An-Najah National University

Faculty of Graduate Studies

CUBIC B-SPLINE FOR SOLUTIONS OF BOUNDARY VALUE PROBLEMS

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Dedication

To my parents, for their unwavering love, support, and encouragement throughout my academic journey. I am forever grateful for the sacrifices you have made and for always believing in me. To my husband, for his love, support, and patience during me work on this thesis. Your constant encouragement and understanding has been invaluable to me. And to my children, for being my source of inspiration and motivation. This thesis is dedicated to all of you, with love and gratitude.

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Declaration

I, the undersigned, declare that I submitted the thesis entitled:

CUBIC B-SPLINE FOR SOLUTIONS OF BOUNDARY VALUE PROBLEMS

I declare that the work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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Date: 15/3/2023

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Abstract

Many of the mathematical models of engineering problems are expressed in terms of Boundary Value Problems (BVP). The Finite Difference Method (FDM) is the most modern method for solving (BVP). This is incredibly helpful in resolving challenging issues involving typical geometrical shapes or boundaries. Another numerical method, the B-spline method, has seen growing application in recent years in engineering research to solve mathematical models.

The main objective of this study is to examine the effectiveness of cubic B-spline functions in addressing boundary value problems. The derivation of linear, quadratic, and cubic B-spline functions is covered at the start of the study. Subsequently, I use cubic B-spline functions to solve second-order linear boundary value problems with non-homogeneous boundary conditions for ordinary differential equations, both in cases where coefficients are constant and where they are variable.

Results from the examples show that the B-spline method leads to lower error rates compared to the Finite Difference Method.

Keywords: Cubic spline, B-spline, differential equations, Boundary value problems, Finite difference methods

Chapter One
Introduction

1.1 Mathematical Problem

If we want to solve an engineering problem (usually of a physical nature), we first have to formulate the problem as a mathematical expression in terms of variables, functions, and equations. Such an expression is known as a mathematical model of the given problem. The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called mathematical modeling or, briefly, modeling. Many of the mathematical models of engineering problems are expressed in terms of Boundary Value Problems (BVP). The Finite Difference Method (FDM) is the most modern method for solving (BVP). This is incredibly helpful in resolving challenging issues involving typical geometrical shapes or boundaries. Another numerical method, the B-spline method, has seen growing application in recent years in engineering research to solve mathematical models. B-spline functions introduce a great importance in several science branches such as numerical analysis, ordinary and partial differential equations, integral equations, and statistical analysis. It has also many applications in science, engineering, economics, biology and medicine,... etc [Zahra, 2016].

1.2 Differential Equation

However, before we can turn to methods of solution, we must first define some basic concepts needed throughout this chapter , all definitions presented below are taken from the following reference: [Harper, 1976]. The first definition that we should cover should be that of differential equation. **A Differential Equations is** any equation with derivatives is referred to as a differential equation, which can be either a full or partial differential equation. **A differential equation's order** is determined by the highest derivative that appears in the equation. After all fractional powers of all derivatives have been eliminated, **the degree of a differential equation** is the power of the highest derivative. **The differential**

equation is referred to as linear if the dependent variable and all of its derivatives appear in the first power without a dependent variable and derivative product. An equation relating the derivatives of a (scalar) function depending on one or more variables is known as a differential equation. For example

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = cos(x) \tag{1.1}$$

is a differential equation that describes the function $u(x)$,which depends only on the variable x.

Types of Differential Equations :

- **Partial Differential Equations (PDE)**.
- **Ordinary Differential Equations (ODE)**.

Partial differential equations contain partial derivatives with regard to two or more independent variables, while **ordinary differential equations** contain complete derivatives (one independent variable). **ODEs** have a lengthy history and are frequently used in a variety of fields. In the 20th century, there has been significant advancement in ODE numerical solution. Numerous novel concepts and intricate techniques for solving ODEs have surfaced, expanding the understanding of numerical methods for doing so. Numerous issues in physics, engineering, biology, and other fields have been addressed by systems of ODEs. All definitions presented above are taken from the following reference:[Chang et al., 2010]. Usually ,but not quite always, the derivatives that emerge in the differential equation can be used to deduce the dependency of the variable [Olver, 2014]. **(PDE)** depends on partial derivatives of many independent variables. Order, functional form, and scalar versus systems of ODEs are the three characteristics that are typically used to categorize differential equations. The initial the equation's order determines the classification. The hierarchy is known, being the highest order derivative in the equation ODEs can also be categorized into linear and nonlinear groupings. A function F, of x , y, and derivatives of y are all given. Then a formula in the form

$F(x, y, y', ..., y^{n-1}) = y^n$ is referred to as **an explicit** n-order ordinary differential equation. **An implicit** ordinary differential equation of order n often has the following form: $F(x, y, y', y''......., y^n) = 0$. **Autonomous** : differential equations don't depend on x. **Linear** : If F can be expressed as a linear combination of the derivatives of y, then a differential equation is said to be linear: $y^n = \sum_{i=0}^{n-1} a_i(x)y^i + r(x)$ where $a_i(x)$ and $r(x)$ are continuous functions of x . The source term is the function $r(x)$, which results in two more significant classifications. **Homogeneous** : If $r(x)$ = 0, and consequently one "automatic" solution is the trivial solution, y = 0. The solution of a linear homogeneous equation is a complementary function, denoted here by y_c. **Non-homogeneous** : If ($r(x) \neq 0$). The specific integral, here represented by y_p, is the additional solution to the complementary function. **Non-linear** : a differential equation that can't be expressed as a linear combination of the derivatives of y . Finding numerical approximations to the solutions of ordinary differential equations is done using numerical methods for (ODEs). The process of using them is referred to as "numerical integration", while this term can also be used to describe computing integrals. All definitions presented above are taken from the following reference: [Harper, 1976].

1.3 Boundary Value Problem (B V P)

An ordinary differential equation with conditions affecting values of the solution and, or its derivatives at two or more places is a boundary value issue in one dimension. The differential equation's order is equal to the number of conditions that are applied. The typical properties of boundary value problems of any physical significance are as follows: (1) The requirements are imposed at two distinct points (2) The solution is only of interest between those two points(3) The independent variable is a space variable, which we shall denote as x. Additionally, we are mostly interested in situations when the differential equation is second order, and linear [Powers, 2009]. As with any physical differential equation, boundary value issues occur in many disciplines of physics. Boundary value problems are frequently used to describe issues with the wave equation, including those involving the identification of normal modes. A boundary value problem would be to determine the temperature of an iron bar at all locations, with one end kept at absolute zero and the other end at the freezing point of water. If a problem depends on both space and time, its value could be specified at a certain point in time or at a specific time in space. The issue is a specific illustration of a boundary value (in one spatial dimension).

Example of a boundary value (in one spatial dimension) is the problem

$Z''(x) + Z(x) = 0$

To be solved for the unknown function $Z(x)$ with the boundary conditions

$$Z(0) = 0, \quad Z(\pi/2) = 2.$$

Without the boundary conditions, the general solution to this equation is

$$Z(x) = A\sin(x) + B\cos(x)$$

From the boundary condition $Z(0) = 0$ one obtains

$$0 = A \cdot 0 + B \cdot 1$$

which implies that $B = 0$.From the boundary condition $Z(\pi/2) = 2$ one finds $2 = A \cdot 1$ and so A=2.

That imposing boundary conditions allowed one to determine a unique solution, which in this case is

$Z(x) = 2\sin(x).$

Unlike initial value problems, boundary value problems may have a single solution, no solutions, or an unlimited number of solutions, despite their seeming innocence.We employ the two boundary conditions to provide two equations that need to be satisfied by the two constants in the general solution when the differential equation in a boundary value problem has a known general solution. These two linear equations, if there is a solution, can be quickly solved if the differential equation is linear[Pandey, 2018].

1.4 Finite Difference Method (FDM)

While there are many numerical methods for solving such boundary value problems, (FDM) is most commonly used. The finite difference method was among the first approaches applied to the numerical solution of differential equations. It was first utilised by Euler, probably in 1768. It is directly applied to the differential form of the governing equations. The principle is to employ a Taylor series expansion. (FDM) use discrete approximations to the space derivatives. where we can use finite difference formulas at evenly spaced grid points to approximate the differential equations. This way, we can transform a differential equation into a system of algebraic equations to solve. The basis of the finite difference method is the replacement of all derivatives occurring by the corresponding difference quotients; this is applicable to any problem in differential equations and the resulting linear system of equation is solved by any standard procedure. These roots are the value of the required solution at the pivotal points. An important advantage of the finite difference methodology is its simplicity. Another advantage is the possibility to easily obtain high-order approximations, and hence to achieve high-order accuracy of the spatial discretisation. On the other hand, because the method requires a structured grid, the range of application is clearly restricted.

Finite Difference Methods for Ordinary BVPs :

The solution of a BVP by finite difference method is accomplished by the following steps[Endeshaw, 2019].

1. Discretizing the continuous solution domain into a discrete finite difference grid.
2. Approximating the exact derivatives in the Ordinary Differential Equation by finite difference approximation at each grid point
3. Substituting finite difference approximation into the Ordinary Differential Equation to obtain algebraic finite difference equation.
4. Solving the resulting system of equations by any standard procedure.

A finite difference method for solving a two-point boundary value problem given by the set of equations.

$$y'' = f(x, y, y'), \quad a \leq x \leq \ b \textit{ where } y(a) = \alpha \quad y(b) = \beta \tag{1.2}$$

Starts by introducing a mesh $x_i = x_0 + ih$, $\quad i = 0, 1, 2, ..., n \quad$ on the interval $[a, b]$, where $h = \frac{b-a}{n}$, followed by replacing the derivative terms in the differential equation with finite difference quotients relative to the mesh. The exact equation.

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x_i} = f\left[x_i, y(x_i), \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_i}\right], i = 1, 2, 3, ..., n-1 \tag{1.3}$$

Is replaced a by difference equation obtained by approximating the derivative terms.

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_i} \quad \text{and} \quad \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x_i}$$

with the central difference quotients

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_i} \approx \frac{y_{i+1} - y_{i-1}}{2h}; \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x_i} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \tag{1.4}$$

Substituting these difference quotients into (1.3) leads to the system of equations.

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = f\left[x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right] \qquad i = 1, 2, 3, ..., n-1 \tag{1.5}$$

where $\quad y_0 = \alpha$, $y_n = \beta$

Equations (1.5) constitute a system of $(n-1)$ equations in the $(n-1)$ unknown values $y(x_i)$ of the solution at the $(n-1)$ interior mesh points x_i , $i = 1, 2, 3, ..., n-1$. For a linear two-points boundary value problem of the general form

$$y'' + p(x)y' + q(x)y = r(x), \qquad a < x < b, \qquad y(a) = \alpha, \qquad y(b) = \beta \tag{1.6}$$

equation (1.3) becomes

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x_i} + p(x_i)\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_i} + q(x_i)y(x_i) = r(x_i) \tag{1.7}$$

Again, substitution of the two difference quotients (1.4) into (1.7) leads to the set of linear equations.

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + p_i \left[\frac{y_{i+1} - y_{i-1}}{2h}\right] + q_i y_i = r_i; \qquad i = 1, 2, 3, ..., n-1$$

where $p_i = p(x_i)$, $q_i = q(x_i)$,$r_i = r(x_i)$ with an $O(h^2)$ local truncation error . Multiplication of each term by h^2 and collecting like terms leads to the difference equation .

$$\left(1 - \frac{h}{2}p_i\right) y_{i-1} + \left(-2 + h^2 q_i\right) y_i + \left(1 + \frac{h}{2}p_i\right) y_{i+1} = h^2 r_i. \tag{1.8}$$

Evaluating (1.8) at each of the $(n-1)$ points x_i, $i = 1, 2, ..., n-1$ and taking note of the prescribed boundary values, finally leads to the linear system of equations can be expressed in the matrix form $MY = b$ [Malaki and Masenge, 2020], where

$$M = \begin{bmatrix} (-2 + h^2 q_1) & (1 + \frac{h}{2}p_1) & 0 & \cdots & \cdots & 0 \\ (1 - \frac{h}{2}p_2) & (-2 + h^2 q_2) & (1 + \frac{h}{2}p_2) & 0 & \cdots & 0 \\ 0 & (1 - \frac{h}{2}p_3) & (-2 + h^2 q_3) & (1 + \frac{h}{2}p_3) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (1 - \frac{h}{2}p_{n-2}) & (-2 + h^2 q_{n-2}) & (1 + \frac{h}{2}p_{n-2}) \\ 0 & \cdots & \cdots & 0 & (1 - \frac{h}{2}p_{n-1}) & (-2 + h^2 q_{n-1}) \end{bmatrix}$$

$$b = \begin{bmatrix} h^2 r_1 - (1 - \frac{h}{2}p_1)\alpha \\ h^2 r_2 \\ h^2 r_3 \\ \vdots \\ h^2 r_{n-2} \\ h^2 r_{n-1} - (1 - \frac{h}{2}p_{n-1})\beta \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix} \tag{1.9}$$

The coefficient matrix M is a tri-diagonal matrix; Y is the vector of unknown solution at the interior grid points and (b) is the right hand side vector which contains the known boundary values and values of the function $r(x)$ at the interior node points [Malaki and Masenge, 2020]. The system has a unique solution, The matrix M in the system is invertible.

1.5 Spline Interpolation

Second order boundary-value issues have been explored and quantitatively resolved by several academics utilizing a variety of methods with various boundary conditions, such as the (FDM), the collocation method, Quantic spline, non-polynomial spline, and shooting method[Endeshaw, 2019]. When numerical considerations are taken into account, the theory of spline functions is a very active area of approximation theory and (BVPs). Spline interpolation is a method of constructing a smooth curve that passes through a set of given data points. The term "spline" comes from the flexible strips of wood or metal that were traditionally used as guides in shipbuilding and drafting. In the context of interpolation, a spline is a piece-wise polynomial function that is used to approximate the behavior of a data-set. That is, instead of fitting a single, high-degree polynomial to all of the values at once, spline interpolation fits low-degree polynomials to discrete subsets of the values. For instance, instead of fitting a single degree-ten polynomial to all of the pairs of ten points, nine cubic polynomials were fitted between each pair. Spline interpolation is frequently used over polynomial interpolation because the interpolation error may be reduced even when using low-degree polynomials for the spline. Runge's phenomenon,which can result in oscillation between points when employing high-degree polynomials for interpolation, is another issue that is avoided by spline interpolation[Hall and Meyer, 1976]. Consider dividing an interval of length $a \leq x \leq b$ into m sub-intervals by placing knots at the locations $x_0, x_1, ..., x_n$ and so on, where $a = x_0 < x_1 < ...... < x_n = b$ Then, a spline function of degree n called $c(x)$ with knots $x_0, x_1, ..., x_n$ is a function having the two following characteristics.

1- $c(x)$ is a polynomial of degree (n) or less in each interval

$x_i \leq x \leq x_{i+1}(i = 1, 2, .., n.)$

2- $c(x)$ is continuous, as are its derivatives of orders $1, 2, ..., n - 1$.

One of the main advantages of spline interpolation is that it can produce smooth curves even when the data points are irregularly spaced or have high levels of noise. Additionally, spline interpolation can be used to estimate the derivatives of the data, which can be useful in various fields such as numerical analysis, computer graphics,and image processing. However, the choice of knots and the smoothness of the spline can be a trade-off and the interpolation may not be accurate in certain situations[Phillips, 2003]. There are several different types of splines, including polynomial splines, natural splines,and cubic splines. Polynomial splines are defined by a set of polynomials that are joined together at certain knots, or breakpoints, in the data-set. Natural splines are a type of polynomial splines that are required to have zero curvature at the endpoints, which ensures a smooth transition between the polynomials. Cubic splines are a specific type of polynomial spline that are defined by cubic polynomials.

1.6 Cubic Spline Interpolation

Cubic splines are a type of mathematical function used to approximate a set of data points and create smooth, continuous curves. They are a specific type of spline, which is a piece-wise polynomial function used to represent smooth curves. A cubic spline is defined by a set of control points and a set of polynomials, each of which is a cubic function. The polynomials are joined together at the control points to form a continuous curve that passes through all of the data points[Bartels et al., 1995]. Cubic splines are commonly used in computer graphics, engineering, and other fields to create smooth, realistic-looking curves. They are also commonly used in numerical analysis to interpolate or approximate a set of data points. The coefficients of each piece-wise polynomial in a spline are fixed between "knots" or joints. Usually, cubicles are utilized. Once the function and its first and second derivatives at each joint are known, the coefficients are then chosen to match.

In general, continuity up to the(n -1)derivative can be obtained with n^{th} degree polynomials. This method allows for the calculation of rates of change and total change over an interval[Richard, 2011]. Although a more robust form could also include unevenly spaced data points, we will only cover splines that interpolate equally spaced data points in this quick introduction. The basic concept of cubic spline interpolation is based on the tool used by engineers to construct curved paths through a collection of points. Weights are affixed to flat surfaces at the connection points of this spline. Then, a flexible strip is twisted across each of these weights to produce an aesthetically attractive curve. In theory, the mathematical spline is comparable. In this instance, the points are numerical data. The coefficients of the cubic polynomials used to interpolate the data are the weights. The line is "bent" by these coefficients so that it smoothly traverses each of the data points without exhibiting any erratic behavior or discontinuities. Fitting a piece-wise function of the form is the main notion[McKinley and Levine, 1998].

$$c(x) = \begin{cases} c_1(x) \quad if \quad x_1 \le x < x_2 \\ c_2(x) \quad if \quad x_2 \le x < x_3) \\ \vdots \\ c_{n-1}(x) \quad if \quad x_{n-1} \le x < x_n \end{cases} \tag{1.10}$$

where c_i is a third degree polynomial defined by

$c_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + d_i(x - x_i) + L_i$ for $i = 1, 2, 3....., n - 1$

The first and second derivatives of these $n - 1$ equations are fundamental to this process,and they are

$c'(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + d_i$

$c''(x) = 6a_i(x - x_i) + 2b_i$ for $i = 1, 2..., n - 1$

Four Characteristics of Cubic Splines[McKinley and Levine, 1998].

1. The data points will all be interpolated using the piece-wise function c(x).

2. On the interval $[x_1, x_n]$, c(x) will be continuous.

3. On the interval $[x_1, x_n]$, $c'(x)$ will be continuous.

4. On the interval $[x_1, x_n]$,$c''(x)$ will be continuous.

Cubic spline functions are crucial in solving boundary value problems numerically. Bickley proposed an alternative method for solving linear two-point BVPs by utilizing the differential equation and the boundary conditions, and cubic spline interpolation to model the solution curve in order to determine the unknown constants. Khan also used cubic spline functions to solve two-point BVPs [Kalyani et al., 2015]. B-splines, also known as basis splines, are a type of mathematical function used in computer graphics and other fields to represent smooth curves. They are a piece-wise polynomial representation that can be replaced by a basis function consisting of such polynomials. A sequential approach of interpolation with varying knots has been shown to reduce rounding error and be computationally efficient.The B-spline function of degree n can support $n + 2$ data points and its derivatives up to order $n - 1$ are continuous.[Panda and Rath, 1995].

1.7 Thesis outline

Building on the concepts introduced in chapter one, chapter two contains two sections. The first section, titled "Notion of the History of B-spline Function," presents the background of the one-dimensional B-spline method. The second section covers the derivations of the cubic B-spline functions, as well as the basic properties of B-spline and the method for obtaining a numerical solution for a non-homogeneous linear second order boundary value problem with non-homogeneous boundary conditions. Chapter three contains two sections, where we demonstrate some numerical results with specific examples of the boundary value problem presented in section one. In section two, the comparison between B-spline method and finite difference methods of BVP's are discussed. Finally, Chapter four presents the conclusions and future directions for research in cubic B-spline methods.

Chapter Two

B-splines

N. Lobachevsky began researching B-splines in the nineteenth century [Farin and Farin, 2002]. Isaac Jacob Schoenberg employed B-splines in his 1946 study to smooth statistical data, which marked the beginning of the contemporary theory of spline approximation[Schoenberg, 1988]. In the context of this work, other mathematicians, like Carl de Boor, who collaborated closely with Schoenberg, were inspired by his work. De Boor (1962, 1972, 1978) developed a recursive definition for splines at the beginning of the 1970s [De Boor, 1972].Bryant and De Boor (1962) investigated the convergence and error bound of spline interpolation[De Boor, 1962]. The field of splines and interpolations has a large number of publications. One of the many topics that attracted a lot of interest was the use of graphic curves and surfaces in numerous research and application domains. The most pertinent publications will only be acknowledged and listed in this thesis. Introduction to Computing with Geometry Notes, a package of online lectures by Professor C.-K. Shene, is a good place to start learning more about the topic [Shene, 2011]. Basis functions, B-spline curves, and surfaces are all thoroughly examined. The notes provide summaries, theories, illustrations, and references. Geometric Modeling with Multivariate B-splines was the subject of Timothy Irwin Mueller's PhD dissertation in 1986 [Mueller, 1986]. In order to define spline surfaces over three, four, five, and six-sided regions, The potential of a B-spline collocation approach for numerically resolving the equations of fluid dynamics was examined by R.W. Johnson et al. in 1999 [Johnson and Landon, 1999]. It is well known that compared to ordinary shape functions, B-splines may resolve complex curves with a far smaller amount of input. Using their approach, a channel flow problem example was resolved. Different kinds of techniques have been used to approximate the solutions of differential equations methods. Solutions to second order boundary problems were proposed by Fang, Tsuchiya, and Yamamoto in 2002. Value issues with homogeneous boundary conditions are solved using three techniques: using a non-singular tri-diagonal matrix's inversion formula, the finite element and finite volume approaches[Fang et al., 2002]. For the heat equation, Farago and Horvath (1999) derived numerical solutions[Faragó and Horváth, 1999]. A finite difference technique equation while Bernstein polynomials have a degree that varies on the number of sub-intervals, the advantage of cubic B-splines is that the polynomials are always of degree three. Cubic B-spline functions were studied in relation to interpolation by Munguia et al.(2014)[Munguia, 2014]. cubic B-spline approach for solving the time-fractional telegraph (Akram et al., 2019)[Akram et al., 2019]. B-spline collocation techniques and their convergence for a class of nonlinear singular boundary value problems that depend on the derivative, Appl.Math.Comput(2019) [Roul and Goura, 2019], Splines, now the fields of mathematics and engineering greatly benefit from the use of splines, particularly B-splines. Splines are often used in computer graphing due to their precision, smoothness, and adaptability[Munguia and Bhatta, 2015].

A B-spline, known as a basis spline, is a spline function with minimal support for a given degree, smoothness, and domain partition in the mathematical discipline of numerical analysis. A linear combination of B-splines of that degree can be used to express any spline function of a particular degree [Zahra, 2016]. The knots on cardinal B-splines are evenly spaced apart. Curve fitting and numerical differentiation of experimental data can both be done using B-splines. It is possible to generate the degree k spline curve f from n controls points $(c_i)_{i=1}^{n}$ and $(n+k+1)$ knots $(x_i)_{i=1}^{n+k+1}$ and written as $f = \sum_{i=1}^{n} c_i B_i^k$ where B_i^k are B-splines. The B-spline function is a collection of flexible bands that traverses a number of so-called control points to produce smooth curves. With the help of several points, these features make it possible to create and control complicated forms and surfaces. We think about the kind of base for which the knots are equally spaced. In the B-spline is referred to as uniform in this situation. According to a fundamental theorem each spline function with the specified degree, smoothness, and domain partition can be visualized as a linear combination of the same B-spline over that same partition, and in terms of degree and smoothness. This attribute shows right away that the matrices that occur in challenges with collocation and interpolation, as well as some characteristics of

B-splines suggest complete positive. As a result, these matrices can Gaussian elimination is used to efficiently store information in a computer and invert it[Zahra, 2016]. If you've ever utilized the curve fitting tools in your preferred software for mathematical analysis, you've definitely seen a selection for spline fitting more specifically, B-spline fitting. A basis spline, often known as a B-spline, is a piece-wise polynomial function having unique characteristics that specify the degree/order of the polynomial. Finding a distinctive polynomial representation of a group of data, whether they are structural points in 3D space or a set of data on a graph, is the goal of employing a B-spline curve. B-splines can generate fairly precise approximations to data, depending on the quantity of data available and the selected order/degree of the polynomial representation. In light of this, they might be helpful in scenarios (such as machine learning and statistics) where a functional transformation does not convert a set of data to a linear trend or when a curved representation of a set of discrete data is sought (e.g., computer graphics and 3D printing). In this thesis, we provide a general review of B-spline curves and some examples of how its beneficial characteristics might be used[Bartels et al., 1995].

2.1 B-spline Curve Components

In order to calculate B-spline curves three things are required[Magoon, 2010]:

1. Knot Vectors.
2. Basis Functions.
3. Control Points.

Knot Vectors : A Knot Vector is a sequence of non-decreasing real numbers that are used to define the shape and location of a spline function. In the context of cubic B-spline, Knot vectors play a crucial role in defining the shape and location of a spline function, particularly in the context of cubic B-spline. The positions of the knots in the knot vector determine the location of the polynomial segments that make up the spline, and the number of knots in the knot vector determines the degree of the polynomial segments and the smoothness of the spline. The spacing of the knots in the knot vector also plays a significant role in determining the smoothness of the spline. For example, a uniform spacing of knots results in a smooth spline, while a non-uniform spacing of knots can result in a spline with more variation and a more complex shape. In computer-aided design and computer graphics, knot vectors are used to define the shape of curves and surfaces. However, there are limitations and challenges that come with working with knot vectors. One common limitation is that the number of knots in the knot vector must be greater than or equal to the number of control points plus the degree of the spline. Additionally, the knots in the knot vector must be non-decreasing, which can be a limitation when trying to achieve a specific shape or form. Furthermore, when using non-uniform knots vectors the shape can be difficult to predict and control, it may lead to more complex shape but also it could result in unpredictable and unwanted shape. In summary, Knot vectors play a crucial role in defining the shape and location of a spline function in the context of cubic B-spline. The spacing and distribution of knots can greatly affect the shape of the spline, and it is important to consider these factors when working with knot vectors. However, there are limitations and challenges that come with working with knot vectors, such as the requirement for a specific number of knots and the non-decreasing nature of the knots.[Kwok et al., 2001].

