An-Najah National University
Faculty of Graduate Studies

The Exact and Asymptotic Parameters of the First Canonical Weight Vector where the Rank of ($\sum_{11}^{-1}\sum_{12}\sum_{22}^{-1}\sum_{21}$) Equal One

By

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Dedication

To My Beloved Family With Love And Respect

Acknowledgment

First of all, I thank my god for all the blessing, he bestowed on me and continues to bestow on me.

I wish to extend my gratitude and appreciation to my supervisor Dr. Farouk for his help and encouragement during the preparation of this thesis. I also wish to thank Dr. Abdelrahim Barham and Dr. Tahseen Al-Mughrabi for accepting to be members of the defense committee and for their valuable remarks.

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Finally, I would like to thank my friends.

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Abstract

This research discusses the stability of the first canonical weight vector from the one-factor structure derivation of the exact formula for the weight's variance and the asymptotic distribution of the weight are emphasized to give theoretical robustness to the concept of stability. The closeness of the exact and asymptotic variances points to the precision of the derivation.

It was proven that the weight $\frac{1}{s} \sim N\left(1, \frac{1}{2(n-1)}\right)$, where the asymptotic variance is $\frac{1}{2(n-1)}$. The greater sample size the less variance of the weight will be, and hence more stable weight.

Chapter One

Modes of Convergence

1.1 Introduction

The most important theoretical results in probability theory are limit theorems of these the most important are those that are classified under the heading of “Central Limit Theorems” which are concerned with determining conditions under which the sum of a large number at random variables has a probability distribution that is approximately normal.

In many problems of probability and statistics we are faced with sequence of random variables like x_1, x_2,…x_n,…(e.g. estimators depending on sample, size n in statistics) and we are expected to find the limit of this sequence and also the distribution (asymptotic distribution) of the limit random variable.

1.2. Limiting distribution

Consider a distribution that depends upon the positive integer n, clearly the distribution function F of that distribution will also depend upon n, we denote this fact by writing the distribution function as F_n and the corresponding P.D.F as F_n. Moreover, to emphasize the fact that we are working with sequences of distribution functions, we place a subscript n on the random variables.

1.2.1. Example

$$F_n(\overline{x}) = \int_{-\infty}^{\overline{x}} \frac{1}{\sqrt{1/n}\sqrt{2\pi}} e^{-nw^2/2} dw$$

Is the distribution function of the mean $\overline{x_n}$ of a random sample of size n (x_1,…, x_n) from a normal distribution with mean zero and variance 1.

$$(\overline{x_n} \sim N(0, \frac{1}{n}))$$

1.2.2. Definition

The sequence $\{x_n\}$ converges to x in distribution ($x_n \xrightarrow{D} x$ as $n \to \infty$) if $F_n(x) \to F(x)$ as $n \to \infty$ ($\lim_{n\to\infty} F_n(x) = F(x)$) for every point x at which F(x) is continuous (we call F(x) the asymptotic distribution of x_n).

1.2.3. Definition

A distribution of the discrete type that has a probability of 1 at a single point has been called a degenerate distribution (a random vector $x \in \Re^n$ is degenerate at point $c \in \Re^n$ if $p(x = c) = 1$).

1.2.4. Example

Let Y_n denote the nth order statistic[1] of X_1, X_2, …X_n from a distribution having p.d.f

$$f(x) = \begin{cases} \dfrac{1}{\theta} & 0 < x < \theta, 0 < \theta < \infty \\ 0 & else \quad where \end{cases}$$

Then the P.D.F of Y_n is

$$g_n(y) = \frac{ny^{n-1}}{\theta^n}, \; 0 < y < \theta$$

And the distribution function of Y_n is

$$F_n(y) = \begin{cases} 0 & y < 0 \\ \displaystyle\int_0^y \frac{nz^{n-1}}{\theta^n} dz = \left(\frac{y}{\theta}\right)^n & 0 \le y < \theta \\ 1 & \theta \le y \end{cases}$$

Then $\lim_{n\to\infty} F_n(y) = \begin{cases} 0 & y < \theta \\ 1 & y \ge \theta \end{cases}$

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[1] See reference 7.

Now, $F(y)=\begin{cases}0 & -\infty<y<\theta \\ 1 & \theta\le y<\infty\end{cases}$

Is a distribution function and $\lim_{n\to\infty} F_n(y)=F(y)$

At each point of continuity of $F(y)$.

1.2.5. Example

Let $\overline{X}_n$ have the distribution function

$$F_n(\bar{x})=\int_{-\infty}^{\bar{x}} \frac{1}{\sqrt{1/n}\sqrt{2n}} e^{-nw^2/2} dw$$

By making the change of variable $v=\sqrt{n}w$ we get

$$F_n(\bar{x})=\int_{-\infty}^{\sqrt{n}\,\bar{x}} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$$

Then ,

$$\lim_{n\to\infty} F_n(\bar{x})=\begin{cases}0 & \bar{x}<0 \\ 1/2 & \bar{x}=0 \\ 1 & \bar{x}>0\end{cases}$$

Now the function $F(\bar{x})=\begin{cases}0 & \bar{x}<0 \\ 1 & \bar{x}\ge 0\end{cases}$

Is a distribution function and $\lim_{n\to\infty} F_n(\bar{x})=F(\bar{x})$ at every point of continuity of $F(\bar{x})$.

Notice:

$\lim_{n\to\infty} F_n(0)=1/2\neq F(0)=1$ But $F(\bar{x})$ isn't continuous at $\bar{x}=0$.

1.3. Stochastic convergence

1.3.1. Definition

A distribution that has a probability of 1 at a single point is said to be degenerate and we say that the distribution converges stochastically to the constant that has a probability of 1.

