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Convexity , Fixed Point Theorems and Walrasian Equilibrium

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Submitted In Partial Fulfillment of the Requirements For The Degree of Master of Mathematics , Faculty of Graduate Studies , At An – Najah National University At Nablus , Palestine.

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Date : 3-2-2002

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Dedication

In loyalty and faithfulness to the souls of my parents, whose memory has been the torch that helped me. I present this work.

Acknowledgements

First of all, I thank my God for all the blessings , He bestowed on me and continues to bestow on me.

I would like to express my deep gratitude to my supervisor: Dr. Abdallah A. Hakawati , An-Najah National University for supervising this thesis and for his continuous help during my graduate education. I also wish to thank Dr.Rimon Y. Jadoun (BirZeit University) for his assistance in this project. No words are enough to express my appreciation for his patience, and continuous encouragement.

I also would like to express my best wishes and gratitude for Dr. Fathi M. Allan (Bir Zeit University) the external examiner, and Dr.Mohammad Najib Ass'ad, An-Najah National University, the internal examiner.

Special thanks to Dr. Omar Abdul-Raziq (MASS) for his help in finding references and suitable material , and for his valuable comments.

Finally ,I would like to express my utmost appreciation to my family, and my friends for all kinds of support, keen interest and concern.

Contents

Abstract

Abstract

In this thesis, I will deal with an application of fixed-point theorem of set valued map [Let X and Y be two subsets of R^n. A set-valued map F from X to Y, is a map that associates with any $x \in X$ a subset $F(x)$ of Y, A fixed point x for F exists if $x \in F(x)$], and convexity to prove the existence of Walrasian Equilibrium under sufficient conditions for both pure exchange economy and private ownership economy. Then I will show how to modify these theorems in more general cases under uncertainty and externalities.

Intr luction:

The goal of general equilibrium theory is to describe and explain economic phenomena pertaining to markets. The most remarkable of the phenomena are related to price, the only information shared among all economic agents. Prices have the suppressing power of enabling individual supplies and demands to coordinate so that resources in the market are allocate efficient ; that is, resources are neither in short supply, nor are they wasted.

Developing a rigorous analysis of this coordination of supply and demand requires the formalization of commodities, prices and economic agents. Thus we have to specify the consumption set X as a subset of, R^L_+ then identify the initial endowment of each consumer and find his best possible choice within his budget set. We assume that the preference relation of the consumer can be represented by a utility function .$u : X \to R$ Finding the appropriate commodity space X especially in case of exhaustible resources is one of the problems discussed in some detail by Geldrop and (Withager) [6] in their, 1999 paper on general equilibrium and resources economic.

Now, we will depend on two principles:

The optimization principle: People try to choose the best bundle of consumption set X_i , that they can afford given their wealth and price of commodities . Firm Yj tries to maximize its profit , and The equilibrium principle: Price adjusts until the amount that people demand of something is equal to the amount that is supplied .

The optimization principle can be written mathematically as:

i- for every j, y_j* maximizes profits in Y_j, that is,

$$p.y_j \leq p.y_j^* \text{ for all } y_j \in Y_j$$

ii- for every i, x_i* is maximal for $\succeq_i$ in the budget set

$$\{x_i \in X_i: p.xi \leq p.w_i + \Sigma_j \theta_{ij} P.y_j^*\}$$

iii- $\Sigma_i x^*_i = \bar{w} + \Sigma_j y_j^*$

where $x_i \in X_i$ the consumption set , $y_j \in Y_J$ the production set ,w_i the initial endowment , θ_{ij} the profit share.

To be able to construct a mathematical model, we need to introduce some microeconomic concepts and definitions , as a background for our work, thus, in chapter one, we state the definitions for preference relation , and its properties such as the convexity of preference relation . In general under this circumstance (convexity) , the demand is a set valued map and not a function as it is the case when we require strict convexity . We also define the consumption and production sets, and then explain both pure exchange economy and private ownership economy, and other definitions and concepts that we need in the next chapters.

In chapter two, we state some theorems and definitions about convexity to be used in the next chapters ,such as convex set[A subset A of R^n is said to be convex, let $x, y \in A$,all points of the form $\alpha x + (1-\alpha)y$ with $0 \leq \alpha \leq 1$ are in A] and separation theorem [let A and B be nonempty convex subset of R^n such that $A \cap B = \phi$ and suppose A is open. Then (1) there exists a closed hyperpalne H separating A and B (2) A lies strictly to one side of H. (3) if B is also open then it separates A and B strictly] .

In chapter three, we state the definition of the set valued map[Let X and Y be two subsets of R^n . A set-valued map F from X to Y, is a map that associates with any $x \in X$ a subset $F(x)$ of Y] providing examples. Also we provide the definitions of two types of set valued maps, the upper and lower semicontinuous maps , with illustrative examples. We have worked in simplifying the proof of some theorems related to the upper and lower semicontinuous set valued maps and provided our interpretation of these results.

At the end of chapter three, we deduce the proof of the fixed point theorem of the set valued map (Kakutani)- [Let A be a nonempty compact and convex subset of R^n and F be convex valued map of $A \to A$ which has a closed graph. Then there exists a fixed point x for F i.e. $x \in F(x)$]from Brower's fixed-point theorem for point to point map (function).

This thesis deals with the general equilibrium analysis as initiated by Walras ,Debreu and Arrow. The presentation depends mainly on the fixed-point theorem and its application to the excess demand set valued map($Z(p): P \to Z$ where $Z=X-\{W\}$. We define $Z(p)= \Sigma_i x^*_i - \Sigma_i \bar{w}$.

Then, in chapter four we prove the existence of Walrasian equilibrium in the case where the preferences of the consumers are rational ,convex and continuous on the consumption sets. We depend on the work of French researchers Oki Nomia (1996) .The proof consists of three steps : In the first step ,we define an auxiliary economy whose consumption ,production sets are bounded, and convex and the auxiliary economy has the same equilibrium prices as the original one; in the second one , we modify the fixed point theorem , which is used in the third part to show the existence of equilibrium in the auxiliary economy, and using the relation between quasiequilibirium and Walrasian equilibrium to reach the existence through out these steps.

Throughout, we emphasize how mathematical rigor can be used to prove (under certain conditions) the existence of Walrasian equilibrium.

Also , in chapter four, we prove mathematically, the properties of demand and supply set valued maps, to be used with the other lemmas and theorems , which we proved in this chapter , in the proof of the existence of Walrasian equilibrium for both pure exchange economy and private ownership economy.

In chapter five we provide the definition of Pareto efficiency, and we show the relationship between the Pareto efficiency (which touch the real world) and Walrasian equilibrium by providing the proof of the first and the second welfare theorems. Then we reach to the external effect to show the application on the real world. Thus we start from the most abstract concepts to end with the noticeable application in life .

Chapter 1

Preliminaries *

1.1Introduction:

In this chapter , we will introduce some concepts and definitions , that we shall use in the next chapters .We will start first with the history of general equilibrium theory since we will deal with the existence of two types of it . We will also illustrate some concepts and definitions such as commoditiy ,consumption set and production set.

We will study preference relation in some detail to show its relation with the utility function . In this chapter we study two types of economies the pure exchange economy and private ownership economy .

1.2 History of general equilibrium theory:

Classical economists had a strong notion of equilibrium. It deals with the idea of decentralization .The best – known description of how equilibrium is achieved is illustrated by Adam Smith's notation of an invisible hand. The nineteenth-century economists including Richard, Mill, Marx and Jevons all recognized a notation of stable equilibrium tendencies in the economy and the importance of the interaction among markets (general equilibrium) without formalizing these notations mathematically.

The supply and demand diagram generally presented for partial equilibrium analysis is known as Marshallion after the treatment of Alfred Marshall (1890).

*This chapter is taken from ch 1,2,3,4,5,15,16,17(Mas – Collell)

Cournot and others in the nineteenth century understood the partial equilibrium ,but they did not formulate a full general equilibrium model. The existence was first successfully taken by Leon Walras, a French economist at the school of Lausanne, Switzerland .His elegant comprehensive treatment appeared in elements of pure Economics in (1874).

Walras set the problem and principal research agenda for all of the twentieth century mathematical general equilibrium theory. The Walrasian model represented the first full recognition of the general equilibrium concept in the literature.

It stated that for N commodities there are N equations $S_i(p_1,\dots,p_n) = D_i(p_1,\dots,p_n)$ in the N unknowns p_i, to prove the existence .

F.Y Edgworth presented the field with new concepts and new tools to analyze them mathematically (1881) .

In the early 1950s, three American mathematicians, Kenneth Arrow, Gerend Deboreu and Lionel Mckenzie, entered the field ; The papers of Arrow and Deboreu and Mikenzie were presented to the 1952 meeting of Economics Society .They shared the same essential modeling insight : A fixed point theorem would lead to a general proof of existence of equilibrium . Additional contributions to the field in this period include Arrow (1951) , restating the essential ideas of Welfare economics in the language of general equilibrium theory ,and Arrow (1953) extending the notation of commodity to include allocation under uncertainty .The body of work was then summarized by Debreu (1959).

The next major step of the general equilibrium theory is the elaboration of the Edgeworth bargaining model in the contribution by Debreu and Scarf (1963).

The role of large numbers in competitive economy was confirmed mathematically by Arrow and Debreu who received Nobel prize in economics for their research in general equilibrium theory (1970) .

Later economists characterized equilibrium as limiting cases of other game theoretic solution concepts ,e .g with the set of fair net trades by David Schmeidler and Karl Vind (1972) , with the Shapley value by Robert J Aumann and Lloyd S . Shapley (1974) with the bargaining set by Andreu Mas-Colell (1989) .

Two approaches were developed in the 1980s ; one relying on "non- smooth " differentials (e . g .Cornet ,1982) and another involving " integral " activity analysis (e . g. Scarf ,1986).

The innovations continued , but much remains to be done in this field .

Now we want to state some concepts, and definitions that we will use in our work

1. 3 Preference Relation and Demand

Commodities :

The decision problem faced by the consumer in a market economy is to choose consumption levels of the various goods and services that are available for purchase in the market. We call these goods and services commodities. We assume for simplicity that the number of commodities is finite and equal L (index by $i=1,2,\ldots,L$).

A commodity vector (bundle) is a list of amounts of different commodities $x=[x_1,\ldots,x_L]$ and can be viewed as a point in R^L , the commodity space.

The Consumption Set:

The consumption set is a subset of the commodity space, denoted by $X \subset R^L$, whose elements are the consumption bundles that the individual can consume given the physical constraints imposed by his environment. We will define the following consumption set.

$$X = R_+^L = \{x_i \in R^L : x_i \geq 0 \; for \; i = 1,\dots L\}$$

The set of all nonnegative bundles of commodities .

Preference Relations:

The objectives of the decision maker are summarized in a prefere ce relation, denoted by $\succeq$, which is a binary relation on the set of alternative X, for $x,y \in X$ allowing the comparison of pairs alternatives $x,y \in X$. We read $x \succeq y$ as "x is at least as good as y" We can derive two other relations on X from $\succeq$:

1- The strict preference relation $\succ$ defined by: $x \succ y$ iff $x \succeq y$ but not $y \succeq x$. We read it as "x is preferred to y".

2- The indifference relation $\sim$ defined by: $x \sim y$ iff $x \succeq y$ and $y \succeq x$. We read it as "x is indifferent to y".

1.1 Definition :The preference relation $\succeq$ is rational if it possesses the following properties:

(i) Completeness: for all $x,y \in X$, we have that $x \succeq y$ or $y \succeq x$.

(ii)Transitivity: for all $x,y,z \in X$, if $x \succeq y$ and $y \succeq z$ then $x \succeq z$.

(In some presentation, the assumption that $\succeq$ is reflexive defined as $x \succeq x$ for all $x \in X$, is added to the completeness and transitivity assumption.)

The assumption that $\succeq$ is complete says that an individual has a well defined preference between any two possible alternatives . If the preference relation were not transitive, there might be sets, which have no best elements.

We also use other assumptions about preferences:

Desirability: we will assume that larger amounts of commodities are preferred to a smaller one, we assume that the consumption of larger amounts of goods is always feasible in principle, that is, if $x \in X$ and $y \geq x$ then $y \in X$.

1.2 Definition :The preference relation $\succeq$ on X is monotone , if $x \in X$ and $y > x$ implies $y \succeq x$.

A weaker desirability assumption than monotonicity, known as local nonsatiation.

1.3 Definition :The preference relation $\succeq$ on X is locally nonsatiated if for every $x \in X$ and every $\varepsilon > 0$ there is $y \in X$ such that $||x-y|| \leq \varepsilon$ and $y \succeq x$.

Given the preference relation $\succeq$ and a consumption bundle x, we can define three related sets of consumption bundles.

i. The indifference set containing the point x is the set of all bundles that are indifferent to x: $A = \{y \in X : y \sim x\}$.

ii. The upper contour set of bundle x is the set of all bundles that are at least as good as x : $B = \{y \in X : y \succeq x\}$.

iii. The lower contour set of x, is the set $C = \{y \in X : x \succeq y\}$.

1.4 Definition :The preference relation $\succeq$ on X is convex if for all $x \in X$, the upper contour set $C = \{y \in X : y \succeq x\}$ is convex, that is, if $y \succeq x$ and $z \succeq x$ then $\alpha y + (1-\alpha) z \succeq x$, for all $\alpha \in [0,1]$.

1.5 Definition : The preference relation $\succeq$ on X is strictly convex if for every $x \in X$, we have that $y \succeq x$, $z \succeq x$ and $y \neq z$ then $\alpha y + (1-\alpha) z \succ x$, for all $\alpha \in (0,1)$.

1.6 Definition :The preference relation $\succeq$ on X is continuous if it is preserved under limits, that is, for any sequence of pairs $\{(x^n, y^n)\}^{\infty}_{n=1}$ with $x^n \succeq y^n$ for all n,$x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$ we have $x \succeq y$.

The consumer preferring each element in the sequence $\{x^n\}$ to the corresponding element in the sequence $\{y^n\}$.

An equivalent way to state this notation of continuity is to say that for all x, the upper contour set, and lower contour set are both closed (include their boundaries).

Competitive Budgets:

First, we suppose that the L commodities are all traded in the market at dollar prices that are publicly quoted. Formally the price vectors represent these prices. $P=[p_1,\dots,p_L] \in R^L$, which gives the dollar cost for a unit of each of the L commodities, for simplicity, her we assume $P>0$, that is, $P_i>0$ for every $i=1,2,\dots L$.

Second, we assume that these prices are beyond the influence of the consumer. The affordability of a consumption bundles depends on two things, the market prices p, and the consumer's wealth level W. the consumption bundles $x \in R^L{}_+$ is affordable if its total cost does not exceed the consumer's wealth level W, that is, if $P.x = p_1 x_1 + \dots + p_L x_L \leq W$.

1.7 Definition: The Walrasian or competitive budget set:

. $B\ (p,W) = \{x \in R^l{}_+ : P.x \leq W\}$ is the set of all feasible consumption bundles for the consumer who faces market prices and has wealth W.

The set $\{x \in R^L + : P.x = W\}$ is called the budget hyperplane. The Walrasian set $B\ (p,W)$ is convex, that is, if bundles x and x' are in $B\ (p,W)$ then the bundle $x'' = \alpha x + (1-\alpha)\, x' \in B\ (p,W),\ \alpha \in [0,1]$.

To see this ; since $\alpha \in [0,1]$ and $x,x' \geq 0$ and $p.x \leq W,\ p.\ x' \leq W$, we have $x'' = \alpha x + (1-\alpha)\, x'$, therefore $P.x'' = \alpha\, P.x + (1-\alpha)\, P.x' \leq W$.

Utility Functions:

In economics , we often describe preference relations by means of a utility function. A utility function $u\ (x)$ assigns a numerical value to each element in X (It is labeled the indifference curve):

1.8Definition:A function $u: X \to R$ is a utility function representing preference relation $\succeq$ if for all $x,y \in X$.

$$x \succeq y \text{ if and only if } u(x) \geq u(y)$$

The utility Maximization Problem: (UMP)

We will study the consumer's design problem, we assume that the consumer has a rational, monotone continuous and locally nonsatiated preference relation, and we take $u(x)$ to be a continuous utility function representing this relation on the consumption set $X=R_+^L$. The consumer's problem of choosing his most preferred consumption bundle given prices $p>0$ and wealth level $W>0$ can now be stated as the following utility maximization problem *(UMP)*: $\underset{x \in X}{Max} \quad u(x)$

$$such\ that\ p.x \leq W$$

that is there exists $x^* \in X \cap B(p,W)$,such that $u(x^*) \geq u(x) for\ all\ x \in X \cap B(p,W)$

In the *UMP*, the consumer chooses a consumption bundle in the Walrasian budget set $B(p,W) = \{x \in R_+^L : p.x \leq W\}$ to maximize his utility level.

Example :_ suppose that the utility function (the Cobb – Douglas utility function) $u(x_1,x_2) = x_1^c x_2^d$ where $c,d > 0$.

Solution : If we take the natural log of utility , the product of the terms will become

$v(x_1,x_2) = c \ln x_1 + d \ln x_2$

The indifference curve is just the set of all x_1 and x_2 such that

$$k = c \ln x_1 + d \ln x_2 \qquad so \qquad x_2 = e^{\frac{k - c \ln x_1}{d}}$$

The marginal rate of substitution (MRS) is given by

$$MRS = \frac{\partial v(x_1,x_2)/\partial x_1}{\partial v(x_1,x_2)/\partial x_2} = \frac{c/x_1}{d/x_2} = \frac{cx_2}{dx_1}.$$

Now the utility maximization problem can be written as

$$\underset{x_1, x_2}{Max}\; c\ln x_1 + d\ln x_2$$
$$\text{such that } p_1x_1 + p_2x_2 \le W$$

Now we should have the MRS = slope of budget line that is

$$MRS = P_1/P_2 \text{ so } cx_2/dx_1 = P_1/P_2 \text{------ } (1)$$

$$p_1x_1 + p_2x_2 = W \text{------- } (2)$$

By solving (1) and (2) we have

$$x_1(W,p) = \frac{c}{c+d}\frac{W}{p_1}, \quad x_2(W,p) = \frac{d}{c+d}\frac{W}{p_2}.$$

Demand set-valued map:

The consumer's demand is a set-valued map $x(p,W)$ (see ch 3 for more on set valued maps) that assigns the set of preferred consumption bundles for each price-wealth pair (p,W). If $x(p,W)$ is single valued, we refer to it as a demand function.

1.9Definition: The Walrasian demand set valued map $x(p,W)$ satisfies Walras's law if for every $p>0$ and $W>0$ we have $p.x = W$ for all $x \in x(p,W)$, that is, the consumer fully expends his resources over his lifetime.

The rule that assigns the set of optimal consumption vectors in the *UMP* to each price-wealth situation $(p,W)>0$ is denoted by $x(p,W) \in R_+^L$ and is known as the Walrasian demand set-valued map.

1.10Definition: The Indirect Utility Function ,for each $(p,W)>0$, the utility, value of the *UMP* is denoted $v(p,W) \in R$. It is equal to $u(x^*)$ for any $x^* \in x(p,W)$.

1.4 Production:

A production vector is a vector $y=(y_1,\ldots,y_L) \in R^L$ that describes the net output of L commodities from a production process. The positive numbers denote the outputs and the negative numbers denotes the inputs. Some element of a production vector may be zero. The production set, is the set of all production vector that constitute feasible plans for the firm, and is denoted by $Y \subset R^L$.

