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Numerical Simulation for Solving Control Systems of High Dimension

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Dedication

I dedicate this thesis to my beloved homeland, to my parents, to my only brother, to my farah,to my doctors,Prof. Naji Qatanani and Dr. Adnan Daraghmeh , to my friends, to everyone who supports and encourages me.

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First of all Thanks to Almighty ALLAH who made it easy for me than i would like to.

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الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

Numerical Simulation for Solving Control Systems of High Dimension

أقر بأن ما اشتملت عليه هذه الرسالة إنما هي نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد، وأن هذا الرسالة ككل أو أي جزء منها لم يقدم من قبل لنيل أي درجة علمية أو بحث علمي لدى أي مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any degree or qualification.

Student's Name: **اسم الطالب: هديل منذر فتحي صوافطة**

Signature: **التوقيع:**

Date: **التاريخ: 26 / 2 / 2020**

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Notations

$\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$: Constant matrices

$\mathbb{R}$: Real plane

$\mathscr{L}$:Laplace transform .

$\mathbb{O}(\mathbb{M})$:Observability Matrix .

$\mathbb{C}(\mathbb{M})$:Controllability Matrix .

$\mathbb{I}_{\ltimes}$:Identity Matrix .

$\hat{\mathbb{H}}$:Hankel Operator .

Σ: Linear System.

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Abstract

In this thesis we have introduced some definitions in control theory including dynamical system, Laplace transformation, Lyapunov equations, stability, controllability, observability and their Gramians, we focus our attention mainly on the numerical methods for solving control systems of high dimension. These methods are: the balanced truncation method and the singular perturbation approximation method.

The mathematical framework of these numerical methods together with their properties will be presented. These numerical methods will be clarified by some numerical examples .

The two methods give us the same error bounds. For the balanced truncation method the error is small at high frequencies and large at low frequencies, but for the singular perturbation approximation we have large error at high frequencies and small error at low frequencies.

Introduction

Control theory is a branch of science and mathematics that plays a major role in nearly every modern precision device. In the classical engineering world, everything from stereos and computers to chemical manufacturing and aircraft utilizes control theory [9].

In a more natural setting, biological systems, even the smallest single-celled creatures, have evolved intricate, life-sustaining feedback mechanisms in the form of biochemical pathways. In the atomic physics laboratory (closer to home), classical control theory plays a crucial experimental role in stabilizing laser frequencies, temperature, and the length of cavities and interferometers. The new field of experimental quantum feedback control is beginning to improve certain precision measurements to their fundamental quantum noise limit [29].

In control engineering, a state-space representation is a mathematical model of a physical system as a set of input, output and state variables related by first-order differential equations. The state of the system can be represented as a vector within that space. To abstract from the number of inputs, outputs and states, these variables are expressed as vectors. Additionally, if the dynamical system is linear, time-invariant, and finite-dimensional, then the differential and algebraic equations may be written in matrix form [4, 18].

The internal state variables are the smallest possible subset of system variables that can represent the entire state of the system at any given time. The minimum number of state variables required to represent a given system, n, is usually equal to the order of the system's defining differential equation [30, 45].

The control design is a study that examines the system settings through feedback, so that the closed loop system behaves as expected with a minimum cost. Control design is one of the central themes in system theory. There are many problems of dynamics in science and engineering which are modelled in term of partial differential equations. The state space formulation for such model requires infinite dimensionality. Design control for such state space is also of infinite dimension. For the purpose of computation and implementation, this is not practical. Therefore, it is important to find a low order controller for the infinite dimensional systems.

The most general state-space representation of a linear system with p inputs, q outputs and n state variables are written in the following form:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where A, B, C and D are matrices, u is an input vector, x is a state vector and y is the output vector [2].

The proposed approach aims at reducing the number of states of the system and lowering the computational effort, with a negligible loss of accuracy in the numerical solution. To do this we have to apply model order reduction. Model order reduction (MOR) is a technique for reducing the computational complexity of mathematical models in numerical simulations.

Many modern mathematical models of real-life processes pose challenges when used in numerical simulations, due to complexity and large size (dimension). Model order reduction aims to lower the computational complexity of such problems, for example, in simulations of large-scale dynamical systems and control systems. By a reduction of the model's associated state space , an approximation to the original model is computed [44].

Methods of model order reduction: Proper orthogonal decomposition, Proper generalized decomposition, Approximate balancing, Reduced basis method, Matrix interpolation, Transfer function interpolation, Piecewise tangential interpolation, Loewner framework, (Empirical) cross Gramian, Krylov subspace methods and Balanced truncation [27].

Balanced truncation is a model reduction technique from robust control theory, is a systematic method for producing simple approximate models of complex linear systems. This technique may have significant applications in physics contexts, and the results suggest it will prove a useful tool for treating large systems in both classical and quantum settings [16].

Model reduction is a major issue for control, optimization and simulation of large-scale systems. In particular for linear time-invariant systems, balanced truncation is a well-established tool for deriving reduced (i.e., low-dimensional) models that have an input-output behavior similar to the original model. The general idea of balanced truncation is to restrict the system onto the subspace of easily controllable and observable states which can be determined by the computing the Hankel singular values associated with the system [10].

Moreover, the method is known to preserve certain properties of the original system such as stability or passivity and gives an error bound that is easily computable [3, 14].

The goal of Balanced truncation is to produce a low dimensional system that has similar response characteristics as the original system with far lower storage requirements and evaluation time. first transform the system to a balanced rep-

resentation and then eliminate (truncate) some of the state variables, stability is preserved and there is an a priori computable error bound for the error system. From a mathematical viewpoint, balancing methods consist of the simultaneous diagonalization of appropriate controllability and observability Gramians, which are positive definite matrices [15, 17, 18].
One model reduction scheme that is well grounded in theory is Balanced Truncation, first introduced by Mullis and Roberts [18] and later in the systems and control literature by Moore [45]. The approximation theory underlying this approach was developed by Glover [17]. Several researchers have recognized the importance of balanced truncation for model reduction because of its theoretical properties [32, 33, 42].

A number of methods have been proposed in the literature to reduced order of infinite dimensional linear time invariant (FDLTI) systems such as balanced truncation (BT) , Hankel norm approximation and singular perturbation approximation (SPA). All these methods give the stable reduced systems and guarantee the upper bound of the error reduction. Although balanced truncation and SPA methods gives the same of the upper bound of error reduction, but the characteristics of both methods are contrary to each other [11].It has been shown that the reduced systems by balanced truncation have a smaller error at high frequencies, and tend to be larger at low frequencies. Moreover, the reduced systems through SPA method behave otherwise, i.e., the error goes to zero at low frequencies and tend to enlarge at high frequencies. The balanced truncation and Hankel norm approximation techniques have been generalized to infinite dimensional systems [5, 36].

Curtain and Glover generalized the balanced truncation techniques to infinite-dimensional systems and the upper bound of the error reduction have been published in [1, 8, 9].

It can be shown that the reduced systems through balanced truncation method in infinite dimensional systems preserve the behavior of the original system in infinite frequency.

This condition is sometimes not desirable in applications. It is therefore necessary to improvise SPA technique such that can be applied to infinite dimensional systems. Many of the properties of the SPA approach FDLTI system can be connected through balanced reciprocal system as shown in [20]. Fatmawati, et al. [35] have generalized reciprocal transformation method to reduce the infinite-dimensional system. This fact motivates generalization SPA method of infinite dimensional systems to obtain reduced order models that have perform well at low frequencies.

We will also focus on solving a model reduction of infinite dimensional by the singular perturbation approximation. The system we used is the stable linear with bounded and finite rank output and input operators such that the balanced truncation can be performed on the system.Moreover, the singular perturbation method is used to decrease the order of the balanced infinite dimensional systems. A reduced-order model can be obtained by setting to zero of derivative all states corresponding to smaller Hankel singular values. To show the effectiveness of the proposed method [12, 13].

Chapter 1

Preliminaries

In this chapter we will discuss some of concepts which we can use in our study. We present the dynamic system and its state and output equation. Recall the Laplace transform and its properties. Discuss the Lyapunov equations. Introduce the basic concepts of controllability, observability and stabilizability. Clarify issues related to these concepts [47].

1.1 State equations for the dynamical system

The concept of the state of a dynamical system indicates to a minimum set of variables, known as state variables, that describe the system and its response to any given set of inputs.To describe a linear dynamical system,we introduce the state space equations which is a set of first-order linear differential equations defined by:

$$\dot{x} = Ax + Bu \tag{1.1}$$

where: $\dot{x} = \dfrac{dx}{dt}$.

In common the experts express the state equations in a vector form, in which the set of n state variables are written as a state vector:

$$x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T \in \mathbb{R}^n$$

we denote the input control function of the dynamical system by:

$$u = u(t) \in \mathbb{R}^m$$

The initial condition of the system is denoted by: $x(t_0) = x_0$

The matrix A is a square matrix of the constant coefficients a_{ij}, which called state (or system) matrix”,with

$$dim[A] = n \times n$$

The matrix B is a matrix of the coefficients b_{ij} that weight the inputs ”iutput matrix” [2, 43].

$$dim[B] = n \times p$$

1.2 Output Equations

The output equation for the linear continuous dynamical system is given as:

$$y = Cx + Du \tag{1.2}$$

where y is a column vector of the output variables, and represents the response of the system. The matrix C is called ”output matrix” which describe the interaction between the system and the outside world, the matrix D is a matrix of constant coefficient that describes the weight of the system input [2, 43].

$$dim[C] = dim[D] = q \times n$$

1.3 Dynamical System

To introduce a dynamical system in new form, we refer to equations(1.1) and (1.2) to have the following equation:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1.3}$$

We can write the dynamical system described by equation (1.3) in general form by using the notation Σ.

Definition 1. *[2, 47] A linear system in state space description is a quadruple of linear maps represented as matrices:*

$$\Sigma = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

The Dimension of the dynamical system is the same of the dimension of the state space :i.e.,

$$Dim(\Sigma) = n \tag{1.4}$$

If $D = 0$, the system written as:

$$\Sigma = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array} \right)$$

and can be represented as:

$$\begin{aligned} \dot{x}(t) =& Ax(t) + Bu(t) \\ y =& Cx(t) \end{aligned} \tag{1.5}$$

System (1.5) can be written in a matrix form as:

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{1.6}$$

where:

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \tag{1.7}$$

is a block matrix.

Definition 2. *Let:*

$$\Sigma = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

be a linear, continuous dynamical system, if it has single input and single output $(m = 1)$ $(p = 1)$*, then* Σ *is called a SISO system. If it has a multiple input and multiple output it is called MIMO [47].*

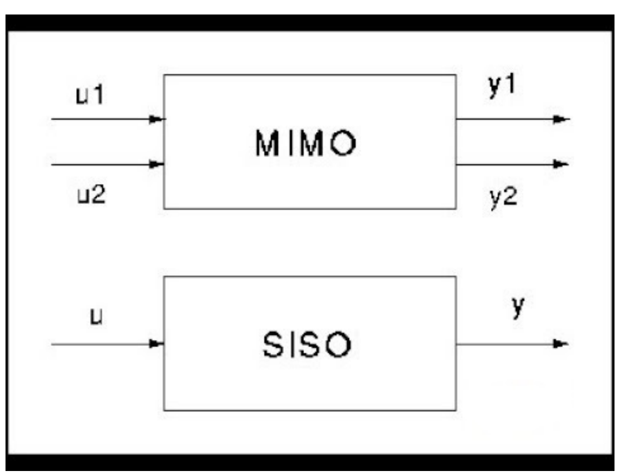


Figure 1.1: MIMO and SISO systems

Figure (1.1) show that the representation of any linear dynamical SISO or MIMO systems.

1.4 Laplace Transformation

In this section we discuss some important properties related to Laplace transformation .

Definition 3. *[5, 23] The Laplace transform of a continuous function of time $f(t)$, denoted by $\mathscr{L}\{f(t)\}$, and defined as:*

$$\mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s) \tag{1.8}$$

where $s = \sigma + i\omega$, σ and ω are real variables.

