An-Najah National University

Faculty of Graduated Studies

Approximation methods of Fractional Derivatives and Their Applications to Fractional Differential Equations

By

Donia Khairi Mohammed Foqhaa

Supervisor

Dr. Anwar Saleh

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This Thesis was defended successfully on 2 / 8 /2017 and approved by

Defense Committee Members	**Signature**
– **Dr. Anwar Saleh /Supervisor**	**…………………**
– **Dr. Sae'd Mallak /External Examiner**	**…………………**
– **Dr. Khaled Adarbeh /Internal Examiner**	**…………………**

Dedication

Dedication to my father, mother, brothers

And

To my husband Ahmad, and my little daughter Zeina

Acknowledgements

I am deeply grateful to my supervisor, Dr. Anwar Saleh for introducing me to the subject of my thesis, and also for his valuable remarks, and for giving me the encouragement and support that I need to complete this work.

Also, I would like to thank my family, and my husband Ahmad very much, because without their patience, support, and most of all love, this work couldn't be finished.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان

Approximation methods of Fractional Derivatives and Their Applications to Fractional Differential Equations

أقر بأن ما شملت عليه الرسالة هو نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد، وأن هذه الرسالة ككل أو أي جزء منها لم يقدم من قبل لنيل أي درجة أو لقب علمي أو بحثي لدى أي مؤسسة علمية أو بحثية

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degrees or qualifications.

Student's Name:	**اسم الطالبة: دنيا خيري محمد فقها**
Signature	**التوقيع:**
Date	**التاريخ: 2/8/2017**

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Fractional Derivatives and Their Applications to Fractional Differential Equations

By
Donia Khairi Mohammed Foqha
Supervisor
Dr. Anwar Saleh

Abstract

Fractional calculus is a field of mathematics which concern on finding derivatives or integrals of non-integer order. The exact birthday of fractional calculus was in september 30, 1695.

Over the years, many mathematicians contributed to this field, and bit by bit the importance of fractional calculus has appeared in many aspects of our life, such as physics, engineering, viscoelasticity, and many others.

In this thesis, we start our work in chapter 1 by focusing on the definition of Riemann-Liouville fractional integral, and finding the fractional integral of any order p for many functions.

In chapter 2, we study the two mostly used definitions of fractional differentiation namely; the Riemann-Liouville and the Caputo definitions, then we make comparison between the two definitions of fractional derivatives. In chapter 3, we focus on solving fractional initial value problems with Caputo operator or with Riemann-Liouville operator using the easiest method which is Laplace transform method.

At the end of this thesis, we focus on some applications of fractional calculus in real life such as tautochrone problem.

Chapter One

Introduction

The concept of differentiation and integration is known to everyone who have studied classical calculus. For example, the first-order derivative of the function $f(t) = t^3$ is $f'(t) = 3t^2$

and the second-order derivative is $f''(t) = 6t$

Similarly, for anti-derivatives, the first-order anti-derivative of the function $f(t) = t^3$ is $\int t^3\, dt = \frac{t^4}{4} + c$

The second-order anti-derivative is:

$$\iint f(t)dt = \frac{t^5}{20} + ct + d$$

Several questions need answers: What happens if the order of differentiation or integration is a fraction? How can we define these new operations? Would the result be convenient or have a meaning or application comparable to that of classical integer order derivative or integration?

The answer of all these questions can be found in the new field of mathematics which is called fractional calculus.

Fractional calculus is a field of mathematics which is more than 300 years old. However, it didn't attract attention until recently. The revival of fractional calculus is due to B. Ross who, shortly after his Ph.D. thesis on fractional calculus, organized the first conference on fractional calculus and it's applications at the university of New Haven in June 1974. This field started to attract many scientists and engineers. A number of studies and papers have been published in famous journals, because they discovered that it has many applications in engineering and science, such as fluid flow, electrical networks, probability, mathematical biology, etc.

The basic mathematical idea of fractional calculus was developed long ago by Leibniz in 1695 when he asked by L'Hopital: "What is the meaning of $\frac{d^n}{dt^n}f(t)$ if $n = \frac{1}{2}$". This question lead to the birth of fractional calculus. Following Leibniz, many famous mathematicians like Laplace, Liouville, Riemann, Fourier, and many others have contributed to the field of fractional calculus.

Fractional calculus is a generalization of the ordinary differentiation and integration of integer order, in the same way that the fractional exponents are outgrowth of exponents of integer value.

At the beginning, the field of fractional calculus did not draw much attention due to the fact that scientists could not find applications of the concept of fractional derivative or integral. Now, there is no adoubt that fractional calculus has become an essential new mathematical tool for solving many problems.

1.1 Historical view

The question that led to the birth of fractional calculus was from a letter which is written by L'Hopital asking Leibniz about the n^{th} derivative of the linear function $f(t) = t$, and what will happen when $n = \frac{1}{2}$ [7]. In general, what would the result be when n is a fraction, then Leibniz replied and wrote "this is an apparent paradox from which, one day, useful consequences will be drawn", and from this first inquisition between L'Hopital and Leibniz the new field of mathematics was called fractional calculus, but in fact, the order of differentiation or integration can be any positive real number.

After L'Hopital and Leibniz first inquisition, the field of fractional calculus has motivated many famous mathematicians, such as Fourier (1820-1822), Lacorix (1819), Riemann (1826-1866), Liouville (1809-1882), Laplace (1812), Caputo, Euler (1730), and many others, and each of them made an effort to make progress in this field of mathematics. Next, we list some major contributions to fractional calculus by famous mathematicians[**15**], [**17**].

In 1812, Laplace defined the fractional derivative by means of an integral.

In 1819, Lacorix was the first mathematician to define the m^{th} fractional derivative using the gamma function. He applied his definition on the function $f(t) = t^n$ in a paper published in 1819. The m^{th}-derivative of $f(t) = t^n$ is:

$$\frac{d^m f(t)}{dt^m} = \frac{n!}{(n-m)!} t^{n-m}, \qquad n \geq m, m \text{ is integer}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-m+1)} t^{n-m}$$

After that, he try to apply this definition when $m = \frac{1}{2}, n = 1$, and he get:

$$\frac{d^{\frac{1}{2}}(t)}{dt^{\frac{1}{2}}} = 2\frac{\sqrt{t}}{\sqrt{\pi}}$$

In 1823, Abel applied fractional calculus in the solution of an integral equation that arised from the formulation of the tautochrone problem.

The tautochrone problem is the problem of determining the shape of the curve such that the time of a discent that a frictionless point mass needs to slide down the curve under the work of gravity is independent of the starting point.

Over the next 10 years (1823 to 1832), no significant progress has been made. In 1832, Liouville was successful in applying his definition of fractional calculus to problems in potential theory.

After that, many mathematicians made very important work in fractional calculus such as Riemann, Grunwald-Letnikove, Caputo in 1967, K.S. Miller, B.Ross in 1993 and many others.

While fractional derivatives can be defined in different ways, we will a dapt the Riemann-Liouville and Caputo definitions.

1.2 Special Functions

In this section we give a brief review of some functions that are important for the fractional calculus. These functions are: the gamma function, the beta function, the error function, the complementary error function, the confluent hyper geometric function, the Mittag-Leffler function, the incomplete gamma function, and the Mellin-Ross function.

1. The gamma function [8], [18], [12], [1]

Definition 1.1: The gamma function is defined as

$$\Gamma(z) = \int_0^{\infty} t^{z-1}\, e^{-t} dt, z \in \mathbb{R} - \{\mathbb{Z}^-\} \qquad (1.1)$$

Here, It is interested to say that the gamma function is a generalization of the factorial function, since for any positive integer n,

$$\Gamma(n) = \int_0^{\infty} t^{n-1}\, e^{-t} dt = (n-1)!$$

Example 1.1:

$$\Gamma(1) = \Gamma(2) = 1$$

$$\Gamma(z+1) = z\Gamma(z)$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(n + 1/2) = \frac{\sqrt{\pi}}{2^n}(2n-1)!\,, n \in \mathbb{N}$$

Figure (1.1) describes the shape of the gamma function for real numbers

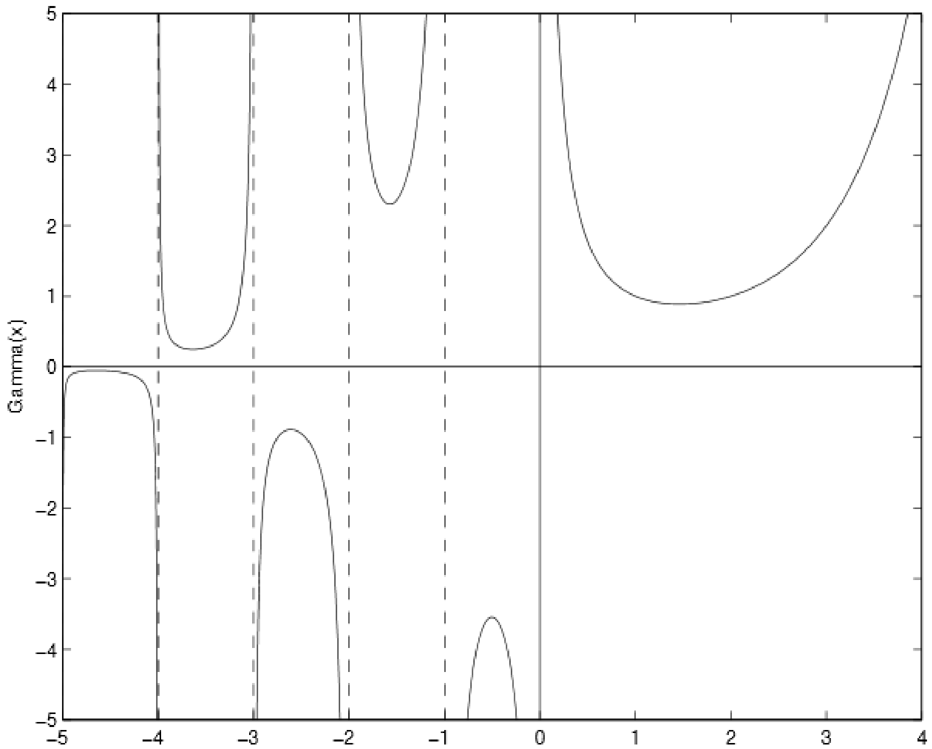

Figure 1.1: Gamma function for real numbers [**9**]

2. The beta function: [12], [7]

Definition 1.2: The beta function is defined as:

$$\beta(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}dt, Re\ x > 0, Re\ y > 0 \tag{1.2}$$

Two important properties of the beta function are:

a. the Beta function is related to the Gamma function through the relation:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{1.3}$$

b. the Beta function is symmetric, which means $\beta(x,y) = \beta(y,x)$

(The proof is straight forward using (1.3))

Example 1.2:

$\beta(2,3) = \Gamma(2)\Gamma(3)/\Gamma(5) = 2/24 = 1/12$

$\beta(3,2) = \Gamma(3)\Gamma(2)/\Gamma(5) = 2/24 = 1/12$

3. The error function (erf): [8], [10], [9]

The error function is considered to be one of the most important functions in fractional calculus and it is given in the following definition:

Definition 1.3:

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx, t \geq 0 \tag{1.4}$$

A function which is closely related to the error function is

The complementary error function (erfc) which is defined as:

Definition 1.4:

$$erfc(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx = 1 - erf(t) \tag{1.5}$$

Figure (1.2) describes the shape of complementary error function for real numbers.

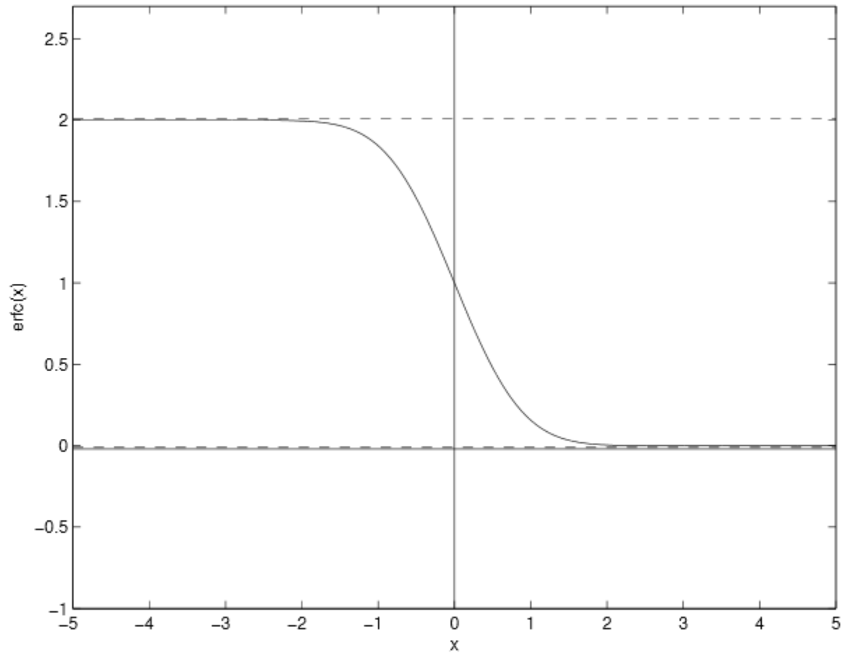

Figure 1.2: The complementary error function [9]

Example 1.3:

a. $erfc(-\infty) = 2$

b. $erfc(0) = 1$

c. $erfc(\infty) = 0$

d. $erfc(-t) = 2 - erfc(t)$

e. $\int_0^\infty erfc(t)dt = \frac{1}{\sqrt{\pi}}$

f. $\int_0^\infty erfc^2(t)dt = \frac{2-\sqrt{2}}{\sqrt{\pi}}$

g. $erf(\infty) = 1, erf(-\infty) = -1$

4. The confluent hyper geometric function: [9]

Definition 1.5: The confluent hyper geometric function is defined as:

$$ {}_1F_1(x,y;z) = \frac{\Gamma(y)}{\Gamma(x)} \sum_{k=0}^{\infty} \frac{\Gamma(x+k)}{\Gamma(y+k)} \frac{z^k}{k!} , \qquad (1.6) $$

where

$x, y, z \in \mathbb{C}, -y \notin \mathbb{N}_0, |z| < \infty$

Remark: for $x = y,$ the definition becomes:

$$ {}_1F_1(x,x;z) = \frac{\Gamma(x)}{\Gamma(x)} \sum_{k=0}^{\infty} \frac{\Gamma(x+k)}{\Gamma(x+k)} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z $$

which is the exponential function. Therefore, the confluent hyper geometric function can be considered as a generalization of the exponential function.

