An-Najah National University

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Clean Like Semi-ring Notions and Trivial Semi-ring Extension

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Dedication

To my beloved parents and my first teachers Osama Jamal and Lamya Nazzal.

To my older brother Anas who has been a source of inspiration for me and supported me no matter what.

To my sister Fatina and her kids Laith and Deema; to my younger brother Mustafa.

I thankfully dedicate this work to you.

I also dedicate this work to my best friends Lamees Othman and Saja Zaidan for their love, support and their ongoing belief in me and ability to reach my potential.

Last but not least I am deeply grateful to all of my teachers, friends and everyone that I ever learnt from.

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A journey of a thousand miles begins with one small step; I just finished the first one.

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الإقرار

أنا الموقعة أدناه مقدمة الرسالة التي تحمل العنوان:

Clean Like Semi-ring Notions and Trivial Semi-ring Extension

أقر بأن ما اشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد، وان هذه الرسالة ككل أو من جزء منها لم يقدم من قبل لنيل أية درجة أو بحث علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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Clean Like Semi-ring Notions and Trivial Semi-ring Extension

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Abstract

Suppose that S is a commutative semiring with unity different than zero and M an S-semimodule. In this thesis, we study the algebraic and the ideal theoretic properties of $S\alpha M$, where $S\alpha M$ denotes the trivial semiring extension (or the expectation of S), providing an analog results to the proved ones in the ring situation. In this thesis, different elements like units, zero divisors and other elements of $S\alpha M$ as well as the special ideals like subtractive ideals, prime ideals and other types of ideals of $S\alpha M$ will be identified. The generalization of some of the clean like notions into the semiring situation will be investigated; this thesis also examines some of their properties and the transfer of these notions in the trivial semiring extension. This thesis also provides an application of semirings in classification system which is considered an important technique in data mining which used to assign every element to specific groups based on the similarities between the referred to elements.

Introduction

In 1934, H.S Vandiver [22] formally defined semirings, which are rings, but without requiring the existence of the additive inverse for each element. The algebraic structure of semirings have lots of practical implications in both math and computer science sectors [10]. They also have implications in data minig sector especially in the classification system [1].

A typical example of a semiring which is not a ring is the set of all non negative integers i.e. natural numbers under the usual addition and multiplication of integers. Another example, which is not trivial, is the set of all the ideals of a given ring under the usual addition and multiplication of ideals. It forms a semiring which is not a ring, since one can add or multiply two ideals but cannot subtract them. There are lots of previous studies providing examples of important sets that have the structure of semirings but do not have the ring structure [22].

Nagata [4] in 1962 introduced the idealization of R-module M over a ring R in order to facilitate interaction between rings and their modules and also to provide families of examples of commutative rings containing zero-divisors (reduced elements). A generalization of idealization from rings to semirings is possible in a very identical way; that is to say if S is a semiring and M is an S-semimodule; then, the set $S\alpha M$

where the underlying group is $S \times M$ and the multiplication is defined by $(a, m)(b, f) = (ab, af + bm)$ for $a, b \in S$ and $m, f \in M$ forms a new semiring that extends S; it is called the trivial semiring extension of S over M [9].

In 1977, Nicholson [21] introduced the notion of the clean rings; he defined a ring to be clean if all its elements are clean, where an element is called clean if it can be written as a sum of a unit and an idempotent. In 2013, Alexander J. Diesl [8] introduced another ring notion called the nil clean ring; he defined a ring to be nil clean if all its elements are nil clean, where the nil clean element is the one that can be written as a sum of a nilpotent and an idempotent. He proves in the same article that the class of nil clean rings is contained in the class of clean rings. Previous studies examines many of the related notions to the clean notion such as weakly clean, strongly clean, etc. Check for example [8, 21].

The concept of incidence semiring is a generalization of the concept of incidence ring [15], which can be cosidered as sets of polynomials over a graph, where the edges are the unknowns and the coefficients are taken from a semiring. The construction of incidence rings has many useful practical implications in different areas such as data mining.

In this thesis, all semirings are assumed to be commutative with unity $1 \neq 0$, where S denotes a semiring, M denotes an S-semimodule, $S\alpha M$ denotes the trivial semiring extension of S by M, G denotes a directed graph and $I_G(S)$ denotes the incidence semiring of G over S.

Chapter one introduces some basic definitions, examples and theorems in semirings, semimodule, trivial ring extension and clean like notions in ring case which will be transformed to semiring case.

Chapter two studies some of the algebraic and the ideal theoretic properties of the trivial semiring extension $S\alpha M$; some special elements and ideals of $S\alpha M$ are identified; local trivial semiring extension is studied. The main idea highlighted in this chapter is how the referred to properties of $S\alpha M$ are related to similar ones of S.

Chapter three presents how to generalize some of the clean like notions into the semiring situation; some of their properties, especially the ones that have an analog in the ring situation are studied; the transfer of these notions in the trivial semiring extension is also examined.

Finally, the incidence semiring and the set of centroids with largest weight which are valuable in the design of centroid based classifiers, that are considered as one of the most common techniques used in data mining models, are studied.

Chapter 1

PRELIMINARIES

Chapter 1

Preliminaries

This chapter presents basic information on semirings and the ideals associated with them. It also provides a summary on clean rings and some related ring theoretic notions.

1.1 Introduction to Semirings

In this section, major definitions and theorems in semiring theory are provided. The main references are [4, 8].

1.1.1 Definitions and Examples

Definition 1.1 (Semiring)**.** A semiring is an algebraic structure, consisting of a non empty set S and two binary operations, addition $(+)$ and multiplication $(.)$ such that the following axioms hold:

1. $(S, +)$ is associative and commutative that is $a+(b+c) = (a+b)+c$ and $a+b = b+a$. Also, there exist an identity element 0 such that $a + 0 = 0 + a = a$ for all $a, b, c \in S$.

2. $(S, .)$ is associative that is $a(bc) = (ab)c$ and there is an identity element, denoted by 1 such that $a1 = a$, for all $a, b, c \in S$.

3. To avoid the trivial case, we assume that $0 \neq 1$.

4. The multiplication is distributive over addition from both sides that is $a(b+c) = (ab) + (ac)$ and $(a+b)c = (ac) + (bc)$.

5. For any $x \in S$, $0x = x0 = 0$.

In other words, semirings are rings, but without requiring the existence of the additive inverse for each element. Semirings were formally defined by H.S Vandiver in 1934 [22], and they have many practical implications in both mathematical and computer science sectors. See for example [10,1].

Definition 1.2. A semiring S is said to be commutative if the multiplication is commutative (i.e $ab = ba$, for $a, b \in S$).

Example 1.3. The set of natural numbers $\mathbb{N}$ under the usual addition and multiplication of integers is a commutative semiring. Also, the set of $n \times n$ matrices with non negative entries $M_n(\mathbb{N})$ form a semiring.

It should be pointed out that all rings are semirings, but the converse is not true as in the case of naturals $\mathbb{N}$.

Definition 1.4 (Division semiring)**.** Let S be a semiring. Then S is said to be a division semiring if every non zero element in S has a multiplicative inverse (i.e. for all $0 \neq x \in S$, there exist $0 \neq y \in S$ such that $xy = 1$).

Definition 1.5 (Semidomain)**.** A semiring S is called a semidomain if $ab = ac$ implies $b = c$ for all $b, c \in S$ and all nonzero $a \in S$ which means that there is no zero-divisors in S.

Definition 1.6 (Semifield)**.** A semiring S is said to be semifield if it is a commutative division semiring. A good example is $\mathbb{Q}^{+} \cup \{0\}$.

Next we introduce the definition of a semimodule.

Definition 1.7 (semimodule)**.** Let S be a semiring. An S-semimodule is a non empty set M on which one have two operations addition and scalar multiplication by elements of S where the following axioms are satisfied:

- The addition is associative, commutative, and has an identity element, denoted 0_M.
- For all $a, b \in S$ and $m, n \in M$:

  1. $(ab)m = a(bm)$
  2. $(a + b)m = am + bm$
  3. $a(m + n) = am + an$
  4. $1m = m$ and $0m = 0_M = a0_M$

Example 1.8. Let M be the set of all functions between a set A and a semiring S where the addition is the usual addition of functions and the scalar multiplication is given by $(a.f)(x) = f(a)f(x)$, for any $a \in A$, $x \in A$ and $f \in M$.

1.1.2 Elements and Ideals

In this section, definitions of special elements and sets of a semiring S are presented.

Definition 1.9. Let x be an element of a semiring S.

1. x is called unit if it has a multiplicative inverse (i.e. there exist $0 \neq y \in S$ such that $xy = 1$). Any non zero element in $\mathbb{R}$ is a unit.

2. x is called nilpotent if $x^n = 0$, for some positive integer n which is called the nilpotency degree. In $\mathbb{Z}_4$, $2^2 = 0$ is nilpotent.

3. x is called additively idempotent if $x + x = x$. An example is 0 in $\mathbb{Z}_2$.

4. x is called multiplicatively idempotent if $x^2 = x$. An example is 1 in $\mathbb{Z}_2$.

5. x is called zero divisor if there is $0 \neq y \in S$ such that $xy = 0$. In $\mathbb{Z}_6$, 3 is a zero divisor since $3.2 = 0$.

6. x is called additively regular if there is $x' \in S$ such that $x+x+x' = x$. In $\mathbb{Z}_5$, $2 + 2 + 3 = 2$.

7. x is called multiplicatively regular if there is $x' \in S$ such that $x^2x' = x$. In $\mathbb{Z}_6$, $3^2.1 = 3$.

Similar to the ring situation, the ideals play a fundamental role in the theory of semirings. The definition of subtractive, prime, maximal, primary and nilpotent ideals is presented next. Recall that a subset I of a ring R is called an ideal if it is closed under subtraction (i.e. $a - b \in I$, for all $a, b \in I$) and absorbs the elements of R (i.e. $ra \in I$, for all $r \in R, a \in I$).

Definition 1.10 (Ideal)**.** Let I be a subset of a semiring S. Then I is called an ideal of S if it is closed under addition (i.e. $x + y \in I$, for all $x, y \in I$) and absorbs the elements of S (i.e $sx \in I$ for any $s \in S$ and $x \in I$). I is called proper ideal if $I \neq S$.

Recall that for two given ideals I and J in S. The addition of $I + J = \{a + b : a \in I, b \in J\}$ and the multiplication of $I.J = \{ab : a \in I, b \in J\}$.

Remark 1.11. The set of all the ideals of a given ring under the usual addition and multiplication of ideals forms a semiring which is not a ring, since one can add or multiply two ideals but can not subtract them.

Let I denotes an ideal in a semiring S. Below some definitions of special ideals.

Definition 1.12.

1. I is called subtractive if whenever $x \in I$ and $x + y \in I$, then $y \in I$. Let S be the semiring of natural numbers with usual addition and multiplication. Then $I = 3\mathbb{N}$ is subtractive ideal.

2. I is called prime if $I \subsetneq S$ and if $xy \in I$, then either $x \in I$ or $y \in I$. An example of prime ideal is $2\mathbb{N}$.

3. I is called maximal if $I \subsetneq S$ and if $I \subseteq J$, either $J = S$ or $J = I$. An example is $2\mathbb{N}$.

4. I is called primary if $xy \in I$ implies $x \in I$ or $y^m \in I$ where $m \in \mathbb{Z}^+$.

5. I is called nilpotent if $I^k = 0$ for some $k \in \mathbb{Z}^+$. The smallest k such that $I^k = 0$ is called the nilpotency degree of a nilpotent ideal I and it is denoted by n.deg(I).

6. I is called nil if all its elements are nilpotent.

The following is the definition of the radical of an ideal I.

Definition 1.13. Let I be an ideal of a semiring S, the radical of I is the set of all $x \in S$ such that $x^n \in I$, for some $n > 0$ and it is denoted by $\sqrt{I}$. So, $\sqrt{I} = \{x \in S : x^n \in I, n \in \mathbb{N}\}$. I is called radical if $\sqrt{I} = I$.