A sufficient condition for

$$\lim_{R\to\infty} \int_0^R f(t)e^{-st}dt \tag{1.9}$$

to be exist for $Re(s) > k$ (Re mean 'the real part of s) is:

$$\lim_{t\to\infty} \mid f(t) \mid e^{-kt} = 0$$

In equation (1.8), $f(t)$ is called the inverse transform of $F(s)$, which written as:

$$\mathscr{L}^{-1}[F(s)] = f(t)$$

Remark 1. *The definition of the Laplace transform $F(s)$ makes no use of the values of $f(t)$ for $t < 0$ [5, 23].*

1.4.1 Some Properties of Laplace Transforms

Suppose that a is a constant and $\mathscr{L}\{f(t)\} = F(s)$, then we have:

(1) $\mathscr{L}\{af(t)\} = a\mathscr{L}\{f(t)\} = aF(s)$.

(2) $\mathscr{L}\{f_1(t) + f_2(t)\} = \mathscr{L}\{f_1(t)\}\mathscr{L}\{f_2(t)\} = F_1(s) + F_2(s)$.

(1) and (2) prove that the operator $\mathscr{L}$ is linear.

(3) If $\mathscr{L}\{f(t)\} = F(s)$ and a is a positive real number, then the Laplace transform of the translated function $f(t-a)$ is given as:

$$\mathscr{L}\{f(t-a)\} = e^{-as}F(s)$$

(4) $\mathscr{L}\ [\frac{d}{dt}f(t)] = sF(s) - f(0)$.

(5) $\mathscr{L}\ [f^n(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - .... - sf^{n-2}(0) - f^{n-1}(0)$.

(6) $\mathscr{L}\ [\int_0^\infty f(\tau)d\tau]$ = $\frac{F(s)}{s}$.

(7)

$$\lim_{s\longrightarrow 0} sF(s) = \lim_{t\longrightarrow\infty} f(t).$$

For more details see [5, 23].

1.5 Lyapunov Equations

In this section we will introduce an important equation in control theory called Lyapunov equation.

Definition 4. *Consider two square matrices $A, M \in R^{n\times n}$, the matrix equations:*

$$AX + XA^T + M = 0 \tag{1.10}$$

is called the Lyapunov Equation [2, 24].

1.6 Stability

In this part we will discuss the concept of stability of continuous linear time-invariant dynamical system.

Definition 5. *[31] "Stable matrix" A matrix A is called stable matrix if and only if all the eigenvalue of A, or Characteristic of A, is have a non-positive real part.*

Definition 6. *[31, 47]"Stable System " The time-invariant continuous linear system*

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.11}$$

is a asymptotically stable if and only if all of the eigenvalues of A have strictly negative real parts.

Definition 7. *[31]"Asymptotically stable matrix" A matrix A is called asymptotically stable matrix if and only if all the Characteristic of A is have strictly negative real part.*

Definition 8. *[47]"The characteristic of a matrix" let A is $n \times n$ matrix, then the characteristic of A are the roots of the characteristic polynomial $\det(\lambda I - A)$.*

Theorem 1. *The system:*

$$\Sigma = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

is a asymptotically stable if and only if for any positive symmetric definite matrix M, there exists a unique positive symmetric definite matrix X s.t:

$$AX + XA^T + M = 0$$

For the proof of the theorem and more details see [2, 24].

Theorem 2. *The time-invariant continuous linear system*

$$\dot{x}(t) = A\ x(t) + Bu(t)$$

is a exponentially stable if and only if it is asymptotically stable.

1.7 Controllability and Observability

The concepts of controllability and observability, introduced by Kalmanf in 1960 are fundamental to modern control theory. The fundamental questions to be answered for a system, in particular for a multi variable system are [19, 41]:

(1) Can a control function be found which will transform the initial state x_0 of a system to some required final state in finite time ?

(2) Can the state of the system be determined by measuring the system output over a finite time interval ?

The two concepts explained are called controllability and observability respectively.

So if the answer is yes to the first question, the system is controllable.

By the same way if the answer is yes to the second question the system is observable.

[5, 19, 41].

1.8 Controllability

In this part of the chapter we introduce a very important concept related to reduced dynamical system.

Definition 9. *[2, 5, 46, 47] Consider a system described by the state equations:*

$$\begin{aligned}\dot{x}(t) =& Ax(t) + Bu(t)\\ y =& Cx(t)\end{aligned} \tag{1.12}$$

where A is $n \times n$, B is $n \times p$ and C is $q \times n$.

The pair (A, B) is said to be controllable if for any initial state $x(0) = x_0, t_1 > 0$ and final state x_1 there exists a input $u(\Delta)$ such that the solution of equation (1.12) satisfies $x(t_1) = x_1$.
Otherwise the system or the pair (A, B) is said to be uncontrollable.
Then the controllability matrix of the system is defined by

$$C(A, B) = \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}$$

where n is a positive integer.

We must notice that the concepts of Controllability and Reachability are equivalent for continuous time invariant systems [2, 47].

1.9 Observability

In this section we will explain the concept of observability

Definition 10. *[2, 5, 46, 47] Consider a system described by the state equations:*

$$\begin{aligned} \dot{x}(t) =& Ax(t) + Bu(t) \\ y =& Cx(t) \end{aligned}$$

where A is $n \times n$, B is $n \times p$ and C is $q \times n$.
the pair (C, A) is said to be observable if for any $t_1 > 0$, the initial state can be determined from the input $u(t)$ and the output $y(t)$ on $[0, t_1]$.
Otherwise the system or the pair (C, A), is said to be unobservable.
Then the observability matrix of the system is defined by:

$$O(C, A) = \begin{pmatrix} C & CA & CA^2 & \ldots & CA^{n-1} \end{pmatrix}^T$$

where n is a positive integer.

1.10 Controllability and Observability Gramian

Definition 11. *[2, 24, 26] Let A be a stable matrix, then the matrix:*

$$W_c = \int_0^\infty e^{At} B B^T e^{A^T t} dt \tag{1.13}$$

is called the controllability Gramian of (A, B).

Definition 12. *[2, 24, 26] Suppose A a stable matrix, then the matrix:*

$$W_0 = \int_0^\infty e^{A^T t} C^T C e^{At} dt \tag{1.14}$$

is called the observability Gramian of (A, C).

Theorem 3. *[24, 26] Let Σ be a stable, continuous-time system and let W_o and W_c be the observability and controllability Gramians of Σ, then W_o and W_c satisfy the continuous time Lyapunov equations:*

$$AW_c + W_c A^T + BB^T = 0$$
$$A^T W_o + W_o A + C^T C = 0$$

Proof. Since Σ is stable then:

$AW_c + W_c A^T + BB^T$

$= A(\int_0^\infty e^{At} BB^T e^{A^T t} dt) + (\int_0^\infty e^{At} BB^T e^{A^T t} dt)A^T + BB^T$

= $\int_0^\infty (Ae^{At} BB^T e^{A^T t} + e^{At} BB^T e^{A^T t} A^T) dt + BB^T$

= $\int_0^\infty (e^{At} BB^T e^{A^T t}) + BB^T$

$=0 - BB^T + BB^T=0$ We can prove the second equation by the same way [26].

$\square$

Definition 13. *[2] A Hermition matrix $X = X^*$is called positive semi-definite (or positive definite) if its eigenvalues are positive.*

The controllability Gramians have the following property that is holds for the continuous-time dynamical system.

$$W_c(t) = W_c^T(t) \geq 0, \forall t > 0$$

Theorem 4. *[2, 23, 26] The pair (A, B) is controllable if and only if W_c is positive definite for any $t > 0$.*

Theorem 5. *[2, 47] The system pair (A, B) is controllable if and only if the controllability matrix*

$$C(A, B) = \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}$$

*has full row rank (i.e., rank(**C(A,B)**) = n).*

Theorem 6. *[2,47] The system Σ is controllable if and only if the controllability matrix $C(A, B)$ has full row rank.*

Theorem 7. *[2,47] The pair (C, A) is observable if and only if the observability Gramian (1.14) is positive definite for any $t > 0$.*

Theorem 8. *[2, 23, 47] The following statements are equivalent .*

1.The pair (A, B),$A \in \mathbb{R}^{n\times n}$,$B \in \mathbb{R}^{n\times m}$is controllable.

2.The rank of the controllability matrix is full i.e., rank $(C(A, B)) = n$.

3.The controllability Gramian is positive semi-definite $W_c(t) > 0$, for some $t > 0$.

Theorem 9. *[2, 23, 47] The following statements are equivalent:*

1.The pair (A, C),$A \in \mathbb{R}^{n\times n}$,$C \in \mathbb{R}^{p\times n}$is observable.

2.The rank of the observability matrix is full i.e., rank $(O(A, C)) = n$.

3.The observability Gramian is positive semi-definite $W_o(t) > 0$, for some $t > 0$.

Chapter 2

Continuous Linear Dynamical System

2.1 State-Space Representation

The most general state-space representation of a linear system with p inputs, q outputs and n state variables has the following form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y = Cx(t)$$

where:

$x(t) \in \mathbb{R}^n$,which is called the 'state vector'.

$y(t) \in \mathbb{R}^q$,which is called the 'output vector'.

$u(t) \in \mathbb{R}^p$,which is called the 'input (or control)vector'.

A is the 'state (or system) matrix' and $dim[A] = n \times n$.

B is the 'intput matrix' and $dim[B] = n \times p$.

C is the 'output matrix' and $dim[C] = q \times n$

$\dot{x}(t) = \frac{d}{dt}x(t)$: denotes the derivative of x with respect to time t.

We call $x(t)$= $[x_1(t), x_2(t), x_3(t), \ldots, x_n(t)]^T \in R^n$ the state vector of the dynamical system.

We denote by $u = u(t) \in R^m$ the input function of the dynamical system.

The initial condition of the system is denoted by:

$$x(t_0) = x_0$$

2.2 Solution of Continuous Linear control System Equations

In this section we will show a methods for solution of continuous linear system

2.2.1 Direct solution of the control equations

Now we consider a linear control system with a state space equation

$$\begin{aligned}\dot{x}(t) =& Ax(t) + Bu(t)\\ y =& Cx(t)\end{aligned}$$

If we multiply both sides with e^{-At} , then we have:

$$e^{-At}\dot{x}(t) = e^{-At}Ax(t) + e^{-At}Bu(t)$$

$$e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}Bu(t)$$

But $e^{-At}\dot{x}(t) - e^{-At}Ax(t) = \frac{d}{dt}[e^{-A\tau}x(\tau)]$, then:

$$\frac{d}{dt}[e^{-A\tau}x(\tau)] = e^{-At}Bu(t)$$

If we integrate both sides on $[0, t]$, then we have:

$$\int_0^t \frac{d}{dt}[e^{-A\tau}x(\tau)]d\tau = \int_0^t [e^{-A\tau}Bu(\tau)]d\tau$$

$$e^{-At}x(t) - x(0) = \int_0^t [e^{-A\tau}Bu(\tau)]d\tau$$

$$x(t) = e^{At}x(0) + e^{At}\int_0^t [e^{-A\tau}Bu(\tau)]d\tau \tag{2.1}$$

Now, if we put $x(t)$ of equation(2.1)in the output equation:

$$y(t) = Cx(t)$$

Then we have:

$$y(t) = Ce^{At}x(0) + Ce^{At}\int_0^t [e^{-A\tau}Bu(\tau)]d\tau \tag{2.2}$$

In our case we have $x(0) = 0$, So we get

$$y(t) = Ce^{At}\int_0^t [e^{-A\tau}Bu(\tau)]d\tau \tag{2.3}$$

2.2.2 Using laplace Transform to solve the dynamical system

Since the control system consists of two equations which are vector matrices, so we can use laplace transform to solve the system [2].