Two important properties of the confluent hyper geometric function are:

$a.$ $\quad {}_1F_1(x,y;0) = 1$

$b.$ $\quad \frac{d}{dz}\, {}_1F_1(x,y;z) = \frac{x}{y}\, {}_1F_1(x+1,y+1;z)$

5. The Mittag-Leffler function: [10], [3], [6]

The Mittag-Leffler function is defined by a Swedish mathematician in 1903, and it is considered to be a generalization of the exponential function just as the confluent hyper geometric function is. It is defined in two ways:

one-parameter function, and a two-parameter function which is introduced by Agarwal.

a. The one-parameter Mittag-Leffler function is defined as:

Definition 1.6:

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \alpha > 0, \alpha \in \mathbb{R}, t \in \mathbb{C} \qquad (1.7)$$

b. The two-parameter Mittag-Leffler function is defined as:

Definition 1.7:

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta > 0, \qquad \alpha, \beta \in \mathbb{R}, t \in \mathbb{C} \qquad (1.8)$$

Example 1.4:

a. $E_{1,1}(t) = e^t$

b. $E_{2,1}\,(t^2) = \cosh(t)$

c. $E_{2,2}(t^2) = \frac{\sinh(t)}{t}$

d. $E_{\frac{1}{2},1}(t) = e^{t^2} erfc(-t)$

Remark: for $\beta = 1$, the two parameter Mittag-Leffler function is reduced to the one- parameter Mittag-Leffler function**.**

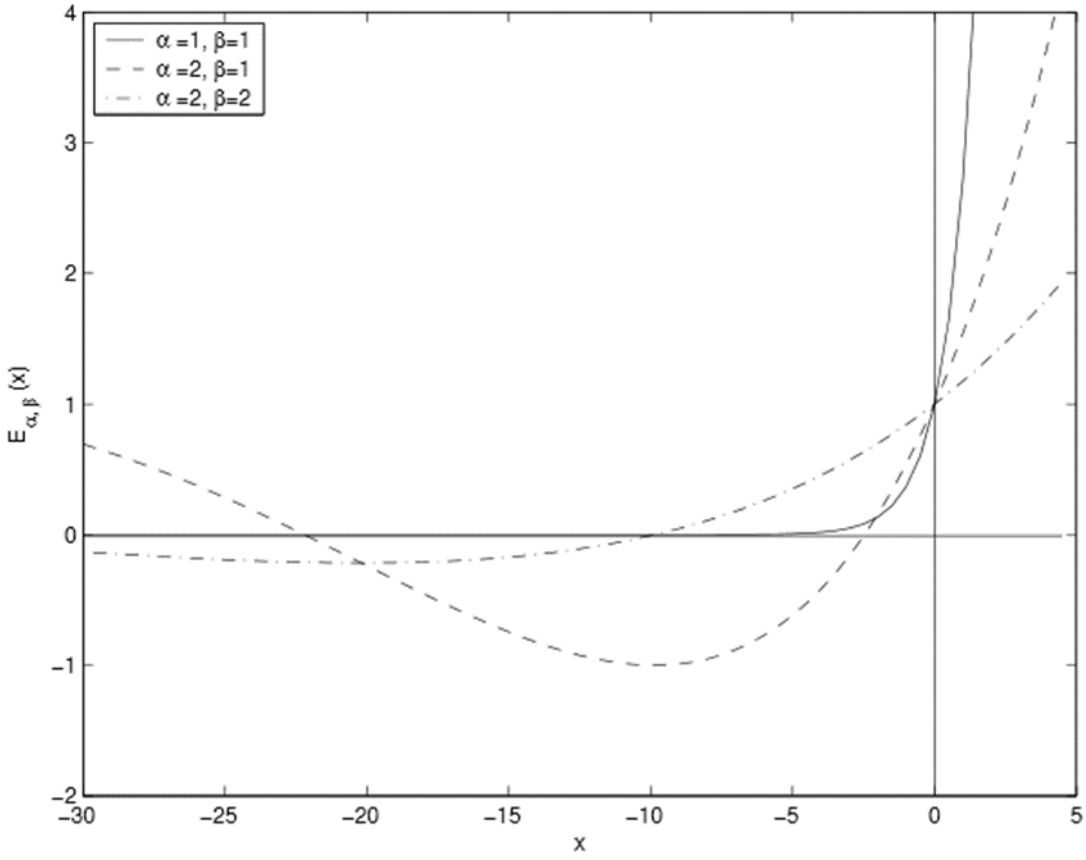

Figure 1.3: The two-parameter function of the Mittag-Leffler type [**9**]

6. The incomplete gamma function: [15]

The incomplete gamma function is significant to fractional calculus and it is defined as:

Definition 1.8:

$$\gamma^*(v,z) = e^{-z}\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(v+k+1)} \tag{1.9}$$

This function is entire function of both z and v, and if $Re\, z > 0$, then it has the integral form:

$$\gamma^*(v,z) = \frac{1}{z^v\Gamma(v)}\int_0^z t^{v-1}e^{-t}\,dt \tag{1.10}$$

Recall that an entire function is a function that is analytic in the entire complex plane.

7. The Mellin-Ross function: [15], [10]

Mellin-Ross function is considered to be one of the essential functions in fractional calculus, and it is related to both the incomplete gamma function and Mittag-Leffler function.

Definition 1.9:

$$E_z(v,a) = z^v e^{az} \gamma^*(v,az) = \frac{1}{\Gamma(v)} \int_0^z t^{v-1} e^{a(z-t)}\, dt \qquad (1.11)$$

1.2 Fractional integration

In this section we will derive the formula for integration of arbitrary real order number and study important properties about this integration. Then, we give examples to illustrate evaluation of fractional integral.

1.2.1 Unification of integer-order derivatives and integrals [18], [15]

Drichlet's formula:

If $G(x,y)$ is jointly continuous on $[a,b] \times [a,b]$, then we know from the elementary theory of functions that:

$$\int_a^b dx \int_a^x G(x,y)dy = \int_a^b dy \int_y^b G(x,y)dx \qquad (1.12)$$

Theorem 1.1: suppose that a function $f(\tau)$ is continuous and integrable in every finite interval (a,t). The function $f(\tau)$ may have an integrable singularity of order $x < 1$ at the point $\tau = a$, which means:

$$\lim_{\tau\to a}(\tau - a)^x f(\tau) = const \neq 0$$

In this case, the integral:

$$f^{(-1)}(t) = \int_a^t f(\tau)\, d\tau$$

exists and has finite value with

$$\lim_{t\to a} f^{(-1)}(t) = 0$$

Proof:

Let $y = \dfrac{\tau - a}{t - a}$

then:

$$\tau = a + (t - a)y$$

and

$$d\tau = (t - a)dy$$

Now,

$$\lim_{t\to a} f^{(-1)}(t) = \lim_{t\to a}\int_a^t f(\tau)d\tau$$

$$= \lim_{t\to a}(\mathrm{t} - \mathrm{a})\int_0^1 f(a + y(t - a))dy$$

Let $\epsilon = t - a$, then:

$$\lim_{t\to a} f^{(-1)}(t) = \lim_{\epsilon\to 0} \epsilon \int_0^1 f(a+y\epsilon)\,dy$$

$$= \lim_{\epsilon\to 0} \epsilon \int_0^1 \frac{(\epsilon y)^x}{(\epsilon y)^x} f(a+y\epsilon)\,dy$$

$$= \lim_{\epsilon\to 0} \epsilon^{1-x} \int_0^1 (\epsilon y)^x f(a+y\epsilon) y^{-x}\,dy = 0$$

Now, since $x < 1$, we can consider the two fold integral $f^{(-2)}(t)$

$$f^{(-2)}(t) = \int_a^t d\tau_1 \int_a^{\tau_1} f(\tau)d\tau$$

Applying Drichlet's formula, we get:

$$f^{(-2)}(t) = \int_a^t f(\tau)d\tau \int_\tau^t d\tau_1 \quad = \int_a^t (t-\tau)f(\tau)d\tau$$

In the same way, we can find the three-fold integral $f^{(-3)}(t)$:

$$f^{(-3)}(t) = \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \int_a^{\tau_2} f(\tau_{3)}\,d\tau_3$$

$$= \int_a^t d\tau_1 \int_a^{\tau_1} (\tau_1 - \tau) f(\tau)d\tau$$

$$= \frac{1}{2}\int_a^t (t-\tau)^2 f(\tau)d\tau$$

and by induction, we reach the well-known Cauchy formula

1.2.2 Cauchy formula: [2]

Let f be a continuous function on the real line, then the n^{th}-repeated integral of f based at a is:

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau \tag{1.13}$$

Remarks:

a**.** For a given integer $n \geq 1$ and any integer $k \geq 0$, we have:

$$f^{(-k-n)}(t) = \frac{1}{\Gamma(n)} D^{-k} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau \tag{1.14}$$

where the symbol D^{-k} , $k \geq 0$ means k iterated integrations.

b**.** For a fixed integer $n \geq 1$, and any integer $k \geq n$, the

$(k-n)^{th}$ −derivative of the function $f(t)$ can be written as:

$$f^{(k-n)}(t) = \frac{1}{\Gamma(n)} D^{k} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau \tag{1.15}$$

where the symbol $D^k, k \geq 0$ denotes k iterated differentiations.

Definition 1.10: Riemann-Liouville fractional integral:

Riemann-Liouville fractional integral is defined as:

$$ {}_aD_t^{-p}f(t) = \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\, f(\tau)d\tau \qquad (1.16)$$

where the symbol ${}_aD_t^{-p}f(t)$ denotes fractional integration of order p. This definition was derived from Cauchy formula, when we replace the integer n by $p > 0,\ p \in \mathbb{R}$

Note: when $a = 0$, we will use the notation $D^{-p}f(t)$ to denote the fractional integration of order p

1.2.3 Properties of Riemann-Liouville fractional integral [18], [12]

We prove three major properties of the fractional integration of arbitrary real order p

Property 1:

$$\lim_{p\to 0} {}_aD_t^{-p}\, f(t) = f(t),$$

and so we can put

$${}_aD_t^0\, f(t) = f(t)$$

Proof: To prove this formula, we will consider the following two cases:

First case: The function $f(t)$ is continuous for $t \geq 0$.

Integration by parts to the formula (1.16)

implies that:

$$ {}_aD_t^{-p}f(t) = \frac{(t-a)^p f(a)}{\Gamma(p+1)} + \frac{1}{\Gamma(p+1)}\int_a^t (t-\tau)^p\, f'(\tau)d\tau $$

Now, take the limit as $\to 0$.

$$ \lim_{p\to 0}\ {}_aD_t^{-p}f(t) = f(a) + \int_a^t f'(\tau)d\tau $$

$$ = f(a) + f(t) - f(a) = f(t) $$

Second case: $f(t)$ is continuous for $t \geq a,\ a \in \mathbb{R}$ then:

$$ {}_aD_t^{-p}f(t) = \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\, f(\tau)d\tau $$

$$ {}_aD_t^{-p}f(t) = \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\, (f(\tau)+f(t)-f(t))d\tau $$

$$ {}_aD_t^{-p}f(t) = \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\, (f(\tau)-f(t))d\tau + \frac{f(t)}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\, d\tau $$

$$ = \frac{1}{\Gamma(p)}\int_a^{t-\delta} (t-\tau)^{p-1}\, (f(\tau)-f(t))d\tau $$

$$ + \frac{1}{\Gamma(p)}\int_{t-\delta}^{t} (t-\tau)^{p-1}\, (f(\tau)-f(t))d\tau + \frac{f(t)(t-a)^p}{\Gamma(p+1)} $$

$$ = I_1 + I_2 + \frac{f(t)(t-a)^p}{\Gamma(p+1)} $$

where,

$$I_1 = \frac{1}{\Gamma(p)} \int_a^{t-\delta} (t-\tau)^{p-1} \,(f(\tau) - f(t))d\tau$$

$$I_2 = \frac{1}{\Gamma(p)} \int_{t-\delta}^{t} (t-\tau)^{p-1} \,(f(\tau) - f(t))d\tau$$

In the integral I_2, $f(t)$ is continuous. Therefore, for every $\delta > 0$, there exists $\varepsilon > 0$ such that $|f(\tau) - f(t)| < \varepsilon$, a bound on I_2 is given as follows:

$$|I_2| < \frac{\varepsilon}{\Gamma(p)} \int_{t-\delta}^{t} (t-\tau)^{p-1} \, d\tau < \frac{\varepsilon \delta^p}{\Gamma(p+1)}$$

Now, as $\delta \to 0$, $\varepsilon \to 0$, then we have $\lim_{\delta \to 0} |I_2| = 0$ when $p \geq 0$

Let us now take an arbitrary $\varepsilon > 0$ and choose δ such that $|I_2| < \varepsilon$ for all $p \geq 0$, then a bound on I_1 can be given by:

$$|I_1| \leq \frac{S}{\Gamma(p)} \int_a^{t-\delta} (t-\tau)^{p-1} d\tau \leq \frac{S}{\Gamma(p+1)} (\delta^p - (t-a)^p)$$

from which it follows that, for fixed $\delta > 0$,

$$\lim_{p \to 0} |I_1| = 0$$

Now, the triangle inequality implies:

$$\left| \, {}_aD_t^{-p} f(t) - f(t) \right| \leq |I_1| + |I_2| + |f(t)| \left| \frac{(t-a)^p}{\Gamma(p+1)} - 1 \right|$$

Therefore, $\lim_{p \to 0} \sup |{}_aD_t{}^{-p} f(t) - f(t)| \leq \varepsilon$

where ε can be chosen as small as we wish. Therefore,

$$\lim_{p\to 0} \sup\left| {}_aD_t{}^{-p} f(t) - f(t) \right| = 0$$

and $\lim_{p\to 0} {}_aD_t{}^{-p} f(t) = f(t)$

Property 2:

$${}_aD_t{}^{-p}\left({}_aD_t{}^{-q} f(t)\right) = {}_aD_t{}^{-p-q} f(t) \tag{1.17}$$