Example 1.14. Let S be the semiring of natural numbers. Then $\sqrt{4\mathbb{N}} = 2\mathbb{N}$ and $\sqrt{5\mathbb{N}} = 5\mathbb{N}$.

Next is the definintion of subtractive and semisubtractive semiring.

Definition 1.15 (Subtractive Semiring)**.** A semiring S is called subtractive if every ideal of S is subtractive. An example is $\mathbb{M}_3(\mathbb{R})$ [14].

Definition 1.16 (Semisubtractive Semiring)**.** A semiring S is called semisubtractive if for any $a \neq b \in S$ there is always some $x \in S$ such that $b + x = a$ or some $y \in S$ such that $a + y = b$. An example is $\mathbb{N} \cup \{0\}$.

Next the definition of quotient semiring is introduced.

Definition 1.17 (Quotient semiring)**.** Let I be an ideal of a commutative semiring S. Then the quotient semiring of S by I is ${}^{\mathcal{S}}/_{\mathcal{I}} = \{s+I : s \in S\}$ where the addition and multiplication are defined as follows:

- $(s_1 + I) + (s_2 + I) = (s_1 + s_2) + I$.
- $(s_1 + I).(s_2 + I) = (s_1 s_2) + I$.

1.1.3 Sets and Ideals

Recall that a nonempty subset C of S is called multiplicatively closed set if $1 \in C$ and if $c_1, c_2 \in C$, then $c_1 c_2 \in C$.

Definition 1.18. A multiplicatively closed set C of S is called saturated if $c_1 c_2 \in C$ means that both $c_1, c_2 \in C$ for all $c_1, c_2 \in S$.

Similar to the ring situation, the following result satisfied.

Proposition 1.19. [19, 11] *Let C be multiplicatively closed set of S and I be an ideal of S that is maximal with respect to $I \cap C = \phi$. Then I is prime ideal of S.*

Proof. Suppose, in the contrary, that I is not prime. So, there is $ab \in I$ such that $a \notin I$ and $b \notin I$. Clearly, $I + Sa$ is a semiideal (because it is sum of two ideals), and hence $I \subsetneq I + Sa$ implies $I + Sa \cap C \neq \phi$. Therefore, there is $c_1 \in C$ such that $c_1 = i_1 + xa$ for $i_1 \in I$, $x \in S$. Similarly, there is $c_2 \in C$ such that $c_2 = i_2 + yb$ for $i_2 \in I$, $y \in S$. Now, $c_1 c_2 = i_1 i_2 + i_1 yb + i_2 xa + xyab$, and hence $c_1 c_2 \in I$. But C being multiplicatively closed set and $c_1, c_2 \in C$ implies $c_1 c_2 \in C$. Thus, $I \cap C \neq \phi$ which is a contradiction. Hence, I is prime ideal of S. $\square$

Proposition 1.20. [20] *C is a saturated multiplicatively closed set of S if and only if its complement is a union of prime ideals of S.*

Proof. Let C be saturated multiplicatively closed set of S and $s \in C^{\complement}$. Then $(s) \cap C = \phi$. Hence, by proposition 1.19, if (s) is prime, then we are done. Otherwise, (s) is a subset of a prime ideal which is also disjoint from C. Thus, every element in $C^{\complement}$ is a subset of prime ideal of S such that it is disjoint from C. Therefore, $C^{\complement} = \cup P_i$ where P_i is prime ideal in S and $i = 1, 2, ...$ Conversely, let $C^{\complement} = \cup P_i$ where P_i is prime ideal in S and $i \in I$. Indeed, let $x, y \in C$ and assume that $xy \notin C$. Then $xy \in \cup P_i$, so $xy \in P_j$ for some $j \in I$. Therefore, either $x \in P_j$ or $y \in P_j$. Implies, either $x \notin C$ or $y \notin C$, and hence a contradiction. Now, suppose that $xy \in C$ and $x, y \notin C$. Then x or y in P_j, and hence $xy \in P_j$. Thus, $xy \notin C$ which is a contradiction. Therefore, C is a saturated multiplicatively closed set of S. □

Corollary 1.21. [18] *The set of zero-divisors $Z(M)$ of M is a union of prime ideals of S.*

Proof. Let $C = S - Z(M)$. Now, we want to show that C is a saturated multiplicatively closed set of S. Indeed, let $a, b \in C$ where $a, b \in S$ implies $a \notin Z(M)$ and $b \notin Z(M)$. Suppose, in the contrary, that $ab \notin C$ which means that $ab \in Z(M)$. Therefore, there is $0 \neq m$ such that $(ab)m = 0$. So, $a(bm) = 0$ implies that either $bm = 0$, and hence $b \in Z(M)$ or $bm \neq 0$, and hence $a \in Z(M)$. In both cases we get a contradiction which implies that $ab \in C$. Conversely, suppose that $ab \in C$. Then $ab \notin Z(M)$ and so both $a \notin Z(M)$ and $b \notin Z(M)$. Now, in the contrary, suppose that $a \in Z(M)$ implies $am = 0$ for some

$m \neq 0$. Hence, $(ab)m = 0$ for some $m \neq 0$. Thus, $ab \in Z(M)$ which is a contradiction. Therefore, C is a saturated multiplicatively closed set of S. Thus, by proposition 1.20, $Z(M)$ is a union of prime ideals of S. □

1.2 Trivial ring extension

This section explains the trivial ring extension which is a construction utilized often in previous studies to solve open questions, conjectures and to provide examples [4].

Definition 1.22 (Trivial ring extension)**.** Let R be a ring and M be an R-module. The trivial ring extension of R by M is the ring $R\alpha M$ where the underlying group is $R \times M$ under the componentwise addition and the multiplication is defined by $(a, m)(b, f) = (ab, af + bm)$.
The ring $R\alpha M$ is also called the idealization of M over R. $R\alpha M$ contains a subring $R\alpha(0)$ which is isomorphic to R. It also contains a nilpotent ideal $(0)\alpha M$ with nilpotency degree 2 which is isomorphic to M.

The trivial ring extension was first introduced in 1962 by Nagata [4] in order to facilitate interaction between rings and their modules and also to provide families of examples and counter examples of commutative rings containing zero-divisors (reduced elements).

Proposition 1.23. [20, 19] *Let R be a commutative ring, I an ideal of R, M an R-module and N a submodule of M. Then the following are satisfied:*

1. *$I\alpha N$ is an ideal of $R\alpha M$ if and only if $IM \subseteq N$.*

2. *The maximal ideal of $R\alpha M$ can be written as $\mathfrak{m}\alpha M$, where $\mathfrak{m}$ is a maximal ideal of R.*

3. *The prime ideals of $R\alpha M$ have the form $\mathfrak{p}\alpha M$, where $\mathfrak{p}$ is prime ideal of R.*

4. *Radical ideals of $R\alpha M$ can be written as $\mathfrak{I}\alpha M$, where $\mathfrak{I}$ is radical ideal of R.*

5. *$\sqrt{I\alpha N} = \sqrt{I}\alpha M$. Hence, $Nil(R\alpha M) = Nil(R)\alpha M$.*

Next, we will see the form of some special elements in $R\alpha M$.

Proposition 1.24. [18, 21] *Let R be commutative ring and M an R-module. Then:*

1. *The set of zero-divisors of $R\alpha M$, denoted by $Z(R\alpha M)$, is given by $Z(R\alpha M) = \{(r, m) : r \in Z(R) \cup Z(M), m \in M\}$.*

2. *The set of units of $R\alpha M$, denoted by $U(R\alpha M)$, is given by $U(R\alpha M) = U(R)\alpha M$.*

3. *The set of idempotent elements of $R\alpha M$, denoted by $Id(R\alpha M)$, is given by $Id(R)\alpha 0$.*

Theorem 1.25. [3] *Let R_1 and R_2 be commutative rings, and let M_i be an R_i-module, $i = 1, 2$. Then $(R_1 \times R_2)\alpha(M_1 \times M_2) \cong (R_1\alpha M_1) \times (R_2\alpha M_2)$.*

Proof. It is easily to show that the map ϕ defined by $\phi((r_1, r_2), (m_1, m_2)) = ((r_1, m_1), (r_2, m_2))$ is an isomorphism. □

Next is the definition of finitely generated module M and a Noetherian ring R.

Definition 1.26. An R-module M is said to be finitely generated if there exist $a_1, a_2, .., a_n \in M$ such that for any $m \in M$ we have $r_1, r_2, .., r_n \in R$ with $m = r_1a_1 + r_2a_2 + .. + r_na_n$.

Definition 1.27. A ring R is a Noetherian ring if it satisfies the ascending chain condition on ideals that is, given an increasing sequence of ideals $I_1 \subseteq ... \subseteq I_k \subseteq ...$ there exist a natural number n such that $I_n = I_{n+k}$, for $k \geq 0$.

Theorem 1.28. *The trivial ring extention $R\alpha M$ is Noetherian if and only if R is Noetherian and M is finitely generated.*

1.3 Clean Like Ring Notions

Here we recall some of the clean like ring notions and some relations among them.

Definition 1.29. Let R be a ring and $r \in R$.

1. r is called clean if there exist a unit u and an idempotent element e such that $r = u + e$.

2. r is called nil clean if there exist a nilpotent element n and an idempotent element e such that $r = n + e$.

Now, we introduce the definitions of clean and nil clean rings.

Definition 1.30. A ring R is called clean if all of its elements are clean. $\mathbb{R}$ is clean ring.

Definition 1.31. A ring R is called nil clean if all of its elements are nil clean. $\mathbb{Z}_2$ is nil clean ring.

Proposition 1.32. *If b is nilpotent element of R. Then $1 - ab$ is a unit for all $a \in R$. In particular, if b is nilpotent, then $1 + b$ is a unit.*

Proof. Let $b^n = 0$, for positive integer n. Now, $(1 - ab)(1 + ab + (ab)^2 + ... + (ab)^{n-1}) = 1$. Thus, $(1 - ab)$ is a unit for all $a \in R$. Now take $a = -1$, we have $1 + b$ is a unit. □

Proposition 1.33. *Every nil clean ring is clean. But the converse is not always true.*

Proof. Let R be nil clean ring and let $r \in R$, $r - 1 \in R$ and $r - 1 = b + e$ where $b^n = 0$, for some positive integer n and $e^2 = e$. Implies $r = e + (b + 1)$ is clean since $(b + 1)$ is a unit by the previous proposition. Thus, R is clean. □

Proposition 1.34. *Let R be a ring and I be a nil ideal of R. Then R is nil clean if and only if $\mathcal{R}/\mathcal{I}$ is nil clean.*

Proposition 1.35. *Any finite product of nil clean rings is nil clean.*

Chapter 2

PROPERTIES OF TRIVIAL SEMIRING EXTENSION

Chapter 2

Properties of Trivial Semiring Extension

This chapter focuses on the algebraic and ideal properties of the trivial semiring extension.
Most of the results provided in this chapter are either a generalization or an enhancement of the results obtained by Peyman Nasehpour in [18].

2.1 Definition and Example

The following proposition includes the definition of the trivial semiring extension.

Proposition 2.1. *Let S be a semiring and M an S-semimodule. Then the set $S\alpha M$ with the componentwise addition and the multiplication:*

$$(s_1, m_1).(s_2, m_2) = (s_1 s_2, s_1 m_2 + s_2 m_1)$$

is a semiring.

The semiring $S\alpha M$ in the previous proposition is called the trivial semiring extension of S by M. It is very clear that this semiring extension generates the trivial ring extension mentioned in the introduction.

