An-Najah National University

Faculty of Graduate Studies

Estimating Numerical Error Bound for Unstable Dynamical Linear Systems

By

Sanad HajMohammed

Supervisor

Dr. Adnan Daraghmeh

Co-Supervisor

Prof. Naji Qatanani

This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics, Faculty of Graduate Studies, An-Najah National University, Nablus - Palestine.

2020

Estimating Numerical Error Bound for Unstable Dynamical Linear Systems

By

Sanad HajMohammed

This Thesis was Defended Successfully on 8/1/2020 and approved by:

Defense Committee Members		**Signature**
1. Dr. Adnan Daraghmeh	**/ Supervisor**	**…….……..…**
2. Prof. Naji Qatanani	**/ Co-Supervisor**	**…….……..…**
3. Dr. Abdel Qader Mustafa	**/ External Examiner**	**...…………….**
4. Dr. Hadi Hamad	**/ Internal Examiner**	**…...………….**

الاهداء

الى روح مفجر ثورة السكاكين

من ضحى بسنين عمره خلف قضبان السجون

من وقف كالجبل الأشم في وجه الاحتلال

في سبيل تحرير فلسطين

الاسير المحرر والشهيد المناضل

أحمد توفيق حج محمد

الشكر والتقدير

الحمد لله رب العالمين والصلاة والسلام على سيد الخلق أجمعين سيدنا محمد عليه أفضل الصلاة وأتم التسليم، وبعد...

فأني أشكر الله سبحانه وتعالي على فضله حيث أتاح لي إنجاز هذا العمل بفضله، فله الحمد أولاً وآخراً.

ثم أشكر أولائك الأخيار الذين مدوا لي يد المساعدة خلال هذه الفترة وفي مقدمتهم أساتذتي الفضلاء المشرفين على هذه الأطروحه الذين غمروني بكرمهم ولم يبخلوا بوقت أو جهد الدكتور عدنان دراغمة والاستاذ الدكتور ناجي قطناني.

كما اشكر أعضاء لجنة المناقشة المحترمين الدكتور عبد القادر مصطفى / جامعة فلسطين التقنية والدكتور هادي حمد / جامعة النجاح الوطنية.

ويوجب عليَ الإعتراف بالفضل أن أشكر والدي العزيز الذي ما برح يذلل لي كل الصعاب ويمهد لي كل الدروب حتى بلوغ الغاية.

وأتقدم بشكري الجزيل لعائلتي وكل الأهل والأصدقاء والزملاء وكل من ساعدني وأعانني على إنجاز هذا البحث، فلهم في النفس منزلة فهم أهل الفضل والخير والشكر.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

Estimating Numerical Error Bound for Unstable Dynamical Linear Systems

أقر بأن ما اشتملت عليه هذه الرسالة إنما هو من نتاج جهدي الخاص باستثناء ما تمت الإشارة إليه حيثما ورد، وان هذه الرسالة ككل، أو أي جزء منها لم يقدم لنيل أي درجة أو لقب علمي أو بحثي لدى أية مؤسسة تعميمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's Name: **اسم الطالب:**

Signature: **التوقيع:**

Date: **التاريخ:**

Table of Contents

Estimating Numerical Error Bound for Unstable Dynamical Linear Systems

By

Sanad Haj Mohammed

Supervisors

Dr. Adnan Daraghmeh

Co-Supervisor

Prof. Naji Qatanani

Abstract

Many of the physical, chemical and engineering applications lead to a system of differential equations or partial differential equations. Some of those applications involve a high order system. In this work, we will present some important analytical and numerical results concerning linear dynamical systems and their applications. We consider the case of unstable linear dynamical systems and our goal is to reduce the order of this system with minimal error bound with zero initial condition. First, we present the stable system and study two approaches to reduced order of stable system, balanced truncation method and singular perturbation approximation method. Then we study the $\mathcal{L}_{2[0,T],ind}$ norm to reduce the order of unstable system. Next, to show the efficiency of these approaches we use MATLAB software to solve an example of stable system by balanced truncation method and another example of unstable system by $\mathcal{L}_{2[0,T],ind}$ norm.

Introduction

The field of control systems has a long history which began with the early desire of humans to take advantage of the materials and forces of nature. A control system is a system, which provides the desired response by controlling the output.

A more formal analysis of the field began with dynamic analysis of the centrifugal governor, conducted by the physicist James Clerk Maxwell in 1868, entitled On Governors. A centrifugal governor was already used to regulate the velocity of windmills [19]. Maxwell's work leads to generate a flurry of interest in the topic, during Maxwell's classmate, Edward John Routh, abstracted Maxwell's results for the general class of linear systems [23]. Independently, Adolf Hurwitz analyzed system stability using differential equations in 1877, resulting in what is now known as the Routh–Hurwitz theorem [22]. A notable application of dynamic control was in the area of manned flight. The Wright brothers made their first successful test flights on December 17, 1903 and were distinguished by their ability to control their flights for substantial periods. Continuous reliable control of the airplane was necessary for flights lasting longer than a few seconds.

During World War II, control theory was becoming an important area of research. Irmgard Flügge-Lotz developed the theory of discontinuous automatic control systems, and applied the bang-bang principle to the development of automatic flight control equipment for aircraft. Other areas of application for discontinuous controls included fire-control systems, guidance systems and electronics. Sometimes, mechanical methods are used to improve the stability of systems; for example, ship stabilizers.

Modeling of chemical, physical or biological phenomena often leads to high-dimensional systems of differential equations, resulting from semi-discretized partial differential equations [15].

This leads to a well-known representation called linear time invariant (LTI) system:

$$\dot{x} = Ax + Bu \qquad\qquad y = Cx + Du \qquad (1)$$

$$x(t_0) = x_0$$

Where $\in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$. The order n of the system ranges from a few tens to several hundred as in control problems for large flexible space structures [9].

A linear system is a mathematical model of a system based on the use of a linear operator. Linear systems typically exhibit features and properties that are much simpler than the nonlinear case. As a mathematical abstraction or idealization, linear systems find important applications in automatic control theory, signal processing and telecommunications.

Linear systems have been under study for long time because of their important applications in physics and engineering. Furthermore, there is no doubt that linear systems will continue to be important subject to study for long time. Finite dimensional linear system was studied prior to 1930s. All this work was for single-input, single-output systems. In the late 1950s, linear systems were extended to multi-input, multi-output systems that become important in many physics and engineering applications. This led to a special work by Bellman and Kellman. This approach has led to a more important details of the structure of finite-dimensional linear systems and to questions of redundancy, minimality, controllability, observability, etc [20, 5, 2].

Linear large-scale systems arise in many practical applications, for instance, in circuit simulations and in control problems where the underlying physical process is modeled by partial differential equations. Model reduction or model order reduction is a mathematical process to find a low-dimensional approximation for a system of equations. The main idea is that a high-dimensional state vector actually belongs to a low-dimensional subspace [1, 3, 4].

In this thesis, our main focus is to reduce the order of unstable system with a minimal error. First we study stable system and reduce its order. In Chapter 1, we discus some definitions and preliminary results. In Chapter 2, we introduce the stable system and study two methods to reduce its order. In Chapter 3, we discuss the unstable system and study a method to reduce its order (this work depends on stable systems). Finally, we show the efficiency of this work by introducing numerical examples and solveing them using MATLAB software.

Chapter One
Preliminaries

In this chapter, we will study some of the theoretical notations of control systems, and then discuss the state-space and the output equation for the dynamical system. We introduce the Laplace transform and some of its properties and discuss the description of the system in terms of its transfer function and the transition matrix. We introduce the basic concepts of controllability, observability and stabilization. Then we present the Lyapunov equations**.**

1.1 State space equation

To study the linear dynamical system one must introduce first the state space equation which is a combination of first order differential equations, given as:

$$\dot{x} = Ax + Bu, \tag{1.1}$$

such that,

$$\dot{x} = \frac{dx}{dt}$$

denotes the derivation of x with respect to time t.

The following equation

$$x(t) = [x_1(t), x_2(t), \dots \dots ., x_n(t)]^T \in \mathbb{R}^n$$

is called the state vector of the system, and

$$u = u(t) \in \mathbb{R}^m$$

the input function.

The initial condition of the system is denoted by:

$X(t_0) = x_0$.

A and B are constant matrices such that $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ [1].

A state vector of the system gives a relationship between the input and the state variables. Now we will illustrate the output equation of a dynamical system.

The output equation for a linear dynamical system is:

$$y = Cx + Du, \tag{1.2}$$

where y is the output function, C and D are constant matrices such that $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{p \times m}$. C is called here as the output map and it is depicting the react between the system and the outside world [1]. D is a matrix that describe the weight of the system input. In our work, we consider the continuous linear time invariant system. Time invariant system means that A, B, C and D are independent with time (constant matrices).

The following two equations with constant coefficients characterize a finite dimensional linear time invariant (FDLTI) dynamical system:

$$\dot{x} = Ax + Bu \tag{1.3}$$

$$y = Cx + Du, \tag{1.4}$$

such that, $x(t) \in \mathbb{R}^n$ representing the state of the system, $x(t_0)$ initial condition, the input of the system is $u(t) \in \mathbb{R}^m$ and the output is $y(t) \in \mathbb{R}^p$. The dimension of this system is n [32].

If the matrix $D = \mathbf{0}$, so the linear system can be described as:

$$\dot{x} = Ax + Bu \tag{1.5}$$

$$y = Cx, \tag{1.6}$$

such that $x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times m}$ and the initial condition is $x(t_0) = x_0$.

The dynamical system given by equations (1.3) and (1.4) can be written in a general form using the symbol Σ_s.

Definition 1.1.1 . [1] A linear system described by state space equation is a quadruple linear maps (matrices):

$$\Sigma_s = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{1.7}$$

The dimension of the system is defined as the dimension of correlating state space; that is:

$$Dim(\Sigma_s) = n. \tag{1.8}$$

In case where $D = \mathbf{0}$, we write the system as:

$$\Sigma_s = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}. \tag{1.9}$$

In this work D is considered to be equal to $\mathbf{0}$.

Definition 1.1.2 . [32] Let

$$\Sigma_{\mathrm{s}} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

be linear continuous dynamical system. Then Σ_s is called a single input single output (SISO) system if it has single input ($m = 1$) and single output ($p = 1$). Otherwise it is called multi input multi output (MIMO) system.

1.2 Stability of continuous time system

In this section we will study the concept of stability of continuous time linear dynamical systems, and we discus some definitions associated to stability.

Definition 1.2.1. [10] A matrix N is said to be stable matrix if all eigenvalues of N have negative real parts (i.e. $R\{\lambda_i(N)\} < 0$).

A continuous time linear system:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

is called bounded input bounded output (BIBO) stable if we get bounded output of any bounded input [4].

Definition 1.2.1. [9] The system

$$\Sigma_s = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

Is said to be asymptotically stable if the real parts of eigenvalues of the matrix A are strictly negative (i.e. $R\{\lambda_i(A)\} < 0$), and it's called stable if $R\{\lambda_i(A)\} \leq 0$, where λ_i denote the eigenvalues of A. Otherwise the system is called unstable system.

1.3 The Laplace transformation

In this section I will discuss Laplace transform. It is very useful transformation making the calculation easier.

Definition 1.3.1. [6, 11] Let $\varphi(t)$ be a real-valued function defined on $t \geq 0$, then the Laplace transformation of $\varphi(t)$ denoted by $\Phi(s)$ is given as:

$$L[\varphi(t)] = \int_0^\infty \varphi(t)e^{-st}dt = \Phi(s), \qquad (1.10)$$

where $s = \sigma + i\beta$, σ and β are real variables.

The inverse Laplace transformation of a function $\Phi(s)$ is the unique function $\varphi(t)$ that is continuous on $[0, \infty)$ and satisfies:

$$L^{-1}[\Phi(s)] = \varphi(t). \qquad (1.11)$$

The following properties are used for computing the Laplace transformation:

Let α be a constant and $\Phi(s) = L[\varphi(t)]$, then:

1. (Linearity) $[\alpha\varphi(t)] = \alpha\Phi(s)$.

2. (super position) $L[\varphi_1(t) + \varphi_2(t)] = \Phi_1(s) + \Phi_2(s)$.

   where $\Phi_1(s) = L[\varphi_1(t)]$ and $\Phi_2(s) = L[\varphi_2(t)]$.

3. (Translation in time) If $\alpha > 0$, then $L[u(t)\varphi(t-\alpha)] = e^{-\alpha s}\Phi(s)$.

4. (Translation in the domain) $L[e^{\alpha t}\varphi(t)] = \Phi(s-\alpha)$.

5. (Real differentiation) Let $\dot{\varphi}(t)$ be the first derivative of $\varphi(t)$, then $L[\dot{\varphi}(t)] = s\Phi(s) - \varphi(0)$.

   Note that: this property can be generalized to the n^{th} derivation:

$$L[\varphi^{n}(t)] = s^{n}\Phi(\mathrm{s}) - \mathrm{s}^{\mathrm{n}-1}\varphi(0) - \mathrm{s}^{\mathrm{n}-2}\dot{\varphi}(0) - \cdots - s\varphi^{(n-2)}(0) - \varphi^{(n-1)}(0)$$

6. (Real integration) $L\left[\int_0^t \varphi(\tau)d\tau\right] = \frac{\Phi(s)}{s}$.

7. (Convolution) $L[\varphi(t) * h(t)] = L[\varphi(t)]L[h(t)]$

$$= \Phi(s)\mathrm{H}(s)$$

where the convolution operation is defined as:

$(\varphi * h)(t) = \int_0^t \varphi(\tau)h(t-\tau)d\tau$. For more details see [11, 31, 6].

1.4 Matrix derivation, integral and exponential

In this section our consideration is look to the derivative and integral of a matrix and studies its properties, then we define the exponential matrix and its impersonation and give the rules for its calculation.