The following are some remarks about production sets:

1- Y is nonempty: The firm has something, it can plan to do.

2- Y is closed: The set include its boundary, if $y^n \rightarrow y$ and $y^n \in Y$ then $y \in Y$.

3- Possibility of inaction: This property says that $0 \in Y$.

4- Free disposal: If a total production has all its outputs null,. $Y\text{-}R^L{}_+ \subset Y$.

5- Irreversibility: Suppose that $y \in Y$ and $y \neq 0$. Then the irreversibility says that $-y \notin Y$.

6- Convexity: This is one of the fundamental assumption of microeconomics. It postulates that the production set Y is convex. That is, if $y, y' \in Y$ and $\alpha \in [0,1]$ then $\alpha y + (1-\alpha)y' \in Y$.

Profit Maximization

Let there is a vector of prices quoted for the L goods denoted by $P = (p^1, \ldots p^L) > 0$, and that there prices are independent of the production plans of the firm. The firm's objective is to maximize its profit, We always assume that the firm's production set Y satisfies the property of nonemptiness, closeness and free disposal.

The profit Maximization Problem states that : Given a price vector p>0 and production vector $y \in R^L$, the profit generated by implementing y is $p.y = \sum_{i=1}^{L} p^i y_i$.

Given the technological constrains represented by its production set Y, the firm's profit maximization problem (PMP) is then

$$\underset{y \in Y}{Max}\ p.y \qquad \text{such that } y \in Y.$$

Given a production set Y, the firm's profit function $\pi(P)$ associates to every P the amount $\pi(P) = Max \{p.y : y \in Y\}$, the value of the solution to the PMP. We defined the firm's supply set-valued map at p, denoted $y\ (P)$, as the set of profit-maximizing vector $s\ (P) = \{y \in Y:\ p.y = \pi(P)\}$. The optimizing vector $y\ (P)$ lies at the point in Y associated with the highest level of profit.

More properties of $\pi(P): P \to R$ and $s\ (P): P \to Y$ will appear in ch 4 .

1.5 Pure Exchange: The Edgeworth Box:

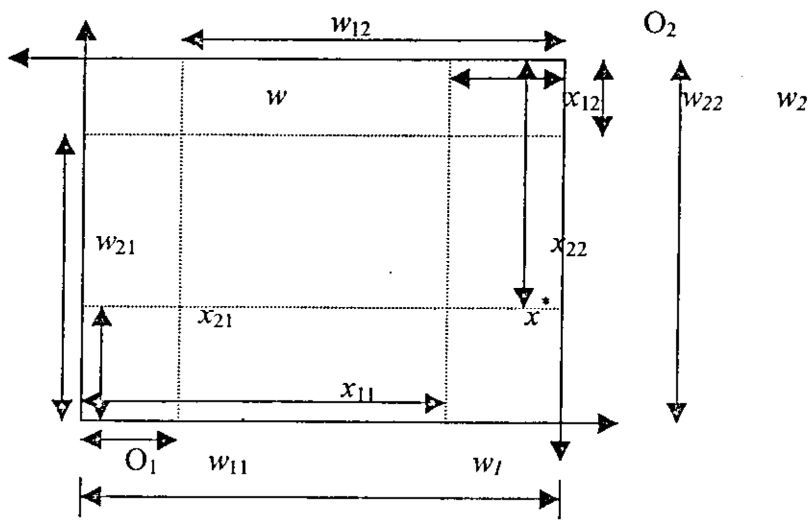
A pure exchange economy: is an economy in which there are no production opportunities. The economic agents of such an economy are consumers who possess initial endowments of commodities. Economic activity consists of trading and consumption. The two consumers are assumed to act as price takers.

Assume that there are two consumers, indexed by i=1,2 and two commodities, indexed by l=1,2, Consumer i's consumption vector is $x_i=(x_{1i}, x_{2i})$, that is, consumer i's consumption of commodity l is x_{li}, the consumer 's consumption set is $\Re_2^+$ and he has preference relation $\succeq$ over consumption vector in this set. Each consumer i is initially endowed with amount $w_{li} \geq 0$ of good l. So consumer i's endowment vector is $w_i=(w_{1i}, w_{2i})$. The total endowment of good l is $\bar{w}_l = w_{l1} + w_{l2}$. An allocation $x \in R^4{}_+$ is an assignment of nonnegative consumption vector to each consumer $x = (x_1, x_2) = ((x_{11},\ x_{21}),\ (x_{12},\ x_{22}))$, we say that an allocation is feasible if $x_{l1}, x_{l2} \leq w_l$,l= 1,2.as in figure 1.1

1.11 Definition : A Walrasian equilibrium for an economy ε is an allocation x^* and a price system $p^* \in R^L{}_+$ such that.

i. $x_i^* \in x_i\ (p^*, p^*.w)$ for all $i\ (i=1,\ldots m)$

ii. x_i^* is a redistribution i.e $\sum_{i=1}^{m} x_i^* = \sum_{i=1}^{m} w_i$



Figur 1.1

1.6 Private Ownership

Assume that an economy is composed of $I > 0$ consumers and $J>0$ firms in which there are L commodities, each consumer $i=1,\dots,I$ is characterized by a consumption set $X_i \subset R^L$ and a preference relation $\succeq$i defined on X_i, these preferences relations are rational. Each firm $j=1,\dots,J$ is characterized by a production set, $Y_j \subset R^L$. We assume that every Y_j is nonempty and closed. The initial resources of commodities in the economy, the economy's endowments are given by a rector $\bar{w}= (\bar{w}_1,\dots, \bar{w}_L) \in R^L$, these are summarized by $(\{(X_i, \succeq_i)\}^I_{i=1}, \{Y_j\}^J_{j=1}, \bar{w})$.

1.12 Definition: An allocation$(x, y) = ((x_1,\dots,x_I),(y_1,\dots,y_J))$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i =1,\dots I$ and a production vector $y_j \in Y_j$ for each firm $j =1,\dots J$. An allocation (x, y) is attainable if $\Sigma_i x_{li} = \bar{w}_l + \Sigma_j Y_{lj}$ for each commodity l. that is, if $\Sigma_i x_i = \bar{w} + \Sigma_j y_j$

We denote the set of attainable allocation by

$A - \{(x,y) \in X_1,\dots X_i \text{ x } Y_i \text{ x}\dots\text{x } Y_j, \Sigma_i x_i = w+ \Sigma_j Y_j\} \subset R^{L(I+J)}$

Consumers trade in the market to maximize their well being and firms pruduce and trade to maximize profit..The wealth of the consumers is derived form individual endowments of commodities and from ownership shares to the profit o the firms, a share $\theta_{ij} \in [0,1]$ of the profit of firm j such that $\Sigma_i\ \theta_{ij}=1$.

1.13 Definition : Given a private ownership economy specified by $(\{(X_i, \succeq_i)\}^I_{i=1}, \{y_j\}^J_{j=1}, \{(w_i, \theta_{i1},\ldots,\theta_{ij})\}^I_{i=1})$ an allocation (x^*,y^*) and a price vector $p=(p_1,\ldots p_L)$ constitute a Walrasian (or competitive) equilibrium if:

i- for every j, y_j^* maximizes profits in Y_j, that is,

$$p.y_j \leq p.y_j^* \text{ for all } y_j \in Y_j$$

ii- for every i, x_i^* is maximal for $\succeq_i$ in the budget set

$$B=\{x_i \in X_i: p.xi \leq p.w_i + \Sigma_j\, \theta_{ij}\, P.y_j^*\}$$

(Saying that xi is maximal for $\succeq_i$ in the set B" means that x_i is a preference-maxin ing choice for consumer i in the set B, that is, $x_i \in B$ and $x_i \succeq_i x'_i$ for all $x'_i \in B$).

iii- $\Sigma_i\, x^*_i = \bar{w} + \Sigma_j\, y_j^*$

condition (i) says that at a Walrasian equilibrium, firms are maximizing their profit given the equilibrium prices P. (ii) says that the consumers are maximizing their well-being given, first the equilibrium prices and second the wealth derived from their holdings the commodities and from there shares of profit (iii) says the market must clear at an equilibrium, that is, all consumers.and firms must be labeled to achieve their desire trade at the going market prices.

Price Equilibrium with Transfers: We have a situation where a social planner is able to carry out redistribution of wealth, and where society's aggregate wealth can therefore be redistributed a money consumers in any desired manner.

1.14 Definition:Given an economy specified by $(\{(X_i, \succeq_i)\}^I_{i=1}, \{y_j\}^J_{j=1}, \bar{w})$ an allocation (x^*,y^*) and a price vector $P=(P_1,\dots P_L)$ constitute a price equilibrium with tranfere if there is an assignment of wealth level $(W_1,\dots,W_2)$ with $\Sigma_i W_i = p.\ \bar{w} + \Sigma_j\ p.y_j^*$ such that:

i- For ever j, y_j^* maximizes profits in Y_j, that is,

$$p.\ y_j \leq p.y_j^* \text{ for all } y_j \in Y_j$$

ii- For every i x_i^* is maximal for $\succeq_i$ in the budget set

$$B=\{x_i \in X_i: p.x_i \leq p.W_i\}$$

iii- $\Sigma_i x_i^* = \bar{w} + \Sigma_j y_j^*$

The Walrasian equilibrium is a especial case of an equilibrium with transfer.

Quasi equilibrium is identical to price equilibrium except that the preference maximization conditions that any thing preferred to x_i^* must cost than wi (i.e "if $x_i \succ_i x_i^*$ then $p.x_i > w_i$) is replaced by the weaker requirement that anything preferred to x_i^* cannot cost less than w_i (i.e "if $x_i \succ_i x_i^*$ then $p.x_i \geq w_i$) .

1.15 Definition: Given an economy specified by $(\{x_i, \succeq_i)\}^I_{i=1}, \{y_j\}^J_{j=1}, \bar{w})$ an allocation (x^*,y^*) and a price vector $P=(P_1,\dots P_L) \neq 0$.

Constitute a price quasi equilibrium with transfers if there is an assignment of wealth levels$(W_1,\dots W_I)$ with $\Sigma_i W_i^* = p.\ \bar{w} + \Sigma_j\ p.y_j$ such that:

i- For every j, y_j^* maximizes profits in Y_j, that is,

$$p.y_j \leq p.y_j^* \text{ for all } y_j \in Y_j$$

ii- For every i, if $x_i \succ x_i^*$ then $p.x \geq w_i$

iii- $\Sigma_i x_i^* = \bar{w} + \Sigma_j\ py_j^*$.

So Any price equilibrium with transfers is a price quisiequilibrium with transfers

1.16 Proposition[(12)page 555]:

Assume that X_i is convex and $\succeq_i$ is continuous. Suppose also that the consumption vector $x_i^* \in X_i$, the price vector p and the wealth w_i are such that $x_i \succ_i x_i^*$ implies $p.x_i \geq W_i$. Then if there is a consumption vector $x'_i \in X_i$ such that $p.x'_i < W_i$ [a cheaper consumption for (p, W_i)] it follows that $x_i \succ_i x_i^*$ implies $p.x_i > W_i$.

Proof: suppose that, contrary to the assertion of the proposition, there is an $x_i \succ_i x_i^*$ with $p.x_i = W_i$. By the cheaper consumption assumption, there exists an $x'_i \in X_i$ such that $p.x'_i < W_i$. Then for all $\alpha \in [0,1)$ we have $\alpha x_i + (1-\alpha) x'_i \in X_i$ and $p.(\alpha x_i + (1-\alpha) x'_i < W_i$. But if α is closed enough to 1 the continuity of $\succeq_i$ implies than $\alpha x_i + (1-\alpha) x_i \succ_i x_i^*$ which constitutes a contradiction because we have then found consumption bundle that is preferred to x_i^* and costs less than W_i.

1.17 proposition[(12)page 555]:

suppose that for every i X_i is convex, $0 \in X_i$, and $\succeq$ is continuous. Then any price quasiequilibrium with transfers that has$(W_1 \dots W_I) > 0$ is price equilibrium with transferees.

Proof: let $x_i \succeq_i x_i^*$ then $p.x_i \geq_i W_i$. Because $0 \in X_i$ let $0 = x'_i$ then $x'_i \in X_i$ such that $p.x'_i < W_i$ since $p.x'_i = 0$ so by previous proposition if $x_i \succ x_i^*$ then $p.x_i > W_i$.

Chapter 2

Convexity

Introduction: A large part of economy especially in microeconomic analysis, which assumes optimizing behavior on the part of economic agents, relies on theorems dealing with a particular type of sets, namely convex sets.

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As we saw in chapter one, we assume that the preference relation is convex. This assumption corresponds with the idea of diminishing marginal rate of substitution (in which the indifference curves have the usual convex to the origin shapes). The convexity (not strict convexity) includes the possibility of the flat segment on the indifference curves that admits perfect substitutability between goods (this opens the possibility of set –valued rather than point – valued demand). The convexity on preference can't hold if the consumption set X, *is* not convex. Convex preference relation and the convexity of consumption and production set are among the sufficient conditions for proving the existence of Walrasian equilibrium. So we will use several theorems related to convexity in our proof, we define the budget set as: $B_{P,w} = \{x \in X : P.x = W\}$, which is called the budget hyperplane. We will use several theorems related to hyperplane in explaining of the relationship between the Walrasian equilibrium and Welfare economy.

2.1 Convex sets: Geometrically, a set is convex if the line segment connecting any two points in the set lies entirely in the set. More formally, we have the following.

2.1 Definition: If $x, y \in E$ where E denotes a vector space over R, the line segment from x to y is the set of all vectors of the form $\alpha x+(1-\alpha)y$ where $0 \leq \alpha \leq 1$. Such a vector is called an internal point of the segment if $0< \alpha <1$. A subset A of E is said to be convex, if given $x, y \in A$,all points of the form $\alpha x +(1-\alpha)y$ with $0 \leq \alpha \leq 1$ are in A

The empty subset and every singleton $\{x\}$, $x \in E$ are convex.

2.2 Propositi.n[(10): page18] Let N and M be convex sets in a vector space then:

(i): $\alpha N = \{x : x = \alpha n, n \in N\}$ is convex for any scalar α.

(ii): $N+M$ is convex.

For proof see[(10) : page 18]

2.3Proposition [(10):page18] Let J be an arbitrary collection of convex sets .Then $\bigcap_{N \in J} N$ is convex.

For proof see[(10) : page 18]

2.4 Definition: Let S be any subset of E, there exist convex supersets of S, the intersection of the family of all convex supersets of S. is a convex set, called the convex hull of S and denoted coS.

The convex hull of S is characterized by the properties (i) $S \subset co\ S$ (ii) $co\ S$ is convex (iii) $S \subset A$ and A is convex, then $co\ S \subset A$. Suoset A of E is convex if and only ıs if $coA = A$

2.5 Theorem [(3):page99] Let E and F be vector spaces over R and let $u: E \to F$ be a linear mapping

(i)If A is a convex subset of E, then $u(A)$ is a convex subset of F

(ii)If B is a convex subset of F, then $u^{-1}(B)$ is a convex subset of E

For proof see [(3) : page 99]

2.6 Theorem (3):page 100] The Cartesian product of a family of nonempty sets is convex if and only if each factor is convex. More precisely, let $(E_i)_{i \in I}$ be a family of vector spaces over R and let $E = \Pi_{i \in I} E_i$ be the product vector space. For each $i \in I$, let A_i be a nonempty subset of E_i and let $A = \Pi_{i \in I} A_i$. Then A convex if and only if every Ai is convex.

For proof see [(3) : page 100]

2.7 Definition: Let E be a TVS over R and let S be any subset of E. there exist a closed convex supersets of S, whose intersection of the family of all closed convex supersets of S is itself a closed convex superset of S called the closed convex hull of S.

2.8 Theorem [(16):page105] If E is a TVS over R and if A is a convex subset of E, then clA is also convex, and so is $(intA)$.

For proof see [(14): page 105]

2.9 Definition: A vector of the form $\sum_1^n \lambda_k x_k$ where $\sum_1^n \lambda_k = 1$ and $0 \leq \lambda_k$ for all k is called a convex combination.

2.10 Theorem [(2):page3] *If* A is a convex subset of E and if $x_1, \ldots, x_n \in A$ then A contain every convex combination of $x_1, \ldots, x_n$.

For proof see [(2) page 3]

2.11 Lemma [(16)page73] *If* x lies in the convex hull of a set $E \subset R^n$ then x is in the convex hull of some subset of E that contains at most $n+1$ points.

For proof see [(16) page 73]

2.12 Theorem (Caratheodary's theorem) [(2):page4] If X is a nonempty subset of R^n then every $x \in coX$ can be expressed as convex combination of at most $n+1$ points of X.

For proof see [(2) page 4]

2.13 Definition : A subset A of E is called a cone with vertex 0, if $\alpha A \subset A$ whenever $\alpha > 0$. More generally a cone with vertex x may be defined as a set of the form $x+A$, where A is a cone with vertex 0 (cone here means cone with vertex 0).

2.14 Theorem [(3):page109] A subset A of E is a convex cone if and only if (i) $\alpha A \subset A$ for all $\alpha > 0$ and (ii) $A+A \subset A$.

For proof see [(3) page 109]

2.2 Hyperplane:

As we have seen , the budget hyperplane plays an essential role in the choice of the consumer, and the hyperplane and the relative theorems .It will be used in the proof of the second Welfare theorem which explains the relationship between the Walrasian equilibrium and Welfare economy.

2.15 Definition : A linear subspace M of E is said to be maximal if (i)$M\neq E$, and (ii) if N is a linear subspace of E such that $M\subset N$, then either $N=M$ or $N=E$

2.16 Definition: A linear variety in E is a set V of the form $V= x+N$ where $x \in E$ and N is a linear subspace of E. thus a linear variety is a translate of some linear subspace.

2.17 Definition: Ahyperplane in E is a linear variety H that is maximal in the sense that (i) $H\neq E$ (ii) if V is a linear variety such that $H\subset V$ then either $V=H$ or $V=E$.

2.18 Theorem[(3):page81] Let H be a linear variety in E. the following condition on H are equivalent.

(a) H is a hyperplane

(b)H is a translate of some maximal linear subspace

(c) There exist a nonzero linear form f and a scalar λ such that $H = \{x: f(x) =\lambda\}$

For proof see [(3) page 81]

(Separation Theorem): The Hahn-Banach theorem is so fundamental for analysis, it is present in several formulations. In its classical version it concerns to the extension of linear form. Another approach to it exploits the connection between linear forms and hyperplanes, in which we are interest.

2.19 Theorem (3) :page112] Let E be a TVS over R let A be nonempty open convex subset E, and let M be a linear variety in E such that $M\cap A= \phi$. Then there exist a closed hyerplane H in E such that $M\subset H$ and $H\cap A= \phi$.

For proof see [(3) page 112]

2.20 Definition: A subset S of a real vector space E is called a half- space if there exist a nonzero linear form g of E and a real number α such that S may be represented in one of the following forms:

(i) $S_1 = \{x: g(x)\leq \alpha\}$.(ii)$S_2= \{x: g(x)\geq \alpha\}$.(iii)$S_3 = \{x: g(x)< \alpha\}$.(iv)$S_4 = \{x: g(x)> \alpha\}$

Let H be a hyperplane determined by g and α that is

$H=\{x: g(x)=\alpha\}$

S_1, S_2 are called the closed half—spaces, S_3 and S_4 the open half spaces determined by H.