This property can be proved by the following series of equations:

$${}_aD_t{}^{-p}\left({}_aD_t{}^{-q} f(t)\right) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} {}_aD_\tau{}^{-q} f(\tau) d\tau$$

$$= \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t (t-\tau)^{p-1} d\tau \int_a^\tau (t-\xi)^{q-1} f(\xi) d\xi$$

Using Drichlet's formula (1.12)

$$= \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t f(\xi) d\xi \int_\xi^t (t-\tau)^{p-1} (\tau-\xi)^{q-1} d\tau \tag{1.18}$$

Let $\tau = \xi + \delta(t-\xi)$

$${}_aD_t{}^{-p}\left({}_aD_t{}^{-q} f(t)\right) =$$

$$\frac{1}{\Gamma(p)\Gamma(q)} \int_a^t f(\xi) d\xi \int_0^1 \left(t - \left(\xi + \delta(t-\xi)\right)\right)^{p-1} \left(\delta(t-\xi)\right)^{q-1} (t-\xi) d\delta$$

$$= \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t f(\xi) d\xi \int_0^1 \left(t - \xi - \delta(t-\xi)\right)^{p-1} (\delta(t-\xi))^{q-1} d\delta$$

$$= \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t f(\xi) d\xi \int_0^1 ((t-\xi)\,(1-\delta))^{p-1} (\delta(t-\xi))^{q-1} d\delta$$

$$= \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t f(\xi) d\xi \int_0^1 (\mathrm{t}-\xi)^{p+q-1} (\delta)^{q-1} (1-\delta)^{p-1} d\delta$$

$$= \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t f(\xi) (\mathrm{t}-\xi)^{p+q-1} \beta(p,q)\, d\xi$$

$$= \frac{1}{\Gamma(p+q)} \int_a^t (t-\xi)^{p+q-1} f(\xi) d\xi = {}_aD_t{}^{-p-q} f(t).$$

Property 3:

We can interchange p and q that appears in property 2, so we have:

$${}_aD_t{}^{-p}\left({}_aD_t{}^{-q} f(t)\right) = {}_aD_t{}^{-q}\left({}_aD_t{}^{-p} f(t)\right) = {}_aD_t{}^{-q-p} f(t)$$

Theorem 1.2: Leibniz rule for fractional integral [14]**:**

In classical calculus, general Leibniz rule gives the n^{th} derivative of a product of two functions in terms of the derivatives of each function, assuming n is non zero integer.

The general Leibniz Rule:

$$D^n(f(t)g(t)) = \sum_{k=0}^{n} \binom{n}{k} [D^k g(t)][D^{n-k} f(t)] \tag{1.19}$$

Now, equation (1.19) can be extended to fractional integration by applying it to fractional operators under the two conditions:

a. $f(t)$ is contiuous on $[0, t]$

b. $g(t)$ is analytic on $[0, t]$

then,

$$D^{-p}\big(f(t)g(t)\big) = \frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1} f(\tau)g(\tau)d\tau \qquad (1.20)$$

Now, using Taylor series, $g(\tau)$ can be written as:

$$g(\tau) = \sum_{k=0}^{\infty}(-1)^k \frac{D^k g(t)}{k!}(t-\tau)^k$$

or

$$g(\tau) = g(t) + \sum_{k=1}^{\infty}(-1)^k \frac{D^k g(t)}{k!}(t-\tau)^k \qquad (1.21)$$

If we substitute (1.21) in (1.20), then we get:

$$D^{-p}\big(f(t)g(t)\big) =$$

$$\frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}\left[f(\tau)\left(g(t) + \sum_{k=1}^{\infty}(-1)^k \frac{D^k g(t)}{k!}(t-\tau)^k\right)\right]d\tau \quad (1.22)$$

Now, since f is continuous on $[0, t]$, and $\int_0^t (t-\tau)^k f(\tau)d\tau$ is bounded on $[0, t]$, then the order of integration and summation can be interchanged, so equation (1.22) will be:

$$D^{-p}\big(f(t)g(t)\big) = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(p+k)}{k!\;\Gamma(p)}[D^k g(t)][D^{-p-k} f(t)]$$

$$= \sum_{k=0}^{\infty} \binom{-p}{k} [D^k g(t)][D^{-p-k} f(t)] \qquad (1.23)$$

where,

$$\binom{-p}{k} = \frac{(-1)^k \Gamma(p+k)}{k!\ \Gamma(p)}$$

Example 1.5: The Riemann-Liouville fractional integral ${}_aD_t^{-p} f(t),\ p > 0$ of the power function $(t-a)^s,\ s \in \mathbb{R}$

$${}_aD_t^{-p}(t-a)^s = \frac{1}{\Gamma(\mathrm{p})} \int_a^t (t-\tau)^{p-1} (\tau-a)^s \, d\tau$$

Now, if we let $\tau = a + \epsilon(t-a)$, then:

$${}_aD_t^{-p}(t-a)^s = \frac{1}{\Gamma(\mathrm{p})} \int_0^1 (t-a-\epsilon(t-a))^{p-1} \epsilon^s (t-a)^{s+1} \, d\epsilon$$

$$= \frac{1}{\Gamma(\mathrm{p})} \int_0^1 (t-a)^{p-1} (1-\epsilon)^{p-1} \epsilon^s (t-a)^{s+1} \, d\epsilon$$

$$= \frac{1}{\Gamma(p)} (t-a)^{s+p} \int_0^1 \epsilon^s \, (1-\epsilon)^{p-1} d\epsilon$$

$$= \frac{1}{\Gamma(p)} (t-a)^{s+p} \beta(s+1, p)$$

$$= (t-a)^{s+p} \frac{\Gamma(s+1)}{\Gamma(s+p+1)} \qquad (1.24)$$

Special case: when $a = 0, s = 0$, then we can replace ${}_0D_t{}^{-p}\ by\ D^{-p}$ and we will have:

$$D^{-p}(t-0)^0 = \frac{t^p}{\Gamma(\text{p}+1)}$$

Example 1.6 [15]**:** The Riemann-Liouville fractional integral of the exponential function $f(t) = e^{\lambda t}$

Let $a = 0, p > 0$, then:

$$D^{-p}e^{\lambda t} = \frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}e^{\lambda\tau}d\tau$$

Now, let $x = t - \tau$, then:

$$D^{-p}e^{\lambda t} = \frac{-e^{\lambda t}}{\Gamma(p)}\int_0^t (x)^{p-1}e^{-\lambda x}dx \tag{1.25}$$

$$= -t^p e^{\lambda t}\gamma^*(p,\lambda t)$$

$$= -E_t(p,\lambda)$$

Example 1.7: [15] The Riemann-Liouville fractional integral of

$f(t) = \cos(\lambda t)\,, a = 0,\ p > 0$

$$D^{-p}\cos(\lambda t) = \frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}\cos(\lambda\tau)d\tau$$

Assuming $x = t - \tau$, implies:

$$D^{-p}\cos(\lambda t) = \frac{-1}{\Gamma(p)}\int_0^t x^{p-1}\cos\lambda(t-x)\,dx = -c_t(p,\lambda) \tag{1.26}$$

Similarly,

$$D^{-p}\sin(\lambda t) = \frac{-1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}\sin(\lambda\tau)\,d\tau$$

$$= \frac{-1}{\Gamma(p)}\int_0^t x^{p-1}\cos\lambda(t-x)\,dx = -\,s_t(p,\lambda) \tag{1.27}$$

Chapter Two

Fractional differentiation

In this chapter, we will talk about fractional differentiation. There are different definitions for fractional derivatives. In this chapter, we will start by studying two of the most popular definitions. Namely, the Riemann-Liouville fractional derivative and the Caputo fractional derivative. A comparison between these two definitions will be made. At the end, we will move to find Riemann-Liouville and Caputo fractional derivative of different functions.

Note: We will use the notation ${}_aD_t^p\big(f(t)\big)$ for Riemaan-Liouville fractional differentiation, and when

$a = 0$, we will delete the lower limits and use the notation $D^p f(t)$ for the Riemann-Liouville fractional di

2.1 Riemann-Liouville fractional derivative: $[\mathbf{19}], [\mathbf{15}]$

Definition 2.1: The Riemann-Liouville fractional derivative of order $\boldsymbol{p}$:

$${}_aD_t^p = \frac{1}{\Gamma(k-p)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{k-p-1} f(\tau)d\tau, \quad k-1 \le p < k\,, p > 0 \quad (2.1)$$

In fact, this definition is an extension of the derivative of integer order.

For $k \geq n$ and $k, n \in \mathbb{N}$, equation (1.15) can be written as:

$$f^{(k-n)}(t) = \frac{1}{\Gamma(n)} D^k \int_a^t (t-\tau)^{n-1} f(\tau)d\tau = {}_aD_t^{(k-n)} f(t) \qquad (2.2)$$

This form of the derivative of an integer order $(k-n)$ provides a chance to extend the notion of differentiation to non-integer order. Here, we can replace the integer n with any real α in formula (2.2), and let α be any real number such that $k-\alpha > 0$. This will give:

$$ {}_aD_t^{k-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau $$

However, for convergence, we need to restrict α to $\alpha > 0$. Moreover**,** for $p = k - \alpha$, the result is the well-known Riemann-Liouville fractional derivative as given in definition 2.1.

Another representation of the Riemann-Liouville fractional derivative is:

$$ {}_aD_t^{p} f(t) = \frac{d^k}{dt^k}\left({}_aD_t^{-(k-p)} f(t)\right), \qquad k-1 \leq p < k \qquad (2.3) $$

This means that Riemann-Liouville fractional derivative is equivalent to find fold integration of order $k-p$ followed by k^{th} −order differentiation.

Note: We use the symbol $D^n f(t)$ which denotes the ordinary derivative of order n, where n is nonnegative integer.

2.1.1 Main properties of Riemann-Liouville fractional derivative: [18], [9], [19]

Property 1:

If $p = k - 1$, then we obtain a conventional integer-order derivative of order $k - 1$:

$$ {}_aD_t^{k-1} f(t) = \frac{d^k}{dt^k}\left({}_aD_t^{-(k-(k-1))} f(t)\right) $$
$$ = \frac{d^k}{dt^k}\left({}_aD_t^{-1} f(t)\right) = f^{(k-1)}(t) $$

Property 2:

For $p = k \geq 1$ and $t > a$ and using (2.3)

$$ {}_aD_t^{k} f(t) = \frac{d^k}{dt^k}\left({}_aD_t^{0} f(t)\right) = \frac{d^k}{dt^k} f(t) = f^{(k)}(t) $$

That means for $t > a$, the Riemann -Liouville fractional derivative of order $p = k > 1$ coincides with the conventional derivative of order k.

Property 3:

The p^{th}-order derivative of the p^{th}-order integral of a function $f(t)$ is the function $f(t)$ which means that:

$$ {}_aD_t^{p}\left({}_aD_t^{-p} f(t)\right) = f(t),\ p > 0,\ t > a \qquad (2.4) $$

Proof:

For $p = n \geq 1$, we have:

$$ {}_aD_t^{n}\left({}_aD_t^{-n} f(t)\right) = \frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau = \frac{d}{dt}\int_a^t f(\tau)d\tau = f(t) $$

For any p with $k - 1 \leq p < k$, we have:

$$\frac{d^k}{dt^k}\left[{}_aD_t^{-k}\,{}_aD_t^{p}\left({}_aD_t^{-p}f(t)\right)\right] = \frac{d^k}{dt^k}\left\{{}_aD_t^{-(k-p)}\left({}_aD_t^{-p}f(t)\right)\right\}$$

But, using the composition rule (1.17), one get:

$$\begin{aligned} {}_aD_t^{-k}f(t) &= {}_aD_t^{-k+(p-p)}f(t) \\ &= {}_aD_t^{-k+p}\left({}_aD_t^{-p}f(t)\right) \\ &= {}_aD_t^{-(k-p)}\left({}_aD_t^{-p}f(t)\right) \end{aligned}$$

Therefore,

$${}_aD_t^{p}\left({}_aD_t^{-p}f(t)\right) = \frac{d^k}{dt^k}\left\{{}_aD_t^{-k}f(t)\right\} = f(t)$$

Remark: In general, when $p, q \geq 0$, then we have:

$${}_aD_t^{p}\left({}_aD_t^{-q}f(t)\right) = {}_aD_t^{p-q}f(t) \qquad (2.5)$$

property4: Commutativity:

Similar to conventional integer-order differentiation and integration, fractional differentiation and integration do not commute.

$${}_aD_t^{-p}\left({}_aD_t^{p}f(t)\right) = f(t) - \sum_{j=1}^{k}\left({}_aD_t^{p-j}f(t)\right)\Big|_{t=a}\frac{(t-a)^{p-j}}{\Gamma(p-j+1)} \qquad (2.6)$$

Note: If we assume that $p, q \geq 0$, then we have:

$${}_aD_t^{-p}\left({}_aD_t^{q}f(t)\right) = {}_aD_t^{q-p}f(t) - \sum_{j=1}^{k}{}_aD_t^{q-j}f(t)\Big|_{t=a}\frac{(t-a)^{p-j}}{\Gamma(p-j+1)} \qquad (2.7)$$

Property 5: Linearity:

$${}_aD_t^p\big(\alpha\, f(t) + \beta\, g(t)\big) = \alpha\, {}_aD_t^p f(t) + \beta\, {}_aD_t^p g(t)$$