Definition 1.4.1. [9, 6] Let $\mathrm{M}(t) = [m_{\mu\gamma}(t)]$ be a square matrix where its entries are functions of time t. Then:

1. The derivative of $\mathrm{M}(t)$ denoted by $\frac{d}{dt}\mathrm{M}(t)$ is:

$$\frac{d}{dt}\mathrm{M}(t) = \dot{\mathrm{M}}(t) = \left(\frac{d}{dt}(m_{\mu\gamma}(t))\right). \tag{1.13}$$

2. The integral of $\mathrm{M}(t)$ is:

$$\int \mathrm{M}(t)dt = \left(\int m_{\mu\gamma}(t)dt\right). \tag{1.14}$$

The derivative or integral of any matrix can be calculated by differentiating or integrating each entry of the matrix. Depending on this definition we have the following rules [6]:

Let α, β, a and b are constant, M and N be matrices. Then:

- $\frac{d}{dt}(\alpha\mathrm{M}) = \alpha\frac{d}{dt}\mathrm{M} = \alpha\dot{\mathrm{M}}$.
- $\frac{d}{dt}(\alpha\mathrm{M} + \beta\mathrm{N}) = \alpha\frac{d}{dt}\mathrm{M} + \beta\frac{d}{dt}\mathrm{N} = \alpha\dot{\mathrm{M}} + \beta\dot{\mathrm{N}}$.
- $\int_a^b \alpha\mathrm{M}dt = \alpha\int_a^b \mathrm{M}dt$, a and b are real numbers.
- $\int_a^b (\alpha\mathrm{M} + \beta\mathrm{N})dt = \alpha\int_a^b \mathrm{M}\,dt + \beta\int_a^b \mathrm{N}\,dt$.
- $\frac{d}{dt}(\mathrm{MN}) = \mathrm{M}\frac{d}{dt}\mathrm{N} + \mathrm{N}\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{M} = \mathrm{M}\dot{\mathrm{N}} + \dot{\mathrm{M}}\mathrm{N}$.

- $\mathrm{M}^0 = I.$
- $\frac{d}{dt}\mathrm{M}^n \neq n\mathrm{M}^{n-1}\frac{d}{dt}\mathrm{M}.$

Definition 1.4.2.[9] Given a square matrix $\mathrm{M} \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}$. Then the matrix exponential of M is denoted by $e^{\mathrm{M}t}$ and it's a square matrix of the same order as M given as:

$$e^{\mathrm{M}t} = \frac{I}{0!} + \frac{\mathrm{M}t}{1!} + \frac{\mathrm{M}^2t^2}{2!} + \frac{\mathrm{M}^3t^3}{3!} + \cdots \qquad (1.15)$$

If M and N are two matrices, α and β are two constants, then the following rules hold for the matrix exponential [9].

- $e^{\mathrm{M}0} = I.$
- $e^{-\mathrm{M}\alpha} = [e^{\mathrm{M}\alpha}]^{-1}.$
- $e^{\mathrm{M}(\alpha+\beta)} = e^{\mathrm{M}\alpha}e^{\mathrm{M}\beta}.$
- $e^{(\mathrm{M}+\mathrm{N})\alpha} = e^{\mathrm{M}\alpha}e^{\mathrm{N}\alpha}$, only when $\mathrm{M}N = \mathrm{NM}$.
- $\frac{d}{dt}e^{\mathrm{M}t} = \mathrm{M}e^{\mathrm{M}t} = e^{\mathrm{M}t}\mathrm{M}.$
- $\int_0^\alpha e^{\mathrm{M}\alpha}d\alpha = \mathrm{M}^{-1}[e^{\mathrm{M}\alpha} - I] = [e^{\mathrm{M}\alpha} - I]\mathrm{M}^{-1}.$

1.5 State transition matrix

In this section we study the concept of state transition matrix and we present some of its properties.

Definition 1.5.1.[11] For a dynamical system, a state transition matrix is a matrix function denoted by $\mathrm{K}(t)$ and defined as:

$$\mathrm{K}(t) = e^{\mathrm{M}t}, \tag{1.16}$$

where M is a matrix.

State transition matrix has the following properties:

- $\mathrm{K}(t_2 - t_1)\mathrm{K}(t_1 - t_0) = \mathrm{K}(t_2 - t_0), \forall t_0, t_1, t_2.$
- $\mathrm{K(t)K(t)K(t)} \cdots \mathrm{K(t)} = \mathrm{K^n(t)}$, n is positive integer.
- $\mathrm{K}^{-1}(t) = \mathrm{K}(-t)$.
- $\mathrm{K}(0) = I$, unity matrix.
- $\mathrm{K(t)}$ is nonsingular for all finite value of t.

For more details see [11].

1.6 Solution of the state and output equation

In this section a study of the solution of the state and output equations of the linear dynamical continuous systems is presented.

To obtain the solution of the state space equation, we consider the following steps:

Multiply both sides of equation (1.5) by e^{-At} giving:

$$e^{-At}\dot{x} = e^{-At} \quad Ax + e^{-At}Bu$$

$$e^{-At}\dot{x} - e^{-At}Ax = e^{-At}Bu$$

$$\frac{d}{dt}[e^{-At}x(t)] = e^{-At}Bu$$

$$\int_{t_0}^{t} \frac{d}{dt}[e^{-At}x(t)]d\tau = \int_{t_0}^{t} e^{-At}Bu(\tau)d\tau$$

$$e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^{t} e^{-At}Bu(\tau)d\tau \qquad (1.17)$$

$$e^{-At}x(t) - e^{-At_0}x_0 = \int_{t_0}^{t} e^{-At}Bu(\tau)d\tau$$

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau, \forall t > t_0.$$

This equation describes the change of state with respect to the input vector $u(t)$ and the initial condition $x(t_0)$.

Suppose $y(t) = Cx(t)$ given in equation (1.6). Then, from the solution of the state equation $x(t)$, the solution of the output equation of the system is:

$$y(t) = Ce^{A(t-t_0)}x_0 + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau. \qquad (1.18)$$

In case when $t_0 = 0$, the solution of the dynamical system becomes:

$$x(t) = e^{At}x_0 + \int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau, \qquad (1.19)$$

$$y(t) = Ce^{At}x_0 + C\int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau. \qquad (1.20)$$

Consider the system given by equations (1.3) and (1.4), it follows that, the solution of the output equation with $D \neq \mathbf{0}$ is given as:

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \qquad (1.21)$$

Equation (1.21) called the convolution equation and the general form of the solution of the system can be represented by this equation.

The system time response is determined by the state $x(t)$, the output $y(t)$, the control input $u(t)$ and the initial condition x_0 for $t \geq 0$.

For zero input control and from equation (1.20), we obtain the response of the system as:

$$y(t) = Ce^{At}x_0. \qquad (1.22)$$

For zero initial condition, the forced response of the dynamical system is determined by the following equation:

$$y(t) = C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \qquad (1.23)$$

Finally, we have the following case known as the impulse response and in this case we set $x_0 = 0$ and define the input control as:

$$u(t) = \delta(t) = \begin{cases} 0\ , & t \neq 0 \\ \infty\ , & t = 0 \end{cases}$$

where $\delta(t)$ is the unit impulse (the Dirac delta) function satisfying the dirac distribution:

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = f(\tau),$$

where f is a continuous function at $t = \tau$.

Now, the impulse response is given as:

$$y(t) = \int_0^t \left(Ce^{A(t-\tau)}B + D\delta(t-\tau)\right)u(\tau)d\tau. \qquad (1.24)$$

The impulse response matrix of the dynamical system is defined as:

$$g(t) = Ce^{At}B + D\delta(t).$$

The relationship between the input and the output with zero initial condition can be described by the convolution equation [32, 10].

$$y(t) = (g * u)(t) = \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau$$

$$= \int_{-\infty}^{t} g(t-\tau)u(\tau)d\tau. \qquad (1.25)$$

1.7 Transfer function of the dynamical system

In this section, we will study the concept of transfer function of dynamical linear system. It's very important property of dynamical system [9, 10, 31].

Let

$$\Sigma_s = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (1.26)$$

be a linear continuous time dynamical system. Using Laplace transformation for the state and output equations of the system (1.5) and (1.6), we obtain:

$$L[\dot{x}] = L[Ax] + L[Bu]$$

this gives

$$sX(s) - x(0) = AX(s) + BU(s)$$

so

$$sX(s) - AX(s) = BU(s) + x(0) \tag{1.27}$$

gives

$$(sI - A)X(s) = BU(s) + x(0)$$

so, obtain

$$X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0).$$

and

$$L[y] = L[Cx],$$

so

$$Y(s) = CX(s). \tag{1.28}$$

We call the matrix $(sI - A)^{-1}$ the function matrix or the transition matrix.

From equations (1.27) and (1.28), we obtain:

$$Y(s) = C(sI - A)^{-1}BU(s) + C(sI - A)^{-1}x(0). \tag{1.29}$$

If we consider zero initial condition, means $x(0) = 0$, then (1.29) becomes:

$$Y(s) = C(sI - A)^{-1}BU(s). \tag{1.30}$$

Definition 1.7.1.[1, 10] The function matrix or transition matrix $H(s)$ from u to y with zero initial condition is given as:

$$Y(s) = H(s)U(s). \quad (1.31)$$

We define $H(s)$ as:

$$H(s) = \frac{Y(s)}{U(s)}. \quad (1.32)$$

If A is stable matrix, then $H(s)$ takes the form:[9]

$$H(s) = (sI - A)^{-1}B \quad (1.33)$$

1.8 Lyapunov equations

In this section, a combination of important equations in control theory named Lyapunov equations is presented.

Definition 1.8.1.[1]The matrix equations

$$ZA + A^T Z = -F \quad (1.34)$$

and

$$AZ + ZA^T = -F \quad (1.35)$$

are called the Lyapunov equations.

where

$$F \in \mathbb{R}^{n\times n}.$$

Theorem 1.8.2. [1, 10](Lyapunov stability theorem)

The system

$$\Sigma_s = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

is asymptotically stable if and only if we have a unique symmetric positive definite matrix Z for any symmetric positive definite matrix F, satisfying the equations:

$$ZA + A^T Z = -F$$

and

$$AZ + ZA^T = -F$$

Proof: ($\Longrightarrow$) Define the matrix Z by:

$$Z = \int_0^\infty e^{A^T t} F e^{At} dt$$

Want to prove that when the system is asymptotic stable, then Z is a unique symmetric positive definite solution of equation (1.34).

If we substitute Z in equation (1.34), then we obtain:

$$ZA + A^T Z = \int_0^\infty e^{A^T t} F e^{At} A dt + \int_0^\infty A^T e^{A^T t} F e^{At} dt$$

$$= \int_0^\infty \frac{d}{dt}\left(e^{A^T t} F e^{At}\right) dt = \left[e^{A^T t} F e^{At}\right]_0^\infty$$

Suppose A is stable, then $e^{A^T t} \rightarrow 0$ as $t \rightarrow \infty$. Thus $ZA + A^T Z = -F$.

So, Z satisfies equation (1.34).

To prove that Z is positive definite, we must show $u^T Z u > 0$ for any non-zero vector u

$$u^T Z u = \int_0^\infty u^T e^{A^T t} F e^{At} u dt$$

since $e^{A^T t}$ and e^{At} are both non-singular and F is positive definite, then we have $u^T Z u > 0$.

Finally we must prove that, Z is unique. Assume we have two solutions Z_1 and Z_2 of equation (1.34), then:

$$A^T(Z_1 - Z_2) + (Z_1 - Z_2)A = 0$$

which implies:

$$e^{A^T t}(A^T(Z_1 - Z_2) + (Z_1 - Z_2)A)e^{At} = 0$$

or

$$\frac{d}{dt}\left[e^{A^T t}(Z_1 - Z_2)e^{At}\right] = 0$$

hence, $e^{A^T t}(Z_1 - Z_2)e^{At}$ is constant matrix for all values of t. Calculating at $t = 0$ and $t = \infty$, we get $(Z_1 - Z_2) = 0$, hence Z is unique solution.

($\Leftarrow$) Conversely, show that, if Z is symmetric positive definite solution of equation (1.34), then A is stable matrix.

Let λ and $\bar{\lambda}$ be an eigenvalue, υ and υ^* be an eigenvector of matrix A. Multiplying (1.34) from left by υ^* and from right by υ, we obtain:

$$\upsilon^* ZA\upsilon + \upsilon^* A^T Z\upsilon = \lambda\upsilon^* Z\upsilon + \bar{\lambda}\upsilon^* Z\upsilon = \left(\lambda + \bar{\lambda}\right)\upsilon^* Z\upsilon = -\upsilon^* F\upsilon$$

Suppose F and Z are both symmetric positive definite, we get $\lambda + \bar{\lambda} < 0$, or $Re(\lambda) < 0$.

Since λ was arbitrary, so A is stable. ■

Solution of Lyapunov equations:

Let A be stable and let F be symmetric positive definite or semi definite, then:

1. The unique solution of Lyapunov equation:

$$ZA + A^T Z = -F$$

is defined as:

$$Z = \int_0^\infty e^{A^T t} F e^{At} dt. \tag{1.36}$$

2. The unique solution of Lyapunov equation:

$$AZ + ZA^T = -F$$

is defined as:

$$Z = \int_0^\infty e^{At} F e^{A^T t} dt. \tag{1.37}$$

For more details see [10].

1.9 Controllability and Observability

In this section we will focus on the concepts of controllability and observability, both are important notation in the study of continuous time linear dynamical systems.

Definition 1.9.1.[9, 10] The system

$$\Sigma_s = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

or the pair (A, B) is called controllable, if for any initial state $x(0) = x_0$,the system can be driven to any final state x_1 by using a piecewise continuous input $u(t)$, such that $x(t_1) = x_1$ where $t_1 > 0$.

Definition 1.9.2.[1, 10] The matrix

$$K\ (A, B) = (B\ AB\ A^2B\ A^3B\ \cdots\ A^{n-1}B) \qquad (1.38)$$

is called the controllability matrix, where n is a positive integer.

Definition 1.9.3.[1, 9, 10] The system

$$\Sigma_s = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

or the pair (C, A) is called observable if, for any $t_1 > 0$, the initial state $x(0)$ can be uniquely found from the time history of the input $u(t)$ and the output $y(t)$ for all t belongs to the interval $[0, t_1]$.

Definition 1.9.4.[1, 10] The matrix

$$E(C,A)=\begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{1.39}$$

is called the observability matrix, where n is a positive integer.

