this result we solved one of the most important second order ODE, called damped harmonic oscillator equation. The result was that Runge_Kutta method was more accurate and effective.

We presented the solutions to this equation using the two numerical methods mentioned, as we found the solutions to this equation when $h = 0.25$, $h = 0.2$, $h = 0.15 and h = 0.1$, then we did the root mean square error (RMSE) for all values of h, and for both methods, and it became clear to us that the decrease in the value of h leads to a decrease in the error between the numerical solution and the exact solution, and this mean that the answer becomes more accurate and closer to the exact solution, and this means that the two methods are consistent.

Keywords: Finite difference method; Runge kutta method; Second order ODEs.

Chapter One
Introduction

Many problems related to mathematical physics and engineering, can be modeled by second order ODEs. In many cases, it is necessary to use numerical methods to solve such equations, and comparing these methods is an important matter to know the precise dynamic and physical behavior. Among the numerical methods that were compared are the (FEM) and the (FDM), which are two famous methods for solving differential equations. (FEM) is a numerical method used to solve partial differential equations. This method is based on dividing the interval into simpler and smaller elements. Expressing each element through a set of equations, and by compiling these equations, the general solution to the equation is obtained. The (FDM) method is a numerical method that uses approximate values of the derivatives and finite differences at separate points in the field. It divides the field into a grid and then replaces the derivatives with their approximate values. This method is easy to implement.

Several studies have compared both of the above mentioned methods, including Smith Et Al, who studied thermal conduction problems, It was found that the (FEM) method gives more accurate results for irregular geometric shapes, but the (FDM) method is more effective for regular geometric shapes.

Jons and Wang also conducted a study on structural analysis and concluded that (FEM) was superior to (FDM) in this type of analysis.

Zhang and Li also presented an investigation on flow problems in fluids and pointed out that (FDM) is easier to implement but faces difficulties in dealing with complex boundary conditions, while (FEM) provided more flexibility and accuracy.

Among these comparisons, two widely used numerical methods, were compared, namely the RK method and Euler methods. Euler's method is known for its simplicity and is used to solve elementary differential equations. Despite its simplicity, it suffers from stability problems and tends to accumulate errors quickly, especially if the equation is difficult.

While the Runge-Kutta method is a set of procedures that provide higher accuracy than the Euler method, the RK method is considered the fourth degree of the most famous Runge-Kutta methods, as it calculates the average values repeatedly to obtain greater accuracy and it is more computationally efficient, which makes it suitable for many applications[1].

In terms of accuracy: The RK method, especially the RK4, is more accurate than Euler's methods. So the Euler can't be used in applications that require greater accuracy.

In terms of computational efficiency: Although the Runge-Kutta method is accurate, it requires additional effort in calculations, especially for high-order variables, which makes this method less efficient in large-scale simulations [1].

In terms of applications: Both of the mentioned methods find applications in several different fields, including engineering, chemistry, and physics. The choice between these two methods depends on the nature of the problem we face, as Euler's method is used for simple problems that do not face mathematical concerns, as for complex problems that require greater accuracy, the Runge_ Kutta method is more efficient [1].

In other studies, the block method and the (FDM) were compared due to their wide spread and frequent use in science, engineering, and other applications. These studies aim to present the strengths and weaknesses of each method in order to make it easier for students to choose the most appropriate between these methods. Numerical methods play an important role in solving problems with difficult or unavailable analytical solutions. Among these numerical methods, the (FDM) and the block method appeared to approximate solutions and obtain results that are very close to the original results, but even though the two methods share the same goal, which is to find solutions to problems. However, they differ significantly in terms of implementation strategies, basic principles, and computational efficiency.

The block method also known as divided Runge_kotta, is a digital mechanism that analyzes the computational field into separate sections, then each block is solved independently and the solutions are constantly updated through communication between the sections. This method has several advantages, including stability. It is flexible in dealing with problems with irregular geometric shapes and is characterized by superior stability properties compared to traditional numerical methods, which makes it suitable for difficult differential problems with multiple physical simulations[2].

On the other hand, the (FDM) relies on approximating derivatives using finite difference approximations, then the original differential equations are converted into a system of algebraic equations. Despite the ease and simplicity of this method, it suffers from numerical instability, and the accuracy of this method depends on the accuracy of the network and boundary conditions. In short, both methods provide distinct advantages and limitations, making them suitable for many numerical and computational tasks, while the block method excels in dealing with complex geometric shapes, while the finite difference method remains a popular choice due to its simplicity and easiness. In the end, the choice between the two methods depends on the requirements of the problem we study [2].

Numerical methods play an important role in solving differential equations. The RK and The (FDM) are among the famous and widely numerical methods used. Comparing the methods aims to know the accuracy and efficiency of each and the possibility of using them in various fields.

In terms of accuracy: The RK method is famous for its high accuracy, as this accuracy is achieved by improving the solution using a series of steps. RK4 provides greater accuracy but requires more calculations. While the accuracy in using FDM method depends on the size of the chosen grid, it can provide accurate solutions with fine grids, but it may introduce numerical dispersion errors, especially in problems with steep gradients[3].

Computational efficiency: Although the RK method is accurate, it may be computationally expensive. The FDM method is often more computationally efficient, but the computational cost may increase significantly for problems related to irregular geometric shapes.

Applicability: The RK method can be widely applied to both ordinary and partial equations, and it is particularly used in problems for which it is impossible to obtain analytical solutions. The FDM method is versatile and widely used. Due to its simplicity and flexibility, it is a popular choice for both industrial applications and academic research.

In general, the choice between these methods depends on the characteristics of the problem we are facing, taking into account the requirements of high accuracy and computational efficiency, knowing that the RK method excels in accuracy, but researchers must carefully evaluate the aforementioned factors to choose the appropriate numerical method for the problem [3].

Such comparisons of numerical methods were discussed by Najmuddin Ahmad and Shiv Charan [4] when they made a comparative study of numerical solutions of second order (ODE), using the shooting method and finite difference method [5] when he made a comparative study on numerical solutions of initial value problems (IVP) for ordinary differential equations (ODE) with Euler and RK methods. And for solving second order ordinary differential equations by using numerical methods, Md.Jahangir Hossain, Md.Shah Alam, and Md.Babul Hossain [6] when they make a study about RK methods for solving second order initial value problem for ODEs, and Dr.Vedavathi Aluri when he make a study about Numerical solution of Damped harmonic oscillator. And for solving generalized oscillator equations, Yufeng Xu and Om p. Agrawal [7]have shown a numerical solutions of generalized oscillator equations and Sudi Mungkasi [8]when they use Finite difference and Runge-Kutta methods for solving vibration problems.

Due to the importance of numerical methods in solving differential equations of all kinds, researchers and scholars have been interested in comparing these methods in terms of accuracy, easy of calculations, and other things.

In another study, kamal Al Khaled and M.Naim Anwar studied the Adomian decomposition method (ADM)in solving second order differential equation, and they compared the results of this method with previosly known results using Quintic c^2 spline integration method. He concluded from his research that (ADM) is easier to implement and more accurate, and showed some of the necessary modifications to solve the oscillatory system and make (ADM) more accuracy [9]. Such comparisons were made in several studies including these studies [10, 11, 12] , many researchers also solved oscillator problems numerically in their study [13, 14] by the numerical methods, and in this study [15], Shaher Momani and Rabha Ibrahim solved oscillator problems analytically by decomposition method.

Negessey Izengaw studied Finite difference method for solving first differential equations and concluded despite the many steps of this method, that it is an effective method for obtaining accurate results. He concluded that it is better for the periods taken to be smaller to obtain more accurate results, and that for the larger steps, it is preferable to use RK method, and in general it is preferable to use close interval in any method.

Ordinary differential equations (ODEs) play a fundamental role in various fields of science and engineering, serving as mathematical models for describing dynamic systems. Second order ODEs, in particular, are very important as they capture the behavior of numerous physical phenomena from electrical circuits to mechanical vibrations. Solving these equations efficiently and accurately is very important to understand the current world problems. Due to the importance of this topic, many researchers have conducted studies on how to solve these second order (ODEs)using numerical methods and investigated their accuracy, including Nur Zahidah Mukhtar, Zanariah Abdul Majed, Fudziah Ismail and Mohammed Suleiman whom studied solving these equations using Block method [16].

Anwar Jaafar Muhammad Jawad conducted a study that included solving applied problems in the form of second order non linear boundary value problems using four numerical methods, RK4, Runge kotta butcher of 6^{th} order, Differential transformation method and the Homotopy perturbation method, Then he compared the result of these methods in terms of efficiency and high accuracy, and he concluded that differential transformation method is the most effective and most accurate [17]. Hazrat ali and M.D.Shafiqul islam used Galerkin finite element method to solve such problems [18].