1.3.2. Theorem

Let $F_n(y)$ be the distribution function of a random variable y_n and let c be a constant independent of n then y_n converges stochastically to c *iff* for every $\in>0$ the $\lim_{n\to\infty} \Pr(|y_n - c| < \in) = 1$

Proof:

First assume

$\lim_{n\to\infty} \Pr(|y_n - c| < \in) = 1$ for every $\in>0$ want to show $\lim_{n\to\infty} F_n(y) = \begin{cases} 0 & y < c \\ 1 & y > c \end{cases}$

Now,

$$\Pr(|y_n - c| < \in) = F_n(c+\in)^- - F_n(c-\in)$$

Where $F_n(c+\in)^-$ is the left-hand limit of $F_n(y)$ at $y = c+\in$

Thus we have:

$$1 = \lim_{n\to\infty} P(|y_n - c| < \in) = \lim_{n\to\infty}\left[F_n(c+\in)^- - F_n(c-\in)\right]$$

Because $0 \le F_n(y) \le 1$ for all values of y and for every n it must be that

$\lim_{n\to\infty} F_n(c-\in) = 0$ And $\lim_{n\to\infty} F_n(c+\in)^- = 1 \quad \forall \in> 0$

=> $\lim_{n\to\infty} F_n(y) = \begin{cases} 0 & y < c \\ 1 & y > c \end{cases}$

On the other hand, assume:

$$\lim_{n\to\infty} F_n(y) = \begin{cases} 0 & y < \in \\ 1 & y > \in \end{cases}$$

Want to show $\lim_{n\to\infty} P(|y_n - c| < \in) = 1 \quad \forall \in > 0$

$$P(|y_n - c| < \in) = F_n(c+\in)^- - F_n(c-\in)$$

But $\lim_{n\to\infty} F_n(c+\in)^- = 1$

and $\lim_{n\to\infty} F_n(c-\in) = 0 \quad \forall \in > 0$

So $\lim_{n\to\infty}\left(|y_n - c| < \in\right) = 1 - 0 = 1$

1.3.3 Example

Let x1,…,xn $\sim N(\mu, \sigma^2)$ => $\overline{x_n} \sim N(\mu, \sigma^2 / n)$

$$P(|\overline{x_n} - \mu| \geq \in) = P(|\overline{x_n} - \mu| \geq \frac{k\sigma}{\sqrt{n}})$$

Where $\in = \frac{k\sigma}{\sqrt{n}} > 0$

$$\lim_{n\to\infty} P\left(|\overline{x_n} - \mu| \geq \frac{k\sigma}{\sqrt{n}}\right) \leq \lim_{n\to\infty} \frac{\sigma^2}{n \in^2} = 0$$

By Chebyshev's inequality [(1)] and $\frac{1}{k^2} = \frac{\sigma^2}{n \in^2}$

=> $P(|x_n - \mu| < \in) = 1$ for every $\in > 0$

Hence, $\overline{x_n}$ converges stochastically to μ.

Remark:

When $\lim_{n\to\infty} P(|y_n - c| < \in) = 1$, we say y_n converges in probability to c which is equivalent to stochastic convergence.

[(1)] See reference 4.

1.3.4 Definition

If $P(\lim_{n\to\infty} y_n = c) = 1$ then we say y_n converges to c with probability 1 (w.p.1) or strong converges.

1.4. Limiting moment - generating functions

1.4.1. Definition

The moment generating function of a random variable denoted by $M(t)$and is defined for every real number t by$M(t) = E(e^{tx})$.

Remark:

$M^{(n)}(t) = E(x^n e^{tx})$ also $M^{(n)}(0) = E(x^n)$

$M'(0) = E(x)$, $M^{(2)}(0) = E(x^2)$

1.4.2. Taylor's theorem

Let $f(x)$ be a function on interval I such that $f^{(n)}(c)$ exists for some real c in I, let $R_n(x)$be the remainder for the n^th^ degree Taylor polynomial at c then $R_n(x)$ is continuous at c, that is $\lim_{n\to\infty} R_n(x) = R_n(c) = 0$

$R_n(x)$ = n^th^ degree Taylor polynomial of $f(x)$ at c

$$= \sum_{J=0}^{n} \frac{(x-c)^J f^{(J)}(c)}{J!}$$

$$R_n(x) = n! \frac{[f(x) - P_n(x)]}{(x-c)^n}, \quad x \neq c$$

$$f(x) = P_n(x) + \frac{(x-c)^n R_n(x)}{n!}$$

$$= f(c) + \ldots + \frac{(x-c)^{n-1}}{(n-1)!} f^{(n-1)}(c) + \frac{(x-c)^n}{n!} [f^{(n)}(c) + R_n(x)]$$

1.4.3. Theorem [1]

Let the random variable y_n has distribution function $F_n(y)$ and the moment-generating function $M(t;n)$ that exists for $-n<t<n$ for all n. If there exists a distribution function $F(y)$, with corresponding moment-generating function $M(t)$, defined for $|t|\le h_1\le h$, such that $\lim_{n\to\infty} M(t;n)=M(t)$ then y_n has a limiting distribution with distribution function $F(y)$.

Remark:

$$\lim_{n\to\infty}[1+\frac{b}{n}+\frac{\psi(n)}{n}]^{cn}=\lim_{n\to\infty}\left(1+\frac{b}{n}\right)^{cn}=e^{bc}$$

Where b and c do not depend on n and where $\lim_{n\to\infty}\psi(n)=0$

1.4.4. Example

$$\lim_{n\to\infty}\left(1-\frac{t^2}{n}+\frac{t^3}{n^{3/2}}\right)^{-n/2}=\lim_{n\to\infty}\left(1-\frac{t^2}{n}+\frac{t^3/\sqrt{n}}{n}\right)^{-n/2}=e^{t^2/2}$$

for every fixed value of t

here $b=-t^2$, $c=-1/2$ and $\psi(n)=\dfrac{t^3}{\sqrt{n}}$

1.4.5. Example

let $Z_n\sim\chi^2{}_{(n)}$, find the limiting distribution of $y_n=\dfrac{(Z_n-n)}{\sqrt{2n}}$.

$$Z_n\sim\chi^2{}_{(n)} \Rightarrow M_{z_n}(t,n)=(1-2t)^{-n/2} \quad , \quad t<\frac{1}{2}$$

we find $M_{y_n}(t,n)=M(t,n)=E(e^{\left(\frac{1}{\sqrt{2n}}(Z_n-n)t\right)})=e^{-nt/\sqrt{2n}}E(e^{(1/\sqrt{2n})Z_nt})$

$$=e^{\left(-(t\sqrt{2/n}\frac{n}{2})\right)(1-\frac{2t}{\sqrt{2n}})^{-n/2}} \quad , \; t<\frac{\sqrt{2n}}{2}$$

$$=$$

$$(e^{t\sqrt{2/n}})^{-n/2}(1-\frac{2t}{\sqrt{2n}})^{-n/2}=\left(e^{t\sqrt{2/n}}-t\sqrt{\frac{2}{n}}e^{t\sqrt{2/n}}\right)^{-n/2},\ t<\sqrt{n/2}.$$

In accordance with Taylor's formula, there exists a number $C(n)$ between 0 and $t\sqrt{2/n}$ such that $e^{t\sqrt{2/n}}=1+t\sqrt{2/n}+\frac{1}{2}(t\sqrt{2/n})^2+\frac{e^{C(n)}}{6}(t\sqrt{2/n})^3$

Substituted we get:

$$M(t,n)=(1-\frac{t^2}{n}+\frac{\psi(n)}{n})^{-n/2}$$

where $\psi(n)=\frac{\sqrt{2}t^3}{3\sqrt{n}}e^{C(n)}-\sqrt{\frac{2}{n}}t^3-\frac{2t^4e^{C(n)}}{3n}$

since $C(n)\to 0$ as $n\to\infty$ then $\lim_{n\to\infty}\psi(n)=0$, for every fixed value of t

hence $\lim_{n\to\infty}M(t,n)=e^{t^2/2}$

so $y_n=(Z_n-n)/\sqrt{2n}\xrightarrow{D}N(0,1)$.

