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Composition operator on de Branges-Rovnyak spaces

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Dedication

This dissertation is dedicated to my parents Fahmi and Fa'eda, who instilled in me the virtues of perseverance and commitment and relentlessly encouraged me to strive for excellence.

I also dedicate this to my brothers Abdallah, Zohdi, Abdelrahman and Mahmoud for all their love, patience, kindness and support.

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الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان

Composition operator on de Branges-Rovnyak spaces

أقر بأن ما شملت عليه الرسالة هو نتاج جهدي الخاص, باستثناء ما تمت الإشارة إليه حيثما ورد, وأن هذه الرسالة ككل أو أي جزء منها لم يقدم من قبل لنيل أي درجة أو لقب علمي أو بحثي لدى أي مؤسسة علمية أو بحثية

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degrees or qualifications.

Student's Name: **اسم الطالبة:**

Signature **التوقيع:**

Date **التاريخ:**

Contents

Notations

$\mathbb{C}$: Complex plane.

$\mathbb{T}$: Unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

$\mathbb{D}$: Unit disc $\{z \in \mathbb{C} : |z| < 1\}$ of complex plane.

T_α: Toeplitz operator.

C_α: Composition operator.

M_α: Multiplication operator.

S: Shift operator.

S^*: Backward shift operator.

m: Lebesgue measure on $\mathbb{T}$ normalized so that $m(\mathbb{T}) = 1$.

$L^p(\mathbb{T}, m)$: Lebesgue space.

$\mathcal{H}(b)$: de Branges-Rovnyak spaces.

Composition operators on De Branges Rovnyak–spaces
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Abstract

If α which is analytic and which maps the unit disc to itself. Then we describe the image of the de Branges-Rovnyak spaces $\mathcal{H}(b)$ under the composition operator C_α for algebraic functions b. In this thesis, we answered partially about what are the conditions on α such that C_α maps $\mathcal{H}(b)$ into itself, which are α's such that $C_\alpha : \mathcal{H}(b) \longrightarrow \mathcal{H}(b)$, and which are α's such that C_α maps $\mathcal{H}(b)$ to $\mathcal{H}(q)$.

Preface

One of the most common examples of the composition operator is the composition operator on Hardy space H^2. It is well-known that C_α maps H^2 into itself boundedly for bounded α on $\mathbb{D}$.

Recently, Emmanuel Fricain, Andreas Hartmann, and William T. Ross, in [7], gave concrete examples of de Branges-Rovnyak spaces with algebraic symbols. We will give more examples of $\mathcal{H}(b)$ with algebraic symbols in section 2.3.

In [19], the authors characterized those α's for which the composition operators map the model space into itself for the finite dimension case, in other words, when b is a finite Blaschke product. Then, in [18] the authors gave some partial results for inner functions. But in our study we will consider the composition operators on $\mathcal{H}(b)$ for certain choices of b in the closed unit ball of H^∞.

Motivated by the fact that: $C_\alpha : \mathcal{H}(b) :\longrightarrow H^2$, we will try to find X as a subspace such that $C_\alpha(\mathcal{H}(b)) \subseteq X$.

In chapter one, we give recapitulation about Hardy space, inner and outer function, reproducing kernel, Toeplitz operator, model space, and composition operators on H^2.

In chapter two, we give an introduction about de Branges-Rovnyak spaces $\mathcal{H}(b)$. We focus here on the case $b = q^\rho$ where q is rational and ρ is any positive number, and on the example which Emmanuel Fricain, Andreas Hartmann, and William T. Ross derived. In section 2.3 we construct a new algebraic example.

In chapter three, we describe the image of the composition operator C_α on $\mathcal{H}(b)$ spaces for some algebraic functions b in the closed unit ball of H^∞.

Chapter 1

Preliminaries

In this chapter we will give recapitulation about Hardy space, inner and outer function, reproducing kernels, Toeplitz operator, model space, and we will introduce the composition operators on H^2.

1.1. Hardy spaces

All definitions and results in this section can be found in [5, 14].

A measurable function defined on $\mathbb{T}$ is said to belong to the space $L^p(\mathbb{T}, m)$ if

$$\int_{\mathbb{T}} |f|^p < \infty.$$

For $1 \leqslant p < \infty$ the Hardy space H^p is defined as the space of all analytic functions f in the unit disk $\mathbb{D}$, for which the norm

$$||f||_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}, \tag{1.1}$$

is finite. The space H^∞ denotes the space of all bounded analytic functions on the open unit disc $\mathbb{D}$ normed by

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

Fatou's theorem says that for functions in H^p for $1 \leqslant p \leqslant \infty$, the radial limit

$$f(e^{it}) = \lim_{r \to 1} f(re^{it}),$$

exists almost everywhere in t, moreover $f(e^{it}) \in L^p(\mathbb{T}, m)$. So, H^p can be regarded as a closed subspace of $L^p(\mathbb{T}, m)$.

For any analytic function f on $\mathbb{D}$ and for every non-negative integer n, let $\hat{f}(n) = f^{(n)}(0)/n!$. Then the series $\sum_{n=0}^{\infty} \hat{f}(n) z^n$ is the Taylor series of f with center at the origin: it converges uniformly on compact subsets of $\mathbb{D}$ to f.

The Hardy space H^2 is of a special interest, it is the collection of all analytic functions f with $\sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$. The norm in (1.1) becomes

$$||f||_2 = \left(\sum_{n=0}^{\infty} |\hat{f}(n)|^2 \right)^{\frac{1}{2}}.$$

The Hardy space H^2 is a Hilbert space with the inner product

$$\langle f, g \rangle_2 = \sum_{n=0}^{\infty} \hat{f}(n) \overline{\hat{g}(n)} \qquad (f, g \in H^2).$$

It is clear that, $H^\infty \subset H^2$. More generally, if $b \in H^\infty$ and $f \in H^2$ then the pointwise product $bf \in H^2$, that is to say H^∞ is the set of all multipliers of H^2.

Functions in H^2 can be factorized in a canonical way, into a Blaschke product, an inner factor, and an outer function. Here our terms.

Definition 1.1. *[5](a) Let $\{z_j\}_{j=1}^{\infty}$ be a sequence in $\mathbb{D}$ such that* $\sum_{j=1}^{\infty}(1 -$

$|z_j|) < \infty$, *then the Blaschke product with zeros* $\{z_j\}_{j=0}^{\infty}$*is a function of the form*

$$B(z) = e^{i\gamma} \prod_{j=1}^{\infty} \frac{z_j}{|z_j|} \frac{z - z_j}{1 - \bar{z}_j z}.$$

Of course, B *is an analytic function on* $\mathbb{C}$ *with zeros of order* m_n *at* z_n*.*

(b) An inner function is an H^∞ *function that has unit modulus almost everywhere on* $\mathbb{T}$*.*

(c) An outer function is a function $f \in H^1$ *which can be written in the form*

$$f(re^{i\theta}) = \alpha exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} k(e^{it})dt\right),$$

for $re^{i\theta} \in \mathbb{D}$*, where* k *is a real-valued integrable function and* $|\alpha| = 1$*.*

It is well-known that non-zero functions in H^1 can be written as $f = B.S.O$, where B is a Blaschke product, S a singular inner function, and O an outer function. This factorization is unique up to constant of modulus 1.

1.2. Reproducing kernel Hilbert spaces

A Hilbert space of functions on $\mathbb{D}$ is called a reproducing kernel space if the point evaluations $f \longmapsto f(\lambda)$ at points of $\mathbb{D}$ are continuous.

Note that [10],

$$|f(\lambda)| \leqslant \sum_{n=0}^{\infty} |a_n||\lambda|^n \leqslant (\sum_{n=0}^{\infty} |a_n|^2)^{\frac{1}{2}} (\sum_{n=0}^{\infty} |\lambda|^2)^{\frac{1}{2}} = \frac{\|f\|}{\sqrt{1 - |\lambda|^2}},$$

which holds for all $\lambda \in \mathbb{D}$ and all f in H^2. It follows that for fixed $\lambda \in \mathbb{D}$

the point evaluation functionals $f \longmapsto f(\lambda)$ are bounded on H^2 and hence, by the Riesz Representation Theorem, f must be of the form

$$f(\lambda) = \langle f, k_\lambda \rangle,$$

for some k_λ in H^2.