Another important comparison between numerical methods is the comparison between The Differential Transformation Method and The Finite Difference Method, as these methods are commonly used in solving differential equations[19].

Applicability: The differential transformation method has been used successfully in a large number of partial and ordinary differential equations, and linear and nonlinear differential equations of different orders and boundary conditions, while the finite difference method is widely used in partial differential equations, especially in physics and engineering problems[19].

Precision: Studies have shown that the differential transformation method usually provides accurate solutions, especially in linear and weak nonlinear problems, while the accuracy of the finite difference method depends on the accuracy of the chosen grid, as it may achieve high accuracy with fine grids, but it suffers from numerical diffusion and dispersion errors.

Computational efficiency: The differential transformation method is computationally efficient, especially in problems with easy solutions, where the required grid points are fewer compared to the finite difference method. In the finite difference method, the computational efficiency is affected by the dimensions of the problem and the size of the computational field. Versatility across a wide range of scientific fields[19].

Dexterity: Both methods show versatility in use in several applications, as the differential transformation method deals with linear and nonlinear problems, and the finite difference method shows versatility in dealing with geometric conditions and shapes, which makes it applicable to a wide range of scientific fields.

Conclusion: Both methods are considered effective and successful methods and have strong points, but the choice between these methods depends on the type of problem we have in terms of accuracy, computational resources, etc [19].

S.E.Fadugba, S.N.ogunyebi and B.O.Falodun conducted a test of second order numerical method to solve initial value problems, to examine both accuracy and stability. During the study, they presented improved modified Euler method, Fadugba and Olaosebikan scheme and the exact solution, they concluded that by varying the step length, there are two order decrease in the values of the final absolute relative errors generated via this method based on this method, it turned out to be consistant, stable and convergent and it had second order accurcy and it turned out to be more accurate than comparable methods[20].

Elena shcherbakove and sergey knyazev made comparison between two numerical methods, Point sources method and the direct collection method [21].

In another study, the researchers focused on comparing two famous numerical methods for solving differential equations numerically, namely the Runge_kotta method and the finite element method, as these methods are considered pivotal in simulating difficult physical phenomena. This study aims to know the strengths and weaknesses of both methods.

Accuracy and Precision: The Runge_kotta method is characterized by high accuracy, especially in solving ordinary differential equations and is widely used in simulating dynamic systems, while the finite element method is used to solve partial differential equations, especially in complex geometric shapes, and its accuracy depends on the accuracy of the chosen mesh and on the choice of elements[22].

Computational efficiency: The Runge_kotta method is computationally long, especially in rigid differential equations, it may take a long time for large-scale problems. While the computational efficiency of the finite element method depends on the quality of the network and the size of the problem, the problem may require computational intensity for complex problems, but efficiency can be enhanced by improving the networks.

Applicability: The Runge_kotta method is widely used in control systems, physical simulations, and time-dependent problems in various engineering and scientific disciplines. While the finite element method is widely used in structural analysis, electromagnetism, fluid dynamics, and heat transfer problems, it excels in dealing with complex geometric shapes.

In conclusion, both methods are considered indispensable tools in numerical analysis, and each has strengths and weaknesses. The choice between both methods depends on the type of problem we have and the accuracy and computational efficiency they require [22]. To know more about Runge kotta methods, Ponalagusamy.R and his partners presented these method in their study[23].

In another study, Salisu Ibrahim and Abdulnasir Isah studied a solution to the seconed order ODE using one of the numerical methods, which is least square method. They studied ordinary differential equations, partial differential equations, fractional differential equations, and more precisely, they studied second order ODEs, wherr they used the L_2 norm to obtain the least approximation error. They obtained the results are very close to the exact solution with the least error, and their study is supported by examples on matlap and mathematica programs [24].

Byakatonda Denis gave us an overview of the analytical and numerical methods for solving ordinary differential equations in his book [25], and he touched on many of the main and commonly used methods numerically and analytically.

Many researchers studied the accuracy of numerical methods including E.N.Trofimets and V.Ya.Trofimets, who studied the accuracy of Euler methods on the Ms Excel program, and it become clear to them that modified Euler method is more accurate than other Euler methods. They explained through their studies that Runge_kotta method is more accurate than all of Euler's methods[26]. Jitendra Binwal, Arvind Maharshi and Anita Mundra used matlap in their study [27].

Many scholars were interested in explaining and displaying some numerical methods and how to use them and explaining the extent of their feasibility. Among them was Sayd Abdul Bashin Osmani who presented three numerical methods, Tyler, Euler and Runge kotta methods are more suitable with a reasonable number of calculation steps and more accurate results than others [28].

To solve high order differential equations numerically, Shaban Gholamtabar and Nouredin parandin used PredictorCorrector method in their study.

There are many books that deal with solving ordinary differential equations using numerical and nalytical methods, with explanations of the applications of these methods. You can refer to these references to know more about numerical methods and their application[29, 30, 31, 32].

Chapter Two
Differential Equations

Why solving differential equations?

For example, on physics, we are usually interested in how things move, with respect to time, space or other variable, but what concerns us is how this movement takes place, and this is what the differential equations describe.

2.1 Ordinary Differential Equations

Definition 1. *An Ordinary differential equation (ODE) is a mathematical equation that contains one or more unknown function and their derivatives with respect to exactly one independent variable. It is called ordinary because it deals with functions of one variable.*

Example:

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + ... + a_n(x)y^n = b(x) \tag{2.1}$$

Applications of ODEs in some fields:

- **Physics**: ODEs are used to explain forces, heat, electromagnetic phonomena vibrations, waves and other things. For example: differential equations used to describe heat transfer in materials.
- **Engenering**: ODEs are used in various fields of engineering, including automotive engineering, control systems engineering and others. For example: in control engineering, ODEs can be used to design automatic systems and examine the systems response to external variables.
- **Biology**: ODEs are used in many sections of biology, including the spread of infections, population growth, the life cycle and nutrien cycles. For example: in spread of infection we use ODEs to create a model that expresses the number of infected individuals, the number of individuals who recover and the number of individuals who could be susceptible to infection.

Types of ODEs accord to linearity :

- **Linear ODEs:**

  Def: differential equation that is defined by a linear polynomial in the unknown function and its derivatives, that is an equation of the form

  $$a_0(x)y + a_1(x)y' + a_2(x)y'' + ... + a_n(x)y^n = b(x) \tag{2.2}$$

  where $a_0(x)......a_n(x)$and $b(x)$ are arbitrary differentiable functions that not need to be linear and $y'.......y^n$ are the derivatives of unknown function y with respect to the variable x . For example:

  1) $y' + 2y = sin(x)$
  2) $y'' + y = e^x$

- **Non Linear ODEs :**

  Def: they are a type of differential equations in which the dependent variable and/or it's derivatives appear with non linear terms. For example:

  1) $(y'')^2 = yx$
  2) $y'y = -x$
  3) $y' = sin(y)$

- **Homogenuos ODEs:**

  Def: An ordinary differential equation is said to be homogeneous if we can write it in this form $F(t, y, y', y'') = 0$

  this means that there are no constant terms. For example:

  1) $t^2y'' + 2ty' - 3y = 0$
  2) $sin(t)y'' + 4y' + y = 0$

- **Non Homogeneous ODEs:**

  Def: A non homogeneous ordinary differential equation is a differential equation where the equation contains the derivatives of an unknown function, and a non_zero function of the independent variable, I mean we can write it as the following:

$$G(t, y, y', y'', \dots) = h(t) \tag{2.3}$$

where:

y is the unknown function of the independent variable t.

y', y'',... derivatives of y respect to t.

$h(t)$ is a non zero function of t. For example :

1) $2t^2y'' + 4ty' + y = cos(t)$

2) $2t^2y'' - 3ty' + y = 2$

2.2 Partial Differential Equations

Definition 2. *Partial differential equation contains one or more dependant variable with respect to two or more independent variables and derivatives.*

These equations are used to explain how functions change with respect to their variables, there are many applications of PDEs in engineering, physics, and other fields, such as fluid flow, quantum mechanics and heat transfer. So, solving these equations help us understand the phenomena surrounding us.

Some examples about types of PDEs:

- **Linear PDEs:**

  Def: a Linear PDE is a PDE where the dependent variable and its partial derivatives appear linearly(have power 1). I mean they not raised to high powers and not appear as a product of eath others. they called linear PDEs. For example:

  1) $\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0$

  2) $\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = x$

- **Qusilinear PDEs:**

  Def: Qusilinear PDEs are a type of PDEs where the constant of highest order is allowed to be a factor of the dependent variable . For example:

  1) $\frac{\partial u}{\partial t} + g(u)\frac{\partial u}{\partial x} = 0$

  where g(u) is a function of the dependent variable u.