1.5. Central limit theorem (C.L.T)

1.5.1. Theorem

Let $x_1,x_2,\ldots,x_n$ denote the item of a random sample from a distribution that has mean μ and positive variance σ^2, then $y_n=(\sum_1^n x_i-n\mu)/\sqrt{n}\sigma=\sqrt{n}(\overline{x_n}-\mu)/\sigma$ has a limiting distribution that is $N(0,1)$

$$\left(\sqrt{n}\left(\overline{x}_n-\mu\right)/\sigma\xrightarrow{D}N(0,1)\right)$$

Proof:

Let $M(t)=E(e^{tx})$, -h< t< h exists.

The function $m(t) = E(e^{t(x-\mu)}) = e^{-\mu t} M(t)$ also exists for –h < t < h, since m(t) is the moment generating function for $(x-\mu)$

m(0) = 1, $m'(0) = E(x-\mu) = 0$ and $m''(0) = E[(x-\mu)^2] = \sigma^2$.

By Taylor's formula there exists a number C between 0 and t such that $m(t) = m(0) + m'(0)t + \frac{m''(c)t^2}{2}$

$$= 1 + \frac{m''(c)t^2}{2} = 1 + \frac{\sigma^2 t^2}{2} + \frac{m''(c)t^2}{2} - \frac{\sigma^2 t^2}{2}$$

$$= 1 + \frac{\sigma^2 t^2}{2} + \frac{m''(c) - \sigma^2}{2} t^2$$

consider $M(t,n) = E(e^{t\left(\frac{2x_i - n\mu}{\sigma\sqrt{n}}\right)} = \left[E(e^{\left(t\frac{x-\mu}{\sigma\sqrt{n}}\right)})\right]^n = \left[m\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$,

$-h < \frac{t}{\sigma\sqrt{n}} < h$.

In m(t) replace t by $\frac{t}{\sigma\sqrt{n}}$ to obtain

$m\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \frac{t^2}{2n} + \frac{\left[m''(c) - \sigma^2\right] \quad t^2}{2n\sigma^2}$ where $0 < c < \frac{t}{\sigma\sqrt{n}}$ with $-h\sigma\sqrt{n} < t < h\sigma\sqrt{n}$

Accordingly,

$$M(t,n) = \left\{1 + \frac{t^2}{2n} + \frac{[m''(c) - \sigma^2]t^2}{2n\sigma^2}\right\}^n$$

since $m''(t)$ is continuous at $t = 0$ and $c \to 0$ as $n \to \infty$

we have $\lim_{n\to\infty}\left[m''(c) - \sigma^2\right] = 0$

so $\lim_{n\to\infty}(M(t,n)) = e^{t^2/2}$ for all values of t.

So $y_n = \sqrt{n}(\bar{x}_n - \mu)/\sigma \xrightarrow{D} N(0,1)$

Remark:

On the central limit theorem (C.L.T)

$\sqrt{n}(\bar{x}_n - \mu) \xrightarrow{D} N(0,\sigma^2)$

1.5.2. Example

If $x_1, x_2, \ldots, x_n$ are bin(r, p) and $S_n = x_1 + x_2 \ldots + x_n$

Find the asymptotic distribution of $\dfrac{S_n - nrp}{\sqrt{nrpq}}$

$E(S_n) = nrp$, $Var(S_n) = nrpq$, $0 < p < 1$

Hence, by C.L.T $\dfrac{S_n - nrp}{\sqrt{nrpq}} \xrightarrow{D} N(0,1)$ and $\sqrt{n}(\bar{x}_n - rp) \xrightarrow{D} N(0, rpq)$

1.6. Some theorems on convergence

1.6.1. Theorem

Let u_n converge stochastically to c, if h(u) is a continuous function at

u = c, then $h(u_n)$ converges stochastically to h(c).

Proof:

Since h(u) is continuous at c, then for each $\in > 0$

$\exists \delta > 0$ Such that if $|u - c| < \delta$ => $|h(u) - h(c)| < \in$

Then if $|h(u) - h(c)| < \in$ => $|u - c| < \delta$

So $\lim_{n\to\infty} P(|h(u_n) - h(c)| > \in) \leq \lim_{n\to\infty} P(|u_n - c| > \delta) = 0$

So $0 \leq \lim_{n\to\infty} P(|h(u_n) - h(c)| > \in) \leq 0$

=> $\lim_{n\to\infty} P(|h(u_n)-h(c)|>\in)=0$, $\forall \in>0$.

1.6.2. Theorem

Let u_n converges stochastically to c, then u_n/c converges stochastically to 1.

Proof:

Let h(u) = u/c then h(u) cont.

=> $h(u_n)=u_n/c \xrightarrow{S} h(c)=c/c=1$.

Note that: $u_n \xrightarrow{S}$ u means that un converges stochastically to u.

1.6.3. Result

Let $u_n \xrightarrow{S} c$ and let $\underline{P(u_n<0)=0}$, $\forall n$

And c>0

Then $\sqrt{u_n} \xrightarrow{S} \sqrt{c}$.

1.6.4. Theorem: (Cramer) [9]

If $\sqrt{n}(X_n-a) \xrightarrow{D} X$ and let g(x) be a function which is differentiable at $x=a$ then,

$\sqrt{n}(g(x_n)-g(a)) \xrightarrow{D} g'(a)X$.

In particular if $\sqrt{n}(x_n-a) \xrightarrow{D} N(0,\sigma^2)$

=> $\sqrt{n}[g(x_n)-g(a)] \xrightarrow{D} N(0,(g'(a))^2\sigma^2)$.