2.21 Definition: Let A be a subset of E. we say that A lies in one side of H if either $A\subset S_1$ or $A\subset S_2$; A lies strictly in one side of H if either $A\subset S_3$ or $A\subset S_4$. Let A,B be subsets of H, we say that H separates A and B if either $A\subset S_1$ and $B\subset S_2$ or $A\subset S_2$ and $B\subset S_1$, we say that H separates A and B strictly if either $A\subset S_3$ and $B\subset S_4$ or $A\subset S_4$ and $B\subset S_3$

2.22 Theorem [(3) :page123] Let H be a hyperplane in a real vector space E and let A be a convex subset of E. Then A lies strictly to one side of H if and only if $A\cap H=\phi$.

For proof see [(3) page 123]

Since the budget line (the hyperplane) separation the consumption set into two sets: the upper set in which the consumer's wealth does not help him to buy from it , and lower set , which is achievable for him. These two sets are considered ,as we have seen , half spaces.

2.23Theorem (3):age124] Let E be a TVS let A and B be nonempty convex subset of E such that $A\cap B=\phi$ and suppose A is open. Then (1) there exists a closed hyperpalne H separating A and B (2) A lies strictly in one side of H. (3) if B is also open then it separates A and B strictly

For proof see [(3) page 124]

Chapter Three

The set valued map:

We have seen , when the upper contour set is strictly convex , we deal with point to point mapping (function) , but if it is convex , there will be a flat area is tangential with the budget line .This make possible for us to deal with the set valued maps that assign point to set . So the demand will not be a function , and the consumer's demand will be a subset of X for a given price . In this chapter , we will deal with two types of set valued maps : upper and lower semicontinuous set valued (correspondence). We will also deduce the proof for the fixed point theorem of set valued maps from Brower's fixed point theorem of function , while making use of several theorems and lemmas.

A vector $x \in A$ is a fixed point of f (.) if $x = f(x)$. [or in the set valued map case if $x \in F(x)$]. That is, the vector is mapped into itself and so it remains fixed.

Therefore , we shall first study the set-valued map and the lower and upper semicontinuous set-valued map , and prove some related theorems .

The central result we hope to conclude is Kakutani's fixed point theorem which states: Let C be a nonempty compact and convex subset of R^n , and F be a convex-valued set valued map of C into C which has a closed graph. Then there exists a fixed point $x^\wedge$ for F i.e $x^\wedge \in F(x^\wedge)$.

3.1 Set-Valued Maps:

We observed at several points in economics, for example in considering demand relations, that the actions taken by an individual are determined by the values of those variables which constitute his economic system . If these values uniquely determine the action to be taken, then are 'point –to- point' relations, which is known as a function.

However, If these values are not uniquely determined the a ion to be taken, then are 'point –to- set' relation or correspondence.

3.1 Definition:

Let X and Y be two subsets of R^n. A set-valued map F from X to Y, is a map that associates with any $x \in X$ a subset $F(x)$ of Y. called the image of F at x.

For example, let $A = B = R$, we might consider $\varphi(x) = \{y: x-1 \leq y \leq x+1\}$. $\varphi(0) =$ [-1,1].

We say that a set valued map F is proper if there exists at least an element $x \in X$ such that $F(x) \neq \phi$. The domain of F is $Dom(F) = \{x \in X \mid F(x) \neq \phi\}$.

A set valued map F is characterized by its graph, the subset of X x Y defined by

$$Graph(F) = \{(x,y) \mid y \in F(x)\}$$

The domain of F is the projection on X of $Graph(F)$.

The image of F, the subset of Y *is* defined by

$$Im(F) = \bigcup_{x \in X} F(x) = \bigcup_{x \in dom(F)} F(x)$$

is the projection on Y of $Graph(F)$.

We shall say that a set-valued map F from X to Y is strict if $Dom(F) = X$.

When $K \subset X$ is a nonempty subset and when F is a strict set-valued map from K to Y, we will extend it to the set valued map F_K from X to Y is defined by

$$F_K(x) = \begin{cases} F(x) & when\ x \in K \\ \phi & when\ x \notin K \end{cases}$$

The domain ($Dom(F_K)$) is K .

Let (Ł) be a property of a subset (for instance ,closed , convex ,compact ,etc) . As a general rule , we shall say that a set – valued map F satisfies the property (Ł) if the graph of F satisfies this property . For example , we shall speak of closed , convex , compact map which is a set – valued map whose graph is closed, convex , compact .

When* denotes an operation on the subsets ,we use the same notation for the operation on set – valued maps which is defined by

$$F_1 * F_2 : x \longrightarrow F_1(x) * F_2(x)$$

We defined in this way $F_1 \cap F_2$, $F_1 \cup F_2$, $F_1 \backslash F_2$ and $F_1 + F_2$

Similarly, if $\propto$ is a map from X to Y we define

$$\propto (F) : x \longrightarrow \propto (F(x))$$

For instance $\overline{F} = cl\,(F) : x \longrightarrow \overline{F}(x)$

$Int\,(F) : x \longrightarrow int\,(F(x))$

$co\,(F) : x \longrightarrow co\,(F(x))$

$\overline{co}\,(F) : x \longrightarrow \overline{co}\,(F(x))$ *and so on.*

We have the following elementary properties:

i. $F(K_1 \cup K_2) = F(K_1) \cup F(K_2)$

ii. $F(K_1 \cap K_2) \subset F(K_1) \cap F(K_2)$

iii. $F(X \backslash K) \supset F(X) \backslash F(K)$

iv. *if* $K_1 \subset K_2$ *then* $F(K_1) \subset F(K_2)$

In the case of single-valued maps (functions) f from X to Y, continuous functions are characterized by two equivalent properties:

a. for any neighborhood $N(f(x_0))$ of $f(x_0)$ there exists a neighborhood $N(x_0)$ of x_0 such that $f(N(x_0)) \subset N(f(x_0))$.

b. For any generalized sequence of elements x_n converging to x_o the sequence $f(x_n)$ converges to $f(x_0)$.

These two properties can be adapted to the case of strict set-valued map from X to Y, and they become.

a. For any neighborhood $N(F(x_0))$ of $F(x_0)$, there exists a neighborhood $N(x_0)$ of x_0 such that $F(N(x_0)) \subset N(F(x_0))$.

b. For any generalized sequence of elements x_n converging to x_o and for any $y_0 \in F(x_n)$ there exists a sequence of elements $y_n \in F(x_n)$ that converges to y_0.

In the case of set-valued maps, these two properties are no longer equivalent. We call upper semicontinuous maps those that satisfy property (a), lower semi continuous maps those that satisfy property (b) and continuous maps those which satisfy both properties (a) and (b).

3.2 Upper semicontinuous set-valued maps

3.2 Definition:

Let F be a set-valued map from X to Y, we say that F is upper semi continuous (in short U.S.C) at $x_o \in X$, if for any neighborhood $N(F(x_o))$ of $F(x_o)$, there exists a neighborhood $N(x_o)$ of x_o such that

$$\text{For all } x \in N(x_o), F(x) \subset N(F(x_o))$$

We say that F is upper semicontinuous if F is upper semicontinuous at every point $x \in X$.

3.3 Example:

Let $\varphi(x)$ be defined as follows $\varphi: R \longrightarrow R$.

$$\varphi(x) = \left\{ \begin{array}{ll} x-4 \leq y \leq x-2 & \text{for } x < 0 \\ -4 \leq y \leq 4 & \text{for } x = 0 \\ x+2 \leq y \leq x+4 & \text{for } x > 0 \end{array} \right\}$$

$\varphi(x)$ is upper semicontinus

3.4 Example:

Let $\varphi(x)$ be defind much as in Example 3.2. but not u.s .c at 0, as follows $\varphi: R \longrightarrow R$.

$$\varphi(x) = \left\{ \begin{array}{ll} x-4 \le y \le x-2 & for\ x < 0 \\ 0 & for\ x = 0 \\ x+2 \le y \le x+4 & for\ x > 0 \end{array} \right\}$$

$\varphi(x)$ is not upper semicontinuous at $x = 0$

3.5 Theorem [(1): page 110]

Let F be an upper semicontinuous set-valued map from X to Y with closed valued. Then F has closed gragh .

Proof: suppose F is U.S.C with closed valued i.e. the image of F is closed therefore $F(x)$ is closed. Want to show that F is closed i.e. its graph is closed.

Now, let us consider a sequence of elements (x_n, y_n) of the graph of F that converges to some $(x,y) \in X \times Y$. since F is upper semi continuous, we can associate to any closed neighborhood $N(F(x))$ an index n_o such that for all $n > n_o$, $y_n \in N(F(x))$. Since $N(F(x))$ is closed y belongs to every neighborhood of $F(x)$.

3.6 Theorem [(1): page 110]

Let F and G be two set-valued maps from X to Y such that for all $x \in X$, $F(x) \cap G(x) \neq \emptyset$. We suppose that

i. F is upper semicontinuous at x_o.

ii. $F(x_o)$ is compact.

iii. G is closed.

Then the set-valued map $F \cap G: x \longrightarrow F(x) \cap G(x)$ is upper semi continuous at x_o.

Proof: Let $N: = N(F(x_o) \cap G(x_o))$ be an open neighborhood of $F(x_o) \cap G(x_o)$ we have to find a neighborhood $N(x_o)$ such that for all $x \in N(x_o)$, $F(x) \cap G(x) \subset N$.

Now. If F $(x_o) \subset N$, since F is upper semi continuous ,we are done. If not, then we have the subset.

$$K := F(x_o) \setminus N$$

K is compact (since $F(x_o)$ is compact). Let $P :=$ *graph* of G.Which is closed .

For any $y \in K$, we have $y \notin G(x_o)$. (since $y \in F(x_o) \setminus N$) and thus $(x_o, y) \notin P$. Since P is closed, there exist open neighborhoods $N_y(x_o)$ and $N(y)$ such that $P \cap (N_y(x_o) \times N(y)) = \varnothing$.

therefore

for all $x \in N_y(x_o)$ $\quad G(x) \cap N(y) = \varnothing$…………………(1)

Since K is compact. It can be covered by n neighborhoods $N(y_i).i=1,2,\ldots,n$. The union $M := \bigcup_{i=1}^{n} N(y_i)$ is a neighborhood of K and MUN is neighborhood of $F(x_o)$. Since F is upper semicontinuous at x_o, there exists a neighborhood $N_o(x_o)$of x_o such that for all $x \in N_o(x_o)$,$F(x) \subset MUN$ …..(2)

We set $N(x_o) := \bigcap_{i=1}^{n} N_o(y_i)(x_o)$, when $x \in N(x_o)$ satisfies properties (1) and (2).

i. $F(x) \subset MUN$

ii. $G(x) \cap M = \varnothing$

Therefore, $F(x) \cap G(x) \subset N$ when $x \in N(x_o)$

3.7 Corollary [(1):page 111]

Let G be a closed set-valued map from X to a compact set Y. Then G is upper semi continuous.

Proof: we can take F to be defined by $F(x) = Y$ for all $x \in X$. Since $F(x_o)$ is compact, and F is upper semi continuous since Y is compact, and G is closed.(by 3.6)

So the set-valued map $F \cap G: x \longrightarrow F(x) \cap G(x) = G(x)$ is upper semi continuous.

3.8 Theorem [(9):page 190]

let F_i be set-valued maps from X to Y $(i=1,\dots,n)$ and are upper semicontinuous at x_o. Then their union i.e the set-valued map $G(x) : x \longrightarrow \bigcup_{i=1}^{n} F_i(x)$ is upper semicontinuous at x_o.

proof: suppose F_i is upper semi continuous at x_o for all $i=1,\dots,n$ so for any $N_i(F_i(x_o))$ neighborhood of $F_i(x_o)$ there exists $N_i(x_o)$ neighborhood of x_o such that for all $x \in N_i(x_o)$.

$F_i(x) \subset N_i(F_i(x_o))$

$N = N(\bigcup_{i=1}^{n} F_i(x_o)) F_i$ be an open neighborhood of $\bigcup_{i=1}^{n} F_i(x_o)$. We have to find a neighborhood $N(x_o)$ of x_o such that for all $x \in N(x_o)$ $\bigcup_{i=1}^{n} F_i(x) \subset N$.

Let $N = \bigcup_{i=1}^{n} N_i(F_i(x_o))$,so N is a neighborhood of $F_i(x_o)$ for every $i, i=1,\dots,n$. Want to show that $N(x_o) = \bigcap_{i=1}^{n} N_i(x_o)$ is the required neighborhood of x_o. Now for any $x \in N(x_o)$,$F_i(x) \subset \bigcup_{i=1}^{n} N_i(F_i(x_o))$, therefore, for all $x \in N(x_o)$, $\bigcup_{i=1}^{n} F_i(x) \subset N(Fi(x_o))$.

In most applications the set-valued maps are compact-valued i.e $F(x)$ is compact subset of Y for every $x \in X$. In this case the general definition of upper semicontinuous in terms of neighborhoods has an easy equivalent formulation in terms of sequences. In many proofs it is easier to work with this alternative definition of upper semi continuous.

3.9 Theorem [(9):190]

The compact-valued set-valued map F from X to Y is upper semi continuous at x_o if and only if for every sequence $\{x_n\}$ converging to $x_o \in X$ and every sequence $\{y_n\}$ with $y_n \in F(x_n)$ there exists a converging subsequence of $\{y_n\}$ whose limit belong to $F(x_o)$.

Proof: suppose that F is upper semi continuous at x_o, we want first to show that the sequence $\{y_n\}$ is bounded and consequently possesses a convergent subsequence. Then we show that the limit of the subsequence belongs to $F(x_o)$.

Since $F(x_o)$ is compact, so it is closed and bounded. Since $F(x_o)$ is bounded, there exists a bounded and open set B containing $F(x_o)$. By the definition of upper semi continuous, there exists a neighborhood $N(x_o)$ of x_o such that $F(z) \subset B$ for every $z \in N(x_o)$. since the sequence $\{x_o\}$ converges to x_o, there exists an integer n_o such that $x_n \in N(x_o)$ for $n \geq n_o$. Consequently we have $F(x_n) \subset B$ therefore $y_n \in B$ for $n \geq n_o$. Thus, the sequence y_n is bounded. Hence, there exists a converging subsequence, say $\{y_{nq}\} \to y_q$.

Assume now that $y \notin F(x_o)$. So there exists a closed neighborhood of $F(x_o)$ not containing the point y. For example the closed ball B_ε around the set $F(x_o)$ with radius $\varepsilon > 0$, where the number ε is smaller than the distance of y to any point $\varepsilon \in F(x_o)$ i.e $B_\varepsilon = \{v \in Y: \inf_{z \in F(xo)} d(z,v) \leq \varepsilon\}$.

Since F is upper semi continuous it follows that for n large enough we have $F(x_n) \subset B_\varepsilon$ and consequently $y_n \in B_\varepsilon$ Since the subsequence $\{y_{nq}\}$converges to y and since B_ε is by assumption, closed it follows that the limit point $y \in B_\varepsilon$. But this is a contradiction since by construction $y \notin B_\varepsilon$

To prove the converse we assume that F is not upper semicontinuous at x_o, i.e there exists a neighborhood $N(F(x_o))$ of $F(x_o)$ such that every neighborhood $N(x_o)$ of x_o contains a point z with $F(z) \not\subset N(F(x_o))$. By choosing the sequence of neighborhoods $B_{1/n}(x_o)$ $(n=1,\dots)$ i.e the balls with center x_o and radius $1/n$, we therefore obtain a sequence $\{x_n\}$ converging to x_o and a sequence $\{y_n\}$ with $y_n \in F(x_n)$ and $y_n \notin N(F(x_n))$. By assumption we know that there exists a converging subsequence of $\{y_n\}$ whose limit belongs to $F(x_o)$. But this is imposible since the set $Y/N(F(x_o))$ is closed. Thus, $y_n \in Y/N(F$

(x_o) for every n implies that the limit point of no converging subsequence of $\{y_n\}$ will belongs to N (F (x_o)) and a boundary of $F(x_o)$ which is contained in $N(F(x_o))$.

The following results, are immediate consequences of the definition of upper semi continuous in terms of sequences. They show that set-theoretic operations like union, product, sum and convex hull applied 'point-wise', preserve upper semicontinuous.

3.10 Proposition [(2):page192]

Let the set-valued map F from X to Y be compact-valued and upper semi continuous. Then, the image

$$F(K) = U_{x \in k} F(x)$$

Of a compact set K is compact.

Proof: let K be compact set , let $N_i(x_i)$ be a neighborhood of x_i, $x_i \in K$.

We have $K \subset \bigcup_{i=1} N_i(x_i)$ but K is compact, therefore we can fined finite cover of K such that $K \subset \bigcup_{i=1}^{n} N_i(x_i)$,let F be upper semi continuous, so for any $N(F(x_i))$ is neighborhood of $F(x_i)$, there exists $N_i(x_i)$ neighborhood of x_i such that for all $x \in N_i(x_i)$ $F(x) \subset N_i(F(x_i))$. But for all $x \in K$, $x \in \bigcup_{i=1}^{n} N_i(x_i)$, so we have

$F(x) \subset \bigcup_{i=1}^{n} N_i(F(x_i))$.

So $F(K) \subset \bigcup_{i=1}^{n} N_i(x_i)$ so $F(K)$ compact.

3.11 Proposition [(9):192]

Let F_i $(i=1,\ldots,k)$ be set-valued maps from X to Y be compact-valued and upper semi continuous at x. Then the product set-valued map $x \longrightarrow \prod_{i=1}^{n} F_i(x)$ from X into the product space Yx…xY is compact-valued and upper semi continuous at x.

Proof: We want to show that for every sequence $\{x_n\}$ converging to x and for every sequence $y_n = \{y_n^1, y_n^2, \ldots, y_n^k\}$ $n=1,\ldots$ with $y_n \in \prod_{i=1}^{k} F_i(x_n)$ $(i=1,\ldots,k)$, there exists a converging subsequence whose limit belongs to the product $F_i(x)$ x...x $F_k(x)$ .We know that the sequence$\{y_n^1, y_n^2, ..., y_n^k\}n=1,...$ in the product space converges to $\{y^1, y^2, ..., y^k\}$ if and only if every sequence $\{y_n^1\}$ $n=1,...$ of it coordinates converges to y^1 .

Let $\{x_n\}$ be sequence converging to x in X, and $\{y_n\} = \{y_n^1, y_n^2, ..., y_n^k\}$ n=1,... with $y_n^1 \in F_i(x_n)$.

We have for every $\{x_n\}$ converging to x and every sequence $\{y_n^1\}$ with $y^1 \in F_i(x_n)$ there exists a converging subsequence of $\{y_n^1\}$ whose limit belongs to $F_i(x)$ since F_i for $i=1,...,K$ is compact-valued upper semicontinuess.

So (by 3.9) and so for the second $\{y_n^1, y_n^2, ..., y_n^k\}n=1,...$ there exits a converging subsequence whose limit belongs to $\prod_{i=1}^{k} F_i(x)$ i.e$\{y_{nq}^1, , y_{nq}^k\} \longrightarrow \{y^1, ..., y^k\} \in \prod_{i=1}^{k} F_i(x)$.since the finite product of compact sets is compact.