Proposition 2.2. *Let $S\alpha M$ be the trivial semiring extension of S by M. Then:*

1. *$S\alpha M$ contains a subsemiring $S\alpha(0)$ that is isomorphic to S.*

2. *$S\alpha M$ contains a nilpotent ideal $(0)\alpha M$ with nilpotency degree 2 that is isomorphic to M.*

3. *$\frac{S\alpha M}{(0)\alpha M} \cong S$. (Proof: Let $\Phi : S\alpha M \to S$ defined by: $\Phi((s,m)) = s$. Then Ker(Φ) $= (0)\alpha M$ and so $\frac{S\alpha M}{(0)\alpha M} \cong S$).*

The following example provides a matrix analog for the trivial semiring extension, as it is the case in the trivial ring extension.

Example 2.3. Let S be a semiring and M an S-semimodule. Then the set of all matrices of the form E $= \begin{pmatrix} s & m \\ 0 & s \end{pmatrix}$ where $s \in S$, $m \in M$ under the usual addition and multiplication of matrices is a semiring which is isomorphic to $S\alpha M$.

Proof. Let $\Phi : E \to S\alpha M$ be defined by: $\phi\left(\begin{pmatrix} s & m \\ 0 & s \end{pmatrix}\right) = (s,m)$ so $\Phi\left(\begin{pmatrix} s_1 & m_1 \\ 0 & s_1 \end{pmatrix} . \begin{pmatrix} s_2 & m_2 \\ 0 & s_2 \end{pmatrix}\right) = \Phi\left(\begin{pmatrix} s_1 & m_1 \\ 0 & s_1 \end{pmatrix}\right) \Phi\left(\begin{pmatrix} s_2 & m_2 \\ 0 & s_2 \end{pmatrix}\right)$ which means that Φ is homomorphisim. It is easy to show that Φ is one to one and onto so it is an isomorphisim. $\square$

2.2 Ideal Properties of Trivial Semiring Extension

This section will introduce some ideal properties of the semiring $S\alpha M$. From now on, S denotes a semiring and M denotes an S-semimodule.

The following theorem generalizes the proposition 1.23.

Theorem 2.4. [18] *Let I be an ideal of S and N an S-subsemimodule of M. Then the following are satisfied:*

1. *$I\alpha N$ is an ideal of $S\alpha M$ if and only if $IM \subseteq N$.*

2. *If $IM \subseteq N$. Then $\sqrt{I\alpha N} = \sqrt{I}\alpha M$.*

3. *$Nil(S\alpha M) = Nil(S)\alpha M$.*

4. *Let J be an ideal of $S\alpha M$ and define L and Q as: L=$\{s \in S : \exists\ m \in M \mid (s, m) \in J\}$ and Q=$\{q \in M : \exists\ s \in S \mid (s, q) \in J\}$. Then:*

   (a) L is an ideal of S.

   (b) Q is an S-subsemimodule of M.

   (c) $LM \subseteq Q$.

   (d) $J \subseteq L\alpha Q$.

5. *$0\alpha M$ is a subset of any prime ideal of $S\alpha M$.*

6. *If J in (4) is a subtractive prime ideal of $S\alpha M$. Then L is subtractive prime ideal of S with J= $L\alpha M$.*

Proof. :

(1) Suppose that $I\alpha N$ is an ideal of $S\alpha M$. We want to show that $IM \subseteq N$. Indeed, let $x \in IM$. Then $x = am$ for some $a \in I, m \in M$. Now, $a \in I$ and N is a subset of M implies that $(a, 0) \in$

$I\alpha N$. But $I\alpha N$ being an ideal of $S\alpha M$ implies that $(0,m)(a,0) = (0,am) \in I\alpha N$. Thus, $am = x \in N$. Conversely, assume that $IM \subseteq N$. We want to show that $I\alpha N$ is an ideal of $S\alpha M$. (i) Let $(a_1,n_1),(a_2,n_2) \in I\alpha N$. Then $(a_1+a_2,n_1+n_2) \in I\alpha N$. (ii) Let (s,m) $\in$ SαM. Then $(s,m)(a,n) = (sa, sn+am)$. But $sa \in I$, $sn \in N$ and $am \in N$ (since $IM \subseteq N$). So, $(sa, sn+am) \in I\alpha N$. Thus, $I\alpha N$ is an ideal of $S\alpha M$.

(2) First, let us show that $(s,m)^n = (s^n, ns^{n-1}m)$ by induction. So, $(s,m)^1 = (s,m)$, $(s,m)^2 = (s^2, 2sm)$. Now, suppose that it is true for n-1, $(s,m)^{n-1} = (s^{n-1},(n-1)s^{n-1-1}m)$. Then for n we have:

$(s,m)^n = (s,m)^{n-1}.(s,m)$

$(s,m)^n = (s^{n-1},(n-1)s^{n-1-1}m).(s,m)$

$(s,m)^n = (s^n, s^{n-1}m + (n-1)ss^{n-1-1}m)$

$(s,m)^n = (s^n, ns^{n-1}m)$

Now, we start the proof of the equality $\sqrt{I\alpha N} = \sqrt{I}\alpha M$ by showing first that $\sqrt{I\alpha N} \subseteq \sqrt{I}\alpha M$. For that, let $(s,m) \in \sqrt{I\alpha N}$. Then there is $n \in \mathbb{N}$ such that $(s,m)^n = (s^n, ns^{n-1}m) \in I\alpha N$ which implies that $s^n \in I$, and hence $s \in \sqrt{I}$. Thus, $(s,m) \in \sqrt{I}\alpha M$. Oppositely, take $(s,m) \in \sqrt{I}\alpha M$. Then there is $n \in \mathbb{N}$ such that $s^n \in I$. Now, $(s,m)^{n+1} = (s^{n+1},(n+1)s^n m)$. Since I is an ideal of S, $s.s^n = s^{n+1} \in I$. Also, $IM \subseteq N$ implies that $(n+1)s^n m \in N$. Thus, $(s,m)^{n+1} \in I\alpha N$, and hence $(s,m) \in \sqrt{I\alpha N}$.

(3) $Nil(S\alpha M) = \sqrt{0\alpha 0} = \sqrt{0}\alpha M = Nil(S)\alpha M.$

(4) (a) Let $s_1, s_2 \in L$. Then there is $m_1, m_2 \in M$ such that (s_1, m_1), $(s_2, m_2) \in J$. But J being an ideal of $S\alpha M$ implies that $(s_1, m_1) + (s_2, m_2) = (s_1 + s_2, m_1 + m_2) \in J$, and hence $s_1 + s_2 \in L$. Now, let $s \in S$ and $s_1 \in L$. Then there is $m \in M$ such that $(s_1, m) \in J$. But J is an ideal of $S\alpha M$ which implies that $(s, m_1).(s_1, m) = (ss_1, sm + s_1m_1) \in J$, and hence $ss_1 \in L$. Thus, L is an ideal of S.

(b) The addition similar to (a). For the scalar multiplication, let $q \in Q$. Then there is $s \in S$ such that (s,q) $\in$ J. Take $s = 0$, then $(0, q) \in J$. Since J is an ideal of $S\alpha M$, we have $(s, m)(0, q) \in J$ implies that $(0, sq) \in J$, and hence $sq \in Q$ where Q is an S-subsemimodule of M.

(c) We want to show that LM $\subseteq$ Q. Indeed, Let $y \in LM$. Then $y = lm_1$ for some $l \in L$ and $m_1 \in M$. Since J is an ideal of $S\alpha M$, then $(0, m_1)(l, m) = (0, lm_1) \in J$ implies that $lm_1 \in Q$.

(d) Since J is an ideal of $S\alpha M$, then $J \subseteq L\alpha Q$.

(5) By (3) $\sqrt{0}\alpha M = Nil(S\alpha M)$ implies that $(0)\alpha M \subseteq Nil(S\alpha M) = \cap\ P$, where P is a prime ideal in SαM.

(6) We want to show that L is subtractive ideal of S. Indeed, let $s_1, s_1 + s_2 \in L$. Then there is $m_1, m_2 \in M$ such that $(s_1, m_1), (s_1 + s_2, m_2) \in J$. Since J is subtractive prime ideal containing $0\alpha M$, then $(s_1 + s_2, m_2) + (0, m_1) = (s_1, m_1) + (s_2, m_2) \in J$ implies that

$(s_2, m_2) \in J$, and hence $s_2 \in L$. Moreover, L is prime ideal of S. To prove that, let $s_1 s_2 \in L$. Then there is $m \in$ M such that $(s_1 s_2, m) \in J$. Now, $(s_1 s_2, m) = (s_1 s_2, 0) + (0, m)$ and J being subtractive ideal implies that $(s_1 s_2, 0) \in J$. Since J is prime ideal and $(s_1 s_2, 0) = (s_1, 0)(s_2, 0)$, then either $(s_1, 0) \in J$ implies $s_1 \in L$ or $(s_2, 0) \in J$ implies $s_2 \in L$. By (d) in (4) $J \subseteq L\alpha M$. Conversely, let $(l, m_1) \in L\alpha M$. Then, there is $m \in M$ such that (l,m) $\in$ J. But J being subtractive prime ideal implies that $(l, m) + (0, m_1) = (l, m + m_1) = (l, m_1) + (0, m) \in J$, and hence $(l, m_1) \in J$.

□

Recall that an ideal I of a ring R is called nil if all the elements of I are nilpotents. It is called nilpotent ideal if there is a positive integer n such that $I^n = 0$. The nilpotency degree of a nilpotent ideal I is denoted by n.deg(I) and defined to be the smallest positive integer k such that $I^k = 0$. The same definitions adapted to the semiring situation and deduce the following corollary from parts 2 and 3 in the previous theorem.

Corollary 2.5. *Let I be an ideal of S and N be a subsemimodule of M. Then the following are satisfied:*

1. *$I\alpha N$ is radical ideal of $S\alpha M$ if and only if I is radical ideal of S.*

2. *$S\alpha M$ is nil semiring if and only if S is nil semiring.*

Proof. :

(1) $I\alpha N$ is radical ideal if and only if $\sqrt{I\alpha N} = I\alpha N$ if and only if $\sqrt{I\alpha N} = \sqrt{I}\alpha M$ (by 2 in theorem 2.4) if and only if $\sqrt{I} = I$ if and only if I is radical ideal of S.

(2) $S\alpha M$ is nil if and only if $Nil(S\alpha M) = S\alpha M$ if and only if $Nil(S)\alpha M = S\alpha M$ if and only if $Nil(S) = S$ if and only if S is nil.

□

The following fact provides the transfer of the nilpotent notion to the trivial semiring extension.

Proposition 2.6. *Let I be an ideal of S and N be a subsemimodule of M. Then the following are satisfied:*

1. *$I\alpha N$ is a nilpotent ideal of $S\alpha M$ if and only if I is a nilpotent ideal of S. Moreover, if $n.deg(I) = m$, then $m \leq n.deg(I\alpha N) \leq m + 1$.*

2. *Particularly, $S\alpha M$ is nilpotent semiring if and only if S is nilpotent and if $n.deg(S) = m$, then $m \leq n.deg(S\alpha M) \leq m + 1$.*

Proof. :

(1) Let $(x, e) \in S\alpha M$. Then $(x, e)^k = (x^k, kx^{k-1}e)$. Assume that I is a nilpotent ideal with $n.deg(I) = m$. Then $(x, e)^{m+1} = (x^{m+1}, (m + 1)x^m e) = (0, 0)$ for any $(x, e) \in I\alpha N$, and hence $I\alpha N$ is nilpotent with nilpotency degree at most $m + 1$. From another side,

if $n.deg(I\alpha N)$ is k, then $(x,0)^k = 0$ for any $x \in I$, and hence $k \geq n.deg(I)$ which forces to have $m \leq n.deg(I\alpha N) \leq m+1$. Conversely, $I\alpha N$ is nilpotent imply that $(I\alpha N)^k = (0,0)$ for some k and particularly $(x,0)^k = (0,0)$ for all $x \in I$ imply I is nilpotent.

(2) It is direct consequence of (1), just take $I = S$ and $N = M$.

□

The following example shows that if $n.deg(I) = m$, then $n.deg(I\alpha N)$ can take both of the values m and $m+1$ according to N.