Three different kinds of knot vectors are employed

Uniform knot vectors: are a type of knot vector that have equal spacing between each knot. In a uniform knot vector, the distance between any two consecutive knots is the same. This type of knot vector is commonly used in B-spline when a smooth spline is desired. With a uniform knot vector, the polynomial segments that make up the spline have the same degree of continuity, resulting in a smooth and continuous spline. Because of this property, uniform knot vectors are often used in computer-aided design and computer graphics to define smooth curves and surfaces. However, it is worth mentioning that using a uniform knot vector does not guarantee that the resulting spline will be optimal for a particular problem, and in some cases using a non-uniform knot vector may be more appropriate[De Boor, 2001].

Example 2.1.1

Here's an example of a uniform knot vector: knots = [0, 1, 2, 3, 4, 5] In this example, the knot vector has 6 knots, and each knot is evenly spaced by a value of 1.

Open-uniform knot vector: is a type of knot vector that is used in B-spline, which combines the features of both open and uniform knot vectors. An open knot vector is one in which the first and last knots are repeated a certain number of times. The number of times the first and last knots are repeated is equal to the degree of the spline. This type of knot vector is commonly used to ensure that the spline starts and ends with a certain degree of continuity. An open-uniform knot vector is an open knot vector where all the knots inside the first and last repeated knots are evenly spaced. This type of knot vector combines the advantages of both open and uniform knot vectors. It ensures that the spline starts and ends with the desired degree of continuity, while also providing a smooth and continuous spline throughout the entire range. The open-uniform knot vector is often used in computer-aided design and computer graphics, where it is important to ensure that the spline starts and ends with a certain degree of continuity while also maintaining a smooth and continuous shape.the degree of continuity in B-splines refers to the smoothness and continuity of the curve at the points where adjacent curve segments meet. It is controlled by the placement of knots in the knot vector.

Example 2.1.2

knots = [0, 0, 1, 2, 3, 4, 5, 6, 7, 7] In this example, the knot vector has 10 knots, and the first and last knots (0 and 7) are repeated twice, which is equal to the degree of the spline. **Non-uniform knot vector**: is a type of knot vector that does not have equal spacing between each knot. In a non-uniform knot vector, the distance between consecutive knots can vary. This type of knot vector is used in B-spline when a more complex or varied shape is desired. With a non-uniform knot vector, the polynomial segments that make up the spline can have different degrees of continuity, resulting in a spline with more variation. Non-uniform knot vectors can be used to create shapes with more complex curvatures, for example, if the knots are placed closer together in certain areas and farther apart in others, it creates a spline with a higher degree of curvature in those areas with closer knots. Also, non-uniform knot vectors can be used to model shapes with sharp changes in the slope, such as a step function. It is worth noting that non-uniform knot vectors can be more difficult to work with compared to uniform knot vectors, as they require more knots to obtain the same degree of smoothness, which can increase the computational cost. However, they are more versatile as they can generate a wide range of shapes and can be used to model a variety of phenomena[Liu, 2003].

Example 2.1.3

knots = [0, 0, 0, 0.5, 1, 1.5, 2, 3, 4, 5, 5, 5, 5] In this example, the knot vector has 13 knots, and the spacing between the knots is not uniform.

Basis Functions :[Munguia and Bhatta, 2015] In this study, we are examining a division of a given interval $[a, b]$ into N segments with mesh size $h = \frac{(b-a)}{N}$, represented by the partition $\triangle_N$: $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$. A piece-wise polynomial function on the interval [a, b] is defined as a spline of degree$k$, if it satisfies the condition $s \in C^{k-1}[a, b]$ and is a polynomial of degree at most k on each sub-interval $[x_i, x_{i+1}]$. The set of all polynomials of degree k, associated with $\triangle_N$ is denoted as $S_k(\triangle_N)$, and it is a linear space with dimension $N + k$. This study focuses on a particular type of spline function called B-splines of degree 3, which were introduced by Carl de Boor in the 1970s and are defined recursively. see [De Boor, 1972]. B-splines of degree zero are defined by:

$$B_i^0(x) = \begin{cases} 1 & \text{if } x_i \leq \ x \ < x_{i+1}, \\ 0 & otherwise. \end{cases} \tag{2.1}$$

B-spline of degree k,$k \in \mathbb{Z}^+$ are defined recursively in terms of B-splines of degree $(k-1)$ by

$$B_i^k(x) = \left(\frac{x - x_i}{x_{i+k} - x_i}\right) B_i^{k-1}(x) + \left(\frac{x_{i+k+1} - x}{x_{{i+k+1} - x_{i+1}}}\right) B_{i+1}^{k-1}(x) \tag{2.2}$$

For $i \in \mathbb{Z}^+$, the B-spline of degree one ,two ,and three are as follows[Phillips, 2003]

(a) Liner B-spline

$$B_i^1(x) = \begin{cases} \frac{x-x_i}{x_{i+1}-x_i} & \text{if } x_i \leq \ x \ < x_{i+1}, \\ \\ \frac{x_{i+2}-x}{x_{i+2}-x_{i+1}} & \text{if } x_{i+1} \leq \ x \ < x_{i+2}, \\ \\ 0 & \text{otherwise .} \end{cases} \tag{2.3}$$

(b) Quadratic B-spline:

$$B_i^2(x) = \begin{cases} \frac{(x-x_i)^2}{(x_{i+2}-x_i)(x_{i+1}-x_i)} & \text{if} \quad x_i \leq \ x \ < x_{i+1}, \\ \\ \frac{(x-x_i)(x_{i+2}-x)}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})} + \frac{(x_{i+3}-x)(x-x_{i+1})}{(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} & \text{if} \quad x_{i+1} \leq \ x \ < x_{i+2}, \\ \\ \frac{(x_{i+3}-x)^2}{(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} & \text{if} \quad x_{i+2} \leq \ x \ < x_{i+3}, \\ \\ 0 & \text{otherwise .} \end{cases} \tag{2.4}$$

(c) Cubic B-spline:

$$B_i^3(x) = \begin{cases} \frac{(x-x_i)^3}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+1}-x_i)} & \text{if } x_i \leq \ x \ < x_{i+1}, \\ \\ \frac{(x-x_i)^2(x_{i+2}-x)}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+2}-x_{i+1})} + \frac{(x-x_i)(x_{i+3}-x)(x-x_{i+1})}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} + \\ \frac{(x_{i+4}-x)(x-x_{i+1})^2}{(x_{i+4}-x_{i+1})(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} & \text{if } x_{i+1} \leq \ x \ < x_{i+2}, \\ \\ \frac{(x-x_i)(x_{i+3}-x)^2}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} + \frac{(x_{i+4}-x)(x-x_{i+1})(x_{i+3}-x)}{(x_{i+4}-x_{i+1})(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} + \\ \frac{(x_{i+4}-x)^2(x-x_{i+2})}{(x_{i+4}-x_{i+1})(x_{i+4}-xi+2)(x_{i+3}-x_{i+2})} & \text{if } x_{i+2} \leq \ x \ < x_{i+3}, \\ \\ \frac{(x_{i+4}-x)^3}{(x_{i+4}-x_{i+1})(x_{i+4}-x_{i+2})(x_{i+4}-x_{i+3})} & \text{if } x_{i+3} \leq \ x \ < x_{i+4}, \\ \\ 0 & \text{otherwise .} \end{cases} \tag{2.5}$$

The previous equation describes a cubic spline with knots x_i, x_{i+1}, x_{i+2}, x_{i+3}, and x_{i+4}. This spline is zero everywhere except the interval [x_i,x_{i+4}). All B-splines of degree 3 have this property, where they are only non-zero within the interval (x_i, x_{i+k+1}). In this particular case, we will only focus on B-splines of degree 3, which we refer to simply as B_i. The knots are evenly spaced, so after including four additional knots, the uniform grid partition $\triangle$ is as follows: $x_{-2} < x_{-1} < x_0 < x_1 < ... < x_{N-1} < x_N < x_{N+1} < x_{N+2}$. The uniform cubic B-spline $B_i(x)$ is defined using (2.5) with a uniform grid spacing of$h = x_{i+1} - x_i$ for any $0 \leq \ i \ \leq N$.

$$B_i(x) = \frac{1}{6h^3} \begin{cases} (x - x_{i-2})^3 & \text{if } x_{i-2} \leq \ x \ < x_{i-1}, \\ \\ -3(x - x_{i-1})^3 + 3h(x - x_{i-1})^2 + \\ 3h^2(x - x_{i-1}) + h^3 & \text{if } x_{i-1} \leq \ x \ < x_i, \\ \\ -3(x_{i+1} - x)^3 + 3h(x_{i+1} - x)^2 + \\ 3h^2(x_{i+1} - x) + h^3 & \text{if } x_i \leq \ x \ < x_{i+1}, \\ \\ (x_{i+2} - x)^3 & \text{if } x_{i+1} \leq \ x \ < x_{i+2}, \\ \\ 0 & \text{otherwise .} \end{cases} \tag{2.6}$$

Example 2.1.4

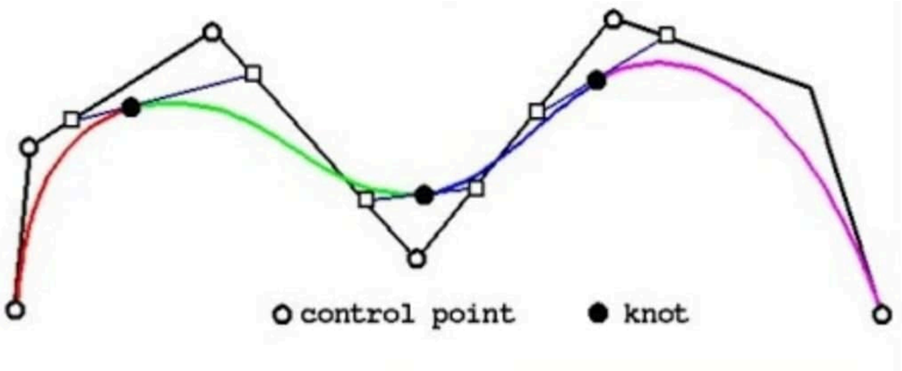
Consider apartition of interval $[-2, 2]$, with $h = 1$

$$B_0(x) = \frac{1}{6} \begin{cases} (x + 2)^3 & \text{if } -2 \leq \ x \ < -1, \\ 4 - 6x^2 - 3x^3 & \text{if } -1 \leq \ x \ < 0, \\ 4 - 6x^2 + 3x^3 & \text{if } 0 \leq \ x \ < 1, \\ (2 - x)^3 & \text{if } 1 \leq \ x \ < 2, \\ 0 & \text{otherwise .} \end{cases} \tag{2.7}$$

Control Point:

The position vector is the final component required to build the B-spline curve after the knot vectors and basis functions have been calculated B_i. The vertices of the defining polygon are located by the coordinates that make up these vectors, which are known as control points. To produce a precise approximation of the intended curve, control points are chosen. In B-spline, control points are used to define the shape of the spline. The control points are a set of points in the plane or in higher dimensions, that are used to determine the position of the polynomial segments that make up the spline. The number of control points and their positions in space determine the shape of the spline. A B-spline is defined by a set of control points, a degree and a knot vector. The control points determine the shape of the spline,and the knot vector determines the position of the polynomial segments. While the knot vector and the degree of the spline are used to determine the smoothness of the spline. The number of control points must be greater than or equal to the degree of the spline. The control points can be chosen in various ways, such as to match the desired shape of the spline or to minimize the difference between the spline and the data points. In addition to determining the shape of the spline, control points also have a significant impact on the accuracy and stability of the B-spline method[Yang et al., 2004]. The more control points that are used, the more accurate the spline will be, but also it will require more computational resources. However, increasing the number of control points can also make the method more sensitive to errors and noise, leading to a more unstable solution. The choice of control points is also important for ensuring the smoothness of the spline. If the control points are chosen such that they are not col-linear or co-planar, the resulting spline will be smooth. However, if the control points are col-linear or co-planar, the resulting spline may be less smooth. Furthermore, the choice of the control points also affects the flexibility of the B-spline method. If the control points are chosen such that they are widely spaced, the resulting spline will be less flexible and will not be able to closely match more complex shapes. On the other hand, if the control points are chosen such that they are closely spaced, the resulting spline will be more flexible and will be able to closely match more complex shapes. Overall, the choice of control points is a trade-off between accuracy, stability, smoothness and flexibility. Finding the optimal choice of control points can be a challenging task, but it is crucial for obtaining a good solution with B-spline method.[Magoon, 2010]

Figure 1
Designing of B-spline curve



What adjustments can be made a shape a B-spline?

1. Reposition the control points.

2. Change the control points.

3. Utilize several control points.

4. Change the order.

5. Modify the knot vector's type.

6. Adjust the knots' relative spacing.

7. Use a knot vector with several knot values[Magoon, 2010]

Next we derive the cubic B-spline method for approximating solutions to second - order liner ordinary BVPs . "

2.2 The cubic B-spline method

Both mathematics and engineering use B-spline functions extensively. In this thesis, we offer a numerical method for leveraging B-splines to solve the BVP of a second-order linear ODE. In this section, we study the use of cubic B-spline to solve second-order linear (BVP) of the form[Munguia and Bhatta, 2015].

$$a_1(x)y'' + a_2(x)y' + a_3(x)y = f(x) \tag{2.8}$$

with boundary conditions

$$y(a) = \alpha, y(b) = \beta \tag{2.9}$$

The given equation represents an approximation of the solution to a (BVP) where $a_1(x) \neq 0$, $a_2(x)$, $a_3(x)$ and $f(x)$ are continuous real-valued functions on the interval [a, b]. To find the solution using cubic B-spline, $Y(x)$ is assumed to be a cubic spline with knots $\triangle$ and is expressed as a linear combination of $B_i(x)$. The solution is represented as

$$Y(x) = \sum_{i=-1}^{N+1} c_i B_i(x) \tag{2.10}$$

where the constants c_i are to be determined and the $B_i(x)$ are defined in (2.6). It is required that (2.10) satisfies our BVP (2.8-2.9) at $x = x_i$, where x_i is an interior point. That is

$$a_1(x_i)Y''(x_i) + a_2(x_i)Y'(x_i) + a_3(x_i)Y(x_i) = f(x_i) \tag{2.11}$$

and the boundary conditions are

$$Y(x_0) = \alpha \quad for \quad x_0 = a, Y(x_N) = \beta \quad for \quad x_N = b$$

From (2.10) , we have

Y(x$_i$) $= c_{i-1}B_{i-1}(x_i) + c_iB_i(x_i) + c_{i+1}B_{i+1}(x_i) + c_{i+2}B_{i+2}(x_i),$

$Y'(x_i) = c_{i-1}B'_{i-1}(x_i) + c_iB'_i(x_i) + c_{i+1}B'_{i+1}(x_i) + c_{i+2}B'_{i+2}(x_i),$

$$Y''(x_i) = c_{i-1}B''_{i-1}(x_i) + c_iB''_i(x_i) + c_{i+1}B''_{i+1}(x_i) + +c_{i+2}B''_{i+2}(x_i), \tag{2.12}$$

and these yield

$$\mathrm{c}_{i-1}[a_1(x_i)B''_{i-1}(x_i)+a_2(x_i)B'_{i-1}(x_i)+a_3(x_i)B_{i-1}(x_i)]$$
$$+\,c_i[a_1(x_i)B''_i(x_i)+a_2(x_i)B'_i(x_i)+a_3(x_i)B_i(x_i)]$$
$$+\,c_{i+1}[a_1(x_i)B''_{i+1}(x_i)+a_2(x_i)B'_{i+1}(x_i)+a_3(x_i)B_{i+1}(x_i)]$$
$$+\,c_{i+2}[a_1(x_i)B''_{i+2}(x_i)+a_2(x_i)B'_{i+2}(x_i)+a_3(x_i)B_{i+2}(x_i)]=f(x_i), \qquad (2.13)$$

also by the properties of cubic B-spline function, we obtain the following

$$B''_{i-1}(x_i)=\tfrac{1}{h^2} \qquad B'_{i-1}(x_i)=-\tfrac{1}{2h}, \qquad B_{i-1}(x_i)=\tfrac{1}{6}$$
$$B''_i(x_i)=-\tfrac{2}{h^2} \qquad B'_i(x_i)=0, \qquad {}''B_i(x_i)=\tfrac{2}{3}$$
$$B''_{i+1}(x_i)=\tfrac{1}{h^2} \qquad B'_{i+1}(x_i)=\tfrac{1}{2h}, \qquad B_{i+1}(x_i)=\tfrac{1}{6}$$
$$B''_{i+2}(x_i)=0 \qquad B'_{i+2}(x_i)=0, \qquad B_{i+2}(x_i)=0 \qquad (2.14)$$

if we combine (2.13) and (2.14) , we obtain

$$\mathrm{c}_{i-1}[6a_1(x_i)-3a_2(x_i)h+a_3(x_i)h^2]+c_i[-12a_1(x_i)+4a_3(x_i)h^2]$$
$$+\,c_{i+1}[6a_1(x_i)+3a_2(x_i)h+a_3(x_i)h^2]=6h^2f(x_i) \qquad (2.15)$$

Now we apply the boundary conditions:

Y(x$_0$) $= c_{-1}B_{-1}(x_0) + c_0B_0(x_0) + c_1B_1(x_0) + c_2B_2(x_0) = \alpha$

$Y(x_N) = c_{N-1}B_{N-1}(x_N) + c_NB_N(x_N) + c_{N+1}B_{N+1}(x_N) + c_{N+2}B_{N+2}(x_N) = \beta$ (2.16)

where the value of $B_i(x)$ at $x = x_0$ and $x = x_N$ are given below

B$_{-1}(x_0) = \frac{1}{6} = B_{N-1}(x_N)$

$B_0(x_0) = \frac{4}{6} = B_N(x_N)$

$B_1(x_0) = \frac{1}{6} = B_{N+1}(x_N)$

$B_2(x_0) = 0 = B_{N+2}(x_N)$ (2.17)

Therefor

$$c_{-1} + 4c_0 + c_1 = 6\alpha \quad (2.18)$$

$$c_{N-1} + 4c_N + c_{N+1} = 6\beta \quad (2.19)$$

Now that we have found all the constant coefficients in (2.15), (2.18), and (2.19) we can write a system of$N + 1$linear equations in $N + 1$ unknowns. This system is represented in (2.20) where the coefficient matrix is an$(N + 1) \times (N + 1)$ matrix. In matrix form $Ac_i = D_i$.

$$\begin{pmatrix} o_1 & o_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ p_1 & q_1 & r_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & p_2 & q_2 & r_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p_{N-2} & q_{N-2} & r_{N-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{N-1} & q_{N-1} & r_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & o_3 & o_4 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-2} \\ c_{N-1} \\ c_N \end{pmatrix} = 6 \begin{pmatrix} z0 \\ h^2 f(x_1) \\ h^2 f(x_2) \\ \vdots \\ h^2 f(x_{N-2}) \\ h^2 f(x_{N-1}) \\ z_N \end{pmatrix} \tag{2.20}$$

where p_i,q_i and r_i are defined below

$$\begin{aligned} \text{p}_i &= 6a_1(x_i) - 3a_2(x_i)h + a_3(x_i)h^2, \\ q_i &= -12a_1(x_i) + 4a_3(x_i)h^2, \\ r_i &= 6a_1(x_i) + 3a_2(x_i)h + a_3(x_i)h^2, \\ o_1 &= q_0 - 4p_0, \\ o_2 &= r_0 - p_0, \\ o_3 &= P_N - r_N, \\ o_4 &= q_N - 4r_N, \\ z_0 &= h^2 f(x_0) - \alpha p_0 \\ z_N &= h^2 f(x_N) - \beta r_N \end{aligned} \tag{2.21}$$

The cubic B-spline approximation for the BVP (2.8-2.9) is obtained using (2.10), where the constant coefficients c_i satisfy the system defined in (2.20).

2.2.1 Basic Properties of B-splines

Smooth curves are produced using the B-spline function, which is made up of a number of flexible bands that are controlled by several points known as control points. These operations are used to construct and control intricate surfaces and forms made up of several points. Shape optimization techniques frequently use the B-spline function [Talebitooti et al., 2015]. A piece-wise polynomial function of degree $k-1$ in the variable x is a B-spline of order k. It is specified across $1 + k$ location x_i also known as knots or breakpoints which must be listed in ascending order x. The B-spline has no effect elsewhere and only has an effect in the region between the first and final of these knots. The knot vector and related B-splines are referred to as "uniform" if each knot is spaced apart by the same amount of time h (where $h = x_{i+1} - x_i$) from its predecessor. A B-spline is a polynomial of degree $k - 1$ for each finite knot interval where it is non-zero. At the knots, a B-spline is a continuous function. The derivatives of the B-spline are continuous up to the derivative of degree $k - 2$ when all of its knots are distinct. Each additional coincident knot reduces the continuity of the derivative order by one if the knots are coincident at a particular value of x. A subset of the knots in a B-spline may be shared, but two B-splines defined over the exact same knots are identical. In other words, a B-knot's spline's are what make it uniquely. However, because B-spline basis functions have local support, algorithms like de Boor's algorithm can construct B-splines without having to evaluate basis functions where they are zero. The FORTRAN-coded method BSPLV, which produces values of the B-splines of order k at x, is closely connected to this relationship. Now let us discuss some significant properties of B-Splines[Khazaei and Karamipour, 2021].

1. $B_i^k(x)$ is a degree k polynomial in x.

2. **Non-negativity**: for all i, k and x, $B_i^k(x)$ is non-negative.

3. At most k+1 degree k basis functions are nonzero on any span $[x_i, x_{i+k+1}]$, namely: $B_{i-k}^k(x)$, $B_{i-k+1}^k(x)$, $B_{i-k+2}^k(x)$,....., $B_i^k(x)$.This property shows that the following basis functions are nonzero on $[x_i, x_{i+k+1})$

B^k_{i-k}(x), $B^k_{i-k+1}(x)$, $B^k_{i-k+2}(x)$,............, $B^k_i(x)$.

4. **Partition of unity**:The sum of all nonzero degree k basis functions on span $[x_i, x_{i+k+1})$ is 1 which states that the sum of these $k+1$ basis functions is 1.

5. **Linear independence**: B-spline functions are linearly independent, which means that any linear combination of them will result in a different function.

6. **Flexibility**: B-splines are flexible in the sense that they can be used to model a wide range of shapes, including smooth and non-smooth shapes, by adjusting the control points, degree, and knot vector.

These properties make B-spline an important tool for modeling, computer-aided design, computer graphics, and numerical analysis.

Chapter Three
Numerical solution and results

3.1 Numerical Solution of BVPs using B-spline Method

This section showcases numerical results for specific examples of second order linear BVPs, which we aim to approximate using cubic B-splines. The derivation of cubic B-spline functions can be found in chapter two, and this section focuses on the process of obtaining numerical solutions and results for non-homogeneous linear second order BVPs with non-homogeneous boundary conditions.

What is Root Mean Square Error (RMSE)? Root mean square error or root mean square deviation is one of the most commonly used measures for evaluating the quality of predictions. It shows how far predictions fall from measured true values using Euclidean distance. To compute RMSE, calculate the residual (difference between prediction and truth) for each data point, compute the norm of residual for each data point, compute the mean of residuals and take the square root of that mean. RMSE is commonly used in supervised learning applications, as RMSE uses and needs true measurements at each predicted data point.

Root mean square error can be expressed as:

$$RMSE = \sqrt{\frac{1}{N}\Big(\sum_{i=1}^{N}(F_i - G_i)^2\Big)}$$

Where F_i is the approximation solution and G_i the exact solution.

Solution is achieved by following procedure steps :

1- Find $a_1(x)$,$a_2(x)$,$a_3(x)$,$f(x)$ using (2.8)

then $p_i, q_i, r_i, o_1, o_2, o_3, o_4, z_0$, and z_N using (2.21)

2- Find the constant coefficients c_i for i=0,1,.......,20 using the system of liner equations (2.20) and the coefficient matrix is 21×21 where $c_i = A^{-1}D_i$

3- Calculate all required basis functions using equations (2.6)

4 - Calculate (RMSE)

Example (3.1.1) :

We consider a linear boundary value problem with constant coefficients "

$$y'' + y' - 6y = x \quad \text{for} \quad 0 < x < 1 \tag{3.1}$$

with boundary conditions

$$y(0) = 0 \qquad\qquad y(1) = 1 \tag{3.2}$$

" The exact solution to boundary value problem is

$$y(x) = \frac{(43 - e^2)e^{-3x} - (43 - e^{-3})e^{2x}}{36(e^{-3} - e^2)} - \frac{1}{6}x - \frac{1}{36} \tag{3.3}$$

We approximate the solution in (3.1) with boundary conditions (3.2) using the cubic B-spline method with N =20 in order to use (2.11), we first need to find the constant coefficients c_i for $i = -1, 0, 1......., 21$ using the system of liner equations (2.20) where the coefficient matrix is 21×21 and using (2.18) and (2.19) to find c_{-1} and c_{21} respectively.