Definition 14. *Let:*

$$x(t) = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^T$$

We define

$$\mathscr{L}\,[x(t)] = \begin{pmatrix} \mathscr{L}\,[x_1(t)] & \mathscr{L}\,[x_2(t)] & \dots & \mathscr{L}\,[x_n(t)] \end{pmatrix}^T$$

$$= \begin{pmatrix} X_1(s) & X_2(s) & \dots & X_n(s) \end{pmatrix}^T = X(s)$$

Now, from the definition we can do:

$$\mathscr{L}\left[\dot{x}_1(t)\right] = sX_1(s) - x_1(0)$$

$$\mathscr{L}\left[\dot{x}_2(t)\right] = sX_2(s) - x_2(0)$$

$$\vdots$$

$$\mathscr{L}\left[\dot{x}_n(t)\right] = sX_n(s) - x_n(0)$$

Then:

$$\mathscr{L}\left[\dot{x}(t)\right] = \begin{pmatrix} \mathscr{L}\left[\dot{x}_1(t)\right] & \mathscr{L}\left[\dot{x}_2(t)\right] & \dots & \mathscr{L}\left[\dot{x}_n(t)\right] \end{pmatrix}^T$$

$$= \begin{pmatrix} sX_1(s) - x_1(0) & sX_2(s) - x_2(0) & \dots & sX_n(s) - x_n(0) \end{pmatrix}^T$$

$$= sX(s) - x(0) \tag{2.4}$$

Taking the Laplace transform of the state space equation

$$\dot{x} = Ax + Bu$$

we obtain:

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$\left[sI - A\right] X(s) = x(0) + BU(s). \tag{2.5}$$

Where $\quad U(s) = \mathscr{L}\left[u(t)\right]$.

Equation (2.5) can be easily solved by assuming that the matrix $[sI - A]$ is non-singular matrix(invertible).

Multiplying both sides of equation (2.5) by the inverse of $[sI - A]$, then we have:

$$X(s) = [sI - A]^{-1}\, x(0) + [sI - A]^{-1}\, BU(s) \tag{2.6}$$

Taking the inverse Laplace transfer for both sides of equation (2.6),we get the following solution of $x(t)$:

$$x(t) = e^{At}x(0) + e^{At}\int_0^t [e^{-A\tau}Bu(\tau)]d\tau \tag{2.7}$$

Now, take the Laplace transform for the output equation, $y(t) = Cx(t)$, then we get:

$$\mathscr{L}\,[y(t)] = \mathscr{L}\,[Cx(t)]$$

$$Y(s) = CX(s)$$

$$Y(s) = C\,[sI - A]^{-1}\, x(0) + C\,[sI - A]^{-1}\, BU(s) \tag{2.8}$$

Taking the inverse Laplace transfer for both sides of equation (2.8), we get the following solution of $y(t)$:

$$y(t) = Ce^{At}x(0) + Ce^{At}\int_0^t [e^{-A\tau}Bu(\tau)]d\tau \tag{2.9}$$

For zero-initial condition $x(0) = 0$, equations (2.7) and (2.9) can be written as:

$$x(t) = e^{At}\int_0^t [e^{-A\tau}Bu(\tau)]d\tau$$

$$y(t) = Ce^{At}\int_0^t [e^{-A\tau}Bu(\tau)]d\tau$$

Remark 2. *From above we notice that the matrix "$[sI - A]^{-1}$ " is called the transition matrix.*

2.3 Transfer Function

The transfer function is a function that gives the output value for each possible value of the input to the system as in the figure (2.1).

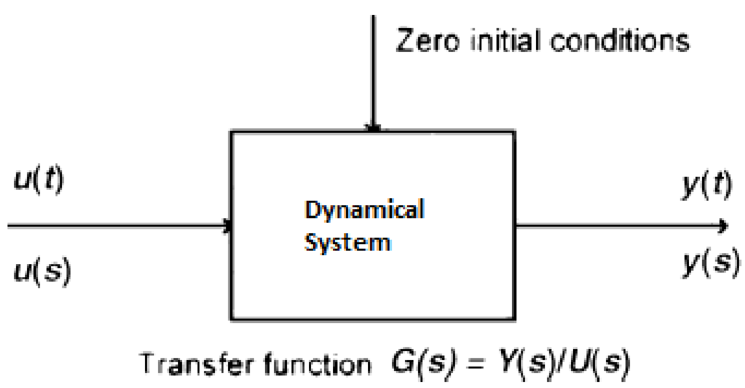


Figure 2.1: The transfer function model

Definition 15. *[23, 47] "The transfer function $G(s)$" Is the ratio of the laplace transform of the output of the system to the laplace transform of the input of the system, where all the initial conditions are zero.*

$$G(s) = \frac{Y(s)}{U(s)} \tag{2.10}$$

Therefore, if A is stable and from the equation (2.10) then the transfer function be:

$$G(s) = C\,[sI - A]^{-1}\,B \tag{2.11}$$

The transfer matrix $G(s)$ in equation (2.10) can be written as:

$$\Sigma = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right) = G(s) = \tfrac{Y(s)}{U(s)}$$

The components of matrix $G(s)$ are proper transfer functions, that is, the degree of numerator is less than or equal to the degree of the denominator:

$$deg(y_{ij}) \leq deg(u)$$

The strict inequality holds if and only if $Dij = 0$:

$$deg(y_{ij}) < deg(u) \iff U_{ij} = 0$$

In this case, the functions are called strictly proper.

To simplify, the matrix of transfer functions $G(s)$ is proper, and it is strictly proper (it has all strictly proper components) if and only if $D = 0$ [6, 34].

2.4 State space realization for transfer function

In this section, we represent the realization for a dynamical system with a transfer function $G(s)$.

Definition 16. *Let $G(s)$ be a proper transfer function, then the state space model (A, B, C) given by:*

$$G(s) = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

is a realization of $G(s)$.

Definition 17 (47)**.** *(A, B, C) is a state space realization of $G(s)$ is said to be a minimal realization of $G(s)$ if $dim(A)$ ia a smallest possible value.*

Theorem 10. *A state space realization (A, B, C) of $G(s)$ is minimal if and only if (C, A) is observable and (A, B) is controllable.*

Theorem 11 (47)**.** *Let (A_1, B_1, C_1) and (A_2, B_2, C_2) be two minimal realization of a real rational transfer function $G(s)$.Furthermore, suppose that C_1,C_2,O_1 and O_2 are the corresponding controllability and observability matrices, respectively. Then there exists a unique non-singular matrix T such that:*

$$A_2 = TA_1T^{-1},\ B_2 = TB_1, C_2 = C_1T^{-1}$$

Moreover, T is given by:

$T = (O_2^T O_2)^{-1} O_2 O_1$ Or: $T^{-1} = C_1 C_2^T (C_2 C_2^T)^{-1}$

The balanced realization method is a numerically reliable method to eliminate the states that are uncontrollable (and/or) unobservable [28, 47].

Chapter 3

Model Order Reduction Of Linear time-invariant systems

3.1 Introduction of Balanced Truncation

Model reduction is a major issue for control of large-scale systems. Especially for linear time-invariant systems, balanced truncation is a well-established tool for deriving reduced models that have an input-output behavior similar to the original model.
A balanced realization of a system is one in which states that are difficult to control, reach and observe. In this case, the minimal energy required to reach a given state is the inverse of the observation energy generated by the same state [22, 39].

3.2 The Controllability and Observability Functions

Definition 18. *[7, 39] The Controllability and Observability Functions of a linear system are defined respectively as:*

$$L_c(x_0) = \underset{\substack{u\in(-\infty,0) \\ x(-\infty)=0, X(0)=x_0}}{min} \frac{1}{2}\int_{-\infty}^{0} ||u(t)||^2 dt \tag{3.1}$$

$$L_o(x_0) = \frac{1}{2}\int_{0}^{\infty} ||y(t)||^2 dt \tag{3.2}$$

$x(0) = x_0, u(t) = 0, 0 \leq t < \infty.$

The value of the controllability function at x_0 is the minimum quantity of control energy we need to reach the state x_0 and the value of the observability function at x_0 is the quantity of output energy generated by the state x_0.

3.3 Balancing for Linear System

In this section we will consider the "Balanced Truncation Method " which is one of the methods that used to find a reduce order model from the original dynamical system.

Definition 19. *[2, 37, 47] Consider a controllable, observable and stable continuous linear input - output system $\Sigma : u \to y$ where $u \in \mathbb{R}^m, y \in \mathbb{R}^p$.*
a) The system Hankel matrix is defined as:

$$\hat{H} = [\hat{H}_{i,j}]$$

where $\hat{H}_{i,j} = H_{i+j-1}$,for $i, j \geq 1$.
If Σ is stable, then the system hankel operator is defined as:

$$H : L_2^m[0, +\infty) \to L_2^p[0, +\infty)$$

$$: \hat{u} \to \hat{y}(t) = \int_0^\infty H(t+\tau)\hat{u}(\tau)d\tau$$

b) The Hankel Singular Values (HSVs)

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$$

of Σ are the square roots of the eigenvalues of the product of W_cW_0 are given *by:*

$$\sigma_i(\Sigma) = \sqrt{\lambda_i(W_c W_0)}$$

The diagonal matrix of The Hankel Singular Values (HSVs) is denoted by:

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \tag{3.3}$$

Definition 20. *[2, 37, 47] The controllable, observable and stable system Σ is balanced if:*

$$W_c = W_o = \Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{pmatrix} \textit{ with } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$$

Here the $\sigma_i^{\prime}s, i = 1, 2, \cdots, n$ are the Hankel Singular Values , i.e. the singular values of the Hankel operator of the system.

Theorem 12. *[7, 22, 39] Consider a Linear system:*

$$\begin{aligned} \dot{x} =& Ax + Bu \\ y =& Cx \end{aligned}$$

where $u \in \mathbb{R}^m$,$x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$, we assume that the system is stable, controllable and absorvable.

Now; the controllability Gramian considered as:

$$W_c = \int_0^{\infty} e^{At} B B^T e^{A^T t} dt$$

and the observability Gramian:

$$W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

Furthermore, W_o and W_c are the unique - positive definite solutions of the Lyapunov equations:

$$AW_c + W_c A^T + BB^T = 0$$
$$A^T W_o + W_o A + C^T C = 0$$

Then the controllability and observability functions can be written as:

$$L_c(x_0) = \frac{1}{2} x_0^T W_c^{-1} x_0$$

and

$$L_o(x_0) = \frac{1}{2} x_0^T W_o x_0$$

The main goal we want is to reduce the order of the model and this thing happens by balancing the system and deleting the states that are difficult to control and observe in other word, which need large amount of control energy and give small amount of energy. The transformation that achieves this goal is called a balancing transformation [2].

Theorem 13. *"Balancing transformation" There exists a state space transfor-*

mation ($x = Tz$) for the linear system in equation(1.3)

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Such that the transformed system is described as:

$$\begin{aligned}\dot{z} &= \overline{A}z + \overline{B}u \\ y &= \overline{C}z\end{aligned} \tag{3.4}$$

where $\overline{A} = T^{-1}AT, \overline{B} = T^{-1}B$ *and* $\overline{C} = CT$.

Proof. Since $x = Tz$,then:

$$\dot{x} = T\dot{z}$$

But:

$$\dot{x} = Ax + Bu$$

$$T^{-1}\dot{x} = T^{-1}Ax + T^{-1}Bu$$

$$T^{-1}\dot{x} = T^{-1}ATz + T^{-1}Bu$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu$$

$$\dot{z} = \overline{A}z + \overline{B}u$$

where:$\overline{A} = T^{-1}AT$ and $\overline{B} = T^{-1}B$.