Example 2.7. $2\mathbb{Z}_8\alpha\mathbb{Z}_8$ is a nilpotent ring (hence) semiring of nilpotency degree 4, while $2\mathbb{Z}_8\alpha 2\mathbb{Z}_8$ ideal of $\mathbb{Z}_8\alpha\mathbb{Z}_8$ with nilpotency degree 3.

Proposition 2.8. *Let $S\alpha M$ be the trivial semiring extension. Then the following are satisfied:*

1. *If U is a maximal ideal of $S\alpha M$ of the form $U = \mathfrak{U}\alpha M$, then $\mathfrak{U}$ is a maximal ideal of S.*

2. *If P is an ideal of $S\alpha M$ of the form $P = \mathfrak{P}\alpha M$, then P is prime ideal of $S\alpha M$ if and only if $\mathfrak{P}$ is prime ideal of S.*

Proof. :

(1) Let U $= \mathfrak{U}\alpha M$ be a maximal ideal of $S\alpha M$ and suppose contrarily that $\mathfrak{U}$ is not maximal ideal of S. Then there is a maximal ideal $\mathfrak{U}'$ of S such that $\mathfrak{U} \subseteq \mathfrak{U}' \subseteq S$, and hence there is an ideal $\mathfrak{U}'\alpha M$ with $U \subseteq \mathfrak{U}'\alpha M \subseteq S\alpha M$ which gives a contradiction.

(2) Let P $= \mathfrak{P}\alpha M$ be prime ideal of $S\alpha M$. Now let $xy \in \mathfrak{P}$. Then $(xy, 0) = (x, 0)(y, 0) \in \mathfrak{P}\alpha M$. But P being prime ideal of $S\alpha M$ implies that either $(x, 0) \in \mathfrak{P}\alpha M$ or $(y, 0) \in \mathfrak{P}\alpha M$. Thus, either $x \in \mathfrak{P}$ or $y \in \mathfrak{P}$, and hence $\mathfrak{P}$ is prime ideal of S. On the other side, let $\mathfrak{P}$ be prime ideal of S and let $(x, m)(y, n) = (xy, xn+ym) \in P$. Since $\mathfrak{P}$ is prime ideal of S, then either $x \in \mathfrak{P}$ or $y \in \mathfrak{P}$. Therefore, either $(x, m) \in P$ or $(y, n) \in P$. Thus, P is prime ideal of $S\alpha M$.

□

Although the form of the prime ideals and the maximal ideals of $S\alpha M$ is not fully determined in the previous theorem. Still in some classes of semirings such as weak Gaussian semiring is possible. Below is the definition of the weak Gaussian semirings followed by the form of the prime (maximal) ideals of $S\alpha M$.

Definition 2.9. S is called a weak Gaussian semiring if each prime ideal of a semiring S is subtractive.

The following is a corollary from proposition 2.8 and the proof is the same.

Corollary 2.10. [18] *If $S\alpha M$ is a weak Gaussian semiring. Then the following statements hold:*

1. *Every prime ideal of the semiring $S\alpha M$ is of the form $P = \mathfrak{P}\alpha M$, where $\mathfrak{P}$ is a subtractive prime ideal of S.*

2. *Every maximal ideal of the semiring $S\alpha M$ is of the form $U = \mathfrak{U}\alpha M$, where $\mathfrak{U}$ is a subtractive maximal ideal of S.*

Recall that an ideal I of a semiring S is called subtractive, if $x, x+y \in I$ implies $y \in I$. A semiring S is subtractive if every proper ideal is subtractive.

Proposition 2.11. *Let S be a semiring. Then S is subtractive if and only if each proper ideal that is generated by 2 elements is subtractive.*

Proof. It is easy to show that if S is subtractive semiring, then every proper ideal is subtractive. Conversely, let I be any proper ideal of S and let $x, x+y \in I$, we want to show that $y \in I$. Assume that I_0 is the ideal generated by $x, x+y$. Then by the hypothesis, I_0 is subtractive and $y \in I_0$. Now, I_0 is a subset of I implies that $y \in I$. Therefore, I is subtractive, and hence S is subtractive. □

Theorem 2.12. [18] *If $S\alpha M$ is a subtractive semiring, then both S and M are subtractive.*

Proof. $S\alpha M$ being subtractive semiring implies that $I\alpha 0 \cong I$, $0\alpha N \cong N$ are subtractive ideals of $S\alpha M$, where I is any ideal of S and N is any S-subsemimodule of M. □

Question If S and M are subtractive. Does that mean that $S\alpha M$ is a subtractive semiring ?

Definition 2.13. A semiring S is called Noetherian if it satisfies the ascending chain condition.

Theorem 2.14. [18] *Let $S\alpha M$ be subtractive. Then $S\alpha M$ is a Noetherian semiring if and only if S is a Noetherian semiring and M is a finitely generated S-semimodule.*

Proof. Suppose that the left hand side is satisfied. Let $I_1 \subseteq I_2 ... \subseteq I_k \subseteq ...$ be ascending chain of ideals of S. Then $I_1\alpha 0 \subseteq I_2\alpha 0 ... \subseteq I_k\alpha 0 \subseteq ...$ is also ascending chain of ideals of $S\alpha M$. But $S\alpha M$ being Noetherian implies that there is $n \in \mathbb{N}$ such that $I_n\alpha 0 = I_{n+k}\alpha 0$ where $k \geq 0$, and hence $I_n = I_{n+k}$ which implies that S is a Noetherian semiring. Now, $0\alpha M$ being ideal of a Noetherian semiring $S\alpha M$ implies that $0\alpha M$ is finitely generated by $(0, m_1), (0, m_2), .., (0, m_n)$. Indeed, $(0, m) = (s_1, m_1')(0, m_1) + ... + (s_n, m_n')(0, m_n)$ where $m \in M$ and $(s_i, m_i') \in S\alpha M$, $i = 1, 2, .., n$. Therefore, $(0, m) = (0, s_1m_1) + ... + (0, s_nm_n)$, and hence $m = s_1m_1 + s_2m_2 + ... + s_nm_n$. Thus, M is a finitely generated S-semimodule. Conversely, let P be prime ideal of the subtractive semiring $S\alpha M$. Then $P = \mathfrak{P}\alpha M$ where $\mathfrak{P}$ is prime ideal of S. But S being Noetherian implies that every prime ideal $\mathfrak{P}$ in S is finitely generated. But M is also finitely generated. Thus, $P = \mathfrak{P}\alpha M$ is finitely generated. Since P is arbitrary, $S\alpha M$ is a Noetherian semiring by Cohen's theorem. □

Definition 2.15. The annihilator of M denoted by $ann(M) = \{a \in S : am = 0, m \in M\}$.

Recall that an ideal I of S is called weakly prime if $ab \in I$ implies that either $a \in I$ or $b \in I$.

The following proposition determines the form of the weakly prime ideals of $S\alpha M$.

Proposition 2.16. [18] *For I being a proper ideal of S. $I\alpha M$ is a weakly pime ideal of $S\alpha M$ if and only if I is weakly prime ideal of S and $ab = 0$ implies that $a, b \in ann(M)$ where $0 \neq a, b \in S$.*

Proof. Suppose that $I\alpha M$ is a weakly prime ideal of $S\alpha M$ and $0 \neq a_1a_2 \in I$. Then $(a_1a_2, 0) \in I\alpha M$. Now, $(a_1a_2, 0) = (a_1, 0)(a_2, 0) \in I\alpha M$ and since $I\alpha M$ is a weakly prime ideal of $S\alpha M$, either $(a_1, 0) \in I\alpha M$ or $(a_2, 0) \in I\alpha M$. Thus, either $a_1 \in I$ or $a_2 \in I$, and hence I is a weakly prime ideal of S. Now, in contrary, suppose that $a \notin ann(M)$. Then there is $m \in M$ such that $am \neq 0$. Since $ab = 0$, $(ab, am) \in I\alpha M$. Therefore, $(ab, am) = (a, 0)(b, m) \in I\alpha M$. But $(a, 0) \notin I\alpha M$ and $(b, m) \notin I\alpha M$, and hence a cotradiction. The same argument shows that $b \in ann(M)$. Conversely, let $(a_1, m_1)(a_2, m_2) \in I\alpha M$ where $a_1, a_2 \in I$ and $m_1, m_2 \in M$. Since I is a weakly prime ideal of S and if $a_1a_2 \neq 0$, implies that either a_1 *or* $a_2 \in I$, and hence either (a_1, m) *or* $(a_2, m) \in I\alpha M$ for any $m \in M$. Now, if $a_1a_2 = 0$ and neither a_1 *nor* $a_2 = 0$. Then by assumption $a_1, a_2 \in ann(M)$, and hence $(a_1, m_1)(a_2, m_2) = (0, 0)$. Thus, $I\alpha M$ is a weakly pime ideal of $S\alpha M$. □

Definition 2.17. [18] Let N be S-subsemimodule of M. Then:

1. The residual of M by N is the subset $\{s \in S : sM \subseteq N\}$ of S and it is denoted by $[N : M]$.

2. The radical of N in M, denoted by $\sqrt{N}$, is the subset $\sqrt{[N:M]}$ of S.

3. N is called primary if $N \neq M$ and $sm \in N$ where $m \notin N$ for all $s \in S$ and $m \in M$ imply that $s^n M \subseteq N$ for some positive integer n.

The following fact insure that $[N:M]$ has the ideal structure.

Proposition 2.18. [18] *Let N be S-subsemimodule of M and I an ideal of S. Then the following are satisfied:*

1. *$[N:M]$ is an ideal of S. In addition, if N is subtractive, then $[N:M]$ is subtractive.*

2. *If N is primary, then $\sqrt{N}$ is a prime ideal of S.*

Proof. :

(1) First, let $s_1, s_2 \in [N:M]$. Then $s_1M, s_2M \subseteq N$ which implies that $s_1M + s_2M \subseteq N$, and hence $(s_1+s_2)M \subseteq N$ implies $(s_1+s_2) \in [N:M]$. Second, let $s_1 \in [N:M]$. Then $s(s_1M) \subseteq N$ for any $s \in S$, and hence $(ss_1)M \subseteq N$ implies $ss_1 \in [N:M]$. Therefore, $[N:M]$ is an ideal of S. Now, let $s_1, s_1+s_2 \in [N:M]$. Then $s_1M \subseteq N$ and $(s_1+s_2)M \subseteq N$. Take $y \in s_2M$, then $y = s_2m$ for some $m \in M$. But $s_1m \in N$ and $(s_1+s_2)m = s_1m + s_2m \in N$. Since N is subtractive, $y = s_2m \in N$. Thus, $s_2M \subseteq N$, and hence $s_2 \in [N:M]$.

(2) Since N is primary, $N \neq M$ implies that $1 \notin \sqrt{N}$, and hence $\sqrt{N}$ is a proper ideal of S. Now, let $nn' \in \sqrt{N}$. Then there is $k \in \mathbb{N}$ such that $(nn')^k M \subseteq N$ implies $n^k n'^k M \subseteq N$. Suppose that $n \notin \sqrt{N}$. Then there is $m \in M$ such that $n^k m \notin N$ implies $n'^k n^k m \in N$. But N being primary and $n^k m \notin N$ implies that there is a natural number L such that $(n'^k)^L M \subseteq N$, and hence $n' \in \sqrt{N}$. Thus, $\sqrt{N}$ is a prime ideal of S.

□

The following theorem describes the primary ideals of $S\alpha M$.