Step 1:

We first need to find $a_1(x)$, $a_2(x)$, $a_3(x)$, $f(x)$ using (2.8)

$a_1(x) = 1$, $a_2(x) = 1$, $a_3(x) = -6$, $f(x) = x$, $\alpha = 0$, $\beta = 1$, $h = .05$

now we need to find p_i, q_i, r_i, o_1, o_2, o_3, o_4, z_0, z_N in (2.21)

p_i = 5.8350

q_i = -12.0600

r_i = 6.1350

o_1 =-35.4 o_2 =0.3000000000000007 o_3 = -0.3000000000000007
o_4 =-36.6 z_0 =0.00 z_n =-6.1325

Step 2:

We find the constant coefficients c_i for $i = 0, 1......., 20$ using the system of liner equations (2.20) where the coefficient matrix is 21×21 where $c_i = A^{-1}D_i$

$$
D_i = 6 \begin{pmatrix} z0 \\ h^2 f(x_1) \\ h^2 f(x_2) \\ h^2 f(x_3) \\ h^2 f(x_4) \\ h^2 f(x_5) \\ h^2 f(x_6) \\ h^2 f(x_7) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ h^2 f(x_{N-3}) \\ h^2 f(x_{N-2}) \\ h^2 f(x_{N-1}) \\ z_N \end{pmatrix} = \begin{pmatrix} 0.000000 \\ 0.000750 \\ 0.001500 \\ 0.002250 \\ 0.003000 \\ 0.003750 \\ 0.004500 \\ 0.005250 \\ 0.006000 \\ 0.006750 \\ 0.007500 \\ 0.008250 \\ 0.009000 \\ 0.009750 \\ 0.010500 \\ 0.011250 \\ 0.012000 \\ 0.012750 \\ 0.013500 \\ 0.014250 \\ -36.795000 \end{pmatrix}
$$

$$A = \begin{bmatrix}
-35.40 & 0.30 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.83 & -12.06 & 6.14 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.30 & -36.60
\end{bmatrix}$$

$$A^{-1} = \begin{bmatrix}
-0.028456 & -0.001257 & -0.001134 & -0.001023 & -0.000922 & -0.000830 & -0.000745 & -0.000669 & -0.000598 & -0.000533 & -0.000473 & -0.000417 & -0.000365 & -0.000315 & -0.000268 & -0.000223 & -0.000179 & -0.000135 & -0.000091 & -0.000046 & -0.000008 \\
-0.024441 & -0.148278 & -0.133831 & -0.120705 & -0.108766 & -0.097891 & -0.087967 & -0.078889 & -0.070561 & -0.062894 & -0.055802 & -0.049206 & -0.043031 & -0.037201 & -0.031646 & -0.026293 & -0.021070 & -0.015904 & -0.010717 & -0.005429 & -0.000910 \\
-0.020981 & -0.127287 & -0.262002 & -0.236306 & -0.212933 & -0.191642 & -0.172213 & -0.154442 & -0.138138 & -0.123127 & -0.109244 & -0.096332 & -0.084241 & -0.072829 & -0.061953 & -0.051473 & -0.041249 & -0.031135 & -0.020981 & -0.010629 & -0.001782 \\
-0.017998 & -0.109189 & -0.224750 & -0.349720 & -0.315129 & -0.283621 & -0.254867 & -0.228565 & -0.204437 & -0.182222 & -0.161675 & -0.142566 & -0.124673 & -0.107783 & -0.091687 & -0.076178 & -0.061046 & -0.046078 & -0.031051 & -0.015730 & -0.002637 \\
-0.015424 & -0.093578 & -0.192617 & -0.299719 & -0.416951 & -0.375262 & -0.337217 & -0.302418 & -0.270494 & -0.241100 & -0.213915 & -0.188631 & -0.164956 & -0.142609 & -0.121312 & -0.100791 & -0.080770 & -0.060966 & -0.041084 & -0.020813 & -0.003489 \\
-0.013203 & -0.080103 & -0.164881 & -0.256561 & -0.356912 & -0.467927 & -0.420488 & -0.377095 & -0.337288 & -0.300636 & -0.266737 & -0.235210 & -0.205689 & -0.177824 & -0.151268 & -0.125680 & -0.100715 & -0.076021 & -0.051229 & -0.025952 & -0.004350 \\
-0.011285 & -0.068462 & -0.140920 & -0.219276 & -0.305044 & -0.399926 & -0.505855 & -0.453653 & -0.405764 & -0.361671 & -0.320890 & -0.282962 & -0.247448 & -0.213925 & -0.181978 & -0.151196 & -0.121163 & -0.091454 & -0.061629 & -0.031221 & -0.005233 \\
-0.009625 & -0.058395 & -0.120198 & -0.187032 & -0.260187 & -0.341117 & -0.431469 & -0.533121 & -0.476844 & -0.425027 & -0.377102 & -0.332530 & -0.290795 & -0.251400 & -0.213857 & -0.177682 & -0.142387 & -0.107475 & -0.072425 & -0.036690 & -0.006150 \\
-0.008188 & -0.049676 & -0.102252 & -0.159108 & -0.221341 & -0.290188 & -0.367050 & -0.453526 & -0.551443 & -0.491520 & -0.436098 & -0.384553 & -0.336289 & -0.290730 & -0.247313 & -0.205479 & -0.164663 & -0.124289 & -0.083755 & -0.042430 & -0.007112 \\
-0.006942 & -0.042113 & -0.086684 & -0.134884 & -0.187642 & -0.246006 & -0.311166 & -0.384476 & -0.467485 & -0.561972 & -0.498606 & -0.439673 & -0.384490 & -0.332402 & -0.282762 & -0.234931 & -0.188265 & -0.142104 & -0.095760 & -0.048512 & -0.008132 \\
-0.005858 & -0.035537 & -0.073149 & -0.113822 & -0.158343 & -0.207594 & -0.262580 & -0.324443 & -0.394491 & -0.474224 & -0.565372 & -0.498547 & -0.435976 & -0.376912 & -0.320625 & -0.266390 & -0.213475 & -0.161132 & -0.108583 & -0.055008 & -0.009221 \\
-0.004913 & -0.029805 & -0.061349 & -0.095461 & -0.132799 & -0.174106 & -0.220221 & -0.272105 & -0.330853 & -0.397724 & -0.474168 & -0.561856 & -0.491339 & -0.424775 & -0.361341 & -0.300218 & -0.240584 & -0.181594 & -0.122372 & -0.061993 & -0.010391 \\
-0.004086 & -0.024789 & -0.051026 & -0.079398 & -0.110453 & -0.144809 & -0.183165 & -0.226318 & -0.275180 & -0.330799 & -0.394380 & -0.467313 & -0.551203 & -0.476529 & -0.405366 & -0.336796 & -0.269896 & -0.203719 & -0.137282 & -0.069546 & -0.011658 \\
-0.003360 & -0.020383 & -0.041956 & -0.065285 & -0.090820 & -0.119069 & -0.150607 & -0.186090 & -0.226267 & -0.272000 & -0.324279 & -0.384248 & -0.453227 & -0.532743 & -0.453185 & -0.376527 & -0.301734 & -0.227751 & -0.153476 & -0.077750 & -0.013033 \\
-0.002718 & -0.016491 & -0.033945 & -0.052820 & -0.073480 & -0.096335 & -0.121851 & -0.150559 & -0.183065 & -0.220066 & -0.262363 & -0.310882 & -0.366691 & -0.431024 & -0.505314 & -0.419838 & -0.336442 & -0.253948 & -0.171130 & -0.086694 & -0.014532 \\
-0.002148 & -0.013032 & -0.026824 & -0.041739 & -0.058065 & -0.076125 & -0.096289 & -0.118974 & -0.144661 & -0.173900 & -0.207324 & -0.245664 & -0.289765 & -0.340603 & -0.399308 & -0.467190 & -0.374388 & -0.282590 & -0.190431 & -0.096471 & -0.016171 \\
-0.001637 & -0.009932 & -0.020445 & -0.031813 & -0.044256 & -0.058021 & -0.073389 & -0.090679 & -0.110257 & -0.132542 & -0.158017 & -0.187239 & -0.220852 & -0.259599 & -0.304343 & -0.356081 & -0.415971 & -0.313977 & -0.211582 & -0.107186 & -0.017967 \\
-0.001175 & -0.007130 & -0.014677 & -0.022838 & -0.031771 & -0.041653 & -0.052686 & -0.065098 & -0.079153 & -0.095151 & -0.113440 & -0.134418 & -0.158549 & -0.186365 & -0.218486 & -0.255629 & -0.298624 & -0.348436 & -0.234803 & -0.118950 & -0.019939 \\
-0.000753 & -0.004570 & -0.009407 & -0.014637 & -0.020363 & -0.026696 & -0.033768 & -0.041723 & -0.050731 & -0.060985 & -0.072707 & -0.086152 & -0.101618 & -0.119446 & -0.140033 & -0.163839 & -0.191396 & -0.223321 & -0.260333 & -0.131883 & -0.022107 \\
-0.000363 & -0.002202 & -0.004532 & -0.007053 & -0.009811 & -0.012863 & -0.016270 & -0.020103 & -0.024443 & -0.029384 & -0.035032 & -0.041510 & -0.048962 & -0.057552 & -0.067471 & -0.078941 & -0.092219 & -0.107601 & -0.125434 & -0.146119 & -0.024493 \\
0.000003 & 0.000018 & 0.000037 & 0.000058 & 0.000080 & 0.000105 & 0.000133 & 0.000165 & 0.000200 & 0.000241 & 0.000287 & 0.000340 & 0.000401 & 0.000472 & 0.000553 & 0.000647 & 0.000756 & 0.000882 & 0.001028 & 0.001198 & -0.027122
\end{bmatrix}$$

we find the constant coefficients c_i for $i = 0, 1......., 20$ using the system of liner equations (2.20) where the coefficient matrix is 21×21

$$c_i = A^{-1}D_i$$

$$c_i = \begin{pmatrix} 0.000234498 \\ 0.027670728 \\ 0.054293511 \\ 0.080655427 \\ 0.107278209 \\ 0.134661904 \\ 0.163293271 \\ 0.193653567 \\ 0.226225882 \\ 0.261502130 \\ 0.299989841 \\ 0.342218834 \\ 0.388747909 \\ 0.440171619 \\ 0.497127249 \\ 0.560302075 \\ 0.630440999 \\ 0.708354660 \\ 0.794928112 \\ 0.891130170 \\ 0.998023523 \end{pmatrix}$$

and using (2.18) and (2.19) to find c_{-1} and c_{21} respectively

c_{-1}= - 0.028608719 c_{21}= 1.116775737

where the constants c_i are to be determined

Step 3:

Calculate all required basis functions using equations(2.6) in this example

$0 < x < 1$ and $h = .05$

$0 \leq x < .05$
B_{-1}=$-1333.33333333333x^3 + 200.0x^2 - 10.0x + 0.166666666666667$
B_0=$4000.0x^3 - 400.0x^2 + 2.31296463463574e^{-15}x + \quad 0.666666666666667$
B_1=$-4000.0 \quad x^3 + 200.0 \quad x^2 + 10.0x + 0.166666666666667$
B_2= 1333.33333333333 x^3
$Y(x) = c_{-1}B_{-1} + c_0B_0 + c_1B_1 + c_2B_2$
$Y(x)$ =0.79138 x^3 - $0.28140x^2$+ $0.56279x$
Y(0) = 0

$0.05 \leq x < 0.10$
B_0=$-1333.33333333333x^3 + 400.0x^2 - 40.0x + 1.33333333333333$
B_1=$4000.0x^3 - 1000.0x^2 + 70.0x - 0.833333333333333$
B_2=$-4000.0x^3 + 800.0x^2 - 40.0x + 0.666666666666667$
B_3=1333.33333333333 x^3 $- 200.0x^2 + 10.0x - 0.166666666666667$
$Y(x) = c_0B_0 + c_1B_1 + c_2B_2 + c_3B_3$
$Y(x)$ =0.73677 x^3 $- 0.27321x^2 + 0.56238x + 0.00001$
Y(.05) = .0275351538

$0.10 \leq x < 0.15$

B_1=$-1333.33333333333x^3 + 600.0x^2 - 90.0x + 4.5$

B_2=$4000.0x^3 - 1600.0x^2 + 200.0x - 7.33333333333333$

B_3=$-4000.0x^3 + 1400.0x^2 - 150.0x + 5.16666666666667$

B_4=$1333.33333333333x^3 - 400.0x^2 + 40.0x - 1.33333333333333$

Y(x)=c$_1 B_1 + c_2 B_2 + c_3 B_3 + c_4 B_4$

$Y(x)$ =0.69564 x^3 $- .26087x^2 + 0.56115x + .00005$

Y(0.1) = 0.0542500333

$0.15 \leq x < 0.2$

B_2=$-1333.33333333333x^3 + 800.0x^2 - 160.0x + 10.6666666666667$

B_3=$4000.0x^3 - 2200.0x^2 + 390.0x - 21.8333333333333$

B_4=$4000.0x^3 - 2200.0x^2 + 390.0x - 21.8333333333333$

B_5=$1333.33333333333x^3 - 600.0x^2 + 90.0x - 4.5$

Y(x)=c$_2 B_2 + c_3 B_3 + c_4 B_4 + c_5 B_5$

$Y(x)$ =0.66673 x^3 $- 0.24786x^2 + 0.55920x + 0.00015$

Y(0.15) = 0.0806989046

$0.20 \leq x < 0.25$

B_3=$-1333.33333333333x^3 + 1000.0x^2 - 250.0x + 20.8333333333333$

B_4=$4000.0x^3 - 2800.0x^2 + 640.0x - 47.3333333333333$

B_5=$-4000.0x^3 + 2600.0x^2 - 550.0x + 38.1666666666667$

B_6=$1333.33333333333x^3 - 800.0x^2 + 160.0x - 10.6666666666667$

Y(x)=c$_3 B_3 + c_4 B_4 + c_5 B_5 + c_6 B_6$

$Y(x)$ =0.64901 x^3 $- 0.23722x^2 + 0.55707x + 0.00029$

Y(0.20) = .1074050277

$0.25 \leq x < 0.30$

B_4=$-1333.33333333333x^3 + 1200.0x^2 - 360.0x + 36.0$

B_5=$4000.0x^3 - 3400.0x^2 + 950.0x - 86.8333333333333$

B_6=$-4000.0x^3 + 3200.0x^2 - 840.0x + 72.6666666666666$

B_7=$1333.33333333333x^3 - 1000.0x^2 + 250.0x - 20.8333333333333$

Y(x)=c$_4 B_4 + c_5 B_5 + c_6 B_6 + c_7 B_7$

$Y(x)$ =0.64168 x$^3 - 0.23172x^2 + 0.55570x + 0.00040$

Y(0.25) = .1348698493

$0.3 \leq x < 0.35$

B_5=$-1333.33333333333x^3 + 1400.0x^2 - 490.0x + 57.1666666666666$

B_6=$4000.0^3 - 4000.0x^2 + 1320.0x - 143.333333333333$

B_7=$-4000.0x^3 + 3800.0x^2 - 1190.0x + 123.166666666667$

B_8=$1333.33333333333x^3 - 1200.0x^2 + 360.0x - 36.0$

Y(x)=c$_5 B_5 + c_6 B_6 + c_7 B_7 + c_8 B_8$

$Y(x)$ =0.64412 x$^3 - 0.23392x^2 + 0.55636x + 0.00034$

Y(0.30) = .1635814257

$0.35 \leq x < 0.4$

B_6=$-1333.33333333333x^3 + 1600.0x^2 - 640.0x + 85.3333333333333$

B_7=$4000.0x^3 - 4600.0x^2 + 1750.0x - 219.833333333333$

B_8=$-4000.0x^3 + 4400.0x^2 - 1600.0x + 192.666666666667$

B_9=$1333.33333333333x^3 - 1400.0x^2 + 490.0x - 57.1666666666666$

Y(x)=c$_6 B_6 + c_7 B_7 + c_8 B_8 + c_9 B_9$

$Y(x) =$ 0.65589 x$^3 - 0.24628x^2 + 0.56068x - 0.00017$

Y(0.35) =0.1940222368

$0.4 \leq x < 0.45$

B_7=$-1333.33333333333x^3 + 1800.0x^2 - 810.0x + 121.5$

B_8=$4000.0x^3 - 5200.0x^2 + 2240.0x - 319.333333333333$

B_9=$-4000.0x^3 + 5000.0x^2 - 2070.0x + 284.166666666667$

B_{10}=$1333.33333333333x^3 - 1600.0x^2 + 640.0x - 85.3333333333333$

Y(x)=c$_7 B_7 + c_8 B_8 + c_9 B_9 + c_{10} B_{10}$

$Y(x) =$ 0.67670 x$^3 - 0.27126x^2 + 0.57067x - 0.00150$

Y(0.4) = .2266765374

$0.45 \leq x < 0.5$

B_8=$-1333.33333333333x^3 + 2000.0x^2 - 1000.0x + 166.666666666667$

B_9=$4000.0x^3 - 5800.0x^2 + 2790.0x - 444.833333333333$

B_{10}=$-4000.0x^3 + 5600.0x^2 - 2600.0x + 400.666666666667$

B_{11}=$1333.33333333333x^3 - 1800.0x^2 + 810.0x - 121.5$

Y(x)=c$_8 B_8 + c_9 B_9 + c_{10} B_{10} + c_{11} B_{11}$

$Y(x) =$ 0.70643 x$^3 - 0.31139x^2 + 0.58873x - 0.00421$

Y(0.45) = 0.2620373739

$0.5 \leq x < 0.55$

B_9=$-1333.33333333333x^3 + 2200.0x^2 - 1210.0x + 221.833333333333$

B_{10}=$4000.0x^3 - 6400.0x^2 + 3400.0x - 599.333333333333$

B_{11}=$-4000.0x^3 + 6200.0x^2 - 3190.0x + 545.166666666667$

B_{12}=$1333.33333333333x^3 - 2000.0x^2 + 1000.0 \quad x - 166.666666666667$

Y(x)=c$_9 B_9 + c_{10} B_{10} + c_{11} B_{11} + c_{12} B_{12}$

$Y(x) =$ 0.74506 x$^3 - 0.36934x^2 + 0.61771x - 0.00904$

Y(0.5) =.3006133878

$0.55 \leq x < 0.6$

B_{10}=$-1333.33333333333x^3 + 2400.0x^2 - 1440.0x + 288.0$

B_{11}=$4000.0x^3 - 7000.0x^2 + 4070.0x - 785.833333333333'$

B_{12}=$-4000.0x^3 + 6800.0x^2 - 3840.0x + 720.666666666667$

B_{13}=$1333.33333333333x^3 - 2200.0x^2 + 1210.0x - 221.833333333333$

Y(x)=c$_{10}B_{10} + c_{11}B_{11} + c_{12}B_{12} + c_{13}B_{13}$

$Y(x) =$ 0.79274 x$^3 - 0.44800x^2 + 0.66097x - 0.01697$

Y(0.55) = .3429355142

$0.6 \leq x < 0.65$

B_{11}=$-1333.33333333333x^3 + 2600.0x^2 - 1690.0x + 366.166666666667$

B_{12}=$4000.0x^3 - 7600.0x^2 + 4800.0x - 1007.33333333333$

B_{13}=$-4000.0x^3 + 7400.0x^2 - 4550.0x + 930.166666666666$

B_{14}=$1333.33333333333x^3 - 2400.0x^2 + 1440.0x - 288.0$

Y(x)=c$_{11}B_{11} + c_{12}B_{12} + c_{13}B_{13} + c_{14}B_{14}$

$Y(x) =$ 0.84971 x$^3 - 0.55056x^2 + 0.72251x - 0.02928$

Y(0.6) = .3895636812

$0.65 \leq x < 0.7$

B_{12}=$-1333.33333333333x^3 + 2800.0x^2 - 1960.0x + 457.333333333333$

B_{13}=$4000.0x^3 - 8200.0x^2 + 5590.0x - 1266.83333333333$

B_{14}=$-4000.0x^3 + 8000.0x^2 - 5320.0x + 1176.66666666667$

B_{15}=$1333.33333333333x^3 - 2600.0x^2 + 1690.0x - 366.166666666667$

Y(x)=c$_{12}B_{12} + c_{13}B_{13} + c_{14}B_{14} + c_{15}B_{15}$

$Y(x) =$0.91637 x$^3 - 0.68053x^2 + 0.80699x - 0.04758$

Y(0.65) = .4410936054

$0.7 \leq x < 0.75$

B_{13}=$-1333.33333333333x^3 + 3000.0x^2 - 2250.0x + 562.5$

B_{14}=$4000.0x^3 - 8800.0x^2 + 6440.0x - 1567.33333333333$

B_{15}=$-4000.0x^3 + 8600.0x^2 - 6150.0x + 1463.16666666667$

B_{16}=$1333.33333333333x^3 - 2800.0x^2 + 1960.0x - 457.333333333333$

Y(x)=c$_{13}B_{13} + c_{14}B_{14} + c_{15}B_{15} + c_{16}B_{16}$

$Y(x) =$ 0.99320 x^3 $- 0.84189x^2 + 0.91994x - 0.07394$

Y(0.7) = .4981637819

$0.75 \leq x < 0.8$

B_{14}=$-1333.33333333333x^3 + 3200.0x^2 - 2560.0x + 682.666666666667$

B_{15}=$4000.0x^3 - 9400.0x^2 + 7350.0x - 1911.83333333333$

B_{16}=$-4000.0x^3 + 9200.0x^2 - 7040.0x + 1792.66666666667$

B_{17}=$1333.33333333333x^3 - 3000.0x^2 + 2250.0x - 562.5$

Y(x)=c$_{14}B_{14} + c_{15}B_{15} + c_{16}B_{16} + c_{17}B_{17}$

$Y(x) =$ 1.08085 x^3 $- 1.03910x^2 + 1.06785x - 0.11091$

Y(0.75) =.5614627582

$0.8 \leq x < 0.85$

B_{15}=$-1333.33333333333x^3 + 3400.0x^2 - 2890.0x + 818.833333333333$

B_{16}=$4000.0x^3 - 10000.0x^2 + 8320.0x - 2303.33333333333$

B_{17}=$-4000.0x^3 + 9800.0x^2 - 7990.0x + 2168.16666666667$

B_{18}=$1333.33333333333x^3 - 3200.0x^2 + 2560.0x - 682.666666666667$

Y(x)=c$_{15}B_{15} + c_{16}B_{16} + c_{17}B_{17} + c_{18}B_{18}$

$Y(x) =$ 1.18007 x^3 $- 1.27723x^2 + 1.25835x - 0.16171$

Y(0.8) =.6317367884

$0.85 \leq x < 0.9$

B_{16}=$-1333.33333333333x^3 + 3600.0x^2 - 3240.0x + 972.0$

B_{17}=$4000.0x^3 - 10600.0x^2 + 9350.0x - 2744.83333333333'$

B_{18}=$-4000.0x^3 + 10400.0x^2 - 9000.0x + 2592.66666666667$

B_{19}=$1333.33333333333x^3 - 3400.0x^2 + 2890.0x - 818.833333333333$

Y(x)= $c_{16}B_{16} + c_{17}B_{17} + c_{18}B_{18} + c_{19}B_{19}$

Y(x)= $1.29175x^3 - 1.56201x^2 + 1.50041x - 0.23030$

Y(0.85) =.7097979583

$0.9 \leq x < 0.95$

B_{17}=$-1333.33333333333x^3 + 3800.0x^2 - 3610.0x + 1143.16666666667$

B_{18}=$4000.0x^3 - 11200.0x^2 + 10440.0x - 3239.33333333333$

B_{19}=$-4000.0x^3 + 11000.0x^2 - 10070.0x + 3069.16666666667$

B_{20}=$1333.33333333333x^3 - 3600.0x^2 + 3240.0x - 972.0$

Y(x)= $c_{17}B_{17} + c_{18}B_{18} + c_{19}B_{19} + c_{20}B_{20}$

Y(x)= $1.41692x^3 - 1.89996x^2 + 1.80457x - 0.32155$

Y(0.9) =.7965328796

$0.95 \leq x < 0.1$

B_{18}=$-1333.33333333333x^3 + 4000.0x^2 - 4000.0x + 1333.33333333333$

B_{19}=$4000.0x^3 - 11800.0x^2 + 11590.0x - 3789.83333333333$

B_{20}=$-4000.0x^3 + 11600.0x^2 - 11200.0x + 3600.66666666667$

B_{21}=$1333.33333333333x^3 - 3800.0x^2 + 3610.0x - 1143.16666666667$

Y(x)=c$_{18}B_{18} + c_{19}B_{19} + c_{20}B_{20} + c_{21}B_{21}$

$Y(x) =$ 1.55675 x^3 $- 2.29849x^2 + 2.18317x - 0.44144$

Y(.95) =.8929120525

The cubic B-spline polynomials are as follows:

$$Y(x) = \begin{cases} 0.79138x^3 - 0.28140x^2 + 0.56279x + 0.00 & , \quad x \in [0.00, 0.05) \\ 0.73677x^3 - 0.27321x^2 + 0.56238x + 0.00001 & , \quad x \in [0.05, 0.10] \\ 0.69564x^3 - .26087x^2 + 0.56115x + .00005 & , \quad x \in [0.10, 0.15) \\ 0.66673x^3 - 0.24786x^2 + 0.55920x + 0.00015 & , \quad x \in [0.15, 0.20) \\ 0.64901x^3 - 0.23722x^2 + 0.55707x + 0.00029 & , \quad x \in [0.20, 0.25) \\ 0.64168x^3 - 0.23172x^2 + 0.55570x + 0.00040 & , \quad x \in [0.25, 0.30) \\ 0.64412x^3 - 0.23392x^2 + 0.55636x + 0.00034 & , \quad x \in [0.30, 0.35) \\ 0.65589x^3 - 0.24628x^2 + 0.56068x - 0.00017 & , \quad x \in [0.35, 0.40) \\ 0.67670x^3 - 0.27126x^2 + 0.57067x - 0.00150 & , \quad x \in [0.4, 0.45) \\ 0.70643x^3 - 0.31139x^2 + 0.58873x - 0.00421 & , \quad x \in [0.45, 0.50) \\ 0.74506x^3 - 0.36934x^2 + 0.61771x - 0.00904 & , \quad x \in [0.50, 0.55) \\ 0.79274x^3 - 0.44800x^2 + 0.66097x - 0.01697 & , \quad x \in [0.55, 0.60) \\ 0.84971x^3 - 0.55056x^2 + 0.72251x - 0.02928 & , \quad x \in [0.60, 0.65) \\ 0.91637x^3 - 0.68053x^2 + 0.80699x - 0.04758 & , \quad x \in [0.65, 0.70) \\ 0.99320x^3 - 0.84189x^2 + 0.91994x - 0.07394 & , \quad x \in [0.70, 0.75) \\ 1.08085x^3 - 1.03910x^2 + 1.06785x - 0.11091 & , \quad x \in [0.75, 0.80) \\ 1.18007x^3 - 1.27723x^2 + 1.25835x - 0.16171 & , \quad x \in [0.80, 0.85) \\ 1.29175x^3 - 1.56201x^2 + 1.50041x - 0.23030 & , \quad x \in [0.85, 0.90) \\ 1.41692x^3 - 1.89996x^2 + 1.80457x - 0.32155 & , \quad x \in [0.90, 0.95) \\ 1.55675x^3 - 2.29849x^2 + 2.18317x - 0.44144 & , \quad x \in [0.95, 1.00)] \end{cases} \tag{3.4}$$