Proof:

$${}_aD_t^p\big(\alpha\, f(t) + \beta\, g(t)\big) = \frac{1}{\Gamma(k-p)}\frac{d^k}{dt^k}\int_a^t \frac{\big(\alpha\, f(t) + \beta\, g(t)\big)}{(t-\tau)^{p-k+1}}\, d\tau$$

$$= \frac{1}{\Gamma(k-p)}\{\frac{d^k}{dt^k}\int_a^t \frac{\big(\alpha\, f(t)\big)}{(t-\tau)^{p-k+1}}\, d\tau \;\; + \frac{d^k}{dt^k}\int_a^t \frac{(\beta\, g(t))}{(t-\tau)^{p-k+1}}\, d\tau \;\}$$

$$= \frac{\alpha}{\Gamma(k-p)}\frac{d^k}{dt^k}\int_a^t \frac{f(t)}{(t-\tau)^{p-k+1}}\, d\tau + \frac{\beta}{\Gamma(k-p)}\frac{d^k}{dt^k}\int_a^t \frac{g(t)}{(t-\tau)^{p-k+1}}\, d\tau$$

$$= \alpha\, {}_aD_t^p f(t) + \beta\, {}_aD_t^p g(t)$$

Property 6: Interpolation:

a. $\lim_{p\to k} {}_aD_t^p f(t) = f^{(k)}(t)$ (2.8)

b. $\lim_{p\to k-1} {}_aD_t^p f(t) = f^{(k-1)}(t)$ (2.9)

Proof:

$${}_aD_t^p f(t) = \frac{1}{\Gamma(k-p)}\frac{d^k}{dt^k}\int_a^t (t-\tau)^{k-p-1} f(\tau)d\tau$$

$$= \frac{1}{\Gamma(k-p)}\frac{d^k}{dt^k}\left(\frac{-f(\tau)(t-\tau)^{k-p}}{k-p}\Bigg|_a^t + \int_a^t \frac{f'(\tau)(t-\tau)^{k-p}}{k-p}d\tau\right)$$

$$= \frac{1}{\Gamma(k-p+1)} \frac{d^k}{dt^k} \left(f(a)t^{k-p} + \int_a^t f'(\tau)(t-\tau)^{k-p}\, d\tau \right)$$

Now, if we take $\lim_{p \to k} {}_aD_t^p f(t)$, $\lim_{p \to k-1} {}_aD_t^p f(t)$ respectively, then we have:

a. $$\lim_{p \to k} {}_aD_t^p f(t) = \frac{d^k}{dt^k} \left(f(a) + \int_a^t f'(\tau) d\tau \right)$$

$$= \frac{d^k}{dt^k} (f(a) + f(t) - f(a)) = f^{(k)}(t)$$

b. $$\lim_{p \to k-1} {}_aD_t^p f(t) = \frac{d^k}{dt^k} \left(f(a)t + \int_a^t f'(\tau)(t-\tau) d\tau \right)$$

$$= \frac{d^k}{dt^k} \left(f(a)t + f(\tau)(t-\tau)|_a^t + \int_a^t f(\tau) d\tau \right)$$

$$= \frac{d^k}{dt^k} \left(f(a)t - f(a)t + \int_a^t f(\tau) d\tau \right) = f^{(k-1)}(t)$$

Property 7: Composition with integer order derivatives:

Suppose that:

$k-1 < p < k,\ k, m \in \mathbb{N}, p \in \mathbb{R}$, then:

$$\frac{d^m}{dt^m} \left({}_aD_t^p f(t) \right) = {}_aD_t^{p+m} f(t) \neq {}_aD_t^p \left(\frac{d^m}{dt^m} f(t) \right) \tag{2.10}$$

Property 8: Composition with fractional derivatives:

Let $m-1 \le p \le m$ and $n-1 \le q < n$, then:

$$ {}_aD_t^p\left({}_aD_t^q f(t)\right) = {}_aD_t^{p+q} f(t) - \sum_{j=1}^{n}\left[{}_aD_t^{q-j}\right]_{t=a}\frac{(t-a)^{-p-j}}{\Gamma(1-p-j)} \tag{2.11} $$

Theorem 2.1: Leibniz rule for Riemann-Liouville fractional derivative [18]:

Let $t > 0,\ p \in \mathbb{R}, n > p > n-1,$ and $n \in \mathbb{N}$. If $f(\tau), g(\tau),$ and all their derivatives are continuous on $[0, t]$, then the following holds:

$$ D^p\big(f(t)g(t)\big) = \sum_{k=0}^{\infty}\binom{p}{k}[D^k g(t)][D^{p-k} f(t)], $$

where $$ \binom{p}{k} = \frac{\Gamma(p+1)}{k!\ \Gamma(p-k+1)} \tag{2.12} $$

Example 2.1: Let $a = 0,\ p = \frac{1}{2}$, and $f(t) = t$. Then, the Riemann-Liouville fractional derivative of order $\frac{1}{2}$ for the function $f(t) = t$ is:

$$ D^{\frac{1}{2}}(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{d}{dt}\int_0^t (t-\tau)^{\frac{-1}{2}}\,\tau\, d\tau, \qquad 0 < p \le 1 $$

Now, if we let $u = t - \tau$, then

$$ \int_0^t (t-\tau)^{\frac{-1}{2}}\tau\, d\tau = -\int_t^0 u^{\frac{-1}{2}}(t-u)du $$

$$ = -\int_t^0 tu^{\frac{-1}{2}}\, du + \int_t^0 u^{\frac{1}{2}}\, du $$

$$= -2t\,u^{\frac{1}{2}}\Big|_t^0 + \frac{2}{3}u^{\frac{3}{2}}\Big|_t^0 = -\left(0 - 2t^{\frac{3}{2}}\right) + \frac{2}{3}\left(0 - t^{\frac{3}{2}}\right)$$

$$= 2t^{\frac{3}{2}} - \frac{2}{3}t^{\frac{3}{2}} = \frac{4}{3}t^{\frac{3}{2}}$$

then:

$$D^{\frac{1}{2}}(t) = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\left(\frac{4}{3}t^{\frac{3}{2}}\right) = \frac{2\sqrt{t}}{\sqrt{\pi}}$$

Example 2.2: Let $a = 0, p = 1, f(t) = t$. Then, the Riemann-Liouville fractional derivative of order 1 for the function $f(t) = t$ is:

$$D^1(t) = \frac{1}{\Gamma(1)}\frac{d^2}{dt^2}\int_0^t (t-\tau)^0\,\tau\,d\tau, \qquad 1 \le p < 2$$

$$= \frac{d^2}{dt^2}\int_0^t \tau\,d\tau = \frac{d^2}{dt^2}\left(\frac{t^2}{2}\right) = 1$$

We must expect this result, because we mentioned that the Riemann-Liouville fractional derivative of order $p = k \ge 1$ coincides with the conventional derivative of order k.

Example 2.3: [18] **Fractional derivative of the power function $(t-a)^s$**

Here, we will show how we can evaluate the Riemann-Liouville fractional derivative ${}_aD_t^p$ of the power function $f(t) = (t-a)^s, s \in \mathbb{R}$

Assuming that $k - 1 \le p < k$ and using :

$${}_aD_t^p f(t) = \frac{d^k}{dt^k}\left({}_aD_t^{-(k-p)} f(t)\right)$$

together with:

$$ {}_aD_t^{-p}(t-a)^s = \frac{\Gamma(1+s)}{\Gamma(1+s+p)}(t-a)^{s+p} $$

then, we get that:

$$ {}_aD_t^{p}(t-a)^s = \frac{\Gamma(1+s)}{\Gamma(1+s-p)}(t-a)^{s-p} \qquad (2.13) $$

and the only condition for $f(t)=(t-a)^s$ is its integrability, which means: $s>-1$

2.2 Caputo fractional derivative [21], [10], [8], [9]

Another definition for the fractional derivative is the following Caputo definition which is denoted by $D_*^p f(t)$

Definition 2.2: Suppose that $p>0$, $t>a=0$, p and $t \in \mathbb{R}$, then

$$ D_*^p f(t) = \begin{cases} \dfrac{1}{\Gamma(k-p)} \displaystyle\int_0^t \dfrac{f(\tau)^{(k)}}{(t-\tau)^{p+1-k}} d\tau & , \quad k-1<p<k, k \in \mathbb{N} \\ \dfrac{d^k}{dt^k} f(t) & , \quad p=k \in \mathbb{N} \end{cases} $$

Alternatively:

$$ D_*^p f(t) = D^{-(k-p)} D^k f(t), \quad k-1 \le p < k \qquad (2.14) $$

This means that finding the Caputo fractional derivative of any function is equivalent to find the k^{th} -order differentiation and then finding the fold integration of order $(k-p)$

The Caputo fractional derivative is considered to be an alternative definition for Riemann- Liouville definition to find fractional derivatives, and it is introduced by the Italian Mathematician Caputo in 1967.

2.2.1 Properties of Caputo fractional derivatives [9], [18]:

Property 1: Interpolation:

Let $k-1<p<k$, $k\in\mathbb{N}, p\in\mathbb{R}$ and $f(t)$ be such that $D_*^p f(t)$ exists, then the following properties for the caputo operator hold:

a. $\lim\limits_{p\longrightarrow k} D_*^p f(t) = f^{(k)}(t)$ (2.15)

b. $\lim\limits_{p\longrightarrow k-1} D_*^p f(t) = f^{(k-1)}(t) - f^{(k-1)}(0)$ (2.16)

Proof:

$$D_*^p f(t) = \frac{1}{\Gamma(k-p)} \int_0^t \frac{f^{(k)}(t)}{(t-\tau)^{p+1-k}} d\tau$$

$$= \frac{1}{\Gamma(k-p)} \left(-f^{(k)}(\tau) \frac{(t-\tau)^{k-p}}{k-p} \bigg|_{\tau=0}^{t} - \int_0^t -f^{(k+1)}(\tau) \frac{(t-\tau)^{k-p}}{k-p} d\tau \right)$$

$$= \frac{1}{\Gamma(k-p+1)} \left(f^{(k)}(0) t^{k-p} + \int_0^t f^{(k+1)}(\tau)(t-\tau)^{k-p} d\tau \right)$$

Now, by taking the limit as $p\longrightarrow k$ and $p\longrightarrow k-1$, respectively, it follows that:

a. $\lim\limits_{p\to k} D_*^p f(t) = \left(f^{(k)}(0) + f^{(k)}(\tau)\big|_{\tau=0}^{t} \right) = f^{(k)}(t)$

b. $\lim\limits_{p\to k-1} D_*^p f(t) = \left(f^{(k)}(0)t + f^{(k)}(\tau)(t-\tau)\big|_{\tau=0}^{t} + \int_0^t f^{(k)}(\tau) d\tau \right)$

$$= f^{(k-1)}(\tau)\big|_{\tau=0}^{t} = f^{(k-1)}(t) - f^{(k-1)}(0)$$

Property 2: Linearity:

The Caputo fractional derivative of a linear combination of functions is the linear combination of the Caputo fractional derivatives of the functions

$$D_*^p\left(\alpha f(t) + \beta g(t)\right) = \alpha D_*^p f(t) + \beta D_*^p g(t)$$

Property 3: Composition with integer order derivatives:

Suppose that:

$k-1 < p < k,\ \ k, m \in \mathbb{N},$ and $p \in \mathbb{R}$, then:

$$D_*^p D_*^m f(t) = D_*^{p+m} f(t) \neq D_*^m D_*^p f(t) \tag{2.17}$$

Theorem 2.2: suppose that $k-1 < p < k, \beta = p-(k-1), \beta \in (0,1)$, $k \in \mathbb{N}$ and $p, \beta \in \mathbb{R}$, then:

$$D_*^p f(t) = D_*^\beta D^{k-1} f(t) = D_*^{p-(k-1)} D^{k-1} f(t)$$

Proof: In the previous property if we replace p by β and substitute $m = k-1$, then we get:

$$D_*^\beta D^{k-1} f(t) = D_*^{\beta+k-1} f(t) = D_*^{p-(k-1)+k-1} f(t) = D_*^p f(t)$$

This theorem means that only derivatives of order $\beta \in (0,1)$ should be considered, because to find the fractional derivative of order p of a function $f(t)$, it is sufficient to find the fractional derivative of order β to $(k-1)^{th}$ -derivative of the function $f(t)$.

Remark: This theorem is true for Riemann-Liouville fractional derivative which means that:

For $k-1 < p < k,\ \beta = p-(k-1), \beta \in (0,1), k \in \mathbb{N}$ and $p, \beta \in \mathbb{R}$

$$D^p f(t) = D^{k-1} D^{p-(k-1)} f(t) = D^{k-1} D^\beta f(t),$$

Property 4:

$$D^{-p}D_*^{p}f(t)=f(t)-\sum_{j=1}^{k}\left(D^{k-j}f(t)\right)\Big|_{t=a=0}\frac{(t-a)^{k-j}}{\Gamma(k-j+1)} \tag{2.18}$$

Proof:

$$D^{-p}D_*^{p}f(t)=D^{-p}\left(D^{-(k-p)}D^{k}f(t)\right)$$

$$=D^{-k}D^{k}f(t)$$

If we use formula (2.6) , we get that:

$$D^{-k}D^{k}f(t)=f(t)-\sum_{j=1}^{k}\left(D^{k-j}f(t)\right)\Big|_{t=a=0}\frac{(t-a)^{k-j}}{\Gamma(k-j+1)}$$

Property 5:

$$D_*^{p}D^{-p}f(t)=f(t)-\sum_{j=1}^{k}\left(D^{k-p-j}f(t)\right)\Big|_{t=a=0}\frac{(t-a)^{k-p-j}}{\Gamma(k-p-j+1)} \tag{2.19}$$

Proof:

$$D_*^{p}D^{-p}f(t)=D^{-(k-p)}D^{k}\left(D^{-p}f(t)\right)=D^{-(k-p)}D^{(k-p)}f(t)$$

If we use formula (2.6), we get that:

$$D^{-(k-p)}D^{(k-p)}f(t)=f(t)-\sum_{j=1}^{k}\left(D^{k-p-j}f(t)\right)\Big|_{t=a=0}\frac{(t-a)^{k-p-j}}{\Gamma(k-p-j+1)}$$

Theorem 2.3: Leibniz rule for Caputo fractional derivative [9]**:**

Let $t>0, p\in\mathbb{R},\ n-1>p>n,\ n\in\mathbb{N}$. If $f(\tau), g(\tau)$, and all their derivatives are continuous on $[0,t]$, then the following holds:

$$D_*^{p}(f(t)g(t))=\sum_{k=0}^{\infty}\binom{p}{k}[D^{k}g(t)][D^{p-k}f(t)]$$

$$-\sum_{k=0}^{n-1}\frac{t^{k-p}}{\Gamma(k+1-p)}\left(f(t)g(t)\right)^{(k)}(0) \tag{2.20}$$

The main advantages of Caputo operator over Riemann-Liouville [9]

We illustrate the main advantage of using the Caputo definition over Riemann-Liouville definition using the following two initial value problems:

a. Riemann-Liouville fractional initial value problem

$D^p y(t) - \gamma y(t) = 0, \quad t > 0, \quad k-1 < p < k$

$[D^{p-n-1} y(t)]_{t=0} = b_n, \quad n = 0,1,\dots\dots,k-1$

b. Caputo fractional initial value problem

$D_*^p y(t) - \gamma y(t) = 0, \quad t > 0, \quad k-1 < p < k$

$y^{(n)}(0) = b_n, \quad n = 0,1,\dots\dots,k-1$

The initial conditions in the initial value problem in (a) are fractional derivatives. We can theoretically solve this initial value problem, but practically the solutions that we have are not useful because there is no significant and useful analysis to this kind of initial conditions.

