An-Najah National University

Faculty of Graduate Studies

Singular Perturbation Approximation Method of Large Scale Dynamical System

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This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics, Faculty of Graduate Studies, An-Najah National University, Nablus, Palestine

2021

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Dedication

This thesis is dedicated to my Parents, my wife, my brothers, my sisters, my nephews, nieces, and for all my friends and lovers in the worldwide.

Acknowledgements

First of all, I would like to express my sincere thanks and gratitude to my main supervisor Prof. Dr. Naji Qatanani for introducing me to the beautiful world of dynamical systems and for his continuous help and support, excellent guidance and encouragement. My sincere and deep gratitude goes also to my co-supervisor Dr. Adnan Daraghmah for his everlasting support and guidance, his invaluable advice and care.

The distinguished personality of my supervisors both at personal and professional levels has contributed to finish my work successfully.

Now, I would like to express my deep obligation to my father, my mother, my wife, brothers, sisters and friends in worldwide. Their consistent moral supports have always been the strong source for me during my Master.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

Singular Perturbation Approximation Method of Large Scale Dynamical System

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Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work and has not been submitted elsewhere for any other degree or qualification.

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Singular Perturbation Approximation Method of Large Scale Dynamical System

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Abstract

In this thesis, we will study balanced model reduction methods for linear dynamical systems. These methods are the Balanced Truncation (BT) and the Singular Perturbation (SP) approximation. The two methods will be implemented to approximate the solution of linear time-invariant (LTI) stable dynamical system with zero initial condition. Some illustrative numerical examples to demonstrate the validity and applicability of these methods will be presented. A comparison between these methods will be carried out.

Introduction

Historical Background

The field of control systems has a long history which began with the early desire of humans to take advantage of the materials and forces of nature. However, there was very little in the way of actual progress made in the field of engineering until the beginning of the Industrial Revolution. Leonhard Euler (1707-1783) discovered a powerful integral transform, but Pierre S Laplace (1749–1827) used the transform to solve complex problems in probability theory [11]. Joseph Fourier (1768-1830) created a special function decomposition called the Fourier Series, that was later generalized into an integral transform, and named in his honor (the Fourier Transform) [1]. The "golden age" of control engineering occurred between 1910-1945, where mass communication methods were being created and two world wars led to many developments in control engineering. During this period, some of the most famous prominent names in control engineering: H. Bode [38] and H. Nyquist [25], created the bulk of what we now call "Classical Control Methods". These methods were based on the results of the Laplace and Fourier Transforms. Modern control methods were introduced in the early 1950's, as a way to bypass some of insufficiency of the classical methods. Rudolf Kalman is famous for his work in modern control theory, and an adaptive controller called the Kalman filter was named in his honor [21]. Modern control methods became increasingly popular after 1957 with the invention of computer, and the start of the space program. Computers created the need for digital control methodologies, and the space program required the creation of some "advanced" control techniques, such as "optimal control" [13, 26], "robust control" [12], and "nonlinear control" [31]. These last subjects, and several more, are still active areas of study among research engineers. By the end of the twentieth century, control has become obvious in everywhere from simple household products (temperature regulation in air-conditioners, thermostats in hot water heaters etc.) to more sophisticated systems such as the family car (which has hundreds of control loops) to large scale systems (such as chemical plants, aircraft, and manufacturing processes).

Reduction of Linear Control Systems

Linear large-scale systems arise in many practical applications, for instance, in circuit simulations and in control problems where the underlying physical process is modeled by partial differential equations [5]. The concept of the dynamical system has motivated a huge amount of research work in recent years such as weather prediction data assimilation, air quality simulations , biological system (e.g. :honey comb vibrations), molecular system, vibration /acoustic system , chemical vapor deposition reactors, microelectromechanical system and optimal cooling, etc.[2].

Originally, model order reduction was developed in the area of systems and control theory, which studies properties of dynamical systems in application for reducing their complexity, while preserving their input-output behavior as much as possible [6].

Balanced Truncation Method

The fundamental methods in the area of model order reduction were published in the eighties and nineties of the last century. In 1981 Moore [23] published the method of truncated balanced realization. Glover [15] in 1984 published his famous paper on the Hankel-norm transform. In 1987 the proper orthogonal decomposition method was proposed by Sirovich [34]. All these methods were developed in the field of systems and control theory. In 1990 the first method related to Krylov subspaces was established in asymptotic waveform evaluation. Then, in 1993, Freund and Feldmann proposed Pade Via Lanczos and showed the relation between the Pade approximation and Krylov spaces [14].