1.6.5. Example

By C.L.T, $\sqrt{n}(\bar{x}_n-\mu) \xrightarrow{D} N(0,\sigma^2)$

What is the asymptotic distribution of $\bar{x}_n^2$.

Solution: let $g(x) = x^2$ => $g'(x) = 2x$ => $g'(\mu) = 2\mu$

$$=> \sqrt{n}\left(\bar{x}_n^2 - \mu^2\right) \xrightarrow{D} N(0, 4\mu^2\sigma^2).$$

Chapter Two

The Exact and Asymptotic Parameters of the First Canonical Weight Vector

2.1. Canonical correlations

We shall be interested in measures of association between two groups of variables. The first group of p variables is represented by the $(p\times1)$ random vector $X^{(1)}$ the second group of q variables is represented by the $(q\times1)$ random vector $X^{(2)}$ and let $p \le q$ also,

$$E(X^{(1)}) = \mu^{(1)} \qquad \mathrm{COV}(X^{(1)}) = \sum_{11}$$

$$E(X^{(2)}) = \mu^{(2)} \qquad \mathrm{COV}(X^{(2)}) = \sum_{22}$$

$$\mathrm{COV}(X^{(1)}, X^{(2)}) = \sum_{12} = \sum_{21}$$

Let

$$X_{((p+q)\times1)} = \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix} = \begin{bmatrix} X_1^{(1)} \\ \cdot \\ \cdot \\ X_p^{(1)} \\ \hline X_1^{(2)} \\ \cdot \\ \cdot \\ X_q^{(2)} \end{bmatrix}$$

So,

$$\mu_{((p+q)\times1)} = E(X) = \begin{bmatrix} E(X^{(1)}) \\ \hline E(X^{(2)}) \end{bmatrix} = \begin{bmatrix} \mu^{(1)} \\ \hline \mu^{(2)} \end{bmatrix}$$

And the covariance matrix

$$\sum_{(p+q)\times(p+q)} = E[(X-\mu)(X-\mu\ )^T]$$

$$\begin{bmatrix} E(X^{(1)}-\mu^{(1)})(X^{(1)}-\mu^{(1)})^T & E(X^{(1)}-\mu^{(1)})(X^{(2)}-\mu^{(2)})^T \\ E(X^{(2)}-\mu^{(2)})(X^{(1)}-\mu^{(1)})^T & E(X^{(2)}-\mu^{(2)})(X^{(2)}-\mu^{(2)})^T \end{bmatrix} = \begin{bmatrix} \sum_{11(p\times p)} & \sum_{12(p\times q)} \\ \sum_{21(q\times p)} & \sum_{22(q\times q)} \end{bmatrix}$$

Now let $U = a^T X^{(1)}$, $V = b^T X^{(2)}$ then

$$Var(U) = a^T Cov(X^{(1)})a = a^T \sum_{11} a$$

$$Var(V) = b^T Cov(X^{(2)})b = b^T \sum_{22} b$$

$$Cov(U,V) = a^T Cov(X^{(1)}, X^{(2)})b = a^T \sum_{12} b$$

We shall seek coefficient vectors a & b such that

$$Corr(U,V) = \frac{a^T \sum_{12} b}{\sqrt{a^T \sum_{11} a}\sqrt{b^T \sum_{22} b}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

is as large as possible.

2.1.1. Definition

The first pair of canonical variables are the pair of linear combinations U_1, V_1 having unit variances which maximize the correlation (1).

The second pair of canonical variables are the pair of linear combinations U_2, V_2 having unit variances which maximize the correlation (1). Among all choices which are uncorrelated with the first pair of canonical variables.

at the k^{th} step.

The k^{th} pair of canonical variable are the linear combinations U_k, V_k having unit variance, which maximize the correlation (1) among all choices which are uncorrelated with the previous (k-1) canonical variable pairs.

The correlation between the k^{th} pair of canonical variable is called the k^{th} canonical correlation.

2.1.2. Result

Suppose $p \le q$ and let the random vectors $X^{(1)}_{p\times 1}$ and $X^{(2)}_{q\times 1}$ have

$Cov(X^{(1)}) = \sum_{11(p\times p)}$, $Cov(X^{(2)}) = \sum_{22(p\times q)}$

& $Cov(X^{(1)}, X^{(2)}) = \sum_{12(p\times q)}$

for coefficient vectors $a_{(p\times 1)}$ & $b_{(q\times 1)}$, form the linear combinations

$U = a^T X^{(1)}$ and $V = b^T X^{(2)}$ then

$$\max_{a,b} \; Corr(U,V) = \rho_1^*$$

attained by the linear combination (first canonical variate pair)

$U_1 = e_1^T \sum_{11}^{-1/2} X^{(1)}$ and $V_1 = f_1^T \sum_{22}^{-1/2} X^{(2)}$

the kth pair of canonical variates k = 2,3...p

$U_k = e_k^T \sum_{11}^{-1/2} X^{(1)}$ and $V_k = f_k^T \sum_{22}^{-1/2} X^{(2)}$

maximizes

$$Corr(U_k, V_k) = \rho_k^*$$

Among those linear combinations uncorrelated with the preceding 1,2,...k-1 , the canonical variables [$\left(\rho_1^{*2} \ge \rho_2^{*2} \ge .... \ge \rho_p^{*2}\right)$ are the Eigen values of $\sum_{11}^{-1/2}\sum_{12}\sum_{22}^{-1}\sum_{21}\sum_{11}^{-1/2}$] & $e_1,...,e_p$ are the associated (p×1) eigen vectors.

(The quantities ρ_1^* , ρ_2^* ,....., ρ_p^* are also the p largest eigen values of the matrix $\sum_{22}^{-1/2}\sum_{21}\sum_{11}^{-1}\sum_{12}\sum_{22}^{-1/2}$ with corresponding (q×1) eigen vector $f_1,...f_p$ each f_i is proportional to $\sum_{22}^{-1/2}\sum_{21}\sum_{11}^{-1/2} e_i$).

The canonical variates have the properties:

$Var(U_k) = var(V_k) = 1$

$Cov(U_k,U_L) = Corr(U_k,U_L) = 0$, $k \neq L$

$Cov(V_k,V_L) = Corr(V_k,V_L) = 0$, $k \neq L$

$Cov(U_k,V_L) = Corr(U_k,V_L) = 0$, $k \neq L$

Where k,L = 1,2,…p.