On the other hand, the initial value problem in (b) involves ideal initial conditions in terms of derivatives of integer order, which have an understood analysis and interpretation as an initial position $y(a)$ at the point $t = a$, the initial velocity $y'(a)$, initial acceleration $y''(a)$ and so on.

The Caputo fractional derivative is more restrictive, because it demands the existence of the n^{th} derivative of the function. Luckily, most functions that appears in applications meet this demand. Later, whenever the Caputo operator is used we can assume that this condition is satisfied.

2.3 Relation between Riemann-Liouville and Caputo operator:[4]

In this section, we state the relation between the two major definitions of fractional derivatives.

Theorem 2.4: Let $t > 0$ $(a = 0), p \in \mathbb{R},$ and $n - 1 < p < n \in \mathbb{N}$, then the following relation between the Riemann-Liouville and the Caputo operators holds:

$$D_*^p f(t) = D^p f(t) - \sum_{k=0}^{n-1} \frac{t^{k-p}}{\Gamma(k+1-p)} f^{(k)}(0) \qquad (2.21)$$

Proof: we will prove this theorem using the well-known Taylor series expansion about the point 0 which is:

$$f(t) = f(0) + tf'(0) + \frac{t^2}{2!} f''(0) + \cdots + \frac{t^{n-1}}{(n-1)!} f^{(n-1)}(0)$$
$$+ R_{n-1}(M_{n-1}) \text{ where } M_{n-1} \text{ between 0 and } t$$

$$= \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0) + R_{n-1} \ (M_{n-1})$$

where the reminder term $R_{n-1}(M_{n-1})$ is given by:

$$R_{n-1}(M_{n-1}) = \int_0^t \frac{f^{(n)}(\tau)(t-\tau)^{n-1}}{(n-1)!} d\tau$$
$$= \frac{1}{\Gamma(n)} \int_0^t f^{(n)}(\tau)(t-\tau)^{n-1} d\tau = D^{-n} f^{(n)}(t)$$

Now, if we use different properties of the Riemann-Liouville fractional derivative such as: the linearity property, representation formula, and the Riemann-Liouville fractional derivative of the power function, then we will have:

$$D^p f(t) = D^p \left(\sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0) + R_{n-1} \right)$$

$$= \sum_{k=0}^{n-1} \frac{D^p t^k}{\Gamma(k+1)} f^{(k)}(0) + D^p R_{n-1}$$

$$= \sum_{k=0}^{n-1} \frac{\Gamma(k+1)}{\Gamma(k-p+1)} \frac{t^{k-p}}{\Gamma(k+1)} f^{(k)}(0) \; + D^p \, D^{-n} \, f^{(n)} \, (t)$$

$$= \sum_{k=0}^{n-1} \frac{t^{k-p}}{\Gamma(k-p+1)} f^{(k)}(0) + \; D^{-n+p} \, f^{(n)} \, (t)$$

$$= \sum_{k=0}^{n-1} \frac{t^{k-p}}{\Gamma(k-p+1)} f^{(k)}(0) + \; D_*^p \, f(t)$$

This means that:

$$D_*^p f(t) = D^p f(t) - \sum_{k=0}^{n-1} \frac{t^{k-p}}{\Gamma(k-p+1)} f^{(k)}(0)$$

Remark: This formula implies that the Caputo and the Riemann-Liouville fractional operator coincide if $f(t)$ together with its first $n-1$ derivatives vanish at $t = 0$

Corollary 2.1: The following relation between the Riemann-Liouville and Caputo fractional derivatives holds : [9]

$$D_*^p f(t) = D^p \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{\text{k!}} f^{(k)}(0) \right) \tag{2.22}$$

Proof:

$$D_*^p f(t) = D^p f(t) - \sum_{k=0}^{n-1} \frac{t^{k-p}}{\Gamma(k-p+1)} f^{(k)}(0)$$

$$= D^p f(t) - \sum_{k=0}^{n-1} \frac{D^p t^k}{\Gamma(k+1)} f^{(k)}(0)$$

$$= D^p \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{\text{k!}} f^{(k)}(0) \right)$$

2.4 Computation of fractional derivatives

In this section we will find the fractional derivative of different functions such as the constant, the power, and the exponential function. Moreover, we will consider the fractional derivatives of the sine and cosine functions.

1. The constant function [18]**:** If we look physically to the fractional derivative of a constant function, we see that it is logically equal to zero. But this doesn't happen in the Riemann-Liouville fractional derivative because,

$$D^p c = \frac{c}{\Gamma(1-p)} t^{-p} \neq 0, \quad c = constant \tag{2.23}$$

Proof:

$$D^p c = \frac{c}{\Gamma(k-p)} D^k \left(\int_0^t (t-\tau)^{k-p-1} d\tau \right)$$

$$= D^k \frac{c\, t^{k-p}}{(\mathrm{k}-\mathrm{p})\Gamma(k-p)}$$

$$= \frac{c}{\Gamma(k-p)} \frac{\Gamma(k-p)}{\Gamma(1-p)} t^{-p} = \frac{c}{\Gamma(1-p)} t^{-p} \neq 0$$

Example 2.4:

$$D^{1/2}(2) = \frac{2}{\Gamma(1/2)} t^{-1/2} = \frac{2}{\sqrt{\pi t}}$$

By contrast, the Caputo fractional derivative of the constant function is equal to zero, that is $D_*^p c = 0$, and this property is one of the advantages of Caputo derivative over the Riemann-Liouville derivative.

Proof:

Let $k-1<p<k, k\in\mathbb{N}$, and applying the definition of the Caputo derivative and since the k^{th} derivative of a constant is equal to zero, it follows that:

$$D_*^p c=\frac{1}{\Gamma(k-p)}\int_a^t \frac{c^{(k)}}{(t-\tau)^{p+1-k}}d\tau=0$$

2. The power function:

Full detailed examination is required for this type of function as it is considered to be very important. To calculate the derivative of this function we need to remember the Taylor expansion:

a. The Riemann-Liouville fractional derivative of the power function satisfies [18], [5]:

$$D^p t^\alpha=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-p+1)}t^{\alpha-p},\qquad k-1<p<k, \alpha>-1, \alpha\in\mathbb{R}\quad (2.24)$$

Proof:

$$D^p t^\alpha= \frac{1}{\Gamma(k-p)}D^k\int_0^t \tau^\alpha(t-\tau)^{k-p-1}d\tau$$

Now, if we take $\tau=\lambda t$ where $0\leq\lambda\leq 1$, then the previous integral becomes:

$$D^p t^\alpha=\frac{1}{\Gamma(k-p)}D^k\left(\int_0^1 \lambda^\alpha t^\alpha t^{k-p-1}(1-\lambda)^{k-p-1}t\ \ d\lambda\right)$$

$$=\frac{1}{\Gamma(k-p)}D^k(t^{k+\alpha-p}\int_0^1 \lambda^\alpha(1-\lambda)^{k-p-1}\,d\lambda)$$

$$=\frac{1}{\Gamma(k-p)}D^k\left(t^{k+\alpha-p}\frac{\Gamma(\alpha+1)\Gamma(k-p)}{\Gamma(k+\alpha-p+1)}\right)$$

$$= \frac{1}{\Gamma(k-p)} \frac{\Gamma(\alpha+1)\Gamma(k-p)}{\Gamma(k+\alpha-p+1)} \frac{\Gamma(k+\alpha-p+1)}{\Gamma(\alpha-p+1)} t^{\alpha-p}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-p+1)} t^{\alpha-p}$$

b. The Caputo fractional derivative of the power function [5] satisfies:

$$D_*^p t^\alpha = \begin{cases} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-p+1)} t^{\alpha-p} = D^p t^\alpha, n-1 < p < n, \alpha > n-1, \alpha \in \mathbb{R} \\ 0 \qquad ,n-1 < p < n, \alpha \le n-1, \alpha \in \mathbb{N} \end{cases}$$

Proof:

The proof of the first case is:

$$D_*^p t^\alpha = \frac{1}{\Gamma(n-p)} \int_0^t \frac{(\tau^\alpha)^{(n)}}{(t-\tau)^{p+1-n}} d\tau$$

$$= \frac{1}{\Gamma(n-p)} \int_0^t \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)} (\tau^{\alpha-n}) (t-\tau)^{n-p-1} d\tau$$

Now, if we use the substitution $\tau = \lambda t$ where $0 \le \lambda \le 1$, then the previous equation becomes:

$$D_*^p t^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)} \int_0^1 (\lambda t)^{\alpha-n} ((1-\lambda)t)^{n-p-1}\, t\, d\lambda$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)} t^{\alpha-p} \int_0^1 (\lambda)^{\alpha-n} (1-\lambda)^{n-p-1}\, d\lambda$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)} t^{\alpha-p}\, \beta(\alpha-n+1, n-p)$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)} t^{\alpha-p}\, \frac{\Gamma(\alpha-n+1)\Gamma(n-p)}{\Gamma(\alpha-p+1)}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-p+1)} t^{\alpha-p} \qquad (2.25)$$

Finally, it is interested to say that we can reach the same result if we use the relation between Riemann-Liouville and Caputo fractional derivative i.e.

$$D_*^p t^\alpha = D^P t^\alpha - \sum_{k=0}^{n-1} \frac{t^{k-p}}{\Gamma(k+1-p)} (t^\alpha)^k \Big|_{t=0}$$

Taking in to account that $(t^\alpha)^k\big|_{t=0} = 0$, for $k \leq n-1 < \alpha$

$$D_*^P t^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-p+1)}\, t^{\alpha-p} - \sum_{k=0}^{n-1} \frac{t^{k-p}}{\Gamma(k+1-p)} 0$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-p+1)}\, t^{\alpha-p}$$

Important comments:

a. We can prove the second case by using the fact that the Caputo fractional derivative of the constant function equal 0 since $(t^\alpha)^{(n)} = \frac{d^n}{dt^n}(t^\alpha) = 0,$

$\alpha \leq n-1, \alpha \in \mathbb{N}$.

b. The case $-1 < \alpha < n-1$, $\alpha \in \mathbb{R}$ is included in Riemann-Liouville definition, but not in Caputo definition. There isn't any formula for the Caputo operator until now.

c. We can note that for $\alpha > n-1$, the Caputo fractional derivative of the power function is a generalization of the integer order derivative of the power function.

At this point, we recall that:

$$(t^\alpha)^{(n)} = (\alpha t^{\alpha-1})^{(n-1)}$$

$$= (\alpha(\alpha-1)t^{\alpha-2})^{(n-2)}$$

$$= \alpha(\alpha-1) \ldots (\alpha-n+1)t^{\alpha-n}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}\, t^{\alpha-n}, n \in \mathbb{N}, \alpha \in \mathbb{R}$$

d. As a special case when $n = 1$, which means $0 < p < 1$. In this case, we have:

$$D_*^P t^\alpha = D^p t^\alpha, \alpha > 0, \alpha \in \mathbb{R}$$

e. We can compute the Riemann fractional derivative of an arbitrary function $f(t)$ by the formula:

$$D^P f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k-p+1)} t^{k-p}$$

Proof: Recall that:

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k$$

then,

$$D^P f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} D^p t^k$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \frac{\Gamma(k+1)}{\Gamma(k-p+1)} t^{k-p} \quad = \sum_{K=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k-p+1)} t^{k-p}$$

Example 2.5: Suppose that $p \in \mathbb{R}$ but $p \notin \mathbb{N}$. Consider the function $f(t) = t^\alpha$ for $= 1$ and $\alpha = 2$.

a. Let $\alpha = 1$ which means $f(t) = t$

For this case, we have:

$$D_*^p(t) = \frac{\Gamma(1+1)}{\Gamma(1-p+1)} t^{1-p}$$

$$D_*^{1/2}(t) = \frac{1}{\Gamma(2-1/2)} t^{1-1/2} = 2\frac{\sqrt{t}}{\sqrt{\pi}}$$

$$D_*^{1/3}(t) = \frac{1}{\Gamma(2-1/3)} t^{1-1/3} \approx \frac{t^{2/3}}{0.9027}$$

$$D_*^{3/4}(t) = \frac{1}{\Gamma(2-3/4)} t^{1-3/4} \approx \frac{t^{1/4}}{0.9064}$$

b. Let $\alpha = 2$ which means $f(t) = t^2$.