Now, we consider two important matrices regarding to the linear dynamical system, the controllability and the observability Gramians. And we will study some theorems related to these matrices.

Definition 1.9.5.[1,10] The matrix

$$W_c=\int_0^\infty e^{At}BB^Te^{A^Tt}dt \tag{1.40}$$

is called the controllability Gramian.

Definition 1.9.6.[1, 10] The matrix

$$W_o=\int_0^\infty e^{A^Tt}C^TCe^{At}dt \tag{1.41}$$

is called the observability Gramian.

The two matrices W_c and W_o are both solutions of the Lyapunov equation, so we have:

$$AW_c+W_cA^T+BB^T=0 \tag{1.42}$$

$$W_oA+A^TW_o+C^TC=0 \tag{1.43}$$

Proposition 1.9.7. [10] Let

$$\Sigma_{\mathrm{s}} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

be a stable system and let W_c and W_o be the controllability and observability Gramians of the system Σ_s. Then W_c and W_o satisfy the continuous time Lyapunov equations:

$$AW_c + W_c A^T + BB^T = 0$$

$$W_o A + A^T W_o + C^T C = 0$$

Proof: Suppose Σ_s is stable, then:

- $AW_c + W_c A^T = \int_0^\infty (Ae^{At}BB^Te^{A^Tt} + e^{At}BB^Te^{A^Tt}A^T)dt$

$$= \int_0^\infty \frac{d}{dt}(e^{At}BB^Te^{A^Tt})dt$$

$$= \left[e^{At}BB^Te^{A^Tt}\right]_0^\infty$$

$$= -BB^T.$$

So,

$$AW_c + W_c A^T + BB^T = 0$$

- $W_o A + A^T W_o = \int_0^\infty (e^{A^Tt}e^{At}A + A^Te^{A^Tt}C^TCe^{At})dt$

$$= \int_0^\infty \frac{d}{dt}(e^{A^Tt}C^TCe^{At})dt$$

$$= \left[e^{A^Tt}C^TCe^{At}\right]_0^\infty$$

$$= -C^TC.$$

Finally,

$$W_o A + A^T W_o + C^T C = 0 \qquad \blacksquare$$

The controllability Gramian satisfies the following property on continuous time dynamical system [1]:

$$W_c(t) = {W_C}^T(t) \geq \mathbf{0}, \forall\ t > 0 \tag{1.44}$$

Theorem 1.9.8.[1] (Controllability Conditions)

The following statements are equivalent:

1. The pair $(A, B), A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ is controllable.
2. The controllable matrix has a full rank (i:e rank $K(A, B) = n$).
3. The controllablility Gramian W_C is positive definite, $W_C(t) > 0, \forall t > 0$.

Theorem 1.9.9.[1] (Observability Conditions)

The following statements are equivalent:

1. The pair $(C, A), C \in \mathbb{R}^{p\times n}, A \in \mathbb{R}^{n\times n}$ is observable.
2. The observable matrix has a full rank (i:e rank $E(C, A) = n$).
3. The observability Gramian W_o is positive definite, $W_o(t) > 0, \forall t > 0$.

1.10 Norms

In this section we will introduce the notation of the norm and introduce some important types of norm.

Definition 1.10.1.[17] Let Q be a vector space over a field F. A norm on Q is a nonnegative function $\|\cdot\|$

$$\|\cdot\|: Q \to \mathbb{R}$$

$$q \to \|q\|$$

such that $\forall\, q, \rho \in Q$ and $\forall \lambda \in F$, then:

1. $\|q\| \geq 0$, and $\|q\| = 0$ if and only if $q = 0$.

2. $\|\lambda q\| = |\lambda|\|q\|$.

3. $\|q + \rho\| \leq \|q\| + \|\rho\|$.

$\|q\|$ is called the norm of q.

Definition 1.10.2.[17] Euclidean norm

let the vector $Q = (q_1, q_2, \dots \dots, q_n) \in \mathbb{R}^n$, then the Euclidean norm of Q, $\|Q\|$, is given as:

$$\|Q\|_2 = \sqrt{q_1^2 + q_2^2 + \cdots + q_n^2} \qquad (1.45)$$

Definition 1.10.3.[17] Taxicab norm

let $Q = (q_1, q_2, \dots \dots, q_n) \in \mathbb{R}^n$ be a vector, then the Taxicab norm of Q is defined as:

$$\|Q\|_1 = \sum_{i=1}^{n} |q_i|. \qquad (1.46)$$

Definition 1.10.4.[17] Maximum norm

let $Q = (q_1, q_2, \dots \dots, q_n) \in \mathbb{R}^n$ be a vector. The maximum norm or infinite norm of Q is define as:

$$\|Q\|_\infty = \max\{|q_1|, |q_2|, \dots, |q_n|\}. \qquad (1.47)$$

If A is a $n \times m$ matrix, then:

$$\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{m} |a_{ij}|. \qquad (1.48)$$

Definition 1.10.5.[17] p-norm

let $p \geq 1$ be a real number, the p-norm (L_p-norm) of vector

$Q = (q_1, q_2, \dots \dots, q_n) \in \mathbb{R}^n$ is given as:

$$\|Q\|_p = \left(\sum_{i=1}^{n} |q_i|^p\right)^{\frac{1}{p}}. \qquad (1.49)$$

Let A be $n \times m$ matrix, then:

$$\|A\|_p = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|^p\right)^{\frac{1}{p}}. \qquad (1.50)$$

If $p = 1$ then p-norm becomes the Taxicab norm, for $p = 2$, p-norm is Euclidean norm and if p approaches ∞ then p-norm approaches infinite norm.

1.11 The amount of energy for controlling and observing state

In this section we discuss an important property of a dynamical system, that is important construction in model reduction to classify states according to their degree of controllability and observability.

Consider the stable, controllable and observable linear system

$$\Sigma_s = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

$$x(0) = x_0$$

The controllability and observability functions of x_0 are defined as follows:

Definition 1.11.1.[9, 7] The controllability function is defined as:

$$L_c(x_0) = \min \frac{1}{2} \int_{-\infty}^{0} \|u(t)\|^2 dt, \qquad (1.51)$$

where $u \in L_2(-\infty, 0), x(-\infty) = 0, x(0) = x_0$

Definition 1.11.2.[9, 7] The observability function is defined as:

$$L_o(x_0) = \frac{1}{2} \int_{0}^{\infty} \|y(t)\|^2 dt, \qquad (1.52)$$

where $x(0) = x_0, u(t) = 0, 0 \leq t < \infty$.

The value of $L_c(x_0)$ is the minimum amount of control energy desired to approach the state x_0 . And the value of $L_o(x_0)$ is the amount of output energy produced by the state x_0 [9].

Theorem 1.11.3.[27] Let

$$W_c = \int_0^\infty e^{At} B B^T e^{A^T t} dt$$

and

$W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt.$

be the controllability and observability Gramains, such that W_c and W_o are the unique positive definite solutions of the Lyapunov equations. Then $L_c(x_0)$ and $L_o(x_0)$ can be written in terms of W_c and W_o to get:

$$L_c(x_0) = \frac{1}{2} x_0^T W_c^{-1} x_0 \tag{1.53}$$

and

$$L_o(x_0) = \frac{1}{2} x_0^T W_o x_0 \tag{1.54}$$

Lemma 1.11.4.[1] Let W_c and W_o be the controllability and observability Gramians of a stable dynamical system Σ_s, then:

1. The lower amount of energy wanted to drive the state of the system from 0 to x_0 is given by $L_c(x_0)$.

2. The greater energy generated by observing the output of the system whose initial state is x_0 is given by $L_o(x_0)$.

Chapter Two
Model Reduction for Stable Systems

In this chapter we will study the model order reduction for stable systems, we focus on the balanced truncation method and in chapter 4 a review of a numerical example is presented to show the efficiency of this method.

2.1 State space realization

In this section we define a property of realization for dynamical systems with a transfer function $H(s)$.

Definition 2.1.1.[32] Assume that $H(s)$ is a real-rational transfer function which is proper, then the state space model $(A, B, C, 0)$ defined as:

$$\Sigma_s(s) = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

is a realization of $H(s)$.

Definition 2.1.2.[32] A state space realization $(A, B, C, 0)$ of $H(s)$ is called minimal realization of $H(s)$ if the matrix A has the smallest possible dimension.

The following theorem gives a description of the minimal realization.

Theorem 2.1.3.[32] A state space realization $(A, B, C, 0)$ of $H(s)$ is said to be minimal if and only if (A, B) is controllable, and (C, A) is observable.

The next property of minimal realization can also be verified.

Theorem 2.1.4.[32] Let $(A_1, B_1, C_1, 0)$ and $(A_2, B_2, C_2, 0)$ be two minimal realizations of a real-rational transfer function $H(s)$, and let K_1, K_2, E_1, and E_2 be the corresponding controllability and observability matrix respectively, then there exists a unique non-singular matrix T such that:

$$A_2 = \mathrm{T}A_1\mathrm{T}^{-1}, B_2 = \mathrm{T}B_1, C_2 = C_1\mathrm{T}^{-1}.$$

Furthermore, T can be specified as:

$$\mathrm{T} = (E_2^T E_2)^{-1} E_2^T E_1 = K_1 K_2^T (K_2 K_2^T)^{-1}.$$

2.2 Balancing for linear system

In this section we introduce one of the most important methods used to obtain a reduce order model from the original dynamical system. This is called the Balanced Truncation method [14].

Suppose the linear time invariant (LTI) continuous system:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

with initial condition $x(0) = x_0$.

The concept of the Balanced Truncation method depends on the controllability and observability Gramians W_c and W_o [9, 26], which are symmetric positive definite solutions of the Lyapunov equations. See [Proposition 1.9.7].

$$AW_c + W_cA^T + BB^T = 0.$$

$$A^TW_o + W_oA + C^TC = 0.$$

To obtain a reduced order model, we balance the system then we omit the states that are hard to control (i.e need large amount of control energy) and hard to observe (i.e produce small amount of energy), these states are not important so they may not influence on the transfer function [9, 21, 15].

Now we introduce the concept of Hankel Singular Values (HSVs) of the dynamical system.

Definition 2.2.1.[1, 9] Let $\Sigma_s = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ be an n-dimensional controllable, observable and stable continuous time system.Then, the Hankel Singular Values

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n \geq 0$$

of Σ_s are defined as the square roots of the eigenvalues of the product W_c and W_o and denoted:

$$\sigma_i(\Sigma_s) = \sqrt{\lambda_i(W_c W_o)}. \tag{2.1}$$

Let Σ denotes the diagonal matrix of the (HSVs):

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}. \tag{2.2}$$

where

$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n \geq 0.$

Definition 2.2.2.[1, 24] The controllable, observable and stable system $\Sigma_s = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ is said to be balanced if

$$W_c = W_o = \Sigma = diag(\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n \geq 0).$$

In the following theorem we show the method of balancing the system by find a coordinate transformation ω such that:

$$\bar{x} = \omega^{-1} x \qquad (2.3)$$

in which the controllability and observability Gramians turn out diagonal and equal.

Theorem 2.2.3.[9] There exists a state space transformation $\bar{x} = \omega^{-1} x$ for the system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

such that, the transformed system

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$

$$\bar{y} = \bar{C}\bar{x} \qquad (2.4)$$

is balanced and $\bar{A} = \omega A \omega^{-1}, \bar{B} = \omega B$ and $\bar{C} = C\omega^{-1}$. For more details see [7, 15].

Let $\overline{H}$ be the transfer function of the balanced system (2.4), then:

$$\overline{H} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{pmatrix} = \begin{pmatrix} \omega A \omega^{-1} & \omega B \\ C\omega^{-1} & 0 \end{pmatrix} \qquad (2.5)$$

Lemma 2.2.4.[32] Let $H(s) = C(sI - A)^{-1}B$ and $\overline{H}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B}$ be the transfer function of the (LTI) system and balanced system respectively, then:

$$\overline{H}(s) = H(s)$$

Proof.

$$\overline{H}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B}$$

$$= C\omega^{-1}(sI - \omega A\omega^{-1})^{-1}\omega B$$

$$= C\omega^{-1}(\omega(sI - A)\omega^{-1})^{-1}\omega B$$

$$= C\omega^{-1}(\omega(sI - A)^{-1}\omega^{-1})\omega B$$

$$= C(sI - A)^{-1}B$$

$$= H(s) \ \blacksquare$$

Let $\overline{W}_c$ and $\overline{W}_o$ be the controllability and observability Gramians of the balance system (2.4). Then we have:

$$\overline{W}_c = \omega^{-1} W_c (\omega^{-1})^T \qquad (2.6)$$

and

$$\overline{W}_o = \omega^T W_o \omega. \qquad (2.7)$$

Suppose the two Gramians $\overline{W}_c$ and $\overline{W}_o$ of balanced system are equal, then:

$$\overline{W}_c = \overline{W}_o = \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}, \tag{2.8}$$

such that

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n \geq 0.$$

The controllability and observability Gramians $\overline{W}_c$ and $\overline{W}_o$ in equation (2.8) satisfy the Lyapunov equations:

$$\bar{A}\Sigma + \Sigma(\bar{A})^T + \bar{B}(\bar{B})^T = 0$$

$$(\bar{A})^T\Sigma + \Sigma\bar{A} + (\bar{C})^T\bar{C} = 0$$

Suppose the two Gramians W_c and W_o are positive definite (or semi-definite), then one can decompose them as:

$$W_c = UU^T$$

$$W_o = LL^T \tag{2.9}$$

where L and U are lower triangular matrix with real and positive diagonal entries, and L^T and U^T denotes the transpose of L and U respectively.

If we do a singular value decomposition of the matrix L^TU, we obtain:

$$L^TU = X\Sigma Y^T = (X_1 \quad X_2)\begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}\begin{pmatrix} Y_1^T \\ Y_2^T \end{pmatrix} \tag{2.10}$$

such that

$\Sigma_1 = \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r,$

and

$\Sigma_2 = \sigma_{r+1} \geq \sigma_{r+2} \geq \cdots \geq \sigma_n.$

The other matrices satisfy

$$X_1^T X_1 = Y_1^T Y_1 = I_{r\times r}$$

and

$$X_2^T X_2 = Y_2^T Y_2 = I_{l\times l}$$

with $l = n - r$ [15].