Proof:

We assume that $\sum_{11}$ & $\sum_{22}$ are non singular (if $\sum_{11}$ or $\sum_{22}$ is singular, variables(s) may be deleted from the appropriate set, and the linear combinations $a^T X^{(1)}$ and $b'X^{(2)}$ can be expressed in terms of the reduced set. The reduced set has a non singular covariance matrix).

Introduce the symmetric square-root matrices $\sum_{11}^{1/2}$ and $\sum_{22}^{1/2}$ with $\sum_{11}=\sum_{11}^{1/2}\sum_{11}^{1/2}$, $\sum_{11}^{-1}=\sum_{11}^{-1/2}\sum_{11}^{-1/2}$

Let $c = \sum_{11}^{1/2} a$, $d = \sum_{22}^{1/2} b$ => $a = \sum_{11}^{-1/2} c$ and $b = \sum_{22}^{-1/2} d$, then

$$Corr(a^T X^{(1)}, b^T X^{(2)}) = \frac{a^T \sum_{12} b}{\sqrt{a^T \sum_{11} a}\sqrt{b^T \sum_{22} b}} = \frac{c^T \sum_{11}^{-1/2} \sum_{12} \sum_{22}^{-1/2} d}{\sqrt{c^T c}\sqrt{d^T d}} \text{…….(2)}$$

By the Cauchy-Schawarz inequality [1]

$$c^T \sum_{11}^{-1/2} \sum_{12} \sum_{22}^{-1/2} d \leq \left(c^T \sum_{11}^{-1/2} \sum_{12} \sum_{22}^{-1} \sum_{21} \sum_{11}^{-1/2} c\right)^{1/2} \left(d^T d\right)^{1/2} \text{……..…(3)}$$

Since is a (p×p) symmetric matrix, then by maximization[2] yields

$$c^T \sum_{11}^{-1/2} \sum_{12} \sum_{22}^{-1} \sum_{21} \sum_{11}^{-1/2} c \leq \lambda_1 c^T c \text{ ……………….. (4)}$$

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(1) See reference 5.
(2) See reference 5.

Where λ_1 is the largest eigen value of $\sum_{11}^{-1/2}\sum_{12}\sum_{22}^{-1}\sum_{21}\sum_{11}^{-1/2}$

(Equality occurs in (4) for c = e_1, a normalized eigen value associated with λ_1, equality also holds in (3) if *d* is proportional to $\sum_{22}^{-1/2}\sum_{21}\sum_{11}^{-1/2}e_1$.

$$\text{Thus} \qquad \max_{a,b}\ Corr\left(a^T X^{(1)}, b^T X^{(2)}\right) = \sqrt{\lambda_1}\ \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

with equality occurring for $a = \sum_{11}^{-1/2} c = \sum_{11}^{-1/2} e_1$ & with *b* proportional to $\sum_{22}^{-1/2}\sum_{22}^{-1/2}\sum_{21}\sum_{11}^{-1/2}e_1$, where the sign is selected to give positive correlation. We take $b = \sum_{22}^{-1/2} f_1$ this last correspondence follows by multiplying both sides of

$$\left(\sum_{11}^{-1/2}\sum_{12}\sum_{22}^{-1}\sum_{21}\sum_{11}^{-1/2}\right) e_1 = \lambda_1 e_1$$

By $\sum_{22}^{-1/2}\sum_{11}\sum_{11}^{-1/2}$ yielding

$$\sum_{22}^{-1/2}\sum_{21}\sum_{11}^{-1}\sum_{12}\sum_{22}^{-1/2}\left(\sum_{22}^{-1/2}\sum_{21}\sum_{11}^{-1/2}e_1\right) = \lambda_1\left(\sum_{22}^{-1/2}\sum_{21}\sum_{11}^{-1/2} e_1\right) \ldots\ldots\ldots\ldots (6)$$

Thus if (λ_1, e_1) is an eigen value-eigen vector pair for $\sum_{11}^{-1/2}\sum_{12}\sum_{22}^{-1}\sum_{21}\sum_{11}^{-1/2}$ or $\sum_{11}^{-1}\sum_{12}\sum_{22}^{-1}\sum_{21}$.

(λ_1, f_1) with f_1 the normalized form of $\sum_{22}^{-1/2}\sum_{21}\sum_{11}^{-1/2}e_1$ is an eigen value-eigen Vector pair for $\sum_{22}^{-1/2}\sum_{21}\sum_{11}^{-1}\sum_{12}\sum_{22}^{-1/2}$.

The sign for f_1 is chosen to give a positive correlation. We have demonstrated that $U_1 = e_1^T \sum_{11}^{-1/2} X^{(1)}$ and $V_1 = f_1^T \sum_{22}^{-1/2} X^{(2)}$ are the first pair of canonical variable and their correlation is $\rho_1^* = \sqrt{\lambda_1}$.

Also,

$$Var(U_1) = e_1^T \sum_{11}^{-1/2}\sum_{11}\sum_{11}^{-1/2} e_1 = e_1^T e = 1$$

Similarly,

$Var(V_1) = f_1^T \sum_{22}^{-1/2} \sum_{22} \sum_{22}^{-1} f_1 = f_1^T f_1 = 1$.

Continuing, we note that U_1 and V_1 an arbitrary linear combination $a^T X^{(1)} = c^T \sum_{11}^{-1/2} X^{(1)}$ are uncorrelated if

$0 = Cov\left(U_1, c^T \sum_{11}^{-1/2} X^{(1)}\right) = e_1^T c = e_1^T c$ or e1 $\perp$ c.

At the k^th^ stage we require c $\perp$ $e_1, e_2 \ldots e_{k-1}$

So by maximization yields:

$c^T \sum_{11}^{-1/2} \sum_{12} \sum_{22}^{-1} \sum_{21} \sum_{11}^{-1/2} c \le \lambda_k c^T c$ for c $\perp$ $e_1, e_2 \ldots e_{k-1}$.

By (3):

$$Corr\,(a^T X^{(1)}, b^T X^{(2)}) = \frac{c^T \sum_{11}^{-1/2} \sum_{12} \sum_{22}^{-1/2} d}{\sqrt{c'c}\sqrt{d'd}} \le$$

$$\frac{\left(c^T \sum_{11}^{-1/2} \sum_{12} \sum_{22}^{-1} \sum_{21} \sum_{11}^{-1/2} c\right)^{1/2} \left(d^T d\right)^{1/2}}{\sqrt{c^T c}\sqrt{d^T d}} \le \frac{\left(\lambda_k c^T c\right)^{1/2}}{\sqrt{c^T c}} = \sqrt{\lambda_k}$$

With equality for $c = e_k$ or $a = \sum_{11}^{-1/2} e_k$ and $b = \sum_{22}^{-1/2} f_k$

Thus, $U_k = e_k^T \sum_{11}^{-1/2} X^{(1)}$ and $V_k = f_k^T \sum_{22}^{-1/2} X^{(2)}$ are the kth canonical pair, & have correlation $\sqrt{\lambda_k} = \rho_k^*$.