Theorem 2.19. [18] *Let I be an ideal of S and $N \neq M$ be a subtractive S-subsemimodule of M. Then:*

1. *I is primary ideal of S if and only if $I\alpha M$ is a primary ideal of $S\alpha M$.*

2. *$I\alpha N$ is primary ideal of $S\alpha M$ if and only if N is a primary S-subsemimodule of M, $IM \subseteq N$ and $\sqrt{I} = \sqrt{N}$.*

Proof. :

(1) Suppose that I is a primary ideal of S, we want to show that $I\alpha M$ is a primary ideal of $S\alpha M$. Indeed, let $(a_1, m_1)(a_2, m_2) \in I\alpha M$ such that $(a_2, m_2) \notin I\alpha M$. Then $a_1 a_2 \in I$ such that $a_2 \notin I$ for $m_2 \in M$. Therefore, there is a natural number n such that $a_1^n \in I$, and hence $(a_1, m_1)^n \in I\alpha M$. Thus, $I\alpha M$ is a primary ideal of

$S\alpha M$. Conversely, suppose that $I\alpha M$ is a primary ideal of $S\alpha M$. Now, let $a_1a_2 \in I$ such that $a_2 \notin I$. Then $(a_1,0)(a_2,0) \in I\alpha M$ where $(a_2,0) \notin I\alpha M$ which implies that there is a natural number n such that $(a_1,0)^n \in I\alpha M$, and hence $a_1^n \in I$. Thus, I is a primary ideal of S.

(2) Suppose that $I\alpha N$ is primary ideal of $S\alpha M$. Indeed, let $sx \in N$ where $x \notin N$. Then $(s,0)(0,x) \in I\alpha N$ such that $(0,x) \notin I\alpha N$ which implies that there is a natural number n such that $(s,0)^n \in I\alpha N$, and hence $s^n \in I$. Therefore, I is a primary ideal of S. Now, by (4) in theorem 2.4, $IM \subseteq N$, and hence $s^nM \subseteq N$. Thus, N is primary S-subsemimodule of M. Next, we want to show that $\sqrt{I} = \sqrt{N}$. Indeed, let $x \in \sqrt{I}$. Then there is $n \in \mathbb{N}$ such that $x^nM \subseteq N$ which implies that $x \in \sqrt{N}$. On the other hand, let $x \in \sqrt{N}$. Then there is $n \in \mathbb{N}$ such that $x^nM \subseteq N$. Now, let $m \in M - N$, then $(x^n,0)(0,m) \in I\alpha N$ such that $(0,m) \notin I\alpha N$. But $I\alpha N$ being primary implies that there is $k \in \mathbb{N}$ such that $(x^n,0)^k \in I\alpha N$, and hence $x^{nk} \in I$ implies $x \in \sqrt{I}$. Thus, $\sqrt{I} = \sqrt{N}$. Conversely, we want to show that $I\alpha N$ is primary ideal of $S\alpha M$. Indeed, let $(a_1,n_1)(a_2,n_2) \in I\alpha N$ such that $(a_2,n_2) \notin I\alpha N$ implies $a_1a_2 \in I$ such that $a_2 \notin I$ for $n_2 \in N$. But I being primary ideal of S implies that there is a natural number k such that $a_1^k \in I$. Since $IM \subseteq N$, then $(a_1,n_1)^{k+1} = (a_1^{k+1},(k+1)a_1^kn_1) \in I\alpha N$, and hence $I\alpha N$ is primary ideal of $S\alpha M$. If $a_2 \in I$, then $n_2 \notin N$. But

N being subtractive and since $a_2n_1 \in N$ implies that $a_1n_2 \in N$. Therefore, there is $k \in \mathbb{N}$ such that $a_1^kM \subseteq N$. Now, following the previous argument, $(a_1, n_1)^{k+1} \in I\alpha N$ implies that $I\alpha N$ is primary ideal of $S\alpha M$.

$\square$

Example 2.20. Since $4\mathbb{N}$ is primary ideal of $\mathbb{N}$. By the previous theorem $4\mathbb{N}\alpha\mathbb{N}$ is a primary ideal of $\mathbb{N}\alpha\mathbb{N}$.

2.3 Special Elements of Trivial Semiring Extension

In this section some special elements of the trivial semiring extension $S\alpha M$ like units, idempotents, zero-divisors and regular elements are studied.

Notation 2.21. Let $U(S)$ be the set of all units of S, $V(M)$ the set of all elements of M having additive inverse, $Z(S)$ the set of all zero-divisors of S and $Z(M)$ the set of all zero divisors of M.

The following theorem provides some special elements of $S\alpha M$ such as units, zero-divisors, idempotents and regular elements.

Theorem 2.22. [18] *Let $S\alpha M$ be the trivial semiring extension of S by M. Then the following are satisfied:*

1. *The set of all units of $S\alpha M$; $U(S\alpha M)$, is the set $U(S)\alpha V(M)$ (i.e. $U(S\alpha M) = U(S)\alpha V(M)$).*

2. *The set of all zero-divisors of $S\alpha M$, denoted by $Z(S\alpha M)$, is the set $\{(s,m) : s \in Z(S) \cup Z(M), m \in M\}$.*

3. *$(s,m) \in S\alpha M$ is an additively idempotent if and only if s and m are additively idempotent elements in S and M respectively.*

4. *$(s,m) \in S\alpha M$ is a multiplicatively idempotent if and only if s is multiplicatively idempotent in S and $sm + sm = m$.*

5. *$(s,m) \in S\alpha M$ is additively regular if and only if s and m are additively regular elements of S and M respectively.*

6. *If $(s,m) \in S\alpha M$ is multiplicatively regular, then s is multiplicatively regular element of S and sm is additively regular element of M.*

Proof. :

(1) Let (s,m) be a unit in $S\alpha M$. Then there is $(s', m') \in S\alpha M$ such that $(s,m)(s',m') = (1,0)$ implies $(ss', sm' + s'm) = (1,0)$, and hence $ss' = 1$ and $(sm' + s'm) = 0$. Now, $s(sm' + s'm) = s^2m' + ss'm = 0$. But $ss' = 1$ implies $s^2m' + m = 0$. Thus, $s \in U(S)$ and $m \in V(M)$, and hence $(s,m) \in U(S)\alpha V(M)$. On the other hand, let $(s,m) \in U(S)\alpha V(M)$. Then $(s,m)(s^{-1}, s^{-2}(-m)) = (1,0)$, and hence (s,m) is a unit in $S\alpha M$.

(2) Let $(s,m) \in Z(S\alpha M)$. Then there is $(0,0) \neq (s',m') \in S\alpha M$ such that $(s,m)(s',m') = (0,0)$ implies $(ss', sm' + s'm) = (0,0)$.

If $s' \neq 0$, then $s \in Z(S)$ implies $s \in Z(S) \cup Z(M)$. Otherwise, if $s' = 0$, then $m' \neq 0$ and $sm' + 0 = 0$ implies $sm' = 0$, and hence $s \in Z(M)$ implies $s \in Z(S) \cup Z(M)$. Conversely, let $s \in Z(S)$. Then there is $0 \neq s' \in S$ where $ss' = 0$ implies $(s, 0)(s', 0) = (0, 0)$, and hence $(s, 0) \in Z(S\alpha M)$. On the other hand, if $s \in Z(M)$, then there is $0 \neq m \in M$ such that $sm = 0$ implies $(s, 0)(0, m) = (0, 0)$, and hence $(s, 0) \in Z(S\alpha M)$. Thus, if $s \in Z(S) \cup Z(M)$, then $(s, 0) \in Z(S\alpha M)$. Now, since $(0, m)^2 = (0, 0)$ for any $m \in M$ is contained in any prime ideal and $(s, 0) \in Z(S\alpha M)$. Then, by corollary 1.21, $(s, 0)$ is contained in some prime ideal of $S\alpha M$. Therefore, $(s, m) = (s, 0) + (0, m) \in Z(S\alpha M)$.

(3) Let (s, m) be an additively idempotent element in $S\alpha M$. Then $(s, m)+(s, m) = (s, m) \in S\alpha M$. Therefore, $(s+s, m+m) = (s, m)$ which implies that s is additively idempotent element of S and m is additively idempotent element of M.

(4) Let (s, m) be a multiplicatively idempotent element in $S\alpha M$. Then $(s, m)^2 = (s, m) \in S\alpha M$. But $(s, m)^2 = (s^2, sm + sm) = (s, m)$, and hence s is multiplicatively idempotent element in S and $sm + sm = m$.

(5) If (s, m) is additively regular element in $S\alpha M$, then there is $(s', m') \in S\alpha M$ such that $(s, m) + (s, m) + (s', m') = (s, m)$ implies that $s + s + s' = s$ and $m + m + m' = m$, and hence s is additively

regular element of S and m is additively regular element of M. Conversely, let s and m be additively regular elements of S and M respectively. Then $s+s+s' = s$ and $m+m+m' = m$ for $s' \in S$ and $m' \in M$, and hence $(s,m)+(s,m)+(s',m') = (s,m)$ which implies that (s,m) is additively regular element in $S\alpha M$.

(6) Let (s,m) be multiplicatively regular element of $S\alpha M$. Then there is $(s',m') \in S\alpha M$ such that $(s,m)^2(s',m') = (s,m)$ implies $(s^2s',$ $s^2m' + ss'm + ss'm) = (s,m)$, and hence $s^2s' = s$ implies that s is multiplicatively regular element of S. Also, $s^2m'+ss'm+ss'm = m$ implies $ss^2m' + s^2s'm + s^2s'm = sm$. Now, $s^2s' = s$, and hence $ss^2m'+sm+sm = sm$. Therefore, sm is additively regular element of M.

□

Example 2.23. 1. $U(\mathbb{N}\alpha\mathbb{N}) = \{(1,0)\}$.

2. $Z(\mathbb{Z}_6\alpha\mathbb{Z}_4) = \{(s,m) : s = \{2,3,4\}, m \in \mathbb{Z}_4\}$.

3. $(0,0)$ is an additively idempotent element in $\mathbb{N}\alpha\mathbb{Z}_2$.

4. The set of multiplicatively idempotents in $\mathbb{Z}_6\alpha\mathbb{Z}_5$ is equal to $\{(0,0),(1,0),(4,0),(3,0),(3,1),(3,2),(3,3),(3,4)\}$.

5. The set of additively regular elements in $\mathbb{Z}_3\alpha\mathbb{Z}_2$ is $\{(s,m) : s \in \mathbb{Z}_3, m \in \mathbb{Z}_2\}$.

6. In $\mathbb{Z}_3\alpha\mathbb{Z}_3$, the set of multiplicatively regular elements is $\{(s,m) : s,m \in \mathbb{Z}_3\}$.

The following corollary handles the conditions under which $S\alpha M$ is a semifield or a semidomain, and other results of the trivial semiring extension.

Corollary 2.24. *Let $S\alpha M$ be the trivial semiring extension. Then the following are satisfied:*

1. *$S\alpha M$ is a semidomain if and only if S is a semidomain and $M = 0$. Similarly, $S\alpha M$ is a semifield if and only if S is a semifield and $M = 0$.*

2. *If 0 is the only additively idempotent element of M, then the multiplicatively idempotent elements of $S\alpha M$ will be of the form $(s,0)$ where s is a multiplicatively idempotent element of S.*

3. *$S\alpha M$ is additively regular if and only if both S and M are additively regular.*

4. *If $S\alpha M$ is multiplicatively regular, then S is multiplicatively regular and M is additively regular.*

Proof. :

(1) $S\alpha M$ is a semidomain implies that there is no zero-divisors of $S\alpha M$. But $Z(S\alpha M) = \{(s,m) : s \in Z(S) \cup Z(M), m \in M\}$ which means

that there is no $s \in Z(S) \cup Z(M)$, and hence S is a semidomain. Suppose, in the contrary, that $M \neq 0$. So, there is $0 \neq m \in M$ such that $(0, m)(0, m) = (0, 0)$ implies that $(0, m)$ is a zero divisor in $S\alpha M$. Thus, a cotradiction. Hence, $M = 0$. Now, $S\alpha M$ is a semifield implies that $U(S\alpha M) = U(S)\alpha V(M) = S\alpha M$. Hence, S is a semifield and since every semifield is a semidomain then $M = 0$.