- **Non Linear PDEs**

  Def: Non linear PDEs are equations in which the dependent variable or its derivatives may appear as non linear. For example: Burger's equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2} \tag{2.4}$$

  where the second term appear as a non linear .

Note:

let

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g \tag{2.5}$$

Where a, b, c, d, e are functions of x,y or u.

linearity of PDE

In the previous equation if all are functions of x or y then it is linear, and if some depends on u then it is quisie linear or non linear.

Homogeneous and nonhomogeneous PDE

In the previous equation, if $g = 0$ then the equation is homogeneous, and if g non zero term then the equation will be non homogeneous.

Some applications on PDEs:

Partial differential equations appear in various science, the most important of which are physics, chemistry, and problems that contain functions with several variables and some of these applications are:

- **Heat Transfer:** heat transfer can be expressed by partial differential equations which expresses the distribution of temperature with respect to time and space.

  As an example of equations that are very important in studying heat transfer is: **heat conduction equation** ,

$$\frac{\partial T}{\partial t} = \alpha \delta^2 T \tag{2.6}$$

  Where T is the temperature, t the time.

$\frac{\partial T}{\partial t}$ is the rate of change of temperature with respect to time

α is the thermal diffusivity of the material

δ^2 is th4e Laplacian operator.

Where in general $\delta^2 T = T_{xx} + T_{yy}$

2.3 Second Order ODEs

Converting a physical problem and building it into a mathematical model is often one of the most difficult parts facing specialists. However, there are several useful steps on which the art of mathematical modeling can be divided:

- **Formulation:** after analyzing and interpreting the physical system, we identify the independent variables and dependent variables, then determine how these variables interact mathematically. Differential equations are usually used to creat this mathematical formula.
- **Solution:** after the mathematical model is ready, we try to solve it, and this requires knowing the nature of the model that we have in order to determine the way in which we will solve the model, either analytical or numerical.
- **Validation and interpretation:** after the solution is ready, the solution must be verified, as there are several questions that come to mind, is the solution logical, is the solution accurate, and does it conform to our expectations about this physical system.

Important note:
One of the most common equations used in building mathematical models of physical problems is second order ODEs.

Definition 3. *A second order ODE it is an equation that contains the second derivative of a dependent variable with respect to an independent variable and takes this form:*

$$G(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}) = 0 \tag{2.7}$$

Where t is the independent variable

x is the dependent variable

$\frac{dx}{dt}$ is the first derivative of x with respect to t

$\frac{d^2x}{dt^2}$ is the Second derivative of x with respect to t

G is the function of these variables and their derivatives.

2.3.1 Characteristics of Second Order ODEs

- **Order:** Second order ODE contains the second derivative of the dependent variable as the highest derivative and has the following formula:

$$\frac{d^2x}{dt^2} + P\frac{dx}{dt} + Qx = R \tag{2.8}$$

  Where x is the dependent variable, t is the independent variable, P,Q and R are functions of t or x.

- **Homogeneous and non homogeneous:** If $R(t) = 0$ then second order ODE is homogeneous, if $R(t)$ non zero then the second order ODE is not homogeneous.

- **Linearity:** The second order ODE is linear if $P(t)$,$Q(t)$ and $R(t)$ are functions of t, or constant terms only.

- **Solution:** Second order ODEs has many types of solutions depending on the nature of the equation and can be solved analytically or numerically.

- **Uniqueness and Existence:** The second order ODE can have a unique solution, and this depends on the nature and characteristics of the equation that we have. There are equations that do not have a solution or have several solutions[33].

Theorem 4. *Let $P(t)$,$Q(t)$ and $R(t)$ be continuous functions on the interval (a,b) and let $t_0 \in$ (a,b), then the second order ODE with initial values*

$x'' + P(t)x' + Q(t)x = R(t)$

$x(t_0) = x_0$,$x'(t_0) = x'_0$

has only one solution on (a,b).

We have three conclusions from this theorem :

1) The solution exists.

2) The solution is unique.

3) The solution exists on (a,b).

2.3.2 Some Physical Problems that Can be Modeled Mathematically Using Second Order ODE

- **Simple mass_spring system with free vibration:** This experiment involves a mass attached to a Spring without the influence of any external force or inhibitors. When this system is moved from it's equilibrium position, it begins to move without being affected by any external force. And to express this movement, this equation is used.

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{2.9}$$

where: m is mass, k is the spring constant, x is the displacement of the mass from its equilibrium position as a function of time t.

- **Simple damped mass_spring system in free vibration:** This experiment involves a mass attached to a spring and a damper. When this system is moved from its equilibrium position, it will swinging back and forth and its speed will gradually decrease due to the damper, to express this movement, this equation is used

$$mx''(t) + cx'(t) + kx(t) = 0 \tag{2.10}$$

where: m is the mass of the object, $x(t)$ is the displacement of the mass from its equilibrium position as a function of time,$x''(t)$ represents the second derivative of x with respect to time, k is the spring constant.

- **An electric motor:** The second order ODE represents the relationship between angular acceleration with respect to current and angular velocity, taking into account the moment of inertia and damping effect is:

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = kI \tag{2.11}$$

where: J is the moment of inertia of the motors, $\frac{d^2\theta}{dt^2}$ angular Acceleration, B is damping cofficient, $\frac{d\theta}{dt}$ Angular velocity, k the motor's torque constant, I is the current supplied to the motor, θ is angular position of the motor.

Chapter Three

Finite Difference Method and Runge_ Kutta Method

3.1 Finite Difference Method (FDM)

Definition 5. *It is a numerical method used to find approximate solutions to differential equations that are difficult to solve using usual methods, especially in the field of physics and engineering.*

The idea of this method is to replace the derivatives in the differential equation that we have, regardless of their type (ordinary or partial), with approximate derivative formulas. We will learn about this method by discussing some examples.

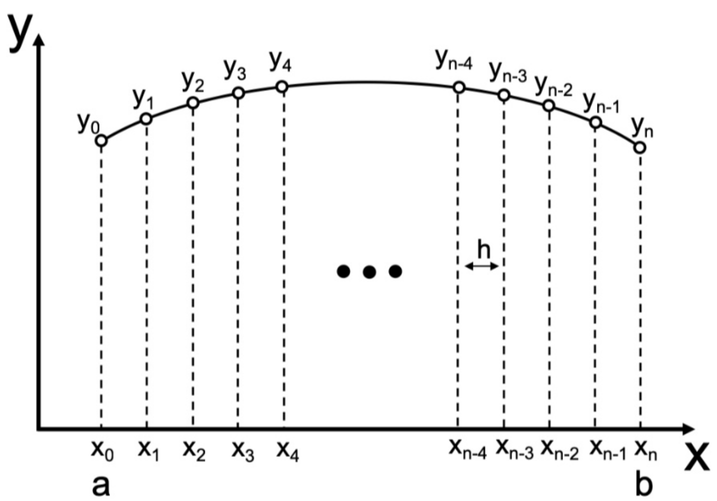
In this research, we will discuss how to use FDM to find approximate solutions to the second order ODE of this form:

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x) \qquad a \leq x \leq b \tag{3.1}$$

Note: in this method we need to know the solution at two different points, say y(a)=c , y(b)=d. And we used to call these solutions a boundary conditions.

Figure 1

Intervels of finite differnce method



The next step is to divide the interval (a,b)into n equal subintervals provided that they have the same length, let's call it h, where $h = \frac{b-a}{n}$ and $x_i = a + ih$.

We will use the notation y_i to denote the value of the function at the i^{th}node, where $y_0 = c$ and $y_n = d$.