Now: $y = Cx$, since $x = Tz$ then:

$$y = CTz$$

$$y = \overline{C}z$$

where: $\overline{C} = CT$ □

The obtained transformed system (3.4) is in balanced form i.e:,

$$\overline{W_c} = \overline{W_o} = \Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{pmatrix} \tag{3.5}$$

$$with: \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$$

are the controllability and observability Gramian of the transformed system(3.4) where:

$$\overline{W}_c = TW_cT^T \tag{3.6}$$

$$\overline{W}_o = T^{-T}W_o \tag{3.7}$$

and since the system is balanced then the two gramians are equal, then:

$$\overline{W_c} = \overline{W_o} = \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \tag{3.8}$$

let $\overline{G}$ be the transfer function of the transformed system in equations (1.3), Then:

$$\begin{aligned} \overline{G} &= \overline{C}\left[sI - \overline{A}\right]^{-1}\overline{B} \\ \overline{G} &= CT\left[sI - T^{-1}AT\right]^{-1}T^{-1}B \end{aligned} \tag{3.9}$$

Lemma 14. *The transfer function of the dynamical system is equal to the transfer function of the transformed system.*

Proof.

$$\begin{aligned} \overline{G} &= \overline{C}\left[sI - \overline{A}\right]^{-1}\overline{B} \\ &= CT\left[sI - T^{-1}AT\right]^{-1}T^{-1}B \\ &= CTT^{-1}\left[sI - A\right]^{-1}TT^{-1}B \\ &= C\left[sI - A\right]^{-1}B \\ &= G \end{aligned}$$

□

The controllability and obsevability gramians in equation (3.8) satisfies the two Lyapunov equations:

$$\overline{A}\Sigma + \Sigma\overline{A}^T + \overline{B}\overline{B}^T = 0 \tag{3.10}$$

$$\overline{A}^T\Sigma + \Sigma\overline{A} + \overline{C}^T\overline{C} = 0 \tag{3.11}$$

Starting from a balanced representation, we will now derive a dimension-reduced system. By positive definiteness we may decompose the two gramians according to:

$$W_c = UU^T$$

$$W_o = LL^T$$

and do a singular value decomposition of the full-rank matrix $L^T U$, i.e:,

$$L^T U = X \Sigma Y^T = \begin{pmatrix} X_1 & X_2 \end{pmatrix} = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} = \begin{pmatrix} Y_1^T \\ Y_2^T \end{pmatrix} \tag{3.12}$$

The partitions:

$$\Sigma_1 = diag(\sigma_1, ..., \sigma_r)$$

and

$$\Sigma_2 = diag(\sigma_{r+1}, ..., \sigma_n)$$

indicates which singular values are important and which are negligible.

The remaining matrices satisfy:

$$X_1^T X_1 = Y_1^T Y_1 = I_{r \times r}$$

and

$$X_2^T X_2 = Y_2^T Y_2 = I_{l \times l}$$

with $l = n - r$ [22].

In terms of the SVD (singular value decomposition), the balancing transformation T and its inverse T^{-1} is given as:

$$T = \Sigma^{\frac{-1}{2}} X^T L^T$$
$$T^{-1} = UY\Sigma^{\frac{-1}{2}} \tag{3.13}$$

For more information see [22].

Definition 21. *The controllability and observability functions of the transformed system (3.4) are defined as:*

$$\overline{L}_c(z_0) = \frac{1}{2} z_0^T \Sigma^{-1} z_0$$

and

$$\overline{L}_o(z_0) = \frac{1}{2} z_0^T \Sigma z_0$$

Now, if $\sigma_i \geq \sigma_{i+1}$for i = 1,2, $\cdots$,n, then the amount of control energy to reach the state z is large for small values of σ_i, and the output energy at z is small for large values of σ_i.
Hence, to reduce the number of states components of the system, we delete the state components x_{j+1} to x_n for which $\sigma_i \geq \sigma_{i+1}$ [38, 39, 47].

3.4 Error bounds for model reduction

A major benefit of doing balanced truncation is that one has a priori error bounds that are close to the lower bound achievable by any reduced-order model [2, 40].To understand these error bounds, we consider linear,time-invariant,finite-dimensional state space systems of the form:

$$\dot{x} = Ax + Bu;$$
$$y = Cx$$
$$x(0) = x_0$$

and consider the transfer function of the balanced system

$$\hat{G}(s) = C(sI - A)^{-1}B$$

which binds the Laplace transform of the input to the Laplace transform of the output (i.e: $\hat{y}(s) = \hat{G}(s)\hat{u}(s)$) .The L_2-induced operator norm of $\hat{G}$ is defined by:

$$\overset{max}{u} \frac{\| G(u) \|_2}{\| u \|_2} = \| G \|_\infty \equiv \overset{max}{\omega} [\sigma_1(\hat{G}(iw))] \tag{3.14}$$

where $\sigma_1(M)$ denotes the maximum singular value of the matrix M. The following error bounds are standard results [2, 40].

Theorem 15. *[2, 28, 40] Any reduced order model G_r with r states must satisfy:*

$$\| G - G_r \|_\infty > \sigma_{i+1} \tag{3.15}$$

where σ_{i+1} is the first neglected Hankel singular value of G. Balanced truncation also guarantees an upper bound of the error:

$$\| G - G_r \|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_i \tag{3.16}$$

which is usually close to the lower bound (3.15) , if the Hankel singular values drop off quickly.

Chapter 4

Singular Perturbation Approximation of Balanced Infinite-Dimensional Systems

4.1 The Reciprocal System of a Linear Continuous Dynamical System

In this part of chapter we introduce the reciprocal system and show some properties of its transfer function.

Definition 22. *(The Reciprocal System):Consider a Linear system*

$$\begin{aligned} \dot{x} =& Ax + Bu \\ y =& Cx \end{aligned} \tag{4.1}$$

Now, let $x_0 = 0$ in (2.1) The reciprocal system is denoted $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ defined as:

$$\begin{aligned} \frac{d\hat{x}}{dt} &= \hat{A}\hat{x} + \hat{B}u\,, \qquad \hat{x}(t_0) = \hat{x}_0 \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u \end{aligned} \tag{4.2}$$

where:

$$\hat{A} = A^{-1}\,, \quad \hat{B} = A^{-1}B\,, \quad \hat{C} = -CA^{-1}\,, \quad \hat{D} = D - CA^{-1}B \tag{4.3}$$

and the initial condition

$$\hat{x}_0 = 0\,. \tag{4.4}$$

We will consider two theorems to show some properties of the reciprocal system .

Theorem 16. *The system* (A, B, C, D) *is stable and balanced with HSVs =* Σ *if and only if the reciprocal system* $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *is stable and balanced with the same HSVs =* Σ *.*

Proof. If the system is balanced, then the coefficients of system satisfy the pair of Lyapunov equations

$$\begin{aligned} AW_c + W_cA^T + BB^T &= 0 \\ W_oA + A^TW_o + C^TC &= 0 \end{aligned} \tag{4.5}$$

with $W_c = \Sigma = W_o$ where $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$.

Multiplying the first equation in (4.5) from the left by A^{-1} and from the right by A^{-T}, then we have the pair $(\hat{A}, \hat{B})$ solves the same Lyapunov equation as the pair (A, B) for $W_c = \Sigma$.

By a similar way, we have the matrix pairs $(\hat{A}, \hat{C})$ and (A, C) solve the same Lyapunov equation for $W_o = \Sigma$. □

Theorem 17. *let the transfer function of the system* (A, B, C, D) *is* G *and the transfer function of the reciprocal system* $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *is* $\hat{G}$ *with zero initial conditions* $\hat{x}_0 = 0$*. Then*

$$\hat{G}(s) = G(1/s)\,. \tag{4.6}$$

Proof. Since $\hat{G}$ is the transfer function of the system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ that is ; $\hat{G} = \hat{D} + \hat{C}(sI - \hat{A})^{-1}\hat{B}$, then

$G(s)$= $C[(sI - A)^{-1} + A^{-1}]B + D$

=$-CA^{-1}\left(\dfrac{1}{s}I - A^{-1}\right)^{-1} A^{-1}B + D$

=$Cs^{-1}(A^{-1} - s^{-1}I)^{-1}A^{-1}B + D$

= $-C(s^{-1}I - A^{-1} + A^{-1})(s^{-1}I - A^{-1})^{-1}A^{-1}B + D$

= $-CA^{-1}(s^{-1}I - A^{-1})^{-1}A^{-1}B + D - CA^{-1}B$

=$\hat{C}(s^{-1}I - \hat{A})^{-1}\hat{B} + \hat{D}$

= $\hat{G}(1/s)$ □

According to theorem (13) we can use the balanced truncation method to the reciprocal system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ and let W_o and W_c be the controllability and observability gramians of the system .

Now, Choose a positive integer r s.t $\sigma_r > \sigma_{r+1}$ with $W_c = W_o = \Sigma$ = diag (Σ_1, Σ_2)

$\Sigma_1 = diag(\sigma_1, ..., \sigma_r)$ and $\Sigma_2 = diag(\sigma_{r+1}, ..., \sigma_n)$ as:

$$\hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix}, \; \hat{B} = \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \end{pmatrix}, \; \hat{C} = \begin{pmatrix} \hat{C}_1 & \hat{C}_2 \end{pmatrix}, \tag{4.7}$$

Then the rth order truncated system of $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is given by $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D})$ with transfer function $\hat{G}^r$. In our study we suppose that HSVs ,σ_i , are all distinct s.t $\sigma_1 > \sigma_2 > ... > \sigma_r > \sigma_{r+1} > ... > 0$. From this we obtain$\Sigma_1 > 0$.

Corollary 18. *The truncated system $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D})$ is balanced and asymptotically stable with gramian Σ_1*

Now, the reduced system

$$\begin{aligned} \frac{d\hat{x}_r}{dt} &= \hat{A}_{11}\hat{x}_r + \hat{B}_1 u\,, \quad \hat{x}_r(0) = 0 \\ \hat{y}_r &= \hat{C}_1\hat{x}_r + \hat{D}_1 u \end{aligned} \tag{4.8}$$

with coefficients

$$
\begin{aligned}
\hat{A}_{11} &= \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \\
\hat{B}_{1} &= \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}\left(B_{1} - A_{12}A_{22}^{-1}B_{2}\right) \\
\hat{C}_{1} &= -\left(C_{1} - C_{2}A_{22}^{-1}A_{21}\right)\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \\
\hat{D}_{1} &= D - CA^{-1}B\,,
\end{aligned}
\tag{4.9}
$$

The transfer function for the reduced system (4.8) is denoted by $\hat{G}_r$ and defined as:

$$\hat{G}_r(s) = \hat{C}_1(sI - \hat{A}_{11})^{-1}\hat{B}_1 + \hat{D} \tag{4.10}$$

Lemma 19. *The error bound is represented as in the following:*

$$\|\hat{G} - \hat{G}_r\|_{\infty} \leq 2\sum_{i=r+1}^{n}\sigma_i \tag{4.11}$$

4.2 Balanced Singular Perturbation Approximation

In this section we discuss another method of solution of the linear system that called singular perturbation approximation method (SPAM) for finite dimensional systems which reduce the dimension of the original system and obtained an error bound. The difference between the two methods that for the balanced truncation method the error is small at high frequencies and large at low frequencies, but for the singular perturbation approximation we have large error at high frequencies and small error at low frequencies [2].

Suppose we have a linear dimensional continuous system (A, B, C, D) described as in the equations:

$$\begin{aligned}\dot{x}(t) =& Ax(t) + Bu(t)\\ y =& Cx(t)\end{aligned} \tag{4.12}$$

Now, our goal is to derive a new equations with reduced dimension from the equations (4.12). We start with the balanced representation of the linear continuous system. The observability and controllability gramians W_o and W_c are positive semi definite and can decompose according to:

$$W_c = UU^T$$

$$W_o = LL^T$$

If we go back to section (3.3) we notice that the balanced gramain Σ is partitioned as:

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \tag{4.13}$$

such that $\Sigma_1 = diag(\sigma_1, ..., \sigma_r)$ and $\Sigma_2 = diag(\sigma_{r+1}, ..., \sigma_n)$ are the two partitions.

Also, as in section (3.3) the balance transformation ”S” that satisfies the below equations:

$$S = UY\Sigma^{\frac{-1}{2}} \tag{4.14}$$

$$S^{-1} = \Sigma^{\frac{-1}{2}} X^T L^T \tag{4.15}$$

Now; if we take $\sigma_r >> \sigma_{r+1}$ and we know from the section (3.3) that the HSVs are invariant, then the reduced dimension system with small parameters can be obtained since ; $\sigma_{r+1} > \sigma_{r+2} > ... > \sigma_n > 0$ [21].