Now,

$$COV\left(U_K, U_L\right) = e_k^T \sum_{11}^{-1/2} \sum_{11} \sum_{11}^{-1/2} e_L = e_k^T e_L = 0 \quad \text{if } k \ne L \le p$$

$$COV\left(V_K, V_L\right) = f_k^T \sum_{22}^{-1/2} \sum_{22} \sum_{22}^{-1/2} f_L = f_k^T f_L = 0 \quad \text{if } k \ne L \le p$$

$$COV\left(U_K, V_L\right) = e_k^T \sum_{11}^{-1/2} \sum_{12} \sum_{22}^{-1/2} f_L = 0 \quad \text{if } k \ne L \le p$$

Since f_L^T is a multiple of $e_L^T \sum_{11}^{-1/2} \sum_{12} \sum_{22}^{-1/2}$.

2.2. Gamma distribution

2.2.1. Definition

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \quad , x > 0$$

This function is called Gamma function and its domain is the set of positive real number.

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx = (n-1)! = (n-1)\Gamma(n-1)$$

2.2.2. Definition

A random variable X is said to have a Gamma distribution if

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^\alpha} (x)^{\alpha-1} e^{-x/\beta} & x \geq 0 \\ 0 & otherwise \end{cases}$$

Now we find the expectation & the variance of the Gamma distribution…

$$M(t) = E(e^{tx}) = \int_0^\infty \frac{1x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-x/\beta} e^{tx} dx$$

$$= \int_0^\infty \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-x\left(\frac{1}{\beta}-t\right)} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1-\beta t}{\beta}\right)} dx$$

let $y = x\left(\frac{1-\beta t}{\beta}\right) \Rightarrow dy = \left(\frac{1-\beta t}{\beta}\right) dx$

and $x = \left(\frac{\beta}{1-\beta t}\right) y$

$$\therefore\ M(t)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\int_0^{\infty}\left(\frac{\beta}{1-\beta t}\right)^{\alpha-1}\left(\frac{\beta}{1-\beta t}\right)e^{-y}dy$$

$$=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\left(\frac{\beta}{1-\beta t}\right)^{\alpha}\int_0^{\infty}y^{\alpha-1}e^{-y}dy$$

But $\int_0^{\infty}y^{\alpha-1}e^{-y}dy=\Gamma(\alpha)$

$$\therefore\ M(t)=\left(\frac{1}{1-\beta t}\right)^{\alpha}=(1-\beta t)^{-\alpha}$$

$$M'(t)=-\alpha(-\beta)(1-\beta t)^{-\alpha-1}=\alpha\beta(1-\beta t)^{-(\alpha+1)}$$

$$M'(0)=\alpha\beta=EX$$

$$EX^2=M''(0)$$

$$M''(t)=-(\alpha+1)(-\beta)(1-\beta t)^{-(\alpha+1)-1}\alpha\beta$$

$$M''(0)=\beta^2\alpha(\alpha+1)$$

$$\therefore\ Var(X)=EX^2-(EX)^2$$

$$=\beta^2\alpha(\alpha+1)-\alpha^2\beta^2=\beta^2\alpha(\alpha+1-\alpha)=\beta^2\alpha\,.$$

2.2.3 Definition

A random variable X is said to have chi-square distribution ($X\sim\chi^2(n)$) with n degrees of freedom if

$$f(x)=\begin{cases}\dfrac{1}{\Gamma(n/2)2^{n/2}}(x)^{n/2-1}e^{-x/2} & x\geq 0\\ 0 & otherwise\end{cases}$$

Remark:

$\chi^2(n)$ is gamma distribution with $\alpha = \frac{n}{2}$ and $\beta = 2$ so $E(\chi) = \alpha\beta = n$ and $\mathrm{var}(\chi) = \alpha\beta^2 = 2n$.

2.2.4. Theorem [1]

Let $x_1, x_2 \ldots x_n$ be a random sample from a distribution that is $N(\mu, \sigma^2)$ then:

a. $\bar{x}$ is $N\left(\mu, \frac{\sigma^2}{n}\right)$

b. $\frac{ns^2}{\sigma^2}$ is $\chi^2_{(n-1)}$

c. $\bar{x}$ and S^2 are independent random variable where $\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$, and $S^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n}$

2.3. The exact and asymptotic parameters of the first canonical weight vector where the rank of ($\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$) = 1

Let $$X = \begin{pmatrix} x_{11} & x_{12} & . & x_{1n} \\ . & . & . & . \\ . & . & . & . \\ x_{p1} & x_{p2} & . & x_{pn} \end{pmatrix}$$

Then sample mean

$$\bar{X} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_1 \\ . \\ \bar{x}_p \end{bmatrix}, \quad \bar{x}_i = \frac{1}{n}\sum_{j=1}^{n} x_{ij} \text{ , i =1,2,}\ldots\ldots\ldots\ldots\text{p.}$$

Sample variance and covariance

$$S_n = \begin{bmatrix} s_{11} & . & s_{1p} \\ . & . & . \\ s_{p1} & . & s_{pp} \end{bmatrix}$$

$$S_{ik} = \frac{1}{n}\sum_{j=1}^{n}\left(x_{ij} - \overline{x_i}\right)\left(x_{kj} - \overline{x}_k\right)$$, i, k =1,2,………..p.

Suppose that p + q variates

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N_{P+q}\left\{\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \sum = \begin{bmatrix} \sum_{11} & \sum_{12} \\ \sum_{21} & \sum_{22} \end{bmatrix}\right\}$$

and that from this normal population n observation vectors have been randomly drawn and their covariance matrix has been partitioned as

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$$

The stability of the canonical weight vectors ∂ resulting from the solution to the homogeneous linear equations

$$\left(s_{12}s_{22}^{-1}s_{21} - \lambda s_{11}\right)\partial = 0$$

Is the focus of this study? This study encompasses one underlying population structure where the matrix ($\sum_{11}^{-1}\sum_{12}\sum_{22}^{-1}\sum_{21}$) assumes a unit rank.