We can represent

$y'' = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}$

$y' = \frac{y_{i+1}-y_{i-1}}{2h}$

$y = y_i$

There for, we can write the main equation in this form:

$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2} + p(x_i)\frac{y_{i+1}-y_{i-1}}{2h} + q(x_i)y_i = r(x_i) \quad i = (1, 2, ..., n-1)$

We can write this equation as:

$2(y_{i+1} - 2y_i + y_{i-1}) + hp_i(y_{i+1} - y_{i-1}) + 2h^2q_iy_i = 2h^2r_i$

Where $p_i = p(x_i)$, $q_i = q(x_i)$, $r_i = r(x_i)$

If we arrange this equation according to $y_1, ..., y_n$, we get:

$(2 + hp_i)y_{i+1} + (2h^2q_i - 4)y_i + (2 - hp_i)yi - 1 = 2h^2r_i$

We can express this equation using matrices as follows:

$A_nY_n = R_n$

Where A_n has size $(n - 1 * n - 1)$

Y_n and R_n have size $(n - 1 * 1)$

$$A_n = \begin{bmatrix} 2h^2q_1 - 4 & 2 + hp_1 & 0 & 0 & \cdots & 0 \\ 2 - hp_2 & 2h^2q_2 - 4 & 2 + hp_2 & 0 & \cdots & 0 \\ 0 & 2 - hp_3 & 2h^2q_3 - 4 & 2 + hp_3 & 0 & \cdots \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & 2h^2q_{n-1} - 4 & 2 + hp_{n-1} & \end{bmatrix}$$

$$Y_n = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

$$R_n = \begin{bmatrix} 2h^2r_1 - (2 - hp_1)a \\ 2h^2r_2 \\ 2h^2r_3 \\ \vdots \\ 2h^2r_{n-2} \\ 2h^2r_{n-1} - (2 + hp_{n-1})b \end{bmatrix}$$

Example 1: If we wanted to find an approximate solution to this boundary value problem

$y'' - (1 - \frac{x}{5})y = x$

$y(1) = 2$ and $y(3) = -1$

in the interval $1 \leq x \leq 3$ let $h = 0.5$

Solution:

let $h = 0.5$, Let $x_i = x_0 + ih$, $x_0 = 1$, $h = 0.5$ and suppose y_i be the approximate value for $y_i = y(x_i)$,

So, if we use finite difference method to solve this problem we get

$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - (1 - \frac{x_i}{5})y_i = x_i \quad i = 1, 2, 3$

With $p(x) = 0$, $q(x) = -(1 - \frac{x}{5})$ and $r(x) = x$

if we arrange this equation according to $y_1, ..., y_n$, we get:

$y_{i+1} - [2 + (\frac{1-x_i}{5})h^2]y_i + y_{i-1} = x_ih^2$

for $i = 1 \quad y_0 - [2 + (1 - \frac{1.5}{5})(.5)^2]y_1 + y_2 = 1.5(.5)^2$

for $i = 2 \quad y_1 - [2 + (1 - \frac{2}{5})(.5)^2]y_2 + y_3 = 2(.5)^2$

for $i = 3 \quad y_2 - [2 + (\frac{1-2.5}{5})(.5)^2]y_3 + y_4 = (2.5)(.5)^2$

if we substitute $y_0 = 2$, $y_n = -1$, we get this linear system and we can solve it as:

$$\begin{bmatrix} -2.175 & 1 & 0 \\ 1 & -2.15 & 1 \\ 0 & 1 & -2.125 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1.625 \\ 0.5 \\ 1.625 \end{bmatrix} \tag{3.2}$$

If we use **matlap** to solve this system, we get:

$y_1 = 0.552$

$y_2 = -0.4244$

$y_3 = -0.9644$

Example 2: Find an approximate solution to the boundary value problem

$u'' + u - x = 0$ with boundary conditions:

$u(0) = u(1) = 0$

$0 \leq x \leq 1$

Solution:

Let $h = 0.25$, so, u'' become after we using finite difference method

$u'' = \frac{u_{i+1}+ui-1-2u_i}{h^2} \qquad i = 1, 2, 3$

When $i = 1 \qquad \frac{u_2+u_0-2u_1}{h^2} + u_1 - h = 0$

$u_2 + u_0 - 2u_1 + h^2u_1 - h^3 = 0 \qquad (1)$

When $i = 2 \qquad \frac{u_3+u_1-2u_2}{h^2} + u_2 - 2h = 0$

$u_3 + u_1 - 2u_2 + h^2u_2 - 2h^3 = 0 \qquad (2)$

When $i = 3 \qquad \frac{u_4+u_2-2u_3}{h^2} + u_3 - 3h = 0$

$u_4 + u_2 - 2u_3 + h^2u_3 - 3h^3 = 0 \qquad (3)$

this is the linear system and we can solve it as:

$$\begin{bmatrix} -2+h^2 & 1 & 0 \\ 1 & -2+h^2 & 1 \\ 0 & 1 & -2+h^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} h^3 \\ 2h^3 \\ 3h^3 \end{bmatrix} \qquad (3.3)$$

If we substitute the value of $h = .25$, then we get by matlap

$u_1 = -0.04427$

$u_2 = -0.07156$

$u_3 = -0.0604$

Example 3: Use the finite difference method to solve

$y'' = y + x(x-4) \qquad 0 \leq x \leq 4$ with boundary conditions:

$y(0) = y(4) = 0$

solution:

Take $h = 1$ then $n = \frac{4-0}{1} = 4$

we know, $y'' = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}$

so, the main equation becomes,

$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2} = y_i + x_i(x_i - 4) \qquad i = 1, 2, 3$

$x_1 = 0 + h = 1 \qquad x_2 = 1 + h = 2 \qquad x_3 = 2 + h = 3$

for $i = 1 \qquad \frac{y_2 - 2y_1 + y_0}{h^2} = y_1 + x_1(x_1 - 4)$

$y_2 - 2y_1 = y_1 - 3$

$-3y_1 + y_2 = -3 \qquad (1)$

for $i = 2 \qquad \frac{y_3 - 2y_2 + y_1}{h^2} = y_2 + x_2(x_2 - 4)$

$y_3 - 2y_2 + y_1 = y_2 + 2(2 - 4)$

$y_1 - 3y_2 + y_3 = -4 \qquad (2)$

for $i = 3 \qquad \frac{y_4 - 2y_3 + y_2}{h^2} = y_3 + x_3(x_3 - 4)$

$0 - 3y_3 + y_2 = 3(-1)$

$y_2 - 3y_3 = -3 \qquad (3)$

by using matrices and matlap to solve these equations, we get,

$$\begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ -3 \end{bmatrix} \tag{3.4}$$

$y_1 = 1.8571$

$y_2 = 2.5714$

$y_3 = 1.8571$

Example 4: Let $u''(t) + \frac{1}{t}u'(t) - \frac{u}{t^2} = 0$ with boundary conditions:

$u(2) = 0.008$ and $u(6.5) = 0.003$

find u as a function of t.

We want to solve this differential equation by finite difference method.

Note: use 4 nodes to solve the problem.

Solution:

since we use four nodes, then we have three intervals so, $h = \frac{6.5-2}{3} = 1.5$, we can write the differential equation as

$u''(t)$ become after we using finite difference method

$u_{tt} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$, u_t becomes $u_t = \frac{u_{i+1} - u_{i-1}}{2h}$

so, we can write the main equation as

$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{1}{t}(\frac{u_{i+1} - u_{i-1}}{2h}) - \frac{u_i}{t^2} = 0$

When $i = 1$ $\quad \frac{u_2+u_0-2u_1}{1.5^2} + \frac{1}{t_1}\frac{(u_2-u_0)}{2*1.5} - \frac{u_1}{t_1^2} = 0$

substitute the value of u_0 and t_1, we get

$1.2142u_2 - 2.1836u_1 = -0.0063$ (1)

When $i = 2$ $\quad \frac{u_3+u_1-2u_2}{1.5^2} + \frac{1}{t_2}\frac{(u_3-u_1)}{2*1.5} - \frac{u_2}{t_2^2} = 0$

substitute the value of u_3 and t_2, we get

$-2.09u_2 + 0.85u_1 = -0.00345$ (2)

we have this linear system and we can solve it as:

$$\begin{bmatrix} -2.1836 & 1.2142 \\ 0.85 & -2.09 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -0.0063 \\ -0.003451!`2 \end{bmatrix} \tag{3.5}$$

we get by MATLAB

$u_1 = 0.0049$

$u_2 = 0.0036$

3.2 The Runge_Kutta Method(RK)

Definition 6. *It is a numerical method used to solve differential equations.*

It is considered an accurate method compared to other simple numerical methods, such as Euler method, because it uses several stages and not one stage as Euler did. Due to it's accuracy, it is widely used in solving physical, dynamic and engineering differential equations. The strategy for this method is to choose the most appropriate arrangement for the method, then taking the problem in steps, estimating the derivatives and combining these derivatives to obtain a close solution to the problem we have frequently encountered. In this research, we will study how this method works in solving the second order ODE.

Note: we will study RK4 to solve the second order ODE.