In fact, it is not hard to show that

$$k_\lambda = \frac{1}{1 - \bar{\lambda} z}, \quad \lambda \in \mathbb{D},$$

which is called the Cauchy-Szegő kernel or, more frequently, the Cauchy kernel. In more general terms, one says that the Cauchy-Szegő kernel is the reproducing kernel for H^2.

1.3. Toeplitz operators

Since H^2 is a closed subspace of L^2 then the orthogonal projection P_{H^2} : $L^2(\mathbb{T}) \longrightarrow H^2$ is

$$P_{H^2}\left(\sum_{n=-\infty}^{\infty} a_n e^{int}\right) = \sum_{n=0}^{\infty} a_n e^{int}.$$

The following is the definition of the Laurent operator.

Definition 1.2. *[20] Let $\varphi \in L^\infty(\mathbb{T})$. Then the Laurent (or multiplication* *operator)* $M_\varphi : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$ *is given by*

$$(M_\varphi f)(e^{it}) = \varphi(e^{it})f(e^{it}).$$

Theorem 1.3. *[1] Let* $\varphi \in L^\infty(\mathbb{T})$. *Then* M_φ *is a bounded operator and its norm is given by* $||M_\varphi|| = ||\varphi||_\infty$.

Definition 1.4. *[20] For* $\varphi \in L^\infty(\mathbb{T})$*, the Toeplitz operator with the symbol* φ *is the operator* $T_\varphi : H^2 \longrightarrow H^2$ *is defined by* $T_\varphi f = P_{H^2}(M_\varphi f)$.

Clearly, since $||P_{H^2}|| = 1$ and $||M_\varphi|| = ||\varphi||_\infty$ we have that T_φ is a bounded operator on H^2 and satisfies $||T_\varphi|| \leqslant ||\varphi||_\infty$. Moreover; one can prove that $||T_\varphi|| = ||\varphi||_\infty$ for every φ. This was shown in [1].

If $\varphi(z) = z$, we will write $S = T_\varphi$ and will call it the shift operator. Its adjoint S^* (which is called the backward shift operator).[1] It acts as

$$S^* f = \frac{f(z) - f(0)}{z}.$$

Note that $S = T_z$, $S^* = T_{\overline{z}}$.

The following are direct properties of the backward shift operator [25]. If f is analytic on the unit disc $\mathbb{D}$, then

$$f(z) = f(0) + zS^* f(z) \quad (z \in \mathbb{D}) \tag{1.2}$$

$$(S^*)^n f(0) = \hat{f}(n) \quad (n = 0, 1, 2 \ldots). \tag{1.3}$$

1.4. Model spaces

In this section we will give the definition of model spaces. The first major contribution to mode spaces occurred in 1970 with the publication of the seminal paper of Douglas, Shapiro, and Shields, see [21]. The study of model spaces is a vast area of research with connections to complex analysis, operator theory and functional analysis.

Definition 1.5. *[9] A subspace $E \subset L^2(\mathbb{T}, m)$ will be called invariant with respect to pointwise multiplication by z if*

$$zE \subset E.$$

Definition 1.6. *[9] If u is an inner function, the corresponding model space K_u is defined to be*

$$K_u := (uH^2)^\perp = \{f \in H^2 : \langle f, uh \rangle = 0 \, \forall h \in H^2\} = H^2 \ominus uH^2.$$

Just as the subspace uH^2 constitutes the non-trivial invariant subspace for the unilateral shift $Sf = zf$ on H^2, the subspace K_u plays an analogous role for the backward shift

$$S^* f = \frac{f(z) - f(0)}{z}.$$

Theorem 1.7. *(A. Beurling, H. Helson) [20] Let $E \subseteq L^2(\mathbb{T}, m)$, with $zE \subset E$, $zE \neq E$. Then there exists a measurable function u (unique* *up to a constant) such that* $|u| = 1$ *a.e on* $\mathbb{T}$ *and* $E = uH^2$.

The function

$$K_\lambda(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \bar{\lambda}z}, \quad \lambda, z \in \mathbb{D},$$

is called the reproducing kernel for K_u.

1.5. Composition operators on Hardy spaces

The composition operator C_α with symbol α is a linear operator defined by $C_\alpha(f)$=$f \circ \alpha$, where $f \circ \alpha$ denotes function composition. The composition operator on the function space is

$$\begin{aligned} C_\alpha : X &\rightarrow Y \\ f &\mapsto f \circ \alpha, \end{aligned}$$

where X and Y are function spaces. One of the most important examples of the composition operator is the composition operator on Hardy space H^2, which appears in [2, 25].

It is well-known that C_α maps H^2 into itself boundedly [25]. This fact is a direct consequence of the next Littlewood's Subordination Principle.

Theorem 1.8. *(Littlewood's Subordination Principle). [25] Suppose* α *is a holomorphic self-map of* $\mathbb{D}$*, with* $\alpha(0) = 0$ *Then for each* $f \in H^2$*,*

$$C_\alpha f \in H^2 \text{ and } ||C_\alpha f|| \leqslant ||f||.$$

Proof. Let f be in H^2,

$$f = \sum_{n=0}^{\infty} \hat{f}(n) z^n.$$

Then,

$$S^* f(z) = \sum_{n=0}^{\infty} \hat{f}(n+1) z^n.$$

Substitute $\alpha(z)$ for z in (1.2), to obtain

$$f(\alpha(z)) = f(0) + \alpha(z)(S^* f)(\alpha(z)) \quad (z \in \mathbb{D}),$$

which is

$$C_\alpha f = f(0) + M_\alpha C_\alpha S^* f. \tag{1.4}$$

Since $\alpha(0) = 0$, the functions on the right hand side of the last equality are orthogonal, hence

$$\begin{aligned} ||C_\alpha f||^2 = |f(0)|^2 \;\; &+ \;\; ||M_\alpha C_\alpha S^* f||^2 \\ &\leqslant \;\; |f(0)|^2 + ||C_\alpha S^* f||^2, \end{aligned} \tag{1.5}$$

where the last inequality follows from $||M_\alpha f|| \leqslant ||f||$, the contraction property of M_α. Now successively substitute $S^* f, (S^*)^2 f, \ldots$ for f in (1.5) to obtain:

$$||C_\alpha S^* f||^2 \leqslant |S^* f(0)|^2 + ||C_\alpha (S^*)^2 f||^2$$

$$||C_\alpha (S^*)^2 f||^2 \leqslant |(S^*)^2 f(0)|^2 + ||C_\alpha (S^*)^3 f||^2$$

$$\vdots \qquad\qquad \vdots$$

$$||C_\alpha(S^*)^n f||^2 \leqslant |(S^*)^n f(0)|^2 + ||C_\alpha(S^*)^{n+1} f||^2$$

Putting all these inequalities together, we get:

$$\|C_\alpha f\|^2 \leqslant \sum_{k=0}^{n} |((S^*)^k f)(0)|^2 + ||C_\alpha(S^*)^{n+1} f||^2 \tag{1.6}$$

for all integers $n \geqslant 1$.