Lemma 2.2.5.[1] (Balancing transformation) Given the controllable, observable and stable system $\Sigma_s = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ and it's Gramians W_c and W_o. Then a balancing transformation is define as:

$$\Gamma = UY\Sigma^{-\frac{1}{2}}. \tag{2.11}$$

$$\Gamma^{-1} = \Sigma^{-\frac{1}{2}} X^T L^T. \tag{2.12}$$

Definition 2.2.6.[9] The controllability and observability functions of the transformed system (2.4) are given as:

$$\bar{L}_c(\bar{x}_0) = \frac{1}{2}\bar{x}_0^T \Sigma^{-1} \bar{x}_0 \tag{2.13}$$

$$\bar{L}_0(\bar{x}_0) = \frac{1}{2}\bar{x}_0^T \Sigma \bar{x}_0 \tag{2.14}$$

The value of $\bar{L}_c(\bar{x}_0)$ is the minimum amount of control energy desired to approach the state $\bar{x}_0$. And the value of $\bar{L}_0(\bar{x}_0)$ is the amount of output energy produced by the state $\bar{x}_0$. If $\sigma_i \gg \sigma_{i+1}$ for $i = 1,2,\cdots,n$, then we need a large amount of control energy to reach the state $\bar{x}_0$ for small values of σ_i, and we have small amount of output energy at $\bar{x}_0$ for large values of σ_i.

To decrease the number of state components of the system, we remove the state components from x_{i+1} to x_n for $\sigma_i \gg \sigma_{i+1}$.

Now, by the following procedure we can obtain balance realization for a minimal realization system $\Sigma_s = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, [32, 9]:

1. Compute W_c and W_o the controllability and observability gramians for the system.

2. Find matrix U, such that $W_c = U^T R$.

3. Diagonalize $U^T W_o U$ to get $W_c = R^T W_o R = \mathrm{L}\Sigma^2 L^T$

4. Let $\omega^{-1} = U^T L \Sigma^{\frac{1}{2}}$

Then

$$\begin{aligned} \omega^{-1} &= U^T L \Sigma^{-\frac{1}{2}} \omega W_c \omega^T \\ &= (\omega^{-1})^T W_o \omega^{-1} \\ &= \Sigma, \end{aligned}$$

and

$$\begin{pmatrix} \omega A \omega^{-1} & \omega B \\ C\omega^{-1} & 0 \end{pmatrix}$$

is balanced.

2.3 Error bounds for linear dynamical systems using balanced truncation

In this section we study an advantageous property of balanced truncation method, it has a prior error bounds that is close to the lower bound achievable by any reduced-order model.

Consider a (LTI) continuous system:

$$\dot{x} = Ax + Bu$$

$$y = Cx \qquad (2.15)$$

$$x(0) = x_0$$

and the transfer function:

$$H(s) = C(sI - A)^{-1}B$$

Assumption 2.3.1.[28] A system $\begin{pmatrix} A & B \\ D & 0 \end{pmatrix}$ is asymptotically stable, (A, B) is controllable and (C, A) is observable.

The controllabilty and observability Gramians (W_c and W_o) are positive semi-definite and satisfy the Lyapunov equations.

By theorem (2.2.3), we have the next balanced system:

$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$

$$\bar{y} = \bar{C}\bar{x} \tag{2.16}$$

such that $\bar{A} = \omega A \omega^{-1}, \bar{B} = \omega B$ and $\bar{C} = C\omega^{-1}$.

Let us partition the balance system $(\bar{A}, \bar{B}, \bar{C})$ and the Gramian Σ as:

$$\bar{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \bar{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \bar{C} = (C_1 \quad C_2), \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$$

where A_{11} and A_{22} are matrices of dimension $r \times r$ and $n - r \times n - r$ respectively, and the other matrices has dimension correspond to the original system.

Assuming the Hankel Singular Values are satisfies $\sigma_r \gg \sigma_{r+1}$, then we delete all the states related to the small Hankel Singular Values ($\sigma_{r+1} \geq \sigma_{r+2} \geq \cdots \geq \sigma_n \geq 0$) to obtain the reduced order model acquired by Balance Truncation method is given by:

$$\dot{x}_r = A_{11}x_r + B_1 u$$

$$y_r = C_1 x_r \tag{2.17}$$

with transfer function:

$$H_r(s) = C_1(sI - A_{11})^{-1}B_1 \tag{2.18}$$

The subsystem (A_{11}, B_1, C_1) is a good approximation of the balanced system $(\bar{A}, \bar{B}, \bar{C})$.

Lemma2.3.2.[18] The subsystems (A_{ii}, B_i, C_i) and $(i = 1,2)$ are balance with Gramians Σ_1 and Σ_2.

Lemma2.3.3.[18] The matrices A_{ii} $(i = 1,2)$ are asymptotically stable (i.e. The real parts of eigenvalues λ_k of A_{ii} $(i = 1,2)$ are less than or equal zero, $R(\lambda_k(A_{ii}) \leq 0), i = 1,2, \forall k)$ if Σ_1 and Σ_2 do not have common entries in the diagonal. Furthermore the subsystem $(A_{ii}, B_i, C_i), (i = 1,2)$ is controllable and observable.

Now, we consider a very important concept in control theory.

We compute the infinite norm $(\|.\|_\infty)$ of the transfer function of the original model and compare the difference with the norm of the transfer function of our reduced order model obtained by Balanced Truncation.

Let $H(s)$ be the transfer function of the balanced system $(\bar{A}, \bar{B}, \bar{C})$ and $H_r(s)$ be the transfer function of the reduced system (A_{11}, B_1, C_1) then the upper bound for the approximation error is given in the following lemma [28, 18].

Lemma 2.3.4. We have

$$\|H - H_r\|_\infty \leq 2\sum_{i=r+1}^{n} \sigma_i \tag{2.19}$$

such that σ_{r+1} is the first neglected (HSV) of $H(s)$.

Lemma 2.3.5.[9, 28] Let y and y_r be the output of the original and reduced system respectively. Then the $\|.\|_2 \equiv \|.\|_{L_2}$ bound of the approximation error between y and y_r is given by:

$$\|y - y_r\|_{L_2} \leq 2 \sum_{i=r+1}^{n} \sigma_i \|u\|_{L_2} \quad (2.20)$$

where u is the input vector.

2.4 Reciprocal system of a linear dynamical system

In this section we will discuss some results and properties of reciprocal system of the balanced realization for the infinite dimensional system.

Consider a linear time-invariant continuous system:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du.$$

Assume the system is balanced with Gramian Σ. Then we have:

$$A\Sigma + \Sigma A^T + BB^T = 0$$

$$A^T\Sigma + \Sigma A + C^T C = 0.$$

Let

$$H(s) = C(sI - A)^{-1}B + D.$$

be the transfer function of the balanced system (A, B, C, D). Then, the reciprocal system $(\tilde{A}, \tilde{B}, \tilde{C}, \widetilde{D})$ of the system (A, B, C, D) is defined as:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$

$$\tilde{y} = \tilde{C}\tilde{x} + \widetilde{D}u.$$

Such that

$$\tilde{A} = A^{-1}$$

$$\tilde{B} = A^{-1}B$$

$$\tilde{C} = -CA^{-1} \qquad (2.21)$$

$$\widetilde{D} = D - CA^{-1}B$$

Remark 2.4.1. If we compute $H(0)$ we get:

$$H(0) = -CA^{-1}B + D = \widetilde{D}$$

Remark 2.4.2.[9] If the matrix A is given as:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Then

$$A^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{12}A_{11}^{-1} \\ -(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}.$$

also

$$A^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}.$$

The transfer function $\widetilde{H}(s)$ of the reciprocal system $(\tilde{A},\tilde{B},\tilde{C},\widetilde{D})$ is given as:

$$\widetilde{H}(s) = \tilde{C}\left(sI - \tilde{A}\right)^{-1}\tilde{B} + \widetilde{D}. \qquad (2.22)$$

The relation between $H(s)$ and $\widetilde{H}(s)$ is given as [3]:

$$H(s) = C(sI - A)^{-1}B + D$$

$$= C(sI - A)^{-1}AA^{-1}B + D$$

$$= C\frac{1}{s}\left(A^{-1} - \frac{1}{s}\right)^{-1} B + D$$

$$= -C\left(\frac{I}{s} - A^{-1} + A^{-1}\right)\left(\frac{I}{s} - A^{-1}\right)^{-1} A^{-1}B + D$$

$$= -CA^{-1}B - CA^{-1}\left(\frac{I}{s} - A^{-1}\right)^{-1} A^{-1}B + D \qquad (2.23)$$

$$= -CA^{-1}\left(\frac{I}{s} - A^{-1}\right)^{-1} A^{-1}B + D - CA^{-1}B$$

$$= \tilde{C}\left(\frac{I}{s} - \tilde{A}\right)^{-1} \tilde{B} + \widetilde{D}$$

$$= \widetilde{H}\left(\frac{1}{s}\right) \ \blacksquare$$

Lemma 2.4.3.[9, 18, 25] Let (A, B, C, D) be balanced minimal realization of a (LTI) system with Gramian Σ. Then the reciprocal system $\left(\tilde{A},\tilde{B},\tilde{C},\widetilde{D}\right)$ is also balanced with the same Gramian Σ.

Proof. Since Σ is solution of the Lyapunove equations:

$$A\Sigma + \Sigma A^T + BB^T = 0$$

$$A^T\Sigma + \Sigma A + C^TC = 0$$

multiply the first equation by A^{-1} from the left and by ${A^{-1}}^{T}$ from the right, we get:

$$A^{-1}(A\Sigma){A^{-1}}^{T} + A^{-1}(\Sigma A^{T}){A^{-1}}^{T} + A^{-1}(BB^{T}){A^{-1}}^{T} = 0,$$

so

$$\Sigma {A^{-1}}^{T} + A^{-1}\Sigma + (A^{-1}B)(A^{-1}B)^{T} = 0$$

by equation (2.21), we obtain:

$$\tilde{A}\Sigma + \Sigma\tilde{A}^{T} + \tilde{B}\tilde{B}^{T} = 0.$$

Multiplying the second equation by ${A^{-1}}^{T}$ from the left and by A^{-1} from the right, we get:

$${A^{-1}}^{T}(A^{T}\Sigma)A^{-1} + {A^{-1}}^{T}(\Sigma A)A^{-1} + {A^{-1}}^{T}(C^{T}C)A^{-1} = 0,$$

then

$$\Sigma A^{-1} + {A^{-1}}^{T}\Sigma + (CA^{-1})^{T}(CA^{-1}) = 0.$$

By equation (2.21), we have:

$$\tilde{A}^{T}\Sigma + \Sigma\tilde{A} + \tilde{C}^{T}\tilde{C} = 0.$$

So, the reciprocal system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is balanced with Gramian Σ. ■

Let us partition the system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ and it's Gramian Σ as:

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \tilde{B} = \begin{pmatrix} \tilde{B}_{1} \\ \tilde{B}_{2} \end{pmatrix}, \tilde{C} = \begin{pmatrix} \tilde{C}_{1} & \tilde{C}_{2} \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{1} & 0 \\ 0 & \Sigma_{2} \end{pmatrix}. \qquad (2.24)$$

Lemma 2.4.4.[18] Suppose the hypothesis of lemma (2.4.3), and the reciprocal system $\left(\tilde{A}, \tilde{B}, \tilde{C}, \widetilde{D}\right)$ be partitioned as above. Then the subsystem $\left(\tilde{A}_{ii}, \tilde{B}_i, \tilde{C}_i, \widetilde{D}\right), (i = 1,2)$ is also internally balanced with Gramian $\Sigma_i, (i = 1,2)$.

Lemma 2.4.5.[18] Consider the hypothesis of lemma (2.4.4), then the matrices $\tilde{A}_{ii}\,, (i = 1,2)$ are asymptotically stable (i.e. The real parts of eigenvalues λ_k of $\tilde{A}_{ii}$ $(i = 1,2)$ are less than or equal zero, $R\left(\lambda_k\left(\tilde{A}_{ii}\right) \leq 0\right), i = 1,2, \forall k)$ if Σ_1 and Σ_2 do not have common entries in the diagonal. Furthermore the subsystem $\left(\tilde{A}_{ii}, \tilde{B}_i, \tilde{C}_i, \widetilde{D}\right), (i = 1,2)$ is controllable and observable.

Applying balanced truncation method on the reciprocal system $\left(\tilde{A}, \tilde{B}, \tilde{C}, \widetilde{D}\right)$, obtain the balance $r \times r$ reduced system $\left(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, \widetilde{D}\right)$, given by the following state space equation:

$$\dot{\tilde{x}} = \tilde{A}_{11}\tilde{x} + \tilde{B}_1\tilde{u}$$

$$\tilde{y} = \tilde{C}_1\tilde{x} + \widetilde{D}u \tag{2.25}$$

By equation (2.22) and remark (2.4.2) we can find the values of $\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1$ and $\widetilde{D}$ as:

$$\begin{aligned}
\tilde{A}_{11} &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} \\
\tilde{B}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}(B_1 - A_{12}A_{22}^{-1}B_2) \\
\tilde{C}_1 &= -(C_1 - C_2A_{22}^{-1}A_{21})(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} \\
\widetilde{D} &= D - CA^{-1}B
\end{aligned} \tag{2.26}$$

The transfer function of reduced system $\left(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, \tilde{D}\right)$ is given as:

$$\tilde{H}_r(s) = \tilde{C}_1\left(sI - \tilde{A}_{11}\right)^{-1}\tilde{B}_1 + \tilde{D} \qquad (2.27)$$

Now, the error bound is given in the next lemma.

Lemma 2.4.6.[18] We have that

$$\left\|\tilde{H} - \tilde{H}_r\right\|_\infty = 2\sum_{i=r+1}^{n}\sigma_i \qquad (2.28)$$

Where σ_i is the (HSVs) of the system.