3.12 Proposition [(2):page 82]

Let F_i be a set-valued map from X into R_n $(i=1,.... k)$ be compact-valued and upper semi continuous at x. Then the sum set valued map

$x: \longrightarrow \sum_{i=1}^{k} F_i(x)$ of X into R^n is compact valued upper semi continuous at x.

Proof: Let the sequence (x_n) be convergent to x and let $y_n \in \sum_{i=1}^{k} F_i(x_n)$ $n=1,...$ Thus the vector of y_n is of the form $y_n = \sum_{i=1}^{k} y_k^i$ where $y_k^i \in F_i(x_n)$ $(i=1,...)$. (By 3.9) since F_i is compact valued upper semi continuous there exists for every sequence $\{y_n^i\}_{n=1,...}$ $(i=1,...,k)$ a converging subsequence whose limit y^i belongs to $F_i(x)$.

consequently, there exists a converging subsequence $\{y_{nq}\}_{q=1,...}$ of $\{y_n\}$ such that the coordinate sequence y_{nq}^i converges to y^i, Hence $\lim y_{nq} = y_1 + ... + y_k \in F_1(x) + ... + F_2(x)$

3.13 Proposition [(2): page 92]

Let F be a set-valued map from X into R^n be upper semi continuous at x. Then the convex hull set-valued map $x: \longrightarrow co\ F(x)$ of X into R^n is compact-valued and upper semi continuous at x.

Proof: Let the sequence $\{x_n\}$ be convergent to x and let $y_n \in co\ (F(x_n))$. By Caratheodory every vector $y_n \in R^m$ can be written as a convex sum of $m+1$ vector in $F(x_n)$ i.e

$y_n = \lambda_n^0 \cdot z_n^0 + ... + \lambda_n^m\ z_n^m$ Where $z_n^i \in F(x_n)$, $\lambda_n^0 + ... + \lambda_n^m = 1$ and $\lambda_n^1 \geq 0$ (By 3. 8)

there exists for every $i=0,...m$ a converging subsequence $\{z_n^i\}_{n=1,...}$ whose limit belongs to $F(x)$. the sequence $\{\lambda_n^i\}_{n=1,...}$ is bounded and therefore possesses a converging subsequence. So there exists a converging subsequence $\{y_{nq}\}$ of $\{y_n\}$ such that the corresponding subsequences $\{z_n^i\}_{q=1,...}$ and $\{\lambda_{nq}\}_{q=1,...}$ are convergent say,

$z_n^i \longrightarrow z^i$ and $\lambda_n^i \longrightarrow \lambda^i$ so we have $\lambda^o + \lambda^i + ... + \lambda^m = 1$, $\lambda^i \geq 0$, Hence

$\lim y_{nq} = \lambda^o z^o + ... + \lambda^m z^m \in co\ F(x)$

3.3 Lower-Semi continuous Set-valued maps

3.14 Definition:

The set-valued map F from X into Y is said to be lower semi continuous at x_o, in short L.S.C if for every open set O in Y with $F(x_o) \cap O \neq \varnothing$ there exists a neighborhood $N(x_o)$ of x_o such that

$$F(x) \cap O \neq \emptyset \text{ for all } x \in N(x_o)$$

The set-valued map is called lower semi continuous if it is lower semi continuous at every point $x \in X$.

Example 3.14: Let $F(x)$ be defined as follows, $F: R \longrightarrow R$. For

$$F(x) = \begin{cases} x-4 \leq y \leq x & x \neq o \\ -3 \leq y \leq -1 & x = o \end{cases}$$

$F(x)$ is lower semi continuous?

Example: Let $F(x)$ be defined as follows $F: R \longrightarrow R$

$$F(x) = \begin{cases} x-4 \leq y \leq x-2 & x < o \\ -4 \leq y \leq 4 & x = o \\ -x+2 \leq y \leq x+4 & x > o \end{cases}$$

$F(x)$ is U.S.C but not l.S.C at $x = o$

As in the case of upper semi continuous, we will study some theorems related to L.S.C.

3.15 Theorem [(1):page197]

Let F_i be lower semi continuous from X into Y $(i=1,\dots,n)$ at x_o. Then their union

$x: \longrightarrow \bigcup_{i=1}^{n} F_i(x)$ is also lower semi continuous at x_o.

Proof: Let F_i *be* lower semi continuous at x_o. If V be an open set in Y with $(\bigcup_{i=1}^{n} F_i\ (x_o)) \cap V \neq \varnothing$ we want to find a neighborhood $N\ (x_o)$ of x_o such that

$$(\bigcup_{i=1}^{n} F_i\ (x_o)) \cap V \neq \varnothing \textit{ for all } x \in N(x_o)$$

since F_i is lower semi continuous at x_o, therefore for all V_i open set with $F_i \cap V_i \neq \varnothing$, there exists $N_i\ (x_o)$ of x_o such that

$$F_i\ (x) \cap V_i \neq \varnothing \textit{ for all } x \in N_i\ (x_o)$$

Let $V = \bigcup_{i=1}^{n} V_i$ with $\bigcup_{i=1}^{n} F_i\ (x_o) \cap V \neq \varnothing$. Want to show that $N\ (x_o) = \bigcap_{i=1}^{n} N_i\ (x_o)$ is a neighborhood of x_o which satisfies the condition. Now for all $x \in N\ (x_o)$, *every* $F_i\ (x)$ such that $F_i\ (x_o) \cap V \neq \varnothing$ therefor for all $x \in N\ (x_o)$, $\bigcup_{i=1}^{n} F_i\ (x_o) \cap V \neq \varnothing$.

The following result characterizes lower semicontinuous in terms of sequences. No compactness assumption is needed here.

3.16Theorem [(9):197]

The set-valued map F from X into Y is lower semi continuous at x_o if and only if for every sequence $\{x_n\}$ converging to x_o and every $y \in F\ (x_o)$ there exists a sequence $\{y_n\}$ cnverging to y with $y_n \in F\ (x_n)$.

i.e every point y corresponding to x can be obtained as a limit of point y_n corresponding to points x_n near to x.

Proof: Let F be lower semi continuous at x_o and let $\{x_n\}$ be convergent to x_o and let $y \in F(x_o)$. For every integer r, let $B_r\ (y)$ denote the ball with radius $1/r$ and center y. since F is lower semi continuous at x_o, there exists for every r a neighborhood $N_r\ (x_o)$ of x_o such that $z \in N_r\ (x_o)$, so $F\ (Z) \cap B_r\ (y) \neq \varnothing$. Since $\{x_n\}$ converges to x_o there is for every r an integer n_r such that $n \geq n_r$ implies $x_n \in N_r(x_o)$. we can assume that $n_r < n_{r+1}$ $(r=1,2,\ldots)$. We now define the desired sequence $\{y_n\}$. for n with $n_r \leq n < n_{r+1}$ choose y_n in the set $F(x_n) \bigcap Br(y)$. This is possible since $n \geq n_r$ implies $x_n \in N_r(x_o)$, which in turn implies that $F(x_n) \bigcap Br(y) \neq \emptyset$.

The sequence $\{y_n\}$, constructed in this way, converges to y since with increasing n the index r also increases and the ball $B_r(y)$ becomes smaller and smaller.

Conversely: assume that F is not lower semi continuous at x_o,i.e there exists an open set V with $V \bigcap F(x_o) \neq \emptyset$ such that every neighborhood $N(x_o)$ of x_o contains a point z with $F(z) \bigcap V = \emptyset$.

Therefor, there exists a sequence $\{x_n\}$ converging to x_o with $F(x_n) \bigcap V = \emptyset$. Now let $y \in V \bigcap F(x_o)$. By assumption, there exists a sequence $\{y_n\}$ converging to y with $y_n \in F(x_n)$. Since V is open and $y \in V$ we have for large enough that $y_n \in V$ thus $F(x_n) \bigcap V \neq \emptyset$ a contradiction.

3.17 Proposition [(9):198]

Let F_i be set-valued maps from X into Y $(i=1,2,\ldots k)$ which are lower semi continuous at x_o. Then The product set-valued map $x: \longrightarrow \prod_{i=1}^{n} F_i\ (x)$ from X into the product space Yx...xY is lower semi continuous at x_o.

Proof: Let the sequence $\{x_n\}$ converge to x_o, and let $y = (y^1, y^2, \ldots, y^k) \in \prod_{i=1}^{n} F(x)$. we want to find a sequence $\{y_n\} = (y_n^1, y_n^2, \ldots, y_n^k)$. Converging to y with $y_n \in \prod_{i=1}^{n} F(x_n)$

Since F_i for $i=1,\dots,k$ is lower semi continuous every $\{x_n\}$ converging to x_o and every $y^i \in F_i(x_o)$ there exists $({}^i y_n)$ converging to y^i with $y^i_n \in F_i(x_n)$.

So (by 3.16.) for every for every $(y^1,\dots,y^k) \in \prod_{i=1}^{n} F_i\ (x_o)$ there exists $(y^1_n, y^2_n, \dots, y^k_n)$ converging to $(y^1,\dots,y^k)$ with $=(y^1_n, y^2_n, \dots, y^k_n) \in \prod_{i=1}^{n} F_i\ (x_o)$.

3.18 Proposition [(2):83]

Let F_i be set-valued maps from X into R^n $(i=1,\dots,k)$ be lower semi continuous at x_o. Then the sum set-valued map $x: \longrightarrow \sum_{i=1}^{k} F_i(x)$ from X into R^n is lower semi continuous at x_o.

Proof: Let the sequence $\{x_n\}$ be convergent to x_o and Let $y \in \sum_{i=1}^{k} F(x_o)$ we want to find a sequence $\{y_n\}$ converging to y with $y_n \in \sum_{i=1}^{k} F(x_n)$.

Since $y \in \sum_{i=1}^{k} F(x_o)$, for the vector of y of the form $y=y^1+\dots+y^k$ we have $y^i \in F^i(x_o)$.

(By 3.16) there exist for every y^i a sequence ${}_n y^i$ $(i= 1,\dots,k)$ converging to y^i with $y^i \in F_i(x_n)$.

Therefore $y_n = \sum_{i=1}^{k} y^i_n$ and $\lim y_n = y^1+\dots+y^k = y$

Since $y^i_n \in F_i(x_n)$ then $y_n \in \sum_{i=1}^{k} F(x_n)$.

So (by 3.16) $\sum_{i=1}^{k} F(x_n)$ is lower semi continuous.

3.19 Proposition [(2):83]

Let F be a set-valued map from X to R^n be lower semi continuous at x_o. Then the convex hull set-valued map $x \longrightarrow co\ F(x)$ is lower semi continuous.

Proof: Let $\{x_n\}$ be a sequence that converge to x_o. Let $y \in co\ F(x_o)$, i.e $y = \lambda^o y^o + \dots + \lambda^n y^n$ where $y^i \in F(x_o)$, $\lambda^o \geq o$ $(i=o,\dots,n)$ and $\sum_{i=1}^{k} \lambda^i = 1$ (by caratheodory). Since F is lower semi continuous at x_o there exists a sequence $\{ y_n^i \}$ converging to y^i with $y_n^i \in F(x_n)$

Hence $y_n = \lambda^o y_n^0 + ... + \lambda^n y_n^n$ belongs to $co\ F(x_n)$ and converge to y.

Note: The intersection of upper semi continuous set-valued maps is upper semi continuous. But the intersection of lower semi continuous set valued maps is, in general not lower semi continuous.

Example: Let $G:R \longrightarrow R$,be defined as in example 3.14,and $F(x) =\{ y ; x-1 \leq y \leq x+1 \}$,which is l.s.c at $x=0$,we have $F(x)$ is l.s.c at $x=0$ *,but* $G \cap F$ is not L.S.C at $x=0$

3.20 Proposition [(2):page83]

Let F be a set-valued map from X into Y if F is lower semi continuous at x_o, Then the closure set-valued map $x \longrightarrow cl\ F(x)$ is lower semi continuous at x_o.

Proof: Suppose that F is lower semicontiunus at x_o. Let V be open set in Y with $clF(x_o) \cap V \neq \emptyset$, we want to find $N(x_o)$ neighborhood of x_o such that for all $x \in N(x_o)$, $cl\ F(x) \cap V \neq \emptyset$.

Since $F(x_o) \subset cl\ F(x_o)$. Let V be an open set in Y such that $cl\ F(x_o) \cap V \neq \emptyset$. Since F is lower semi continuous at x_o, therefore for all $x \in N(x_o)$, $clF(x) \cap V \neq \emptyset$.

3.4 Kakutani's Fixed Point Theorem:

We now need an extension of the Brouwer's Fixed-Point Theorem to the context of the set-valued maps.

We will first state Brouwer's fixed point theorem for functions.

For proof see Ekeland (1)

3.21 Brouwer's fixed point theorem(1):

Let A be a nonempty compact and convex subset of R^n and f a continuous mapping of A into A.

Then

There exists a fixed point x^o for f i.e $x^o = f(x^o)$.

3.22 Kakutani's fixed point theorem(9):

Let A be a nonempty compact and convex subset of R^n and F be a convex-valued set-valued map of A into A which has a closed graph. Then

There exists a fixed point x^o for F i.e $x^o \in F(x^o)$.

Note: A fixed point x^o for a function say that $x^o = f(x^o)$ while a fixed point x^o of set valued map saya that $x^o \in F(x^o)$.

Before we show the way to generalize Brouwer's result to that of kakutani, we will first prove the following Lemma.

3.23 Lemma [(9):page202]

Let F be a convex-valued set-valued map of X into Y where X and Y are compact subsets of R^n. Then for every ε>o there exists a lower semi continuous and convex-valued set-valued map F^δ of X into Y such that $Graph\,(F^\delta) \subset B\varepsilon\,(Graph\,(F))$. Where $B\varepsilon$ is the ball with radius ε.

Proof: For every $\delta > o$ define the set-valued map F^δ by

$$x: \longrightarrow F^\delta(x) = co \bigcup_{d(z,x)<\delta \;\, z\in X} F(z)$$

Since x: $\longrightarrow \bigcup_{d(z,x)<\delta \;\, z\in X} F(z)$ is lower semi continuous let $UF(z) \cap V \neq \varnothing$.

Let $N_\delta(z) = \{x : d\,(x, z) < \delta\}$ be the regular neighborhood of z so, for every $x' \in N_\delta(z)$ $UF(x') \cap V \neq \varnothing$, where V is open set in Y).

Now (by 3.19) F^δ is lower semi continuous. We want to show that for δ small enough we have $Graph\,(F^\delta) \subset B\varepsilon\,(Graph\,(F))$.

Assume on the contrary that there is an ε?>o such that for every δ>o. We have *Graph* (F^{δ}) $\not\subset B\varepsilon$ *(Graph (F))*. Then there exists a sequence (x_n, y_n) in X x Y with $y_n = \sum_{i=1}^{k} \lambda^i y_n^i$ where $y_n^i \in F(z_n)$, $\lambda_n^i \geq$ $\sum_{i=1}^{k} \lambda_n^i = 1$ and $d\ (z_n, x_n) \leq 1/n$ such that the *dist [* (y_n, x_n), *Graph (F))* $> \varepsilon$ for every n. Since X and Y are assumed to be compact, we can assume that $\lambda_n^i \to \lambda^i$

$x_n \longrightarrow x$, $y_n^i \longrightarrow y_n^1$, $z_n \longrightarrow z{=}x$, so $y = \sum_{i=1}^{k} \lambda^i y^i$ where $\lambda^i \geq o$,

$\sum_{i=1}^{k} \lambda^i = 1$ and $(x, y^i) \in clGraph\ (F)$.

since $F(x)$ is convex, every combination in it, so $(x,y) \in cl\ Graph\ (F)$ which contradicts that *dist [(x,y), Graph (F)]>* $\varepsilon \geq 0$ for all n.

3.24 Lemma [(9):202]

(Existence of a continuous selection). Let F be a closed and convex valued lower semi continuous set valued map of X into R^n. Then there exists a continuous function f such that $f(x) \in F(x)$ for every $x \in X$.

Proof: We want first to show that for every ε>o there exists a continuous function such that

$$dist\ [f(x), F(x)] < \varepsilon \qquad \text{for every } x \in X$$

For every $y \in R^n$ consider the set

$$U_y = \{x \in X : dist\ [y, F(x)] < \varepsilon\}$$

The sets U_y are open, since the set-valued map F is lower semi continuous, and the set U_y is equal to

$$\{x \in X : F(x)\} \bigcap \{z \in R^n : d(z,y) < \varepsilon\} \neq \varnothing$$

The family $\{U_y\}_{y\in Rn}$ covers the compact set X and therefore there is a finite sub-covering, say

$U_1 = U_{y1}, \ldots, U_r = U_{yr}$

Define the function $a_i(x) = dist(x, X/U_i)$ $(i=1,\ldots,r)$.

$$\beta_i(x) = \propto_i(x)/\sum_{k=1}^{r} \propto_k(x) \qquad , x \in X.$$

Thus, $\beta_i(x)=o$ if and only if $x \notin U_1$, $o \le \beta_i(x) \le 1$ and $\sum_{i=1}^{k} \beta_i(x)=1$. (Since if $x \notin U_i$ then $\propto i(x) = o$ i.e $dist=o$) Now define $f(x) = \sum_{i=r}^{r} \beta_i(x)\, y_i$ $x \in X$.

The function f is continuous since $B_i(x)$ $(i=1,\ldots,r)$ are continuous by definition, $f(x)$ is a convex combination of those points y_i $(i=1,\ldots,r)$ for which the corresponding set U_i contain point x.

Now, by definition of the sets U_i we have $y_i \in B\varepsilon\{F(x)\}$.and consequently $f(x) \in B\varepsilon\{F(x)\}$.

We want to show that there exists a sequence of functions which converges uniformly to a selection f of F.

Let f be a continuous function such that $dist[f(x), F(x)] < ½$ consider the set-valued map F_2 defind by

$$x: \quad F(x) \cap \{y: d\{y, f_1(x)\} < ½\}$$

The set valued map F_2 is convex-valued and lower semi continuous

Since $\{y: d(y, f_1(x)\} < ½\}$ is an open set with $F(x) \cap \{y: d(y, f_1(x)\} < ½\} \neq \emptyset$.

Since F is L.S.C so for every $x' \in N(x)$ ………..(*)

But $F(x') \cap \{y: d(y, f_1, (x)\} < ½\} \neq \emptyset$. So let $\{y: d(y, f_1, (x)\} < 1\}$ be open set such that

$F_1(x) \cap \{y/d(y, f_1, (x)\} < 1\} \neq \emptyset$. Let $N(x)$ (*) be neighborhood of x so for every $x' \in N(x)$ $F_1(x^1) \cap \{y: d(y, f_1, (x)\} < 1\} \neq \emptyset$.

By the first part of the proof there exists a continuous function f_2 such that $dist\,[f_2(x), F_2(x) < \frac{1}{2}^2$.

By this way we define inductively the sequence $\{f_n\}$ given $f_1, \ldots, f_{n-1}$ we defined f_n as continuous function with property that $dist\,[f_n(x), F_{n-1}(x)\} < \frac{1}{2}^n$ where the set valued map F_n is defined by $F_n(x) = F(x) \bigcap \{y : d\{y, f_{n-1}, (x)\} < \frac{1}{2}^{n-1}\}$

The sequence defined in this way has the property $[f_n(x), F(x) < \frac{1}{2}^n$ .and $d\,(f_n(x), f_{n+1}, (x)\} < 3/2^{n+3}$ for every $x \in X$. Thus the sequence $\{f_n\}$ is converge $f(x)$ for $x \in X$.