If f is a polynomial of degree n. Then, $(S^\star)^{n+1} f = 0$, so (1.6) becomes

$$\begin{aligned} ||C_\alpha||^2 &\leqslant \sum_{k=0}^{n} |((S^\star)^k f)(0)|^2 \\ &= \sum_{k=0}^{n} |\hat{f}(k)|^2 \\ &= ||f||^2. \end{aligned}$$

Suppose f is not a polynomial. Define the sequence f_n to be n-th partial sum of the Taylor series of f. Since $f_n \longrightarrow f$ in the norm of H^2 so we get $f_n \circ \alpha \longrightarrow f \circ \alpha$ uniformly. So for all fixed $0 < r < 1$ we get

$$\begin{aligned} \mathcal{M}_2(f \circ \alpha, r) &= \lim_{n \longrightarrow \infty} \mathcal{M}_2(f_n \circ \alpha, r) \\ &\leqslant \limsup_{n \longrightarrow \infty} ||f_n \circ \alpha|| \\ &\leqslant \limsup_{n \longrightarrow \infty} ||f_n|| \qquad (||f_n \circ \alpha|| \leqslant ||f_n||) \\ &\leqslant ||f||. \qquad\qquad (||f_n|| \leqslant ||f||). \end{aligned}$$

To complete the proof, let r tend to 1, where

$$\mathcal{M}_2^2(f,r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 r^{2n}.$$

□

For general α we have the following theorem.

Theorem 1.9. *(Littlewood's Theorem). [25] Suppose α is a holomorphic self-map of $\mathbb{D}$. Then C_α is a bounded operator on H^2, and*

$$||C_\alpha|| \leqslant \sqrt{\frac{1+|\alpha(0)|}{1-|\alpha(0)|}}.$$

Littlewood's original proof showed that every composition operator takes H^2 boundedly into itself, so it is essential in the theory of composition operators.

The composition operator is intensively studied on various spaces in the past decades; the list of references is too long, for example [2, 8, 11, 12, 13, 15, 16, 17, 18, 19, 24, 25, 26].

The composition operator is a concrete example in the operator theory. So, numerous classical theorems can be understood in the language of the composition operator.

Chapter 2

de Branges-Rovnyak spaces

2.1. Introduction to de Branges-Rovnyak spaces

In this section we will introduce the de Branges-Rovnyak spaces $\mathcal{H}(b)$, where b is any function in H^∞ such that $b = \{g \in H^\infty : ||g||_\infty \leqslant 1\}$. It is $S^\star$-invariant inside H^2. Our references [6, 23] contain the full treatment of $\mathcal{H}(b)$ spaces, so we will rely on their contents in this section.

The $\mathcal{H}(b)$ spaces have a role to play in many questions in function theory, operator theory, and in the model theory [3, 4]. In spite of all this, the exact contents of the de Branges-Rovnyak spaces $\mathcal{H}(b)$ remain unclear.

A function b in H^∞ with $||b||_\infty \leqslant 1$ is called non-extreme if $\log(1-|b|)$ is integrable. If b a non-extreme point of $\mathcal{H}(b)$, then there exists an outer function a (this function is unique) in the closed unit ball of H^∞ , we call it the Pythagorean mate for b, such that a is positive at the origin and $|a|^2 + |b|^2 = 1$ $a.e.$ on $\mathbb{T}$. The pair (a, b) is called the Pythagorean pair.

de Branges-Rovnyak space $\mathcal{H}(b)$ may be defined in many different equivalent ways. It is defined [23] as the reproducing kernel Hilbert space of the reproducing kernel

$$k^b_\lambda(z) := \frac{1-\overline{b(\lambda)}b(z)}{1-\bar{\lambda}z}, \quad \lambda, z \in \mathbb{D}.$$

Define $f(\lambda) = \langle f, k^b_\lambda\rangle_b$ for all f in $\mathcal{H}(b)$ and for all λ in the unit disc $\mathbb{D}$, where $\langle ., . \rangle_b$ is the inner product of $\mathcal{H}(b)$.

Another definition of $\mathcal{H}(b)$ is the range space of $(I - T_b T_{\bar{b}})^{1/2} H^2$ equipped with the norm which makes $(I - T_b T_{\bar{b}})^{1/2}$ a partial isometry, where T_b is the Toeplitz operator on H^2.

If $||b||_\infty < 1$, then $\mathcal{H}(b)$ is nothing but a renorming version of H^2.

Suppose T and F are self-adjoint operators on Hilbert space H^2, we say that $T \geqslant F$ if $\langle Tf, f \rangle \geqslant \langle Ff, f \rangle$, for all $f \in H^2$.

The following property of Toeplitz operator was originally proved in [1], we present the proof for the sake of completeness.

Proposition 2.1. *let α be bounded and analytic on $\mathbb{D}$ then*

$$T_\alpha T_{\bar{\alpha}} \leqslant T_{\bar{\alpha}} T_\alpha.$$

Proof. Let $f \in H^2$ then

$$\begin{aligned}
\langle T_\alpha T_{\bar{\alpha}} f, f \rangle &= \langle T_{\bar{\alpha}} f, T_{\bar{\alpha}} f \rangle \\
&= ||P(\bar{\alpha} f)||^2_{H^2} \\
&\leqslant ||\bar{\alpha} f||^2_{L^2} \\
&= ||\alpha f||^2_{L^2} \\
&= ||\alpha f||^2_{H^2} \\
&= ||T_\alpha f||^2_{H^2} \\
&= \langle T_{\bar{\alpha}} T_\alpha f, f \rangle.
\end{aligned}$$

□

For any b in the closed unit ball of H^∞ we can define the space $\mathcal{H}(\bar{b})$ as the range space $(I - T_{\bar{b}}T_b)^{1/2}H^2$. Proposition 2.1. implies that $I - T_{\bar{b}}T_b \leqslant I - T_bT_{\bar{b}}$, that also implies that The space $\mathcal{H}(\bar{b})$ is contained in $\mathcal{H}(b)$, moreover, the inclusion map is contractive.

Also there are two important function spaces in $\mathcal{H}(b)$:

$$\mathcal{M}(\alpha) := T_\alpha H^2 \quad and \quad \mathcal{M}(\bar{\alpha}) := T_{\bar{\alpha}} H^2,$$

where b is a non-extreme point and α is the Pythagorean mate of b.

2.2. $\mathcal{H}(b)$ spaces with algebraic symbols

In the previous section we presented the de Branges-Rovnyak spaces $\mathcal{H}(b)$ for any b in the closed unit ball of H^∞. In this section we will study the $\mathcal{H}(b)$ space when $b = q^\rho$, where q is a rational outer function in the unit ball of H^∞ and $\rho \geqslant 0$.

Authors in [7] characterized de Branges–Rovnyak spaces with these algebraic symbols .

If b is rational, or $b = q^\rho$, $\rho \geqslant 0$, then we obtain the following complete description of $\mathcal{H}(b)$ involving the derivatives of the reproducing kernels.

The notation v^ℓ_{ρ,ξ_i} in the following theorem

$$\frac{d^\ell}{d\bar{\xi_i}^\ell} k^{q^\rho}_{\xi_i}(z) = \frac{d^\ell}{d\bar{\xi}_i^\ell}\left(\frac{1 - \overline{q^\rho(\xi_i)}q^\rho(z)}{1 - \bar{\xi}_i z}\right), \quad z \in \mathbb{D} \ \ and \ \ \xi_i \in \mathbb{T}. \qquad (2.1)$$

Theorem 2.2. *[7] Suppose q is a rational outer function in the closed unit ball of H^∞ and $\rho > 0$. Then*

(1) $\mathcal{H}(q^\rho) = \mathcal{H}(q)$ as sets.

(2) If a is the Pythagorean mate for q and the zeros $\xi_1, \ldots, \xi_m$ on $\mathbb{T}$ of a are distinct with corresponding multiplicities $n_1, \ldots, n_m$, then

(i) the functions $v^\ell_{\rho,i} := v^\ell_{\rho,\xi_i}$ belong to $\mathcal{H}(q^\rho)$ for $1 \leqslant i \leqslant m$ and $0 \leqslant \ell \leqslant n_i - 1$. Moreover, all of them are orthogonal to

$$aH^2 = \left(\prod_{i=1}^{m}(z-\xi_i)^{n_i}\right) H^2.$$

(ii) $\mathcal{H}(q^\rho)$ is equal to

$$\left(\prod_{i=1}^{m}(z-\xi_i)^{n_i}\right) H^2 \oplus \bigvee\{v^\ell_{\rho,i} : 0 \leqslant \ell \leqslant n_i - 1, 1 \leqslant i \leqslant m\},$$

where the orthogonality is with respect to the inner product in $\mathcal{H}(q^\rho)$.