2.5 Singular perturbation approximation method

In previous sections we studied a balanced truncation schema to reduce the order of the system, and acquired an error bound. Then we introduce reciprocal system and extend the error bound to reduce order of reciprocal system.

In this section we introduce another method to reduce the original system which is called the singular perturbation approximation method (SPAM). Balanced truncation method and singular perturbation approximation method give us the same error bounds. For the balanced truncation method the error is small at high frequencies and large at low frequencies, but for the singular perturbation approximation we have large error at high frequencies and small error at low frequencies. Our goal is to find the error bound for the reduced order model using the singular perturbation approximation. To obtain this error bound, we discuss the relationship between the reduced model of the reciprocal system and the reduced model when we use the singular perturbation method.

Consider the system:

$$\dot{x} = Ax + Bu$$

$$y = Cx. \tag{2.29}$$

The controlabillity and observability Gramians W_c and W_o are both positive semi-definite and can write as:

$$W_c = UU^T$$

$$W_o = LL^T.$$

The balanced Gramian Σ is partition as:

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}.$$

Σ_1 and Σ_2 represent the important singular values that we are interested in and the unimportant singular values which we want to neglected [26, 24].

Also, the balanced transformation ω satisfies the equation:

$$\omega = UY\Sigma^{-\frac{1}{2}}$$

$$\omega^{-1} = \Sigma^{-\frac{1}{2}}X^T L^T.$$

Suppose $\sigma_r \gg \sigma_{r+1}$. And HSVs are coordinate invariant (i.e. which remains unchanged, after operations for transformations of a certain type are applied to the objects), since $\sigma_{r+1} > \sigma_{r+2} > \cdots > \sigma_n > 0$. Then we can obtain a reduced order system with small parameter [15].

To check where the small parameters Σ_2 enter the equation, replacing Σ_2 by $\varepsilon\Sigma_2$. The small HSVs are named uniformly according to equation:

$$(\sigma_{r+1}, \sigma_{r+2}, \cdots, \sigma_n) \rightarrow \varepsilon(\sigma_{r+1}, \sigma_{r+2}, \cdots, \sigma_n), \varepsilon > 0.$$

We change the coordinate using balanced transformation $\omega(\varepsilon)$ such that:

$$x \rightarrow \omega(\varepsilon)x$$

Let $\omega^{-1}(\varepsilon) = \Omega(\varepsilon)$. Then partition the balance matrices as:

$$\omega(\varepsilon) = \begin{pmatrix} \omega_{11} & \frac{1}{\sqrt{\varepsilon}}\omega_{12} \\ \omega_{21} & \frac{1}{\sqrt{\varepsilon}}\omega_{22} \end{pmatrix} \tag{2.30}$$

and

$$\Omega(\varepsilon) = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \frac{1}{\sqrt{\varepsilon}}\Omega_{21} & \frac{1}{\sqrt{\varepsilon}}\Omega_{22} \end{pmatrix}. \tag{2.31}$$

which give rise to the balanced coefficients and they are given as:

$$\begin{aligned} \hat{A}(\epsilon) &= \Omega(\varepsilon)A\omega(\varepsilon) \\ &= \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \frac{1}{\sqrt{\varepsilon}}\Omega_{21} & \frac{1}{\sqrt{\varepsilon}}\Omega_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \omega_{11} & \frac{1}{\sqrt{\varepsilon}}\omega_{12} \\ \omega_{21} & \frac{1}{\sqrt{\varepsilon}}\omega_{22} \end{pmatrix} \end{aligned} \tag{2.32}$$

$$= \begin{pmatrix} \hat{A}_{11} & \frac{1}{\sqrt{\varepsilon}}\hat{A}_{12} \\ \frac{1}{\sqrt{\varepsilon}}\hat{A}_{21} & \frac{1}{\varepsilon}\hat{A}_{22} \end{pmatrix}.$$

$$\hat{B}(\varepsilon) = \Omega(\varepsilon)B$$

$$= \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \frac{1}{\sqrt{\varepsilon}}\Omega_{21} & \frac{1}{\sqrt{\varepsilon}}\Omega_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \tag{2.33}$$

$$= \begin{pmatrix} \hat{B}_1 \\ \frac{1}{\sqrt{\varepsilon}}\hat{B}_2 \end{pmatrix}.$$

And

$$\hat{C}(\varepsilon) = C\omega(\varepsilon)$$

$$= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} \omega_{11} & \frac{1}{\sqrt{\varepsilon}}\omega_{12} \\ \omega_{21} & \frac{1}{\sqrt{\varepsilon}}\omega_{22} \end{pmatrix} \tag{2.34}$$

$$= \begin{pmatrix} \hat{C}_1 & \frac{1}{\sqrt{\varepsilon}}\hat{C}_2 \end{pmatrix}.$$

Set $\varepsilon = 1$ In equation (2.32), then $\hat{A} = \Omega(1)A\omega(1)$ represent the balanced matrix A.

The balancing transformation can given as:

$$\Gamma(\varepsilon) = \mathrm{I}(\varepsilon)\omega(1)$$

$$\Omega(\varepsilon) = \Omega(1)\mathrm{I}(\varepsilon),$$

such that $\mathrm{I}(\varepsilon)$ denotes the diagonal matrix:

$$\mathrm{I}(\varepsilon) = \begin{pmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{\varepsilon}} I \end{pmatrix}.$$

In the next steps we omit the tilde from the balanced matrices, in order to have the following matrices:

$$A = \begin{pmatrix} A_{11} & \frac{1}{\sqrt{\varepsilon}} A_{11} \\ \frac{1}{\sqrt{\varepsilon}} A_{21} & \frac{1}{\varepsilon} A_{22} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ \frac{1}{\sqrt{\varepsilon}} B_2 \end{pmatrix}, C = \begin{pmatrix} C_1 & \frac{1}{\sqrt{\varepsilon}} C_2 \end{pmatrix}.$$

Let's define the new variable $z = (z_1, z_2)$ which can be balanced by balance transformation $\Omega(\varepsilon)$ and written in balance form as:

$$z = \Omega(\varepsilon)x.$$

Now, our dynamical linear system described in equation (2.29) turns in to the singular perturbed system of equations:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & \frac{1}{\sqrt{\varepsilon}} A_{11} \\ \frac{1}{\sqrt{\varepsilon}} A_{21} & \frac{1}{\varepsilon} A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{1}{\sqrt{\varepsilon}} B_2 \end{pmatrix} u \tag{2.35}$$

$$y = \begin{pmatrix} C_1 & \frac{1}{\sqrt{\varepsilon}} C_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Equations (2.35) can written as:

$$\dot{z}_1 = A_{11} z_1 + \frac{1}{\sqrt{\varepsilon}} A_{12} z_2 + B_1 u$$

$$\dot{z}_2 = \frac{1}{\sqrt{\varepsilon}} A_{21} z_1 + \frac{1}{\varepsilon} A_{22} z_2 + B_2 u \tag{2.36}$$

$$y = C_1 z_1 + \frac{1}{\sqrt{\varepsilon}} C_2 z_2.$$

The variable z_2 is scaled as:

$$z_2 \to \sqrt{\varepsilon} z_2$$

equations (2.36) become:

$$\dot{z}_1 = A_{11}z_1 + A_{12}z_2 + B_1u$$

$$\varepsilon\dot{z}_2 = A_{21}z_1 + \frac{1}{\varepsilon}A_{22}z_2 + B_2u \tag{2.37}$$

$$y = C_1z_1 + C_2z_2.$$

In matrix form:

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\end{pmatrix} = \begin{pmatrix}A_{11} & A_{11}\\ \frac{1}{\varepsilon}A_{21} & \frac{1}{\varepsilon}A_{22}\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix} + \begin{pmatrix}B_1\\ \frac{1}{\varepsilon}B_2\end{pmatrix}u \tag{2.38}$$

$$y = \begin{pmatrix}C_1 & C_2\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix},$$

where the sub-matrices $A_{11}, A_{12}, \cdots$ are in balanced form [15]. The scalar ε exemplifies all the small parameters to be neglected [16].

To reduce the order of the system and obtain a reduced order model, set the singular perturbation $\varepsilon = 0$.

The linear dynamical system has a multi-time behavior caused by the singular perturbation and this yields the slow and fast variable of the system [16].

Assumption 2.5.1.[16] The block matrix A_{22} is invertible and stable, i.e.

$$Re\{\lambda_i(A_{22}) < 0\}$$

Assumption 2.5.2.[16] The equation

$$\varepsilon \dot{z}_2 = A_{21}z_1 + A_{22}z_2 + B_2u \tag{2.39}$$

has distinct roots when $\varepsilon = 0$.

According to the assumptions (2.5.1) and (2.5.2) and equation (2.37), set $\varepsilon = 0$ then the roots of equation (2.39) denoted by $\bar{z}_2$ given as:

$$\bar{z}_2 = -A_{22}^{-1}A_{21}\bar{z}_1 - A_{22}^{-1}B_2u. \tag{2.40}$$

If we substitute the value of $\bar{z}_2$ in the first part of equation (2.37), we obtain the reduced order model represented by the following state-space equations:

$$\dot{\bar{z}}_1 = \bar{A}\bar{z}_1 + \bar{B}u$$

$$\bar{y} = \bar{C}\bar{z}_1 + \bar{D}u \tag{2.41}$$

$$\bar{z}_1(0) = z_1(0).$$

where

$$\bar{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}$$

$$\bar{B} = B_1 - A_{12}A_{22}^{-1}B_2 \tag{2.42}$$

$$\bar{C} = C_1 - C_2A_{22}^{-1}A_{21}$$

$$\bar{D} = -C_2A_{22}^{-1}B_2.$$

Denote by $\overline{H}$ the transfer function of the reduced system describe in equation (2.41), then:

$$\overline{H}(s) = \bar{C}(sI - \bar{A})^{-1}\overline{B} + \overline{D}. \tag{2.43}$$

From the definition of the reduced reciprocal system (2.25) and the two equations (2.26) and (2.42), we obtain the following:

$$\begin{aligned}
\tilde{A} &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} \\
&= (\bar{A})^{-1} \\
\tilde{B}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}(B_1 - A_{12}A_{22}^{-1}B_2) \\
&= (\bar{A})^{-1}\overline{B} \qquad (2.44) \\
\tilde{C}_1 &= -(C_1 - C_2A_{22}^{-1}A_{21})(B_1 - A_{12}A_{22}^{-1}B_2)^{-1} \\
&= \bar{C}(\bar{A})^{-1} \\
\widetilde{D} &= \overline{D} - \bar{C}(\bar{A})^{-1}\overline{B}.
\end{aligned}$$

The relation between $\overline{H}(s)$ and $\widetilde{H}_r(s)$ is given as [9]:

$$\begin{aligned}
\overline{H}(s) &= \bar{C}(sI - \bar{A})^{-1}\overline{B} + \overline{D} \\
&= \bar{C}\left(\frac{1}{s}\right)\left(I - \frac{I}{s}\bar{A}\right)^{-1}\overline{B} + \overline{D} \\
&= \bar{C}\left(\frac{1}{s}\right)\left((\bar{A})^{-1}\bar{A} - \frac{I}{s}\bar{A}\right)^{-1}\overline{B} + \overline{D} \\
&= \bar{C}\left(\frac{1}{s}\right)\left((\bar{A})^{-1} - \frac{I}{s}\right)^{-1}(\bar{A})^{-1}\overline{B} + \overline{D} \\
&= -\bar{C}\left(\frac{I}{s} - (\bar{A})^{-1} + \bar{A}\right)\left(\frac{I}{s} - (\bar{A})^{-1}\right)^{-1}(\bar{A})^{-1}\overline{B} + \overline{D}
\end{aligned}$$

$$= -\bar{C}(\bar{A})^{-1}(\bar{B}) - \bar{C}\left(\frac{I}{s} - (\bar{A})^{-1}\right)^{-1}(\bar{A})^{-1}\bar{B} + \bar{D} \quad (2.45)$$

$$= -\bar{C}(\bar{A})^{-1}\left(\frac{I}{s} - (\bar{A})^{-1}\right)^{-1}(\bar{A})^{-1}\bar{B} + \bar{D} - \bar{C}(\bar{A})^{-1}\bar{B}$$

$$= \tilde{C}\left(\frac{I}{s} - \tilde{A}_{11}\right)^{-1}\tilde{B}_1 + \widetilde{D}$$

$$= \widetilde{H}_r\left(\frac{1}{s}\right) \blacksquare$$

Theorem 2.5.3.[25] The reduced order model $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ obtained by Singular Perturbation Approximation Method (SPAM) is balanced with Gramian Σ_1 and its asymptotically stable.

Proof. By lemma (2.4.4) the reduced system $\left(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, \widetilde{D}\right)$ is balanced with Gramian Σ_1 that satisfy:

$$\tilde{A}_{11}\Sigma_1 + \Sigma_1\tilde{A}_{11}^T + \tilde{B}_1\tilde{B}_1^T = 0$$

$$\tilde{A}_{11}^T\Sigma_1 + \Sigma_1\tilde{A}_{11} + \tilde{C}_1^T\tilde{C}_1 = 0$$

multiplying the first equation by $\tilde{A}_{11}^{-1}$ from left and by $\left(\tilde{A}_{11}^{-1}\right)^T$ from right, and multiplying the second equation by $\left(\tilde{A}_{11}^{-1}\right)^T$ from left and by $\tilde{A}_{11}^{-1}$ from right yields:

$$\tilde{A}_{11}^{-1}\left(\tilde{A}_{11}\Sigma_1\right)\left(\tilde{A}_{11}^{-1}\right)^T + \tilde{A}_{11}^{-1}\left(\Sigma_1\tilde{A}_{11}^T\right)\left(\tilde{A}_{11}^{-1}\right)^T + \tilde{A}_{11}^{-1}(\tilde{B}_1\tilde{B}_1^T)\left(\tilde{A}_{11}^{-1}\right)^T = 0$$

$$\Sigma_1\left(\tilde{A}_{11}^{-1}\right)^T + \tilde{A}_{11}^{-1}\Sigma_1 + \left(\tilde{A}_{11}^{-1}\tilde{B}_1\right)\left(\tilde{A}_{11}^{-1}\tilde{B}_1\right)^T = 0$$

by equation (2.45), we obtain:

$$\bar{A}\Sigma_1 + \Sigma_1\bar{A}^T + \bar{B}\bar{B}^T = 0$$

and

$$\left(\tilde{A}_{11}^{-1}\right)^T\left(\tilde{A}_{11}^T\Sigma_1\right)\tilde{A}_{11}^{-1} + \left(\tilde{A}_{11}^{-1}\right)^T\left(\Sigma_1\tilde{A}_{11}\right)\tilde{A}_{11}^{-1} + \left(\tilde{A}_{11}^{-1}\right)^T\left(\tilde{C}_1^T\tilde{C}_1\right)\tilde{A}_{11}^{-1} = 0$$

$$\Sigma_1\tilde{A}_{11}^{-1} + \left(\tilde{A}_{11}^{-1}\right)^T\Sigma_1 + \left(\tilde{C}_1\tilde{A}_{11}^{-1}\right)^T\left(\tilde{C}_1\tilde{A}_{11}^{-1}\right) = 0$$

In the same way, by equation (2.45), we obtain:

$$\bar{A}^T\Sigma_1 + \Sigma_1\bar{A} + \bar{C}^T\bar{C} = 0$$

which implies that $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is balanced with gramian Σ_1.