To use RK4 we need to write the second order ODE as a system of two first order ODEs. We will consider this example to reach the algorithm of this method:

Example 1: Given $y''(x) + x^2y'(x) - 2xy(x) = 1$ with initial values $y(0) = 1$, $y'(0) = 0$, evaluate $y(0.1)$ using Runge _ Kotta method of order 4. **Solution:**

$y''(x) + x^2y'(x) - 2xy(x) = 1$, $x_0 = 0, y_0 = 1, y'_0 = 0$

We want to write this equation as two first order ODEs

first step:

Put $y'(x) = z$ and differentiation with respect to x, we obtain $y''(x) = z'(x)$

so, $z'(x) + x^2 z - 2xy = 1$

the two equations are:

$y'(x) = z$ this is $f(x, y, z)$

$z'(x) = 1 + 2xy - x^2 z$ this is $g(x, y, z)$

Let $h = 0.1$

Second step:

now we want to evaluate these slopes:

$k_1 = hf(x_0, y_0, z_0)$

$l_1 = hg(x_0, y_0, z_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

$l_2 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$

$l_3 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$

$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$

So,

$k_1 = (0.1)f(0, 1, 0) = (0.1) * 0 = 0$

$l_1 = (0.1)g(0, 1, 0) = (0.1) * (1 + 2(0)(1) - (0)^2(0)) = 0.1$

$k_2 = (0.1)f(0.05, 1, 0.05) = (0.1) * (0.05) = 0.005$

$l_2 = (0.1)g(0.05, 1, 0.05) = (0.1) * (1 + 2(0.05)(1) - (0.05)^2(0.05)) = 0.11$

$k_3 = (0.1)f(0.05, 1.0025, 0.055) = (0.1) * (0.055) = 0.0055$

$l_3 = (0.1)g(0.05, 1.0025, 0.055) = (0.1) * (1 + 2(0.05)(1.0025) - (0.05)^2(0.055)) = 0.11001$

$k_4 = (0.1)f(0.1, 1.0055, 0.11004) = (0.1) * (0.11004) = 0.011$

$l_4 = (0.1)g(0.1, 1.0055, 0.11004) = (0.1) * (1 + 2(0.1)(1.0055) - (0.1)^2(0.11004)) = 0.12000$

Third step:

$y_(x_i) = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y(0.1) = 1 + \frac{1}{6}(0 + 2(0.005) + 2(0.0055) + 0.011 = 1.0053$

Note: for another value of y_{i+1}, we calculate

$z(0.1) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

then we continue in the same steps[9].

Example 2: Given $y'' - xy' - y = 0$ with the initial condition $y(0) = 1, y'(0) = 0$, compute $y(0.2)$ and $y'(0.2)$ using Runge_kutta method of order 4?

solution: let $h = 0.2$

$y(x_0 + h) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$y'(x_0 + h) = y'_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$

the main equation is $y'' = xy' + y$ such that

$y(0) = 1 and y'(0) = 0$

Put $y' = z$ so, $y'' = z'(x)$

we have two linear differential equations:

the first one is: $z'(x) = xz + y \qquad y(0) = 1, z(0) = 0$

the second equation is: $\frac{\partial y}{\partial x} = z \qquad x_0 = 0, y_0 = 1, z_0 = 0$

$f(x, y, z) = z \ , \ g(x, y, z) = xz + y$

$k_1 = hf(x_0, y_0, z_0) = (0.2)f(0, 1, 0) = 0$

$l_1 = hg(x_0, y_0, z_0) = (0.2)g(0, 1, 0) = 0.2$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) = (0.2)f(0.1, 1, 0.1) = 0.02$

$l_2 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) = (0.2)g(0.1, 1, 0.1) = 0.202$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) = (0.2)f(0.1, 1.01, 0.101) = 0.0202$

$l_3 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) = (0.2)g(0.1, 1.01, 0.101) = 0.204$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = (0.2)f(0.2, 1.0202, 0.204) = 0.0408$

$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3) = (0.2)g(0.2, 1.0202, 0.204) = 0.2122$

$y(0.2) = 1 + \frac{1}{6}[0 + 2(0.02) + 2(0.20202) + 0.0408] = 1.0202$

$y'(0.2) = 0 + \frac{1}{6}[0.2 + 2(0.202) + 2(0.204) + 0.2122] = 0.204$

Example 3: By Runge_ kutta method, solve $y''(x) = x(y'(x))^2 - y^2$, for $x = 0.2$

correct to four decimal places, using the initial conditions $y = 1$ and $y' = 0$ when $x = 0$

solution:

Given equation $y''(x) = x(y'(x))^2 - y^2$, put $y'(x) = z$, and differentiate w.r.t x, we obtain $y''(x) =$

The given equation becomes:

$z'(x) = xz^2 - y^2$ with $y = 1$, $z = 0$ when $x = 0$

$y'(x) = z \qquad f(x, y, z)$

$z'(x) = xz^2 - y^2 \qquad g(x, y, z)$

$y(x_0 + h) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$x_0 + h = 0.2, \quad$ so $\quad h = 0.2 \quad$ when $\quad x_0 = 0$

$k_1 = (0.2)f(0, 1, 0) = (0.2) * 0 = 0$

$l_1 = (0.2)g(0, 1, 0) = 0.2[(0) * (0)^2 - (1)^2] = -0.2$

$k_2 = (0.2)f(0.1, 1, -0.1) = (0.2) * (-0.1) = -0.02$

$l_2 = (0.2)g(0.1, 1, -0.1) = 0.2[(0.1) * (-0.1)^2 - (1)^2] = -0.1998$

$k_3 = (0.2)f(0.1, 0.99, -0.0999) = (0.2) * (-0.0999) = -0.01998$

$l_3 = (0.2)g(0.1, 0.99, -0.0999) = 0.2[(0.1) * (-0.0999)^2 - (0.99)^2] = -0.195$

$k_4 = (0.2)f(0.2, 0.98002, -0.1958) = (0.2) * (-0.1958) = -0.03916$

$l_4 = (0.2)g(0.2, 0.98002, -0.958) = 0.2[(0.2) * (-0.958)^2 - (0.98002)^2] = -0.19055$

Now, we can evaluate $y(0.2)$

$y(0.2) = 1 + \frac{1}{6}[0 + 2(-0.02) + 2(-0.01998) + (-0.03916)]$

$y(0.2) = 0.9801$

3.2.1 Solving Equations by RK4 Method on Python Program

If we want to solve differential equations by Runge_Kutta method on python, we can use this code:

Example: solve this differential equation $f(x, y, y') = -2y' - 10y$, by RK4 method on python program?

solution:

by using python program we get:

```
import numpy as np
```

```
def runge_kutta_ode():
#Define the secondorder ODE: y" = f(x, y, y')
def f(x, y, y_prime):
return -2 * y_prime - 10 *y #Example ODE
# Initial conditions and range of x
x0 = 0 # Initial x
y0 = 4 # Initial y
y_prime0 = -4 # Initial y'
x_end = 1 # Ending x value
h = 0.1 # Step size
# Number of iterations
num_iterations = int((x_end - x0) / h)
# Initialize arrays to store values
x_values = np.zeros(num_iterations + 1)
y_values = np.zeros(num_iterations + 1)
y_prime_values = np.zeros(num_iterations + 1)
# Initial values x_values[0] = x0
y_values[0] = y0
y_prime_values[0] = y_prime0
# Fourth-order RungeKutta method
for i in range(num_iterations):
k1 = h * y_prime_values[i]
l1 = h * f(x_values[i], y_values[i], y_prime_values[i])
k2 = h * (y_prime_values[i] + 0.5 * l1)
l2 = h * f(x_values[i] + 0.5 * h, y_values[i] + 0.5 * k1, y_prime_values[i] + 0.5 * l1)
k3 = h * (y_prime_values[i] + 0.5 * l2)
l3 = h * f(x_values[i] + 0.5 * h, y_values[i] + 0.5 * k2, y_prime_values[i] + 0.5 * l2)
k4 = h * (y_prime_values[i] + l3)
l4 = h * f(x_values[i] + h, y_values[i] + k3, y_prime_values[i] + l3)
y_values[i + 1] = y_values[i] + (1 / 6) * (k1 + 2 * k2 + 2 * k3 + k4)
y_prime_values[i + 1] = y_prime_values[i] + (1 / 6) * (l1 + 2 * l2 + 2 * l3 + l4)
```

x_values[i + 1] = x_values[i] + h

return x_values, y_values, y_prime_values

Run the function

If we run this code we get:

x_ values: [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.]

y_ values: [4, 3.4578, 2.70308417, 1.84219662, 0.97175178, 0.17171423, -0.49877085, -1.00292051, -1.32556738, -1.47060564, -1.45718659]

y'_ values: [-4, -6.6666, -8.25077849, -8.8063151, -8.46969413, -7.43284977, -5.91586692, -4.14209914, -2.31750202, -0.61527555, 0.83376644]

Chapter Four

Damped Harmonic Oscillator Equation

4.1 Damped Oscillation

Definition 7. *A decrease in the amplitude of the oscillating system due to energy loss, as the amplitude decreases with respect to time.*

When an external force acts an on oscillator and reduces it's movement, periodic movements with contradictory amplitudes are damped by simple harmonic motion.An example of this damped motion is the yo_yo.