Writing $v^\ell_i = v^\ell_{1,i}$, the theorem above implies that $\mathcal{H}(q^\rho)$ is equal to

$$\left(\prod_{i=1}^{m}(z-\xi_i)^{n_i}\right) H^2 \dotplus \bigvee\{v^\ell_i(z) : 0 \leqslant \ell \leqslant n_i - 1, 1 \leqslant i \leqslant m\},$$

where $\dotplus$ denotes the (direct) sum and not necessarily orthogonal in the inner product of $\mathcal{H}(q^\rho)$.

Corollary 2.3. *[7] Suppose q is a polynomial outer function of degree t in the closed unit ball of H^∞. Let (a, q) be a Pythagorean pair. Let M be the number of zeros of a on $\mathbb{T}$ counted with multiplicities. Then the following* *are equivalent:*

(1) $\mathcal{H}(q) = \mathcal{M}(a) \oplus \mathcal{P}_{M-1}$

(2) $M = t$.

Proof. Suppose $\mathcal{H}(q) = \mathcal{M}(a) \oplus \mathcal{P}_{M-1}$, we want to show that $M = t$.

Suppose that $\xi_1, \xi_2, \ldots, \xi_m$ are distinct zeros of a on $\mathbb{T}$ with corresponding multiplicities $n_1, n_2, \ldots, n_m$. So $M = \sum_{i=1}^{m} n_i$.

By Theorem(2.2) we get

$$\mathcal{H}(q) = \mathcal{M}(a) \oplus \bigvee\{v_i^\ell : 0 \leqslant \ell \leqslant n_i - 1, 1 \leqslant i \leqslant m\},$$

where $v_i^\ell = v_{1,\xi_i}^\ell \in \mathcal{H}(q)$. Let $v_i^\ell(z) = \ell! \frac{\omega_i^\ell(z)}{(1-\bar{\xi}_i z)^{\ell+1}}$,(we can prove the previous equation by Leibniz's rule) where

$$\omega_i^\ell(z) = z^\ell \left(1 - \overline{q(\xi_i)} q(z)\right) - q(z) \sum_{k=1}^{\ell} \frac{1}{k!} \overline{q^{(k)}(\xi_i)} z^{\ell-k} (1 - \bar{\xi}_i z)^k,$$

and notice that ω_i^ℓ is a polynomial of degree less than or equal to $t + \ell$ and ω_i^0 is a polynomial of degree $t + \ell$. Because $v_i^\ell \in \mathcal{H}(q) \subset H^2$ we can write $\omega_i^\ell(z)$ as $(1 - \bar{\xi}_i z)^{\ell+1} \tilde{\omega}_i^\ell(z)$, where $\tilde{\omega}_i^\ell$ is a polynomial of degree less than or equal to $t - 1$ and $\tilde{\omega}_i^0$ is a polynomial of degree $t - 1$.

Hence $v_i^\ell = \tilde{\omega}_i^\ell(z)$ is a polynomial of degree less than or equal to $t - 1$ and of degree exactly equal to $t - 1$ when $\ell = 0$.

But if $\mathcal{H}(q) = \mathcal{M}(a) \oplus \mathcal{P}_{M-1}$, then we get

$$\bigvee\{v_i^\ell : 0 \leqslant \ell \leqslant n_i - 1, 1 \leqslant i \leqslant m\} = \mathcal{P}_{M-1},$$

Thus $v_i^0 \in \mathcal{P}_{M-1}$, and so $t \leqslant M$.

On the other side, because v_i^ℓ is a polynomial of degree less than or equal to $t-1$, then $v_i^\ell \in \mathcal{P}_{t-1}$, thus $\mathcal{P}_{M-1} \subset \mathcal{P}_{t-1}$, and $M \leqslant t$. So $M = t$.

Conversely, suppose that $t = M$. Then, by the same proof as above, we get $\bigvee\{v_i^\ell : 0 \leqslant \ell \leqslant n_i - 1, 1 \leqslant i \leqslant m\} \subset \mathcal{P}_{M-1}$.
Also $\bigvee\{v_i^\ell : 0 \leqslant \ell \leqslant n_i - 1, 1 \leqslant i \leqslant m\}$ and $\mathcal{P}_{M-1}$ have the same dimension. Indeed, we can form polynomials $p_{i,\ell}$ s.t

$$p_{i,\ell}^{\ell}(\xi_i) = 1, \quad p_{i,\ell}^{(k)}(\xi_i) = 0, \; 1 \leqslant k \leqslant n_i - 1, k \neq \ell.$$

Because polynomials are clearly in $\mathcal{H}(q)$, then

$$\langle p_{i,\ell}, v_i^k \rangle_q = p_{i,\ell}^k(\xi_i) = \begin{cases} 1 & if \; k = \ell \\ 0 & otherwise. \end{cases}$$

That implies that v_i^ℓ, $0 \leqslant \ell \leqslant n_i - 1, 1 \leqslant i \leqslant m$, are linearly independent. Hence,

$$\dim \bigvee\{v_i^\ell : 0 \leqslant \ell \leqslant n_i - 1, 1 \leqslant i \leqslant m\} = \sum_{i=1}^{m} n_i = M.$$

Hence we get

$$\bigvee\{v_i^\ell : 0 \leqslant \ell \leqslant n_i - 1, 1 \leqslant i \leqslant m\} = \mathcal{P}_{M-1}.$$

□

Theorem 2.4. *(Fejér-Riesz Theorem) [22] A trigonometric polynomial* $s(e^{it}) = \sum_{-m}^{m} c_j e^{ijt}$ *which assumes real and nonnegative values for all real* t *is expressible in the form*

$$s(e^{it}) = |p(e^{it})|^2$$

for some polynomial $p(z) = \sum_0^m a_j z^j$. *The polynomial* $p(z)$ *can be chosen so that it has no roots in* $\mathbb{D}$, *and then it is unique except for a multiplicative constant of modulus one.*

That proof is based on the fact that $s(e^{it}) = |p(e^{it})|^2$ satisfies $\overline{s(1/\bar{z})} = s(z)$. If $c_n \neq 0$, then $w(z) = z^n s(z)$ is a polynomial of degree $2n$ with $w(0) \neq 0$. The roots of $w(z)$ occur in pairs $\varphi, 1/\bar{\varphi}$. It follows that

$$s(z) = c \prod_{j=1}^{m} (z - \varphi_j)(z^{-1} - \bar{\varphi}_j)$$

for some $c > 0$ and where $\varphi_1, \ldots, \varphi_m$ satisfy $|\varphi_j| \geqslant 1$ for $1 \leqslant j \leqslant m$. The desired polynomial p is

$$p(z) = \sqrt{c} \prod_{j=1}^{m} (z - \varphi_j).$$

Example 2.5. *[7] Let*

$$q(z) = \frac{1}{2}(1 + z),$$

and note that q *is rational outer and* $||q||_\infty \leqslant 1$ *on* $\mathbb{D}$.*Also note that* $a(z) = \frac{1}{2}(1 - z)$ *is the Pythagorean mate for* q, *and* $z = 1$ *is the only zero of* a *and* *the zero is of order* 1*. A computation leads us to*

$$v^0_{1,1}(z) = \frac{1-\overline{q(1)}q(z)}{1-z} = \frac{1}{2}.$$

By Theorem 2.2

$$\mathcal{H}(q) = (z-1)H^2 \oplus \mathbb{C}.$$

Because $\mathcal{H}(q^\rho) = \mathcal{H}(q)$ *for* $\rho > 0$ *we get*

$$\mathcal{H}(q^\rho) = (z-1)H^2 \dotplus \mathbb{C} = (z-1)H^2 \oplus \mathbb{C}\frac{1-\left(\frac{1+z}{2}\right)^\rho}{1-z}.$$