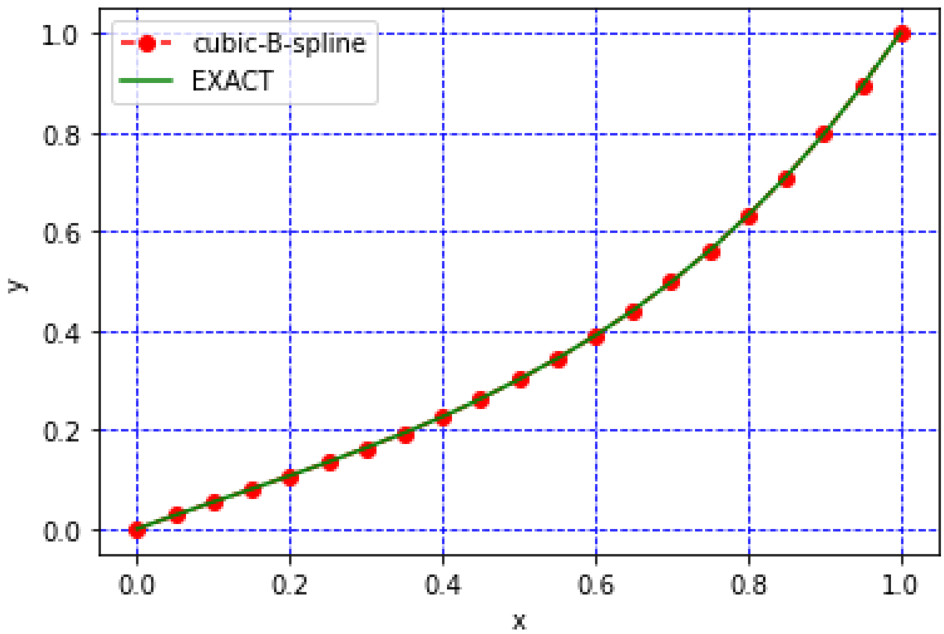
Table 1

The Cubic B-spline results example (3.1.1)

x_i	Exact	Cubic B-spline
0.00	0.0000000000	0.0000000000
0.05	0.0275370031	0.0275351538
0.10	0.0542570003	0.0542500333
0.15	0.0807133503	0.0806989046
0.20	0.1074285617	0.1074050277
0.25	0.1349034523	0.1348698493
0.30	0.1636255435	0.1635814257
0.35	0.1940768511	0.1940222368
0.40	0.2267412146	0.2266765374
0.45	0.2621112965	0.2620373739
0.50	0.3006953693	0.3006133878
0.55	0.3430239998	0.3429355142
0.60	0.3896567348	0.3895636812
0.65	0.4411888843	0.4410936054
0.70	0.4982584988	0.4981637819
0.75	0.5615536314	0.5614627582
0.80	0.6318199790	0.6317367884
0.85	0.7098689951	0.7097979583
0.90	0.7965865702	0.7965328796
0.95	0.8929423791	0.8929120525
1.00	1.0000000000	1.0000000000
$\boldsymbol{RMSE} = 6.395571348794568 * 10^{-5}$		

Figure 1

The exact and cubic B-spline solutions of example (3.1.1)



Example (3.1.2) :

We consider a linear boundary value problem with constant coefficients.

$$y'' + 2y' + 5y = 6\cos(2x) - 7\sin(2x) \quad \text{for} \quad 0 < x < \frac{\pi}{4} \tag{3.5}$$

with boundary conditions

$$y(0) = 4 \qquad\qquad y(\frac{\pi}{4}) = 1 \tag{3.6}$$

The exact solution to boundary value problem is

$$y(x) = 2(1 + e^{-x})\, cos(2x) + \sin(2x) \tag{3.7}$$

We approximate the solution in (3.5) with boundary conditions (3.6) using the cubic B-spline method with N=20 in order to use (2.11), we first need to find the constant coefficients c_i for $i = -1, 0, 1......., 21$using the system of liner equations (2.20) where the coefficient matrix is 21×21 and using (2.18) and (2.19) to find c_{-1} and c_{21} respectively

.

Step 1:

We first need to find $a_1(x)$, $a_2(x)$,$a_3(x)$, $f(x)$ using (2.9)

$a_1(x) = 1$, $a_2(x) = 2$,$a_3(x) = 5$, $f(x) = 6\cos(2x) - 7\sin(2x)$ $\alpha = 4$ $\beta = 1$, h=$\frac{\pi}{80}$ using (2.11)

Now we need to find p_i , q_i , r_i , o_1 , o_2 , o_3 , o_4 , z_0 z_N in (2.21)

p_i =

5.77209118 5.77209118 5.77209118 5.77209118 5.77209118 5.77209118

5.77209118 5.77209118 5.77209118 5.77209118 5.77209118 5.77209118

5.77209118 5.77209118 5.77209118 5.77209118 5.77209118 5.77209118

5.77209118 5.77209118 5.77209118

q_i = -11.96915749 -11.96915749 -11.96915749 -11.96915749
-11.96915749 -11.96915749 -11.96915749 -11.96915749
-11.96915749 -11.96915749 -11.96915749 -11.96915749
-11.96915749 -11.96915749 -11.96915749 -11.96915749
-11.96915749 -11.96915749 -11.96915749 -11.96915749 -11.96915749

r_i = 6.2433301 6.2433301 6.2433301 6.2433301 6.2433301 6.2433301
6.2433301 6.2433301 6.2433301 6.2433301 6.2433301 6.2433301 6.2433301
6.2433301 6.2433301 6.2433301 6.2433301 6.2433301 6.2433301 6.2433301
6.2433301

o_1 = -35.05752220 o_2 =0.4712389
o_3 = -0.4712389
o_4 =-36.94247779 z_0 = -23.07911196
z_n =-6.25412496

Step 2:

Find the constant coefficients c_i for i = 0, 1 ....,20 using the system of liner equations (2.20) where the coefficient matrix is 21×21

$$D_i = 6 \begin{pmatrix} z0 \\ h^2 f(x_1) \\ h^2 f(x_2) \\ h^2 f(x_3) \\ h^2 f(x_4) \\ h^2 f(x_5) \\ h^2 f(x_6) \\ h^2 f(x_7) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ h^2 f(x_{N-3}) \\ h^2 f(x_{N-2}) \\ h^2 f(x_{N-1}) \\ z_N \end{pmatrix} = \begin{pmatrix} -138.47467178130270326619 \\ 0.05026364678596995095 \\ 0.04470087671901810944 \\ 0.03886251088254361052 \\ 0.03278454474338198671 \\ 0.02650445098391000093 \\ 0.02006094847049791005 \\ 0.01349376353882288451 \\ 0.00684338506779929526 \\ 0.00015081485217626853 \\ -0.00654268518716144380 \\ -0.01319584739665671629 \\ -0.01976765281888705206 \\ -0.02621758408807733920 \\ -0.03250587523313323846 \\ -0.03859375684807649087 \\ -0.04444369511832486908 \\ -0.05001962322914045578 \\ -0.05528716372953578623 \\ -0.06021384048069099160 \\ -37.52474974362766602098 \end{pmatrix}$$

$$A = \begin{bmatrix}
-35.058 & 0.471 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.772 & -11.969 & 6.243 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & -0.471 & -36.942
\end{bmatrix}$$

$$A^{-1} = \begin{bmatrix}
-0.029 & -0.002 & -0.002 & -0.002 & -0.002 & -0.002 & -0.002 & -0.002 & -0.002 & -0.002 & -0.002 & -0.002 & -0.002 & -0.002 & -0.002 & -0.002 & -0.001 & -0.001 & -0.001 & -0.000 & -0.000 \\
-0.028 & -0.168 & -0.173 & -0.177 & -0.180 & -0.182 & -0.183 & -0.182 & -0.180 & -0.176 & -0.170 & -0.162 & -0.153 & -0.141 & -0.127 & -0.112 & -0.094 & -0.074 & -0.051 & -0.026 & -0.004 \\
-0.026 & -0.160 & -0.330 & -0.338 & -0.344 & -0.347 & -0.348 & -0.347 & -0.342 & -0.334 & -0.323 & -0.309 & -0.291 & -0.269 & -0.243 & -0.213 & -0.178 & -0.140 & -0.097 & -0.050 & -0.009 \\
-0.025 & -0.152 & -0.312 & -0.484 & -0.492 & -0.497 & -0.499 & -0.496 & -0.490 & -0.479 & -0.463 & -0.442 & -0.416 & -0.384 & -0.347 & -0.304 & -0.255 & -0.200 & -0.139 & -0.072 & -0.012 \\
-0.023 & -0.143 & -0.294 & -0.455 & -0.626 & -0.632 & -0.634 & -0.631 & -0.623 & -0.608 & -0.588 & -0.562 & -0.529 & -0.489 & -0.442 & -0.387 & -0.325 & -0.255 & -0.177 & -0.092 & -0.016 \\
-0.022 & -0.133 & -0.274 & -0.425 & -0.584 & -0.752 & -0.754 & -0.751 & -0.741 & -0.724 & -0.700 & -0.669 & -0.629 & -0.582 & -0.525 & -0.460 & -0.386 & -0.303 & -0.211 & -0.109 & -0.018 \\
-0.020 & -0.124 & -0.254 & -0.394 & -0.542 & -0.697 & -0.860 & -0.856 & -0.845 & -0.826 & -0.798 & -0.762 & -0.717 & -0.663 & -0.599 & -0.525 & -0.441 & -0.346 & -0.240 & -0.125 & -0.021 \\
-0.019 & -0.114 & -0.234 & -0.363 & -0.499 & -0.642 & -0.791 & -0.947 & -0.934 & -0.913 & -0.883 & -0.843 & -0.794 & -0.734 & -0.663 & -0.581 & -0.487 & -0.382 & -0.266 & -0.138 & -0.023 \\
-0.017 & -0.104 & -0.214 & -0.331 & -0.455 & -0.585 & -0.722 & -0.864 & -1.010 & -0.988 & -0.955 & -0.912 & -0.858 & -0.793 & -0.717 & -0.628 & -0.527 & -0.414 & -0.288 & -0.149 & -0.025 \\
-0.015 & -0.094 & -0.193 & -0.299 & -0.411 & -0.529 & -0.652 & -0.781 & -0.913 & -1.049 & -1.014 & -0.969 & -0.912 & -0.843 & -0.761 & -0.667 & -0.560 & -0.439 & -0.305 & -0.158 & -0.027 \\
-0.014 & -0.084 & -0.173 & -0.267 & -0.367 & -0.473 & -0.583 & -0.698 & -0.816 & -0.938 & -1.062 & -1.014 & -0.954 & -0.882 & -0.797 & -0.698 & -0.586 & -0.460 & -0.320 & -0.166 & -0.028 \\
-0.012 & -0.074 & -0.152 & -0.236 & -0.324 & -0.418 & -0.515 & -0.616 & -0.721 & -0.828 & -0.937 & -1.048 & -0.986 & -0.912 & -0.824 & -0.722 & -0.606 & -0.475 & -0.331 & -0.171 & -0.029 \\
-0.011 & -0.064 & -0.133 & -0.205 & -0.282 & -0.363 & -0.448 & -0.536 & -0.627 & -0.720 & -0.816 & -0.912 & -1.009 & -0.933 & -0.842 & -0.738 & -0.620 & -0.486 & -0.338 & -0.175 & -0.030 \\
-0.009 & -0.055 & -0.113 & -0.175 & -0.241 & -0.310 & -0.383 & -0.458 & -0.536 & -0.616 & -0.697 & -0.779 & -0.862 & -0.945 & -0.854 & -0.748 & -0.628 & -0.493 & -0.343 & -0.178 & -0.030 \\
-0.008 & -0.046 & -0.095 & -0.146 & -0.201 & -0.259 & -0.320 & -0.383 & -0.447 & -0.514 & -0.582 & -0.651 & -0.720 & -0.789 & -0.858 & -0.751 & -0.631 & -0.495 & -0.344 & -0.178 & -0.030 \\
-0.006 & -0.037 & -0.077 & -0.119 & -0.163 & -0.210 & -0.259 & -0.310 & -0.363 & -0.416 & -0.472 & -0.527 & -0.583 & -0.639 & -0.695 & -0.749 & -0.629 & -0.493 & -0.343 & -0.178 & -0.030 \\
-0.005 & -0.029 & -0.059 & -0.092 & -0.127 & -0.163 & -0.201 & -0.240 & -0.281 & -0.323 & -0.366 & -0.409 & -0.453 & -0.496 & -0.539 & -0.581 & -0.622 & -0.488 & -0.340 & -0.176 & -0.030 \\
-0.003 & -0.021 & -0.043 & -0.067 & -0.092 & -0.118 & -0.146 & -0.174 & -0.204 & -0.234 & -0.265 & -0.297 & -0.328 & -0.360 & -0.391 & -0.422 & -0.451 & -0.480 & -0.334 & -0.173 & -0.029 \\
-0.002 & -0.013 & -0.028 & -0.043 & -0.059 & -0.076 & -0.094 & -0.112 & -0.131 & -0.151 & -0.171 & -0.191 & -0.211 & -0.231 & -0.251 & -0.271 & -0.290 & -0.309 & -0.326 & -0.169 & -0.029 \\
-0.001 & -0.006 & -0.013 & -0.021 & -0.028 & -0.036 & -0.045 & -0.054 & -0.063 & -0.072 & -0.082 & -0.091 & -0.101 & -0.111 & -0.120 & -0.130 & -0.139 & -0.148 & -0.156 & -0.164 & -0.028 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.001 & 0.001 & 0.001 & 0.001 & 0.001 & 0.001 & 0.001 & 0.001 & 0.002 & 0.002 & 0.002 & 0.002 & 0.002 & 0.002 & -0.027
\end{bmatrix}$$

We find the constant coefficients c_i for $i = 0, 1......., 20$ using the system of liner equations (2.20) where the coefficient matrix is 21×21

$$c_i = A^{-1}D_i$$

$$c_i = \begin{pmatrix} 4.003597990 \\ 3.992780821 \\ 3.961243944 \\ 3.909893955 \\ 3.839671695 \\ 3.751548462 \\ 3.646522513 \\ 3.525615838 \\ 3.389871153 \\ 3.240349099 \\ 3.078125595 \\ 2.904289322 \\ 2.719939298 \\ 2.526182533 \\ 2.324131701 \\ 2.114902848 \\ 1.899613069 \\ 1.679378165 \\ 1.455310254 \\ 1.228515306 \\ 1.000090621 \end{pmatrix}$$

And using (2.18) and (2.19) to find c_{-1} and c_{21} respectively

c_{-1}=3.992827220 c_{21}= 0.771122212

where the constants c_i are to be determined

Step 3:

Calculate all required basis functions using equations (2.6) in this example

$0 < x < \frac{\pi}{4}$ and $h = \frac{\pi}{80}$

$0 \leq x < \frac{\pi}{80}$
$B_{-1} = -2752.13093829969x^3 + 324.227787655481x^2 - 12.7323954473516x + 0.166666666666667$
$B_0 = 8256.39281489907x^3 - 648.455575310962x^2 - 4.77418614763055 \cdot 10^{-15}x + 0.666666666666667$
$B_1 = -8256.39281489907x^3 + 324.227787655481x^2 + 12.7323954473516x + 0.166666666666667$
$B_2 = 2752.13093829969x^3$
$Y(x) = c_{-1}B_{-1} + c_0B_0 + c_1B_1 + c_2B_2$
$Y(x) = 2.38948\, x^3 - 6.99941x^2 - 0.00059x + 4.00000$
$Y(0) = 4$

$\frac{\pi}{80} \leq x < \frac{\pi}{40}$
$B_0 = -2752.13093829969x^3 + 648.455575310962x^2 - 50.9295817894065x + 1.33333333333333$
$B_1 = 8256.39281489907x^3 - 1621.1389382774x^2 + 89.1267681314614x - 0.833333333333334$
$B_2 = -8256.39281489907x^3 + 1296.91115062192x^2 - 50.9295817894065x + 0.666666666666667$
$B_3 = 2752.13093829969x^3 - 324.227787655481x^2 + 12.7323954473516x - 0.166666666666667$
$Y(x) = c_0B_0 + c_1B_1 + c_2B_2 + c_3B_3$
$Y(x) = 2.49507\, x^3 - 7.01185x^2 - 0.00010x + 3.99999$
$Y(\frac{\pi}{80}) = 3.9893275364$

$\frac{\pi}{40} \leq x < \frac{3\pi}{80}$

B_1=$-2752.13093829969x^3 + 972.683362966443x^2 - 114.591559026165x + 4.5$

B_2=$8256.39281489907x^3 - 2593.82230124385x^2 + 254.647908947033x - 7.33333333333333$

B_3=$-8256.39281489907x^3 + 2269.59451358837x^2 - 190.985931710274x + 5.16666666666667$

B_4= $2752.13093829969x^3 - 648.455575310962x^2 + 50.9295817894065x - 1.33333333333333$

Y(x)=$c_1B_1 + c_2B_2 + c_3B_3 + c_4B_4$

Y(x)= $2.58932x^3 - 7.03406x^2 + 0.00164x + 3.99995$

Y($\frac{\pi}{40}$) = 3.9579417589

$\frac{3\pi}{80} \leq x < \frac{\pi}{20}$

B_2=$-2752.13093829969x^3 + 1296.91115062192x^2 - 203.718327157626x + 10.6666666666667$

B_3=$8256.39281489907x^3 - 3566.50566421029x^2 + 496.563422446713x - 21.8333333333333$

B_4=$-8256.39281489907x^3 + 3242.27787655481x^2 - 407.436654315252x + 16.6666666666667$

B_5= $2752.13093829969x^3 - 972.683362966443x^2 + 114.591559026165x - 4.5$

Y(x)=$c_2B_2 + c_3B_3 + c_4B_4 + c_5B_5$

$Y(x)$ =2.67314 x$^3 - 7.063680x^2 + 0.005132x + 3.99981$

Y($\frac{3\pi}{80}$) =3.9067485765

$\frac{\pi}{20} \leq x < \frac{5\pi}{80}$

$B_3 = -2752.13093829969x^3 + 1621.1389382774x^2 - 318.309886183791x + 20.8333333333333$

$B_4 = 8256.39281489907x^3 - 4539.18902717673x^2 + 814.873308630504x - 47.3333333333333$

$B_5 = -8256.39281489907x^3 + 4214.96123952125x^2 - 700.281749604339x + 38.1666666666667$

$B_6 = 2752.13093829969x^3 - 1296.91115062192x^2 + 203.718327157626x - 10.6666666666667$

Y(x)=$c_3B_3 + c_4B_4 + c_5B_5 + c_6B_6$

$Y(x) = 2.74733\ \text{x}^3 - 7.098643x^2 + 0.01062x + 3.999523$

Y($\frac{\pi}{20}$) =3.8366881995

$\frac{5\pi}{80} \leq x < \frac{3\pi}{40}$

$B_4 = -2752.13093829969x^3 + 1945.36672593289x^2 - 458.366236104659x + 36.0$

$B_5 = 8256.39281489907x^3 - 5511.87239014317x^2 + 1209.5775674984x - 86.8333333333334$

$B_6 = -8256.39281489907x^3 + 5187.64460248769x^2 - 1069.52121757754x + 72.6666666666667$

$B_7 = 2752.13093829969x^3 - 1621.1389382774x^2 + 318.309886183791x - 20.8333333333333$

Y(x)=$c_4B_4 + c_5B_5 + c_6B_6 + c_7B_7$

$Y(x) = 2.812650\ \text{x}^3 - 7.13712x^2 + 0.01818x + 3.99903$

Y($\frac{5\pi}{80}$) = 3.7487313426

$\frac{3\pi}{40} \leq x < \frac{7\pi}{80}$

$B_5 = -2752.13093829969x^3 + 2269.59451358837x^2 - 623.88737692023x + 57.1666666666667$

$B_6 = 8256.39281489907x^3 - 6484.55575310962x^2 + 1680.67619905041x - 143.333333333333$

$B_7 = -8256.39281489907x^3 + 6160.32796545414x^2 - 1515.15505823484x + 123.166666666667$

$B_8 = 2752.13093829969x^3 - 1945.36672593289x^2 + 458.366236104659x - 36.0$

Y(x)=$c_5B_5 + c_6B_6 + c_7B_7 + c_8B_8$

$Y(x) = 2.869690$ x$^3 - 7.17744x^2 + 0.02768x + 3.99828$

Y($\frac{3\pi}{40}$) = 3.6438757255

$\frac{7\pi}{80} \leq x < \frac{\pi}{10}$

$B_6 = -2752.13093829969x^3 + 2593.82230124385x^2 - 814.873308630504x + 85.3333333333333$

$B_7 = 8256.39281489907x^3 - 7457.23911607606x^2 + 2228.16920328653x - 219.833333333333$

$B_8 = -8256.39281489907x^3 + 7133.01132842058x^2 - 2037.18327157626x + 192.666666666667$

$B_9 = 2752.13093829969x^3 - 2269.59451358837x^2 + 623.88737692023x - 57.1666666666667$

Y(x)=$c_6B_6 + c_7B_7 + c_8B_8 + c_9B_9$

$Y(x) = 2.91902$ x$^3 - 7.218120x^2 + 0.038861x + 3.99726$

Y($\frac{7\pi}{80}$) =3.5231428361

$\frac{\pi}{10} \leq x < \frac{9\pi}{80}$

$B_7 = -2752.13093829969x^3 + 2918.05008889933x^2 - 1031.32403123548x + 121.5$

$B_8 = 8256.39281489907x^3 - 8429.92247904251x^2 + 2852.05658020676x - 319.333333333333$

$B_9 = -8256.39281489907x^3 + 8105.69469138702x^2 - 2635.60585760179x + 284.166666666667$

$B_{10} = 2752.13093829969x^3 - 2593.82230124385x^2 + 814.873308630504x - 85.3333333333333$

Y(x)=$c_7B_7 + c_8B_8 + c_9B_9 + c_{10}B_{10}$

$Y(x) = 2.96107\ \mathrm{x}^3 - 7.25775x^2 + 0.05131x + 3.99595$

Y($\frac{\pi}{10}$) =3.3875749246

$\frac{9\pi}{80} \leq x < \frac{\pi}{8}$

$B_8 = -2752.13093829969x^3 + 3242.27787655481x^2 - 1273.23954473516x + 166.666666666667$

$B_9 = 8256.39281489907x^3 - 9402.60584200894x^2 + 3552.3383298111x - 444.833333333333$

$B_{10} = -8256.39281489907x^3 + 9078.37805435346x^2 - 3310.42281631142x + 400.666666666667$

$B_{11} = 2752.13093829969x^3 - 2918.05008889933x^2 + 1031.32403123548x - 121.5$

Y(x)=$c_8B_8 + c_9B_9 + c_{10}B_{10} + c_{11}B_{11}$

$Y(x) = 2.99619\ \mathrm{x}^3 - 7.29499x^2 + 0.06447x + 3.99440$

Y($\frac{9\pi}{80}$) =3.2382321907

$\frac{\pi}{8} \leq x < \frac{11\pi}{80}$

B_9=$-2752.13093829969x^3 + 3566.50566421029x^2 - 1540.61984912955x + 221.833333333333$

B_{10}=$8256.39281489907x^3 - 10375.2892049754x^2 + 4329.01445209955x - 599.333333333333$

B_{11}=$-8256.39281489907x^3 + 10051.0614173199x^2 - 4061.63414770517x + 545.166666666667$

B_{12}= $2752.13093829969x^3 - 3242.27787655481x^2 + 1273.23954473516x - 166.666666666667$

Y(x)=$c_9B_9 + c_{10}B_{10} + c_{11}B_{11} + c_{12}B_{12}$

$Y(x) = 3.02465\ \text{x}^3 - 7.32851x^2 + 0.07764x + 3.99268$

Y($\frac{\pi}{8}$) =3.0761901337

$\frac{11\pi}{80} \leq x < \frac{3\pi}{20}$

B_{10}=$-2752.13093829969x^3+3890.73345186577x^2-1833.46494441863x+288.0$

B_{11}=$8256.39281489907x^3 - 11347.9725679418x^2 + 5182.08494707211x - 785.833333333333$

B_{12}=$-8256.39281489907x^3 + 11023.7447802863x^2 - 4889.23985178302x + 720.666666666666$