Note: The damping force is always opposite to the direction of movement of the body and depends on it's Velocity, where damping force is proportional to velocity in a direct relationship.

Mathematically:

$$F_d = -cv \tag{4.1}$$

where:

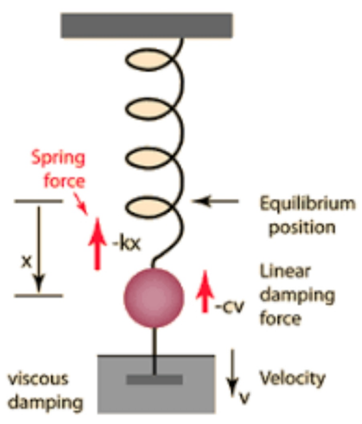
F_d: damping force.

c: damping force constant.

v: velocity of oscillator. Where the negative sign indicates that the direction of the oscil-

Figure 2

Damped Oscillation



lator's velocity and the damping force are opposite to each other.

The acting forces are acting such as damping force and restoring force.

Restoring force can be expressed as:

$$F_r = -Kx \tag{4.2}$$

where:

x: is the displacement of the practical oscillating on the spring from it's equilibrium position.

K: spring constant.

F_r: restoring force.

Where the negative sign indicates that the restoring force and acceleration are in the opposite direction from displacement. Where the restoring force is the force applied to the practical to restore his balanced position.

So, the net force acting on the body is:

$F_{net} = F_r + F_d$

$$F_{net} = -cv - Kx \tag{4.3}$$

Newton's 2^{nd} law of motion says:

$$F_{net} = ma = m\frac{\partial^2 x}{\partial t^2} \tag{4.4}$$

So, when we solve equation (4.3) and (4.4) we get

$$mx''(t) = -cv - Kx \tag{4.5}$$

Or

$$mx''(t) + cx'(t) + Kx(t) = 0$$

Where:

m is the mass of the body.

K is the spring constant.

c is the damping force coefficient.

$x(t)$ represents the position of the body w.r.t time [7].

This equation is called damped harmonic oscillator equation. This equation is a homogeneous 2^{nd} (ODE), and we want to solve this equation by using two numerical methods, (Runge_ Kotta method)and(Finite difference method).

Figure 3

Damped oscillator

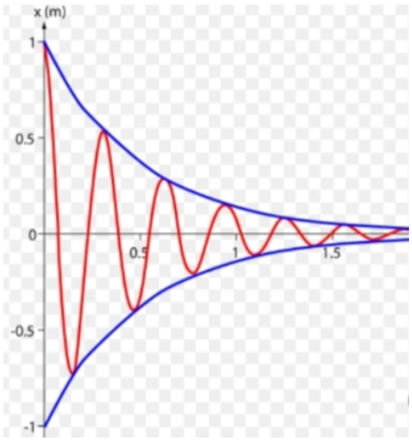
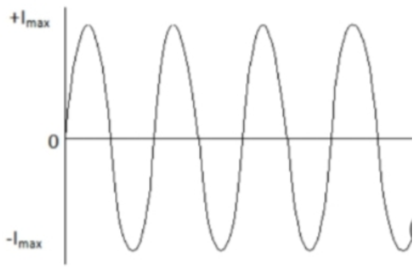


Figure 4

Un Damped Oscillator



These two images express the change in amplitude in the case of damped vibration and undamped vibration.

4.2 Analytical and Numerical Solution to the Damped Harmonic Equation

In this section, we will solve the equation in an analytical way to obtain the exact solution. Then we will solve it using two numerical methods, the Finite difference method and Runge_Kutta method, and then we will compare the results for each method.

4.2.1 Solving Damped Harmonic Oscillator Equation Analytically

Question: consider the damped oscillator of the form

$y'' + 2y' + 10y = 0$

if $y(0) = 4$,$y'(0) = -4$, $y(1) = -1.4568$

$h = 0.25$, find the solution for this equation.

Solution:

By solving this homogeneous differential equation, the exact solution for this equation is

$y(t) = e^{-t}[2e^{3it} + 2e^{-3it}] = 4e^{-t}cos(3t)$

4.2.2 Solving Damped Harmonic Oscillator Equation by Finite Difference Method

Question: consider the damped oscillator of the form

$y'' + 2y' + 10y = 0$

if $y(0) = 4$,$y'(0) = -4$, $y(1) = -1.4568$,$h = 0.25$, find the solution by finite difference method?

Solution:

By (FDM) we can write the ODE as follows:

$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 2\frac{y_{i+1} - y_{i-1}}{2h} + 10y_i = 0$

for $i = 1$: $\frac{y_2 - 2y_1 + y_0}{0.0625} + 2\frac{y_2 - y_0}{0.5} + 10y_1 = 0$

but $y_0 = 4$, so,

$y_2 - 2y_1 + 4 + 0.25y_2 - 1 + 0.625y_1 = 0$

$1.25y_2 - 1.375y_1 + 3 = 0 \quad (1)$

for $i = 2$: $\frac{y_3 - 2y_2 + y_1}{0.0625} + 2\frac{y_3 - y_1}{0.5} + 10y_2 = 0$

$y_3 - 2y_2 + y_1 + 0.25y_3 - 0.25y_1 + 0.625y_2 = 0$

$1.25y_3 - 1.375y_2 + 0.75y_1 = 0 \quad (2)$

for $i = 3$: $\frac{y_4 - 2y_3 + y_2}{0.0625} + 2\frac{y_4 - y_2}{0.5} + 10y_3 = 0$

but $y_4 = -1.4568$

$0.75y_2 - 1.375y_3 = 1.821 \quad (3)$

If we use matrices to solve this system we get:

$$\begin{bmatrix} -1.375 & 1.25 & 0 \\ 0.75 & -1.375 & 1.25 \\ 0 & 0.75 & -1.375 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 1.821 \end{bmatrix} \tag{4.6}$$

by using MATLAB, we get:

$y_1 = 7.3433$

$y_2 = 5.6776$

$y_3 = 1.8394$

Note: in exact solution

$y_1 = 0.6545$

$y_2 = -1.6800$

$y_3 = -2.2407$

If we take $h = 0.1$

By (FDM) we can write the ODE as follows:

$\frac{y_{i+1}-2y_i+y_{i-1}}{0.1^2} + 2\frac{y_{i+1}-y_{i-1}}{2(0.1)} + 10y_i = 0$

we can write this equation as:

$1.1y_{i+1} - 1.9yi + 0.9y_{i-1} = 0$

if $i = 1 \quad 1.1y_2 - 1.9y_1 = -3.6$

if $i = 2 \quad 1.1y_3 - 1.9y_2 + 0.9y_1 = 0$

if $i = 3 \quad 1.1y_4 - 1.9y_3 + 0.9y_2 = 0$

if $i = 4 \quad 1.1y_5 - 1.9y_4 + 0.9y_3 = 0$

if $i = 5 \quad 1.1y_6 - 1.9y_5 + 0.9y_4 = 0$

if $i = 6 \quad 1.1y_7 - 1.9y_6 + 0.9y_5 = 0$

if $i = 7 \quad 1.1y_8 - 1.9y_7 + 0.9y_6 = 0$

if $i = 8 \quad 1.1y_9 - 1.9y_?8 + 0.9y_7 = 0$

if $i = 9 \quad -1.9y_9 + 0.9y_8 = 1.6025$

If we use matrices to solve this system we get:

$$
\begin{bmatrix}
-1.9 & 1.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.9 & -1.9 & 1.1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.9 & -1.9 & 1.1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.9 & -1.9 & 1.1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.9 & -1.9 & 1.1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.9 & -1.9 & 1.1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.9 & -1.9 & 1.1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.9 & -1.9 & 1.1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9 & -1.9
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{bmatrix}
=
\begin{bmatrix} -3.6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.6025 \end{bmatrix}
\tag{4.7}
$$

by using MATLAB we get:

$y_1 = 3.4468$

$y_2 = 2.6809$

$y_3 = 1.8105$

$y_4 = 0.9337$

$y_5 = 0.1315$

$y_6 = -0.5368$

$y_7 = -1.0348$

$y_8 = -1.3482$

$y_9 = -1.4820$

Note: in exact solution

$y_1 = 3.4577$

$y_2 = 2.7029$

$y_3 = 1.8420$

$y_4 = 0.9716$

$y_5 = 0.1716$

$y_6 = -0.4988$

$y_7 = -1.0028$

$y_8 = -1.3253$

$y_9 = -1.4703$

If we take $h = 0.15$ then $y(1.05) = -1.3997$ in exact solution.