Example 2.6. *[7] Suppose* $a(z) = c\prod_{i=1}^m (z-\xi_i)^{n_i}$*, where the* ξ_i*'s are distinct points of* $\mathbb{T}$*,* $n_i \in \mathbb{N}$ *and* $c \in \mathbb{C}$ *such that* $||a||_\infty \leqslant 1$*. Let* (a, q) *be a Pythagorean pair. By Theorem 2.4 we get that q is a polynomial of degree* $M := \sum_{i=1}^m n_i$*. Then, according to Corollary* 2.3*, we have*

$$\mathcal{H}(q) = \prod_{i=1}^m (z-\xi_i)^{n_i} H^2 \oplus \mathcal{P}_{M-1}.$$

Example 2.7. *[7] Let*

$$q(z) = \frac{1}{2}(1-z)(1+z).$$

Note that q *is rational outer and* $||q||_\infty \leqslant 1$*. A computation leads us to*

$$1 - |q(e^{it})|^2 = \frac{1}{4}e^{-2it} + \frac{1}{4}e^{2it} + \frac{1}{2}.$$

Define

$$s(z) = \frac{z^{-2}}{4} + \frac{z^2}{4} + \frac{1}{2}$$

and

$$w(z) = z^2 s(z) = \frac{z^4}{4} + \frac{z^2}{2} + \frac{1}{4} = \frac{1}{4}(z-i)^2(z+i)^2.$$

The zeros occur in pair $i = \frac{1}{\bar{i}}$ *and* $-i = \frac{1}{\overline{-i}}$ *as the proof of Theorem 2.4. Thus*

$$a(z) = \frac{1}{2}(z-i)(z+i).$$

The function a *has two zeros,* $z = i$ *and* $z = -i$*, and both of them are of order one. And*

$$v_{1,i}^0(z) = \frac{1}{2i}(z+i), \;\; v_{1,-i}^0(z) = \frac{1}{2i}(z-i),$$

so we get

$$\mathcal{H}(q) = (z-i)(z+i)H^2 \oplus \bigvee\{z+i, z-i\}.$$

Because $\mathcal{H}(q^\rho) = \mathcal{H}(q)$ *we get*

$$\mathcal{H}(q^\rho) = (z-i)(z+i)H^2 \dot{+} \bigvee\{z+i, z-i\}.$$

Example 2.8. *[7] Let*

$$q(z) = \frac{1}{4}(z+1)^2.$$

Note that q *in the unit ball of* H^∞ *and is rational outer. By the proof of*

Theorem 2.4 note that

$$1-|q(e^{it})|^2=-\frac{e^{-it}}{4}-\frac{e^{it}}{4}-\frac{e^{-2it}}{16}-\frac{e^{2it}}{16}+\frac{5}{8}.$$

Define

$$s(z)=-\frac{z^2}{16}-\frac{1}{16z^2}-\frac{z}{4}-\frac{1}{4z}+\frac{5}{8}$$

and

$$w(z)=z^2s(z)=-\frac{z^4}{16}-\frac{z^3}{4}+\frac{5z^2}{8}-\frac{z}{4}-\frac{1}{16}=-\frac{1}{16}(-1+z)^2(1+6z+z^2).$$

The zeros of s are at $z=1, z=-3-2\sqrt{2}\approx-5.82843, z=-3+2\sqrt{2}\approx-0.171573$. *Thus (by Theorem 2.4)* $a(z)=c(z-1)$, *for some c. There are one zero of a at* $z=1$ *with order 1. And*

$$v^0_{1,1}(z)=\frac{z+3}{4}.$$

so

$$\mathcal{H}(q)=(z-1)H^2\oplus\mathbb{C}(Z+3).$$

Also

$$\mathcal{H}(q^\rho)=(z-1)H^2\dot{+}\mathbb{C}(z+3).$$

2.3. More examples of $\mathcal{H}(b)$

In the previous section we saw the functions of $\mathcal{H}(b)$ spaces with algebraic symbols which Fricain et al. derived it in [7], in this section we will construct some more examples of functions in the de Branges–Rovnyak spaces.

Example 2.9. *Let*

$$q(z) = \frac{1}{2}(1 - iz)$$

Note that q *is rational outer and* $\|q\|_\infty \leqslant 1$*. Let*

$$a(z) = \frac{1}{2}(iz + 1),\ then$$

a *is the Pythagorean mate for* q *on* $\mathbb{T}$*. Indeed,* $a(0) > 0$ *and*

$$\begin{aligned} |a|^2 + |q|^2 &= |\frac{1}{2}(iz+1)|^2 + |\frac{1}{2}(1-iz)|^2 \\ &= \frac{1}{4}|i\cos\theta - \sin\theta + 1|^2 + \frac{1}{4}|1 - i\cos\theta + \sin\theta|^2 \\ &= \frac{1}{4}(2\sin\theta + 2) + \frac{1}{4}(2 - 2\sin\theta) \\ &= 1. \end{aligned}$$

And a *has only zero* $z = i$ *and is of order one. It is easy to see that*

$$v^0_{1,i}(Z) = \frac{1 - \overline{q(i)}q(z)}{1 + iz} = \frac{1}{2}.$$

In this case by Theorem 2.2

$$\mathcal{H}(q) = (z - i)H^2 \oplus \mathbb{C}.$$

Because, $\mathcal{H}(q^{\rho}) = \mathcal{H}(q)$ *for* $r > 0$ *we have*

$$\mathcal{H}(q^{\rho}) = (z-i)H^2 \dotplus \mathbb{C}.$$

In the previous example we defined $q(z) = \frac{1}{2}(1-iz)$ and $a(z) = \frac{1}{2}(iz+1)$, but in the following example we replace q by a.

Example 2.10. *Let*

$$q(z) = \frac{1}{2}(iz+1) \quad and \quad a(z) = \frac{1}{2}(1-iz).$$

Note that q is rational outer and $||q||_{\infty} \leqslant 1$, $a(0) > 0$, *thus* a *is the Pythagorean mate for q on* $\mathbb{T}$. *Notice that a has one zero of order one at* $z = -i$.

$v^{0}_{1,-i}(z) = \frac{1-\overline{q(-i)}q(z)}{1-iz} = \frac{1}{2}$, *hence according to Theorem* 2.2, *we get a new example of* $\mathcal{H}(b)$ *spaces*

$$\mathcal{H}(q) = (z+i)H^2 \oplus \mathbb{C}.$$

Because, $\mathcal{H}(q^{\rho}) = \mathcal{H}(q)$ *for* $r > 0$, *we have*

$$\mathcal{H}(q^{\rho}) = (z+i)H^2 \dotplus \mathbb{C}.$$

We present the following new examples on $\mathcal{H}(b)$ by changing q and a in Examples 2.5, 2.7.

Example 2.11. *Let*

$$q(z) = \frac{1}{2}(1-z)$$

Note that $||q||_\infty \leqslant 1$ *and is rational outer, by using Example* 2.5 $a(z) = \frac{1}{2}(1+z)$ *and* $a(0) > 0$, *thus* $a(z)$ *is the Pythagorean mate for q, and* $a(z)$ *has one zero of order one at* $z = -1$. *A computation leads us to*

$$v^0_{1,-1}(z) = \frac{1 - \overline{q(-1)}q(z)}{1+z} = \frac{1}{2}.$$

Hence

$$\mathcal{H}(q) = (z+1)H^2 \oplus \mathbb{C}.$$

Because $\mathcal{H}(q^\rho) = \mathcal{H}(q)$, *we have*

$$\mathcal{H}(q^\rho) = (z+1)H^2 \dot{+} \mathbb{C}.$$

Example 2.12. *Let*

$$q(z) = \frac{1}{2}(z-i)(z+i).$$

Note that $||q||_\infty \leqslant 1$ *and is rational outer. According to Example* 2.7 *let* $a(z) = \frac{1}{2}(1-z)(1+z)$, $a(0) > 0$, *hence a is the Pythagorean mate for q, and a has two zeros of order one at* $z = 1, z = -1$. *A computation leads us to*

$$v^0_{1,1}(z) = \frac{1}{2}(1+z) \quad and \quad v^0_{1,-1}(z) = \frac{1}{2}(1-z).$$

In this case by Theorem 2.2

$$\mathcal{H}(q) = (z-1)(z+1)H^2 \oplus \bigvee\{1+z, 1-z\}.$$

Because $\mathcal{H}(q^\rho) = \mathcal{H}(q)$*, we have*

$$\mathcal{H}(q^\rho) = (z-1)(z+1)H^2 \dot{+} \bigvee\{1+z, 1-z\}.$$

Chapter 3

Images of composition operators

We know that for any analytic function α with $\alpha : \mathbb{D} \longrightarrow \mathbb{D}$

$$C_\alpha : \mathcal{H}(b) \longrightarrow H^2.$$

In this chapter we will try to describe the image of the de Branges-Rovnyak spaces under the composition operator, in other words, we will try to find $X \subseteq H^2$ such that $C_\alpha(\mathcal{H}(b)) \subseteq X$.