Now

We just proved the existence of a continuous function whose graph in near the graph of a convex-valued set-valued' map. Therefore kakutani's theorem follows from Brouwer's theorem, since we show that for every ε>o there exists a continuous functions $f_n : A \rightarrow A$ such that the graph of f_n of the function f is contained in the ε ball around the graph of F of the set-valued map F. since as we showed above that for every $x \in X$

$$dist\,[\{x, f(x)\}, Graph\,(F)] = inf_{z \in Graph\,F}\, d\,[\{x, f(x)\}, z] < \in$$

since f_n is continuous so if $x_n \rightarrow x$ and $f_n(x_n) \rightarrow y$ then $(x,y) \in Graph\,F$ i.e $\in F(x)$

If x_n is a fixed point of f_n so every limit point of the sequence $\{x_n\}$ is a fixed point of F.

Example: Let $F : [0,1] \rightarrow [0,1]$

Let $F(x) = \{1 - x/2\}$ for $0 < x \leq 0.5$

$F(0.5) = [0.25, 0.75]$

$F(x) = \{x/2\}$ for $1 \geq x > 0.5$

Where F is u.s.c and convex valued .The fixed point $x=0.5$

Chapter Four

The Existence of Walrasian Equilibrium

We will show how mathematical concepts introduced in the previous chapters, can be used to establish the existence of Walrasian equilibrium, in both pure exchange economy and a private ownership economy , with finite number of agents and commodities under sufficient conditions of the convexity of a preference relation and consumption and production sets, and the boundness of attainable set . The proof consists of three steps: in the first we define an auxiliary economy, whose consumption sets and production sets are bounded, and convex , the auxiliary economy has the same equilibrium price as the original one .In the second step we modify the fixed point theorem which is used in the third part to show the existence of equilibrium in the auxiliary economy.

4.1Basic Result for Pure Exchange Economy:

We begin by studying the case of a pure exchange economy. We take $X_i = R^L_+$ and assume at the outset that each consumer's preferences relation is continuous, convex and monotone, locally nonsatiated. We also assume that $\Sigma_i\, w_i > o$.

The vector $x_i\ (p,p.\ w_i) - w_i \in R^L$, lists consumer i's net or excess demand for each good over the amount that he possesses in his endowment vector w_i.

In fact, we will consider the price space as confined to strictly positive prices to forbid the unbounded demand by consumer. If a good is desired and price is zero, consumers would demand an unbounded large quantity of this good, if one agent possess only this good his income will be zero.

Before we prove our main result, we will pave by the following theorems.

4.1 Theorem[(9): page 151]

suppose that the rational preference relation $\succeq_i$ is continuous monotonic, locally non satiated defined on $X_i = R^L_+$. Then the Walrasian demand set-valued map $x_i\ (p,p.w_i)$ possesse the following properties:

1- The demand set $x_i\ (p,p.w_i)$ is non-empty and compact for every $p>o$.

2- The demand set-valued map $x_i\ (p,p.w_i)$ is homogeneous of degree zero in prices (i.e for every $p>0$ and $\lambda>0$ one has $x_i\ (p,p.w_i) = x_i\ (\lambda p,\ \lambda p.w_i)$

3- The demand set-valued map $x_i\ (p,p.w_i)$ is upper semicontinuous at every $p>0$.

4- If the preference relation $\succeq_i$ is convex then the demand set $x_i\ (p,p.w_i)$ is convex.

5- If the preference relation $\succeq_i$ is strictly convex then $x_i\ (p,p.w_i)$ is a continuous function in p.

6- For every price vector $p>o$ and $x \in x_i\ (p,p.wi)$ one has $p.x=p.w_i$ (Walras'law)

Before the proof of theorem 4.1 we will prove this lemma:

4.2lemma [(9):page 43] let A be a compact subset of X and let $\succeq_i$ be a continuous transitive complete preference relation on X, then there exists a best element $x^* \in A$ (*i.e* $x^* \succeq_i x$ for all $x \in A$) and the set $A\ \succeq_i$ of all best elements is compact.

proof: For every $a \in A$ consider the 'not worse than a' set $\{x \in A: x \succeq_i a\} = A_a$. By the continuity of the preference relation the set A_a is closed. The set $A_\succeq$ of best elements in A is then $\bigcap_{a\in A} A_a = A\succeq_i$,as an intersection of closed sets, thus the set $A_\geq$ is a closed , since $A_\geq$ is closed , subset of a compact set so $A\succeq_i$ is compact . We now want to show that it is non empty. For this, we observe that for any finite set $\{ a_1, \dots a_q \}$ of points in A. since $A_{ai} = \{x \in A:\ x \succeq_i a_i\},\ (i=1,\dots,q)$ then . $\bigcap_{i=1}^{q} A_{a_i} \neq \phi$ Indeed, by transitivity of $\succeq_i$ there is a best element in the finite set $\{a_1,\dots a_q\}$, say a_q. But then $A_{a_q} \subset A_{ai}$ for all $i=1,\dots q$ .Hence $\bigcap_{i=1}^{q} A_{ai} = A_{a_q}$ and this set is nonempty since $a_q \in A_{a_q}$. Now, since the intersection of any collection of closed subsets of a compact set is nonempty if the intersection of every finite part of the collection is nonempty. Therefore $A_{\succeq} = \bigcap_{a\in A} A_a$ is nonempty.

Now the Proof of theorem 4.1:

1. If $p>0$ and $w_i>0$ then the budget set $B(p,w)=\{x\in R^L{}_+ : p.x \le P.w_i\}$ is nonempty, bounded and closed, thus compact. Since the preference relation is continuous it follows (by 4.2) that $x_i\ (p,p.w_i)$ is nonempty and compact since $x_i\ (p,p.w_i) = \{x\in B\ (p,w_i) : x \succeq_i y \text{ for all } y\in B\ (p,w_i)\}$.

2. For homogeneity, note that for any scalar $\lambda>0$.

$\{x\in R^L{}_+ : \lambda p.x \le \lambda\, p.w_i\} = \{x\in R^L{}_+ : p.x \le p.w_i\}$ so $xi\ (p,p.w_i) = x_i\ (\lambda p, \lambda p.w_i)$

3. To show that $x_i\ (p,p.w_i)$ is upper semicontinuous, at $p>0$ (by 3.9) we have for every sequence $\{p_n\}\to p$ and for every $x_n \in x_i\ (p_n,p_n.w_i)$ there is a converging subsequence of $\{x_n\}$ whose limit belongs to $x_i\ (p,p.w_i)$ since $p>0$ we have $p_n>0$ for $\overline{n}$ large enough. Let

$$\gamma = inf\{p_n ; n\ge \overline{n}\}$$

so $\gamma>0$. since $x_n \in B\ (p_n, w_i)$ we have

$$0<x_n\,\gamma \le p.w_i\ ,\ 0<x_n \le p.w_i/\gamma$$

Thus, the sequence $\{x_n\}$ is bounded and therefore there is a convergent subsequence which we also write $\{x_n\}\to x\hat{}$, $x\hat{} \in B\ (p,w_i)$, since $p_n.x_n \le p_n.w_i$ and $p_n\to p$ implies $p.\ x\hat{} \le p.w_i$.

Now we want to show that $x^{\wedge} \in x_i\ (p,p.w_i)$ i.e for every $y \in B\ (p,w_i)$ it follows that $x^{\wedge} \succeq_i y$. In the trivial case $w_i=0$ and hence $p.w_i=0$ the vector 0 is the only element in the budget set and thus the best element. Now, assume that y is such that $p.y<p.w_i$, then for n large enough we have $p_n.y<p_n.w_i$. Since $x_n \in x_i\ (p,p.w_i)$ it follows that $x_n \succeq_i y$. Yet the preference relation is continuous and $\{x_n\} \to x^{\wedge}$, it follows that $x^{\wedge} \succeq_i y$. Thus we have shown that every vector y which costs less than P.w_i is not better than $x^{\wedge}$. On the other hand ,suppose y is such that $p.y=p.wi$. Since $p.w_i>0$, then we can find a sequence $\{x_n\}$ converging to y with $p.x_n<p.w_i$. Hence by the above discussion, $x \succeq_i x_n$ and by continuity of $\succeq_i$ we obtain $x^{\wedge} \succeq_i y$..

4. $x_i\ (p,p.w_i) = \bigcap_{z \in B(p,wi)} \{x \in B\ (p,w_i): x \succeq_i z\}$

By the definition of convex preference the set $\{x \in R^L_{++} : x \succeq_i z\}$is convex. Since the budget set is convex, it follows that the set $\{x \in B\ (p,w_i): x \succeq_i z\}$ is convex, and consequently the intersection of all these sets, that is the demand set $x_i\ (p,p.w_i)$ is itself convex.

5. Let x and $y \in x_i\ (p,p.w_i)$ with $x \neq y$, then $x \sim y$. Thus by the assumption of strictly convex we have $\frac{1}{2}\,(x+y) \succ x$, but since $\frac{1}{2}\,(x+y) \in B\ (p,w_i)$ this is a contradiction to $x \in x_i\ (p,p.w_i)$.Therefore, the demand set-valued map is actually a function. But for a function which is upper semi continuous means continuity.
6. Walras law follows from local nonsatiation. If $p.x<p.wi$ for some $x \in x_i\ (p,p.w_i)$ then there must exist another consumption bundle y sufficiently close to x with both $p.y<p.w_i$ and $y \succ x$ but this would contradict that x is optimal.

Note: all these properties hold for total demand

$x(p,p.w) = \sum_{i=1}^{m} x_i\ (p,p.w_i)$ where m stands for consumers.By using Theorem (3.12).

One of the most useful Theorems employed in mathematical economic is the maximum Theorem which deals with the case where a continuous real-valued function is being maximized over a compact set which varies continuously with some parameter vector. The set of solutions is u.s.c with compact values, and the value of the maximized function varies continuously with the parameters.

4.3 Theorem[(2):page 84]

(Maximum Theorem) Let $X\subset R^m$, $Y\subset R^k$ and let $\alpha:X\to Y$ be a compact-valued set-valued map, Let $f:Y\to R$ be a continuous function. Define the set-valued map $u:X\to Y$ by $u(x)=\{y\in\alpha(x):y\ \ max\, f$ on $\alpha(x)\}$, and the function $g:X\to R$ by $g(x)=f(y)$ for $y\in u(x)$. If α is continuous at x then u is closed and u.s.c at x and g is a continuous at x Furthermore u is a compact-valued.

Proof: since α is compact-valued, u is nonempty and compact valued. and since f is continuous in a compact set so it has amaximizer It suffices to show that u is closed at x, for then $u=\alpha\cap u$ and (by 3.6) , u is u.s.c at x. Let $x_n\to x, y_n\to y, y_n\in u(x_n)$. We want to show that $y\in u(x)$ and $g(x_n)\to g(x)$. since α is u.s.c and compact-valued (by 3.9) $y\in\alpha(x)$. suppose $y\notin u(x)$. Then there is a $z\in\alpha(x)$ with $f(z)>f(y)$. since α is l.s.c at x (by 3.16) there is a sequence $\{z_n\}$ such that $z_n\to z$ and $z_n\in\alpha(z_n)$. Since $z_n\to z$, $y_n\to y$ and $f(z)>f(y)$, the continuity of f implies that eventually $f(z_n)>f(y_n)$ contradicting $y_n\in u(x_n)$. Now $g(x_n)=f(y_n)\to f(y)=g(x)$, so g is a continuous at x.

4.4 Definition:

Let f be a real-valued function defined on a convex set $X\subset R^n$. Then f is convex , if $f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$ for all $x\,y\in X$ and all λ such that $0\leq\lambda\leq 1$.

4.5 Theorem [(2): page 17]

Let f be a convex function defined on a convex set $X \subset R^n$. The set of points defined as : $V=\{x \in X: f(x)) \leq \alpha, \alpha \in R\}$, is a convex set.

Proof: Le $x,y \in V$ and consider the point $\lambda x + (1-\lambda)y$ *for* $\lambda \in [0.1]$, since f is convex ; $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$, but $f(x) \leq \alpha$ and $f(y) \leq \alpha$, so that for $0 \leq \lambda \leq 1$, we have $\lambda f(x) + (1-\lambda) f(y) \leq \lambda \alpha + (1-\lambda) \alpha = \alpha$, then $f(\lambda x + (1-\lambda)y) \leq \alpha$. But this implies that $\lambda x + (1-\lambda)y \in V$ for $\lambda \in [0.1]$ and so V is convex

4.6 Definition: Let f be a real valued function defined on a convex set $X \subset R^n$. Then f is concave if $f(\lambda x + (1-\lambda)y \geq \lambda f(x) + (1-\lambda) f(y)$ for all $x,y \in X$ and all $\lambda \in [0,1]$.

Now we want to show that $V=\{x \in X: f(x) \geq \alpha, \alpha \in R\}$ is convex .

Let $x,y \in V$ since f is a concave , $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)y \geq \lambda \alpha + (1-\lambda)\alpha = \alpha$ so $f(\lambda x + (1-\lambda)y) \geq \alpha$ for $0 \leq \lambda \leq 1$ so V is convex.

4.2 the Existence of Walrasian Equilibrium for Pure Exchange:

We have the following primitive concepts of this model, the commodity space $X = R_+^L$, a set m of agents each has the following characteristics of consumption set $X=R^L{}_+$, preference $\succeq_i$ and initial endowment , $w \in X = R_+^L$.

From these we derive two concepts .

1- The budget set $B(p,w) = \{ x \in R^L{}_+ : p.x \leq p.w \}$

2- The demand set $x(p,w) = \{x \in B\ (p,w) : x \succeq_i y$ for all $y \in B\ (p,w)\}$

The above concepts lead us to introduce the idea of competitive or Walrasian equilibrium. Recall that the principle is that there is a given price system. At these prices individual chooses the vectors , that he most prefers in his budget set. If the total demand equals the total supply of all goods at these prices , then this is said to be a Walrasian equilibrium.

4.7 Definition: A Walrasian equilibrium for an economy ε is an allocation x^* and a price system $p^* \in R^L{}_+$ such that.

i. $x_i^* \in x_i\,(p^*, p^*.w)$ for all i $(i=1,\dots m)$

ii. x_i^* is a redistribution for ε i.e $\sum_{i=1}^{m} x_i^* = \sum_{i=1}^{m} w_i$

4.8 Definition: A set valued map $Z(p)$ from P to Z where $Z=X-\{w\}$ defined by $Z(P) = \sum_{i=1}^{n} x_i \;(p, w_i) - \sum_{i=1}^{n} w_i$ is called the excess demand set valued map.

Note first that if $p \in P$ and if $t>0$, we have $x_i\;(tp,\, tw_i)=x_i\,(p,w_i)$. In other words, if all prices of a price system in P are multiplied by the same positive number, the sets of optimal actions of various agents are unchanged. Hence $tp \in P$ and $Z\,(tp) = Z\,(p)$.

Note also that $x_i\,(p,w_i)$ which is chosen by the agents for a price system p in P satisfies the wealth constraints:

$$p\,x_i\,(p,w_i) \leq p.w_i \text{ for every } i.$$

$$P\sum_{i=1}^{n} x_i\,(p,w_i) \leq p \sum_{i=1}^{m} w_i \text{ so } p\,(\sum_{i=1}^{m} (x_i\,(p,w_i)-w_i) \leq 0$$

i.e $P\,Z\,(p) \leq 0$

4.9 Example: A price vector P^o is an equilibrium price vector if $Z(P^o) \leq 0,\; P^o \geq 0$, and if $Z_k(P^o) < 0$ then $P^o{}_k = 0,$ where Z is a vector of aggregate excess demands. Suppose the aggregate excess demand function is continuous, homogeneous of degree zero [Z_k $(\lambda p) = Z_k(p)$ for all $\lambda > 0$] and Walras' Law holds [$\Sigma_k Z_k\,(p)\,p_k = o$ for all $p \geq 0$].

Then an equilibrium price vector exists, as we shall now show.

Since the excess demand function (here the preference relation is strictly convex)is homogeneous we can restrict attention at price vector in the unit simplex $P= \{p\in R^n: \Sigma_k\ p_k=1, p_k\ \geq 0$ for all $k\}$. P is closed, bounded and convex. Now consider the continuous mapping $f:P\rightarrow P$ defined by $f_k\ (p)= \{p_k + max\ [0, z_R\ (p)]\} / \{1+\Sigma_k\ max\ [0,z_k\ (p)]\}$, for all k. By Brouwer's Theorem f has a fixed point, there exists a $p^o\in P$ such that $p^o=f(P^o)$, i.e $p^o_k = \{p^o_k + max\ [0,z_k\ (p^o)]\} / \{1+\Sigma_k\ max\ [0,z_k\ (p^o)]\}$ for all k. so $p^o_k\ \{\Sigma_k\ max\ [0,z_k(p^o)]\} = max\ [0,z_k\ (p^o)]\}$ for all k. now $\Sigma_k\ z_k\ (p^o)\ p^o_k\ \{\Sigma_k\ max\ [o,z_k(p^o)]\} = \Sigma_k\ z_k(p^o) max\ [0,z_k(p^o)]\}$. From Walras Law $\Sigma_k\ z_k(p^o).\ p^o_k=0$. Thus $\Sigma_k\ z_k(p^o)\ max\ [0,z_k(p^o)]\}=0$.

Each term in the sum is greater than or equal to zero, since each term is either 0or $[z_k(p^o)]^2$. But is any term were strictly greater than zero the equality could not hold. Hence, we have $z_k(p^o)\leq 0$ for all k. finally, we need to show the if $z_k(p^o)<0$ then $p^o_k=0$. suppose $z_k(p^o)<0$, From Walras'law $\Sigma_i\ z_i\ (p^o).\ p^o = {}_i\Sigma_{i\neq k}\ z_i\ (p^o).\ p_i + z_k\ (p^o)\ p^o_k=0$. But $z_i\ (p^o)<0$ and $(p_i{}^o)\geq 0$ for all i, so that $\Sigma_{i\neq k}\ z_i\ (p^o)\ p_i\leq 0$. Hence, we must have $z_k\ (p^o).\ p^o_k\geq 0$, otherwise Walras'Law would be violated. But if $z_k\ (p^o)<0$ then $p^o_k\leq 0$. But $p^o_k\geq 0$ Hence, $p^o_k=0$.

4.10Theorem [(9): page 162]

Let ε be an economy with a convex, monotone preference relation and $X_i = R^L{}_+$, $\Sigma_i\ w_i>0$. Then there exists $p^*\in R^L$ with $p^*>0$ such that $0\in Z(p^*)$.

Now we want to prove this Lemma .

4.11 Lemma [(9): page 162]

Let X be a closed and convex subset of the unit simplex P in R^L. assume that the set-valued map ψ of X into R^L has the following properties:

1. ψ is bounded, i.e there exists abounded set B in R^{L} such that $\psi(p) \subset B$ for every $p \in X$.

2 . the graph of ψ is closed.

3 . $\psi(p)$ is convex for every $p \in X$.

4. for every $p \in X$ and every $z \in \psi(p)$ we have $p.z \leq 0$.

Then there exist $p^* \in X$ and $z^* \in \psi(p^*)$ such that $p.z^* \leq 0$ for every $p \in X$.