(2) $sm + sm = s^2m + s^2m = s(sm + sm)$. But $sm + sm = m$ implies $sm + sm = sm$. If the only additive idempotent element of M is 0, then $sm = 0$, and hence $m = 0$. Thus, the multiplicatively idempotent elements of $S\alpha M$ are of the form $(s, 0)$ where s is a multiplicatively idempotent element of S.

(3) The proof is straightforward by taking arbitrary elements and complete the arugument as was done in the previous theorem.

(4) Suppose that $S\alpha M$ is multiplicatively regular and let (s, m) be arbitrary element in $S\alpha M$. Then, by the previous theorem, s is multiplicatively regular element in S. Therefore, S is multiplicatively regular. Also, sm is an additively regular element in M. But $1.m = m$ is an additively regular element in M. Thus, M is additively regular.

$\square$

Since the only additively idempotent element of $\mathbb{N}$ is the zero, then by the previous corollary the multiplicatively idempotents of $\mathbb{N}\alpha\mathbb{N}[x]$ are only $(0,0)$ and $(1,0)$.

2.4 Local Trivial Semiring Extension

This section is devoted to answer the question; when $S\alpha M$ is local semiring. As it is the case of the rings, local semirings are semirings with one maximal ideal. The following theorem insures that $S\alpha M$ is local when S is.

Theorem 2.25. *If S is a local semiring and M is an S-semimodule, then $E = S\alpha M$ is a local semiring.*

Proof. We want to show that $E - U(E)$ is an ideal of E. For addition, let $(s_1, m_1), (s_2, m_2) \in E - U(E)$. Since $U(E) = U(S)\alpha V(M)$, either both $s_1, s_2 \notin U(S)$ or at least one of them not in $U(S)$. If $s_1, s_2 \notin U(S)$ and since S is local semiring, $S - U(S)$ is an ideal of S, and hence $s_1 + s_2 \notin U(S)$. Therefore, $(s_1, m_1) + (s_2, m_2) \in E - U(E)$. Now, if s_1 or s_2 in $U(S)$, then either m_1 or m_2 not in $V(M)$. Take for example $s_1 \in U(S)$. Then $m_1 \notin V(M)$. Now, suppose that $m_1 + m_2 \in V(M)$. Indeed, there is $a \in M$ such that $(m_1 + m_2) + a = 0$. Therefore, $m_1 + (m_2 + a) = 0$ implies that $m_1 \in V(M)$ which is a contradiction. Hence, $m_1 + m_2 \notin V(M)$. Thus, $(s_1, m_1) + (s_2, m_2) \in E - U(E)$. For multiplication, let $(s, m) \in E - U(E)$. Then either $s \notin U(S)$ or $m \notin V(M)$. So, if $s \notin U(S)$, then for $(s', m')(s, m) = (s's, s'm + sm')$, $s's \notin U(S)$. Now, suppose that $s's \in U(S)$. Then there is $0 \neq a \in S$ such that $(ss')a = 1$, and hence $s(s'a) = 1$. Thus, $s \in U(S)$ which is a contradiction. Therefore, $(s's, s'm + sm') \notin U(E)$, and hence $(s', m')(s, m) \in E - U(E)$. Now, if $s \in U(S)$, then $m \notin V(M)$. Now, $(s', m')(s, m) = (s's, s'm + sm')$. So, either $s' \notin U(S)$ implies $ss' \notin U(S)$ or $s' \in U(S)$, and hence $s'm \notin V(M)$. Suppose, in the contrary, that $s'm \in V(M)$. Then, there is $a \in M$ such that $s'm + a = 0$. Implies $m + s'^{-1}a = 0$ which means $m \in V(M)$, and hence a contradiction. Thus, $s'm \notin V(M)$. Also, $s'm + sm' \notin V(M)$. Therefore, $(s', m')(s, m) \in E - U(E)$. Thus, $E - U(E)$ is an ideal of E which implies that $E = S\alpha M$ is a local semiring. □

Corollary 2.26. [18] *Let F be a semifield and M be F-semimodule. Then $B = F\alpha M$ is a local semiring.*

Proof. Since F is a semifield, it is local semiring. By the previous theorem $B = F\alpha M$ is local semiring. □

Chapter 3

CLEAN LIKE SEMIRING NOTIONS AND THEIR TRANSFER IN THE TRIVIAL SEMIRING EXTENSION

Chapter 3

Clean Like Semiring Notions and their Transfer in the Trivial Semiring Extension

This chapter generalizes some of the clean like ring theoritic notions into the semiring situation and study some of their properties and then transfer these notions in the trivial semiring extension.

3.1 Clean Like Semiring Notions

The following definitions are a generalization of the clean ring notions to the semirings.

Definition 3.1. A semiring S is called clean if every element $s \in S$, $s = u + e$, for some unit u and an idempotent e.

Next is the definitions of some clean semiring notions.

Definition 3.2. Let S be a semiring. Then:

1. S is weakly clean if for each $s \in S$ either $s = u + e$ or $u = s + e$, for some unit u and an idempotent e.

2. S is almost clean if every element of the semiring can be written as the sum of a non-zero-divisor and an idempotent.

3. S is nil clean if for each $s \in S$, $s = n+e$, for some nilpotent element n and an idempotent e.

4. S is said to be weakly nil clean if for each $s \in S$ either $s = m + e$ or $m = s + e$, for some nilpotent element m and an idempotent e.

Remark 3.3. Let R be a ring. Then:

1. Clean semiring generalizes the clean ring.

2. Nil clean semiring generalizes the nil clean ring.

Proof. The proof is straightforward. □

Now, some propositions that study the relations among the mentioned rings will be proved. First, we recall that a semiring S is called semisubtractive if for any $a \neq b \in S$ there is always some $x \in S$ such that $b + x = a$ or some $y \in S$ such that $a + y = b$.

Remark 3.4. [10] Every subtractive subset of a semiring S is semisubtractive.

Proposition 3.5. *Every subtractive nil clean semiring is clean.*

Proof. Let S be subtractive nil clean semiring and let $s \in S$. Since subtractive implies semisubtractive. Then there is $y \in S$ such that $1+y = s$. Now, $y = e + m$ where e is an idempotent element in S and m is nilpotent element. So, $1 + y = s = e + m + 1$. To show that s is clean, we want to show that $m + 1$ is a unit. Suppose that $m + 1$ is not a unit, then there exist prime ideal p of S such that $m+1 \in p$. But $m^n = 0$ for some positive integer n, and hence $m \in p$. But since p is subtractive, then $1 \in p$ which is a cotradiction. Thus, $m+1$ is a unit. Therefore, s is clean. Thus, S is clean. $\square$

Question Does every nil clean imply clean semiring ?

Proposition 3.6. *Let S be a semiring and I be a nil ideal of S. Then S is nil clean if and only if $\mathcal{S}/\mathcal{I}$ is nil clean.*

Proof. Define $s+I \in \mathcal{S}/\mathcal{I}$. Then, $(a+e)+I \in \mathcal{S}/\mathcal{I}$ implies $(a+I)+(e+I) \in \mathcal{S}/\mathcal{I}$. Now, $(a+I)^n = 0+I = I$ and $(e+I)^2 = e^2+I = e+I$. Thus, $(a+I)$ is nilpotent and $(e+I)$ is idempotent, and hence $\mathcal{S}/\mathcal{I}$ is nil clean. Conversely, let $\mathcal{S}/\mathcal{I}$ be nil clean and take $s \in S$. Then $s+I \in \mathcal{S}/\mathcal{I}$, and so $s+I = (s_1+I)+(s_2+I)$ where s_1+I is nilpotent for some positive integer n and s_2+I is idempotent. Now, $(s_1+I)^n = s_1^n + I = I$. Thus, $s_1^n = 0$, and hence $s_1 \in S$ is nilpotent element. Also, $(s_2+I)^2 = s_2^2+I = s_2+I$. Hence, $s_2 \in S$ is an idempotent element. Therefore, $s = s_1+s_2$ is nil clean element in S, and hence S is nil clean. $\square$

Proposition 3.7. *Any finite direct product of nil clean semirings is nil clean.*

Proof. Let $E = S_1 \times S_2 \times S_3 \times ..... \times S_n$ be a direct product of nil clean semirings and let $s = (s_1, s_2, ..., s_n) \in E$. Now, $s = (a_1+e_1, a_2+e_2, ..., a_n+e_n)$ implies $s = (a_1, a_2, ..., a_n) + (e_1, e_2, ..., e_n)$. But $a_i, i = 1, 2, .., n$ being nilpotent elements implies that $a_1^{m_1} = 0, ..., a_n^{m_n} = 0$. Thus,

$(a_1, a_2, .., a_n)^{m_1 m_2 .. m_n} = (0, 0, .., 0)$ and $(e_1, e_2, .., e_n)^2 = (e_1, e_2, .., e_n)$. Therefore, s is nil clean element of E. Thus, E is nil clean. □

Proposition 3.8. *Let S be a semiring and I be a nil ideal of S Then the following are satisfied:*

1. *S is weakly nil clean if and only if ${}^{\mathcal{S}}/_{\mathcal{I}}$ is weakly nil clean.*
2. *Any finite direct product of weakly nil clean semirings is weakly nil clean.*

Proof. The proof of both statements is the same as the proof in the previous propositions. □

3.2 Transfer Clean Like Notions in the Trivial Semiring Extension

This section provides the transfer of the clean like notions in the trivial semiring extension. Actually these results are a generalization of the results in [8,18].

Theorem 3.9. *$S\alpha M$ is clean if and only if S is clean and $V(M) = M$.*

Proof. Suppose that $S\alpha M$ is clean and s be an arbitrary element of S. Then $(s, 0) \in S\alpha M$ implies $(s, 0) = (s_1, e_1) + (s_2, e_2)$ where (s_1, e_1) is a unit and (s_2, e_2) is an idempotent. Now, (s_1, e_1) being a unit implies that there is $(s_1', e_1') \in S\alpha M$ such that $(s_1, e_1)(s_1', e_1') = (s_1 s_1', s_1 e_1' + s_1' e_1) = (1, 0)$, and hence $s_1 s_1' = 1$. Thus, s_1 is a unit in S. Next, since (s_2, e_2)

is an idempotent element of $S\alpha M$, $(s_2, e_2)^2 = (s_2^2, 2s_2e_2) = (s_2, e_2)$, and hence $s_2^2 = s_2$. Therefore, s_2 is idempotent element of S, and hence S is clean. Conversely, let $(s, m) \in S\alpha M$. Since S is clean, $s = u + e$ where u is a unit in S and e is an idempotent element of S. Therefore, $(s, m) = (u, m) + (e, 0)$. Now, $u \in U(S)$ and $m \in V(M)$ implies $(u, m) \in U(S\alpha M)$. Also, $(e, 0)^2 = (e^2, 0) = (e, 0)$, and hence $(e, 0)$ is an idempotent element of $S\alpha M$. Thus, $S\alpha M$ is clean. □

Theorem 3.10. *$S\alpha M$ is weakly clean if and only if S is weakly clean and $V(M) = M$.*

Proof. Now, $S\alpha M$ is weakly clean means that every element $(s, m) \in S\alpha M$ can either be written as $(s, m) = (u, m) + (e, n)$ or as $(s, m) + (e, n) = (u, m)$ where (u, m) is a unit in $S\alpha M$ and (e, n) is an idempotent element in $S\alpha M$. Now, in both cases, the proof will be the same as the proof in the previous theorem. □

Theorem 3.11. *$S\alpha M$ is almost clean if and only if S is almost clean.*

Proof. Let $S\alpha M$ be almost clean and $s \in S$. Then $(s, 0) \in S\alpha M$. But $S\alpha M$ being almost clean implies $(s, 0) = (t, x) + (e, n)$ where $(t, x) \notin Z(S\alpha M)$ implies that $t \notin Z(S) \cup Z(M)$ and (e, n) is an idempotent element of $S\alpha M$. Thus, $(e, n)^2 = (e^2, 2en) = (e, n)$ and so $e^2 = e$. Therefore, e is an idemotent element of S. Thus, S is almost clean. Conversely, let S be almost clean, $s \in S$ and $m \in M$. Then $s = t + e$ such that $t \notin Z(S) \cup Z(M)$ and e is an idemotent element of S. So,