B_{13}= $2752.13093829969x^3 - 3566.50566421029x^2 + 1540.61984912955x - 221.833333333333$

Y(x)=$c_{10}B_{10} + c_{11}B_{11} + c_{12}B_{12} + c_{13}B_{13}$

$Y(x) = 3.04663\ \text{x}^3 - 7.35700x^2 + 0.08994x + 3.99091$

Y($\frac{11\pi}{80}$) =2.9025370300

$\frac{3\pi}{20} \leq x < \frac{13\pi}{80}$

B_{11}=$-2752.13093829969x^3 + 4214.96123952125x^2 - 2151.77483060243x + 366.166666666667$

B_{12}=$8256.39281489907x^3 - 12320.6559309083x^2 + 6111.54981472878x - 1007.33333333333$

B_{13}=$-8256.39281489907x^3 + 11996.4281432528x^2 - 5793.23992854499x + 930.166666666667$

B_{14}= $2752.13093829969x^3 - 3890.73345186577x^2 + 1833.46494441863x - 288.0$

Y(x)=$c_{11}B_{11} + c_{12}B_{12} + c_{13}B_{13} + c_{14}B_{14}$

$Y(x) = 3.06223\ x^3 - 7.37906x^2 + 0.10034x + 3.98928$

Y($\frac{3\pi}{20}$) =2.7183715079

$\frac{13\pi}{80} \leq x < \frac{7\pi}{40}$

B_{12}=$-2752.13093829969x^3 + 4539.18902717673x^2 - 2495.54950768092x + 457.333333333333$

B_{13}=$8256.39281489907x^3 - 13293.3392938747x^2 + 7117.40905506956x - 1266.83333333333$

B_{14}=$-8256.39281489907x^3 + 12969.1115062192x^2 - 6773.63437799107x + 1176.66666666667$

B_{15}= $2752.13093829969x^3 - 4214.96123952125x^2 + 2151.77483060243x - 366.166666666667$

Y(x)=$c_{12}B_{12} + c_{13}B_{13} + c_{14}B_{14} + c_{15}B_{15}$

$Y(x) = 3.07150\ x^3 - 7.39325x^2 + 0.10758x + 3.98804$

Y($\frac{13\pi}{80}$) =2.5248001883

$\frac{7\pi}{40} \leq x < \frac{15\pi}{80}$

B_{13}=$-2752.13093829969x^3+4863.41681483221x^2-2864.78897565412x+562.5$

B_{14}=$8256.39281489907x^3 - 14266.0226568412x^2 + 8199.66266809445x - 1567.33333333333$

B_{15}=$-8256.39281489907x^3 + 13941.7948691857x^2 - 7830.42320012125x + 1463.16666666667$

B_{16}= $2752.13093829969x^3 - 4539.18902717673x^2 + 2495.54950768092x - 457.333333333333$

Y(x)=$c_{13}B_{13} + c_{14}B_{14} + c_{15}B_{15} + c_{16}B_{16}$

$Y(x) = 3.07439$ x^3 $- 7.39802x^2 + 0.11021x + 3.98756$

Y($\frac{7\pi}{40}$) =2.3229353642

$\frac{15\pi}{80} \leq x < \frac{\pi}{5}$

B_{14}=$-2752.13093829969x^3 + 5187.64460248769x^2 - 3259.49323452202x + 682.666666666667$

B_{15}=$8256.39281489907x^3 - 15238.7060198076x^2 + 9358.31065380344x - 1911.83333333333$

B_{16}=$-8256.39281489907x^3 + 14914.4782321521x^2 - 8963.60639493555x + 1792.66666666667$

B_{17}= $2752.13093829969x^3 - 4863.41681483221x^2 + 2864.78897565412x - 562.5$

Y(x)=$c_{14}B_{14} + c_{15}B_{15} + c_{16}B_{16} + c_{17}B_{17}$

$Y(x) = 3.07083$ x^3 $- 7.39173x^2 + 0.10650x + 3.98829$

Y($\frac{15\pi}{80}$) =2.1138926936

$\frac{\pi}{5} \leq x < \frac{17\pi}{80}$

B_{15}=$-2752.13093829969x^3 + 5511.87239014317x^2 - 3679.66228428462x + 818.833333333333$

B_{16}=$8256.39281489907x^3 - 16211.389382774x^2 + 10593.3530121966x - 2303.33333333333$

B_{17}=$-8256.39281489907x^3 + 15887.1615951186x^2 - 10173.183962434x + 2168.16666666667$

B_{18}= $2752.13093829969x^3 - 5187.64460248769x^2 + 3259.49323452202x - 682.666666666667$

Y(x)=$c_{15}B_{15} + c_{16}B_{16} + c_{17}B_{17} + c_{18}B_{18}$

$Y(x) = 3.06069$ x$^3 - 7.37262x^2 + 0.094493x + 3.99080$

Y($\frac{\pi}{5}$) =1.8987888813

$\frac{17\pi}{80} \leq x < \frac{9\pi}{40}$

B_{16}=$-2752.13093829969x^3+5836.10017779866x^2-4125.29612494193x+972.0$

B_{17}=$8256.39281489907x^3 - 17184.0727457405x^2 + 11904.7897432738x - 2744.83333333333$

B_{18}=$-8256.39281489907x^3 + 16859.844958085x^2 - 11459.1559026165x + 2592.66666666667$

B_{19}= $2752.13093829969x^3 - 5511.87239014317x^2 + 3679.66228428462x - 818.833333333333$

Y(x)=$c_{16}B_{16} + c_{17}B_{17} + c_{18}B_{18} + c_{19}B_{19}$

$Y(x) = 3.04378$ x$^3 - 7.33874x^2 + 0.07188x + 3.99584$

Y($\frac{17\pi}{80}$) =1.6787393307

$\frac{9\pi}{40} \leq x < \frac{19\pi}{80}$

B_{17}=$-2752.13093829969x^3 + 6160.32796545414x^2 - 4596.39475649394x + 1143.16666666667$

B_{18}=$8256.39281489907x^3 - 18156.7561087069x^2 + 13292.6208470351x - 3239.33333333333'$

B_{19}=$-8256.39281489907x^3 + 17832.5283210514x^2 - 12821.5222154831x + 3069.16666666667'$

B_{20}= $2752.13093829969x^3 - 5836.10017779866x^2 + 4125.29612494193x - 972.0$

Y(x)=$c_{17}B_{17} + c_{18}B_{18} + c_{19}B_{19} + c_{20}B_{20}$

$Y(x)$ = 3.01991 x^3 $- 7.28813x^2 + 0.03610x + 4.00427$

Y($\frac{9\pi}{40}$) =1.4548557478

$\frac{19\pi}{80} \leq x < \frac{\pi}{4}$

B_{18}=$-2752.13093829969x^3 + 6484.55575310962x^2 - 5092.95817894065x + 1333.33333333333$

B_{19}=$8256.39281489907x^3 - 19129.4394716734x^2 + 14756.8463234805x - 3789.83333333333$

B_{20}=$-8256.39281489907x^3 + 18805.2116840179x^2 - 14260.2829010338x + 3600.66666666667$

B_{21}= $2752.13093829969x^3 - 6160.32796545414x^2 + 4596.39475649394x - 1143.16666666667$

Y(x)=$c_{18}B_{18} + c_{19}B_{19} + c_{20}B_{20} + c_{21}B_{21}$

$Y(x)$ = 2.98885 x^3 $- 7.21861x^2 - 0.01577x + 4.01717$

Y($\frac{19\pi}{80}$) = 1.2282436831

The cubic B-spline polynomials are as follows:

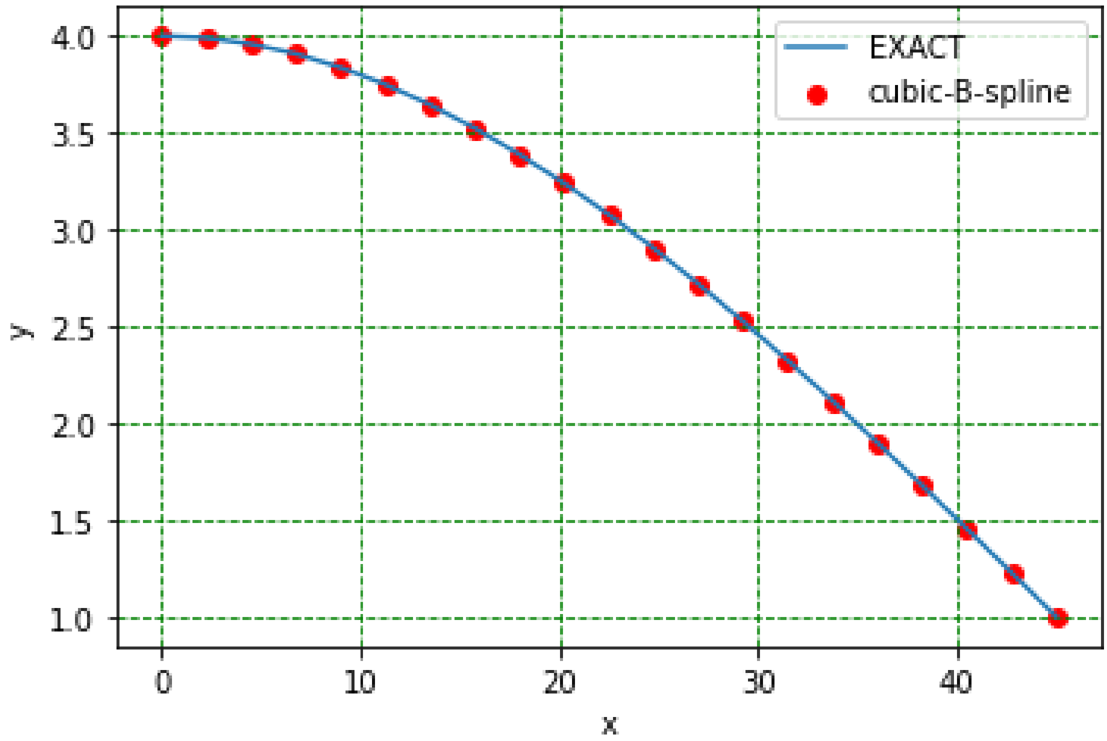
$$
Y(x)= \begin{cases}
2.38948x^3 - 6.99941x^2 - 0.00059x + 4.00000 & , \quad [0. \frac{\pi}{80}) \\
2.49507x^3 - 7.01185x^2 - 0.00010x + 3.99999 & , \quad [\frac{\pi}{80}, \frac{\pi}{40}) \\
2.58932x^3 - 7.03406x^2 + 0.00164x + 3.99995 & , \quad [\frac{\pi}{40}, \frac{3\pi}{80}) \\
2.67314x^3 - 7.063680x^2 + 0.005132x + 3.99981 & , \quad [\frac{3\pi}{80}, \frac{\pi}{20}) \\
2.74733x^3 - 7.098643x^2 + 0.01062x + 3.999523 & , \quad [\frac{\pi}{20}, \frac{5\pi}{80}) \\
2.812650x^3 - 7.13712x^2 + 0.01818x + 3.99903 & , \quad [\frac{5\pi}{80}, \frac{3\pi}{40}) \\
2.869690x^3 - 7.17744x^2 + 0.02768x + 3.99828 & , \quad [\frac{3\pi}{40}, \frac{7\pi}{80}) \\
2.91902x^3 - 7.218120x^2 + 0.03886x + 3.99726 & , \quad [\frac{7\pi}{80}, \frac{\pi}{10}) \\
2.96107x^3 - 7.25775x^2 + 0.05131x + 3.99595 & , \quad [\frac{\pi}{10}, \frac{9\pi}{80}) \\
2.99619x^3 - 7.29499x^2 + 0.06447x + 3.99440 & , \quad [\frac{9\pi}{80}, \frac{\pi}{8}) \\
3.02465x^3 - 7.32851x^2 + 0.07764x + 3.99268 & , \quad [\frac{\pi}{8}, \frac{11\pi}{80}) \\
3.04663x^3 - 7.35700x^2 + 0.08994x + 3.99091 & , \quad [\frac{11\pi}{80}, \frac{3\pi}{20}) \\
3.06223x^3 - 7.37906x^2 + 0.10034x + 3.98928 & , \quad [\frac{3\pi}{20}, \frac{13\pi}{80}) \\
3.07150x^3 - 7.39325x^2 + 0.10758x + 3.98804 & , \quad [\frac{13\pi}{80}, \frac{7\pi}{40}) \\
3.07439x^3 - 7.39802x^2 + 0.11021x + 3.98756 & , \quad [\frac{7\pi}{40}, \frac{15\pi}{80}) \\
3.07083x^3 - 7.39173x^2 + 0.10650x + 3.98829 & , \quad [\frac{15\pi}{80}, \frac{\pi}{5}) \\
3.06069x^3 - 7.37262x^2 + 0.09448x + 3.99081 & , \quad [\frac{\pi}{5}, \frac{17\pi}{80}) \\
3.04378x^3 - 7.33874x^2 + 0.07188x + 3.99584 & , \quad [\frac{17\pi}{80}, \frac{9\pi}{40}) \\
3.01991x^3 - 7.28813x^2 + 0.03610x + 4.00427 & , \quad [\frac{9\pi}{40}, \frac{19\pi}{80}) \\
2.98885x^3 - 7.21861x^2 - 0.01577x + 4.01717 & , \quad [\frac{19\pi}{80}, \frac{\pi}{4})
\end{cases} \qquad (3.8)
$$

Table 2
The Cubic B-spline results example (3.1.2)

x_i	Cubic B-spline	Exact
0.00	4.0000000000	4.0000000000
$\frac{\pi}{80}$	3.9893275364	3.9893481701
$\frac{\pi}{40}$	3.9579417589	3.9579782444
$\frac{3\pi}{80}$	3.9067485765	3.9067967056
$\frac{\pi}{20}$	3.8366881995	3.8367442991
$\frac{5\pi}{80}$	3.7487313426	3.7487922376
$\frac{3\pi}{40}$	3.6438757255	3.6439387036
$\frac{7\pi}{80}$	3.5231428361	3.5232056151
$\frac{\pi}{10}$	3.3875749246	3.3876356206
$\frac{9\pi}{80}$	3.2382321907	3.2382892895
$\frac{\pi}{8}$	3.0761901337	3.0762424637
$\frac{11\pi}{80}$	2.9025370300	2.9025837374
$\frac{3\pi}{20}$	2.7183715079	2.7184120337
$\frac{13\pi}{80}$	2.5248001883	2.5248342470
$\frac{7\pi}{40}$	2.3229353642	2.3229629245
$\frac{15\pi}{80}$	2.1138926936	2.1139139602
$\frac{\pi}{5}$	1.8987888813	1.8988042779
$\frac{17\pi}{80}$	1.6787393307	1.6787494845
$\frac{9\pi}{40}$	1.4548557478	1.4548614743
$\frac{19\pi}{80}$	1.2282436831	1.2282459716
$\frac{\pi}{4}$	1.0000000000	1.0000000000
$\boldsymbol{RMSE} = 4.188842800472591 * 10^{-5}$		

Figure 2

The exact and cubic B-spline solutions of example (3.1.2)



<u>Example(3.1.3) :</u>

We consider the following linear boundary value problem with variable coefficients

$$x^2y'' + 3xy' + 3y = 0 \quad \text{for} \quad 1 < x < 2 \tag{3.9}$$

with boundary conditions

$$y(1) = 5 \qquad\qquad y(2) = 0 \tag{3.10}$$

The exact solution to boundary value problem is

$$y(x) = \frac{5}{x}[\cos(\sqrt{2}lnx) - \cot(\sqrt{2}ln2)\sin(\sqrt{2}lnx)] \tag{3.11}$$

We approximate the solution in (3.9) with boundary conditions (3.10) using the cubic B-spline method with$N$=20 in order to use (2.11), we first need to find the constant coefficients c_i for $i = -1, 0, 1......., 21$ using the system of liner equations (2.20) where the coefficient matrix is 21×21 and using (2.18) and (2.19) to find c_{-1} and c_{21} respectively .

Step 1:

We first need to find $a_1(x)$, $a_2(x)$, $a_3(x)$, $f(x)$ using (2.8)

$a_1(x) = x^2$, $a_2(x) = 3x$, $a_3(x) = 3$, $f(x) = 0$, $h = .05$ $\alpha = 5$ $\beta = 0$ using (2.11)

now we need to find p_i, q_i, r_i, o_1, o_2, o_3, o_4, z_0, z_N in (2.21)

. p_i = 5.5575, 6.15 , 6.7725, 7.425 , 8.1075, 8.82 , 9.5625, 10.335 , 11.137, 11.97 , 12.8325 ,13.725 , 14.6475, 15.6 , 16.5825 ,17.595, 18.6375 ,19.71 20.8125, 21.945, 23.1075

q_i = -11.97 , -13.200 , -14.490, -15.8400, -17.250000, -18.72000, -20.2500 , -21.84000, -23.4900000, -25.200, -26.97000, -28.8000, -30.6900, -32.640000, -34.6500, -36.7200, -38.8500, -41.04000, -43.29000, -45.60000, -47.97000000

r_i = 6.45750000, 7.0950, 7.76250 , 8.460, 9.18750000, 9.94500 10.732500, 11.550000, 12.3975000, 13.275000, 14.18250, 15.12000, 16.0875000, 17.08500000, 18.112500, 19.170000, 20.25750, 21.3750000, 22.522500, 23.7000000, 24.9075000

o_1 = -34.2 o_2 =0.9000000000000004 o_3 = -1.7999999999999972

o_4 =-147.60000000000014 z_0 = -27.7875 z_n = 0.0

Step 2: We find the constant coefficients c_i for i = 0, 1 ....,20 using the system of liner equations (2.20) where the coefficient matrix is 21×21

$$
\mathrm{D}_i{=}6 \begin{pmatrix} z0 \\ h^2 f(x_1) \\ h^2 f(x_2) \\ h^2 f(x_3) \\ h^2 f(x_4) \\ h^2 f(x_5) \\ h^2 f(x_6) \\ h^2 f(x_7) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ h^2 f(x_{N-3}) \\ h^2 f(x_{N-2}) \\ h^2 f(x_{N-1}) \\ z_N \end{pmatrix} = \begin{pmatrix} -166.725 \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}
$$

$$
A=\begin{bmatrix}
-34.200 & 0.900 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
6.150 & -13.200 & 7.095 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 6.773 & -14.490 & 7.763 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 7.425 & -15.840 & 8.460 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 8.108 & -17.250 & 9.188 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 8.820 & -18.720 & 9.945 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 9.563 & -20.250 & 10.733 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 10.335 & -21.840 & 11.550 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 11.138 & -23.490 & 12.398 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 11.970 & -25.200 & 13.275 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 12.833 & -26.970 & 14.183 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 13.725 & -28.800 & 15.120 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 14.648 & -30.690 & 16.088 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 15.600 & -32.640 & 17.085 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 16.583 & -34.650 & 18.113 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 17.595 & -36.720 & 19.170 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 18.638 & -38.850 & 20.258 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 19.710 & -41.040 & 21.375 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 20.813 & -43.290 & 22.523 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 21.945 & -45.600 & 23.700 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.00 & 0.000 & 0.000 & 0.000 & -1.80 & -147.6
\end{bmatrix}
$$

$$A^{-1} = \begin{bmatrix}
-0.030 & -0.004 & -0.004 & -0.004 & -0.003 & -0.003 & -0.003 & -0.003 & -0.002 & -0.002 & -0.002 & -0.002 & -0.002 & -0.001 & -0.001 & -0.001 & -0.001 & -0.001 & -0.000 & -0.000 & -0.000 \\
-0.027 & -0.151 & -0.143 & -0.135 & -0.126 & -0.118 & -0.109 & -0.101 & -0.092 & -0.084 & -0.076 & -0.067 & -0.059 & -0.051 & -0.043 & -0.036 & -0.028 & -0.021 & -0.014 & -0.007 & -0.001 \\
-0.025 & -0.136 & -0.263 & -0.247 & -0.232 & -0.216 & -0.201 & -0.185 & -0.170 & -0.154 & -0.139 & -0.124 & -0.109 & -0.094 & -0.080 & -0.066 & -0.052 & -0.038 & -0.025 & -0.012 & -0.002 \\
-0.022 & -0.123 & -0.237 & -0.344 & -0.323 & -0.301 & -0.279 & -0.258 & -0.236 & -0.214 & -0.193 & -0.172 & -0.151 & -0.131 & -0.111 & -0.091 & -0.072 & -0.054 & -0.035 & -0.017 & -0.003 \\
-0.020 & -0.110 & -0.213 & -0.309 & -0.401 & -0.374 & -0.347 & -0.320 & -0.293 & -0.266 & -0.240 & -0.214 & -0.188 & -0.163 & -0.138 & -0.114 & -0.090 & -0.066 & -0.044 & -0.021 & -0.003 \\
-0.018 & -0.099 & -0.190 & -0.277 & -0.359 & -0.436 & -0.405 & -0.373 & -0.342 & -0.311 & -0.280 & -0.249 & -0.219 & -0.190 & -0.161 & -0.133 & -0.105 & -0.078 & -0.051 & -0.025 & -0.004 \\
-0.016 & -0.088 & -0.170 & -0.247 & -0.320 & -0.389 & -0.454 & -0.419 & -0.384 & -0.349 & -0.314 & -0.280 & -0.246 & -0.213 & -0.181 & -0.149 & -0.118 & -0.087 & -0.057 & -0.028 & -0.005 \\
-0.014 & -0.078 & -0.151 & -0.219 & -0.284 & -0.346 & -0.403 & -0.458 & -0.419 & -0.381 & -0.343 & -0.306 & -0.269 & -0.233 & -0.197 & -0.163 & -0.129 & -0.095 & -0.063 & -0.031 & -0.005 \\
-0.012 & -0.069 & -0.133 & -0.194 & -0.251 & -0.305 & -0.356 & -0.404 & -0.450 & -0.409 & -0.368 & -0.328 & -0.289 & -0.250 & -0.212 & -0.174 & -0.138 & -0.102 & -0.067 & -0.033 & -0.005 \\
-0.011 & -0.061 & -0.117 & -0.170 & -0.220 & -0.268 & -0.313 & -0.355 & -0.395 & -0.432 & -0.389 & -0.347 & -0.305 & -0.264 & -0.224 & -0.184 & -0.146 & -0.108 & -0.071 & -0.035 & -0.006 \\
-0.010 & -0.053 & -0.102 & -0.148 & -0.192 & -0.233 & -0.272 & -0.309 & -0.344 & -0.376 & -0.407 & -0.362 & -0.319 & -0.276 & -0.234 & -0.193 & -0.152 & -0.113 & -0.074 & -0.036 & -0.006 \\
-0.008 & -0.046 & -0.088 & -0.128 & -0.165 & -0.201 & -0.235 & -0.266 & -0.296 & -0.324 & -0.351 & -0.375 & -0.330 & -0.286 & -0.242 & -0.200 & -0.158 & -0.117 & -0.077 & -0.038 & -0.006 \\
-0.007 & -0.039 & -0.075 & -0.109 & -0.141 & -0.171 & -0.200 & -0.227 & -0.253 & -0.276 & -0.299 & -0.320 & -0.340 & -0.294 & -0.249 & -0.205 & -0.162 & -0.120 & -0.079 & -0.039 & -0.006 \\
-0.006 & -0.033 & -0.063 & -0.091 & -0.118 & -0.144 & -0.168 & -0.191 & -0.212 & -0.232 & -0.251 & -0.268 & -0.285 & -0.301 & -0.255 & -0.210 & -0.166 & -0.123 & -0.081 & -0.040 & -0.006 \\
-0.005 & -0.027 & -0.052 & -0.075 & -0.097 & -0.118 & -0.138 & -0.157 & -0.174 & -0.191 & -0.206 & -0.221 & -0.234 & -0.247 & -0.259 & -0.213 & -0.169 & -0.125 & -0.082 & -0.040 & -0.006 \\
-0.004 & -0.021 & -0.041 & -0.060 & -0.078 & -0.095 & -0.111 & -0.125 & -0.140 & -0.153 & -0.165 & -0.177 & -0.188 & -0.198 & -0.207 & -0.216 & -0.171 & -0.127 & -0.083 & -0.041 & -0.007 \\
-0.003 & -0.016 & -0.032 & -0.046 & -0.060 & -0.073 & -0.085 & -0.096 & -0.107 & -0.117 & -0.127 & -0.136 & -0.144 & -0.152 & -0.159 & -0.166 & -0.173 & -0.128 & -0.084 & -0.041 & -0.007 \\
-0.002 & -0.012 & -0.023 & -0.033 & -0.043 & -0.052 & -0.061 & -0.069 & -0.077 & -0.085 & -0.091 & -0.098 & -0.104 & -0.109 & -0.115 & -0.120 & -0.124 & -0.129 & -0.085 & -0.041 & -0.007 \\
-0.001 & -0.008 & -0.015 & -0.021 & -0.028 & -0.034 & -0.039 & -0.044 & -0.049 & -0.054 & -0.058 & -0.063 & -0.066 & -0.070 & -0.073 & -0.077 & -0.080 & -0.082 & -0.085 & -0.042 & -0.007 \\
-0.001 & -0.004 & -0.007 & -0.010 & -0.013 & -0.016 & -0.019 & -0.021 & -0.024 & -0.026 & -0.028 & -0.030 & -0.032 & -0.034 & -0.035 & -0.037 & -0.038 & -0.039 & -0.041 & -0.042 & -0.007 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.001 & -0.007
\end{bmatrix}$$