Proof: we can assume that the bounded set B in (i) is convex. For every vector $z \in B$, we consider the set $u(z)$ of maximizers of $p.z$ in X, i.e $u(z)=\{p \in X: p.z = max_{q \in X}\, q.z\}$. Since P is nonempty and compact, $u(z)$ is nonempty. The set-valued map $z \rightarrow u(z)$ from B to P is u.s.c and convex.

Define a function $g: P \rightarrow R$ by $g(p) = z.p$ which is continuous and define the set-valued map $\alpha: B \rightarrow P$ by $\alpha(z)=P$. As a constant mapping α is a continuous set-valued map. Moreover compact valued since P is compact. we have $u(z)=\{p \in X: p.z = max_{q \in X}\, q.z\}$ which can be expressed as $u(z)=\{p \in \alpha(z): p$ maximizes g on $\alpha(z)\}$. Then by the maximum theorem we have $u(z)$ u.s.c.

By theorem (4.5) $u(x)$ is convex for either $z=0$ and the $u(z)$ is P itself or $z \neq 0$ and then $u(z)$ is the intersection of P and $\{p: p.z = max_{q \in X}\, q.z\}$. consider now the correspondence $\varnothing: X\text{x}B \rightarrow X\text{x}B$ given by $\varnothing(p,z) \rightarrow u(z) \times \psi(p)$. Then the set $X\text{x}B$ is nonempty, compact and convex since X and B have the same properties. The correspondence is $\varnothing$ *is* u.s.c as well as u and ψ (by 3.11). For all $x= (p,z) \in X\text{x}B$ the set $\varnothing(p,z)$ is nonempty, convex, and of closed gragh since $\varnothing$ is u.s.c (by 3.5)as $u(z)$ and $\psi(p)$ have these properties. Now all conditions of kakutati's fixed point theorem satisfies. Therefore there exists afixed point $x^* \in \varnothing(x^*)$,$x^* = (p^*,z^*) \in \varnothing(p^*,z^*)$ Thus $(p^*,z^*) \in u(z^*) \times \psi(p^*)$ or $p^* \in u(z^*)$ and $z^* \in \psi(p^*)$.

From $p^* \in u(z^*)$ it follows that $p.z^* \leq p^*.z^*$ for all $p \in X$. But from $z^* \in \eta(p^*)$ it follows by assumption (iv) that $p.z^* \leq 0$ for every $p \in X$.

Now

$Z(p)$ is not defined if some $p^l=0$, since demand becomes infinite .Thus $Z(p)$ only satisfies the conditions of (4.1) if $p>0$. Therefore we take a sequence of X, say, X_n with $p>0$ and apply the (4.11) to each of the X_n and show that $0 \in Z(p^*)$ where p^* is the limit of the sequence $\{p_n{}^*\}$.

Since Z is bounded below .We can see from Theorem (4.1)(3) that $x_i\ (p,p.w_i)$ is u.s.c at every $P>0$. We have also $Z_i(p)= x_i\ (p,p.w_i)- w_i$,so $Z(p)$is u.s.c. (by4.1) $x_i(p.p.w_i)$is closed ,thus $Z_i\ (p)$ is closed ,now (by 3.5) $Z\ (p)$ has closed graph .

Since $x_i\ (p,p,\ w_i)$is convex (by 4. 1) so is $Z_i\ (p)$. Now for every $p \in B\ (p,w)$, since the consumer is confined to price and wealth so $p.x_i\ (\ p,p.w_i)- pw_i \leq 0$.

So $pZ\ (p) \leq 0$ .Now the conditions of Theorem 4.11 are satisfied .

Proof the Theorem 4.10:

Consider the sequence of sets X_n as follows $X_n=\{p \in P:p^L \geq 1/n,\ L=1,\dots,L\}$ $(n \geq 1)$. $\bigcup_n X_n\ =P$. Now the correspondence Z in bounded below , for each X_n the conditions of lemma 4.2.5 are satisfied , since for every $z \in Z(p)\ p.z \leq 0$. Thus, for each n there exist vectors $p_n{}^* \in X_n$ and $z_n{}^* \in Z\ (p_n{}^*)$ such that $p.z_n{}^* \leq 0$ for all $p \in X_n$.

Now assume that the sequence $\{p_n{}^*\}$ converges, say to p^*. Furthermore the sequence $\{z_n{}^*\}$ is bounded, since $\bar{p}.z_n{}^* \leq 0$ for some arbitrary $\bar{p} \in X_n$, $\bar{p}>0$, and Z in bounded below. Assume without loss of generality, that $\{z_n{}^*\}$ converges to z^*. But $p^*>0$ otherwise $\{z_n{}^*\}$ would be unbounded , by the u.s.c of Z we obtain $z^* \in Z(p^*)$.

Thus, we have $p^*.z^*=0$ (Walras' Law). Since $p.z^* \leq 0$ for all $p \in P$ we have $z^* \leq 0$ and so $p^*.z^*=0$ and $p^*>0$ implies $z^*=0\ \in Z(p^*)$.

4.3Basic Result for Private Ownership Economy:

We consider an economy with a positive finite number L of commodities, a positive finite number m of consumers and a positive finite number n of producers. We denote by X_i the consumption set of the i.th consumer $(i=1,\dots,m)$, by $w_i \in R^l$ his initial endowment vector and by $\succeq_i$ his preference relation on X_i, For all $x_i \in X_i$, let $P_i\ (x_i)=\{x'_i \in X_i:\ x'_i \succ_i x_i\}$ be the set of consumption space which are preferred to x_i. We let w be the total initial endowment vector, that is, $w=\sum_{i=1}^{m} w_i$.the technological possibilities of the j-th producer $(j=1,\dots,n)$ are represented by a subset Y_j of R^l. We denote by Y the total production set of the economy, that is, $Y=\sum_{j=1}^{n} Y_j$.

For all $(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}$, the real number θ_{ij} denotes the share of the i-th consumer in the profit of the j-th producer.

An economy ε is a collection

$\varepsilon = \{(X_i, \succeq_i, w_i)_{i=1,\dots,m}, (Y_j)_{j=1,\dots,n} (\theta_{ij})_{i=1,\dots,m\, j=1,\dots,n}\}$

Throughout the rest of this chapter, we will study economies where consumers own the resources and control the producers. A complete description of a private ownership economy ε therefore consists of:

For each consumer, his consumption set X_i, his preference relation $\succeq i$ his endowment w_i (satisfying $\sum_{i=1}^{m} w_i=w$), and his share $(\theta i_1,\dots \theta_{ij})$ (satisfying $\theta_{ij} \geq 0$ and $\sum_{i=1}^{m} \theta_{ij}=1$ for every j). for each producer, his production set Y_j.

Consider a private ownership economy ε.It is assumed that the firm j chooses a production space which maximizes its profit relative to $p\in P$, that is, firm j chooses $y_j\in Y_j$ which is a maximum for the set $p\ Y_j$. In general, this determines a point-set mapping $s_j:P\rightarrow Y_j$ defined by $s_j\ (p)=\{y_j:y_j\in Y_j,p.\ y=max\ pY_j\}$ s_j is the supply set-valued map of firm j. for the economy as a whole, the aggregate supply set-valued map is defind by $S(p) = \sum_{j=1}^{n} s_i\ (p)$.

A profit function for firm j is a mapping $\pi_j:p\rightarrow R$ defined by $\pi_j(p)=max\ pY_j$.

4.12 Proposition [(2):page 89]

suppose that $\pi\ (.)$ is the profit function of a production set Y and $s(.)$ is the associated supply set-valued map. Assume also that Y is closed and satisfies the free disposal property, Y_j is compact for every j and convex. Then

1 $\pi(.)$ is homogeneous of degree one.

2 $s(.)$ is homogeneous of degree zero.

3 $s_j\ (.)$ is u.s.c and $\pi_j(.)$ is continuous for each j.

4 $S(.)$ is u.s.c.

4 $s_j\ (.)$ and $s(.)$ are convex sets.

Proof:

1 $\pi(p) = max\ \{p.y:y\in Y\}$

$$\pi(\alpha p) = max\ \{\alpha p.y:y\in Y\} = \alpha\ max\ \{p.y:y\in Y\}\ \alpha>o$$

$$\alpha\pi(p) = \alpha max\ \{p.y:y\in Y\} = max\ \{\alpha p.y:y\in Y\}\ \pi(\alpha p)$$

2 $s(p)=\{y\in Y:p.y = \pi\ (p)\}$we have $\pi(\alpha p) = \alpha\pi(p)$

Then $s(\alpha p) = \{y\in Y:\alpha p.y = \pi(\alpha p)\} =$

$s(p)=\{y\in Y:p.y=\pi(p)\}$

so $s(p)=s(\alpha p)$

3 Define the function $g:Y\rightarrow R$ by $g(y_j) = p.y_j$ which is continuous ,and defines the set-valued map $\alpha:P\rightarrow Y_j$ by $\alpha(p)=Y_j$. As a constant mapping α which is continuous and set-valued map. Moreover, α is compact-valued, since Y_j is a compact. Firm j's supply set-valued map is $S_j\ (p) =\{Y_j:y_j\in Y,\ p.y = max\ pY_j\}$, which can be expressed as $S_j\ (p)=\{y_j\in\alpha(p):y_j\ maximizes\ g\ on\ \alpha(p)\}$. Firm j's profit function is $\pi_j\ (p)=max\ \ pY_j$ which can be expressed as $\pi_j\ (p)=g\ (y_j)$ for $y_j\in S_j\ (p)$. Then, from the Maximum Theorem, we have S_j is a closed and u.s.c and π_j is a continuos. Further more S_j is a compact valued.

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4 (By.3.12)

5 (By 4.5)

we will now formally define the notions of Walrasian equilibrium and quasi-equilibrium.

4.13 Definition:

A Walrasian equilibrium of the economy ε is an element $(x_i^*,\ y_i^*, p^*)$ of $R^{l(m+n+1)}$ such that $p^* \neq 0$, and

i- for all $i=1,\ldots,m,\ x_i^*$ is a greatest element of $\succeq_i$ in

$$Bi=\{x^*\in X_i:\ p^*.\ x_i \leq p^*\ (\sum_{i=1}^{n}\ \theta_{ij}\, y_j^* + w_i).$$

That is $x_i^* \in B_i^*$ and $x_i^* \succeq_i x_i$ for all $x_i\in B_i^*$

ii- for all $j=1,\ldots,n,\ y_j^* \in Y_j$ and for all $y_j\in Y_j,\ p^*.y_j^* \geq p^*.y_j$

$$iii- \sum_{i=1}^{m} x_i^* = \sum_{i=1}^{n} y_j^* + w$$

A quasi-equilibrium of the economy is an element $(x_i^*, y_j^*, p^*) \in R^{l(m+n+l)}$ such that $p^* \neq 0$ which satisfies conditions (b) and (c) together with.

(q-a) for all $i=1,\dots,m$, $x_i^* \in B_i^*$ and for all $x_i \in Pi(x_i^*)$

$$p^*. x_i \geq p^* \left(\sum_{j=1}^{n} \theta_{ij} y_j^* + w_i \right).$$

(x_i, y_j) of ε is attainable if $x_i \in X_i$ for every i, $y_j \in Y_j$ for every j, $\sum_{j=1}^{m} x_i - \sum_{j=1}^{n} y_j = w$. The set of attainable states of ε is denoted by A.

Given an economy ε, a consumption x_i for the i-th consumer is said to be attainable if there is an attainable state whose component corresponding to that consumer is x_i. The set of his attainable consumption's is called his attainable consumption set, and denoted $\hat{X}_i$. An attainable production for the i-the producer and his attainable production set $\hat{Y}_j$ are similarly defined. According to the definition $\hat{X}_i$ (resp Y_j.) is the projection of A on the space R^l containing X_i (resp Y_j) $A = \{(x_i, y_j) \in (\prod_{i=1}^{m} X_i) \times (\Pi_{j=1}^{n} Yj) .: \sum_{i=1}^{n} x_i = (\sum_{i=1}^{n} y_j^* + w_i)$.

The properties of $\hat{X}_i$ and $\hat{Y}_j$ are immediately derived from those of A. For example, if A is bounded, or compact or convex every $\hat{X}_i$ and every $\hat{Y}_j$ is respectively bounded, compact convex.

We set the following assumptions which describes the framework here.

Assumption1:

For all $i=1,\ldots,m$, X_i is a closed, convex subset of R^l and $\succeq_i$ on X_i which is rational and closed [for $x_i \in X_i$, the sets $\{x\hat{}_i \in X_i: \hat{}x_i \succeq_i x_i\}$ is closed) and convex, and locally nonstaiated on $\hat{X}_i$ [for $x_i \in X\hat{}_i: x_i \in \overline{Pi}\,(x_i)$].

Assumption2:

Y is closed and convex, note that the individual production sets are not assumed to be convex. Y_j is closed for all i.

Assumption3:

A is bounded. The attainable set.

Assumption4:

Survival assumption(for all $i=1 \ldots,m$,$X_i \bigcap (\{w_i\} + \sum_{j=1}^{n} \theta_{ij} Y_j) \neq \varnothing$.

It is satisfied if $0 \in Y_j$ for every j and $w_i \in X_i$ for all i so this implies that A is not empty.

4.4 The existence ofWalrasian Equilibirium For Pravite Ownership Economy:

4.14 Theorem [15:page 4]:

The economy ε has a quasi-equilibrium under assumption (1-4). Further more, the quasi-equilibrium (x_i^*, y_i^*, p^*) is a Walras equilibrium if it satisfies the following condition.

$$Inf\, p.^* X_i < p^* \left(\sum_{i=1}^{n} \theta_{ij} y_j^* + w_i \right) \text{ for all } i=i,\ldots,m.$$

In chapter 1 we see when quasi-equilibrium is Walras equilibrium.

Proof:

.We want to build an auxiliary economy with bounded consumption and production sets such that one can construct an equilibrium of the original economy from an equilibrium of this auxiliary economy. From assumption (4), for every $i=1,\dots,m$ there exists $(\hat{x}_i, \hat{y}_j) \in X_i \times {}^n\prod_{j=1}^{n} Y_j$ such that $\hat{x}_i = w_i + \Sigma\theta_{ij}\, \hat{y}'_j$ Since $X_i \bigcap \{w_i\} + \sum_{i=1}^{n} \theta_{ij}\, Y_j) \neq \varnothing$.

From assumption (3) A is bounded and we have X_i and Y_j are closed so A is closed thus A is compact, therefore the sets $\hat{X}_i$ and $\hat{Y}_j$ are bounded for every i,j. Thus we can choose a closed ball B in R^l of center o and radius $r>0$ sufficiently large so that the following holds: $\hat{X}_i \subset int\, B$ and $\hat{x}_i \in B$ for every $i=1,\dots,m$ $\hat{x}_i \in X_i$, but we have $\hat{x}_i = w_i + \sum_{j1}^{n} \theta_{ij}\, \hat{y}'_j$, by summing up over i one gets $\sum_{i=1}^{m} \hat{x}_i = \sum_{j=1}^{m} (\sum_{j=1}^{n} \theta_{ij}\, \hat{y}'_j) + w = \sum_{j=1}^{n} (\sum_{i=1}^{n} \theta_{ij}\, \hat{y}'_j) + w$. But for all j, $\sum_{j=1}^{n} \theta_{ij}\, \hat{y}'_j \in co\, Y_j$ (see chapter 2).

Consequently $\sum_{i=1}^{n} (\sum_{j=1}^{m} \theta_{ij}\, \hat{y}'_j)$ belongs to $\sum_{j=1}^{n} co\, Y_j \subset Y$ since Y is convex by assumption (2). Hence there exists $\alpha_j \in \prod_{j-1}^{n} Y_j$ such that ${}^n\Sigma_{j=1} (\sum_{j=1}^{m} \theta_{ij}\, \hat{y}_j) = {}^n\Sigma_{j=1}\, \alpha_j$. One gets that $\sum_{j=1}^{m} \hat{x}_i = \sum_{j=1}^{m} \alpha j + \sum_{j=1}^{m} w_i$ which implies that $(x_i, \alpha_j) \in A$, so for every i, $\hat{x}_i \in \hat{X}_i$.

$\hat{Y}_j \in int\ B$ and $\hat{y}'_j \in B$ for every $j=1,\dots,n$ and every $i=1,\dots,m$

We define now an auxiliary economy ε^B having the same commodity space, m consumers, n producers as the initial economy ε. The characteristics of the agents are as follows. The consumption sets of the i-th consumer is $X_i^B = X_i \cap B$ and his preferences are the restriction of $\succeq_i$ to $X^B{}_i$. His initial endowments are w_i. The production set of the j-th producer is $Y_j^B = (\overline{co}Y_j) \cap B$ and the share of the i-th consumer in the proof of the j-th producer is θ_{ij}.

So the consumption sets and production set are bounded and closed so ,they are compact.

We want now to show the relation ship between equilibrium of ε and ε^B

4.15 Proposition[15: page5]:

Under the assumptions of (Theorem 4.4.1) if (x_i^*,y_j^*,p^*) is a Walras equilibrium (resp a quasi-equilibrium) of ε then it is a Walras equilibrium (resp a quasi-equilibrium) of ε^B.

Conversely if (x_i^*,y_i^*,p^*) is a Walras equilibrium (resp quasi-equilibrium) of ε^B then there exists $\alpha_i^* \in \prod_{j-1}^n Y_j$ such that (x_i^*,α_i^*,p^*) is a Walres equilibrium (resp quasi-equilibrium) of ε.

Proof: Let (x_i^*,y_i^*,p^*) be a Walras equilibrium (resp quasi-equilibrium) of ε. Since it satisfies condition (c) of Definition (4.14) for every i,$x_i^* \in \hat{X}_i$ and for every j,$y_j^* \in \hat{Y}_j$. From the choice of the ball B, therefore $x_i^* \in X_i \cap B$ so for every i,$x_i^* \in X_i^B$ and for every j,$y_j^* \in Y_j^B$. since i,$x_i^* \in X_i^B$ and $y_j^* \in Y_j^B$ and $w \in \varepsilon^B$ then $\Sigma_{i=1}^m x_i^* = w + {}^n\Sigma_{j=1} y_j^*$

Hence, for every i, condition (a) (resp (a-q) of Definition (4.13) is satisfied since $P_i^B(x_i^*) = Pi(x_i^*) \bigcap$,' $\subset P_i(x_i^*)$.

So for all $i=1,\dots,m$, x_i^* is a greate. element of $\succeq_i$ in

$$B^B{}_i^* = \{x_i \in X_i \bigcap B : P^*.x_i \leq P^*. (\sum_{i=1}^{n} \theta_{ij} y_j^* + w_i)\}.$$

Now, we want to show that condition (b) is also satisfied . Since for all j and all $y_j \in Y_j$, $P^*.y_j \leq p^*.y_j^*$, for all $y \in Y$. $P.y \leq p^*. \sum_{i=1}^{n} y_j^*$ let $j_o \in \{1,\dots,n\}$ and $y_{jo} \in Y_{jo}{}^B$

since Y.

Is convex $y_{jo} + \Sigma_{j \neq yo} y_j^* \in Y$ and consequently $p^*. (y_{jo} + \Sigma_{j \neq jo} y_j^*) \leq P^*. \sum_{i=1}^{n} y_j^*$. Hence

$p^*. y_{jo} \leq p^*.y_{jo}^*$.