By (FDM) we can write the ODE as follows:

$\frac{y_{i+1}-2y_i+y_{i-1}}{0.15^2} + 2\frac{y_{i+1}-y_{i-1}}{2(0.15)} + 10y_i = 0$

we can write this equation as:

$1.15y_{i+1} - 1.775yi + 0.85y_{i-1} = 0$

if $i = 1$ $\quad 1.15y_2 - 1.775y_1 = -3.4$

if $i = 2$ $\quad 1.15y_3 - 1.775y_2 + 0.85y_1 = 0$

if $i = 3$ $\quad 1.15y_4 - 1.775y_3 + 0.85y_2 = 0$

if $i = 4$ $\quad 1.15y_5 - 1.775y_4 + 0.85y_3 = 0$

if $i = 5$ $\quad 1.15y_6 - 1.775y_5 + 0.85y_4 = 0$

if $i = 6$ $\quad -1.775y_6 + 0.85y_5 = 1.6096$

If we use matrices to solve this system we get:

$$\begin{bmatrix} -1.775 & 1.15 & 0 & 0 & 0 & 0 \\ 0.85 & -1.775 & 1.15 & 0 & 0 & 0 \\ 0 & .85 & -1.775 & 1.15 & 0 & 0 \\ 0 & 0 & 0.85 & -1.775 & 1.15 & 0 \\ 0 & 0 & 0 & 0.85 & -1.775 & 1.15 \\ 0 & 0 & 0 & 0 & .0.85 & -1.775 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} -3.4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.6096 \end{bmatrix} \quad (4.8)$$

by using matlap we get:

$y_1 = 3.3543$

$y_2 = 2.2207$

$y_3 = 0.9484$

$y_4 = -0.1776$

$y_5 = -0.9715$

$y_6 = -1.3738$

Note: in exact solution

$y_1 = 3.1001$

$y_2 = 1.8420$

$y_3 = 0.5586$

$y_4 = -0.4988$

$y_5 = -1.1869$

$y_6 = -1.4703$

If we take $h = 0.2$

By (FDM) we can write the ODE as follows:

$\frac{y_{i+1}-2y_i+y_{i-1}}{0.2^2} + 2\frac{y_{i+1}-y_{i-1}}{2(0.2)} + 10y_i = 0$

we can write this equation as:

$1.2y_{i+1} - 1.6yi + 0.8y_{i-1} = 0$

if $i = 1 \quad 1.2y_2 - 1.6y_1 = -3.2$

if $i = 2 \quad 1.2y_3 - 1.6y_2 + 0.8y_1 = 0$

if $i = 3 \quad 1.2y_4 - 1.6y_3 + 0.8y_2 = 0$

if $i = 4 \quad -1.6y_4 + 0.8y_3 = 1.7482$

If we use matrices to solve this system we get:

$$\begin{bmatrix} -1.6 & 1.2 & 0 & 0 \\ 0.8 & -1.6 & 1.2 & 0 \\ 0 & 0.8 & -1.6 & 1.2 \\ 0 & 0 & 0.8 & -1.6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -3.2 \\ 0 \\ 0 \\ 1.7482 \end{bmatrix} \tag{4.9}$$

by using matlap we get:

$y_1 = 2.5025$

$y_2 = 0.6700$

$y_3 = -0.7750$

$y_4 = -1.4800$

Note: in exact solution

$y_1 = 2.7029$

$y_2 = 0.9716$

$y_3 = -0.4988$

$y_4 = -1.325$

4.2.3 Solving Damped Harmonic Oscillator Equation by Runge_Kotta Method

Question: consider the damped oscillator of the form $y'' + 2y' + 10y = 0$, if $y(0) = 4$,$y'(0) = -4$, $y(1) = -1.4568$,$h = 0.25$, find the solution by runge_kotta mathod?

solution:

let $\frac{\partial y}{\partial x} = z$ this is $f(x, y, z)$

$\frac{\partial z}{\partial x} = -2z - 10y$ this is $g(x, y, z)$

we want to get these slopes:

$k_1 = hf(x_0, y_0, z_0)$

$l_1 = hg(x_0, y_0, z_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

$l_2 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$

$l_3 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$ $k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$

$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$

if we use Payton program to solve this problem we get:

```
import numpy as np
def runge_kutta_ode():
def f(x, y, y_prime):
return -2 * y_prime - 10 *y
x0 = 0
y0 = 4
y_prime0 = -4
x_end = 1 #
h = 0.25
num_iterations = int((x_end - x0) / h)
x_values = np.zeros(num_iterations + 1)
y_values = np.zeros(num_iterations + 1)
y_prime_values = np.zeros(num_iterations + 1)
x_values[0] = x0
```

```
y_values[0] = y0
y_prime_values[0] = y_prime0
for i in range(num_iterations):
k1 = h * y_prime_values[i]
l1 = h * f(x_values[i], y_values[i], y_prime_values[i])
k2 = h * (y_prime_values[i] + 0.5 * l1)
l2 = h * f(x_values[i] + 0.5 * h, y_values[i] + 0.5 * k1, y_prime_values[i] + 0.5 * l1)
k3 = h * (y_prime_values[i] + 0.5 * l2)
l3 = h * f(x_values[i] + 0.5 * h, y_values[i] + 0.5 * k2, y_prime_values[i] + 0.5 * l2)
k4 = h * (y_prime_values[i] + l3)
l4 = h * f(x_values[i] + h, y_values[i] + k3, y_prime_values[i] + l3)
y_values[i + 1] = y_values[i] + (1 / 6) * (k1 + 2 * k2 + 2 * k3 + k4)
y_prime_values[i + 1] = y_prime_values[i] + (1 / 6) * (l1 + 2 * l2 + 2 * l3 + l4)
x_values[i + 1] = x_values[i] + h
return x_values, y_values, y_prime_values
```
the results is:

$x\ values : [0, 0.25, 0.5, 0.75, 1]$

$y\ values : [4, 2.2890625, 0.18104553, -1.18846235, -1.47061838]$

$y'\ values : [-4, -8.6640625, -7.47743225, -3.27555028, 0.81012927]$

If we take $h = 0.1$

Solution:

we will change h value in Payton code, then we get:

$x\ values : [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1]$

$y\ values : [4,\ 3.4578,\ 2.70308417,\ 1.84219662, 0.97175178,\ 0.17171423,$
$-0.49877085,\ -1.00292051, -1.32556738,\ -1.47060564,\ -1.45718659]$

$y'\ values : [-4,\ -6.6666,\ -8.25077849,\ -8.8063151b,\ -8.46969413$
$, -7.43284977,\ -5.91586692,\ -4.14209914,\ -2.31750202,$
$-0.61527555,\ 0.83376644]$

If we take $h = 0.15$

Solution:

we will change h value in Payton code, then we get:

$x\ values$: $[0, 0.15, 0.3, 0.45, 0.6, 0.75, 0.9]$

$y\ values$: $[4,\ 3.1008625,\ 1.84313562,\ 0.5594985,$
$-0.49854767,\ -1.1876276,\ -1.471844]$

$y'\ values$: $[-4,\ -7.5936625,\ -8.80891314,\ -8.029688,$
$-5.92088867,\ -3.22885101,\ -0.6179359]$

If we take $h = 0.2$

Solution:

we will change h value in Payton code, then we get:

$x\ values$: $[0, 0.2, 0.4, 0.6, 0.8, 1]$

$y\ values$: $[4,\ 2.70613333,\ 0.97553436,\ -0.49723557,\ -1.32787504,$
$-1.46280425]$

$y'\ values$: $[-4,\ -8.25493333,\ -8.48343068,\ -5.93536782,$
$-2.33423039,\ 0.82729611]$

4.2.4 Calculate the RMSE for the Two Numerical Methods

The Root Mean Squared Error (RMSE) is one of the two main performance indicators for a regression model. It measures the average difference between values predicted by a model and the actual values. It provides an estimation of how well the model is able to predict the target value (accuracy).

The lower the value of the Root Mean Squared Error, the better the model is. A perfect model (a hypothetic model that would always predict the exact expected value) would have a Root Mean Squared Error value of 0.

In this subsection we will calculate the RMSE, for the finite difference method and the runge_kutta method at different values of h.