3.1. Composition operator on $\mathcal{H}(b)$ into itself

In this section we will study the composition operator C_α on $\mathcal{H}(b) = (z - \zeta)H^2 \oplus \mathbb{C}$ where α is analytic and maps the unit disc $\mathbb{D}$ into itself, $\zeta \in \mathbb{T}$.

We have constructed sufficient conditions on α such that C_α maps $\mathcal{H}(b)$ into itself in the following theorem.

Theorem 3.1. *If $\mathcal{H}(b) = (z - \zeta)H^2 \oplus \mathbb{C}$, and if $\alpha(\zeta) = \zeta$ then*

$$C_\alpha : \mathcal{H}(b) \longrightarrow \mathcal{H}(b).$$

where α is an analytic function which maps the unit disc $\mathbb{D}$ into itself, and $\zeta \in \mathbb{T}$.

Proof. Let $f \in \mathcal{H}(b)$ such that $f = (z - \zeta)g + c$, $g \in H^2$ and c is constant, then

$$\begin{aligned} f \circ \alpha &= (\alpha(z) - \zeta)g(\alpha(z)) + c \\ &= (z - \zeta)h(z)g(\alpha(z)) + c \in \mathcal{H}(b), \end{aligned}$$

by Theorem 1.8 $g(\alpha(z)) \in H^2$, and $h(z) \in H^\infty$. □

The following is an application on the previous theorem.

Example 3.2. *Let* $b = \frac{1}{2}(1 + z)$ *and suppose that* $\alpha(1) = 1$, *then*

$$C_\alpha : \mathcal{H}(b) \longrightarrow \mathcal{H}(b),$$

where $\mathcal{H}(b) = (z - 1)H^2 \oplus \mathbb{C}$. *Indeed, suppose*

$$\begin{aligned} f &= (z - 1)g + c \\ C_\alpha f &= (\alpha(z) - 1)g + c \\ &= (z - 1)hg + c \in \mathcal{H}(b). \end{aligned}$$

Theorem 3.3. *Suppose* $\mathcal{H}(b) = (z - \zeta)H^2 \oplus \mathbb{C}$, $\zeta \in \mathbb{T}$, *and* $b = \frac{1}{c}(1 + \gamma z)$ *such that* γ *is constant. Let* α *be a rational analytic function, such that* α *maps the unit disc* $\mathbb{D}$ *into itself, then*

$$C_\alpha : \mathcal{H}(b) \longrightarrow \mathcal{H}(b)$$

if and only if $\alpha(\zeta) = \zeta$.

Proof. Suppose $C_\alpha : \mathcal{H}(b) \longrightarrow \mathcal{H}(b)$. Let

$$\begin{aligned} f = k_0^b &= 1 - \overline{b(0)}b(z) \\ &= 1 - \frac{1}{c^2}(1+\gamma z) \\ &= 1 - \frac{1}{c^2} - \frac{1}{c^2}\gamma z \\ &= 1 - \frac{1}{c^2} - \frac{\gamma}{c^2}(z - \zeta + \zeta) \\ &= \frac{-\gamma}{c^2}(z-\zeta) + 1 - \frac{\zeta\gamma + 1}{c^2} \in \mathcal{H}(b), \end{aligned}$$

then

$$C_\alpha f = \frac{-\gamma}{c^2}(\alpha(z) - \zeta) + 1 - \frac{\zeta\gamma+1}{c^2} \in \mathcal{H}(b),$$

therefore, we can write $(\alpha(z)-\zeta)$ as $(z-\zeta)h$, where $h \in H^2$, thus $\alpha(\zeta) = \zeta$.

Conversely, suppose $\alpha(\zeta) = \zeta$, by Theorem 3.1 we get

$C_\alpha : \mathcal{H}(b) \longrightarrow \mathcal{H}(b)$. □

3.2. Composition operator on $\mathcal{H}(\frac{1}{2}(1+z))$

In this section we will study the composition operator C_α on $\mathcal{H}(b)$ where the α is the finite Blaschke product $B(z)$, and $b = \frac{1}{2}(1+z)$.

Theorem 3.4. *If* $b = \frac{1}{2}(z+1)$ *and* $B(z) = \left(\frac{a-z}{1-\bar{a}z}\right)^2$ *then*

$$C_B : \mathcal{H}(\frac{1}{2}(1+z)) \longrightarrow \mathcal{H}(\frac{1}{2}(z-i)(z+i))$$

where a is real and $|a| \leqslant 1$.

Proof. Let $f \in \mathcal{H}(\frac{1}{2}(1+z))$ such that $f = (z-1)g + c$ where $g \in H^2$, $c \in \mathbb{C}$ then

$$
\begin{aligned}
C_B f = f(B(z)) &= (B(z)-1)g + c \\
&= \left(\left(\frac{a-z}{1-\bar{a}z}\right)^2 - 1\right) g + c \\
&= \left(\frac{(a-z)^2 - (1-\bar{a}z)^2}{(1-\bar{a}z)^2}\right) g + c \\
&= \left(\frac{z^2(1-\bar{a}^2) + z(2\bar{a} - 2a) + a^2 - 1}{(1-\bar{a}z)^2}\right) g + c \\
&= \left(\frac{z^2(1-a^2) - (1-a^2)}{(1-az)^2}\right) g + c \;\; (since\, a\, is\, real) \\
&= (z^2-1)\frac{(1-a^2)g}{(1-az)^2} + c \\
&= (z-1)(z+1)\frac{(1-a^2)g}{(1-az)^2} + c \\
&\in \mathcal{H}(\frac{1}{2}(z-i)(z+i)) = (z-1)(z+1)H^2 \oplus \bigvee\{1+z, 1-z\}.
\end{aligned}
$$

□

If $a = i$, then the expression $\left(\frac{z^2(1-\bar{a}^2)+z(2\bar{a}-2a)+a^2-1}{(1-\bar{a}z)^2}\right) g + c$ in the previous proof becomes

$$\begin{aligned}
\frac{2z^2-4iz-2}{(1+iz)^2}g+c &= 2g\frac{(z^2-2iz-1)}{(1+iz)^2}+c \\
&= 2g\frac{(z-i)^2}{(1+iz)^2}+c \\
&= 2g\frac{\bar{i}i(z-i)^2}{(1+iz)^2}+c \\
&= 2g\frac{\bar{i}(iz+1)(z-i)}{(1+iz)^2}+c \\
&= (z-i)\frac{-i2g}{(1+iz)}+c \\
&\in \mathcal{H}(\frac{1}{2}(1-iz)) = (z-i)H^2 \oplus \mathbb{C}
\end{aligned}$$

Similarly for $a=-i$ we get

$$C_B : \mathcal{H}(\frac{1}{2}(1+z)) \longrightarrow \mathcal{H}(\frac{1}{2}(1+iz)).$$