Conversely, let (x_i^*, y_j^*, p^*) be a Walras equilibrium (resp-quasi-equilibrium) of ε^B. Since Y is convex, $\sum_{i=1}^{n} y_j^* \in Y$. Consequently there exists $\alpha_i^* \in \prod_{j-1}^{n} Y_j$ such that $\sum_{j=1}^{n} y_j^* = \sum_{i=1}^{n} \alpha_j^*$ hence $\sum_{i=1}^{m} x_i^* = \sum_{i=1}^{n} \alpha_j^* + w$. so condition (c) of Definition (4.13) is satisfied

$\alpha_j^* \in \hat{Y}_j$ for every j, hence $\alpha j^* \in Y_j^B$, therefore $p^*. \alpha_j^* \leq p^*.y_j^*$ But $\sum_{i=1}^{n} \alpha_j^* = \sum_{i=1}^{n} y_j^*$ one has $p^*. \alpha_j^* = p^*.y_j^*$ for every j.

We want to show that condition (b) is satisfied. Let $j \in \{1,\dots,m\}$ and $y_j \in Y_j$. Since $\alpha_j^* \in \hat{Y}_j \in int\ B$ there exists $t > o$ such that $(1-t)\ \alpha_j^* + ty_j \in coY_j \bigcap B$, therefore $p^*.$ $(1-t)\ \alpha_j^* + ty_j \leq p^*.y_j^* = p.\ \alpha_j^*$, so $p^* y_j \leq p^* \alpha_j^*$.

Now condition (a) (resp (a-q) is satisfied. x_i^* is a greater element of $\succeq_i$ in B_i^* if and only if $x_i^* \in B_i^*$ and for all $x_i \in P_i(x_i^*)$, $p^*.(\sum_{j=1}^{n} \theta_{ij} y_j^* + w_i) < p^*.x_i$. Let $i \in \{1,\dots,m\}$ and $x_i \in Pi(x_i^*)$ from assumption (1) $P_i(x_i^*)$ is convex (let $x_i, x_i' \in Pi(x_i^*)$ and $0 \leq t \leq 1$. Assume without loss of generality that $x_i' \succeq_i x_i$. Since $\succeq_i$ is convex so $tx_i+(1-t)x_i' \succeq_i x_i$. From the transitivity of $\succeq_i$, $x_i^* i \leq x_i i \leq tx_i+(1-t)x_i'$ so $tx_i+(1-t)x_i' \in P_i(x_i^*)$.and $x_i^* \in cl\ P_i(x_i^*)$, therefor , for all, $0 \leq t \leq 1$, $(1-t)x_i^*+tx_i \in P_i(x_i^*)$.Since $x_i^* \in \hat{X}_i \subset int\ B$ for t small enough, $(1-t)x_i^*+tx_i \in B$, so $p^*(\sum_{j=1}^{n} \theta_{ij} y_j^*+w_i) >$ (resp $(\succeq_i)$ $p^*.1-t)x_i^*+tx_i)$. Since $p^*.x_i^* \leq p^*.(\sum_{j=1}^{n} \theta_{ij} y_j^*+w_i)$, so $p^*.(\sum_{j=1}^{n} \theta_{ij} y_j^*+w_i)$ < (resp $\leq$) $p^*.x_i$.

Now the excess demand for a private ownership economy like the pure exchange , the only difference is the $x_i\ (p,(p.w_i+\sum_{j=1}^{n} \theta_{ij}\pi_j(p))$ and its supply is $\sum_{j=1}^{n} s_j(p)$ and $\sum_{i=1}^{m} w_i$.

So $Z(p) = \sum_{i=1}^{m} (x_i(p,p.w_i) + \sum_{j=1}^{n} \pi_j(p))\,\theta_{ij}) - \sum_{j=1}^{n} s_j(p) - \sum_{i=1}^{m} w_i$

And $Z = X-Y-\{w\}$.

4.16 Theorem[15: page 6]:

Let B be a closed ball of center 0 in R^l and let Z be an upper semicontinuous set-valued map from B to R^l with nonempty, convex, compact values such that.

For all $p \in \partial B$, $Sup\ P.Z(p) \leq 0$(Walras'Law).

Then there exists $p^* \in B$ such that $0 \in Z(p^*)$.

[let F be an upper semicontinuous set valued map from B to B with nonempty, convex, compact values. Let us consider the set-valued map Z defined by $Z(x)=F(x)-\{x\}$. If $x \in \partial B$, then for all $z \in Z(x)$, there exists $y \in F(x)$ such that $z=y-x$, Therefore $x.z=x.y-x.x=x.y-r^2$. since $y \in F(x) \subset B$, so $|x-y| \leq |x| \, |y| \, | \leq r^2$. Thus $x.z$-$x.y-r^2 \leq 0$. So Z satisfies the assumption of (4.4.3) which implies that there exists $\bar{x}^* \in B$ such that $0 \in Z(x^*)$, $0 \in F(x^*)-x^*$ then x^* is a fixed point of F].

Proof of Theorem (4.13) :(Since Z is upper semicontinuous with compact-values and B is compact, the set $U_{p \in B}\ Z(p)$ is compact by (3.5), and let it contained in some nonempty, convex, compact subset K of R^l.

Let $u(z)$ (see proof lemma (4.11) be the set of $p \in B$ which maximize $p.z$ on B. Since B is nonempty compact, $u(z)$ is nonempty and the set-valued map u from $K \subset R^l$ is u.s.c (as $s\cdot(p)$) .Now consider the set valued map F from Bxk to itself defined by.

$F(p,z)=u(z)\mathrm{x}Z(p)$. The set Bxk is nonempty, compact and convex since B and k have the same properties-the set-valued map $F(p,z)$ is u.s.c (by 3.11)

for all $x=(p,z) \in Bxk$, the set $F(p,z)$ is nonempty and convex and closed graph (by 3.5) as $u(z)$ and $Z(p)$ have the same properties. Now all the conditions of kakutani's Fixed point Theorem satisfied, so that F has a fixed point x^* ,so$x^* \in F(x^*)$ .Thus $x^* = (p^*,z^*)$ $\in F\ (p^*,z^*) = u(z^*) \mathrm{x}\, z(p^*)$ or $p^* \in u(z^*)$ and $z \in z(p^*)$.

From $p^* \in u(z^*)$ then for $q \in B$ $q.\ z^* \leq p^*.z^*$.

From $z^* \in z(p^*)$ from Walras'Law $p^*.z^* \leq 0$

Assume first that $p^* \in int\ B$, then the linear mappir · $q \to q.z^*$ reaches its maximum in the interior of B which implies that this mapping is constant in B i.e $z^*=0$.

Assume that $p^* \in \partial B$ then for all $q \in B$, $q.z^* \leq p^*.z^*$ from Walras'Law, one has $p^*.z^* \leq 0$, hence for all, $q.z^* \leq 0$ and p>o for som p then $z^*=0$.

We now want to show that ε^B has a quasi-equilibrium by using the fact that the excess demand set-valued map Z^B satisfies the previous theorem.

For all $j=1,\ldots$, let $\pi_j^B:B \to R$ be the profit function and let s_j^B be the supply set valued map from $B \to B^B$. defined by.

$$\pi_j^B\ (p)=max\ \{p.y_j:y_j \in Y_j^B\}$$

$$s_j^B\ (p)=\{y \in Y_j^B : p.y` \leq p.y : for\ all\ y` \in Y_j^B\}."$$

4.17 Lemma[15: page 7]: For every $j=1,\ldots,n$ the mapping π_j^B is continuous and the set valued map ε_j^B is u.s.c with nonempty, convex, compact values.

Proof: (by Maximum Theorem and by (4.5) and by (4.12).

We now define the quasi-demand set-valued map. For every $i=1,\ldots m$ and for every $(p,W) \in R^l xR$ we define the following sets:

$$B_i^B\ (p,W) = \{x_i \in X_i^B : p.x_i \leq W\}$$

$$x_i^B\ (p,W) = \{x_i \in B_i^B(p,W) : for\ all\ x_i` \in P_i^B(x_i),\ p.x_i` \geq W\}.$$

4.18 Lemma[15: page 7]: For every $i=1,\ldots,m$, the set-valued map x_i^B is u.s.c with nonempty, convex, compact values on the set $\{(p,w) \in R^l xR : B_i^B\ (p,W) \neq \emptyset\}$.

Proof: we want first to show that x_i^B has nonempty values. Let $(p,W) \in R^l x\ R$ such that $B_i^B\ (p,W) \neq \emptyset$. For all $x_i \in X_i^B$, we denote by $\tilde{p}i\ (x_i)$ the set $\{x_i \in X_i^B : x_i' \succeq_i x_i\}$. We want to prove that $\bigcap_{x_i \in B_i B(p,w)} \bar{P}_i(x_i^k) \cap B_i^B(p,w) \neq \emptyset$. Since the sets are closed for all $x_i \in B_i^B(p,W)$ $[B_i^B(p,W)=\{x_i \in X_i^B : p.x_i \leq W\}]$, and the set $B_i^B(p,w)$ is compact (it is closed and bounded). It suffices to prove that for all finite subset $(x_i^k)_{k\in\{1,\dots,q\}}$ of $B_i^B(p,W)$, $\bigcap_{k\in\{1,\dots,q\}} \bar{P}_i(x_i^k) \cap B_i^B(p,W) \neq \emptyset$.

. If $q=1$, x_i belongs to $\bar{P}_i(x_i) \cap B_i^B(p,W)$ for all $x_i \in B_i^B(p,W)$, if $q=2$, the fact that $\geq_i$ is complete implies that either $x_i^2 \succeq_i x_i^1$ or $x_i^1 \succeq_i x_i^2$ Hence either $\bigcap_{k\in\{1,2\}} \bar{P}_i(x_i^k) \cap B_i^B(p,W) = \bar{P}_i(x_i^2)$) $\bigcap B_i^B p,W)$ or $\bigcap_{k=\{1,2\}} \bar{P}_i(x_i^1) \cap B_i^B(p,W)$. So the

interaction is nonempty. In general case, by induction on q, that there exists $ko \in \{1,\dots,q\}$ such that for all $k \in \{1,\dots q\}$ $x_i^{ko} \succeq_i x_i^k$. So $\bigcap_{k\in\{1,\dots,q\}} \bar{P}i(x_i^k) \cap b_i^B(p,W) = \bar{P}i(x_i^k) \cap B_i^B(p,W)$ thus the intersection is nonempty. We want to show that the element $\bar{x_i} \in \bigcap_{x_i \in B_i^B(p,W)} \bar{P}_i(x_i)\ B_i^B \cap (p,W)$ belongs to $x_i^B(p,W)$. Indeed for all $x_i \in B_i^B(p,W)$. $\bar{x_i} \succ_i x_i$ but $P_i^B(\bar{x_i}) = \{x_i \in X_i^B : x_i \succeq_i x_i\}$, so $P_i^B(\bar{x_i})\ B_i^B \cap (p,W) = \emptyset$.

We want now to show that x_i^B is convex valued .Let x_i, $x_i' \in x_i^B(p,W$. Let $x_i^t = tx_i + (1-t)x_i'$. Since $x_i, x_i' \in x_i^B(p,W)$ and $B_i^B(p,W)$ is convex then $x_i^t \in x_i^B(p,W)$. Thus if $x_i^t \notin x_i^B(p,W)$ then there exists $\alpha \in P_i^B(x_i^t)$ such that $p.\alpha_t < W$ (since $x_i^B(p,W) = \{x_i \in x_i^B(p,W):$ for all $x_i' \in x_i^B(x_i): p.x_i' \geq W\}$) (and we have $x_i^t \in x_i^B(p,W)$). Since $\succeq_i$ is convex then either $\alpha_i \in (P_i^B(x_i)$ or $\alpha_i \in P_i^B(x_i')$ but together with $p.\alpha_i < W$ this contradicts the fact that x_i and $x_i' \in x_i^B(p,W)$.

We want to shown now that the graph of x_i^B is closed, which implies that x_i^B is u.s.c and compact valued. Let $(p^n, W^n, x_i^n) \in R^l \times R \times X_i^B$ be sequence converging to

$(p, \propto, x_i)$ such that $B_i^B(p,W)\neq\emptyset$ and $x_i^n \in x_i^B(p^n, W^n)$ for all $n.x_i$ is an element of B_i^B (p,W) which is closed. Thus if $x_i \notin x_i^B\ (p,W)$ then there exists $\alpha_i \in p_i^B\ (x_i)$ such that $p.\alpha_i < \propto$. From the closedness of $\succeq_i$ implies that $\alpha_i \in p_i^B\ (x_i^v)$ for v large enough. For v large enough one has $p^n.\alpha_i < W^n$ But this contradicts the fact the $x_i^n \in x_i^B\ (x^n \cdot W^n)$

We now consider the excess demand set-valued map Z^B from B to R^l defined as

$$Z^B(p)=\sum_{i=1}^{m} x_i^B(p,W(p)) - \sum_{j=1}^{n} s_j^B(p)-\{w\}$$

Where $W(p)=p.w+\sum_{j=1}^{n}\theta_{ij}\,\pi_J^B(p)+1/m(r-p)$.

4.19 Lemma[15 : page 8]:

The set valued map Z^B is u.s.c set valued map from B to R^l with nonempty, convex compact values and satisfies [Walras'law] for all $p\in \partial B$ $\sup Z^B(p)\leq 0$.

Proof: since $x_i^B\ (p,W(p))$ and $s_j^B\ (p)$ are convex and compact and u.s.c so that $Z^B\ (p)$. we want to show that $Z^B\ (p)$ has nonempty values. It suffices to show that $x_i^B\ (p,W(p))$ has nonempty values for all i which equivalent to $B_i^B\ (p,W(p)) \neq \emptyset$. Let $p\in B$. Then from the choice of the ball B, one has $\pi_J^B\ (p)\geq p.\,{}^iy_j$. Hence $W(p)=p.w+\sum_{j=1}^{n}\theta_{ij}\,\pi_J^B$

$$(p)+1/m\,(r-|p|)\geq p.w+\sum_{j=1}^{n}\theta_{ij}\,p.\,{}^iy_j=p.\,\hat{x}_i.$$

Since $\hat{x}_i \in X_i^B$, one obtains $\hat{x}_i \in B_i^B\ (p,W(p))\neq\emptyset$.

We want to show that for all $P \in B$, for all $\alpha \in Z^B(p)$, $p.\alpha \leq 0$, $\alpha = \sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_j - w$

with $x_i \in B_i^B(p,p,w_i + \sum_{j=1}^{n} \theta_{ij}\, \pi_J^B(p))$ for every i and $y_j \in {}^B s_j(p)$ for every j. From the

definition of x_i^B one has $p.\sum_{i=1}^{m} x_i \leq \sum_{i=1}^{m} (p.w_i + \sum_{j=1}^{n} \theta_{ij}\, \pi_J^B(p)) = p.w + \sum_{j=1}^{n} \pi_J^B$

$(p)\,.\,)\,(\sum_{i=1}^{m} \theta_{ij} = 1)$.

From he definition of the mapping π_J^B and the set-valued map S_J^B $[\pi_J^B(p) = \sup$

$\{p.y_j / y_j \in Y_j^B\}$, $S_J^B(p) = \{y \in Y_j^B : p.y' \leq p.y, \text{for all } y \in Y_j^B\}]$, so p. $\pi_J^B(p) = p.y_j$ for every j.

so $p.\sum_{i=1}^{m} x_i \leq p.(w + \sum_{j=1}^{n} y_j)$ so $p.\sum_{i=1}^{m} x_i - p.(w + \sum_{j=1}^{n} y_j) \leq 0$. Thus $p.\alpha \leq 0$.

We now want to apply Theorem (4.14) to the set-valued map Z^B. Consequently, there exists $p^* \in B$ such that $0 \in Z^B(p)$.

That is $(x_i^*, y_j^*, p^*) \in \prod_{i=1}^{m} X_i^B \times \prod_{j=1}^{n} Y_j^B$ such that

1-for all $i = 1,\ldots,m$, $x_i \in x_i^B(p^*, W(p^*))$

2-for every $j = 1,\ldots,n$ $y_j^* \in s^B{}_j(p^*)$.

3- $\sum_{i=1}^{m} x_i^* = \sum_{j=1}^{n} y_j^* + w$.

from (i), (ii), (iii) $(x_i^*, y_j^* p^*)$ is a quasi-equilibrium of ε^B. is a consequence of the two following claims.

Claim 1: For all $i=1,\dots,m$, $p^*.x_i^* = W(p^*)$

Proof: from the definition of x_i^B, *one has* $p^*.x_i^* \leq W(p^*)$. Assume on the contrary *that* $p^*.x_i^* < W(p^*)$. Since $x_i^* \in \hat{X}_i \subset int\ B$ and $\succeq_i$ is locally nonsatiated on $\hat{X}_i$ there *exists* $x_i \in P_i(x_i^*) \cap B$ such that $p^*.x_i < W(p)$. This contradicts the fact the $x_i^* \in x_i^B(p, W)$.

Claim2: $|p| = r$

Proof: from previous claim one has $p^*. \sum_{i=1}^{m} x_i^* = p^*.w + \sum_{j=1}^{n} \pi^n{}_{j=1}(p) + 1/m\,(r-|p|)$.

From the definition of π_J^B and s_j^B, one has $p^*.y_j^* = \pi_J^B(p^*)$ for every j. From (iii) $p^*.$

$$\sum_{i=1}^{m} x_i^* = p^*.w + p^*. \sum_{j=1}^{n} y_j^* = p^*.w + \sum_{j=1}^{n} \pi_J^B(p).$$

So $r - |p^*|$

4.5 Equilibrium under uncertainty:

One can investigate the new problem arising from taking into account uncertainty with the general equilibrium model. The State – preference approach to uncertainty was introduced by Arrow (1953)and Deboreu (1959) and itself is taken easily to Walrasian general equilibrium theory. In the general context when multiple agents are endowed with state – contingent commodities, while they can trade among themselves . Thus " a competitive equilibrium " is a set of state – contingent prices and state – contingent commodities which satisfies all agents utility maximizing choices and clean state contingent markets.

In order to approach the idea of a general equilibrium with – contingent markets. It might be useful to recall the individual optimum of a single agent. Letting S be the set of states and assuming the existence of state independent utility functions, then we have the maximization problem.

$$Max U = \sum_{s \in S} \pi_S u(x_s)$$

$s.t$

$$\sum_{s \in S} p_s x_s \leq \sum_{s \in s} p_s e_s$$

Where x_s is a state – contingent commodity bundle, P_s a set of state – contingent prices . π_s is the probability of even x_s to occur . This yields the result that for any commodity and two state $s, s^{*} \in S$

$$\pi_s u(x_{is}) / p_{is} = \pi_{s*} u*(x_{is}) / p_{is*}$$

Which was termed the fundamental theorem of risk bearing.

All the standard results of Arrow – Debrue general equilibrium theory (i. e the existence of equilibrium, Pareto optimality, etc) apply without fail in this state economy .

Chapter 5

Welfare Properties of Walrasian Equilibrium & Conclusion

Introduction

Since Adam smith's evocation of an invisible hand, market equilibrium has been supposed not only to clear markets , but also to achieve an efficient allocation of resources. Therefore in this chapter, we will define a very general efficiency concept, Pareto efficiency. Then we are going to state and prove the two major results relating to market equilibrium and efficient allocation, which are considered to be the most important results in welfare economics: The first and the second Fundamental Theorems of welfare Economics.