We find the constant coefficients c_i for $i = 0, 1, ..., 20$ using the system of liner equations (2.20) where the coefficient matrix is 21×21

$$c_i = A^{-1} D_i$$

$$c_i = \begin{pmatrix} 4.994073274 \\ 4.524784412 \\ 4.089302834 \\ 3.685654832 \\ 3.311784751 \\ 2.965642493 \\ 2.645237403 \\ 2.348670027 \\ 2.074149336 \\ 1.820000442 \\ 1.584666183 \\ 1.366704832 \\ 1.164785437 \\ 0.977681805 \\ 0.804265806 \\ 0.643500436 \\ 0.494432924 \\ 0.356188065 \\ 0.227961883 \\ 0.109015687 \\ -0.001329460 \end{pmatrix}$$

and using (2.18) and (2.19) to find c_{-1} and c_{21} respectively

c_{-1}=5.498922492 c_{21}= -0.103697849

where the constants c_i are to be determined

Step 3:

Calculate all required basis functions using equations (2.6) in this example

$1 \leq x < 1.05$

B_{-1}=$-1333.33333333333x^3 + 4200.0x^2 - 4410.0x + 1543.5$

B_0=$4000.0x^3 - 12400.0x^2 + 12800.0x - 4399.33333333333$

B_1= $-4000.0x^3 + 12200.0x^2 - 12390.0x + 4190.16666666667$

B_2= $1333.33333333333x^3 - 4000.0x^2 + 4000.0x - 1333.33333333333$

Y(x)=$c_{-1}B_{-1} + c_0B_0 + c_1B_1 + c_2B_2$

$Y(x) =$ - 2.33743 $x^3 + 14.12436x^2 - 30.97781x + 24.190881$

Y(1) = 5

$1.05 \leq x < 1.10$

B_0=$-1333.33333333333x^3 + 4400.0x^2 - 4840.0x + 1774.66666666667$

B_1=$4000.0x^3 - 13000.0x^2 + 14070.0x - 5070.83333333333$

B_2= $-4000.0x^3 + 12800.0x^2 - 13640.0x + 4840.66666666667$

B_3= $1333.33333333333x^3 - 4200.0x^2 + 4410.0x - 1543.5$

Y(x)=$c_0B_0 + c_1B_1 + c_2B_2 + c_3B_3$

$Y(x) =$ - 2.63161 $x^3 + 15.05103x^2 - 31.95081x + 24.53143$

Y(1.05) =4.5304189596

$1.10 \leq x < 1.15$

B_1=$-1333.33333333333x^3 + 4600.0x^2 - 5290.0x + 2027.83333333333$

B_2=$4000.0x^3 - 13600.0x^2 + 15400.0x - 5807.33333333333$

B_3= $-4000.0x^3 + 13400.0x^2 - 14950.0x + 5555.16666666667$

B_4= $-4000.0x^3 + 13400.0x^2 - 14950.0x + 5555.16666666667$

Y(x)=$c_1B_1 + c_2B_2 + c_3B_3 + c_4B_4$

$Y(x)$ = - 2.74087 $x^3 + 15.41159x^2 - 32.34743944x + 24.67686$

Y(1.10) = 4.0946084301

$1.15 \leq x < 1.2$

B_2=$-1333.33333333333x^3 + 4800.0x^2 - 5760.0x + 2304.0$

B_3=$4000.0x^3 - 14200.0x^2 + 16790.0x - 6611.83333333333$

B_4= $-4000.0x^3 + 14000.0x^2 - 16320.0x + 6336.6666666666$

B_5= $1333.33333333333x^3 - 4600.0x^2 + 5290.0x - 2027.83333333333$

Y(x)=$c_2B_2 + c_3B_3 + c_4B_4 + c_5B_5$

$Y(x)$ = - 2.73346 $x^3 + 15.38604x^2 - 32.31805x + 24.66559$

Y(1.15) =3.6906178188

$1.20 \leq x < 1.25$

B_3=$-1333.33333333333x^3 + 5000.0x^2 - 6250.0x + 2604.16666666667$

B_4=$4000.0x^3 - 14800.0x^2 + 18240.0x - 7487.33333333333$

B_5= $-4000.0x^3 + 14600.0x^2 - 17750.0x + 7188.16666666666$

B_6= $1333.33333333333x^3 - 4800.0x^2 + 5760.0x - 2304.0$

Y(x)=$c_3B_3 + c_4B_4 + c_5B_5 + c_6B_6$

$Y(x)$ = - 2.65421 $x^3 + 15.10071x^2 - 31.97566x + 24.52864$

Y(1.2) =3.3164060550

$$1.25 \leq x < 1.30$$

B_4=$-1333.33333333333x^3 + 5200.0x^2 - 6760.0x + 2929.33333333333$

B_5=$4000.0x^3 - 15400.0x^2 + 19750.0x - 8436.83333333333$

B_6= $-4000.0x^3 + 15200.0x^2 - 19240.0x + 8112.66666666667$

B_7= $1333.33333333333x^3 - 5000.0x^2 + 6250.0x - 2604.16666666667$

Y(x)=c$_4B_4 + c_5B_5 + c_6B_6 + c_7B_7$

$Y(x)$ = - 2.53260 x^3 $+ 14.64470x^2 - 31.40564x + 24.29113$

Y(1.25) = 2.9699320212

$$1.30 \leq x < 1.35$$

B_5=$-1333.33333333333x^3 + 5400.0x^2 - 7290.0x + 3280.5$

B_6=$4000.0x^3 - 16000.0x^2 + 21320.0x - 9463.33333333333$

B_7= $-4000.0x^3 + 15800.0x^2 - 20790.0x + 9113.16666666666$

B_8= $1333.33333333333x^3 - 5200.0x^2 + 6760.0x - 2929.33333333333$

Y(x)=c$_5B_5 + c_6B_6 + c_7B_7 + c_8B_8$

$Y(x)$ = - 2.38804 x^3 $+ 14.08090x^2 - 30.67270x + 23.97353$

Y(1.30) = 2.6492103552

$$1.35 \leq x < 1.40$$

B_6=$-1333.33333333333x^3 + 5600.0x^2 - 7840.0x + 3658.66666666667$

B_7=$4000.0x^3 - 16600.0x^2 + 22950.0x - 10569.8333333333$

B_8= $-4000.0x^3 + 16400.0x^2 - 22400.0x + 10192.6666666667$

B_9= $1333.33333333333x^3 - 5400.0x^2 + 7290.0x - 3280.5$

Y(x)=c$_6B_6 + c_7B_7 + c_8B_8 + c_9B_9$

$Y(x)$ = - 2.23318 x^3 $+ 13.45373x^2 - 29.82604x + 23.59252$

Y(1.35) = 2.3523444743

$1.40 \leq x < 1.45$

B_7=$-1333.33333333333x^3 + 5800.0x^2 - 8410.0x + 4064.83333333333$

B_8=$4000.0x^3 - 17200.0x^2 + 24640.0x - 11759.3333333333$

B_9= $-4000.0x^3 + 17000.0x^2 - 24070.0x + 11354.1666666667'$

B_{10}= $1333.33333333333x^3 - 5600.0x^2 + 7840.0x - 3658.66666666667$

Y(x)=$c_7 B_7 + c_8 B_8 + c_9 B_9 + c_{10} B_{10}$

$Y(x) =$ - 2.07621 $x^3 + 12.79446x^2 - 28.90305x + 23.16180$

Y(1.40) = 2.0775446353

$1.45 \leq x < 1.50$

B_8=$-1333.33333333333x^3 + 6000.0x^2 - 9000.0x + 4500.0$

B_9=$4000.0x^3 - 17800.0x^2 + 26390.0x - 13034.8333333333$

B_{10}= $-4000.0x^3 + 17600.0x^2 - 25800.0x + 12600.6666666667$

B_{11}= $1333.33333333333x^3 - 5800.0x^2 + 8410.0x - 4064.83333333333$

Y(x)=$c_8 B_8 + c_9 B_9 + c_{10} B_{10} + c_{11} B_{11}$

$Y(x) = -1.92230x^3 + 12.12495x^2 - 27.93225x + 22.69258$

$Y(1.45) = 1.8231362142$

$1.50 \leq x < 1.55$

B_9=$-1333.33333333333x^3 + 6200.0x^2 - 9610.0x + 4965.16666666667$

B_{10}=$4000.0x^3 - 18400.0x^2 + 28200.0x - 14399.3333333333$

B_{11}= $-4000.0x^3 + 18200.0x^2 - 27590.0x + 13935.1666666667$

B_{12}= $1333.33333333333x^3 - 6000.0x^2 + 9000.0x - 4500.0$

Y(x)=$c_9 B_9 + c_{10} B_{10} + c_{11} B_{11} + c_{12} B_{12}$

Y(x)= $-1.77460x^3 + 11.46030x^2 - 26.93528x + 22.19409$

Y(1.50) = 1.5875616676

$1.55 \leq x < 1.60$

B_{10}=$-1333.33333333333x^3 + 6400.0x^2 - 10240.0x + 5461.33333333333$

B_{11}=$4000.0x^3 - 19000.0x^2 + 30070.0x - 15855.8333333333$

B_{12}= $-4000.0x^3 + 18800.0x^2 - 29440.0x + 15360.6666666667$

B_{13}= $1333.33333333333x^3 - 6200.0x^2 + 9610.0x - 4965.16666666667$

Y(x)=$c_{10}B_{10} + c_{11}B_{11} + c_{12}B_{12} + c_{13}B_{13}$

Y(x)= $-1.63492x^3 + 10.81079x^2 - 25.92853x + 21.67394$

Y(1.55) =1.3693784917

$1.60 \leq x < 1.65$

B_{11}=$-1333.33333333333x^3 + 6600.0x^2 - 10890.0x + 5989.5$

B_{12}=$4000.0x^3 - 19600.0x^2 + 32000.0x - 17407.3333333333$

B_{13}= $-4000.0x^3 + 19400.0x^2 - 31350.0x + 16880.1666666667$

B_{14}= $1333.33333333333x^3 - 6400.0x^2 + 10240.0x - 5461.33333333333$

Y(x)=$c_{11}B_{11} + c_{12}B_{12} + c_{13}B_{13} + c_{14}B_{14}$

Y(x)= $' - 1.50417x^3 + 10.18318x^2 - 24.92437x + 21.13839$

Y(1.60) =1.1672547313

$1.65 \leq x < 1.70$

B_{12}=$-1333.33333333333x^3 + 6800.0x^2 - 11560.0x + 6550.66666666666$

B_{13}=$4000.0x^3 - 20200.0x^2 + 33990.0x - 19056.8333333333$

B_{14}= $-4000.0x^3 + 20000.0x^2 - 33320.0x + 18496.6666666667$

B_{15}= $1333.33333333333x^3 - 6600.0x^2 + 10890.0x - 5989.5$

Y(x)=$c_{12}B_{12} + c_{13}B_{13} + c_{14}B_{14} + c_{15}B_{15}$

Y(x)= $-1.38267x^3 + 9.58175x^2 - 23.93200x + 20.59259$

Y(1.65) = 0.9799630775

$1.70 \leq x < 1.75$

B_{13}=$-1333.33333333333x^3 + 7000.0x^2 - 12250.0x + 7145.83333333333$

B_{14}=$4000.0x^3 - 20800.0x^2 + 36040.0x - 20807.3333333333$

B_{15}= $-4000.0x^3 + 20600.0x^2 - 35350.0x + 20213.1666666667$

B_{16}= $1333.33333333333x^3 - 6800.0x^2 + 11560.0x - 6550.66666666666$

Y(x)=$c_{13}B_{13} + c_{14}B_{14} + c_{15}B_{15} + c_{16}B_{16}$

Y(x)= $-1.27036x^3 + 9.00897x^2 - 22.95827x + 20.04081$

Y(1.70) =0.8063742444

$1.75 \leq x < 1.80$

B_{14}=$-1333.33333333333x^3 + 7200.0x^2 - 12960.0x + 7776.0$

B_{15}=$4000.0x^3 - 21400.0x^2 + 38150.0x - 22661.8333333333$

B_{16}= $-4000.0x^3 + 21200.0x^2 - 37440.0x + 22032.6666666667$

B_{17}= $1333.33333333333x^3 - 7000.0x^2 + 12250.0x - 7145.83333333333$

Y(x)=$c_{14}B_{14} + c_{15}B_{15} + c_{16}B_{16} + c_{17}B_{17}$

Y(x)= $-1.16694x^3 + 8.46601x^2 - 22.00809x + 19.48653$

Y(1.75) =0.6454500792

$1.80 \leq x < 1.85$

B_{15}=$-1333.33333333333x^3 + 7400.0x^2 - 13690.0x + 8442.16666666667$

B_{16}=$4000.0x^3 - 22000.0x^2 + 40320.0x - 24623.3333333333$

B_{17}= $-4000.0x^3 + 21800.0x^2 - 39590.0x + 23958.1666666667$

B_{18}= $1333.33333333333x^3 - 7200.0x^2 + 12960.0x - 7776.0$

Y(x)=$c_{15}B_{15} + c_{16}B_{16} + c_{17}B_{17} + c_{18}B_{18}$

Y(x)= $-1.07197x^3 + 7.95315x^2 - 21.08496x + 18.93265$

Y(1.80) =0.4962366996

$1.85 \leq x < 1.90$

B_{16}=$-1333.33333333333x^3 + 7600.0x^2 - 14440.0x + 9145.33333333333$

B_{17}=$4000.0x^3 - 22600.0x^2 + 42550.0x - 26694.8333333333$

B_{18}= $-4000.0x^3 + 22400.0x^2 - 41800.0x + 25992.6666666667$

B_{19}= $1333.33333333333x^3 - 7400.0x^2 + 13690.0x - 8442.16666666667$

Y(x)=$c_{16}B_{16} + c_{17}B_{17} + c_{18}B_{18} + c_{19}B_{19}$

Y(x)= $-0.98492x^3 + 7.47005x^2 - 20.19121x + 18.38151$

Y(1.85) =0.3578578444

$1.90 \leq x < 1.95$

B_{17}=$-1333.33333333333x^3 + 7800.0x^2 - 15210.0x + 9886.5$

B_{18}=$4000.0x^3 - 23200.0x^2 + 44840.0x - 28879.3333333333$

B_{19}= $-4000.0x^3 + 23000.0x^2 - 44070.0x + 28139.1666666667$

B_{20}= $1333.33333333333x^3 - 7600.0x^2 + 14440.0x - 9145.33333333333$

Y(x)=$c_{17}B_{17} + c_{18}B_{18} + c_{19}B_{19} + c_{20}B_{20}$

Y(x)= $-0.90525x^3 + 7.01592x^2 - 19.32836x + 17.83504$

Y(1.9) = 0.2295085473

$1.95 \leq x < 2.00$

B_{18}=$-1333.33333333333x^3 + 8000.0x^2 - 16000.0x + 10666.6666666667$

B_{19}=$4000.0x^3 - 23800.0x^2 + 47190.0x - 31179.8333333333$

B_{20}= $-4000.0x^3 + 23600.0x^2 - 46400.0x + 30400.6666666667$

B_{21}= $1333.33333333333x^3 - 7800.0x^2 + 15210.0x - 9886.5$

Y(x)=$c_{18}B_{18} + c_{19}B_{19} + c_{20}B_{20} + c_{21}B_{21}$

Y(x)= $-0.83239x^3 + 6.58969x^2 - 18.49721x + 17.29479$

Y(1.95) = 0.1104491952

The cubic B-spline polynomials are as follows:

$$
Y(x) = \begin{cases}
-2.33743x^3 + 14.12436x^2 - 30.97781x + 24.190881, & x \in [1.00, 1.05) \\
-2.63161x^3 + 15.05103x^2 - 31.95081x + 24.53143, & x \in [1.05, 1.10) \\
-2.74087x^3 + 15.41159x^2 - 32.34743944x + 24.67686, & x \in [1.10, 1.15) \\
-2.73346x^3 + 15.38604x^2 - 32.31805x + 24.66559, & x \in [1.15, 1.20) \\
-2.65421x^3 + 15.10071x^2 - 31.97566x + 24.52864, & x \in [1.20, 1.25) \\
-2.53260x^3 + 14.64470x^2 - 31.40564x + 24.29113, & x \in [1.25, 1.30) \\
-2.38804x^3 + 14.08090x^2 - 30.67270x + 23.97353, & x \in [1.30, 1.35) \\
-2.23318x^3 + 13.45373x^2 - 29.82604x + 23.59252, & x \in [1.35, 1.40) \\
-2.07621x^3 + 12.79446x^2 - 28.90305x + 23.16180, & x \in [1.40, 1.45) \\
-1.92230x^3 + 12.12495x^2 - 27.93225x + 22.69258. & x \in [1.45, 1.50) \\
-1.77460x^3 + 11.46030x^2 - 26.93528x + 22.19409, & x \in [1.50, 1.55) \\
-1.63492x^3 + 10.81079x^2 - 25.92853x + 21.67394, & x \in [1.55, 1.60) \\
-1.50417x^3 + 10.18318x^2 - 24.92437x + 21.13839, & x \in [1.60, 16.5) \\
-1.38267x^3 + 9.58175x^2 - 23.93200x + 20.59259, & x \in [1.65, 1.70) \\
-1.27036x^3 + 9.00897x^2 - 22.95827x + 20.04081, & x \in [1.70, 1.75) \\
-1.16694x^3 + 8.46601x^2 - 22.00809x + 19.48653, & x \in [1.75, 1.80) \\
-1.07197x^3 + 7.95315x^2 - 21.08496x + 18.93265, & x \in [1.80, 1.85) \\
-0.98492x^3 + 7.47005x^2 - 20.19121x + 18.38151, & x \in [1.85, 1.90) \\
-0.90525x^3 + 7.01592x^2 - 19.32836x + 17.83504, & x \in [1.90, 1.95) \\
-0.83239x^3 + 6.58969x^2 - 18.49721x + 17.29479, & x \in [1.90, 1.95)
\end{cases} \tag{3.12}
$$

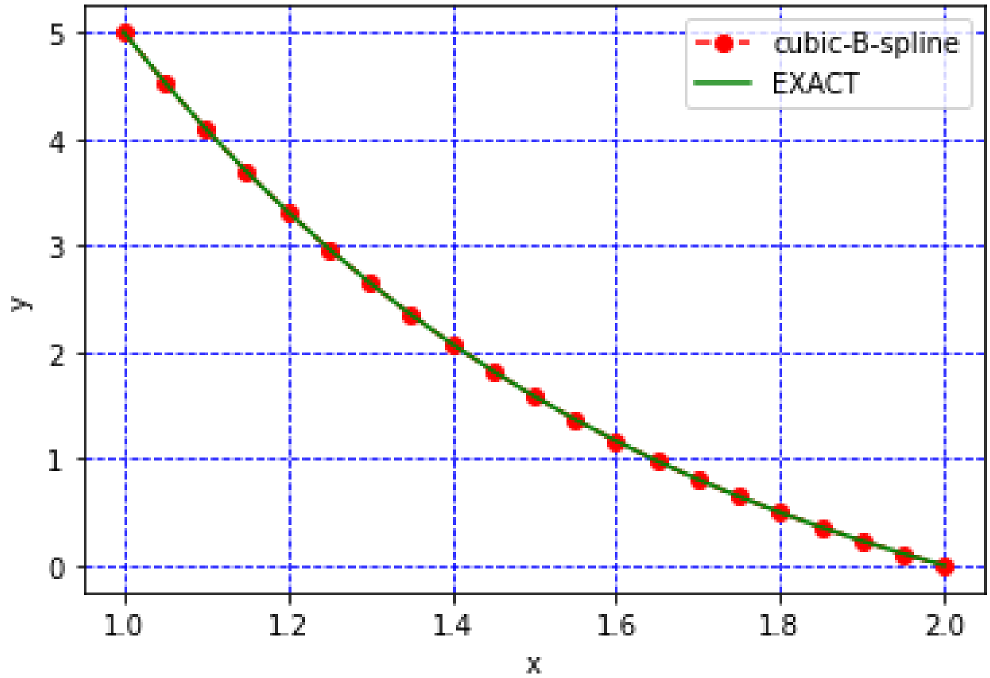
Table 3

The Cubic B-spline results example (3.1.3)

x_i	Cubic B-spline	Exact
1.00	5.0000000000	5.0000000000
1.05	4.5304189596	4.5304962322
1.10	4.0946084301	4.0947693502
1.15	3.6906178188	3.6908571967
1.20	3.3164060550	3.3167126115
1.25	2.9699320212	2.9702917258
1.30	2.6492103552	2.6496084276
1.35	2.3523444743	2.3527665477
1.40	2.07754463537	2.0779773959
1.45	1.8231362142	1.8235677149
1.50	1.5875616676	1.5879814418
1.55	1.3693784917	1.3697775482
1.60	1.1672547313	1.1676254805
1.65	0.9799630775	0.9802992219
1.70	0.8063742444	0.8066706529
1.75	0.6454500792	0.6457026575
1.80	0.4962366996	0.4964422651
1.85	0.3578578444	0.3580140078
1.90	0.2295085473	0.2296136048
1.95	0.1104491952	0.1105020313
2.00	0.0000000000	0.0000000000
$\boldsymbol{RMSE} = 0.0003033682405584666$		

Figure 3

The exact and cubic B-spline solutions of example (3.1.3)



3.2 Comparison Between B-spline Method and FDM's of BVPs

In this section, the comparison is held between the FDM's and B-spline method for solving BVPs. The FDM is commonly used to solve BVPs[Widjaja et al., 2005]

Example (3.2.1):

$$y'' + y' - 6y = x \quad \text{for} \quad 0 < x < 1 \tag{3.13}$$

with boundary conditions

$$y(0) = 0 \qquad\qquad y(1) = 1 \tag{3.14}$$

The exact solution to boundary value problem is

$$y(x) = \frac{(43 - e^2)e^{-3x} - (43 - e^{-3})e^{2x}}{36(e^{-3} - e^2)} - \frac{1}{6}x - \frac{1}{36} \tag{3.15}$$

We approximate the solution in (3.13) with boundary conditions (3.14) using the The FDM method with $N = 20$ in order to use (1.9)

Solution : From the boundary condition at $x = 0$, we obtain

x_0= 0 ; $\quad y_0 = 0$

x_{20} = 1 ; $\quad y_{20} = 1$

$x_1 = .05$, $x_2 = .1$, ..., $x_{19} = .95$

and $h = \frac{b-a}{N} = \frac{1-0}{20}$ =.05

we find the y_i for $i = 1, 2, 3......., 19$ using the system of liner equations (1.9) where the coefficient matrix is 19×19 , $MY = b$

Step 1 :

We first need to find matrix M

$$
M = \begin{bmatrix}
-2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015 & 1.025 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.975 & -2.015
\end{bmatrix}
$$

Step 2 :

We find the constant coefficients b_i **for** $i = 1, 2, ..., 19$ **using (1.9)**

$$\mathrm{b} = \begin{pmatrix} h^2 r_1 - (1 - \frac{h}{2} p_1)\alpha \\ h^2 r_2 \\ h^2 r_3 \\ h^2 r_4 \\ h^2 r_5 \\ h^2 r_6 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ h^2 r_{n-4} \\ h^2 r_{n-3} \\ h^2 r_{n-2} \\ h^2 r_{n-1} - (1 - \frac{h}{2} p_{n-1})\beta \end{pmatrix} = \begin{pmatrix} 0.000125 \\ 0.000250 \\ 0.000375 \\ 0.000500 \\ 0.000625 \\ 0.000750 \\ 0.000875 \\ 0.001000 \\ 0.001125 \\ 0.001250 \\ 0.001375 \\ 0.001500 \\ 0.001625 \\ 0.001750 \\ 0.001875 \\ 0.002000 \\ 0.002125 \\ 0.002250 \\ -0.972625 \end{pmatrix}$$