If $a = id$ and $|d| < 1$, then the expression $\left(\frac{z^2(1-\bar{a})+z(2\bar{a}-2a)+a^2-1}{(1-\bar{a}z)^2}\right) g + c$ from the previous proof becomes

$$\begin{aligned}
&= \frac{z^2(1+d^2)+z(-4id)-(1+d^2)}{(1+idz)^2}g+c \\
&= \left(z^2-\frac{4id}{1+d^2}z-1\right)\frac{1+d^2}{(1+idz)^2}g+c \\
(if\, d=\frac{1}{2}) &= \left(z^2-\frac{8i}{5}z-1\right)\frac{5g}{4(1+iz/2)^2}+c \\
&= \left(z-\frac{4i-3}{5}\right)\left(z-\frac{4i+3}{5}\right)\frac{5g}{4(1+iz/2)^2}+c
\end{aligned}$$

note that $\frac{4i-3}{5}$ and $\frac{4i+3}{5}$ lie on $\mathbb{T}$ because $|\frac{4i-3}{5}| = 1$, $|\frac{4i+3}{5}| = 1$. Ac-

cording to Example 2.5 there is q such that q is a Pythagorean mate for $c\left(z-\frac{4i-3}{5}\right)\left(z-\frac{4i+3}{5}\right)$, where c is suitable constant, and according to Corollary 2.2, we have $\mathcal{H}(q)=\left(z-\frac{4i-3}{5}\right)\left(z-\frac{4i+3}{5}\right)H^2\oplus\mathcal{P}_1$.
Consequently $\left(z-\frac{4i-3}{5}\right)\left(z-\frac{4i+3}{5}\right)\frac{(-5)}{16(1+iz/2)^2}+\frac{1}{2}\in\mathcal{H}(q)$ and $\mathcal{H}(q)=\left(z-\frac{4i-3}{5}\right)\left(z-\frac{4i+3}{5}\right)H^2\oplus\mathcal{P}_1$.

In general, if a is complex with $|a|\leqslant 1$ it is not necessarily true that C_B maps $\mathcal{H}(\frac{1}{2}(1+z))$ into $\mathcal{H}(\frac{1}{2}(z-i)(z+i))$. But it maps $\mathcal{H}(\frac{1}{2}(1+z))$ to another $\mathcal{H}(b)$, but until now we do not know what is b.

Theorem 3.5. *If* $b=\frac{1}{2}(z+1)$ *and* $B(z)=\frac{z-a_1}{1-\overline{a_1}z}\cdot\frac{z-a_2}{1-\overline{a_2}z}$ *then*

$$C_B:\mathcal{H}(\frac{1}{2}(z+1))\longrightarrow\mathcal{H}(\frac{1}{2}(z-i)(z+i))$$

where a_1, a_2 *are real and* $|a_1|\leqslant 1$, $|a_2|\leqslant 1$.

Proof. Let $f\in\mathcal{H}(\frac{1}{2}(1+z))$ such that $f=(z-1)g+c$ where $g\in H^2$, $c\in\mathbb{C}$ then

$$\begin{aligned}
C_B f = f(B(z)) &= (B(z)-1)g + c \\
&= \left(\frac{z-a_1}{1-\overline{a_1}z}\cdot\frac{z-a_2}{1-\overline{a_2}z} - 1\right) g + c \\
&= \frac{z^2(1-\bar{a_1}\bar{a_2}) + z(-a_1 - a_2 + \bar{a_1} + \bar{a_2}) + a_1a_2 - 1}{(1-\overline{a_1}z)(1-\overline{a_2}z)} g + c \\
&= \frac{z^2(1-a_1a_2) - (1-a_1a_2)}{(1-a_1z)(1-a_2z)} g + c \quad (since\, a\, is\, real) \\
&= \frac{(z^2-1)(1-a_1a_2)}{(1-a_1z)(1-a_2z)} g + c \\
&= (z-1)(z+1)\frac{(1-a_1a_2)g}{(1-a_1z)(1-a_2z)} + c \\
&\in \mathcal{H}(\frac{1}{2}(z-i)(z+i)).
\end{aligned}$$

Theorem 3.6. *If $b = \frac{1}{2}(1+z)$ and $B(z) = \left(\frac{a-z}{1-\bar{a}z}\right)$ then*

$$C_B : \mathcal{H}(\frac{1}{2}(1+z)) \longrightarrow \mathcal{H}(\frac{1}{2}(1-z))$$

where a is real and $|a| \leqslant 1$.

Proof. Let $f \in \mathcal{H}(\frac{1}{2}(1+z))$ such that $f = (z-1)g + c$ where $g \in H^2$,

$c \in \mathbb{C}$ then

$$\begin{aligned} C_B f = f(B(z)) &= (B(z)-1)g + c \\ &= \left(\frac{a-z}{1-\bar{a}z} - 1\right) g + c \\ &= \left(\frac{a-z-1+\bar{a}z}{1-\bar{a}z}\right) g + c \\ &= \left(\frac{z(\bar{a}-1)+(a-1)}{1-\bar{a}z}\right) g + c \\ &= (z+1)\left(\frac{a-1}{1-az}\right) g + c \quad (since\, a\, is\, real) \\ &\in \mathcal{H}(\frac{1}{2}(1-z)) = (z+1)H^2 \oplus \mathbb{C}. \end{aligned}$$

□

But if a is complex, then is not guaranteed C_B maps $\mathcal{H}(\frac{1}{2}(1+z))$ into $\mathcal{H}(\frac{1}{2}(1-z))$. For example if $a = i$, C_B maps $\mathcal{H}(\frac{1}{2}(1+z))$ to $\mathcal{H}(\frac{1}{2}(1-iz))$. Indeed, $\left(\frac{z(\bar{a}-1)+(a-1)}{1-\bar{a}z}\right) g + c$ in the previous proof becomes

$$\begin{aligned} &= \left(\frac{z(-i-1)+(i-1)}{1+iz}\right) g + c \\ &= \left(\frac{z(-i-1)+\bar{i}(-i-1)}{1+iz}\right) g + c \\ &= (z-i)\frac{(-i-1)g}{(1+iz)} + c \\ &\in \mathcal{H}(\frac{1}{2}(1-iz)) = (z-i)H^2 \oplus \mathbb{C}. \end{aligned}$$

Similarly for $a = -i$, $C_B : \mathcal{H}(\frac{1}{2}(1+z)) \longrightarrow \mathcal{H}(\frac{1}{2}(1+iz))$.

3.3. Composition operator on $\mathcal{H}(\frac{1}{2}(1-z))$

In the previous section we studied the composition operator C_α on $\mathcal{H}(b)$ where α is finite Blaschke product $B(z)$, and $b = \frac{1}{2}(1+z)$, in this section we will replace $b = \frac{1}{2}(1+z)$ by $b = \frac{1}{2}(1-z)$.

Theorem 3.7. *If* $b = \frac{1}{2}(1-z)$ *and* $B(z) = \left(\frac{a-z}{1-\bar{a}z}\right)$ *then*

$$C_B : \mathcal{H}(\frac{1}{2}(1-z)) \longrightarrow \mathcal{H}(\frac{1}{2}(1+z))$$

where a *is real and* $|a| \leqslant 1$.

Proof. Let $f \in \mathcal{H}(\frac{1}{2}(1-z))$ such that $f = (z+1)g + c$ where $g \in H^2$, $c \in \mathbb{C}$ then

$$\begin{aligned}
C_B f = f(B(z)) &= (B(z)+1)g + c \\
&= \left(\frac{a-z}{1-\bar{a}z} + 1\right) g + c \\
&= \left(\frac{a-z+1-\bar{a}z}{1-\bar{a}z}\right) g + c \\
&= \left(\frac{z(-\bar{a}-1) - (-a-1)}{1-\bar{a}z}\right) g + c \\
&= (z-1)\left(\frac{-a-1}{(1-az)}\right) g + c \quad (since\, a\, is\, real) \\
&\in \mathcal{H}(\frac{1}{2}(1+z)) = (z-1)H^2 \oplus \mathbb{C}.
\end{aligned}$$

□

But if a is complex, then is not guaranteed C_B maps $\mathcal{H}(\frac{1}{2}(1-z))$ into

$\mathcal{H}(\frac{1}{2}(1+z))$. For example if $a = i$, C_B maps $\mathcal{H}(\frac{1}{2}(1-z))$ to $\mathcal{H}(\frac{1}{2}(1-iz))$. Indeed, $\left(\frac{z(-\bar{a}-1)-(-a-1)}{1-\bar{a}z}\right) g + c$ becomes

$$
\begin{aligned}
&= \left(\frac{z(-1+i)+(i+1)}{1+iz}\right) g + c \\
&= \left(\frac{z(-1+i)+\bar{i}(-1+i)}{1+iz}\right) g + c \\
&= (z-i)\frac{(-1+i)}{(1+iz)} g + c \\
&\in \mathcal{H}(\frac{1}{2}(1-iz)) = (z-i)H^2 \oplus \mathbb{C}.
\end{aligned}
$$

Similarly for $a = -i$, $C_B : \mathcal{H}(\frac{1}{2}(1z)) \longrightarrow \mathcal{H}(\frac{1}{2}(1+iz))$.