The RMSE results for the two numarical methods

- For the table 1

  the RMSE= $\sqrt{\frac{\sum(E)^2}{N}}$

  $= \sqrt{\frac{0.4734}{3}}$

  $= 0.3972$

- For the table 2

  the RMSE= $\sqrt{\frac{\sum(E)^2}{N}}$

  $= \sqrt{\frac{0.2311}{4}}$

  $= 0.2402$

- For the table 3

  the RMSE= $\sqrt{\frac{\sum(E)^2}{N}}$

  $= \sqrt{\frac{0.5171}{6}}$

  $= 0.2937$

- For the table 4

  the RMSE= $\sqrt{\frac{\sum(E)^2}{N}}$

  $= \sqrt{\frac{0.0074}{9}}$

  $= 0.028$

- For the table 5

  the RMSE= $\sqrt{\frac{\sum(E)^2}{N}}$

  $= \sqrt{\frac{0.000272}{4}}$

  $= 0.0082$

- For the table 6

  The RMSE $= \sqrt{\frac{\sum(E)^2}{N}}$

  $= \sqrt{\frac{0.000064}{5}}$

  $= 0.0035$

- For the table 7

  the RMSE= $\sqrt{\frac{\sum(E)^2}{N}}$

$= \sqrt{\frac{0.00000508}{6}}$
$= 0.0009$

- For the table 8
  the RMSE$= \sqrt{\frac{\sum(E)^2}{N}}$
  $= \sqrt{\frac{0.000000}{10}}$
  $= 0.0001$

Firstly: finite difference method

Table 1

RMSE When $h = 0.25$

x_value	Exact value	Numerical value	Error	(Error)^2
0.25	2.27940	$.6545$	1.6249	2.6402
0.50	0.17160	-1.68	1.8516	3.4285
0.75	-1.1869	2.2407	1.0538	1.1105

Table 2

RMSE When $h = 0.2$

x_value	Exact value	Numerical value	Error	(Error)^2
0.2	2.70290	2.5025	0.2004	0.0401
0.4	0.97160	0.6700	0.3016	0.0909
0.6	-0.4988	-0.775	0.2762	0.0762
0.8	-1.3253	-1.480	0.1547	0.0239

Table 3

RMSE When $h = 0.15$

x_value	Exact value	Numerical value	Error	(Error)^2
0.15	3.10010	3.35430	-0.2542	0.0646
0.3	1.84200	2.22070	-0.3787	0.1434
0.45	0.55860	0.94840	-0.3898	0.1519
0.6	-0.4988	-0.1776	-0.3212	0.1031
0.75	-1.1869	-0.9751	-0.2118	0.0448
0.9	-1.4703	-1.3738	-0.0965	0.0093

Table 4

RMSE When $h = 0.1$

x_value	Exact value	Numerical value	Error	(Error)^2
0.1	3.4577	3.4468	0.0109	0.0001
0.2	2.7029	2.6809	0.022	0.0004
0.3	1.8420	1.810	0.0315	0.0009
0.4	0.9716	0.9337	0.0379	0.0014
0.5	0.1716	0.1315	0.0401	0.0016
0.6	−0.4988	−0.5368	0.038	0.0014
0.7	−1.0028	−1.0348	0.032	0.0010
0.8	−1.3253	−1.3482	0.0229	0.0005
0.9	−1.4703	−1.4820	0.0117	0.0.0001

secondly: runge_kotta method

Table 5

RMSE When $h = 0.25$

x_value	Exact value	Numerical value	Error	(Error)^2
0.25	2.2794	2.289	−0.0096	0.00009
0.5	0.1716	0.1810	−0.0094	0.00008
0.75	−1.1869	−1.1884	0.0015	0.000002
1	−1.4568	−1.4706	0.0138	0.0001

Table 6

RMSE When $h = 0.2$

x_value	Exact value	Numerical value	Error	(Error)^2
0.2	2.7029	2.7061	−0.0032	0.00001
0.4	0.9716	0.9755	−0.0039	0.00001
0.6	−0.4988	−0.4972	−0.0016	0.000002
0.8	−1.3253	−1.3278	00.0025	0.000006
1	−1.4568	−1.4628	0.006	0.000036

Table 7

RMSE When $h = 0.15$

x_value	Exact value	Numerical value	Error	(Error)^2
0.15	3.1001	3.10086	−0.00076	0.0000005
0.3	1.8420	1.8431	−0.0011	0.0000012
0.45	0.5586	0.5594	−0.0008	0.0000006
0.6	−0.4988	−0.4985	0.0003	0.00000009
0.75	−1.1869	−1.1876	0.0007	0.00000049
0.9	−1.4703	−1.4718	0.0015	0.0000022

Table 8

RMSE When $h = 0.1$

x_value	Exact value	Numerical value	Error	(Error)^2
0.1	3.4577	3.4578	-0.0001	0.00000001
0.2	2.7029	2.703	-0.0001	0.00000001
0.3	1.8420	1.8421	-0.0001	0.00000001
0.4	0.9716	0.9717	-0.0001	0.00000001
0.5	0.1716	0.1717	-0.0001	0.00000001
0.6	-0.4988	-0.49877	-0.0001	0.00000001
0.7	-1.0028	-1.0029	0.0001	0.00000001
0.8	-1.3253	-1.3255	0.0002	0.00000004
0.9	-1.4703	-1.4706	0.0003	$0.0.00000009$
1	-1.4568	-1.4571	0.0003	0.00000009

4.3 Conclusion

In this research, we conducted a study and comparison between two famous numerical methods for solving ordinary and partial differential equations, which are finite difference method (FDM) and runge_kotta method (RK4), and we chose damped harmonic oscillator equation for our study, since it is a seconed_degree_physical differential equation.

We presented the solutions to this equation using the two numerical methods mentioned, as we found the solutions to this equation when $h = 0.25, h = 0.2, h = 0.15andh = 0.1$, then we did the root mean square error (RMSE) for all values of h, and for both methods, and it became clear to us that the decrease in the value of h leads to a decrease in the error between the numerical solution and the exact solution, and this mean that the answer becomes more accurate and closer to the exact solution, and this means that the two methods are consistent.

By comparing the RMSE values for both methods, we notice that the RMSE for the runge_kutta method is less than the RMSE for finite difference method at corresponding values of h. This indicates that the runge_kutta method is more accurate and reaches an exact solution.

Abbreviations

Abbreviations	Meaning
FEM	Finite Element Method.
FDM	Finite Difference Method.
RK	Runge_ Kutta Method
RK4	Runge_ Kutta Method of order four.
ODE	Ordinary Differential Equation.
IVP	Initial Value Problem.
ADM	Adomian Decomposition Method.
PDE	Partial Differential Equation.
RMSE	Root Mean Square Error.

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جامعة النَّجاح الوطنيَّة
كليَّة الدّراسات العليا

المقارنه العدديه لطرق حل المعادلات التفاضليه العاديه من الدرجه الثانيه

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عكرمه سامر فتحي علي

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قُدمت هذه الرسالة استكمالاً لمتطلبات الحصول على درجة الماجستير في الرياضيات، من كلية الدّراسات العليا، في جامعة النجاح الوطنية، نابلس – فلسطين.

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الملخص

في هذه الدراسة تم اختبار طريقتين عدديتين لحل معادلات تفاضلية عادية من الدرجة الثانية، وهما طريقة الفروق المحدودة وطريقة رونج كوتا. تهدف الدراسة إلى معرفة أي الطريقتين أكثر كفاءة ودقة. للحصول على هذه النتيجة قمنا بحل واحدة من أهم معادلات تفاضلية عادية من الدرجة الثانية، تسمى معادلة المذبذب التوافقي المخفف. وكانت النتيجة أن طريقة رونج كوتا كانت أكثر دقة وفعالية.

لقد قدمنا الحلول لهذه المعادلة باستخدام الطريقتين العدديتين المذكورتين،حيث وجدنا الحلول لهذه المعادلة عند $h = 0.25$، $h = 0.2$، $h = 0.15$ و$h = 0.1$،ثم قمنا بحساب خطأ الجذر التربيعي المتوسط (RMSE) لجميع قيم h، وللطريقتين،واتضح لنا أن انخفاض قيمة h يؤدي إلى انخفاض الخطأبين الحل العددي والحل الدقيق، وهذا يعني أن الإجابةتصبح أكثر دقة وأقرب إلى الحل الدقيق، وهذا يعني أن الطريقتينمتوافقتان.

الكلمات المفتاحيه: طريقه الفروق المحدوده، طريقه رانج كوتا، المعادلات التفاضليه العاديه من الدرجه الثانيه.