Step 3: We need to calculate the inverse of M

$$M^{-1} = \begin{bmatrix}
-0.881 & -0.795 & -0.717 & -0.647 & -0.582 & -0.523 & -0.469 & -0.420 & -0.374 & -0.332 & -0.293 & -0.256 & -0.221 & -0.188 & -0.156 & -0.125 & -0.095 & -0.064 & -0.033 \\
-0.757 & -1.564 & -1.410 & -1.271 & -1.144 & -1.028 & -0.922 & -0.825 & -0.735 & -0.652 & -0.575 & -0.503 & -0.435 & -0.370 & -0.308 & -0.247 & -0.186 & -0.126 & -0.064 \\
-0.649 & -1.342 & -2.090 & -1.884 & -1.695 & -1.524 & -1.366 & -1.222 & -1.090 & -0.967 & -0.853 & -0.746 & -0.645 & -0.549 & -0.456 & -0.365 & -0.276 & -0.186 & -0.095 \\
-0.556 & -1.150 & -1.792 & -2.494 & -2.245 & -2.017 & -1.809 & -1.618 & -1.443 & -1.280 & -1.129 & -0.987 & -0.854 & -0.726 & -0.604 & -0.484 & -0.365 & -0.247 & -0.125 \\
-0.476 & -0.985 & -1.534 & -2.135 & -2.800 & -2.516 & -2.257 & -2.019 & -1.800 & -1.597 & -1.408 & -1.232 & -1.065 & -0.906 & -0.753 & -0.604 & -0.456 & -0.308 & -0.156 \\
-0.407 & -0.842 & -1.311 & -1.825 & -2.394 & -3.028 & -2.716 & -2.429 & -2.165 & -1.921 & -1.695 & -1.482 & -1.281 & -1.090 & -0.906 & -0.726 & -0.549 & -0.370 & -0.188 \\
-0.347 & -0.718 & -1.119 & -1.557 & -2.042 & -2.583 & -3.192 & -2.855 & -2.545 & -2.258 & -1.992 & -1.742 & -1.506 & -1.281 & -1.065 & -0.854 & -0.645 & -0.435 & -0.221 \\
-0.296 & -0.611 & -0.952 & -1.325 & -1.738 & -2.198 & -2.716 & -3.302 & -2.944 & -2.612 & -2.303 & -2.015 & -1.742 & -1.482 & -1.232 & -0.987 & -0.746 & -0.503 & -0.256 \\
-0.251 & -0.518 & -0.807 & -1.123 & -1.473 & -1.864 & -2.303 & -2.800 & -3.366 & -2.986 & -2.634 & -2.303 & -1.992 & -1.695 & -1.408 & -1.129 & -0.853 & -0.575 & -0.293 \\
-0.212 & -0.437 & -0.681 & -0.948 & -1.244 & -1.573 & -1.944 & -2.363 & -2.841 & -3.386 & -2.986 & -2.612 & -2.258 & -1.921 & -1.597 & -1.280 & -0.967 & -0.652 & -0.332 \\
-0.177 & -0.367 & -0.572 & -0.795 & -1.043 & -1.320 & -1.631 & -1.983 & -2.383 & -2.841 & -3.366 & -2.944 & -2.545 & -2.165 & -1.800 & -1.443 & -1.090 & -0.735 & -0.374 \\
-0.148 & -0.305 & -0.475 & -0.662 & -0.868 & -1.098 & -1.357 & -1.649 & -1.983 & -2.363 & -2.800 & -3.302 & -2.855 & -2.429 & -2.019 & -1.618 & -1.222 & -0.825 & -0.420 \\
-0.121 & -0.251 & -0.391 & -0.544 & -0.714 & -0.903 & -1.116 & -1.357 & -1.631 & -1.944 & -2.303 & -2.716 & -3.192 & -2.716 & -2.257 & -1.809 & -1.366 & -0.922 & -0.469 \\
-0.098 & -0.203 & -0.316 & -0.441 & -0.578 & -0.731 & -0.903 & -1.098 & -1.320 & -1.573 & -1.864 & -2.198 & -2.583 & -3.028 & -2.516 & -2.017 & -1.524 & -1.028 & -0.523 \\
-0.078 & -0.161 & -0.250 & -0.348 & -0.457 & -0.578 & -0.714 & -0.868 & -1.043 & -1.244 & -1.473 & -1.738 & -2.042 & -2.394 & -2.800 & -2.245 & -1.695 & -1.144 & -0.582 \\
-0.059 & -0.122 & -0.191 & -0.266 & -0.348 & -0.441 & -0.544 & -0.662 & -0.795 & -0.948 & -1.123 & -1.325 & -1.557 & -1.825 & -2.135 & -2.494 & -1.884 & -1.271 & -0.647 \\
-0.043 & -0.088 & -0.137 & -0.191 & -0.250 & -0.316 & -0.391 & -0.475 & -0.572 & -0.681 & -0.807 & -0.952 & -1.119 & -1.311 & -1.534 & -1.792 & -2.090 & -1.410 & -0.717 \\
-0.027 & -0.056 & -0.088 & -0.122 & -0.161 & -0.203 & -0.251 & -0.305 & -0.367 & -0.437 & -0.518 & -0.611 & -0.718 & -0.842 & -0.985 & -1.150 & -1.342 & -1.564 & -0.795 \\
-0.013 & -0.027 & -0.043 & -0.059 & -0.078 & -0.098 & -0.121 & -0.148 & -0.177 & -0.212 & -0.251 & -0.296 & -0.347 & -0.407 & -0.476 & -0.556 & -0.649 & -0.757 & -0.881
\end{bmatrix}$$

Step 4 : **We can multiply both sides by the inverse of M = , to give** $Y = M^{-1}b$

$$\begin{cases} y_1 = 0.02594585 \quad , \quad x_1 = .05 \\ y_2 = 0.05112769 \quad , \quad x_2 = .1 \\ y_3 = 0.07607327 \quad , \quad x_3 = .15 \\ y_4 = 0.10128110 \quad , \quad x_4 = .2 \\ y_5 = 0.12722926 \quad , \quad x_5 = .25 \\ y_6 = 0.15438329 \quad , \quad x_6 = .3 \\ y_7 = 0.18320372 \quad , \quad x_7 = .35 \\ y_8 = 0.21415296 \quad , \quad x_8 = .4 \\ y_9 = 0.24770204 \quad , \quad x_9 = .45 \\ y_{10} = 0.28433704 \quad , \quad x_{10} = .5 \\ y_{11} = 0.32456552 \quad , \quad x_{11} = .55 \\ y_{12} = 0.36892283 \quad , \quad x_{12} = .6 \\ y_{13} = 0.41797865 \quad , \quad x_{13} = .65 \\ y_{14} = 0.47234364 \quad , \quad x_{14} = .7 \\ y_{15} = 0.53267634 \quad , \quad x_{15} = .75 \\ y_{16} = 0.59969051 \quad , \quad x_{16} = .8 \\ y_{17} = 0.67416287 \quad , \quad x_{17} = .85 \\ y_{18} = 0.75694141 \quad , \quad x_{18} = .9 \\ y_{19} = 0.84895428 \quad , \quad x_{19} = .95 \end{cases}$$

(Finite difference method) the $\boldsymbol{RMSE} = 0.021905015483815963$

but in the B-spline $the\boldsymbol{RMSE} = 6.395571348794568 * 10^{-5}$

Table 4

The values of FDM in example (3.2.1)

x_i	FDM	Exact	Cubic B-spline
0.00	0.0000000000	0.0000000000	0.0000000000
0.05	0.02594585	0.0275370031	0.0275351538
0.10	0.05112769	0.0542570003	0.0542500333
0.15	0.07607327	0.0807133503	0.0806989046
0.20	0.10128110	0.1074285617	0.1074050277
0.25	0.12722926	0.1349034523	0.1348698493
0.30	0.15438329	0.1636255435	0.1635814257
0.35	0.18320372	0.1940768511	0.1940222368
0.40	0.21415296	0.2267412146	0.2266765374
0.45	0.24770204	0.2621112965	0.2620373739
0.50	0.28433704	0.3006953693	0.3006133878
0.55	0.32456552	0.3430239998	0.3429355142
0.60	0.36892283	0.3896567348	0.3895636812
0.65	0.41797865	0.4411888843	0.4410936054
0.70	0.47234364	0.4982584988	0.4981637819
0.75	0.53267634	0.5615536314	0.5614627582
0.80	0.59969051	0.6318199790	0.6317367884
0.85	0.67416287	0.7098689951	0.7097979583
0.90	0.75694141	0.7965865702	0.7965328796
0.95	0.84895428	0.8929423791	0.8929120525
1.00	1.0000000000	1.0000000000	1.0000000000

Figure 4

The exact and FDM solutions of example (3.2.1)

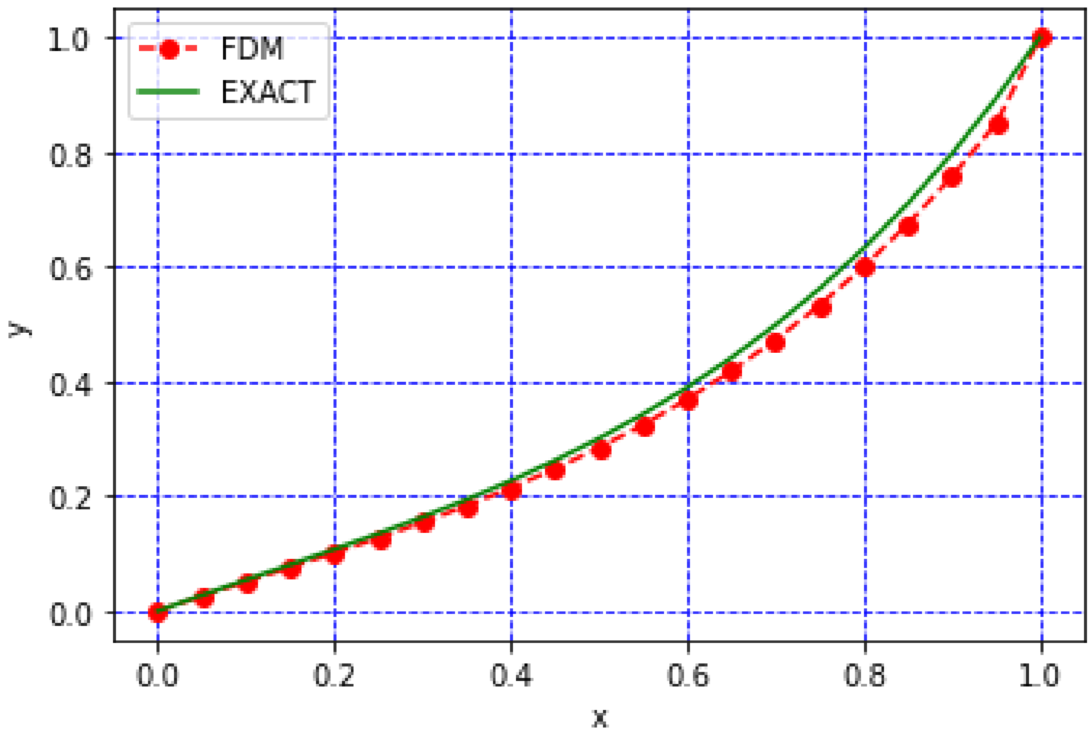
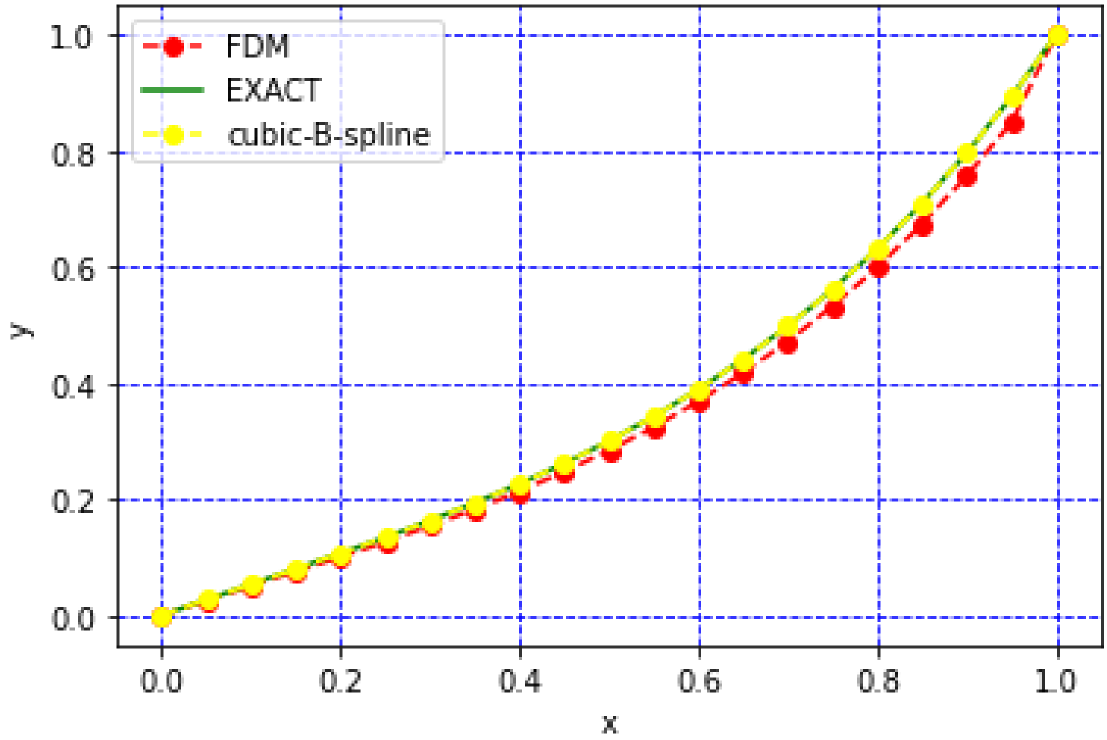


Figure 5

FDM and cubic B-spline solutions of example (3.2.1)



<u>Example (3.2.2) :</u>

$$y'' + 2y' + 5y = 6\cos(2x) - 7\sin(2x) \quad \text{for} \quad 0 < x < \frac{\pi}{4} \tag{3.16}$$

with boundary conditions

$$y(0) = 4 \qquad\qquad y(\frac{\pi}{4}) = 1 \tag{3.17}$$

The exact solution to boundary value problem is

$$y(x) = 2(1 + e^{-x})\cos(2x) + \sin(2x) \tag{3.18}$$

We approximate the solution in (3.16) with boundary conditions (3.17) using the The FDM method with $N = 20$ in order to use (1.9)

Solution : From the boundary condition at x= 0 , we obtain

$x_0 = 0$; $y_0 = 4$

x_{20} =$\frac{\pi}{4}$; $y_{20} = 1$

x_1 = $\frac{\pi}{80}$, $x_2 = \frac{\pi}{40}$, ..., $x_{19} = \frac{19\pi}{80}$

and $h = \frac{b-a}{N} = \frac{\frac{\pi}{4}-0}{20}$ =$\frac{\pi}{80}$

we find the y_i for $i = 1, 2, 3$......., 19using the system of liner equations (1.9) where the coefficient matrix is19×19 , $MY = b$ **Step 1 :**

We first need to find matrix M

$$M = \begin{bmatrix}
-1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992 & 1.039 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.992
\end{bmatrix}$$

Step 2 : We find the constant coefficients b_i for i = 1 ,2, ...,19 using (1.9)

$$
\mathrm{b} = \begin{pmatrix} h^2 r_1 - (1 - \frac{h}{2}p_1)\alpha \\ h^2 r_2 \\ h^2 r_3 \\ h^2 r_4 \\ h^2 r_5 \\ h^2 r_6 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ h^2 r_{n-4} \\ h^2 r_{n-3} \\ h^2 r_{n-2} \\ h^2 r_{n-1} - (1 - \frac{h}{2}p_{n-1})\beta \end{pmatrix} = \begin{pmatrix} -3.8345430929 \\ 0.0074501461 \\ 0.0064770851 \\ 0.0054640908 \\ 0.0044174085 \\ 0.0033434914 \\ 0.0022489606 \\ 0.0011405642 \\ 0.0000251358 \\ -0.0010904475 \\ -0.0021993079 \\ -0.0032946088 \\ -0.0043695973 \\ -0.0054176459 \\ -0.0064322928 \\ -0.0074072825 \\ -0.0083366039 \\ -0.0092145273 \\ -0.9707657319 \end{pmatrix}
$$

Step 3: We need to calculate the inverse of M

$$
M^{-1} = \begin{bmatrix}
-0.998 & -1.028 & -1.053 & -1.071 & -1.082 & -1.086 & -1.081 & -1.066 & -1.043 & -1.008 & -0.963 & -0.907 & -0.838 & -0.758 & -0.664 & -0.558 & -0.438 & -0.305 & -0.159 \\
-0.950 & -1.971 & -2.018 & -2.053 & -2.075 & -2.081 & -2.071 & -2.044 & -1.998 & -1.933 & -1.847 & -1.738 & -1.607 & -1.452 & -1.273 & -1.069 & -0.840 & -0.586 & -0.305 \\
-0.900 & -1.866 & -2.896 & -2.946 & -2.977 & -2.986 & -2.972 & -2.933 & -2.867 & -2.773 & -2.649 & -2.494 & -2.306 & -2.084 & -1.827 & -1.534 & -1.206 & -0.840 & -0.438 \\
-0.846 & -1.755 & -2.723 & -3.750 & -3.789 & -3.800 & -3.783 & -3.733 & -3.649 & -3.530 & -3.372 & -3.174 & -2.935 & -2.652 & -2.325 & -1.953 & -1.534 & -1.069 & -0.558 \\
-0.790 & -1.639 & -2.544 & -3.502 & -4.511 & -4.525 & -4.504 & -4.445 & -4.345 & -4.203 & -4.015 & -3.779 & -3.494 & -3.158 & -2.768 & -2.325 & -1.827 & -1.273 & -0.664 \\
-0.733 & -1.520 & -2.359 & -3.248 & -4.183 & -5.161 & -5.137 & -5.070 & -4.956 & -4.794 & -4.579 & -4.311 & -3.985 & -3.602 & -3.158 & -2.652 & -2.084 & -1.452 & -0.758 \\
-0.674 & -1.398 & -2.170 & -2.988 & -3.849 & -4.749 & -5.684 & -5.610 & -5.484 & -5.304 & -5.067 & -4.770 & -4.410 & -3.985 & -3.494 & -2.935 & -2.306 & -1.607 & -0.838 \\
-0.615 & -1.276 & -1.980 & -2.726 & -3.511 & -4.332 & -5.186 & -6.068 & -5.932 & -5.737 & -5.481 & -5.159 & -4.770 & -4.311 & -3.779 & -3.174 & -2.494 & -1.738 & -0.907 \\
-0.556 & -1.153 & -1.789 & -2.464 & -3.173 & -3.915 & -4.687 & -5.483 & -6.301 & -6.095 & -5.822 & -5.481 & -5.067 & -4.579 & -4.015 & -3.372 & -2.649 & -1.847 & -0.963 \\
-0.497 & -1.031 & -1.600 & -2.203 & -2.837 & -3.501 & -4.190 & -4.903 & -5.634 & -6.380 & -6.095 & -5.737 & -5.304 & -4.794 & -4.203 & -3.530 & -2.773 & -1.933 & -1.008 \\
-0.439 & -0.910 & -1.413 & -1.945 & -2.506 & -3.091 & -3.700 & -4.330 & -4.975 & -5.634 & -6.301 & -5.932 & -5.484 & -4.956 & -4.345 & -3.649 & -2.867 & -1.998 & -1.043 \\
-0.382 & -0.792 & -1.230 & -1.693 & -2.180 & -2.690 & -3.220 & -3.768 & -4.330 & -4.903 & -5.483 & -6.068 & -5.610 & -5.070 & -4.445 & -3.733 & -2.933 & -2.044 & -1.066 \\
-0.327 & -0.677 & -1.051 & -1.447 & -1.863 & -2.299 & -2.752 & -3.220 & -3.700 & -4.190 & -4.687 & -5.186 & -5.684 & -5.137 & -4.504 & -3.783 & -2.972 & -2.071 & -1.081 \\
-0.273 & -0.566 & -0.878 & -1.209 & -1.557 & -1.921 & -2.299 & -2.690 & -3.091 & -3.501 & -3.915 & -4.332 & -4.749 & -5.161 & -4.525 & -3.800 & -2.986 & -2.081 & -1.086 \\
-0.221 & -0.458 & -0.712 & -0.980 & -1.262 & -1.557 & -1.863 & -2.180 & -2.506 & -2.837 & -3.173 & -3.511 & -3.849 & -4.183 & -4.511 & -3.789 & -2.977 & -2.075 & -1.082 \\
-0.172 & -0.356 & -0.552 & -0.761 & -0.980 & -1.209 & -1.447 & -1.693 & -1.945 & -2.203 & -2.464 & -2.726 & -2.988 & -3.248 & -3.502 & -3.750 & -2.946 & -2.053 & -1.071 \\
-0.125 & -0.259 & -0.401 & -0.552 & -0.712 & -0.878 & -1.051 & -1.230 & -1.413 & -1.600 & -1.789 & -1.980 & -2.170 & -2.359 & -2.544 & -2.723 & -2.896 & -2.018 & -1.053 \\
-0.080 & -0.167 & -0.259 & -0.356 & -0.458 & -0.566 & -0.677 & -0.792 & -0.910 & -1.031 & -1.153 & -1.276 & -1.398 & -1.520 & -1.639 & -1.755 & -1.866 & -1.971 & -1.028 \\
-0.039 & -0.080 & -0.125 & -0.172 & -0.221 & -0.273 & -0.327 & -0.382 & -0.439 & -0.497 & -0.556 & -0.615 & -0.674 & -0.733 & -0.790 & -0.846 & -0.900 & -0.950 & -0.998
\end{bmatrix}
$$

Step 4 :

We can multiply both sides by the inverse of M , to give $Y = M^{-1}b$

$$\begin{cases} y_1 = 3.97707309 \quad , \quad x_1 = \frac{\pi}{80} \\ y_2 = 3.93443255 \quad , \quad x_2 = \frac{\pi}{40} \\ y_3 = 3.87299245 \quad , \quad x_3 = \frac{3\pi}{80} \\ y_4 = 3.79369305 \quad , \quad x_4 = \frac{\pi}{20} \\ y_5 = 3.69749765 \quad , \quad x_5 = \frac{5\pi}{80} \\ y_6 = 3.58538968 \quad , \quad x_6 = \frac{3\pi}{40} \\ y_7 = 3.45837011 \quad , \quad x_7 = \frac{7\pi}{80} \\ y_8 = 3.31745507 \quad , \quad x_8 = \frac{\pi}{10} \\ y_9 = 3.16367362 \quad , \quad x_9 = \frac{9\pi}{80} \\ y_{10} = 2.99806579 \quad , \quad x_{10} = \frac{\pi}{8} \\ y_{11} = 2.82168058 \quad , \quad x_{11} = \frac{11\pi}{80} \\ y_{12} = 2.63557414 \quad , \quad x_{12} = \frac{3\pi}{20} \\ y_{13} = 2.44080799 \quad , \quad x_{13} = \frac{13\pi}{80} \\ y_{14} = 2.23844722 \quad , \quad x_{14} = \frac{7\pi}{40} \\ y_{15} = 2.02955869 \quad , \quad x_{15} = \frac{15\pi}{80} \\ y_{16} = 1.81520921 \quad , \quad x_{16} = \frac{\pi}{5} \\ y_{17} = 1.59646365 \quad , \quad x_{17} = \frac{17\pi}{80} \\ y_{18} = 1.37438296 \quad , \quad x_{18} = \frac{9\pi}{40} \\ y_{19} = 1.15002209 \quad , \quad x_{19} = \frac{19\pi}{80} \end{cases}$$

(Finite difference method example(3.2.2))

$\boldsymbol{RMSE} = 0.067694094$

but in the B-spline the $\boldsymbol{RMSE} = 4.188842800472591 * 10^{-5}$

Table 5
The values of FDM in example (3.2.2)

x_i	FDM	Cubic B-spline	Exact
0.00	4.0000000000	4.0000000000	4.0000000000
$\frac{\pi}{80}$	3.97707309	3.9893275364	3.9893481701
$\frac{\pi}{40}$	3.93443255	3.9579417589	3.9579782444
$\frac{3\pi}{80}$	3.87299245	3.9067485765	3.9067967056
$\frac{\pi}{20}$	3.79369305	3.8366881995	3.8367442991
$\frac{5\pi}{80}$	3.69749765	3.7487313426	3.7487922376
$\frac{3\pi}{40}$	3.58538968	3.6438757255	3.6439387036
$\frac{7\pi}{80}$	3.45837011	3.5231428361	3.5232056151
$\frac{\pi}{10}$	3.31745507	3.3875749246	3.3876356206
$\frac{9\pi}{80}$	3.16367362	3.2382321907	3.2382892895
$\frac{\pi}{8}$	2.99806579	3.0761901337	3.0762424637
$\frac{11\pi}{80}$	2.82168058	2.9025370300	2.9025837374
$\frac{3\pi}{20}$	2.63557414	2.7183715079	2.7184120337
$\frac{13\pi}{80}$	2.44080799	2.5248001883	2.5248342470
$\frac{7\pi}{40}$	2.23844722	2.3229353642	2.3229629245
$\frac{15\pi}{80}$	2.02955869	2.1138926936	2.1139139602
$\frac{\pi}{5}$	1.81520921	1.8987888813	1.8988042779
$\frac{17\pi}{80}$	1.59646365	1.6787393307	1.6787494845
$\frac{9\pi}{40}$	1.37438296	1.4548557478	1.4548614743
$\frac{19\pi}{80}$	1.15002209	1.2282436831	1.2282459716
$\frac{\pi}{4}$	1.0000000000	1.0000000000	1.0000000000

Figure 6

The exact and FDM solutions of example (3.2.2)