Suppose $\tilde{A}_{11}$ is asymptotically stable. Since $\bar{A} = \tilde{A}_{11}^{-1}$, then the eigenvalues of $\bar{A}$ is $\bar{\lambda}_i = \frac{1}{\lambda_i}$, where λ_i is the eigenvalue of $\tilde{A}_{11}$ for $i = 1,2,\cdots,n$.

We conclude that $R\{\bar{\lambda}_i(\bar{A})\} < 0$, which means the reduced order model by singular perturbation approximation is asymptotically stable.■

In the following theorem we show the characterization of the error bound of reduced model obtained by singular perturbation approximation method.

Theorem 2.5.4.[25] Let $H(s)$ and $\bar{H}_r(s)$ be the transfer functions of the original system and r^{th} order system obtained by singular perturbation approximation method respectively, then:

$$\|H(s) - \bar{H}(s)\|_\infty \leq 2\sum_{i=r+1}^{n}\sigma_i \tag{2.46}$$

Proof. By equations (2.23) and (2.45) and lemma (2.4.6), using the triangle inequality we obtain:

$$\|H(s) - \bar{H}(s)\|_\infty$$
$$= \left\| H(s) - \tilde{H}\left(\frac{1}{s}\right) + \tilde{H}\left(\frac{1}{s}\right) - \tilde{H}_r\left(\frac{1}{s}\right) + \tilde{H}_r\left(\frac{1}{s}\right) - \bar{H}(s) \right\|_\infty$$
$$\leq \left\| H(s) - \tilde{H}\left(\frac{1}{s}\right) \right\|_\infty + \left\| \tilde{H}\left(\frac{1}{s}\right) - \tilde{H}_r\left(\frac{1}{s}\right) \right\|_\infty + \left\| \tilde{H}_r\left(\frac{1}{s}\right) - \bar{H}(s) \right\|_\infty$$
$$\leq \left\| \tilde{H}\left(\frac{1}{s}\right) - \tilde{H}_r\left(\frac{1}{s}\right) \right\|_\infty$$
$$\leq 2\sum_{i=r+1}^{n} \sigma_i \ \blacksquare$$

Chapter Three
Model Reduction for Unstable Systems

In this chapter we study an order reduction model for unstable finite dimensional linear system. And we will generate the error bound of reduced order unstable system. Our main focus in this chapter to study reduced order model for unstable system by using $\mathcal{L}_2[0,T]$-induced norm approach [30].

3.1 Notations and preliminary results

In this section we introduce some basic notations and definitions related to the $\mathcal{L}_2[0,T]$-induced norm of a finite dimensional LTI system.

Let $\mathcal{L}_2[0,T]$ denote the space of vectors valued real functions essentially bounded in the interval $[0,T]$, equipped with the norm:

$$\|f\|^2_{\mathcal{L}_2[0,T]} = \int_0^T f^T(t)\, f(t) dt. \tag{3.1}$$

Let $\mathcal{L}$ represent the space of LTI system regarding to bounded factor in $\mathcal{L}_2[0,T]$.

Definition 3.1.1.[30] The induced norm of an operator $E \in \mathcal{L}$, is given by:

$$\|E\|_{\mathcal{L}_2[0,T],ind} = sup \frac{\|Eu\|_{\mathcal{L}_2[0,T]}}{\|u\|_{\mathcal{L}_2[0,T]}}, \tag{3.2}$$

such that $\|u\|_{\mathcal{L}_2[0,T]} \neq 0$.

Indeed, the $\mathcal{L}_2[0,\infty)$-induced norm of LTI stable factor H agrees with the top value of its frequency response, i.e.:

$$\|H\|_{\mathcal{L}_2[0,\infty),ind} = \|H\|_\infty. \tag{3.3}$$

Lemma 3.1.2.[30, 13] (Bounded Real) Suppose a finite dimensional, strictly proper LTI stable system with state space realization is given as:

$\Sigma_s = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$.

Then, the following are equivalent:

1. The $\mathcal{L}_2[0,T]$-induced earning is bounded by $\delta > 0$.

$$\|H\|_{\mathcal{L}_2[0,T],ind} < \delta$$

2. The next linear matrix inequality recognize a positive definite solution $X > 0$:

$$\begin{pmatrix} A^T X + XA + C^T C & XB \\ B^T X & -\delta^2 I \end{pmatrix} < 0. \tag{3.4}$$

For the proof of this lemma see [13].

Lemma 3.1.2.[30] (Bounded Real lemma, Differential Version) Consider a strictly proper, finite dimensional, not necessarily stable LTI system Σ_s with state space realization given as:

$\Sigma_s = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$.

Assume that, the following differential matrix inequality admits a positive definite solution $X(t)$ for all $t \in [0, T]$:

$$\begin{pmatrix} A^T X + XA + \dot{X} + C^T C & XB \\ B^T X & -\delta^2 I \end{pmatrix} < 0. \tag{3.5}$$

Let $u \in \mathcal{L}_2[0, T]$ denote an arbitrary input and y the corresponding output. Then the following holds:

$$\int_0^T y^T y \, dt < \delta^2 \int_0^T u^T u \, dt \tag{3.6}$$

Corollary 3.1.4.[30] If the inequality (3.6) is satisfied, then:

$$\|H\|_{\mathcal{L}_2[0,T],ind} < \delta$$

3.2 Model reduction by $\boldsymbol{\mathcal{L}_2[0, T]}$-induced norm

In this section we study the bound on the $\mathcal{L}_2[0, T]$-induced norm by simply computation of infinite norm ($\|.\|_\infty$) of a shifted system obtained from the system under consideration. Then we utilize this bound to solve the problem of model order reduction for unstable system over finite horizon.

Officially, the $\mathcal{L}_2[0, T]$-induced norm of a given LTI factor H is tantamount to the $\mathcal{L}_2[0, \infty)$-induced norm of a time-varying system with convolution kernel:

$$H(t, \tau) = V(t)H(t - \tau). \tag{3.7}$$

where $V(\cdot)$ is the step window function given as:

$$V(t) = \begin{cases} 1\,, 0 \leq t \leq T \\ 0\,, otherwise \end{cases}. \tag{3.8}$$

However, there are no active computations for implement this calculation. So, one can represents the step window by the exponential window defined as e^{-at}, where the time constant a is satisfy that $e^{-at} \ll 1$ for $t > T$.

The advantage of this approach is that, the resulting kernel $e^{-at}H(t)$ can be related with the new LTI factor whose frequency response is a shifted version of the frequency response of the original system. If a is chosen such that this new LTI operator is stable, computing its $\mathcal{L}_{2[0,\infty)}$-induced norm (its infinite norm $\|.\|_{\infty}$) is now a standard problem.

Theorem 3.2.1.[30] Suppose a finite dimensional, strictly proper LTI not necessarily stable system with state space realization is given as:

$$\Sigma_{\mathrm{s}} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

If there exist a satisfying

$$\Sigma_{\mathrm{s}_a} = \begin{pmatrix} A - aI & B \\ C & 0 \end{pmatrix}$$

is stable with $\|H_a\|_{\infty} < \delta$, then the following holds:

$$\|H\|_{\mathcal{L}_2[0,T],ind} < \delta e^{aT} \tag{3.9}$$

Proof. By hypothesis $\|e^{aT}H_a\|_\infty < \delta e^{aT}$, by lemma (3.1.2) there exist $X_a > 0$ such that:

$$\begin{pmatrix} A_a^T X_a + X_a A_a + e^{aT} C^T C e^{aT} & X_a B \\ B^T X_a & -\delta^2 e^{2aT} I \end{pmatrix} < 0, \quad (3.10)$$

where $A_a = A - aI$.

Let we define for $t \in [0, T]$, $X(t) = e^{-2at} X_a$. Multiply (3.11) by e^{-2at}, we obtain:

$$0 > \begin{pmatrix} A_a^T X_a e^{-2at} + e^{-2at} X_a A_a + e^{2a(T-t)} C^T C & e^{-2at} X_a B \\ B^T X_a e^{-2at} & -\delta^2 e^{2a(T-t)} I \end{pmatrix}$$

$$= \begin{pmatrix} A^T X + XA + \dot{X} + e^{2a(T-t)} C^T C & XB \\ B^T X & -\delta^2 e^{2a(T-t)} I \end{pmatrix}$$

$$\geq \begin{pmatrix} A^T X + XA + \dot{X} + C^T C & XB \\ B^T X & -\delta^2 e^{2aT} I \end{pmatrix}$$

such that, the last inequality holds when $a > 0$ and $t \leq T$.

The proof can be completed depending on lemma (3.1.3) and corollary (3.1.4).■

Now, consider an unstable system, we want to obtain approximations in the $\mathcal{L}_2$-induced norm on a finite time interval.

This can be done using the following simple algorithm [30].

1. As input, take a state space realization $H(s) = C(sI - A)^{-1}B + D$ and a constant $a \in \mathbb{R}^+$ such that $H(s + a) \in \|.\|_\infty$.

2. Find a stable reduced order system $\Sigma_{s_{r,a}} = \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix}$ to the system $\Sigma_{s_a} = \begin{pmatrix} A + aI & B \\ C & D \end{pmatrix}$.
3. Use the system $\Sigma_{s_r} = \begin{pmatrix} A_r + aI & B_r \\ C_r & D_r \end{pmatrix}$ as an approximation to the original system in $[0, T]$.

Remark 3.2.2.[30] Suppose the system Σ_{s_a} is stable, then its reduced version $\Sigma_{s_{r,a}}$ can be obtained using standard model reduction techniques, for example, using balanced truncations, which gives the error bound:

$$\left\|H_a - H_{r,a}\right\|_\infty \leq 2\sum_{i=r+1}^{n} \sigma_{i,a}, \tag{3.11}$$

where $\sigma_{i,a}$ denotes the Hankel singular values for Σ_{s_a}.

By theorem (3.2.1) we get:

$$\|H - H_r\|_{\mathcal{L}_2[0,T],ind} \leq 2e^{aT} \sum_{i=r+1}^{n} \sigma_{i,a}. \tag{3.12}$$

Chapter Four
Numerical examples

In this chapter we study two numerical examples of a low order model, first one is example of stable system and the second example is example of unstable system, and using MATLAB software we compute the error bound of reduced system.

4.1 Mass-spring damping system

In this section we consider an example of stable system and we apply balanced truncation method to reduce the order of this system. As an application of engineering system, we study the mass-spring damping system.

Suppose $m_i, i = 1,2,\cdots,n$ are masses characterized by figure (4.1)

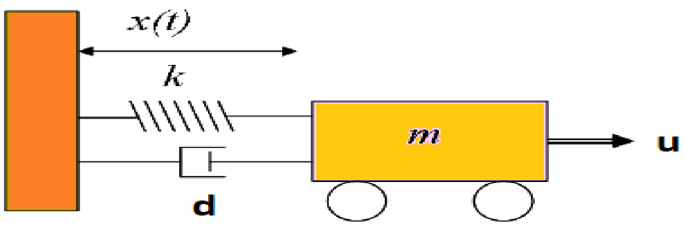

Figure 4.1: Multi mass-spring damping system

such that x_i are the position of the mass m_i respectively, and k_i and d_i are constants describing the stiffness and damping of the springs.

By applying Newton's second law on the masses m_i we obtain the following differential equation:

$$m_i\ddot{x}_i - d_i\dot{x}_{i-1} + (d_i + d_{i+1})\dot{x}_i - d_{i+1}\dot{x}_{i-1} - k_i x_{i-1} + k_{i+1}(x_{i+1} - x_i) = bu$$
(4.1)

such that $i = 1,2,\cdots,n$, and $b = \begin{cases} 1 \ , i = n \\ 0 \ , i \neq n \end{cases}$, while $x_0 = 0$ for $i = 1$ and

$x_{n+1} = k_{n+1} = d_{n+1} = 0$ for $i = n$.

Equation (4.1) can be expressed in matrix form as:

$$M\ddot{x} + D\dot{x} + Kx = Ju \qquad (4.2)$$

Such that

$$M = \begin{pmatrix} m_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & m_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & m_n \end{pmatrix}_{n\times n}$$

is called the mass matrix of the system.

$$D = \begin{pmatrix} d_1 + d_2 & -d_2 & 0 & \cdots & \cdots & 0 \\ -d_2 & d_2 + d_3 & -d_3 & 0 & \cdots & 0 \\ 0 & -d_3 & d_3 + d_4 & \ddots & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -d_n \\ 0 & 0 & 0 & \cdots & -d_n & d_n \end{pmatrix}_{n\times n}$$

is the damping matrix.

And

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & 0 & \dots & \dots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 & \dots & 0 \\ 0 & -k_3 & k_3 + k_4 & \ddots & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -k_n \\ 0 & 0 & 0 & \dots & -k_n & k_n \end{pmatrix}_{n \times n}$$

is the stiffness matrix.

The vector

$$J = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}_{n \times 1}$$

describes the number of controllers acting on masses.