For this case, we have:

$$D_*^P(t^2) = \frac{\Gamma(2+1)}{\Gamma(2-p+1)}\ t^{2-p} = \frac{2}{\Gamma(3-p)} t^{2-p}, k-1 < p < k$$

$$D_*^{1/2}(t^2) = \frac{2}{\Gamma(3-1/2)} t^{2-1/2} = \frac{2}{\Gamma(5/2)} \sqrt{t^3} \approx 1.5\sqrt{t^3}$$

$$D_*^{1/3}(t^2) = \frac{2}{\Gamma(3-1/3)} t^{2-1/3} = \frac{2}{\Gamma(8/3)} t^{5/3} \approx 1.33 t^{5/3}$$

$$D_*^{3/4}(t^2) = \frac{2}{\Gamma(3-3/4)} t^{2-3/4} = \frac{2}{\Gamma(9/4)} t^{5/4} \approx 1.77 t^{5/4}$$

3. The exponential function: Now, we consider the exponential function $e^{\lambda t}$.

a. The Riemann- Liouville fractional derivative of the exponential function [9] satisfies:

$$D^p\left(e^{\lambda t}\right) = t^{-p} E_{1,1-p}(\lambda t), n-1 \leq\ p\ < n\ , p \in \mathbb{R}\ , \lambda \in \mathbb{C} \text{ and } \in \mathbb{N} \quad (2.26)$$

Proof: Recall that:

$$e^{\lambda t} = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}, \text{then:}$$

$$D^p(e^{\lambda t}) = \sum_{k=0}^{\infty} \frac{\lambda^k D^p(t^k)}{k!} = \sum_{k=0}^{\infty} \frac{\lambda^k t^{k-p}}{k!} \frac{\Gamma(k+1)}{\Gamma(k-p+1)}$$

$$= t^{-p} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{\Gamma(k-p+1)} = t^{-p} E(\lambda t)_{1,1-p}$$

where $E_{\alpha,\beta}(z)$ is the two –parameter function of the Mittag-Leffler type.

b. The Caputo fractional derivative of the exponential function has the form [9]:

$$D_*^p(e^{\lambda t}) = \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k+n-p}}{\Gamma(k+1+n-p)} = \lambda^n t^{n-p} E_{1,n-p+1}(\lambda t) \qquad (2.27)$$

Proof: To prove (2.27), we need the relation between Riemann-Liouville and Caputo factional derivative. Also, we need the Riemann-Liouville fractional derivative of the exponential function which is,

$$D^p(e^{\lambda t}) = t^{-p} E_{1,1-p}(\lambda t)$$

Now, for the Caputo fractional derivative

$$D_*^P e^{\lambda t} = D^p e^{\lambda t} - \sum_{k=0}^{n-1} \frac{t^{k-p}}{\Gamma(k+1-p)} \left(e^{\lambda t}\right)^{(k)}(0)$$

$$= t^{-p} E_{1,1-P}(\lambda t) - \sum_{k=0}^{n-1} \frac{t^{k-p}}{\Gamma(k+1-p)} \lambda^k$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda t)^k t^{-p}}{\Gamma(k+1-p)} - \sum_{k=0}^{n-1} \frac{t^{k-p}}{\Gamma(k+1-p)} \lambda^k$$

$$= \sum_{k=n}^{\infty} \frac{\lambda^k t^{k-p}}{\Gamma(k+1-p)} = \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k+n-p}}{\Gamma(k+n+1-p)}$$

$$= \lambda^n t^{n-p} E_{1,n-p+1}(\lambda t)$$

Theorem 2.5 [9]**:** Let $n-1<p<n,$ and $s\in\mathbb{Z}, s>-n.$ Then:

$$D_*^P e^{\lambda t} = D_*^{p+s} e^{\lambda t}$$

This means that only the value after the decimal point or the order of differentiation is important when we want to compute the Caputo fractional derivative of the exponential function. For example, for

$p=0.8,\ p=1.8,$ and $p=7.8$. The same result is obtained, i.e.

$$D_*^{0.8}(e^t) = D_*^{1.8}(e^t) = D_*^{7.8}(e^t) = \sqrt[5]{t}E_{1,1.2}(t)$$

Moreover, the p^{th}-order ($n-1<p<n$) fractional derivatives of the exponential function e^t are moving from e^t-1 to e^t as p taking values from $n-1$ to n.

Now, we can evaluate the limits $\lim_{p\to n} D_*^p\,(e^t)$, $\lim_{p\to n-1} D_*^p(\,e^t)$ respectively:

$$\lim_{p\to n} D_*^p\,(e^t) = \lim_{p\to n} t^{n-p}E_{1,n-p+1}(t) = E_{1,1}\,(t)$$

$$= e^t = D^n(e^t)$$

$$\lim_{p\to n-1} D_*^p\,(e^t) = \lim_{p\to n-1} t^{n-p}E_{1,n-p+1}(t) \ = tE_{1,2}(t)$$

$$= t\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty}\frac{t^{k+1}}{\Gamma(k+2)} = \sum_{k=1}^{\infty}\frac{t^k}{\Gamma(k+1)}$$

$$= e^t - 1 \ = D^{n-1}(e^t) - \ D^{n-1}(e^t)|_{\,t=0}$$

4. The cosine and sine functions:

We will compute the Riemann-Liouville and Caputo fractional derivatives for the cosine and the sine functions [9]:

a. The Riemann- Liouville fractional derivative of the cosine and the sine functions.

Suppose that: $\lambda \in \mathbb{C},\ p \in \mathbb{R},\ n \in \mathbb{N},\ n-1 < p < n$, then:

$$D^p(cos\lambda t) = \frac{1}{2}t^{-p}\left(E_{1,1-p}(i\lambda t) + E_{1,1-p}(-i\lambda t)\right) \tag{2.28}$$

$$D^p(sin\lambda t) = -\frac{\mathrm{i}}{2}t^{-p}\left(E_{1,1-p}(i\lambda t) - E_{1,1-p}(-i\lambda t)\right) \tag{2.29}$$

b. The Caputo fractional derivative of the cosine and sine function is:

$$D_*^p(sin\lambda t) = -\frac{1}{2}i(i\lambda)^n t^{n-p}(E_{1,n-p+1}(i\lambda t) - (-1)^n E_{1,n-p+1}(-i\lambda t))$$

$$D_*^p(cos\lambda t) = \frac{1}{2}(i\lambda)^n t^{n-p}\left(E_{1,n-p+1}(i\lambda t) + (-1)^n E_{1,n-p+1}(-i\lambda t)\right)$$

Proof: Recall that:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, z \in \mathbb{C}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, z \in \mathbb{C}$$

We prove the Caputo fractional derivative of cosine function, and in the same way we can prove the other fractional derivatives.

$$D_*^p(cos\lambda t) = D_*^p\left(\frac{e^{i\lambda t} + e^{-i\lambda t}}{2}\right)$$

$$= \frac{1}{2}\left(D_*^P(e^{i\lambda t}) + D_*^P(e^{-i\lambda t})\right)$$

$$= \frac{1}{2}\left((i\lambda)^n t^{n-p}E_{1,n-p+1}(i\lambda t) + (-i\lambda)^n t^{n-p}E_{1,n-p+1}(-i\lambda t)\right)$$

$$= \frac{1}{2}(i\lambda)^n t^{n-p}\left(E_{1,n-p+1}(i\lambda t) + (-1)^n E_{1,n-p+1}(-i\lambda t)\right) \tag{2.30}$$

Basic comparison between Riemann-Liouville and Caputo derivative:

Here, we will summarize the basic differences that appears in this chapter, and put them in the following table:

Property	Riemann-Liouville	Caputo
Representation	$D^p f(t) = D^k D^{-(k-p)} f(t)$	$D_*^p f(t)$ $= D^{-(k-p)} D^k f(t)$
Interpolation	$\lim\limits_{p \to k} D^p f(t) = f^{(k)}(t)$ $\lim\limits_{p \to k-1} D^p f(t) = f^{(k-1)}(t)$	$\lim\limits_{p \to k} D_*^p f(t) = f^{(k)}(t)$ $\lim\limits_{p \to k-1} D_*^p f(t) =$ $f^{(k-1)}(t) - f^{(k-1)}(0)$
Composition with integer order derivative	$D^m\left(D^p f(t)\right)$ $= D^{p+m} f(t)$ $\neq D^p\left(D^m f(t)\right)$	$D_*^p D_*^m f(t)$ $= D_*^{p+m} f(t)$ $\neq D_*^m D_*^p f(t)$
Fractional derivative of constant function	$D^p(c) = \frac{c}{\Gamma(1-p)} t^{-p}$ $\neq 0$	0
Fractional derivative of exponential function	$D^p\left(e^{\lambda t}\right) = t^{-p} E_{1,1-p}(\lambda t)$	$D_*^p\left(e^{\lambda t}\right)$ $= \lambda^n t^{n-p} E_{1,n-p+1}(\lambda t)$
Fractional derivative of sine function	$D^p(sin\lambda t)$ $= -\frac{\mathrm{i}}{2} t^{-p}\left(E_{1,1-p}(i\lambda t)\right.$ $\left.- E_{1,1-p}(-i\lambda t)\right)$	$D_*^p(sin\lambda t)$ $= -\frac{1}{2} i(i\lambda)^n t^{n-p}(E_{1,n-p}$ $- (-1)^n E_{1,n-p+1}(-i\lambda t))$
Fractional derivative of cosine function	$D^p(cos\lambda t)$ $= \frac{1}{2} t^{-p}\left(E_{1,1-p}(i\lambda t)\right.$ $\left.+ E_{1,1-p}(-i\lambda t)\right)$	$D_*^p(cos\lambda t)$ $= \frac{1}{2}(i\lambda)^n t^{n-p}\left(E_{1,n-p+1}\right.$ $\left.+ (-1)^n E_{1,n-p+1}(-i\lambda t)\right)$

Chapter Three

Fractional Ordinary Differential Equations

3.1 Basic effects on Laplace transform

In this section, we overview important information about Laplace transform, and we find Laplace transform of Riemann-Liouville integral, Riemann-Liouville fractional derivative, and Caputo fractional derivative. Laplace transform can be used to solve fractional linear ordinary differential equations. In fact, it is widely known and very effective method in solving fractional differential equations.

We start by giving an overview of Laplace transform.

1. Let f be a function defined on $[0, \infty)$,then Laplace transform of a function $f(t)$ is defined as: [13], [18]

$$L\{f(t); s\} = F(s) = \int_0^\infty e^{-st} f(t) dt \tag{3.1}$$

The previous integral converges when the function $f(t)$ is of exponential order which means that [18]:

$e^{-\alpha t}|f(t)| \leq M,$ for all $t > T, M, T \in \mathbb{R}^+$

2. The inverse Laplace transform is defined as: [8]

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s) ds, \gamma \in \mathbb{R} \tag{3.2}$$

This formula is seldom used in practice, but it is important for theoretical use. In practice, the transformation of some elementary functions together with the properties of Laplace transform are used to find the Laplace transform of other functions. See table 3.1

3. the convolution of two functions $f(t)$ and $g(t)$ defined on $[0,\infty]$ is:

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

Table 3.1: Laplace transform of some special functions [8]

$f(t)$	$L\{f(t)\}$
$e^{at}, a \in \mathbb{R}$	$\frac{1}{s-a}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
kt^n	$\frac{k\Gamma(n)}{s^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$f(t) * g(t)$	$L\{f(t);s\}L\{g(t);s\}$
$af(t)+bg(t)$	$aL\{f(t);s\}+bL\{g(t);s\}$
$f^{(n)}(t)$	$s^nF(s)-\sum_{k=0}^{n-1} s^{n-k-1}f^{(k)}(0) = s^nF(s)-\sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0)$
$t^{\alpha k+\beta-1}E_{\alpha,\beta}{}^{(k)}(at^\alpha)$	$\frac{k!\,s^{\alpha-\beta}}{(s^\alpha-a)^{k+1}}$

3.2 Laplace transform of fractional derivatives and integrals

In this section, we find Laplace transform of Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, and Caputo fractional derivative

1. Laplace transform of Riemann-Liouville fractional integral is:

$$L\{D^{-p}f(t);s\} = s^{-p}F(s) \tag{3.4}$$

Proof [10]: We know that

$$D^{-p}f(t) = \frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}f(\tau)d\tau = \frac{1}{\Gamma(p)}t^{p-1} * f(t)$$

then:

$$L\left\{\frac{1}{\Gamma(p)}t^{p-1} * f(t);s\right\} = L\left\{\frac{1}{\Gamma(p)}t^{p-1};s\right\}L\{f(t);s\}$$

$$= \frac{1}{s^p}F(s) = s^{-p}F(s)$$

2. **Laplace transform of Riemann-Liouville fractional derivative is:**

$$L\{D^p f(t);s\} = s^p F(s) - \sum_{k=0}^{n-1} s^k\left[D^{(p-k-1)}f(t)\right]_{t=0} \tag{3.5}$$

Proof [18]: Assume that:

$$D^p f(t) = g^{(n)}(t) \tag{3.6}$$

then, we can show that:

$$g(t) = D^{-(n-p)}f(t) = \frac{1}{\Gamma(n-p)}\int_0^t (t-\tau)^{n-p-1}f(\tau)d\tau$$

If we take Laplace to equation (3.6), then we get:

$$L\{D^p f(t); s\} = L\{g^{(n)}(t); s\} = s^n G(s) - \sum_{k=0}^{n-1} s^k g^{(n-k-1)}(0) \qquad (3.7)$$

Now, we find $G(s)$

$$G(s) = L\{g(t); s\} = L\{D^{-(n-p)} f(t)\} = s^{-(n-p)} F(s) \qquad (3.8)$$

also,

$$g^{(n-k-1)}(t) = \frac{d^{(n-k-1)}}{dt^{(n-k-1)}} D^{-(n-p)} f(t)$$

$$= D^{p-k-1} f(t) \qquad (3.9)$$

Substitute equations (3.8), (3.9) in the formula (3.7), then we get:

$$L\{D^p f(t); s\} = s^p F(s) - \sum_{k=0}^{n-1} s^k \left[D^{(p-k-1)} f(t)\right]_{t=0}$$

Note: It is interested here to compare formula (3.5) with the formula of Laplace transform of integer order derivative, and if we do that, then the result will be that the two formulas are the same, but we change the order of differentiation and replace n by p.

3. Laplace transform of Caputo fractional derivative is[18]:

$$L\{D_*^p f(t); s\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0) \qquad (3.10)$$

Proof: The definition of Caputo fractional derivative which is:

$$D_*^p f(t) = D^{-(n-p)} g(t), \qquad g(t) = f^{(n)}(t)$$

If we take Laplace transform to both sides and use formula (3.4) then we get:

$$L\{D_*^p f(t); s\} = L\{D^{-(n-p)} g(t); s\} = s^{-(n-p)} G(s) \tag{3.11}$$

where,

$$G(s) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0) \tag{3.12}$$

Substitute(3.12) in (3.11), then we get that:

$$L\{D_*^p f(t); s\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0)$$

3.3 Solving Caputo and Riemann-Liouville initial value problems:

In this section we will try to find a general solution for a specific Caputo and Riemann initial value problem using the most common and easiest method which is Laplace transform method. Moreover, we will compare the solution of the fractional derivative with the solution of classical initial value problem of integer order and see the result. At the end of this section we will take different examples as special cases and find their solution.