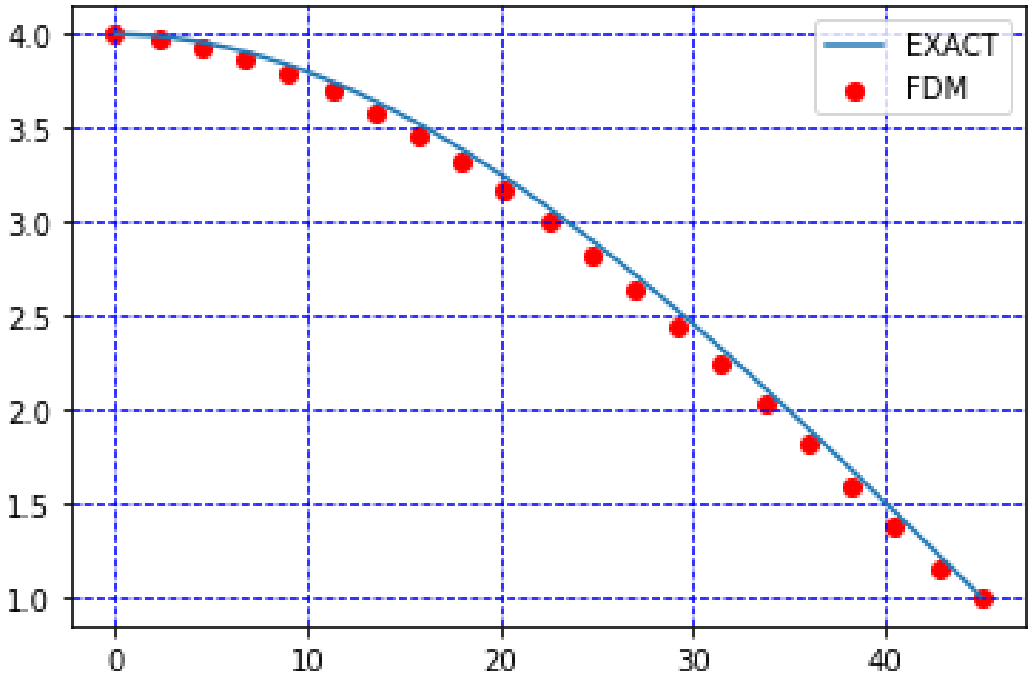
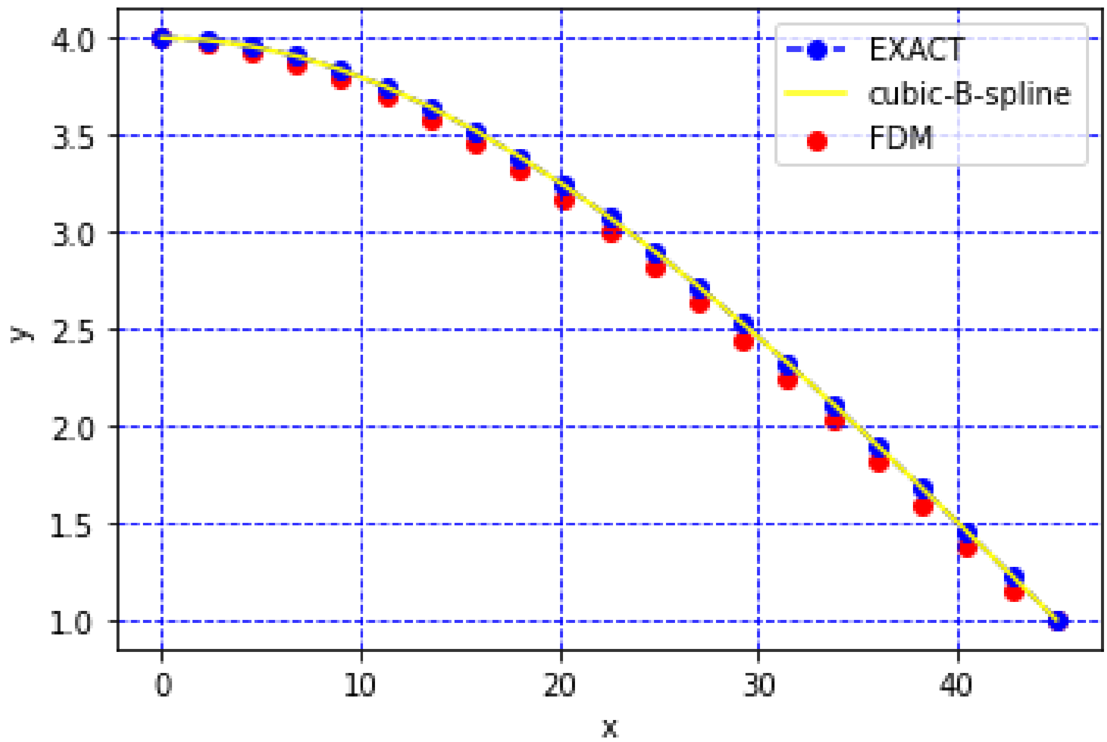


Figure 7

FDM and cubic B-spline solutions of example (3.2.2)



Example(3.2.3) :

$$x^2y'' + 3xy' + 3y = 0 \quad \text{for} \quad 1 < x < 2 \tag{3.19}$$

with boundary conditions

$$y(1) = 5 \qquad\qquad y(2) = 0 \tag{3.20}$$

The exact solution to boundary value problem is

$$y(x) = \frac{5}{x}[\cos(\sqrt{2}lnx - \cot(\sqrt{2}ln2)sin(\sqrt{2}lnx)] \tag{3.21}$$

We approximate the solution in (3.19) with boundary conditions (3.20) using the The FDM method with $N = 20$in order to use (1.9)

Solution : From the boundary condition at $x = 0$, we obtain

x_0= 1 ; $y_0 = 5$

x_{20} = 2 ; $y_{20} = 0$

$x_1 = 1.05$, $x_2 = 1.10$, ..., $x_{19} = 1.95$

and $h = \frac{b-a}{N} = \frac{2-1}{20}$ =.05

We find the y_i for i = 1, 2,3 ....,19 using the system of liner equations (1.9) where the coefficient matrix is 19×19 , $MY = b$

Step 1 :

We first need to find matrix M

$$
M = \begin{bmatrix}
-1.993 & 1.071 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.932 & -1.994 & 1.068 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.935 & -1.994 & 1.065 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.938 & -1.995 & 1.062 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.940 & -1.995 & 1.060 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.942 & -1.996 & 1.058 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.944 & -1.996 & 1.056 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.946 & -1.996 & 1.054 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.948 & -1.996 & 1.052 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.950 & -1.997 & 1.050 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.952 & -1.997 & 1.048 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.953 & -1.997 & 1.047 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.955 & -1.997 & 1.045 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.956 & -1.997 & 1.044 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.957 & -1.998 & 1.043 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.958 & -1.998 & 1.042 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.959 & -1.998 & 1.041 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.961 & -1.998 & 1.039 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.962 & -1.998
\end{bmatrix}
$$

Step 2 :

We find the constant coefficients b_i for i = 1 ,2, ...,19 using (1.9)

$$\text{b} = \begin{pmatrix} h^2r_1 - (1 - \frac{h}{2}p_1)\alpha \\ h^2r_2 \\ h^2r_3 \\ h^2r_4 \\ h^2r_5 \\ h^2r_6 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ h^2r_{n-4} \\ h^2r_{n-3} \\ h^2r_{n-2} \\ h^2r_{n-1} - (1 - \frac{h}{2}p_{n-1})\beta \end{pmatrix} = \begin{pmatrix} -4.6428571429 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \end{pmatrix}$$

Step 3: We need to calculate the inverse of M

$$M^{-1} = \begin{bmatrix}
-0.976 & -1.014 & -1.044 & -1.066 & -1.079 & -1.083 & -1.077 & -1.061 & -1.034 & -0.997 & -0.949 & -0.890 & -0.819 & -0.737 & -0.644 & -0.539 & -0.422 & -0.293 & -0.152 \\
-0.882 & -1.886 & -1.943 & -1.983 & -2.008 & -2.014 & -2.003 & -1.973 & -1.924 & -1.855 & -1.765 & -1.655 & -1.524 & -1.371 & -1.197 & -1.002 & -0.784 & -0.545 & -0.284 \\
-0.795 & -1.700 & -2.715 & -2.772 & -2.806 & -2.815 & -2.800 & -2.758 & -2.689 & -2.592 & -2.467 & -2.313 & -2.130 & -1.917 & -1.674 & -1.400 & -1.096 & -0.762 & -0.396 \\
-0.714 & -1.528 & -2.440 & -3.449 & -3.492 & -3.503 & -3.484 & -3.432 & -3.346 & -3.225 & -3.070 & -2.878 & -2.650 & -2.385 & -2.082 & -1.742 & -1.364 & -0.948 & -0.493 \\
-0.640 & -1.368 & -2.185 & -3.089 & -4.079 & -4.093 & -4.070 & -4.009 & -3.909 & -3.768 & -3.586 & -3.363 & -3.096 & -2.786 & -2.433 & -2.036 & -1.594 & -1.107 & -0.576 \\
-0.570 & -1.220 & -1.949 & -2.755 & -3.639 & -4.598 & -4.572 & -4.503 & -4.391 & -4.233 & -4.028 & -3.777 & -3.478 & -3.130 & -2.733 & -2.286 & -1.790 & -1.244 & -0.647 \\
-0.507 & -1.083 & -1.730 & -2.447 & -3.231 & -4.082 & -5.000 & -4.925 & -4.802 & -4.629 & -4.405 & -4.130 & -3.803 & -3.423 & -2.989 & -2.500 & -1.958 & -1.360 & -0.708 \\
-0.447 & -0.957 & -1.528 & -2.161 & -2.853 & -3.605 & -4.416 & -5.283 & -5.150 & -4.965 & -4.725 & -4.431 & -4.079 & -3.671 & -3.206 & -2.682 & -2.100 & -1.459 & -0.759 \\
-0.393 & -0.840 & -1.341 & -1.896 & -2.504 & -3.164 & -3.875 & -4.636 & -5.445 & -5.249 & -4.996 & -4.684 & -4.313 & -3.881 & -3.389 & -2.835 & -2.220 & -1.542 & -0.802 \\
-0.342 & -0.731 & -1.168 & -1.651 & -2.180 & -2.755 & -3.374 & -4.037 & -4.741 & -5.487 & -5.223 & -4.897 & -4.509 & -4.058 & -3.543 & -2.964 & -2.321 & -1.612 & -0.839 \\
-0.295 & -0.631 & -1.007 & -1.424 & -1.881 & -2.376 & -2.910 & -3.482 & -4.090 & -4.733 & -5.411 & -5.074 & -4.672 & -4.204 & -3.671 & -3.071 & -2.405 & -1.671 & -0.869 \\
-0.251 & -0.538 & -0.859 & -1.214 & -1.603 & -2.026 & -2.481 & -2.968 & -3.486 & -4.035 & -4.613 & -5.219 & -4.805 & -4.325 & -3.776 & -3.159 & -2.474 & -1.719 & -0.894 \\
-0.211 & -0.451 & -0.721 & -1.019 & -1.346 & -1.701 & -2.083 & -2.492 & -2.927 & -3.387 & -3.872 & -4.382 & -4.914 & -4.422 & -3.861 & -3.231 & -2.529 & -1.757 & -0.914 \\
-0.174 & -0.371 & -0.593 & -0.839 & -1.108 & -1.399 & -1.714 & -2.050 & -2.408 & -2.787 & -3.186 & -3.605 & -4.043 & -4.500 & -3.929 & -3.287 & -2.574 & -1.788 & -0.930 \\
-0.139 & -0.297 & -0.475 & -0.671 & -0.886 & -1.120 & -1.372 & -1.641 & -1.928 & -2.231 & -2.551 & -2.886 & -3.237 & -3.602 & -3.981 & -3.331 & -2.608 & -1.812 & -0.943 \\
-0.107 & -0.229 & -0.365 & -0.516 & -0.682 & -0.861 & -1.055 & -1.262 & -1.482 & -1.715 & -1.961 & -2.219 & -2.488 & -2.769 & -3.061 & -3.363 & -2.633 & -1.829 & -0.952 \\
-0.077 & -0.165 & -0.263 & -0.372 & -0.492 & -0.621 & -0.761 & -0.910 & -1.069 & -1.237 & -1.414 & -1.600 & -1.794 & -1.997 & -2.207 & -2.425 & -2.650 & -1.841 & -0.958 \\
-0.049 & -0.106 & -0.169 & -0.239 & -0.315 & -0.398 & -0.488 & -0.584 & -0.685 & -0.793 & -0.907 & -1.026 & -1.151 & -1.281 & -1.416 & -1.555 & -1.700 & -1.849 & -0.962 \\
-0.024 & -0.051 & -0.081 & -0.115 & -0.152 & -0.192 & -0.235 & -0.281 & -0.330 & -0.382 & -0.436 & -0.494 & -0.554 & -0.616 & -0.681 & -0.749 & -0.818 & -0.890 & -0.963
\end{bmatrix}$$

Step 4 : We can multiply both sides by the inverse of M , to give

$Y = M^{-1}b$

$$\begin{cases} y_1 = 4.53011442 & , \quad x_1 = 1.05 \\ y_2 = 4.09411762 & , \quad x_2 = 1.10 \\ y_3 = 3.69002302 & , \quad x_3 = 1.15 \\ y_4 = 3.31576419 & , \quad x_4 = 1.20 \\ y_5 = 2.96928207 & , \quad x_5 = 1.25 \\ y_6 = 2.64857853 & , \quad x_6 = 1.30 \\ y_7 = 2.35174792 & , \quad x_7 = 1.35 \\ y_8 = 2.07699402 & , \quad x_8 = 1.40 \\ y_9 = 1.82263764 & , \quad x_9 = 1.45 \\ y_{10} = 1.58711799 & , \quad x_{10} = 1.50 \\ y_{11} = 1.36899032 & , \quad x_{11} = 1.55 \\ y_{12} = 1.16692111 & , \quad x_{12} = 1.60 \\ y_{13} = 0.97968201 & , \quad x_{13} = 1.65 \\ y_{14} = 0.80614307 & , \quad x_{14} = 1.70 \\ y_{15} = 0.64526572 & , \quad x_{15} = 1.75 \\ y_{16} = 0.49609586 & , \quad x_{16} = 1.80 \\ y_{17} = 0.35775716 & , \quad x_{17} = 1.85 \\ y_{18} = 0.22944466 & , \quad x_{18} = 1.90 \\ y_{19} = 0.11041883 & , \quad x_{19} = 1.95 \end{cases}$$

Table 6
The values of FDM in example (3.2.3)

x_i	FDM	Cubic B-spline	Exact
1.00	5.0000000000	5.0000000000	5.0000000000
1.05	4.53011442	4.5304189596	4.5304962322
1.10	4.09411762	4.0946084301	4.0947693502
1.15	3.69002302	3.6906178188	3.6908571967
1.20	3.31576419	3.3164060550	3.3167126115
1.25	2.96928207	2.9699320212	2.9702917258
1.30	2.64857853	2.6492103552	2.6496084276
1.35	2.35174792	2.3523444743	2.3527665477
1.40	2.07699402	2.07754463537	2.0779773959
1.45	1.82263764	1.8231362142	1.8235677149
1.50	1.58711799	1.5875616676	1.5879814418
1.55	1.36899032	1.3693784917	1.3697775482
1.60	1.16692111	1.1672547313	1.1676254805
1.65	0.97968201	0.9799630775	0.9802992219
1.70	0.80614307	0.8063742444	0.8066706529
1.75	0.64526572	0.6454500792	0.6457026575
1.80	0.49609586	0.4962366996	0.4964422651
1.85	0.35775716	0.3578578444	0.3580140078
1.90	0.22944466	0.2295085473	0.2296136048
1.95	0.11041883	0.1104491952	0.1105020313
2.00	0.0000000000	0.0000000000	0.0000000000

(Finite difference method) the $\boldsymbol{RMSE} = 0.000708826722616754$

but in the B-spline the $\boldsymbol{RMSE} = 0.0003033682405584666$

Figure 8

The exact and FDM solutions of example (3.2.3)

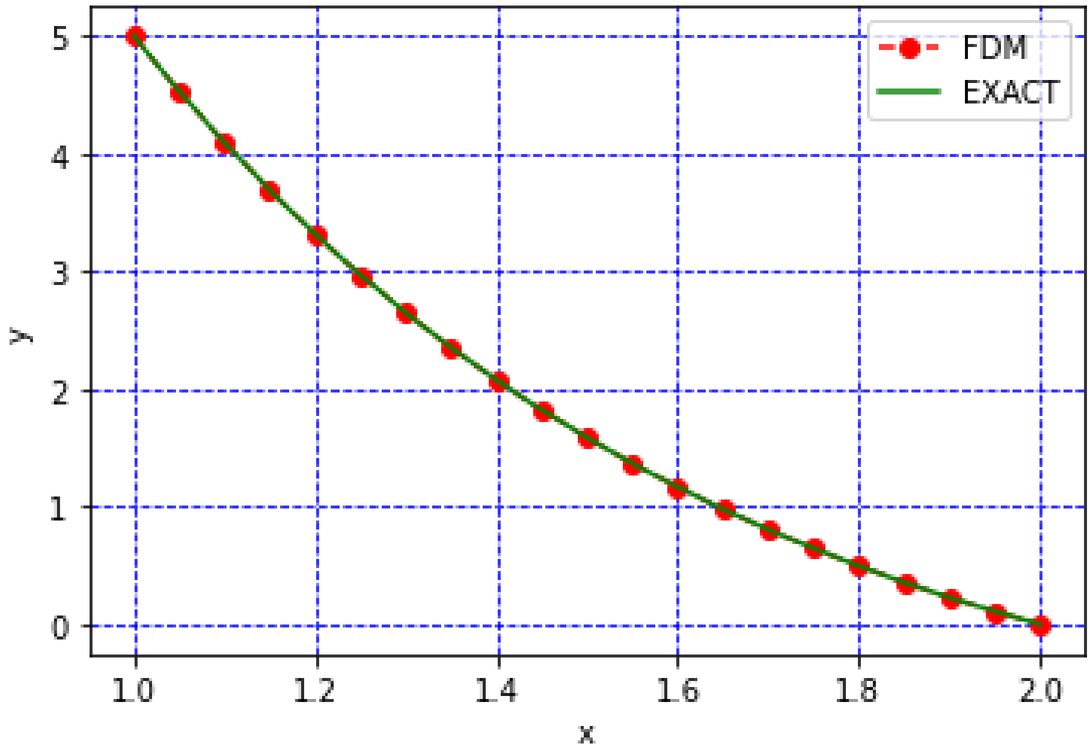
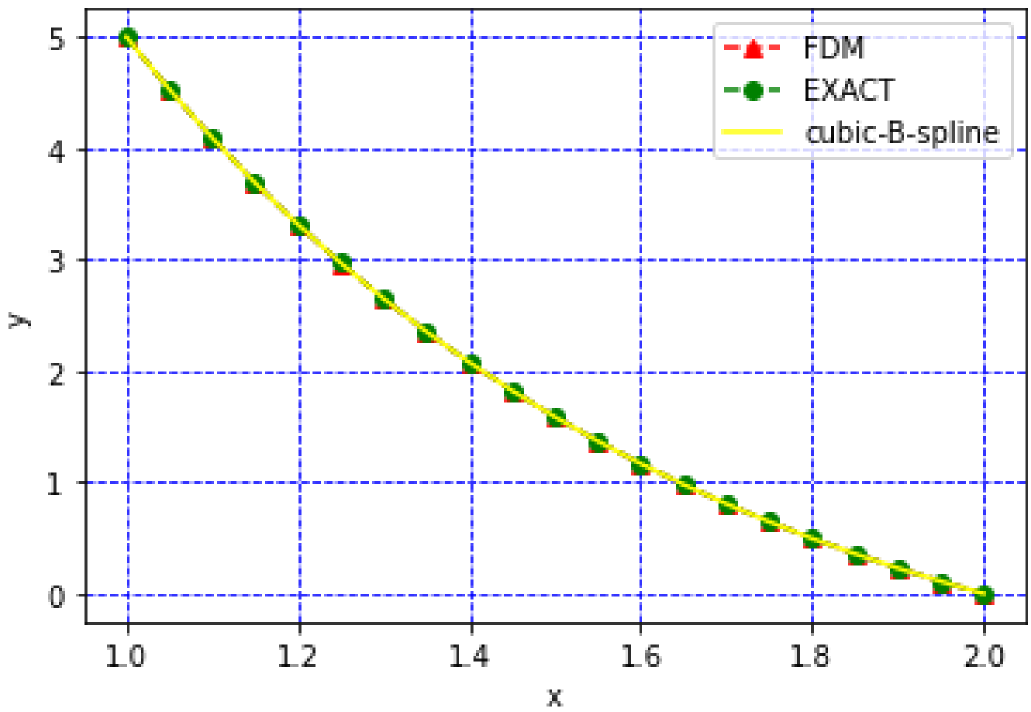


Figure 9

FDM and cubic B-spline solutions of example (3.2.3)



Chapter Four
Conclusions

In this thesis, we have explored the application of B-spline interpolation for the numerical solution of ordinary differential equations . The cubic B-spline was utilized as the method of choice and its performance was evaluated through a series of numerical experiments. These experiments were conducted to compare the effectiveness of the B-spline interpolation method with that of the traditional finite difference method. The results of these experiments were tabulated in Table 4.1 and it was observed that the B-spline method yielded lower errors in comparison to the finite difference method. The Root Mean Square Error (RMSE) of the B-spline method was found to be significantly lower than that of the finite difference method for all test examples. It is well established that B-spline interpolation is a robust and effective method for approximating smooth functions, as demonstrated in previous studies such as[Caglar et al., 2006]. The results of this research further support this claim and indicate that the B-spline interpolation method is a superior choice for the numerical solution of ordinary differential equations. In addition to its effectiveness, the implementation of the B-spline method was found to be simple and efficient. The method provided results that were comparable to other techniques, and the spline function produced can be used to obtain the solution at any point in the range, unlike the finite difference methods, which only approximation the solution at chosen knots. Future research in this field can focus on developing more advanced techniques for error analysis and convergence of B-spline methods, as well as investigating the use of B-spline methods for solving nonlinear boundary value problems and partial differential equations. Additionally, the application of B-spline methods to real-world problems in engineering or physics can demonstrate the practical effectiveness of the method.

4.1 Challenges and Future Research Directions

Cubic B-spline methods have been widely used for solving boundary value problems in various fields of science and engineering. However, during the course of our study, we

Table 1
Comparison between B-spline methods and FDM's for ODE's

A number of example	(finite difference method) RMSE	Cubic B-spline RMSE
(3.1.1),(3.2.1)	0.021905015483815963	6.395571348794568* 10^{-5}
(3.1.2)(3.2.2)	0.067694094	4.188842800472591 *10^{-5}
(3.1.3)(3.2.3)	0.000708826722616754	0.0003033682405584666

have also encountered a number of limitations and challenges that need to be addressed in future research.

Some of the challenges that we encountered during our research include:

1. Handling of non-smooth solutions: B-spline methods are known to be effective for approximating smooth solutions; however, they may not perform well for non-smooth solutions. Further research is needed to develop techniques to handle non-smooth solutions effectively.

2. Handling of high-dimensional problems: As the number of dimensions increases, the computational complexity of B-spline methods also increases. Developing techniques to handle high-dimensional problems efficiently is an ongoing challenge in the field.

3. Adaptive refinement: B-spline methods requires a large number of knots to achieve high accuracy, which can lead to a large computational burden. Developing adaptive refinement techniques to reduce the number of knots while maintaining accuracy is an important area of research.

4. Handling the large-scale problem: As the size of the problem increases, the computational time needed to solve it becomes a major challenge. A parallel or distributed computation technique is needed to handle large-scale BVPs using B-spline methods.

5. Solution at any point in the range: B-spline methods are able to produce spline functions that can be used to obtain the solution at any point in the range, but

this feature is not always necessary. Developing techniques to optimize a B-spline method for specific applications is an important area of research.

These methods have been shown to be an effective and efficient approach for approximating solutions to a wide range of problems. However, there are still many areas where further research is needed to improve the methodologies and their applications. In this thesis, we have investigated the use of cubic B-spline methods for solving boundary value problems and have evaluated their performance through a series of numerical experiments. The results of these experiments have shown that cubic B-spline methods are an effective and efficient approach for approximating solutions to a wide range of problems. Based on the computational results in this thesis, we suggest several areas for further research in this field. These include:

1. Investigating the use of cubic B-spline methods for solving nonlinear boundary value problems and partial differential equations.

2. Developing new types of cubic B-spline functions, such as non-uniform B-splines or B-splines with variable degrees, to improve the flexibility and accuracy of the method.

3. Applying cubic B-spline methods to real-world problems, such as in engineering or physics, to demonstrate the practical effectiveness of the method.

4. Developing more efficient algorithms for solving boundary value problems using cubic B-spline methods, such as those based on adaptive refinement or multiresolution techniques.

5. Exploring the relationship between cubic B-spline methods and other numerical methods, such as finite element methods or spectral methods, to identify potential areas for further integration and collaboration.

6. Investigating the use of deep learning techniques to improve the accuracy and efficiency of cubic B-spline methods for solving boundary value problems.

7. Developing parallel and distributed computation techniques for solving large-scale boundary value problems using cubic B-spline methods.

8. Developing new techniques for the efficient computation of derivatives and integrals of cubic B-spline functions.

The chapter concludes the research presented in this thesis. We have investigated the use of cubic B-spline methods for solving boundary value problems and have evaluated their performance through a series of numerical experiments. The results of these experiments have shown that cubic B-spline methods are effective and efficient approach for approximating solutions to a wide range of problems. We have discussed in detail the various techniques and methodologies used in our research and have demonstrated their effectiveness through numerical examples. We have also highlighted the limitations and challenges encountered during our research and have suggested areas for further research to improve the methodologies and their applications.

In summary, this thesis has demonstrated the effectiveness of cubic B-spline methods for solving boundary value problems and has provided a foundation for future research in this field. We hope that our research will be useful for the scientific community and will contribute to the development of more advanced and efficient methods for solving boundary value problems.

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جامعة النجاح الوطنية
كليـــة الدراســـــات العليــــا

الشرائح -ب التكعيبية لحلول مسائل القيم الحدية

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الملخص

كثير من الظواهر الفيزيائية والهندسية الطبيعية لا تظهر إلا على شكل أنظمة رياضية، وتظهر تحديداً كمسائل القيمة الحدية.

في هذه الرسالة تمّ حلّ المسائل الحدية باستخدام طريقة شرائح ب- التكعيبية، حيث شُرحت بعض المفاهيم الأساسية التي سنحتاجها في أثناء إيجاد حلّ للمسائل الحدية ووُضّحت في بداية الرسالة، ثم تمّ توضيح طريقة الفروق المحددة وأهميتها، وطريقة الحلّ، كما وتمّ توضيح طريقة شرائح ب- التكعيبية بحلّ المسائل الحدية، وتوضيح خصائصها وتقديم طريقة الحلّ، وفي القسم الأخير من الرسالة تمّ حلّ مجموعة متنوعة من المسائل الحدية باستخدام طريقة شرائح ب- التكعيبية، وطريقة الفروق المحددة.

وتم مقارنة النتائج التي أظهرت تفوقاً بطريقة شرائح ب- التكعيبية، من خلال إيجاد الجذر التربيعي لمتوسط مربعات الخطأ، ورسم النتائج ومقارنتها مع الحلّ المثالي للمعادلات.

في ضوء النتائج أوصت الدراسة باستخدام طريقة شرائح ب-التكعيبية في حل المعادلات الحدية.

الكلمات المفتاحية: المعادلات الحدية، شرائح ب-التكعبية ، الفروق المحددة ، متوسط مربعات الخطأ