$(s, m) = (t, m) + (e, 0)$. Now, (t, m) is a non-zero-divisor and $(e, 0)$ is an idempotent element of $S\alpha M$. Thus, $S\alpha M$ is almost clean. $\square$

Theorem 3.12. *$S\alpha M$ is nil clean if and only if S is nil clean.*

Proof. Let $(x, 0) \in S\alpha M$ for some $x \in S$. Since $S\alpha M$ is nil clean, $(x, 0)$ is the sum of nilpotent element (x_1, e_1) and an idempotent element (x_2, e_2). Therefore, $(x_1, e_1)^n = (0, 0)$ for some $n \in \mathbb{N}$ which implies that $(x_1^n, nx_1^{n-1}e_1) = (0, 0)$, and hence $x_1^n = 0$. Thus, x_1 is a nilpotent element of S. Now, $(x_2, e_2)^2 = (x_2, e_2) = (x_2^2, 2x_2e_2)$ which implies that $x_2^2 = x_2$. Therefore, x_2 is an idempotent element of S, and hence $x = x_1 + x_2$. Thus, S is nil clean. Conversely, Let $(x, m) \in S\alpha M$ be an arbitrary element. Then $(x, m) = (x_1, 0) + (0, m) + (x_2, 0)$. But $0\alpha M$ being a nilpotent ideal of $S\alpha M$ implies that $(0, m)$ is a nilpotent element of $S\alpha M$. Now, $x \in S$ and S is nil clean. Then $x = x_1 + x_2$ where x_1 is an idempotent element of S and x_2 is a nilpotent element of S which implies that $(x_1, 0)$ is an idempotent and $(x_2, 0)$ is a nilpotent elements of S. Since both $(x_2, 0), (0, m)$ are nilpotent elements of S, then $(x_2, 0) + (0, m)$ is nilpotent, and hence $S\alpha M$ is nil clean. $\square$

Chapter 4

INCIDENCE SEMIRINGS AND AN APPLICATION OF THEM IN DATA MINING

Chapter 4

Incidence Semirings and An Application of them in Data Mining

This chapter presents the incidence semirings, which are polynomials over a graph. It also suggests a practical implication for them in one of the data mining techniques, which is the classification system. Indeed, a vauluable set of centroids with largest weight which regarded as ideals in incidence semirings will be examined. These sets are used in the design of centroid-based classifiers, as well as for the design of multiple classifers which compines several individual classifiers.

4.1 An Application of Incidence Semirings in Data Mining

Definition 4.1. Data mining is a process of finding useful patterns from large amount of data. It contains several techniques, algorithms and it can be adapted in several organizations to improve their businesses and reach excellent results based on data [6].

One of the data mining techniques is classification, which is defined as follows:

Definition 4.2. Classification is one of the most commonly applied data mining techniques; it assigns items in a collection to several categories or classes. The goal of classification is to accurately predict the class for each item in the data.

Each classification process starts with feature extraction and representation of data in a standard vector space S^n, where S can be semifield. Each centriod-based classifier selects special elements called centroids, denoted by $c_1, c_2, ..., c_k \in S^n$ and every c_i defines its class $N(c_i)$ which contains every vector v where c_i is the nearest centroid of v. Examples of this method can be found in [7,24]. Multiple classifiers are used to combine individual initial classifiers. One of the methods used to design multiple classifiers is to design several simpler initial classifiers, and then combining them into one multiple classification model with several classes which has the ability of correcting errors for individual classifiers. Examples of this method can be found in [25,23].

Example 4.3. An example of classification model is credit risk which can be used in the banking sector to identify which customers are at high risk and shouldn't be qualified to get loans.

Concepts that determines when we consider a classifier with centroid set C to be efficient are introduced next.

Definition 4.4. Let C be a class of centroid set in S^n. Then:

1. The weight $wt(v)$ of $v \in S^n$ is the number of non-zero components of v.

2. The weight of the set $C \subseteq S^n$ is the minimum weight of non-zero elements in C.

3. For a finite semiring S. The information rate of a class set $C \in S^m$ defined by $\log_{|S|} \frac{|C|}{m}$ which reflects the proportion of output of the individual initial classifiers which used to produce the outcomes of the multiple classification.

The following definition highlights conditions that makes the classifier with class set C efficient.

Definition 4.5. For a classifier with a class set C to be efficient, C must satisfy:

1. C must have large weight.

2. The information rate of C must be large.

3. C has a small number of generators.

4. The classes of centroid set for each initial classifier should be different.

Addition research related to these properties can be found in [17,16].

So, the aim is to form sets of centroids with large weights and small number of generators.

4.2 Incidence Semiring and the set of centroids with largest weight

This section presents the concept of incidence semiring and its relation with semiring. Also, the form of the set of centroid in the incidence semiring will be introduced. Furthermore, Concepts from graph theory will be utilized and reflected to incidence semirings.

Definition 4.6. Directed graph G, denoted by, $G = (V, E)$ is the set of vertices and edges, without multiple edges but possibly with loops.

The following is the definition of incidence semiring.

Definition 4.7. [1] The incidence semiring of G over S, denoted by $I_G(S)$, is the set consisting of zero and all finite sums $\sum_{i=1}^{n} s_i(u_i, v_i)$, such that $n \geq 1, s_i \in S, (u_i, v_i) \in E$, where the addition is the standard addition and the multiplication satisfied the distributive law and the rule that for all $(u_1, v_1), (u_2, v_2) \in E$, we have:

1. $(u_1, v_1) \cdot (u_2, v_2) = \begin{cases} (u_1, v_2) & \text{if } v_1 = u_2 \text{ and } (u_1, v_2) \in E, \\ 0 & \text{otherwise} \end{cases}$

   In particular, we have:

2. $\sum_{e \in E} s_e e + \sum_{e \in E} s'_e e = \sum_{e \in E} (s_e + s'_e) e$

3. $(\sum_{e \in E} s_e e)(\sum_{g \in E} s'_g g) = \sum_{e,g \in E} (s_e s'_g) eg$

Remark 4.8. The concept of incidence semiring is a generalization of the concept of incidence ring, see [15].

The definition of balanced graph is introduced next.

Definition 4.9. Let $v_1, v_2, v_3, v_4 \in V$ such that $(v_1, v_2), (v_2, v_3), (v_3, v_4)$, $(v_1, v_4) \in E$. Then, the graph G is balanced if we have, $(v_1, v_3) \in E$ if and only if $(v_2, v_4) \in E$.

The following proposition will lead to the relation between incidence semirings and semirings.

Proposition 4.10. *The multiplication is associative in the incidence semiring $I_G(S)$ if and only if G is balanced graph.*

Proof. Suppose that G is balanced graph. Now, since the distributive law holds for the incidence semiring $I_G(S)$ it is sufficies to show that for elements $a, b, c \in I_G(S)$ of the form $a = (v_1, v_2), b = (v_2, v_3), c = (v_3, v_4)$, where $v_1, v_2, v_3, v_4 \in V$ (because an arbitrary element $s = \sum_{i=1}^{n} s_i(u_i, v_i) \in I_G(S)$ can be written as $s = s_1(v_1, v_2) + s_2(v_2, v_3) + s_3(v_3, v_4)$). Now, if $(v_1, v_4) \notin E$, then $a(bc) = 0 = (ab)c$, and hence multiplication is associative. On the other hand, if $(v_1, v_4) \in E$ and if $(v_1, v_3) \in E$, then $(v_2, v_4) \in E$, since G is balanced. Thus, $a(bc) = (v_1, v_4) = (ab)c$. Now, if $(v_1, v_3) \notin E$, then $(v_2, v_4) \notin E$ and so $a(bc) = (ab)c = 0$. Therefore, multiplication is associative. Conversely, suppose that the multiplication is associative and suppose, on the contrary, that G is not balanced graph. Indeed, let $v_1, v_2, v_3, v_4 \in V$ and $(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_1, v_4) \in E$ such that $(v_1, v_3) \in E$ but $(v_2, v_4) \notin E$. Hence, $a(bc) = 0$ but $(ab)c = (v_1, v_4)$ which contradicts the associative property. Thus, G is balanced. $\square$

Corollary 4.11. *If G is balanced graph. Then incidence semiring $I_G(S)$ is a semiring.*

The form of the set of centroids in the incidence semiring $I_G(S)$ will be introduced next, but first we have the following proposition:

Proposition 4.12. *The additive semigroup of $I_G(S)$ is isomporphic to S^n, $n = |E|$ (where $|E|$ denotes the number of edges of a graph G).*

Proof. The proof is easy by taking the map $\Phi : I_G(S) \to S^n$ defined by: $\Phi(\sum_{i=1}^{n} s_i(u_i, v_i)) = (s_1, s_2, .., s_n)$. Clearly, Φ is homomorphisim, one to one and onto. Thus, $(I_G(S), +) \cong S^n$. $\square$

Thus, multiplication can be identified on S^n as in $I_G(S)$ and the set of centroids is identified as subset generated in $I_G(S)$.

Definition 4.13. Every set of elements $c_1, ..., c_k \in I_G(S)$ generates the centroid set $C(c_1, ..., c_k) = \{\sum_{j=1}^{m} l_{1,j} c_1 r_{1,j} + ... + \sum_{j=1}^{m} l_{k,j} c_k r_{k,j} \mid l_{i,j}, r_{i,j} \in I_G(S) \cup \{1\}\}$. The centroid set $C(c_1, .., c_k)$ is called the ideal generated by $c_1, ..., c_k$.

Next, some concepts related to graph theory will be recalled.

Definition 4.14. Let G be a directed graph. Then:

1. $E_{in} = \{(u, v) \in E \mid \exists\ w \in V \mid (w, u), (w, v) \in E\}$.

2. $E_{out} = \{(u, v) \in E \mid \exists\ w \in V \mid (u, w), (v, w) \in E\}$.

3. $G_{in} = G(V, E \setminus E_{in})$ is the subgraph of $G = (V, E)$ with the same set of vertices and the set $E \setminus E_{in}$ of edges.

4. $G_{out} = G(V, E \setminus E_{out})$ is the subgraph of $G = (V, E)$ with the same set of vertices and the set $E \setminus E_{out}$ of edges.

5. $W_Z = |E \setminus (E_{in} \cup E_{out})|$.

6. For every vertex $v \in V$, $In(v) = \{w \in V \mid (w, v) \in E\}$.

7. For every vertex $v \in V$, $Out(v) = \{w \in V \mid (v, w) \in E\}$.

Two special sets of edges are introduced next.

Definition 4.15. [1] $\mathfrak{P}_k$ is the set of all pairs (F, v), where $v \in V$, $F \subseteq In(v)$ such that $|F| = k$ and $(u, v) \notin E_{in}$ for all $u \in F$. Also, $Out(v) \cap Out(u_1) = Out(v) \cap Out(u_2)$ for all $u_1, u_2 \in F$. We denote W_L as the largest positive integer such that $\mathfrak{P}_{W_L}$ is not empty or zero.

Similarly to the definition of $\mathfrak{P}_k$, we have:

Definition 4.16. [1] $\mathfrak{Q}_k$ is the set of all pairs (v, F), where $v \in V$, $F \subseteq Out(v)$ such that $|F| = k$ and $(v, u) \notin E_{out}$ for all $u \in F$. Also, $In(v) \cap In(u_1) = In(v) \cap In(u_2)$ for all $u_1, u_2 \in F$. We denote W_R as the largest positive integer such that $\mathfrak{Q}_{W_R}$ is not empty or zero.

Now, we will identify three sets of elements from $I_G(S)$.