1. Caputo initial value problem

$$D_*^p y(t) - \lambda y(t) = at,\ t > 0,\ n-1 < p < n \tag{3.13}$$

$$y^{(k)}(0) = b_k, \qquad k = 0,1,\dots\dots,n-1$$

The solution of problem (3.13) is:

$$y(t)=\sum_{k=0}^{n-1}\left(t^k y^{(k)}(0)E_{p,k+1}(\lambda t^p)\right)+at^{p+1}E_{p,p+2}(\lambda t^p) \qquad (3.14)$$

where $E_{\alpha,\beta}(z)$ is the two-parameter function of Mittag-Leffler type

Proof:

If we take Laplace transform of the problem (3.13), then we get:

$$s^p y(s)-\sum_{k=0}^{n-1}s^{p-k-1}\,y^{(k)}(0)-\lambda y(s)=\frac{a}{s^2}$$

$$y(s)[s^p-\lambda]=\frac{a}{s^2}+\sum_{k=0}^{n-1}s^{p-k-1}\,y^{(k)}(0)$$

$$y(s)=\frac{as^{-2}}{s^p-\lambda}+\frac{\sum_{k=0}^{n-1}s^{p-k-1}\,y^{(k)}(0)}{s^p-\lambda}$$

$$=L\left\{at^{p+1}E_{p,p+2}(\lambda t^p);s\right\}+\sum_{k=0}^{n-1}\ y^{(k)}(0)L\left\{t^K E_{p,K+1}(\lambda t^p);s\right\}$$

So,

$$y(t)=at^{p+1}E_{p,p+2}(\lambda t^p)+\sum_{k=0}^{n-1}\ y^{(k)}(0)t^K E_{p,k+1}(\lambda t^p)$$

Now, we illustrate the relation between the solution of fractional IVP and the classical IVP of integer order through the following example:

Example 3.1:

Concider the initial value problem:

$$D_*^p y(t)-\lambda y(t)=at, \qquad t>0, \qquad 1<p<2 \qquad (3.15)$$

$$y(0)=b_0, \qquad y'(0)=b_1$$

According to (3.14) the solution is:

$$y(t) = y(0)E_{p,1}(\lambda t^p) + ty'(0)E_{p,2}(\lambda t^p) + at^{p+1}E_{p,p+2}(\lambda t^p) \qquad (3.16)$$

$$\lim_{p\to 2} y(t) = y(0)E_{2,1}(\lambda t^2) + ty'(0)E_{2,2}(\lambda t^2) + at^3E_{2,4}(\lambda t^2)$$

$$= y(0)\cosh\left(\sqrt{\lambda}t\right) + \frac{y'(0)}{\sqrt{\lambda}}\sinh\left(\sqrt{\lambda}t\right) + at^3E_{2,4}(\lambda t^2)$$

Now, the question is: what is the relation between the solution of the IVP (3.15) and the solution of the problem:

$$y''(t) - \lambda y(t) = at \qquad (3.17)$$

$$y(0) = b_0, \qquad y'(0) = b_1$$

Of course, we can solve this ordinary differential equation by different methods, one of these is Laplace transform.

$$s^2y(s) - sy(0) - y'(0) - \lambda y(s) = \frac{a}{s^2}$$

$$y(s) = \frac{as^{-2}}{s^2-\lambda} + \frac{sy(0)}{s^2-\lambda} + \frac{y'(0)}{s^2-\lambda}$$

$$y(t) = at^3E_{2,4}(\lambda t^2) + y(0)\cosh\left(\sqrt{\lambda}t\right) + \frac{y'(0)}{\sqrt{\lambda}}\sinh\left(\sqrt{\lambda}t\right)$$

If we compare the solution of formula (3.17) with the solution of formula (3.15) , we see that $\lim_{p\to 2} y(t)$ in fractional case coincides with the solution of (3.17), and this is because of the property of Caputo fractional derivative which is $\lim_{p\to n} D_*^p f(t) = f^{(n)}(t)$

Now, consider the case when $2 < p < 3$ and compare the solutions.

$$D_*^p y(t) - \lambda y(t) = at,\ t > 0,\ \ 2 < p < 3 \tag{3.18}$$

$$y(0) = b_0, \qquad y'(0) = b_1, \quad y''(0) = b_2$$

with the solution of the IVP

$$y''(t) - \lambda y(t) = at + y''(0) \tag{3.19}$$

$$y(0) = b_0, \qquad y'(0) = b_1$$

If we solve (3.18), we get:

$$y(t) = y(0)E_{p,1}(\lambda t^p) + ty'(0)E_{p,2}(\lambda t^p) + at^{p+1}E_{p,p+2}(\lambda t^p) + y''(0)t^2E_{p,3}(\lambda t^p) \tag{3.20}$$

On the other hand, if we solve (3.19) we get:

$$y(t) = y(0)E_{2,1}(\lambda t^2) + ty'(0)E_{2,2}(\lambda t^2) + at^3E_{2,4}(\lambda t^2) + y''(0)t^2E_{2,3}(\lambda t^2) \tag{3.21}$$

Obviously, when we take $\lim_{p\to 2} y(t)$ in equation (3.20), the result coincides with the solution of the problem (3.19) not with the solution of (3.17) and that is because:

$$\lim_{p\to n-1} D_*^p f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$$

Example 3.2: solve the problem:

$$D_*^p y(t) - y(t) = 0, \qquad 0 < p < 1, \qquad y(0) = 1 \tag{3.22}$$

Comparing this problem with formula (3.13), we find they are the same, but here $\lambda = 1, a = 0, n = 1$

According to (3.14), the solution will be:

$$y(t) = \sum_{k=0}^{n-1} \left(t^k y^{(k)}(0) E_{p,k+1}(t^p) \right) = E_{p,1}(t^p) \tag{3.23}$$

Now, $\lim_{p \to 1} y(t) = E_{1,1}(t\ ) = e^t$

Let us compare $\lim_{p \to 1} y(t)$ with the solution of the classical problem:

$y'(t) - y(t) = 0, \quad y(0) = 1$

Trivially, the solution of this problem is $y(t) = e^t = \lim_{p \to 1} y(t)$ in the fractional case.

2. Riemann-Liouville initial value problem:

$$D^p y(t) - \lambda y(t) = at, \qquad t > 0, \qquad n - 1 < p < n$$

$$D^{p-k-1} y(t)|_{t=0} = b_k, k = 0,1,2, \ldots, n-1 \tag{3.24}$$

The solution of problem (3.24) is:

$$y(t) = \sum_{k=0}^{n-1} \left(b_k t^{p-k-1} E_{p,p-k}(\lambda t^p) \right) + a t^{p+1} E_{p,p+2}(\lambda t^p) \tag{3.25}$$

Proof:

Taking Laplace transform for the problem (3.24), we get:

$$s^p y(s) - \lambda y(s) = \frac{a}{s^2} + \sum_{k=0}^{n-1} b_k s^k$$

then:

$$y(s) = \frac{a s^{-2}}{s^p - \lambda} + \sum_{k=0}^{n-1} \frac{b_k s^k}{s^p - \lambda}$$

$$= L\left\{ a t^{p+1} E_{p,p+2}(\lambda t^p); s \right\} + \sum_{k=0}^{n-1} \ b_k L\left\{ t^{p-k-1} E_{p,p-k}(\lambda t^p); s \right\}$$

so,

$$\mathrm{y(t)} = at^{p+1}E_{p,p+2}(\lambda t^p) + \sum_{k=0}^{n-1} b_k t^{p-k-1}E_{p,p-k}(\lambda t^p)$$

Now, the aim is: comparing the solution of the IVP in example 3.3 with problem (3.17):

Example 3.3: Consider the IVP:

Solve the problem:

$$D^p y(t) - \lambda y(t) = at,\ t > 0,\ 1 < p < 2$$

$$D^{p-k-1}y(t)|_{t=0} = b_k, k = 0,1$$

According to (3.25)

$$y(t) = b_0 t^{p-1}E_{p,p}(\lambda t^p) + b_1 t^{p-2}E_{p,p-1}(\lambda t^p) + at^{p+1}E_{p,p+2}(\lambda t^p)$$

If we take

$$\lim_{p\to 2} y(t) = y(0)E_{2,1}(\lambda t^2) + ty'(0)E_{2,2}(\lambda t^2) + at^3E_{2,4}(\lambda t^2)$$

then, we see that $\lim_{p\to 2} y(t)$ coincides with the solution of the problem (3.17)

Example 3.4: Consider the IVP:

$$D^p y(t) - y(t) = 0, \qquad t > 0, \qquad 0 < p < 1$$

$$D^{p-1}y(t)|_{t=0} = 1$$

According to (3.25), the solution is:

$$y(t) = t^{p-1} E_{p,p}(t^p)$$

$$\lim_{p \to 1} y(t) = E_{1,1}(t) = e^t$$

which coincides with the solution of the problem:

$$y'(t) - y(t) = 0, \quad y(0) = 1$$

Chapter Four

Different applications of fractional calculus

During the second half of twentieth century, considerable amounts of researches in fractional calculus was done due to its apparent applications in different fields. For example, it has applications in physics, engineering, signal processing, viscoelasticity, fluid mechanics, biology, electro chemistry, and many others.

Nowadays, there is no a doubt that fractional calculus has become an exciting new mathematical method to solve many problems in diverse fields of our life.

In this chapter, we use fractional calculus to solve problems which was modeled using fractional differential equations, and solve these differential equations using Laplace transform.

We will consider five applications, which are the tautochrone problem, the fractional damped simple harmonic oscillator, the decay-growth problem, modeling speech signals using fractional calculus and viscoelasticity.

We will start with the historical example, namely, the tautochrone problem.

4.1 Tautochrone problem [15], [11]

This historical example was studied in nineteenth century by Abel, and it is one of the famous examples which was solved using fractional calculus. The problem concern on finding a curve in xy-plane such that the time needed for a particle to slide down the curve until reach the lowest point is independent of its initial placement on the curve, assuming no friction.

Let us start by fixing the lowest point of the curve C at the origin, and the position of the curve is in the first quadrant of the plane, and let us denote the initial point by (x^{*}, y^{*}), and any point between (0,0), and (x^{*}, y^{*}) by (Q, Z), and let S be the arch length measured a long C from (0,0) to an arbitrary point (Q, Z) on C, and let β be the angle of inclination, then:

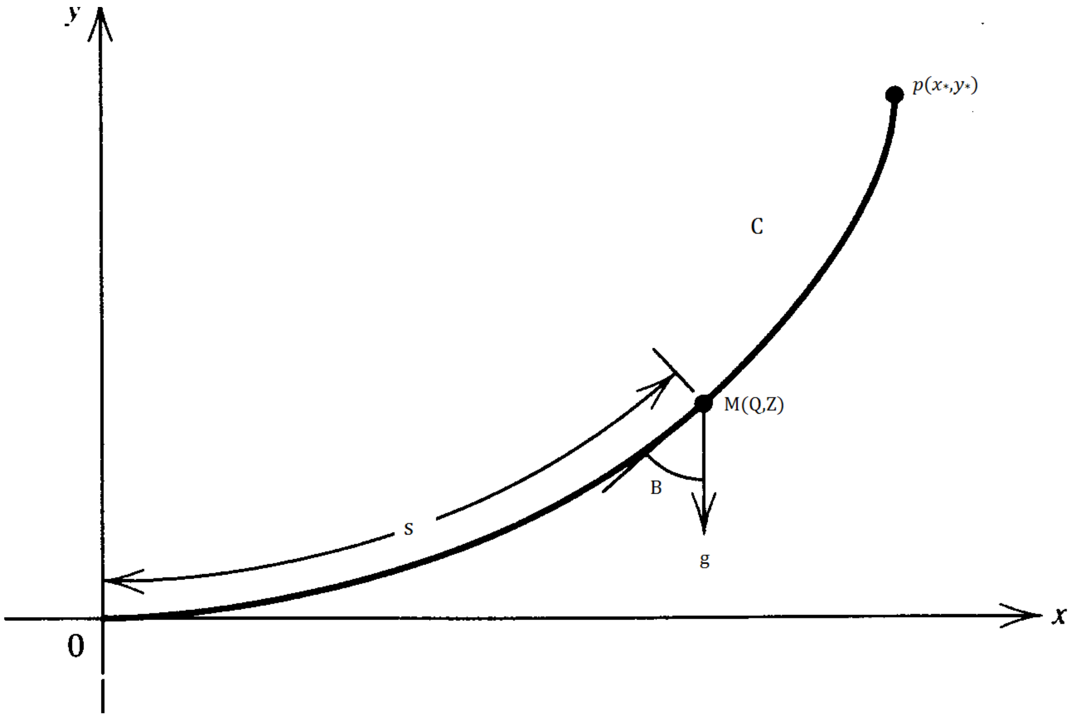

Figure 4.1 [15]

$$\frac{dZ}{dS} = cos\beta \tag{4.1}$$

and

$$\frac{d^2S}{dt^2} = -g\ cos\beta \tag{4.2}$$

where, g is the gravitational acceleration

Now, if we substitute (4.1) in (4.2), then we get:

$$\frac{d^2S}{dt^2} = -g\ cos\beta \tag{4.3}$$

Moreover, if we use the conservation of energy law, we get:

The energy at the initial point $(x^*, y^*$ equals the energy at the point (Q, Z), which means:

$$mgy^* = mgZ + \frac{1}{2}m\left(\frac{dS}{dt}\right)^2 \tag{4.4}$$

Or

$$-\sqrt{2g(y^* - Z)} = \frac{dS}{dt} \tag{4.5}$$

Note that in (4.5) we take the negative square root, because as t increase, S decrease. Now, integration of (4.5) will give us:

$$T = -\frac{1}{\sqrt{2g}}\int_p^o \frac{1}{\sqrt{y^* - Z}}ds \tag{4.6}$$

But, the arch length S is a function of Z, say $S = h(Z)$, and h depends on the shape of the curve C. Therefore,

$$T = -\frac{1}{\sqrt{2g}}\int_{y^*}^{o} (y^* - Z)^{\frac{-1}{2}}\, h'(Z)dZ$$

or

$$\sqrt{2g}T = \int_0^{y^*} (y^* - Z)^{\frac{-1}{2}}\, h'(Z)dZ, \text{where } h'(z) = \frac{dS}{dZ} \tag{4.7}$$