To find the state space representation of this system, let

$$\dot{x} = q$$

and

$$\ddot{x} = \dot{q}.$$

Suppose M^{-1} exists, then by these equations and equation (4.2), we have:

$$\dot{x} = q$$

$$\dot{q} = -M^{-1}Kx - M^{-1}Dq + M^{-1}Ju. \qquad (4.3)$$

In matrix representation

$$\begin{pmatrix} \dot{x} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix} \begin{pmatrix} x \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ -M^{-1}J \end{pmatrix} u. \qquad (4.4)$$

Let

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}$$

be of size $(2n \times 2n)$ and

$$B = \begin{pmatrix} 0 \\ -M^{-1}J \end{pmatrix}$$

of size $(2n \times 1)$. Then the state space representation for this system is

$$\dot{X} = AX + Bu. \qquad (4.5)$$

where $X = \begin{pmatrix} x \\ q \end{pmatrix}$ is the state vector of dimension$(2n \times 1)$.

Now, we consider a mass-spring damping system of five masses.

First, we calculate the Hankel singular values of the system. Figure (4.2) represents the Hankel singular values of mass-spring damping system of size (N=20).

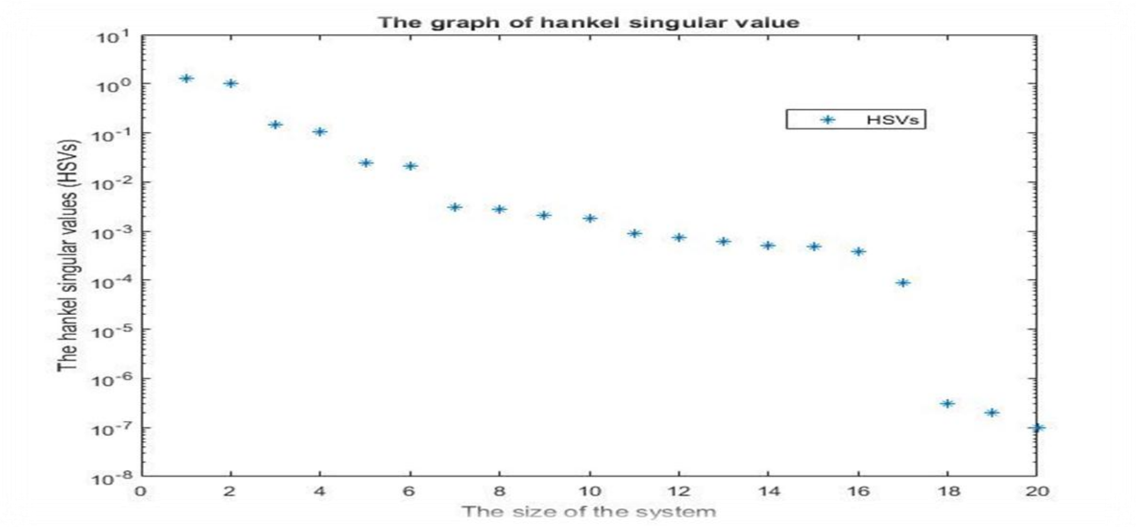

Figure 4.2: Graph of HSVs

Now, by applying the balanced truncation method for this system with zero initial condition obtained a reduce order system, and compute $\|.\|_{\infty}$ bound of the approximation error given in section (2.3).

We take a system of size $N = 20$ and reduce its order to obtain reduced order system of size $r = 2$.

Figure (4.3) represents the maximum singular value decomposition (MSVD) of $(H - H_r)$ and the error bound $2\sum_{i=r+1}^{20} \sigma_i$. We see obviously that the balanced truncation method yields a reduced order model with smaller error at high frequencies and larger error at low frequencies.

Table (4.1) shows the value of $\|H-H_r\|_\infty$ and the error bound $2\sum_{i=r+1}^{20}\sigma_i$.

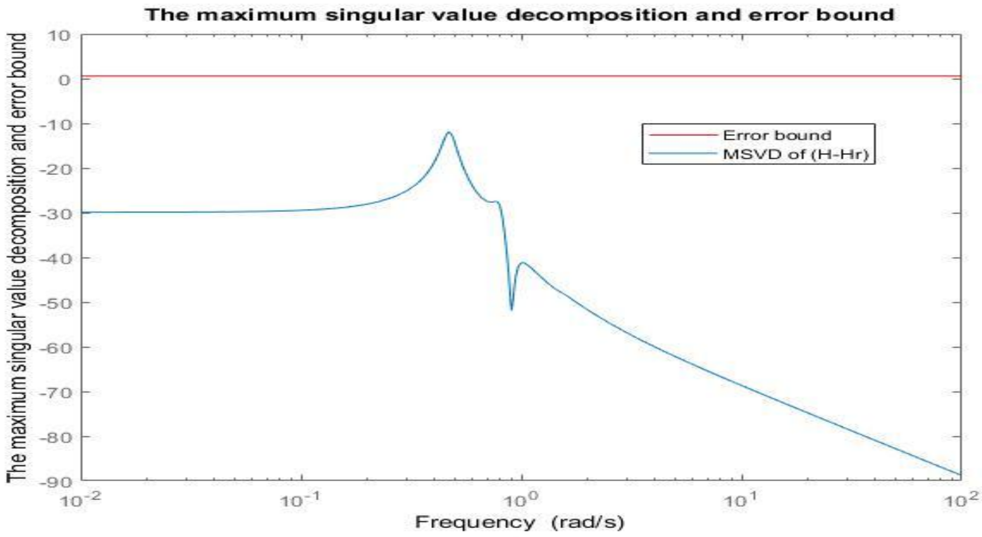

Figure 4.3: The MSVD and the error bound

Table 4.1: $\|H-H_r\|_\infty$ and the error bound

r	$\|H-H_r\|_\infty$	$2\sum_{i=r+1}^{20}\sigma_i$
2	0.254	0.5668
4	0.0303	0.0762
6	0.0012	0.0152
8	2.1264×10^{-4}	0.0078
10	3.2416×10^{-5}	0.0046
12	4.0012×10^{-5}	0.0022
14	6.3145×10^{-6}	3.806×10^{-4}

Next, in figure (4.4), we show the plot of the outputs y and y_r of the original and reduced systems respectively and their differences $(y-y_r)$ and error bound by applying balanced truncation method on the numerical example.

And in table (4.2) we show the $\|.\|_{L_2}$ bound of the approximation error between the output y and y_r of the original and reduced systems respectively and the error bound.

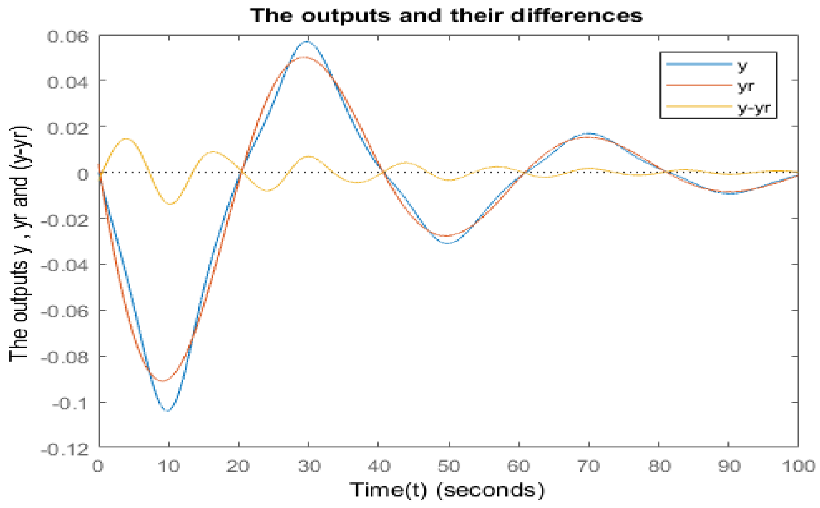

Figure 4.4: The outputs using balanced truncation.

Table 4.2: The L_2 of $y - y_r$ and the error bound.

r	$\|y - y_r\|_{L_2}$	$2\sum_{i=r+1}^{20} \sigma_i \, \|u\|$
2	0.0266	0.6801
4	0.0043	0.0917
6	0.00022	0.0182
8	7.6992×10^{-7}	0.0093
10	5.213×10^{-8}	0.0055
12	2.015×10^{-13}	0.0026
14	3.412×10^{-15}	4.5672×10^{-4}

4.2 RC-Circuit

In this section we discuss an example of unstable dynamical system. We will study an RC-filter, RC-network and resistor–capacitor circuit (RC-circuit). The RC-circuit shown in figure (4.5) is a collection of resistors and capacitors

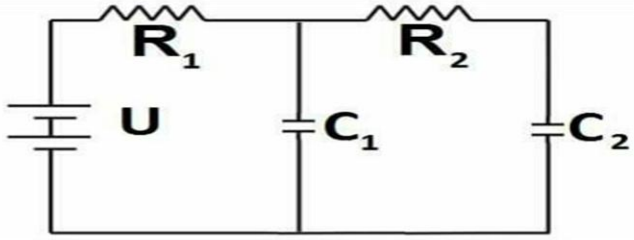

Figure 4.5: Simple RC-circuit

driven by a voltage or current exporter. RC-circuits may be used to filter a signal by blocking confirmed frequencies and passing others.

Suppose we denote by i_ρ and v_ρ the current and voltage through the capacitor C_ρ, let U denote the voltage source in the circuit and I (the output of the system) is the current across a resistor $n, \rho = 1,2,\cdots,n$.

Let $i_\rho = C_\rho \frac{d}{dt} v_\rho(t)$

$$e_1: \frac{U-v_1}{R_1} + i_1 + \frac{v_2-v_1}{R_2} = 0 \rightarrow \frac{d}{dt} v_1(t) = \frac{v_1-U}{C_1R_1} + \frac{v_1-v_2}{C_1R_2}$$

$$e_2: \frac{v_2-v_1}{R_2} + i_2 + \frac{v_3-v_2}{R_3} = 0 \rightarrow \frac{d}{dt} v_2(t) = \frac{v_1-v_2}{C_2R_2} + \frac{v_2-v_3}{C_2R_3}$$

$$\vdots$$

$$e_\rho: \frac{v_\rho - v_{\rho-1}}{R_\rho} + i_\rho + \frac{v_{\rho+1} - v_\rho}{R_{\rho+1}} = 0 \rightarrow \frac{d}{dt} v_\rho(t) = \frac{v_{\rho-1} - v_\rho}{C_\rho R_\rho} + \frac{v_\rho - v_{\rho+1}}{C_\rho R_{\rho+1}}$$

$$\vdots$$

$$e_n: \frac{v_n-v_{n-1}}{R_n} + i_n = 0 \rightarrow \frac{d}{dt} v_n(t) = \frac{v_{n-1}-v_n}{C_nR_n}$$

$$I = \frac{v_n-v_{n-1}}{R_n}.$$

This system can be written in state space representation as:

$$\dot{V} = AV + BU$$

$$I = CV,$$

where:

$$A = \begin{pmatrix}
\frac{1}{C_1R_2}+\frac{1}{C_1R_1} & -\frac{1}{C_1R_2} & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\frac{1}{C_2R_2} & -\frac{1}{C_2R_2}+\frac{1}{C_2R_3} & -\frac{1}{C_2R_3} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \frac{1}{C_3R_3} & -\frac{1}{C_3R_3}+\frac{1}{C_3R_4} & -\frac{1}{C_3R_4} & 0 & \cdots & \cdots & 0 \\
0 & 0 & \frac{1}{C_4R_4} & \ddots & \ddots & 0 & \cdots & 0 \\
\vdots & 0 & 0 & \ddots & \ddots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{C_nR_n} & -\frac{1}{C_nR_n}
\end{pmatrix}_{n\times n}$$

$$B = \begin{pmatrix}
-\frac{1}{C_1R_1} \\
0 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
0
\end{pmatrix}_{n\times 1}$$

and

$$C = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 & -\frac{1}{R_n} & \frac{1}{R_n} \end{pmatrix}_{1\times n}$$

We consider an RC-circuit of size $(N = 20)$ such that the input is the voltage source and the output is the current across the resistor number 20.

Suppose the system Σ_s describe the RC-circuit is unstable dynamical linear system, we use the $\mathcal{L}_{2[0,T],ind}$ norm approach that described in section (3.2).

Let Σ_{s_a} be the shifted system from the original system Σ_s such that Σ_{s_a} is stable system.

Firstly, show the (HSVs) of the shifted system Σ_{s_a}, figure (4.6).

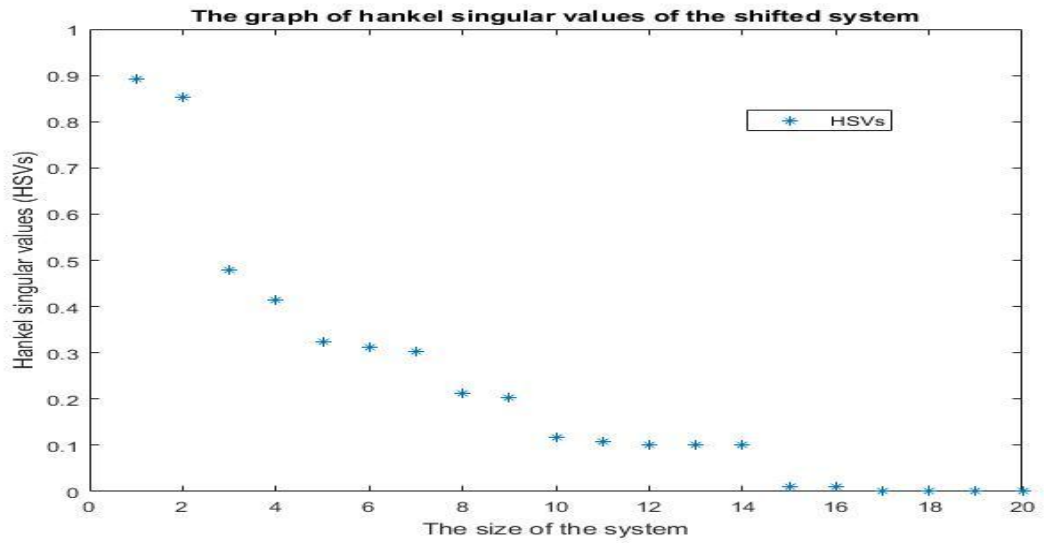

Figure 4.6: Plot of HSVs of the shifted system

Now by applying balanced truncation method on the system Σ_{s_a} we obtain a reduced system $\Sigma_{s_{r,a}}$ of order $r = 2$, shift back $\Sigma_{s_{r,a}}$ by a to obtain Σ_{s_r} as an approximation to the original system in the interval $[0, T]$.