Definition 4.17. Let $I_G(S)$ be the incidence semiring of a graph G over a semiring S. Then:

1. $\mathfrak{g}_Z$ is the set of all elements $x = \sum s_{u,v}(u,v) \in I_G(S)$ such that $s_{u,v} \neq 0 \in S$ and $(u,v) \in E \setminus (E_{in} \cup E_{out})$.

2. $\mathfrak{g}_L$ is the set of all elements $x = \sum_{u \in F} s_u(u,v) \in I_{G_{in}}(S)$ for all pairs $(F,v) \in \mathfrak{P}_{W_L}$ and $0 \neq s_u \in S$ for all $u \in F$.

3. $\mathfrak{g}_R$ is the set of all elements $x = \sum_{u \in F} s_u(v,u) \in I_{G_{out}}(S)$ for all pairs $(v,F) \in \mathfrak{Q}_{W_R}$ and $0 \neq s_u \in S$ for all $u \in F$.

Next is the left and right annihilator of a semiring S.

Definition 4.18. For any semiring S, we have:

1. The left annihilator of S is the set $Ann_l(S) = \{x \in S \mid xS = 0\}$.

2. The right annihilator of S is the set $Ann_r(S) = \{x \in S \mid Sx = 0\}$.

Lemma 4.19. [1] Let S be a semidomain and G be balanced graph. Then the following are satisfied:

1. $Ann_r(I_G(S)) = I_{G_{in}}(S)$.

2. $Ann_l(I_G(S)) = I_{G_{out}}(S)$.

The first one only will be proved since the same process can be followed to prove the second one.

Proof. Let $x \in I_{G_{in}}(S)$. Then $x = \sum_{i=1}^{m} s_i(u_i,v_i)$, where $s_i \in S$ and $(u_i,v_i) \in E \setminus E_{in}$. Now, we want to show that $I_G(S)x = 0$. Indeed, let $(a,b) \in E$ and suppose, on the contrary, $(a,b)x \neq 0$. Then,

$1(a,b)(u_i,v_i) \neq 0$ for some i. Thus, $b = u_i$ and $(a,v_i) \in E$. Hence, $(u_i,v_i) \in E_{in}$ which is a contradiction. Thus, $x \in Ann_r(I_G(S))$. Conversely, let $x \in Ann_r(I_G(S))$. Then, $x = \sum_{i=1}^{m} s_i(u_i,v_i)$, where $s_i \in S$, $(u_i,v_i) \in E$. Now, suppose that $(u_i,v_i) \in E_{in}$. Then, there exist $w \in V$ such that $(w,u_i),(w,v_i) \in E$. Hence, $1(w,u_i)(u_i,v_i) = (w,v_i) \neq 0 \in E$. But $x \in Ann_r(I_G(S))$ implies $(w,u_i)x = 0$ for any multiple in x. Thus, $(u_i,v_i) \notin E_{in}$, and hence $x \in I_{G_{in}}(S)$. Therefore, $Ann_r(I_G(S)) = I_{G_{in}}(S)$. $\square$

Remark 4.20. Every semiring possesses a finitely generated ideal with the largest weight among all ideals.

The next theorem is the description of the centroid set $C(c_1,..,c_k)$ with the largest weight in $I_G(S)$.

Theorem 4.21. [1] *Let S be a semidomain, G is balanced graph and let $C = C(c_1,..,c_k)$ be an ideal with the largest weight in $I_G(S)$. Then the following conditions hold:*

1. *$wt(C(x)) = wt(x) = W_Z$, for all $x \in \mathfrak{g}_Z$.*

2. *$wt(C(x)) = wt(x) = W_L$, for all $x \in \mathfrak{g}_L$.*

3. *$wt(C(x)) = wt(x) = W_R$, for all $x \in \mathfrak{g}_R$.*

4. *C contains an element x in the union of $\mathfrak{g}_Z, \mathfrak{g}_L, \mathfrak{g}_R$ such that $wt(x) = wt(C)$.*

5. *$wt(C) = max\{1, W_Z, W_L, W_R\}$.*

Proof. :

(1) Let $x \in \mathfrak{g}_Z$. Then, $x = \sum s_{u,v}(u_i, v_i) \in I_G(S)$, where $(u_i, v_i) \in E \setminus (E_{in} \cup E_{out})$ and $0 \neq s_{u,v} \in S$. Obviously, $wt(x) = |E \setminus (E_{in} \cup E_{out})| = W_Z$. Next, we want to show that $wt(C(x)) = wt(x)$. Indeed, let $0 \neq y \in C(x)$. Then $y = \sum_{j=1}^{k} l_j x r_j$, where $l_j, r_j \in I_G(S)$. Since $E \setminus (E_{in} \cup E_{out}) = (E \setminus E_{in}) \cap (E \setminus E_{out})$, then by the previous lemma all sums of the form $l_j x r_j$ where $l_j \in I_G(S)$ or $r_j \in I_G(S)$ is equal to zero. Thus, we will assume that $l_j = r_j = 1$, and hence $C(x) = \mathbb{N}x$. Now, S is a semidomain, implies that for every $n \in \mathbb{N}$ such that $nx \neq 0$, we have that $wt(nx) = wt(x)$ (otherwise, we will have a non-zero element $n1 \in S$ such that $n1$ is a zero divisor which contradicts that S is a semidomain). Now, since n is arbitrary, then $wt(C(x)) = wt(x) = W_Z$, where $x \in \mathfrak{g}_Z$.

(2) Let $x \in \mathfrak{g}_L$. Then, $x = \sum_{f \in F} s_f(f, v) \in I_{G_{in}}(S)$, where $(F, v) \in \mathfrak{P}_{W_L}$, $0 \neq s_f \in S$ for all $f \in F$. Since S is a semidomain and by the definition of $\mathfrak{P}_{W_L}$, we have $wt(x) = |F| = W_L$. Next, we want to show that $wt(C(x)) = wt(x)$. Indeed, let $0 \neq y \in C(x)$, we want to show that $wt(y) \geq wt(x)$. Now, $y = \sum_{j=1}^{k} l_j x r_j$, where $l_j, r_j \in I_G(S)$. By the previous lemma, all sums of the form $l_j x$ is equal to zero for all $l_j \in I_G(S)$. Hence, we will assume that all $l_j = 1$. Now, $I_G(S) = \sum_{(u,w) \in E} S(u, w)$, applying the distributive law. Then every $1 \neq r_j \in \cup_{(u,w) \in E} S(u, w)$. Also, since $x r_j \neq 0$ then $r_j \in \cup_{(v,w) \in E} S(v, w)$. Now, the definition of $\mathfrak{P}_{W_L}$ implies

$Out(v) \cap Out(f)$ is the same set T for all $f \in F$. Since $xr_j \neq 0$, then all $r_j \in \cup_{w\in T} S(v,w)$ (otherwise, suppose that $(v,w) \in E$ such that $w \notin Out(f)$ implies $(f,w) \notin E$, and hence $xr_j = 0$, which is a contradiction). Now, since S is semidomain, then $wt(xr_j) = wt(x)$ for each $r_j \in \cup_{w\in T} S(v,w)$. This and the distributive law implies that $wt(y) \geq wt(x)$. Therefore,$wt(C(x)) = wt(x) = W_L$, where $x \in \mathfrak{g}_L$.

(3) The proof is the same as the previous one.

(4) Let $wt(C) > 1$ and let $0 \neq x \in C$ with minimal weight in C. Now, suppose that $x \notin Ann_r(I_G(S)) \cup Ann_l(I_G(S))$. Then, by lemma, $x \notin I_{G_{in}}(S) \cup I_{G_{out}}(S)$. Indeed, there exist $(a,b),(c,d) \in E$ such that $(a,b)x \neq 0$ and $x(c,d) \neq 0$. Hence, $(a,b)x(c,d) \neq 0$, implies $wt((a,b)x(c,d)) = 1$. But $(a,b)x(c,d)$ being an element in C, implies by (1) that $wt(C) = 1$ which contradicts the assumption that $wt(C) > 1$. Thus, this case is not a possible case. Next, suppose that $x \in Ann_r(I_G(S)) \setminus Ann_l(I_G(S))$. Then, there exist $(v,w) \in E$ such that $x(v,w) \neq 0$. Clearly, $wt(x(v,w)) \leq wt(x)$. But by the minimality of x and since $x(v,w) \in C$ we have $wt(x(v,w)) = wt(x)$. Hence, there exist a subset $F \subseteq In(v)$ such that $x = \sum_{f\in F} s_f(f,v)$, where $0 \neq s_f \in S$. Now, $|F| = wt(x)$. Furthermore, from lamma, $x \in I_{G_{in}}(S)$ implies $(f,v) \notin E_{in}$ for all $f \in F$. Now, $Out(v) \cap Out(f_1) = Out(v) \cap Out(f_2)$ (otherwise, suppose that

there is $w \in V$ such that $(f_1, v)(v, w) \neq 0$ but $(f_2, v)(v, w) = 0$ implies $(f_2, w) \notin E$, and hence $wt(x(v, w)) < wt(x)$ which contradicts the minimality of weight of x). Thus, $(F, v) \in \mathfrak{P}_{|F|}$. Now, by the minimality of weight of x we have $wt(C) = wt(x) = |F|$. Condition (2) shows that $wt(x) \geq W_L$. But, by the definition of W_L, $wt(x) = W_L$. Thus, $|F| = W_L$, $(F, v) \in \mathfrak{P}_{W_L}$ and (2) implies $x \in \mathfrak{g}_L$. Therefore, in this case we have $x \in \cup(\mathfrak{g}_Z, \mathfrak{g}_L, \mathfrak{g}_R)$ such that $wt(x) = wt(C)$. For the case that $x \in Ann_l(I_G(S)) \setminus Ann_r(I_G(S))$, the proof is the same of the previous case. Finally, suppose that $x \in Ann_r(I_G(S)) \cap Ann_l(I_G(S))$. By lemma, $x \in I_{G_{in}}(S) \cap I_{G_{out}}(S)$. Now, the maximality of weight of C and condition (1) implies that $wt(C) = W_Z$. Thus, $x \in \mathfrak{g}_Z$. Since $wt(C) = wt(x)$, then there exist $x \in \cup(\mathfrak{g}_Z, \mathfrak{g}_L, \mathfrak{g}_R)$ such that $wt(x) = wt(C)$.

(5) By (4) we have $wt(C) \leq max\{1, W_Z, W_L, W_R\}$. But because of the maximality of weight of C and the previous conditions we have $wt(C) = max\{1, W_Z, W_L, W_R\}$.

□

To sum up, a full description implemented to the set of centroids with largest weight which plays an important role in the design of centroid-based classification system that can be considered as one of the main functions of data mining techniques.

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أشباه الحلقات المماثلة للنظيفة والتوسعة شبه الحلقية البدهية

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قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات، بكلية الدراسات العليا، في جامعة النجاح الوطنية، نابلس، فلسطين.

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الملخص

يناقش الجزء الأول من الأطروحة الخواص الجبرية و والخصائص النظرية للمثاليات للتوسعة شبه الحلقية البدهية التي توفر نتائج تناظرية لتلك المثبتة في التوسعة الحلقية البدهية.

تم تحديد العناصر المختلفة مثل الوحدات، قواسم الصفر وغيرها من العناصر بالإضافة الى دراسة مثاليات الطرح والمثاليات الأولية للتوسعة شبه الحلقية البدهية.

الجزء الثاني يدرس تعميم بعض المفاهيم المتعلقة بالحلقات المماثلة للنظيفة لحالة أشباه الحلقات ودراسة بعض خصائصها ثم تعميم هذه المفاهيم على التوسعة شبه الحلقية البدهية.

يقدم الجزء الأخير تطبيقا حيويا لتوظيف أشباه الحلقات البدهية في نظام التصنيف الذي يتم فيه توزيع مجموعة من العناصر على عدة مجموعات بناءا على الصفات المشتركة لهذه العناصر والذي يعتبر من أكثر التقنيات المستخدمة في التنقيب عن البيانات.