Theorem 3.8. *If* $b = \frac{1}{2}(1-z)$ *and* $B(z) = z\left(\frac{a-z}{1-\bar{a}z}\right)$ *then*

$$C_B : \mathcal{H}(\frac{1}{2}(1-z)) \longrightarrow \mathcal{H}(\frac{1}{2}(z-i)(z+i))$$

where a *is real and* $|a| \leqslant 1$.

Proof. Let $f \in \mathcal{H}(\frac{1}{2}(1-z))$ such that $f = (z+1)g + c$ where $g \in H^2$,

$c \in \mathbb{C}$ then

$$
\begin{aligned}
C_B f = f(B(z)) &= (B(z)+1)g + c \\
&= \left(z\frac{a-z}{1-\bar{a}z} + 1\right) g + c \\
&= \left(\frac{az - z^2 + 1 - \bar{a}z}{1-\bar{a}z}\right) g + c \\
&= \left(\frac{-z^2 + (a-\bar{a})z + 1}{1-\bar{a}z}\right) g + c \\
&= \left(\frac{1-z^2}{1-az}\right) g + c \quad (since\, a\, is\, real) \\
&= (z-1)(z+1)\frac{-g}{(1-az)} + c \\
&\in \mathcal{H}(\frac{1}{2}(z-i)(z+i)) = (z-1)(z+1)H^2 \oplus \mathbb{C}.
\end{aligned}
$$

□

3.4. Composition operator on $\mathcal{H}(\frac{1}{2}(z-i)(z+i))$

In this section we will study C_α on $\mathcal{H}(\frac{1}{2}(z-i)(z+i))$ when α is rational.

Theorem 3.9. *Let α be an analytic function which maps the unit disc into itself, let $b = \frac{1}{2}(z-i)(z+i)$, then $C_\alpha : \mathcal{H}(\frac{1}{2}(z-i)(z+i)) \longrightarrow \mathcal{H}(\frac{1}{2}(1+z))$ iff $\alpha(1) = 1$, where $\mathcal{H}(\frac{1}{2}(z-i)(z+i)) = (z-1)(z+1)H^2 \oplus \bigvee\{1+z, 1-z\}$*

Proof. Suppose $C_\alpha : \mathcal{H}(\frac{1}{2}(z-i)(z+i)) \longrightarrow \mathcal{H}(\frac{1}{2}(1+z))$. Let

$$\begin{aligned} f = k_0^b &= 1 - \frac{1}{4}(z+i)(z-i) \\ &= 1 - \frac{1}{4}(z^2+1) \\ &= \frac{3}{4} - \frac{z^2}{4} \\ &= \frac{3}{4} - \frac{((z-1)+1)((z+1)-1)}{4} \\ &= \frac{-1}{4}(z-1)(z+1) + \frac{1}{2} \in \mathcal{H}(b), \end{aligned}$$

then

$$\begin{aligned} C_\alpha f = f(\alpha(z)) &= \frac{-1}{4}(\alpha(z)-1)(\alpha(z)+1) + \frac{1}{2} \\ &= \frac{-1}{4}(\alpha^2(z)-1) + \frac{1}{2} \end{aligned}$$

but since $\frac{-1}{4}(\alpha^2(z)-1) + \frac{1}{2}$ belong to $\mathcal{H}(\frac{1}{2}(1+z))$,

thus $(\alpha^2(z)-1) = (z-1)h$ where $h \in H^2$. Hence $\alpha(1) = 1$.

Conversely, suppose $\alpha(1) = 1$, then

$$\begin{aligned} C_\alpha f = f(\alpha(z)) &= \frac{-1}{4}(\alpha(z)-1)(\alpha(z)+1) + \frac{1}{2} \\ &= \frac{-1}{4}(\alpha^2(z)-1) + \frac{1}{2} \\ since\, \alpha(1) = 1 & \\ &= \frac{-1}{4}(z-1)g + \frac{1}{2} \in \mathcal{H}(\frac{1}{2}(1+z)), \end{aligned}$$

where $g \in H^2$. □

Theorem 3.10. *Let* $b = \mathcal{H}(\frac{1}{2}(z-i)(z+i))$, *let* $\alpha(z) = B(z) = \frac{a-z}{1-\bar{a}z}$, *then*

$$C_B : \mathcal{H}(\frac{1}{2}(z-i)(z+i)) \longrightarrow \mathcal{H}(\frac{1}{2}(z-i)(z+i))$$

where a *is real such that* $|a| \leqslant 1$.

Proof. Let $f = k_0^b = \frac{-1}{4}(z-1)(z+1) + \frac{1}{2} \in \mathcal{H}(b)$, then

$$\begin{aligned}
C_B f = f(B(z)) &= \frac{-1}{4}(B(z)-1)(B(z)+1) + \frac{1}{2} \\
&= \frac{-1}{4}(B(z)^2 - 1) + \frac{1}{2} \\
&= \frac{-1}{4}\left(\left(\frac{a-z}{1-\bar{a}z}\right)^2 - 1\right) + \frac{1}{2} \\
&= \frac{-1}{4}\left(\left(\frac{a-z}{1-\bar{a}z}\right)^2 - 1\right) + \frac{1}{2} \\
&= \frac{-1}{4}\left(\frac{(a-z)^2 - (1-\bar{a}z)^2}{(1-\bar{a}z)^2}\right) + \frac{1}{2} \\
&= \frac{-1}{4}\left(\frac{z^2(1-\bar{a}) + z(2\bar{a} - 2a) + a^2 - 1}{(1-\bar{a}z)^2}\right) + \frac{1}{2} \\
&= \frac{-1}{4}\left(\frac{z^2(1-a^2) - (1-a^2)}{(1-az)^2}\right) + \frac{1}{2} \quad (since\, a\, is\, real) \\
&= \frac{-1}{4}(z^2-1)\frac{(1-a^2)}{(1-az)^2} + \frac{1}{2} \\
&= \frac{-1}{4}(z-1)(z+1)\frac{(1-a^2)}{(1-az)^2} + \frac{1}{2} \\
&\in \mathcal{H}(\frac{1}{2}(z-i)(z+i)).
\end{aligned}$$

□

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مؤثرات التركيب على فضاءات دي برانجز روفنياك

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الملخص

فضاءات دي برانجز روفنياك هي محتواة في فضاء هيلبرت، وتلعب دوراً مهماً في بعض فروع الرياضيات المختلفة ويوجد العديد من الكتب و الأوراق العلمية تتحدث عنها، بالرغم من ذلك يبقى تكوين هذه الفضاءات غير واضح حتى الآن.

في عام 2016 صدرت ورقة علمية توصل بعض العلماء بها إلى أمثلة جبرية ملموسة عن هذه الفضاءات.

بدورنا قمنا نحن بالانطلاق من هناك بأخذ هذه الأمثلة ودراسة مؤثرات التركيب عليها وتحديد مجال فضاءات دي برانجز روفنياك تحت هذه المؤثرات.

أيضاً قمنا بوضع بعض الشروط على مؤثرات التركيب ليكون المجال الناتج لفضاءات دي براجز تحت هذه المؤثرات هو نفسه فضاءات دي برانجز روفنياك.