Now, replacing $h'(y^*)$ by $f(y^*)$ and using fractional calculus, we can write (4.7) as:

$$\frac{\sqrt{2g}}{\Gamma(1/2)}T = D^{-1/2}f(y^*) \tag{4.8}$$

where the right-hand side is Riemann-Liouville fractional integral, so we can solve (4.8) by applying the fractional operator $D^{1/2}$ to both sides, then we have:

$$D^{1/2}\sqrt{\frac{2g}{\pi}}T = f(y^*) \tag{4.9}$$

but,

$$D^{1/2}T = \frac{T}{\sqrt{\pi y^*}}$$

so,

$$f(y^*) = \sqrt{\frac{2g}{\pi}} \times \frac{T}{\sqrt{\pi y^*}} = \frac{\sqrt{2g}T\ (y^*)^{1/2}}{\pi} \tag{4.10}$$

Next, we want to solve the second part of the problem, which is finding the equation of C. We will start with:

$$f(y^*) = h'(y^*) = \frac{dS}{dy^*} = \sqrt{1 + \left(\frac{dx^*}{dy^*}\right)^2} \tag{4.11}$$

so,

$$\frac{dx^*}{dy^*} = \sqrt{f^2(y^*) - 1}$$

or

$$x^* = \int_0^{y^*} \sqrt{\frac{2gT^2}{\pi^2 Z} - 1}\ dZ + C \tag{4.12}$$

But, $C = 0$ because at the origin $x^* = y^* = 0$

Let

$$m = \frac{gT^2}{\pi^2}$$

and assume that $Z = 2m\ sin^2 Q,$ then (4.12) will be:

$$x^* = 4m \int_0^{\alpha} cos^2\, Q dQ \tag{4.13}$$

where

$$\alpha = arc\ sin \sqrt{\frac{y^*}{2m}}$$

If we solve (4.13), then we will have:

$$x^* = 2m(\alpha + \frac{1}{2}\sin 2\alpha)$$

$$y^* = 2msin^2\alpha$$

Finally, if we make the last substitution $\theta = 2\alpha$, then we will have:

$$x^* = m(\theta + \sin\theta)\,, y^* = m(1 - cos\theta) \tag{4.14}$$

4.2 The fractional damped simple harmonic oscillator [9]

The fractional value problem of the damped simple harmonic oscillator is obtained from the classical one, but we replace $y'(t)$ in the classical case by Caputo fractional derivative of order p where $0 < p < 1$, and we must specify two initial conditions are needed to achieve unique solution for the problem:

The problem is:

$$y''(t) + bD_*^p y(t) + \omega^2 y(t) = f(t), \qquad 0 < p < 1 \tag{4.15}$$

The solution of this problem is:

$$y(t) = a_0 y_0 - \frac{a_1}{\omega^2} y_0'(t) - \frac{1}{\omega^2}\int_0^t y_0'(t-\tau) f(\tau) d\tau \tag{4.16}$$

where,

$$y_0(t) = L^{-1}\left\{\frac{s + bs^{p-1}}{s^2 + bs^p + \omega^2}; t\right\} \tag{4.17}$$

proof:

Applying the Laplace transform to both sides of (4.15), we get:

$$s^2 Y(s) - \sum_{k=0}^{1} s^{2-k-1} y^{(k)}(0) + b\big(s^p Y(s) - s^{p-1} y(0)\big) + \omega Y(s) = F(s)$$

Now, if we solve the previous equation with respect to $y(s)$, then the result will be:

$$Y(s) = \frac{s + bs^{p-1}}{s^2 + bs^p + \omega^2} y(0) + \frac{y'(0)}{s^2 + bs^p + \omega^2} + \frac{F(s)}{s^2 + bs^p + \omega^2}$$

$$= \frac{s + bs^{p-1}}{s^2 + bs^p + \omega^2} a_0 + \frac{a_1}{s^2 + bs^p + \omega^2} + \frac{F(s)}{s^2 + bs^p + \omega^2} \qquad (4.18)$$

Let:

$$y_0(t) = L^{-1}\left\{\frac{s + bs^{p-1}}{s^2 + bs^p + \omega^2}; t\right\}$$

also, from properties of Laplace transform

$$\lim_{s\to\infty} sY_0(s) = y_0(0) = \lim_{s\to\infty} \frac{s(s + bs^{p-1})}{s^2 + bs^p + \omega^2} = 1 \qquad (4.19)$$

Moreover, $L\{y_0'(t); s\} = sY_0(s) - y_0(0)$

also,

$$\frac{1}{s^2 + bs^p + \omega^2} = \frac{-1}{\omega^2}\left(\frac{s(s + bs^{p-1})}{s^2 + bs^p + \omega^2} - 1\right)$$

$$= \frac{-1}{\omega^2}(sY_0(s) - y_0(0)) = \frac{-1}{\omega^2} L\{y_0'(t); s\} \qquad (4.20)$$

so,

$$L^{-1}\left\{\frac{1}{s^2 + bs^p + \omega^2}; t\right\} = \frac{-1}{\omega^2} y_0'(t) \qquad (4.21)$$

also,

$$\frac{1}{s^2 + bs^p + \omega^2} F(s) = \frac{-1}{\omega^2} L\{y_0'(t); s\} L\{f(t); s\}$$

$$= \frac{-1}{\omega^2} L\{y_0'(t) * f(t); s\} = \frac{-1}{\omega^2} \int_0^t y_0'(t - \tau) f(\tau) d\tau \qquad (4.22)$$

At the end, if we take the inverse Laplace transform to both sides of (4.18), we get:

$$y(t) = L^{-1}\{Y(s);t\}$$

$$= L^{-1}\left\{\frac{s+bs^{p-1}}{s^2+bs^p+\omega^2}a_0 + \frac{a_1}{s^2+bs^p+\omega^2} + \frac{F(s)}{s^2+bs^p+\omega^2}\right\}$$

$$= a_0 y_0(t) - \frac{a_1}{\omega^2}y_0'(t) - \frac{1}{\omega^2}\int_0^t y_0'(t-\tau)f(\tau)d\tau$$

4.3 A decay-growth problem [10]

Radium decays to radon, and radon decays to polonium. Suppose there is a sample which contain pure radium, but when time is passing, radium will decay to radon and the sample will contain radium and radon, but how much radium and radon will the sample contain at time t?

Let $y_1(t)$ be the number of radium atoms at time t, and $y_2(t)$ be the number of radon atoms at time t, let the initial number of radium atoms be $y_1(0) = n$. Let a and b be the decay constants of radium and radon respectively.

then we will have:

$$\frac{d}{dt}y_1(t) = -ay_1(t), \qquad y_1(0) = n \tag{4.23}$$

But the rate which radium is decaying is the rate which radon is creating, and also radon is decaying at the rate $by_2(t)$, so we will have:

$$\frac{d}{dt}y_2(t) = ay_1(t) - by_2(t), \qquad y_2(0) = 0 \tag{4.24}$$

Now, let us solve (4.23). If we take Laplace transform to (4.23), then we will have:

$$sY_1(s) - y_1(0) = -aY_1(s) \qquad (4.25)$$

Solve for $Y_1(s)$, then we will get:

$$Y_1(s) = \frac{n}{s+a}$$

so,

$$y_1(t) = ne^{-at} \qquad (4.26)$$

Now, let us move to solve problem (4.24) using Laplace transform method:

$$sY_2(s) - y_2(0) = \frac{an}{s+a} - bY_2(s) \qquad (4.27)$$

Solving for $Y_2(s)$, then we will get:

$$Y_2(s) = \frac{a\,n}{(s+b)(s+a)} \qquad (4.28)$$

Using partial fractional method, then we can write $Y_2(s)$ as:

$$Y_2(s) = \frac{an}{(a-b)(s+b)} + \frac{an}{(b-a)(s+a)} \qquad (4.29)$$

then,

$$L^{-1}\{Y_2(s);t\} = y_2(t) = \ a\frac{ne^{-bt}}{(a-b)} + a\frac{ne^{-at}}{(b-a)}$$

Now, let us see how this problem can be modeled using fractional derivatives, and what will the solution be when the order of differentiation is $p, 0 < p < 1$

The problem in fractional case is:

$$D^p y_1(t) = -ay_1(t),\ D^{1-p} y_1(0) = n, \qquad 0 < p < 1 \tag{4.30}$$

$$D^p y_2(t) = ay_1(t) - by_2(t),\ D^{1-p} y_2(0) = 0, \qquad 0 < p < 1 \tag{4.31}$$

To solve (4.30), we will use the solution that appear in equation (3.25), so the solution of (4.30) will be:

$$y_1(t) = nt^{p-1} E_{p,p}(-at^p) \tag{4.32}$$

To solve (4.31), we will apply Laplace transform to both sides of (4.31), then we will have:

$$s^p Y_2(s) - D^{p-1} y_2(0) = \frac{an}{s^p + a} - bY_2(s) \tag{4.33}$$

Solve for $Y_2(s)$, then we will have:

$$Y_2(s) = \frac{a\,n}{(s^p + b)(s^p + a)}$$

Using partial fractions, $Y_2(s)$ will be:

$$Y_2(s) = \frac{an}{(a-b)(s^p + b)} + \frac{an}{(b-a)(s^p + a)} \tag{4.34}$$

so,

$$L^{-1}\{Y_2(s); t\} = L^{-1}\left\{\frac{an}{(a-b)(s^p + b)} + \frac{an}{(b-a)(s^p + a)}; t\right\}$$

$$y_2(t) = \frac{an}{b-a} E_{p,p}(-at^p) + \frac{an}{a-b} E_{p,p}(-bt^p) \tag{4.35}$$

Now, what the result will be when we take $\lim_{p\to 1} y_1(t)$ and $\lim_{p\to 1} y_2(t)$ respectively

$\lim_{p\to 1} y_1(t) = ne^{-at}$ which is the solution of (4.23)

$\lim_{p\to 1} y_2(t) = a\frac{ne^{-bt}}{(a-b)} + a\frac{ne^{-at}}{(b-a)}$ which is the solution of (4.24)

Finally, we must notice that these results confirm convergence theorems.

4.4 Modeling of speech signals using fractional calculus [2]

Approach of speech signal which is modeled using fractional calculus is called a novel approach. This approach is contrasted with the integer order approach which is called the celebrated linear procedure coding (LPC) approach.

It is observed through numerical simulations that by using fractional integral as basis function, the speech signal can be modeled accurately.

4.5 Viscoelasticity [19]

Viscoelasticity is very important field which contains many applications of fractional differential and integral operators.

The relation between stress and strain for solids is appear in Hook's law which is:

$$\sigma(t) = e\varepsilon(t) \tag{4.36}$$

and for fluids is appear in Newton law which is:

$$\sigma(t) = \gamma\frac{d\varepsilon}{dt}, E, \gamma \text{ are constants}, \sigma \text{ is the stress}, \varepsilon \text{ is the strain} \tag{4.37}$$

Also, these functions satisfy the fractional differential equations:

$$D^p\sigma(t) = \frac{\Gamma(1-p)}{\Gamma(1-2p)}t^{-p}\sigma(t) \tag{4.38}$$

$$D^p\varepsilon(t) = \Gamma(1+p)t^{-p}\varepsilon(t) \tag{4.39}$$

Conclusion

In this thesis, we study different definitions of fractional derivatives, which is Riemann-Liouville and Caputo fractional derivative. Also, we see that fractional derivatives and integrals are generalization for those of integer order. Moreover, we try to find fractional derivatives and integrals of some elementary functions, and we see that different problems in our life can be modeled using fractional calculus, and the results that we obtain are more accurate than models with integer order derivatives or integrals, so we can try to solve many problems in real life by developing models using fractional derivatives or fractional integrals.

In the end, we hope and predict that researches in this subject will be active and promising since there are different questions which is still without any accurate answer. For example, until now there isn't any formula to find the Caputo fractional derivative $D_*^p t^\alpha$ in the case $n-1 > p > n$, and $-1 < \alpha < n-1$, $\alpha \in \mathbb{R}$. However, this case is included in Riemann-Liouville definition.

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جامعة النجاح الوطنية

كلية الدراسات العليا

طرق تقريب المشتقات الكسرية وتطبيقاتها في حل المعادلات التفاضلية الكسرية

اعداد

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اشراف

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قدمت هذه الاطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات لكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس-فلسطين

2017

ب

طرق تقريب المشتقات الكسرية وتطبيقاتها في حل المعادلات التفاضلية الكسرية

اعداد

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الملخص

ان حساب التفاضل والتكامل الكسري هو فرع من فروع الرياضيات والذي يهتم بايجاد المشتقات والتكاملات الكسرية. لقد كان الميلاد الحقيقي لهذا الفرع في الثلاثين من شهر سبتمبر في عام 1695

على مر السنين، العديد من الرياضيين درسوا هذا الموضوع ولهذا ظهرت اهمية حساب التفاضل والتكامل الكسري في مجالات عدة اهمها الفيزياء والهندسة ولزوجة المواد وغيرها من المجالات.

في هذه الاطروحة سنبدأ بدراسة بعض التعريفات المهمة مثل التكامل الكسري لريمان ليوفلي وايجاد ناتج التكامل الكسري للعديد من الاقترانات المهمة.

في الفصل الثاني تم التركيز على دراسة تعريفات مختلفة للمشتقات الكسرية اهمها مفهوم ريمان ليوفلي ومفهوم كابيتو للمشتقات الكسرية ثم اجراء مقارنة بينهما.

في الفصل الثالث تم دراسة المعادلات التفاضلية التي تحوي مشتقات كسرية من كلا النوعين: سواء ريمان ليوفلي او كابيتو وتم استخدام تحويل لابلاس في حل هذه المعادلات.

في نهاية هذه الاطروحة تم دراسة العديد من التطبيقات لحساب التفاضل والتكامل الكسري في الحياة اليومية.