To see where the new small parameter enter the equation, we replace the parameter Σ_2 by $\epsilon\Sigma_2$ i.e: the small HSVs are scaled uniformly according to the below equation:

$$(\sigma_{r+1}, \sigma_{r+2}, ..., \sigma_n) \longmapsto \epsilon(\sigma_{r+1}, \sigma_{r+2}, ..., \sigma_n); \epsilon > 0$$

let we use the balance transformation $S(\epsilon)$ to change the coordinate s.t:

$$x \longmapsto S(\epsilon)x$$

If we let $S^{-1}(\epsilon) = T(\epsilon)$, then the balanced matrices are partitioned as in the following form:

$$S(\epsilon) = \begin{pmatrix} S_{11} & \frac{1}{\sqrt{\epsilon}} S_{12} \\ S_{21} & \frac{1}{\sqrt{\epsilon}} S_{22} \end{pmatrix} \tag{4.16}$$

and the inverse

$$T(\epsilon) = \begin{pmatrix} T_{11} & T_{12} \\ \frac{1}{\sqrt{\epsilon}} T_{21} & \frac{1}{\sqrt{\epsilon}} T_{22} \end{pmatrix} \tag{4.17}$$

If we use the balance transformation described as in equations (4.16) (4.17), then a new balance coefficient is obtained and written as:

$$\begin{aligned}\tilde{A}(\epsilon) &= T(\epsilon)AS(\epsilon)\\ &= \begin{pmatrix} T_{11} & T_{12} \\ \frac{1}{\sqrt{\epsilon}}T_{21} & \frac{1}{\sqrt{\epsilon}}T_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} S_{11} & \frac{1}{\sqrt{\epsilon}}S_{12} \\ S_{21} & \frac{1}{\sqrt{\epsilon}}S_{22} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{A}_{11} & \frac{1}{\sqrt{\epsilon}}\tilde{A}_{12} \\ \frac{1}{\sqrt{\epsilon}}\tilde{A}_{21} & \frac{1}{\epsilon}\tilde{A}_{22} \end{pmatrix}\end{aligned} \tag{4.18}$$

$$\begin{aligned}\tilde{B}(\epsilon) &= T(\epsilon)B\\ &= \begin{pmatrix} T_{11} & T_{12} \\ \frac{1}{\sqrt{\epsilon}}T_{21} & \frac{1}{\sqrt{\epsilon}}T_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{B}_1 \\ \frac{1}{\sqrt{\epsilon}}\tilde{B}_2 \end{pmatrix}\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\tilde{C}(\epsilon) &= CS(\epsilon)\\ &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} S_{11} & \frac{1}{\sqrt{\epsilon}}S_{12} \\ S_{21} & \frac{1}{\sqrt{\epsilon}}S_{22} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{C}_1 & \frac{1}{\sqrt{\epsilon}}\tilde{C}_2 \end{pmatrix}\end{aligned} \tag{4.20}$$

Now, If we take $\epsilon = 1$ in equation (4.18), then the value of $\tilde{A} = T(1)AS(1)$ which is the balance matrix A [21].

We can write the balancing transformations in the following form

$$S(\epsilon) = S(1)\chi(\epsilon)$$

and

$$T(\epsilon) = \chi(\epsilon)T(1)$$

where:

$$\chi(\epsilon) = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{\epsilon}}\mathbf{I} \end{pmatrix}$$

Now, we omit the tilde from the balanced matrices, to have the following matrices:

$$A = \begin{pmatrix} A_{11} & \frac{1}{\sqrt{\epsilon}}A_{12} \\ \frac{1}{\sqrt{\epsilon}}A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ \frac{1}{\sqrt{\epsilon}}B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & \frac{1}{\sqrt{\epsilon}}C_2 \end{pmatrix}$$

Let us define the new variable $q = (q_1, q_2)$ such that q can be balanced by using the balance transformation $T(\epsilon)$ and we write q in the balance form as:

$$q = T(\epsilon)x$$

Now, the linear dynamical system in equation (4.18) is converted to the singular perturbation system that is described in the following equation:

$$\begin{aligned}
\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} &= \begin{pmatrix} A_{11} & \frac{1}{\sqrt{\epsilon}}A_{12} \\ \frac{1}{\sqrt{\epsilon}}A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{1}{\sqrt{\epsilon}}B_2 \end{pmatrix} u \\
y &= \begin{pmatrix} C_1 & \frac{1}{\sqrt{\epsilon}}C_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}
\end{aligned} \tag{4.21}$$

Equation (4.21) can be written in another way:

$$\begin{aligned}
\dot{q}_1 &= A_{11}q_1 + \frac{1}{\sqrt{\epsilon}}A_{12}q_2 + B_1 u \\
\dot{q}_2 &= \frac{1}{\sqrt{\epsilon}}A_{21}q_1 + \frac{1}{\epsilon}A_{22}q_2 + \frac{1}{\sqrt{\epsilon}}B_2 u \\
y &= C_1 q_1 + \frac{1}{\sqrt{\epsilon}}C_2 q_2
\end{aligned} \tag{4.22}$$

the variable q_2 is scaled as

$$q_2 \longmapsto \sqrt{\epsilon}q_2$$

then equation (4.22) will be:

$$\begin{aligned}
\dot{q}_1 &= A_{11}q_1 + A_{12}q_2 + B_1 u \\
\epsilon\dot{q}_2 &= A_{21}q_1 + A_{22}q_2 + B_2 u \\
y &= C_1 q_1 + C_2 q_2
\end{aligned} \tag{4.23}$$

This system can be written in matrix form as:

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{pmatrix} u$$
$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{4.24}$$

where the block matrices $A_{11}, A_{12}, \ldots$ are in the balanced form and ϵ is a very small positive scalar that represent all small parameters to be declined if its equal to 0 [21, 25].

To reduce the dimension of the original system and obtain a reduced order model, we put the singular perturbation $\epsilon = 0$.

Now, to apply the singular perturbation approximation and obtain a reduced order model, we introduce two assumption [25]:

Assumption 1. *The matrix A_{22} is invertible and stable. i.e,*

$$\Re\{\lambda(A_{22})\} < 0$$

Assumption 2. *The equation below has a distinct root when we set $\epsilon = 0$.*

$$\epsilon\dot{q}_2 = A_{21}q_1 + A_{22}q_2 + B_2u \tag{4.25}$$

In our system described by equation (4.23), the slow variable is q_1 and the fast one is q_2.

According to the two assumptions (1),(2) and from equation (4.25), if we set

$\epsilon = 0$, then the root of equation (4.25) denoted by $\overline{q}_2$ is given as:

$$\overline{q_2} = -A_{22}^{-1}A_{21}\overline{x} - A_{22}^{-1}B_2u \tag{4.26}$$

If we set the value of $\overline{q}_2$ in the first part of equation (4.23), we obtain the reduced order model represented by:

$$\begin{aligned} \dot{\overline{q}}_1 &= \overline{A}\overline{q}_1 + \overline{B}u \\ \overline{y} &= \overline{C}\overline{q}_1 + \overline{D}u \\ \overline{q}_1(0) &= q_1(0) \end{aligned} \tag{4.27}$$

where

$$\begin{aligned} \overline{A} &= A_{11} - A_{12}A_{22}^{-1}A_{21} \\ \overline{B} &= B_1 - A_{12}A_{22}^{-1}B_2 \\ \overline{C} &= C_1 - C_2A_{22}^{-1}A_{21} \\ \overline{D} &= -C_2A_{22}^{-1}B_2 \end{aligned} \tag{4.28}$$

Let $\overline{G}$ be the transfer function of the reduced order model in equation (4.27), then:

$$\overline{G}(s) = \overline{C}(sI - \overline{A})^{-1}\overline{B} + \overline{D} \tag{4.29}$$

From the definition of the reduced reciprocal system (4.2) and the two equations

(4.28) and (4.9), we have:

$$
\begin{aligned}
\hat{A}_{11} &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} \\
&= (\overline{A})^{-1} \\
\hat{B}_{1} &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}(B_1 - A_{12}A_{22}^{-1}B_2) \\
&= (\overline{A})^{-1}\overline{B} \\
\hat{C}_{1} &= (C_1 - C_2A_{22}^{-1}A_{21})(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} \\
&= -\overline{C}(\overline{A})^{-1} \\
\hat{D} &= \overline{D} - \overline{C}(\overline{A})^{-1}\overline{B}
\end{aligned}
\tag{4.30}
$$

by using (4.30), we can connect the transfer functions of $\overline{G}(s)$ and $\hat{G}_r(s)$ and written as:

$$
\begin{aligned}
\overline{G}(s) &= \overline{C}\left(sI - \overline{A}\right)^{-1}\overline{B} + \overline{D} \\
&= \overline{C}\left(\frac{1}{s}\right)\left(I - \frac{1}{s}\overline{A}\right)^{-1}\overline{B} + \overline{D} \\
&= \overline{C}\left(\frac{1}{s}\right)\left((\overline{A})^{-1}\overline{A} - \frac{1}{s}\overline{A}\right)^{-1}\overline{B} + \overline{D} \\
&= \overline{C}\left(\frac{1}{s}\right)\left((\overline{A})^{-1} - \frac{I}{s}\right)^{-1}(\overline{A})^{-1}\overline{B} + \overline{D} \\
&= -\overline{C}\left(\frac{I}{s} - (\overline{A})^{-1} + \overline{A}\right)\left(\frac{I}{s} - (\overline{A})^{-1}\right)^{-1}(\overline{A})^{-1}\overline{B} + \overline{D} \\
&= -\overline{C}(\overline{A})^{-1}\overline{B} - \overline{C}\left(\frac{I}{s} - (\overline{A})^{-1}\right)^{-1}(\overline{A})^{-1}\overline{B} + \overline{D} \\
&= -\overline{C}(\overline{A})^{-1}\left(\frac{I}{s} - (\overline{A})^{-1}\right)^{-1}(\overline{A})^{-1}\overline{B} + \overline{D} - \overline{C}(\overline{A})^{-1}\overline{B} \\
&= \hat{C}_1\left(\frac{I}{s} - \hat{A}_{11}\right)^{-1}\hat{B}_1 + \hat{D} \\
&= \hat{G}_r(\frac{1}{s})
\end{aligned}
\tag{4.31}
$$

Since the dynamic system (A, B, C, D) is balanced and asymptotically stable and we have the balanced gramian $\Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix}$, we introduce the following theorem for balancing of the reduced system $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$.