A more elaborate and scientific assessment of the one-factor weight stability is the establishment of a concrete and theoretical formulation of the weight's variance in question. Central to this formulation is the derivation of the exact variance and the asymptotic distribution of the weight.

Suppose that we have a one-factor population structure and that the observation vectors have been transformed so that their distribution has mean vector $\mu = \begin{bmatrix} 0 \\ \cdot \\ 0 \end{bmatrix}$ and variance-covariance matrix $\sum^{*} = \begin{bmatrix} I_p & \Delta \\ \Delta^T & I_q \end{bmatrix}$,

$$\Delta = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 0 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & 0 \end{bmatrix}_{p \times q} ,I_n = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & 1 \end{bmatrix}_{nxn}$$

The one-factor solution resulting from the canonical correlation analysis on $\sum^{*}$ is $\rho_1 = 1$ and its corresponding canonical weight vector is $A_1 = \begin{bmatrix} 1 \\ . \\ 0 \\ 0 \end{bmatrix}$.

The sample estimate of A_i resulting from the canonical correlation analysis on S^* the estimate of $\sum^{*}$ is $\partial_1^* = \begin{bmatrix} \partial_{11} \\ \varepsilon_1 \\ . \\ \varepsilon_p \end{bmatrix}$ where $\varepsilon_i, i = 1, ...p$ are infinitesimal. But since $\partial_1^{*T} S_{11}^{*} \partial_1^{*} = 1$, then $\partial_{11}^{*\,2} .s^2 = 1$ or $\partial_{11}^{*} = \frac{1}{s}$ where s is the standard deviation of the sample drawn. Knowing that $(n-1)s^2 \sim \chi^2_{(n-1)}$, it is of interest to derive the exact variance and the asymptotic distribution of $\frac{1}{s}$ as a theoretical evidence for assessing the stability of the weight of the first variable on the first canonical variate.

2.3.A. Derivation of the exact formula for Var (1/s):

Consider $w = (n-1)s^2 \sim \chi^2_{(n-1)}$ so $\frac{1}{s} = \left(\frac{n-1}{w}\right)^{1/2}$

$$E\left(\frac{1}{s}\right) = E\left(\frac{n-1}{w}\right)^{\frac{1}{2}} = E\left(\frac{1}{w^{1/2}}\right)\sqrt{n-1}$$

$$= \sqrt{n-1}\int_0^\infty \frac{1}{w^{1/2}} \frac{1}{2^{\left(\frac{n-1}{2}\right)}\Gamma\left(\frac{n-1}{2}\right)} w^{\left(\frac{n-1}{2}\right)-1} e^{-w/2} dw$$

$$= \frac{\sqrt{n-1}}{2^{\left(\frac{n-1}{2}\right)}\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty w w^{-3/2} w^{\left(\frac{n-1}{2}\right)-1} e^{-w/2} dw$$

$$= \frac{\sqrt{n-1}}{2^{\left(\frac{n-1}{2}\right)}\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty w w^{\left(\frac{n-4}{2}\right)} e^{-w/2} dw \left[\frac{\Gamma\left(\frac{n-4}{2}\right)2^{\left(\frac{n-4}{2}\right)}}{\Gamma\left(\frac{n-4}{2}\right)2^{\left(\frac{n-4}{2}\right)}}\right]$$

$$= \frac{\sqrt{n-1}\ \ \Gamma\left(\frac{n-4}{2}\right)\ \ 2^{\left(\frac{n-4}{2}\right)}}{2^{\left(\frac{n-4}{2}\right)}\Gamma\left(\frac{n-1}{2}\right)} \quad E\left(w^*\right)$$

Where $w^* \sim \chi^2(n-4)$

So $$E\left(\frac{1}{s}\right) = \frac{\sqrt{n-1}\ \ \Gamma\left(\frac{n-4}{2}\right)\ \ 2^{\left(\frac{n-4}{2}\right)}\ \ (n-4)}{\Gamma\left(\frac{n-1}{2}\right)\ \ 2^{\left(\frac{n-1}{2}\right)}}$$

$$= \sqrt{\frac{n-1}{2}} \; \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

Similarly,

$$E\left(\frac{1}{s^2}\right) = (n-1)E\left(\frac{1}{w}\right)$$

$$= (n-1)\int_0^\infty \frac{1}{w} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)2^{\left(\frac{n-1}{2}\right)}} w^{\left(\frac{n-1}{2}\right)-1} e^{-w/2} dw$$

$$= \frac{(n-1)}{2^{\left(\frac{n-1}{2}\right)}\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty w^{\left(\frac{n-3}{2}\right)-1} e^{-w/2} dw$$

$$= \frac{(n-1)}{2} \frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \frac{\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-3}{2}\right)}{\left(\frac{n-3}{2}\right)\Gamma\left(\frac{n-3}{2}\right)} = \frac{n-1}{n-3}$$

so Var $\left(\frac{1}{s}\right) = \mathrm{E}\left(\frac{1}{s}\right)^2 - \left(E\left(\frac{1}{s}\right)\right)^2$

$$\mathrm{Var}\left(\frac{1}{s}\right) = \left(\frac{n-1}{n-3}\right) - \left(\sqrt{\frac{n-1}{2}} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right)^2$$

2.3.B. Derivation of the asymptotic distribution of $\frac{1}{s}$

The following result and its proof are essential to the asymptotic distribution of $\frac{1}{s}$.

2.3. B1 Lemma

let $X_n \sim \chi^2_{(n)}$ then the limiting distribution of the random variable $Y_n = \frac{(X_n - \mu)}{\sigma}$ is N(0,1), where the mean $(\mu = n)$ and the variance $(\sigma = \sqrt{2n}\,)$.