In figure (4.7) we show the plot of the maximum singular value decomposition of $(H - H_r)$ and the error bound.

Table (4.3) contains the value of $\|H - H_r\|_{\mathcal{L}_{2[0,T],ind}}$ and value of error bound $2e^{aT}\sum_{i=r+1}^{10}\sigma_{a,i}$.

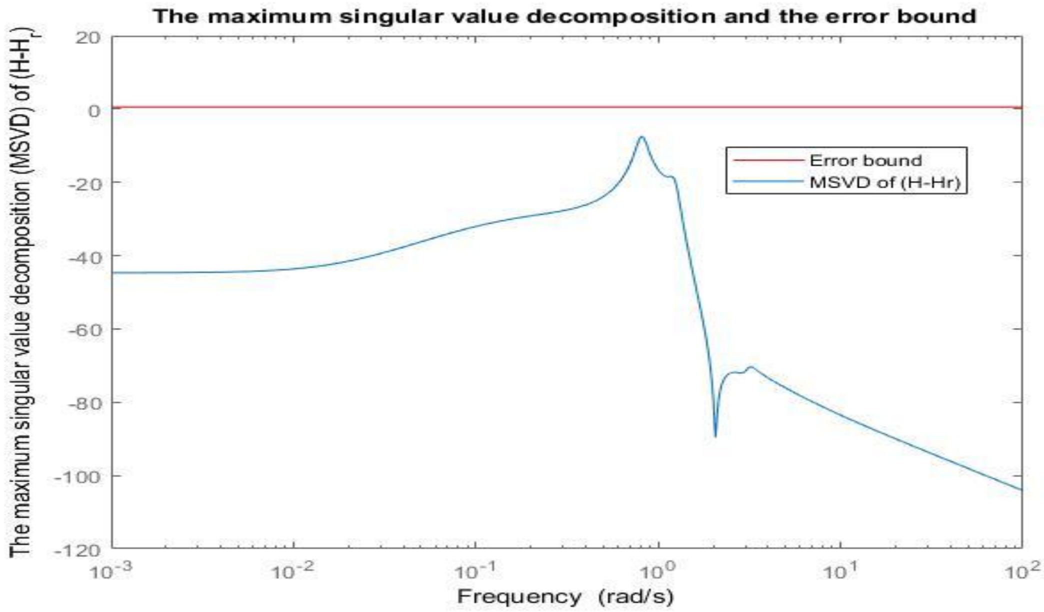

Figure 4.7: The MSVD and the error bound.

Table 4.3: $\|H - H_r\|_{\mathcal{L}_{2[0,T],ind}}$ and the error bound.

r	$\|H - H_r\|_{\mathcal{L}_{2[0,T],ind}}$	$2e^{aT}\sum_{i=r+1}^{20}\sigma_{a,i}$
2	0.1509	1.5862
4	0.0167	1.0748
6	0.0059	0.7099
8	0.00021	0.4298
10	4.2112×10^{-4}	0.2542
12	2.5404×10^{-5}	0.0434

Next, we want to calculate approximation error by L_2 between the outputs y and y_r of the original and the reduced systems respectively.

Figure (4.8) shows the outputs, y_r and $y - y_r$, and table(4.4) contains the value of $\|y - y_r\|_{L_2}$ and the error bound.

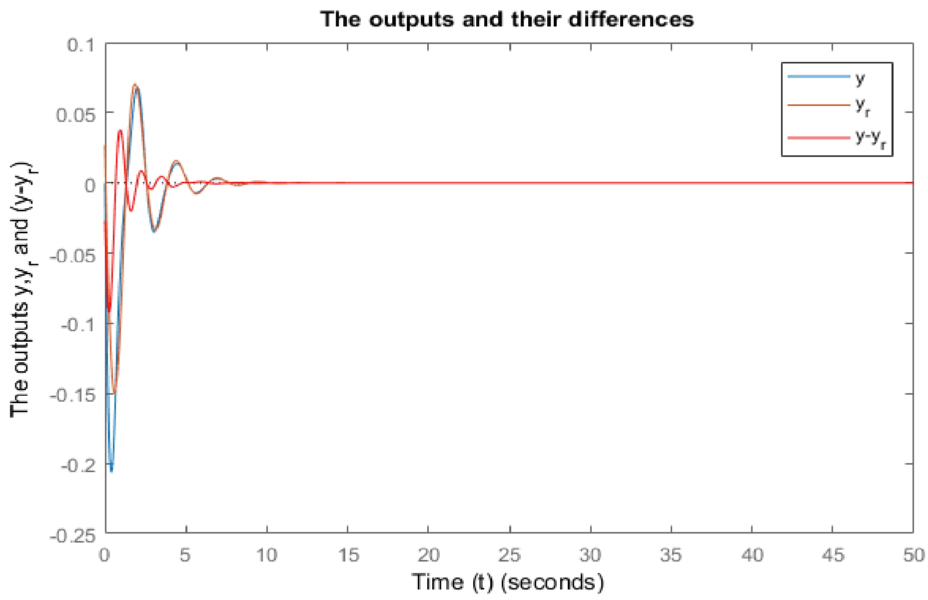

Figure 4.8: The outputs and their difference.

Table 4.2: The L_2 of $y - y_r$ and the error bound.

r	$\|y - y_r\|_{L_2}$	$2e^{aT}\sum_{i=r+1}^{20}\sigma_{a,i}\,\|u\|$
2	2.9012×10^{-4}	1.7452
4	3.5415×10^{-6}	11822
6	7.2115×10^{-7}	0.7809
8	4.1245×10^{-9}	0.4728
10	6.1452×10^{-11}	0.2796
12	4.2348×10^{-15}	0.0478

References

[1] A. C. Antoulas, **Approximation of large-scale dynamical systems, Advances in design and control**, Society for Industrial and Applied Mathematics, 2005.

[2] U. Baur, P. Benner and L. Feng, **Model order reduction for linear and nonlinear systems,** a system-theoretic perspective, 2014.

[3] S. C. Beeler, G. M. Kepler, H. T. Tran and H.T. Banks, **reduced order modeling and control of thin_lm growth in an HPCVD reactor**, Technical Report CRSC-00-33, Center for Research in Scientific Computation, North Carolina State University, 1999.

[4] P. Benner, **Solving large-scale control problems**, Control Syst. Magazine,vol.24, pp. 44-59, 2004.

[5] A. Bensoussan and G. Blankenship, **Singular perturbations in stochastic control, in Singular Perturbations and Asymptotic Analysis in Control Systems**, Springer, pp. 171-260, 1987.

[6] D. N. Burghes and A. Graham, **Control and Optimal Control Theories with Applications**, Wood head Publishing, 2004.

[7] R. Condon, Marissa; Ivanov, ***Nonlinear systems algebraic gramians and model reduction***, **COMPEL Int J for Computation and Maths in Electrical and Electronic Eng**, vol.24, pp. 202-219, 2005.

[8] R.F.Curtain and H.Zwart, **An Introduction to Infnite-Dimensional Systems,** Springer-Verlag. New York, 1995.

[9] A.Daraghmeh, Model Order Reduction of Linear Control Systems: **Comparison of Balance Truncation and Singular Perturbation Approximation with Application to Optimal Control**, Freien Universitat Berlin, 2016.

[10] B. Datta, **Numerical Methods for Linear Control Systems**, Academic Press, 2003.

[11] J. J. D'Azzo, C. H. Houpis, and S. N. Sheldon, **Linear Control System Analysis and Design with Matlab**, Control engineering series, vol. 14, M. Dekker, 5th ed, 2003.

[12] D. F. Enns, **Model reduction with balanced realizations: an error bound and a frequency weighted generalization**, Proc. of the 23rd IEEE Conf. on Decision and Control, pp. 369-373, 1984.

[13] P. Gahinet and P. Apkarian, ***A Linear Matrix Inequality Approach***

to H1 Optimization Problems, **Int. J. Robust and Nonlinear Ctrl**., vol.4, pp. 421-448, 1994.

[14] S.Gugercin and A.C.Antouias, **A survey of model reduction by balanced truncation and some new results**, 2004.

[15] C. Hartmann, B. Schäfer-Bung, and A. Thöns-Zueva, ***Balanced averaging of bilinear systems with applications to stochastic control***, **SIAM Journal on Control and Optimization**, vol.51, pp. 2356-2378, 2013.

[16] P.V. Kokotovic, **Applications of singular perturbation techniques to control problems,** SIAM review, vol.26, pp. 501-550, 1984.

[17] E. Kreyszig, **Introductory functional analysis with applications**, 1978.

[18] Y. Liu and B. D. O. Anderson, ***Singular perturbation approximation of balanced systems,*** **International Journal of Control**, vol.50, pp. 1379-1405, 1989.

[19] J.C.Maxwell, **Proceedings of the Royal Society of London**, pp.270-283, 1868.

[20] S. K. Mitra and M. P. Ekstrom, **Two-dimensional digital signal processing,** Hutchinson Ross Publishing Company, vol. 20, 1978.

[21] N. Monshizadeh, H. L. Trentelman, and M. Kanat Camlibel, **Stability and synchronization preserving model reduction of multi-agent systems**, Systems and Control Letters, vol.62, pp. 1-10, 2013.

[22] E.J.Routh, **A Treatise on the Stability of a Given State of Motion, Particularly Steady Motion: Particularly Steady Motion**, Macmillan and co,1877.

[23] E.J.Routh and A.T.Fuller, **Stability of motion**,Taylor and Francis, 1975.

[24] C. W. ROWLEY, *Model reduction for fluids, using balanced proper orthogonal decomposition*, **International Journal of Bifurcation and Chaos**, vol.15, pp. 997-1013, 2005.

[25] R. Saragih and Fatmawati, ***Singular perturbation approximation of balanced infnite-dimentional system***, **International Journal of Control and Automation**, vol.6, pp. 409-420, 2013

[26] B. Schäfer-Bung, C. Hartmann, B. Schmidt, and C. ***Schütte, Dimension reduction by balanced truncation: Application to light-induced control of open quantum systems,*** **The Journal of chemical physics**, vol.135, 2011.

[27] J. Scherpen, **Balancing for nonlinear systems**, Systems and Control Letters, vol.21, pp. 143-153, 1993.

[28] D. Skataric and N. Kovacevic-Ratkovic, **The system order reduction via balancing in view of the method of singular perturbation**, FME Transactions, vol.38, pp. 181-187, 2010.

[29] S. Skogestad and I. **Postlethwaite, Multivariable Feedback Control,** John Wiley and Sons, Chichester,1996.

[30] M. Sznaier, A.C. Doherty, M. Barahona, H. **Mabuchi and J.C. Doyle, A new bound of the $\mathcal{L}_{2[0,T]}$ −induced norm and applications to model reduction,** IEEE Proceedings of 2002 American Control Conference - Anchorage, AK, USA,8-5-2002 - 10-5-2002.

[31] D. Xue, Y. Chen, and D. P. Atherton, **Linear feedback control: Analysis and Design with MATLAB**, Advances in Design and Control, Society for Industrial and Applied Mathematics, 2007.

[32] K. Zhou, J. C. Doyle, and K. Glover, **Robust and Optimal Control,**

Prentice Hall, 1998.

جامعة النجاح الوطنية

كلية الدراسات العليا

تقدير حدود الخطأ العددي لنظام ديناميكي خطي غير ثابت

إعداد

سند قاسم توفيق حج محمد

إشراف

د. عدنان دراغمة

أ. د. ناجي قطناني

قدمت هذه الأطروحة استكمالاً لمتطلبات الحصول على درجة الماجستير في الرياضيات، بكلية الدراسات العليا، في جامعة النجاح الوطنية، نابلس – فلسطين.

2020

تقدير حدود الخطأ العددي لنظام ديناميكي خطي غير ثابت

إعداد

سند قاسم توفيق حج محمد

إشراف

د. عدنان دراغمة

أ.د. ناجي قطناني

الملخص

إن نظرية التحكم تعتبر من المواضيع القديمة التي ظهرت مع بداية حياة الإنسان ورغبته في إدارة ما حوله واستغلال موارد الطبيعة، إن دراسة التطبيقات الهندسية والفيزيائية والكيميائية وغيرها يقودنا إلى أنظمة من المعادلات التفاضلية العادية و الجزئية والتي تكون في الغالب ذات رتبة عالية.

لتسهيل التعامل مع هكذا أنظمة ولتشغيلها بأقل تكلفة ممكنة والحصول على أفضل المخرجات نحتاج إلى تقليل رتبتها بحيث نستثني منها المتغيرات الأقل تأثيرا للحصول على نظام جديد ذو رتبة صغيرة بأقل خطأ ممكن.

وقد ظهر في هذا المجال العديد من الأبحاث والدراسات القيمة حيث إننا في هذا البحث ركزنا على حالة النظام الديناميكي الخطي الغير ثابت وحاولنا تخفيض رتبته والحصول على النظام الجديد بأقل خطأ.

في البداية درسنا النظام الثابت – أو المستقر– ولتخفيض رتبته استخدمنا طريقة الاقتطاع الثابت وطريقة الاضطراب المفرد وبالاعتماد عليه قمنا بدراسة النظام الغير ثابت ودرسنا احد الطرق الفعالة لتخفيض رتبة النظام الغير مستقر والتي تقوم على تحويله الى نظام مستقر واستخدام أحد الطرق السابقة، ولتوضيح فعالية هذه الطرق قمنا بدراسة أمثلة عددية من واقع التطبيقات الحياتية وطبقنا عليها الطرق التي قمنا بدراستها سابقا وقد أظهرت النتائج جدوى استخدام هذه الطرق.