Theorem 20. *[35] The reduced order model $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ by singular perturbation approximation is balanced with Σ_1 and asymptotically stable.*

Proof. [35] We know from corollary (18) that the reduced system $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D})$ is balanced with Σ_1 which satisfy the Lypunove equations

$$\hat{A}_{11}\Sigma_1 + \Sigma_1\hat{A}_{11}^T + \hat{B}_1\hat{B}_1^T = 0 \tag{4.32}$$

$$\hat{A}_{11}^T\Sigma_1 + \Sigma_1\hat{A}_{11} + \hat{C}_1^T\hat{C}_1 = 0 \tag{4.33}$$

Multiplying equation (4.32) from the right by $\hat{A}_{11}^{-1}$ and from the left by $\hat{A}_{11}^{-T}$ to have

$$\begin{aligned} \hat{A}_{11}^{-1}(\hat{A}_{11}\Sigma_1)\hat{A}_{11}^{-T} + \hat{A}_{11}^{-1}(\Sigma_1\hat{A}_{11}^T)\hat{A}_{11}^{-T} + A^{-1}(\hat{B}_1\hat{B}_1^T)A^{-T} &= 0 \\ \Sigma_1\hat{A}_{11}^{-T} + \hat{A}_{11}^{-1}\Sigma_1 + (\hat{A}_{11}^{-1}\hat{B}_1)(\hat{A}_{11}^{-1}\hat{B}_1)^T &= 0 \end{aligned} \tag{4.34}$$

From equation (4.30) we have:

$\hat{A}_{11} = (\overline{A})^{-1}$

$\hat{B}_1 = (\overline{A})^{-1}\overline{B}$

$\hat{C}_1 = -\overline{C}(\overline{A})^{-1}$

$\hat{D} = \overline{D} - \overline{C}(\overline{A})^{-1}\overline{B}$ Substitute the values above into equation (4.34) then we get:

$$\overline{A}\Sigma_1 + \Sigma_1\overline{A}^T + \overline{BB}^T = 0$$

If the second equation (4.33) is multiplied by $\hat{A}_{11}^{-T}$ from the right and by $\hat{A}_{11}^{-1}$ from the left, then we get

$$\begin{aligned} \hat{A}_{11}^{-T}(\hat{A}_{11}^T\Sigma_1)\hat{A}_{11}^{-1} + \hat{A}_{11}^{-T}(\Sigma_1\hat{A}_{11})\hat{A}_{11}^{-1} + \hat{A}_{11}^{-T}(\hat{C}_1^T\hat{C}_1)\hat{A}_{11}^{-1} = 0 \\ \Sigma_1\hat{A}_{11}^{-1} + \hat{A}_{11}^{-T}\Sigma_1 + (\hat{C}_1\hat{A}_{11}^{-1})^T(\hat{C}_1\hat{A}_{11}^{-1}) = 0 \end{aligned} \tag{4.35}$$

In the same way from equation (4.30) and (4.35), we have

$$\overline{A}^T\Sigma_1 + \Sigma_1\overline{A} + \overline{C}^T\overline{C} = 0$$

Which implies that our reduced system $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ is balanced with gramian Σ_1.

Since $\hat{A}_{11}$ is asymptotically stable .i.e., $\Re\{\lambda(\hat{A}_{11})\} < 0$, where λ is an eigenvalue of $\hat{A}_{11}$

Since $(\hat{A}_{11})^{-1} = \overline{A}$ then the of $\overline{A}$ is $\frac{1}{\lambda}$ so we conclude $\Re\{\lambda_i(\hat{A}_{11})\} < 0$ which mean the reduced order system by singular perturbation approximation $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ is asymptotically stable . $\square$

Theorem 21. *Let $\overline{G}$ be the transfer function of the r th order model by balanced singular perturbation approximation $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ of G . Then we have*

$$\|G - \overline{G}_r\|_\infty \leq 2\sum_{r=i+1}^{n} \sigma_i \tag{4.36}$$

Proof. From Equations (4.31),(4.2) and Lemma (19)and by using the triangle inequality, we have

$$
\begin{aligned}
\|G(s) - \overline{G}(s)\|_\infty &= \|G(s) - \hat{G}(\frac{1}{s}) + \hat{G}(\frac{1}{s}) - \hat{G}_r(\frac{1}{s}) + \hat{G}_r(\frac{1}{s}) - \overline{G}(s)\|_\infty \\
&\leq \|G(s) - \hat{G}(\frac{1}{s})\|_\infty + \|\hat{G}(\frac{1}{s}) - \hat{G}_r(\frac{1}{s})\|_\infty + \|\hat{G}_r(\frac{1}{s}) - \overline{G}(s)\|_\infty \\
&\leq \|\hat{G}(\frac{1}{s}) - \hat{G}_r(\frac{1}{s})\|_\infty \\
&\leq 2\sum_{i=r+1}^{n} \sigma_i
\end{aligned}
$$

$\square$

Chapter 5

Numerical Example

In this chapter we will introduce some examples to describe the action of the control system, such as mass - spring damping system and RLC Circuit.

5.1 Mass Spring Damping system (MSD)

In this section we will use the mass spring damping system as example to show the behavior of the control system.we start with one mass and one spring ; suppose that m is the mass that showed in Figure [5.1] Where:

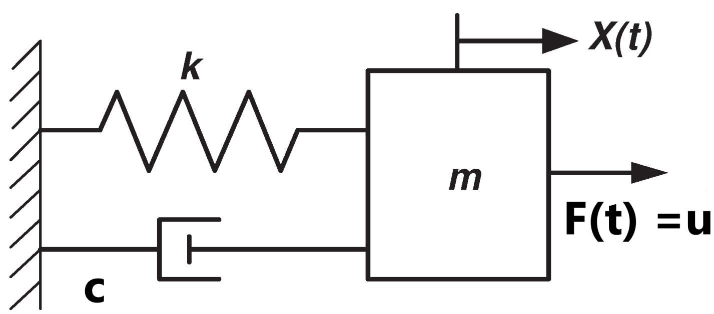


Figure 5.1: One Mass - spring Damper

K : Is the spring constant

C : viscous damping coefficient

x : position of mass at time

u : Force input

Now we want to obtain the differential equation for the mass m:

applying Hook's low spring:

$$F_{spring} = F_s = -ks$$

$$F_{damping} = F_d = c \text{ (velocity)} = c\frac{dx}{dt} = cx'$$

By applying Newton's law , then we have:

$$
\begin{aligned}
\sum F &= ma \\
u(t) - c\frac{dx}{dt} - kx &= mc\frac{d^2x}{dt^2} \\
u(t) - c\dot{x} - kx &= m\ddot{x}
\end{aligned}
\tag{5.1}
$$

By rearranging the above equation , then we get:

$$m\ddot{x} + c\dot{x} + kx = u \tag{5.2}$$

Divide both sides by m , then the equation become:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}u \tag{5.3}$$

Now, let:

$x_1(t) = x(t)$ where: $x_1(t)$: position of the mass (m).

$x_2(t) = \dot{x}(t) = \dot{x}_1(t)$ where:$x_2(t)$:derivative of $x_1(t)$ which it's a velocity of the mass .

Substitute $x_1(t) = x(t)$ and $\dot{x}_2(t) = \ddot{x}(t)$ into equation (5.3) , then we have:

$$
\begin{aligned}
\dot{x}_2 + \frac{c}{m}x_2 + \frac{k}{m}x_1 &= \frac{1}{m}u \\
\dot{x}_2 = -\frac{c}{m}x_2 - \frac{k}{m}x_1 &+ \frac{1}{m}u
\end{aligned}
\tag{5.4}
$$

Finally , The equation (5.4) is the differential equation for the mass m. We can put the set of equations into matrix form as:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u \tag{5.5}$$

With the output equation:

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.6}$$

Then:

The MSD system has the differential equation:

$$m\ddot{x} + c\dot{x} + kx = u \tag{5.7}$$

which has the state space representation:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5.8}$$

if we compare the state representation with MSD system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u \tag{5.9}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.10}$$

Then we have:

A= $\begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{pmatrix}$ which is called the state matrix and the dim of A= $n \times n$

B= $\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$ which is called the input matrix and the dim of B= $n \times r$

C = $\begin{pmatrix} 1 & 0 \end{pmatrix}$ which is called output matrix and the dim of C = $m \times n$

D= 0_{matrix}which is called feed through matrix and

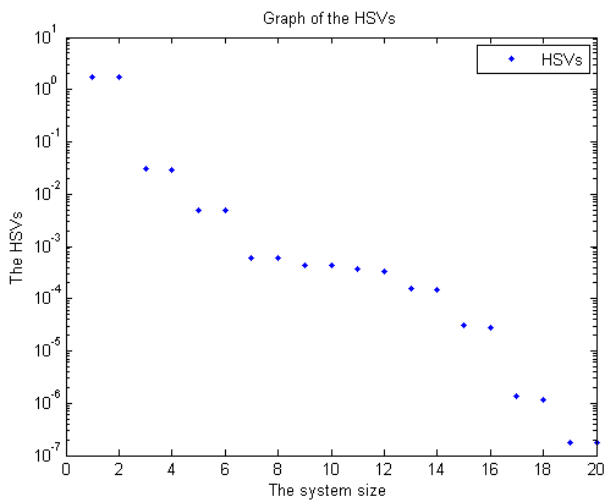
the dim of D = $m \times r$

5.1.1 The balanced truncation (BT) Method

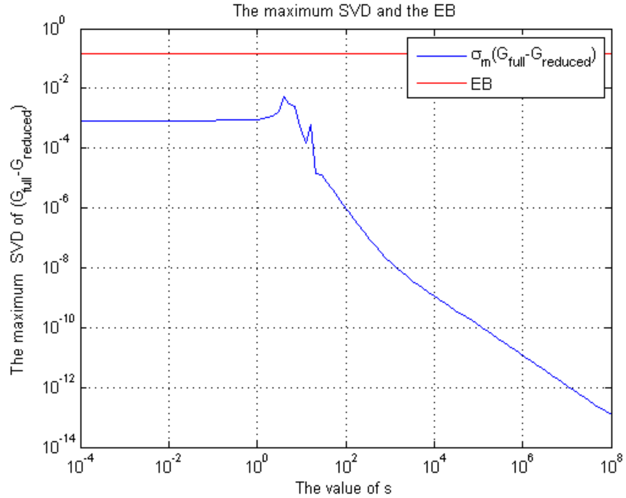
In this section we include all results obtained by the method we explain, namely; the balanced truncation (BT) method to determine the order of the reduced models.

We start by finding the HSVs of the dynamical system explained in section(5.1). Figure (5.2a)represent the Hankel singular values (HSVs) for the mass-spring damping system for BT.

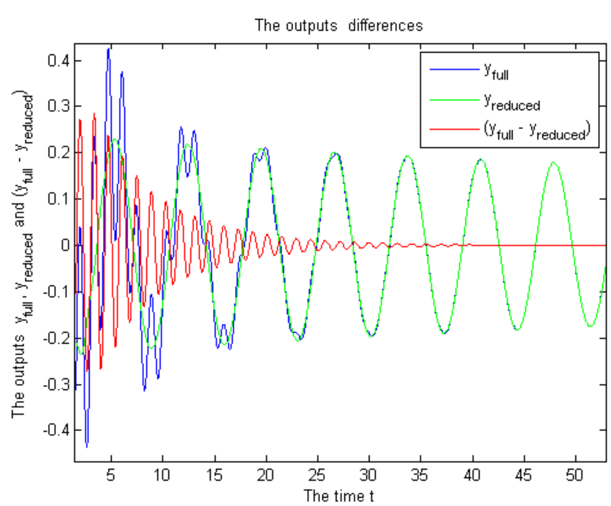
For testing purposes, we apply the balanced truncation method for the example with zero-initial condition and compute the H^∞ bound of the approximation error described in section (3.4) equation (3.16). The size of the reduced model is $r = 2$. Figure (5.2b) shows the maximum singular value decomposition (MSVD) σ_{max} of $(G - G_r)$, where G is the transfer function of the original system and G_r is the transfer function of the reduced order model, and the error bound is $2 \sum_{j=r+1}^{10} \sigma_j$.



(a) HSVs of the mass-spring damping for BT



(b) The maximum singular value decomposition and the error bound for the mass-spring damping for balanced truncation method



(c) The outputs of the mass-spring damping for BT

Figure 5.2: The Numerical results for MSD system by BT method

Figures (5.2c) contain the output Y of the original system,the output Y_r of the reduced model and the difference $Y - Y_r$. For the mass spring damping, let $r = 2$ the L_2 norm of $(Y - Y_r)$ can be computed for different r . Table (5.1) contains the values of $||G - G_r||_\infty$ and $||Y - Y_r||_\infty$ and $2 \sum_{j=r+1}^{10} \sigma_j$ computed for various values of r by applying the balanced truncation for the mass-spring damping system.