Proof:

The moment-generating function of Y_n is

$$M(t;n) = E\left\{\exp\left[t\left(\frac{X_n - n}{\sqrt{2n}}\right)\right]\right\}$$

$$= \exp\left(-\frac{tn}{\sqrt{2n}}\right)E\left(\exp\left(\frac{tX_n}{\sqrt{2n}}\right)\right)$$

$$= \exp\left[-t\sqrt{\frac{2}{n}}\left(\frac{n}{2}\right)\right]\left(1-2\frac{t}{\sqrt{2n}}\right)^{-\frac{n}{2}}, \; t < \frac{\sqrt{2n}}{2}$$

The expression for the moment-generating function of Y_n can be written as

$$M(t;n) = \left(\exp\left[t\sqrt{\frac{2}{n}}\right] - t\sqrt{\frac{2}{n}}\exp\left[t\sqrt{\frac{2}{n}}\right]\right)^{-n/2}, \quad t < \sqrt{\frac{n}{2}}$$

By using the Taylor expansion, there exists a number *C(n)* between 0 and $t\sqrt{2/n}$, such that

$$\exp\left[t\sqrt{\frac{2}{n}}\right] = 1 + t\sqrt{\frac{2}{n}} + \frac{1}{2}\left(t\sqrt{\frac{2}{n}}\right)^2 + \frac{e^{C(n)}}{6}\left(t\sqrt{\frac{2}{n}}\right)^3$$

Substituting this expression for $e^{t\sqrt{\frac{2}{n}}}$ in the last expression for M(t;n), it is seen that

$$M(t;n) = \left(1 - \frac{t^2}{n} + \frac{R(n)}{n}\right)^{-n/2}$$

Where

$$R(n) = \frac{\sqrt{2}\ \ t^3\, e^{C(n)}}{3\sqrt{n}} - \frac{\sqrt{2}\ \ t^3}{\sqrt{n}} - \frac{2t^4 e^{C(n)}}{3n}$$

Since $C(n) \to 0$ as n→ ∞, then $\lim_{n\to\infty} R(n) = 0$ for every fixed value of *t*.

Therefore,

$\lim_{n\to\infty} M(t;n) = e^{t^2/2}$ which implies that $Y_n = \dfrac{X_n - n}{\sqrt{2n}}$ has a limiting normal distribution with mean zero and variance 1.

By using lemma 2.3.B1 and the fact that $(n-1)s^2 \sim \chi^2_{(n-1)}$ implies

$$\frac{1}{\sqrt{2(n-1)}}\left[(n-1)s^2 - (n-1)\right] \sim N(0,1)$$

therefore,

$$\frac{(n-1)}{\sqrt{2(n-1)}}\left(s^2 - 1\right) \sim N(0,1)$$

Or,

$$\left(s^2 - 1\right) \sim N\left(0, \frac{2}{n-1}\right)$$

let $f(s^2) = \frac{1}{s}$ where the taylor series expansion exist and hence the conditions for the δ-method are satisfied ,

Hence $\dfrac{f(s^2) - f(1)}{f^{'}(1)} \sim N\left(0, \dfrac{2}{n-1}\right)$

Or $f(s^2) \quad \sim N\left(f(1), \frac{2}{n-1}\left[f^{'}(1)\right]^2 \right)$

But s. $f(s^2)$=1/s ;

Let s^2=x

Then $\sqrt{x} \,.\, f(x)$ =1

And $\sqrt{x} \,.\, f^{'}(x)x^{'} + f(x)\,\frac{1}{2\sqrt{x}}x^{'} = 0$

So $\sqrt{x}\; f^{'}(x)$= - $f(x)\,\frac{1}{2\sqrt{x}}$

Or $f^{'}(x)$= - $\frac{f(x)}{2x}$

Now let $g(x)=\frac{1}{\sqrt{x}}$ => $g'(x)=\frac{-1}{2x^{3/2}}$

=> g'(1) = -1/2

$\therefore \quad (g'(1))^2 = \frac{1}{4}$

$(g'(1))^2 \quad \mathrm{var}(s^2)=\frac{1}{4}\left(\frac{2}{n-1}\right)=\frac{1}{2(n-1)}$

So by Cramer Theorem,

$\left(\frac{1}{s}-1\right)\xrightarrow{D} N\left(0, \frac{1}{2(n-1)}\right)$ => $\frac{1}{s} \sim N\left(1, \frac{1}{2(n-1)}\right)$

So the asymptotic variance of $\frac{1}{s}$ is $\frac{1}{2(n-1)}$

Although the stability of the one-factor structure can be explained parsimoniously on the grounds that almost all variation could be explained along a single continuum rather than in a multidimensional space, the above is a theoretical as well as a concrete empirical evidence of stability.

A look at the distribution of $\frac{1}{s}$ the sample weight of the one-factor structure reveals that $\frac{1}{s} \sim N\left(1, \frac{1}{2(n-1)}\right)$

Where the mean is one and the variance is inversely proportional to sample size. This points to the fact that the larger the sample, the more stable the prediction.

In this situation where the rank of **(** $\sum_{11}^{-1}\sum_{12}\sum_{22}^{-1}\sum_{21}$) =1 is sufficient to account for the variation between the variants, the weights are uniquely determined.

But when the rank is greater than 1 neither the varieties nor their weights are defined uniquely and hence instability arises .In this situation researchers resort to rotational techniques to get fair solution.

2.3.C. Comparison between the exact variances and the asymptotic variances of ($\frac{1}{s}$) .

n	**Exact variances of ($\frac{1}{s}$) =** $\left(\frac{n-1}{n-3}\right)$ - $\left(\sqrt{\frac{n-1}{2}}\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right)^2$	**Asymptotic variances of ($\frac{1}{s}$) =** $\left(\frac{1}{2(n-1)}\right)$
31	0.016627	0.0166667
62	0.0081664	0.0081967
124	0.004056	0.0040650
248	0.0020301	0.0020243
296	0.0010141	0.00010101
992	0.0005075	0.0005045
1984	0.0002524	0.0002521

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Appendices

جامعة النجاح الوطنية
كلية الدراسات العليا

المعلمات المنضبطة والتقاربية (Asymptotic) للوزن الاول بحيث ان رتبة المصفوفة ($\sum_{11}^{-1}\sum_{12}\sum_{22}^{-1}\sum_{21}$) تساوي واحد

إعداد
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إشراف
الدكتور فاروق السعد

قدمت هذه الاطروحة استكمالا لمتطلبات درجة الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس، فلسطين.

2003

المعلمات المنضبطة والتقاربية (Asymptotic) للوزن الاول بحيث ان رتبة المصفوفة
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الملخص

هدف هذا البحث هو شرح ثبات الوزن الاول (Canonical) المستخرج من بنية العامل الواحد. استخراج المعادلة المضبوطة لتباين الوزن والتوزيع التقاربي (Asymptotic) للوزن لتعطي قـوة نظرية لمبدأ الثبات. قرب التباين المضبوط والتقاربي يشير الى دقة الاستخلاص.

لقد برهن بان الوزن 1/s موزع طبيعي وسطه يساوي 1 وتباينه التقـاربي يسـاوي $\frac{1}{2(n-1)}$. فكلما كبر حجم العينة يقل تباين الوزن وبذلك يكون ثابتاً.