5.1 Definitions:

5.1 Definition:

An allocation x in the Edgeworth box is Pareto optimal if there is no other allocation x' in the Edgeworth box with $x'_i \succeq_i x_i$ for i=1,2 and $x'_i \succ_i x_i$ for some i.

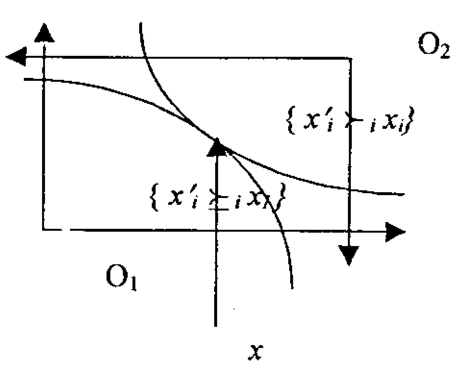

Figure 5.1

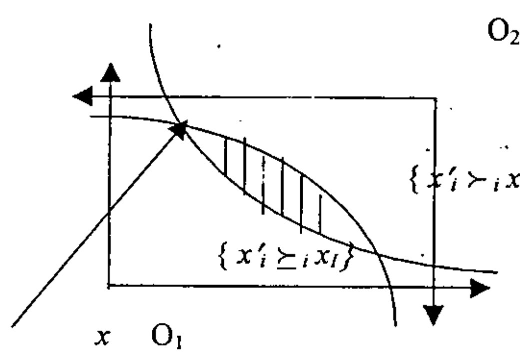

Figure 5.2

figure 5.1 allocation x is Pareto optimal. In figure 5.2 is not Pareto optimal

Equilibrium and its basic Welfare properties:

Assume that an economy is composed of $I>0$ consumers and $J>0$ firms in which there are L commodities, each consumer $i=1,\dots,I$ is characterized by a consumption set $X_i \subset R^L$ and a preference relation $\succeq_i$ defined on X_i, these preferences are rational. Each firm j=1,...,J is characterized by a production set, $Y_j \subset R^L$. We assume that every Y_j is nonempty and closed. The initial resources of commodities in the economy, the economy's endowments are given by a vector $\bar{w}= (\bar{w}_1,\dots, \bar{w}_L) \in R^L$, these are summarized by $(\{(X_i, \succeq_i)\}^I_{I=1}, \{Y_j\}^J_{J=1}, \bar{w})$.

5.2 Definition:

An allocation $(x, y) = (x_1,\dots,x_I, y_1,\dots,y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i=1,\dots I$ and a production vector $y_j \in Y_j$ for each firm $j=1,\dots J$. An allocation (x , y) is attainable if $\Sigma_i x_{li} = \bar{w}_l + \Sigma_j Y_{lj}$ for each commodity l. that is, if

$\Sigma_i x_i = \bar{w} + \Sigma_j y_j$

We denote the set of attainable allocation by

$A = \{ (x,y) \in X_1,\dots X_i \times Y_i \times\dots\times Y_j\ \Sigma_i x_i = w+ \Sigma_j Y_j\} \subset R^{L(I+J)}$

5.3Definition:

A attainable allocation (x,y) is Pareto optimal if there is no other allocaton (x',y') $\in A$ that are Pareto dominates it, that is if there is no attainable allocation (x',y') such that $x' \succeq_i x_i$ for all i and $x' \succ_i x_i$ for some i.

An allocation is Pareto optimal if there is no waste :it is impossible to make any one strictly better off without making other one worse off.

5.2 Fundamental Theorems

We want now to show the relationship between the Pareto optimal and equilibrium throughout two fundamental theorems ,the first one shows that if there is a price equilibrium then there is a Pareto optimal and the second shows the conditions under which the Pareto optimal is price equilibrium

5.4 Proposition [(14):page 549]

(First fundamental Theorem of Welfare Economics) If preferences are locally nonsatiated and if (x^*,y^*,p) is a price equilibrium with transfers, then the allocation (x^*,y^*) is Pareto optimal. In particular, any Walrasian equilibrium allocation is Pareto optimal.

Proof: suppose that (x^*,y^*,p) is a price equilibrium with transfers and that the associated wealth levels are $(W_1,\ldots W_I)$ and that $\Sigma_i W_i=p.\ \bar{w}+\Sigma_j p.y_j^*$.

(ii) from (1.14) implies that if $x_i \succ_i x_i^*$ then $p.x_i > W_i$. That is any thing that is strictly preferred by consumer i to x_i^* must be un affordable to him. The significance of the local nonsatiation condition for the purpose at hand is that with it (1.3) implies an additional property.

$$\text{If } x_i \succeq_i x_i^* \text{ then } P.x_i \geq W_i$$

That is, any thing that is at least as good as x_i^* is at best just affordable. Now consider an allocation (x,y) that Pareto dominates (x^*,y^*). That is, $x_i \succeq_i x_i^*$ for all i and $x_i \succ_i x_i^*$ for some i, But we have if $x_i \succeq_i x_i^*$ then $p.x_i \geq W_i$ for all I and we have if $x_i \succ_i x_i^*$ then $p.x_i > W_i$ for some i, so

$\Sigma_i p.x_i > \Sigma_i W_i = p.\ \bar{w} + \Sigma_j p.y_j^*$

Moreever since y_j^* is profit maximizing for firm j at price vec: :r p, we have p. we have $p. \bar{w} + \Sigma_j p.y_j^* \geq p. \bar{w} + \Sigma_j p.y_j$. Thus

$$\Sigma_i p.x_i > p. \bar{w} + \Sigma_j p.y_j$$

So then (x,y) cannot be attainable, Indeed $\Sigma_i x_i = \bar{w} + \Sigma_j y_j$ implies $\Sigma_j p.x_i = p. \bar{w} + \Sigma_j p.y_j$ which contradict (*)

So we conclude that the equilibrium allocation $(x^*.y^*)$ must be Pareto optimal. At any feasible allocation (x,y), the total c· st of the consumption bundles $(x_1,\ldots,x_I)$ evaluated at prices P, must be equal to the social wealth at those prices, $p. \bar{w} + \Sigma_j p.y_j$. More over, because preferences are locally nonsatiated, if (x,y) Pareto dominates (x^*,y^*) then the total cost of consumption bundle $(x1,\ldots,xi)$at price p and therefor the social wealth at these prices must exceed the total cost of the equilibrium consumption allocation $P. (\Sigma_i x_i^*) = p. \bar{w} + \Sigma_j p.y_j^*$. But by the profit maximization there are no production attainable that attainable v .lue of social wealth at prices p that excess of $p. \bar{w} + \Sigma_j p.y_j^*$.

The second Fundamental Theorem of Welfare Economics:

This theorem gives conditions under which a Pareto optimal allocat'ɔn can be supported as price equilibrium with transfers. We will first show 'hat if all preferences and technologies are convex any Pareto optimal allocation can be achieved as a price quasi equilibrium with transfers.

5.5 Proposition [(14):page552]

(Second Fundamental Theorem of welfare Economics) consider an economy specified by $(\{X_i, \succeq_i\}^I_{i=1}, \{y_j\}^J_{j=1}, \bar{w})$ and suppose that every Y_j is convex and every preference relatic n $\succeq_i$ is convex [i.e the set $\{x'_i \in X_i: x'_i \succeq_i x_i\}$ is convex for every $x_i \in X_i$]

and locally nonstiated. Then, for every pareto optimal allocation (x^*,y^*) there is a price vector $P= (P_1,\dots P_L)\neq 0$ such that (x^*,y^*,P) is a price quasiequilibrium with transfers.

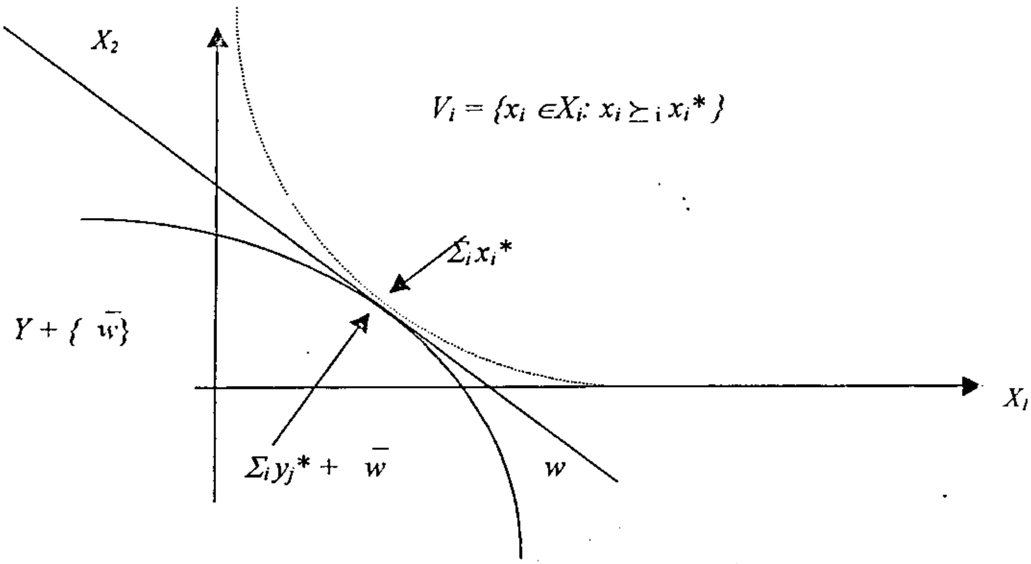

Figure 5.3

Proof. we begin by defining, for every i, the set V_i of consumption preferred to x_i^*, that is $V_i = \{x_i \in X_i: x_i \succeq_i x_i^*\} \in R^L$. Then we define

$V - \Sigma_i V_i \{\Sigma_i x_i \in R^L: x_1 \in V_1,\dots x_1 \in V_I\}$ and

$Y - \Sigma_j Y_j = \{\Sigma_j y_j \in R^L: y_1 \in Y_1,\dots y_J \in Y_J\}$.

We want to show that every set Vi is convex: Suppose that $x_i \succ_{i\,i} x_i^*$. Take $0 \leq \alpha \leq 1$. We want to prove that $\alpha x_i + (1-\alpha) x'_i \succ_{i\,i} x_i^*$.

Because preferences are complete, we can assume without loss of generality that $x_i \succeq_i x'_i$. Therefore by convexity of preferences, we have $\alpha x_i + (1-\alpha)\ x'_i \succeq_i x'_i$, which by transitivity yields the desired conclusion $\alpha\ x_i + (1-\alpha)\ x'_i \succ_i x_i^*$.

We want to show that V and $Y + \{\bar{w}\}$ are convex. Since their sum of convex sets is convex (see ch.2). Now want to show that $V \cap (Y + \{\bar{w}\}) = \emptyset$. If there is a vector in both V and in $Y + \{\bar{w}\}$, then his will mean that with a given endowments and technologies it will be possible to produce an aggregate vector that can be used to give every consumer i a consumption bundle that is preferred to x_i^*. Want now to show that there is $P = (P_1, \dots P_L) \neq 0$ and a number r such that $p.z \geq r$ for every $z \in V$ and $p.z \leq r$ for every $z \in Y + \{\bar{w}\}$ (see ch2).

We want to show If $x_i \succeq_i x_i^*$ for every I then $p. (\Sigma_i x_i) \geq r$: suppose that $x_i \succeq_i x_i^*$ for every i. By local nonatiation, for each consumer i there is a consumption bundle $\hat{x}_i$ arbitrarily close to x_i such that $\hat{x}_i \succ_i x_i$ and therefore $\hat{x}_i \in V_i$. Hence $\Sigma_j \hat{x}_i \in V$ and so $P. (\Sigma_i \hat{x}_i) \geq r$. geometrically, want we have done here is show that the set $\Sigma_i \{x_i \in X_i : x_i \succeq_i x_i^*\}$ is contained in the closure of V. want to show now $P. (\Sigma_i x_j^*) = P. (\bar{w} + \Sigma_j y_j^*) = r$: Because in previous step we have $p. (\Sigma_i x_i^*) \geq r$. but $\Sigma_i x_i^* = \Sigma_i y_j^* + \bar{w} \in Y + \{\bar{w}\}$ then $p.(\Sigma_i x_i^*) \leq r$ So (p. $(\Sigma_i x^*) = r$. Since $\Sigma_i x_i^* = \bar{w} + \Sigma_j y_j = r$ thus $p(\bar{w} + \Sigma_j y_j^*) = r$. For any firm j and $y_j \in Y_j$ we have $y_j + \Sigma_{h \neq j} y_n^* \in Y$. therefor $p.(\bar{w} + y_j + \Sigma_{h \neq j} y_h^*) \leq r = p. (\bar{w} + y_j^* + \Sigma_{h \neq j} y_h^*) \leq r$ $= p. (\bar{w} + y_j^* + \Sigma_{h \neq j} y_h^*)$. Hence $p. y_j \leq p.y_j^*$ ....(1)

.let any $x_i \succ_i x_i^*$ since we have $p. (x_i + (\bar{w} + \Sigma_{k \neq i} x_k^*) \geq r = P. (x_i^* + \Sigma_{k \neq i} x_k^*)$ *then* $p.x_i \geq P.x_i^*$ ....(2).

(3)from the definition of pareto efficiency.

So we showed that the three condition of Walrasian equilibrium are satisfied.

5.3External effects:-

We say that an economic situation involves a consumption externality if one consumer cares directly about the other agent production or consumption.

The crucial feature of externalities is that there are goods that people care about that which not sold in markets such as loud music, smoking, and pollution.

Up until now, we assumed that each agent could make consumption or production decisions without worrying about what other agents were doing , so we can achieve Pareto efficiently.

But in the real world externalities are present and the market will not necessarily be Pareto efficient . However, there are other social institutions such as the legal system, or government intervention that can mimic the market mechanism to some degree and thereby achieve Pareto efficiency.

To illustrate some main considerations, we will imagine two roommates, A and B, who have preferences over money and smoking. We suppose that both consumers like money, but that A likes to smoke and B likes clear air.

Let B has a right to clear air; let us suppose that A has $100 and so does B .It may happen that B would prefer to trade some of his right to clear air for some money .It looks like an auctioneer calling out prices and asking how much each agent would be willing to buy at those prices. When he manages to find a price where supply equal demand, thus the market is then clear, and Pareto efficiency is achieved which reflects the fact that equilibrium (Walrasian equilibrium) implies Pareto efficiency.

Let us take an example of a car's pollution, and how we can deal with the right of clear environment. Experts estimate the cost of the effect of a car's pollution on human is about $1450 over the lifetime of the car. This means that every one who buys a car must pay an extra $1450.

Now, let us assume that there is a right of pollution for two firms . Each has a right to emit 100 tons of nitrogen oxide in a year. If one of them reduces the pollution to 80 tons ,cost will increase as a result . The other firm will have a right of 20 extra tons of this emissions if the first sells them to it .

There are a lot of examples about externalities ,such as the policy of pollution trading sea surface empress oil spill, pollution permits carbon trading, pesticides trust and the economics of tobacco, taxation .The internet site on externalities deals with such issues.

Conclusion

In this thesis ,I presented an application of fixed-point theorems to private ownership firms. The mathematical question asked is whether we could relax the conditions of these theorems (that of continuity and convexity of the correspondence). There is a lot of outgoing research on this field and one can consult the Internet site.

From economic theory, we have shown the conditions under which exchange equilibrium exists and that under perfect competition; such equilibrium is Pareto efficient. Can we generalize such a result to other sectors of economy like the public sector and if there exists equilibrium is it Pareto efficient ?In general we have seen that in presence of externalities, such equilibrium is not Pareto efficient.

S .Y .Wu [22 [at University of Iowa generalizes such theorems to three-sector economy: private profit firm, private non – -profit firms and public sector. The model he selected to carry out the task of obtaining a general equilibrium is a two – period temporary equilibrium. In our case we studied a static model while his model introduces the concept of time.

Another generalization of our model introduces uncertainty. The site [23] which deals with general equilibrium with state contingent markets considers such a model and its applications to financial market. Such a model can be checked by econometric methods as done by E. Malinvaud [13] .

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List of Symbols and conventions

Symbol	Term	Page
A	The attainable Set	16,58
$B(p,W)$	The Walrasian Budget Set	10
$B^{B}(p,W)$	The Walrasian Budget Set of An Auxiliary Economy	65
$Co\,S$	The Convex hull of The Set S	20
$Dom(F)$	The Domain of F	25
$Graph\,(F)$	The Graph of F	25
$Im(F)$	The Image of F	25
LSC	The Lower Semicontinuous Set Valued Map	35
P	The price of commodity	10
PMP	The Firm's Profit Maximization Problem	13
p_s	The State Contingent Price	70
$s(p)$	The Firm's Supply Set Valued Map	14,56
$s^{B}(p)$	The Suppiy of An Auxiliary Economy	65
TVS	The Topological Vector Space	20
UMP	The Utility Maximization Problem	11
USC	The Upper Semicontinuous Set Valued Map	27
$u(x)$	The Utility Function	10
$v(p,W)$	The Indirect Utility Function	12
W	The Consumer's Wealth Level	10
w	The consumer's Endowment vector	14

X	The Consumption Set	7
$\hat{X}_i$	The Attainable Consumption Set	58
x	The commodity vector	7
$x(p,W)$	The Walrasian Demand Set Valued Map	12
$x^B(p,W)$	The Walrasian Demand of An Auxiliary Economy	65
x_s	The State Contingent Commodity Bundle	70
Y	The Production Set	13
$\hat{Y}_j$	The Attainable Production Set	58
y	The production vector	12
$Z(p)$	The Excess Demand Set Valued Map	51
$Z^B(p)$	The excess Demand of An Auxiliary Economy	65
π_s	The Probability of Event x_s to Occur	70
$\succeq$	The Preference relation	8
$\succ$	The Strict Preference Relation	8
$\sim$	The Indifference Preference Relation	8
$\pi(p)$	The Firm's Profit Function	14,56
ε	The Initial Economy	61
ε^B	An Auxiliary Economy	61
$\pi_j^B(p)$	The Profit Function of An Auxiliary Economy	65

الخلاصة

نوقش خلال هذه الأطروحة مبدآن هامان في الاقتصاد ، الأول : مبدأ التوازن العام ، أي أن العرض على كل البضائع يكون مساويا للطلب عليها ، والثاني : هو أن المستهلك يسعى اقتصاديا للوصول إلى ما هو افضل بالنسبة له ،ضمن أسعار السلع في السوق والثروة التي يمتلكها ، والشركات تسعى لتحقيق أقصى ربح .

وقد صيغت هذه المبادئ رياضيا لربط هذه المفاهيم الاقتصادية ، ونظريات هامة في الرياضيات ، حيث نبين كيفية استخدام نظريات النقطة الثابتة للاقترانات المتصلة (Set Valued map) التي نوقشت بشكل تفصيلي خلال هذه الأطروحة ، لإثبات وجود التوازن العام كما عرفه (Walras) ، كما أننا استخدمنا (The Banach Separation Theorem) لا ثبات أن نقطة التوازن هذه هي الأفضل بالنسبة للأفراد والشركات.

خلال هذه الأطروحة طبقت هذه المبادئ على نوعين من التعامل الاقتصادي ، الأول : في حالة (The Pure Exchange Economy) أي تداول السلع في السوق ، أما الثاني: فهو اقتصاد الملكية الخاصة للأفراد (The Private Ownership Economy) ، وقد تعرضنا بشكل ثانوي إلى التدخلات الخارجية (The externality) ، وتأثيرها على التوازن العام.