Table 5.1: The H^∞ norm of $(G - G_r)$ and the error bound For mass-spring damping system (BT).

r	$\|\|G - G_r\|\|_\infty$	$\|\|Y - Y_r\|\|_\infty$	$2 \sum_{j=r+1}^{n} \sigma_j$
1	3.5016	6.5955e-06	3.6523
2	0.0055	2.1036e-10	0.1450
3	0.0593	1.4866e-08	0.0849
4	0.0018	1.8108e-11	0.0259
5	0.0097	1.2979e-10	0.0159
6	5.9742e-04	8.6645e-13	0.0063
7	0.0013	1.3055e-10	0.0051
8	5.1416e-04	1.0127e-12	0.0039
9	8.1211e-04	6.5479e-12	0.0030
10	4.7972e-04	4.9314e-13	0.0021
11	6.8686e-04	1.0212e-10	0.0014
15	5.5890e-05	7.3895e-13	6.1358e-05

5.1.2 The singular perturbation approximation (SPA) method

In this section we include all results obtained by the method we explain, namely; the singular perturbation approximation (SPA) method to determine the order of the reduced models.

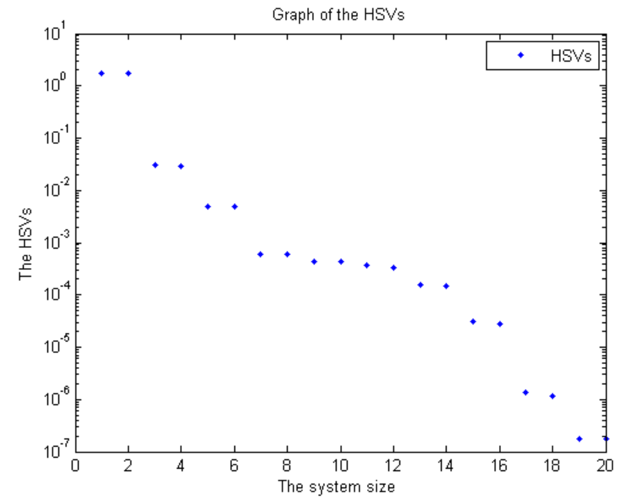
We first start by computing the HSVs of the dynamical system explained in section(5.1).

Figure (5.3a) represent the Hankel singular values (HSVs) for the mass-spring damping system for SPA.

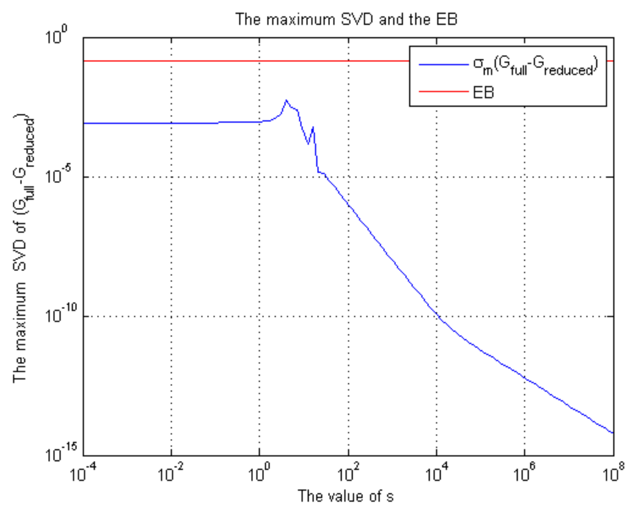
For testing purposes, we apply the singular perturbation approximation method for the example with zero-initial condition and compute the H^{∞} bound of the approximation error described in section (3.4) equation (3.16). The size of the reduced model is $r = 2$.

Figure (5.3b) shows the maximum singular value decomposition (MSVD) σ_{max} of $(G - G_r)$, where G is the transfer function of the original system and G_r is the transfer function of the reduced order model, and the error bound is $2 \sum_{j=r+1}^{10} \sigma_j$.

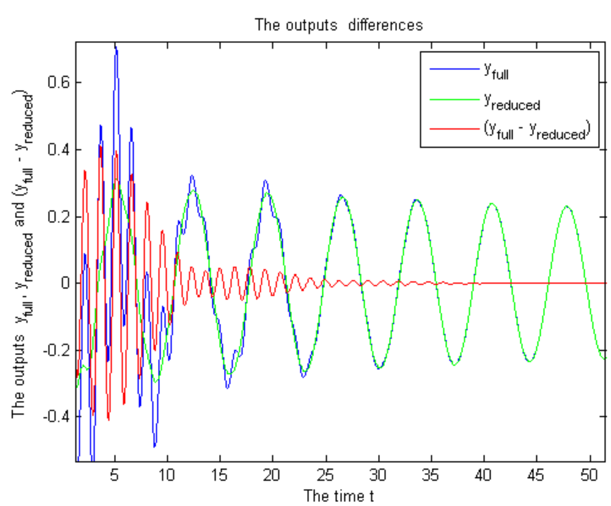
Figures (5.3c) contain the output Y of the original system,the output Y_r of the reduced model and the difference $Y - Y_r$. For the mass spring damping, let $r = 2$ the L_2 norm of $(Y - Y_r)$ can be computed for different r .



(a) HSVs of the mass-spring damping for SPA



(b) The maximum singular value decomposition and the error bound for the mass-spring damping for Singular Perturbation Approximation Method



(c) The outputs of the mass-spring damping for SPA

Figure 5.3: The Numerical results for MSD system by SPA method

Table (5.2) contains the values of $||G-G_r||_\infty$ and $||Y-Y_r||_\infty$ and $2\sum_{j=r+1}^{10}\sigma_j$ computed for various values of r by applying the singular perturbation approximation for the mass-spring damping system.

Table 5.2: The H^∞ norm of $(G-G_r)$ and the error bound for mass-spring damping system (SPA).

r	$\|\|G-G_r\|\|_\infty$	$\|\|Y-Y_r\|\|_\infty$	$2\sum_{j=r+1}^{n}\sigma_j$
1	3.5686	2.7306e-04	3.6523
2	0.0055	1.0539e-12	0.1450
3	0.0633	6.3772e-13	0.0849
4	0.0018	1.9532e-14	0.0259
5	0.0108	3.2569e-14	0.0159
6	5.9392e-04	8.1803e-16	0.0063
7	0.0017	8.2902e-16	0.0051
8	5.1546e-04	5.8767e-16	0.0039
9	8.8417e-04	1.2325e-15	0.0030
10	4.9656e-04	6.8749e-16	0.0021
11	7.4632e-04	1.5845e-15	0.0014
15	7.5905e-05	3.4062e-18	6.1358e-05

5.2 RLC Circuit

In this section we will introduce another example which explain the action of the control system .

RLC circuit shown in Figure [5.4] is an electrical circuit. Consisting of:

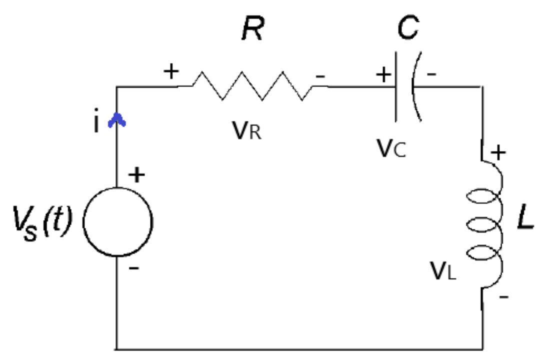


Figure 5.4: RLC Circuit

R : a resistor of resistance .

L : a coil od inductance .

C : a capacitor of capacitance .

V_s : a voltage source .

We want to derive the state equations and the output equations in the RLC circuit. The current and voltage equations governing the circuit are:

$$v = iR + v_L + v_C \tag{5.11}$$

where:

$$i = C\frac{dv_C}{dt} \tag{5.12}$$

and

$$v_L = L\frac{di}{dt} \tag{5.13}$$

Now by using equations (5.11),(5.12) and (5.13) we have:

$$v = iR + v_L + v_C$$

But $v_L = L\frac{di}{dt}$ by using equation (5.13) then:

$$v = iR + L\frac{di}{dt} + v_C$$

Now:

$$\frac{di}{dt} = \frac{1}{L}v - \frac{R}{L}i - \frac{1}{L}v_C \tag{5.14}$$

From equation (5.12)

$$\frac{dv_C}{dt} = \frac{1}{C}i \tag{5.15}$$

Now by substitute the states: $x_1 = v_c$ and $x_2 = i$ and the output equations $y_1 = v_c = x_1$ and $y_2 = Ri = Rx_2$ and $u_1 = v$ in the equations (5.14)(5.15), then we get:

$$\dot{x}_2 = \frac{1}{L}u - \frac{R}{L}x_2 - \frac{1}{L}x_1 \tag{5.16}$$

$$\dot{x}_1 = \frac{dv_C}{dt} = \frac{1}{C}i = \frac{1}{C}x_2 \tag{5.17}$$

Finally, The equations (5.16) (5.17) is the differential equations for the RLC circuit.

We can put the set of equations into matrix form as:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{C} \\ \frac{-1}{L} & \frac{-R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} u \tag{5.18}$$

With the output equation:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.19}$$

Then, the RLC circuit system has the state space representation:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5.20}$$

If we compare the state representation with RLC circuit system:

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{C} \\ \frac{-1}{L} & \frac{-R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} u \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned} \tag{5.21}$$

Then we have:

A= $\begin{pmatrix} 0 & \dfrac{1}{C} \\ \frac{-1}{L} & \frac{-R}{L} \end{pmatrix}$, B= $\begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix}$, C = $\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$

5.2.1 The balanced truncation (BT) Method

In this section we show all results obtained by the balanced truncation (BT) method to determine the order of the reduced models for RLC Circuit.

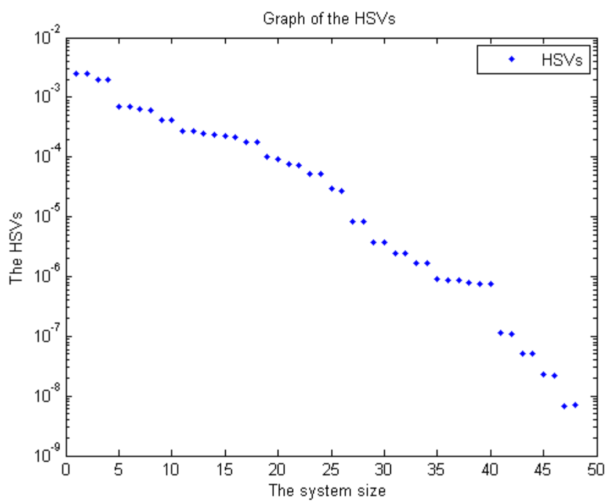
We first start by computing the HSVs of the dynamical system explained in section(5.2).

Figure (5.5a)represent the Hankel singular values (HSVs) for the RLC Circuit for BT.

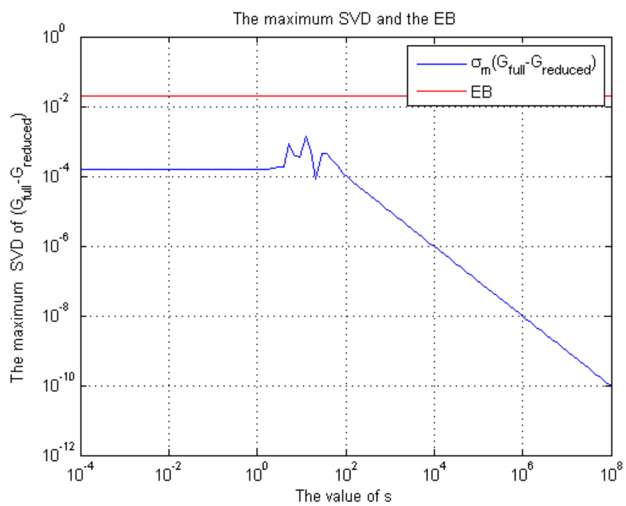
For testing purposes, we apply the balanced truncation method for the example with zero-initial condition and compute the H^{∞} bound of the approximation error described in section (3.4) equation (3.16). The size of the reduced model is $r = 2$.

Figure (5.5b) shows the maximum singular value decomposition (MSVD) σ_{max} of $(G - G_r)$, where G is the transfer function of the original system and G_r is the transfer function of the reduced order model, and the error bound is $2 \sum_{j=r+1}^{10} \sigma_j$.

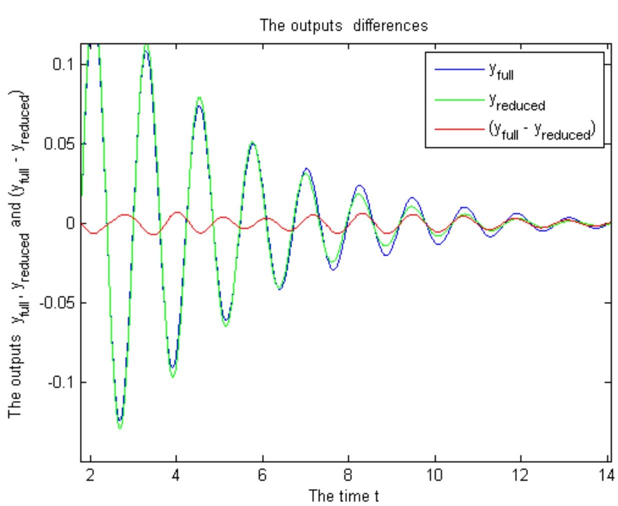
Figures (5.5c) contain the output Y of the original system,the output Y_r of the reduced model and the difference $Y - Y_r$. For the RLC Circuit, let $r = 2$ the L_2 norm of $(Y - Y_r)$ can be computed for different r .



(a) HSVs of the RLC Circuit for BT



(b) The maximum singular value decomposition and the error bound for the RLC Circuit for balanced truncation method



(c) The outputs of the RLC Circuit for BT

Figure 5.5: The Numerical results for RLC Circuit system by BT method

Table 5.3: The H^∞ norm of $(G - G_r)$ and the error bound for the RLC Circuit system (BT).

r	$\|\|G - G_r\|\|_\infty$	$\|\|Y - Y_r\|\|_\infty$	$2 \sum_{j=r+1}^{10} \sigma_j$
1	0.0050	$3.1058 \cdot 10^{-9}$	0.0243
2	0.0014	$1.7772 \cdot 10^{-37}$	0.0194
3	0.0034	$3.9663 \cdot 10^{-12}$	0.0156
4	$8.0706 \cdot 10^{-4}$	$1.7772 \cdot 10^{-37}$	0.0117
5	0.0016	$2.6244 \cdot 10^{-10}$	0.0103
6	$8.1972 \cdot 10^{-4}$	$1.7772 \cdot 10^{-37}$	0.0089
7	0.0011	$4.4054 \cdot 10^{-31}$	0.0076
8	$3.0929 \cdot 10^{-4}$	$1.9912 \cdot 10^{-37}$	0.0064
9	$7.3967 \cdot 10^{-4}$	$5.6526 \cdot 10^{-23}$	0.0055
10	$2.8312 \cdot 10^{-4}$	$1.9050 \cdot 10^{-37}$	0.0047
15	$3.8855 \cdot 10^{-4}$	$2.7679 \cdot 10^{-32}$	0.0022
20	$1.3856 \cdot 10^{-4}$	$2.5175 \cdot 10^{-38}$	$6.8905 \cdot 10^{-4}$

Table (5.3) contains the values of $||G - G_r||_\infty$ and $||Y - Y_r||_\infty$ and $2 \sum_{j=r+1}^{10} \sigma_j$ computed for various values of r by applying the balanced truncation for the RLC Circuit system.

5.2.2 The singular perturbation approximation (SPA) method

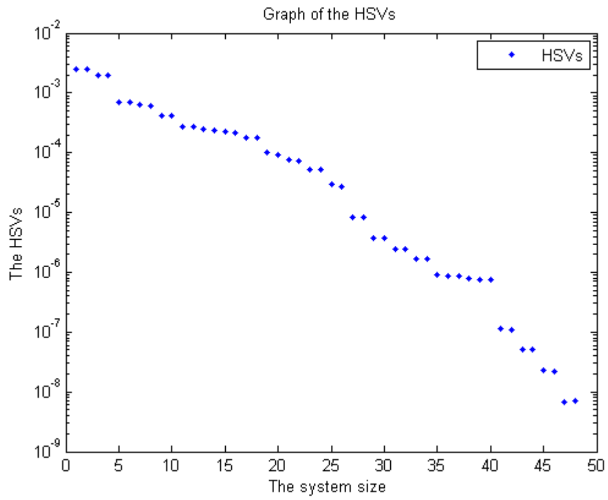
In this section we show all results obtained by the Singular Perturbation Approximation (SPA) method to determine the order of the reduced models for RLC Circuit.

We first start by computing the HSVs of the dynamical system explained in section(5.2). Figure (5.6a)represent the Hankel singular values (HSVs) for the RLC Circuit for SPA.

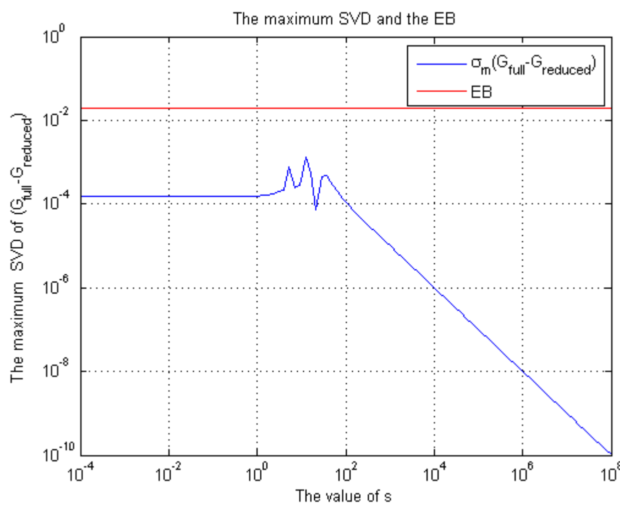
For testing purposes, we apply the singular perturbation approximation method for the example with zero-initial condition and compute the H^{∞} bound of the approximation error described in section (3.4) equation (3.16). The size of the reduced model is $r = 2$.

Figure (5.6b) shows the maximum singular value decomposition (MSVD) σ_{max} of $(G - G_r)$, where G is the transfer function of the original system and G_r is the transfer function of the reduced order model, and the error bound is $2 \sum\limits_{j=r+1}^{10} \sigma_j$.

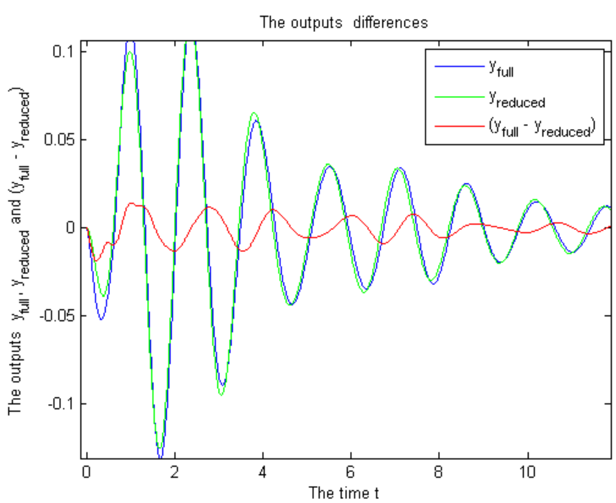
Figures (5.6c) contain the output Y of the original system,the output Y_r of the reduced model and the difference $Y - Y_r$. For the RLC Circuit, let $r = 2$ the L_2 norm of $(Y - Y_r)$ can be computed for different r .



(a) HSVs of the RLC Circuit for SPA



(b) The maximum singular value decomposition and the error bound for the RLC Circuit for Singular Perturbation Approximation Method



(c) The outputs of the RLC Circuit for SPA

Figure 5.6: The Numerical results for RLC Circuit system by SPA method

Table (5.4) contains the values of $||G-G_r||_\infty$ and $||Y-Y_r||_\infty$ and $2\sum_{j=r+1}^{10}\sigma_j$ computed for various values of r by applying the singular perturbation approximation for the RLC Circuit system.

Table 5.4: The H^∞ norm of $(G-G_r)$ and the error bound for the RLC Circuit system (SPA).

r	$\|\|G-G_r\|\|_\infty$	$\|\|Y-Y_r\|\|_\infty$	$2\sum_{j=r+1}^{10}\sigma_j$
1	0.0096	$2.1648\cdot 10^{-20}$	0.0243
2	0.0014	$3.7147\cdot 10^{-23}$	0.0194
3	0.0034	$1.1278\cdot 10^{-20}$	0.0156
4	$6.7785\cdot 10^{-4}$	$2.3938\cdot 10^{-23}$	0.0117
5	0.0017	$1.7610\cdot 10^{-21}$	0.0103
6	$6.7444\cdot 10^{-4}$	$2.6064\cdot 10^{-23}$	0.0089
7	0.0015	$7.7775\cdot 10^{-22}$	0.0076
8	$3.2073\cdot 10^{-4}$	$1.3447\cdot 10^{-23}$	0.0064
9	$8.8473\cdot 10^{-4}$	$3.3785\cdot 10^{-22}$	0.0055
10	$3.1997\cdot 10^{-4}$	$1.0582\cdot 10^{-23}$	0.0047
15	$4.6035\cdot 10^{-4}$	$9.6521\cdot 10^{-23}$	0.0022
20	$1.3474\cdot 10^{-4}$	$1.5070\cdot 10^{-25}$	$6.8905\cdot 10^{-4}$

5.3 Conclusion

In this thesis, we have shown that direct Balanced truncation reduction and the slow singular perturbation approximation of a stable internally balanced continuous-time system are two fully compatible model order reduction techniques, in the sense that both methods yield a minimal, stable, and balanced reduced order system with the same L_2-norm frequency error bound on the reduction.

We provide theoretical tools such as concepts for reachability and observability, which are necessary for balancing related model order reduction of linear differential equations.

We have also shown that though the upper bound for both methods is the same, the actual frequency errors of these two reduction methods are quite different.

The direct Balanced truncation reduction tends to have smaller errors at high frequencies and larger errors at low frequencies while the singular perturbation approximation will have larger errors at high frequencies and smaller errors at low frequencies.

Finally, we derive error bounds for both BT and SPA and provide numerical results for a specific example which support the theory.

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الحلول العددية لانظمة التحكم ذات الابعاد الكبيرة

إعداد

هديل منذر فتحي صوافطة

إشراف

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د. عدنان دراغمة

الملخص

في هذه الرسالة، تم التركيز على المعالجة العددية لانظمة التحكم ذات الابعاد الكبيرة بسبب أهميتها الهائلة في العديد من التطبيقات في مختلف المجالات.

نظرية التحكم هي فرع من فروع الرياضيات والعلوم وهي تؤدي دورًا رئيسيًا في كل الأجهزة الحديثة وعالم الهندسة وتدخل في كل المجالات من أجهزة الاستريو وأجهزة الكمبيوتر إلى التصنيع الكيميائي والطائرة.

بعد ان تناولنا المفاهيم الاساسية في نظرية التحكم من النظام الديناميكي ومعادلته وخصائصه من الثبات والقدرة على التحكم والمشاهدة، قمنا بالتركيز على الطرق العددية لحل النظام ذا الابعاد الكبيرة، وهي: طريقة الاقتطاع المتوازن وطريقة تقريب الاضطراب المفرد.

سيتم عرض الإطار الرياضي لهذه الطرق العددية مع خصائصها. سيتم توضيح هذه الطرق العددية من خلال بعض الأمثلة العددية.

تم توضيح هذه الطرق العددية عن طريق حل بعض الأمثلة العددية وتم إجراء مقارنة بينها. أظهرت النتائج العددية بوضوح أنه بالنسبة لطريقة الاقتطاع المتوازنة، يكون الخطأ صغيرًا عند الترددات العالية وكبير عند الترددات المنخفضة، ولكن بالنسبة لتقريب الاضطراب المفرد، يكون الخطأ كبير عند الترددات العالية والخطأ الصغير عند الترددات المنخفضة.

جامعة النجاح الوطنية
كلية الدراسات العليا

الحلول العددية لانظمة التحكم ذات الابعاد الكبيرة

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قدمت هذه الرسالة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية، نابلس–فلسطين.

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