A variety of methods have been used to solve the linear time-invariant systems, these are: Hankel norm approximation, Balanced truncation, and singular perturbation approximation [27-30].

Model order reduction has its roots in the area of systems and control theory. Within this area, methods have been developed which differ considerably from the Krylov methods [39].

Balanced truncation is a model order reduction technique from robust control theory, that is symmetric method for producing simple approximate models of complex linear system with many input or many output terminals.

Applying balanced truncation method makes it necessary to balance the system, which is equivalent to finding the controllability and observability Gramian of the system in a special diagonal form [9, 40]. The Cholesky factors of these Gramians are efficiently computable as solutions of dual Lyapunov equations for systems with only few inputs and outputs [16-20].

Singular Perturbation Method

The singular perturbation approximation technique for model reduction is related to the direct truncation technique when the system model to be reduced is stable, minimal, and internally balanced [22, 24].

The balancing transformation is such transformation of the state space vector that makes both the controllability and the observability gramians become identical and diagonal [35-37]. Assuming that the original system is controllable and observable, a balanced system will also be controllable and observable as the balancing transformation preserves controllability and observability of the system [32, 33].

In this work we consider the linear time-invariant system

$$\dot{x} = Ax + Bu\ ,$$

$$y = Cx + Du\ , \qquad (1)$$

where x is a state vector, u is an input vector, y is an output vector, A is a dynamic matrix with dimension $n\times n$, B is an input matrix with dimension $n\times r$, C is an output matrix with dimension $m\times n$, and D is a matrix with dimension $m\times r$. For simplicity, the initial condition is assumed to be zero, i.e., $x(0) = 0$. Furthermore, we restrict our attention to stable system, that is, for all eigenvalues of the system λj, we have that $\mathrm{Re}(\lambda j) \leq 0$.

Linear large-scale systems arise in many practical applications [3, 4], for instance, in circuit simulations and in control problems where the underlying physical process is modeled by partial differential equations. For these problems we are interested in constructing reduced-order system

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u,$$

$$y = \hat{L}^T \hat{x}, \qquad (2)$$

where $\hat{A} = W^T A V \in R^{r\times r}$, $\hat{B} = W^T B \in R^{r\times n}$, $\hat{L} = V^T L \in R^{r\times q}$, and $r \ll n$, with good approximation properties for equation (1).

In this thesis, we will study balanced model reduction methods for linear dynamical systems. These methods are the Balanced Truncation (BT) and the Singular Perturbation (SP) approximation. The two methods will be implemented to approximate the solution of linear time-invariant (LTI) stable dynamical system with zero initial condition. Some illustrative numerical examples to demonstrate the validity and applicability of these methods will be presented. A comparison between these methods will be carried out.

This thesis is organized as follows :

In chapter 1, we present some important definitions and theorems related to stable linear dynamical system. The proposed Balanced Truncation Method (BTM) to reduce the dimension of a linear control system together with its associated error bound is presented in chapter 2. Chapter 3 addresses the error bound associated with the Singular Perturbation Approximation (SPA). Finally, some numerical examples to illustrate the BTM and SPA are presented in chapter 4.

Chapter One

Preliminaries

Before we embark on the study of Balanced Truncation method, there are number of introductory ideas and theories need to be discussed. We present the linear time invariant (LTI) system for the dynamical system, then we introduce the basic concepts concerned with the (LTI) system involves Stability, Controllability and Observability.

1.1 linear time invariant system

For a system to be considered as LTI system it must satisfy two properties, linearity and time invariance.

A system Y that maps an input u (t) to an output c (t) is a linear system if and only if

$$\alpha_1 c_1(t) + \alpha_2 c_2(t) = Y[\alpha_1 u_1(t) + \alpha_2 u_2(t)] \tag{1.1}$$

where α is a constant.

A system Y that maps an input u (t) to an output c (t) is a time – invariant system if and only if

$$c\ (t - to) = Y[u(t - to)] \tag{1.2}$$

In the other words, the system is a time invariant if and only if the system coefficient do not depend on time [25].

To describe the finite dimension linear time invariant dynamical system (the dimension of the system is equal to n) we introduce these two equations with constant coefficients

$$\dot{x} = Ax + Bu \tag{1.3}$$

$$y = Cx + Du \tag{1.4}$$

Equation (1.3) is the state space equation, where

$$\dot{x} = \frac{dx}{dt}$$

is the derivative with respect to time t.

We call $x(t) = [x_1(t), x_2(t), x_3(t), \dots, x_n(t)]^T \in \mathbb{R}^n$

the state vector of the system.

Let

$$u = u(t) \in \mathbb{R}^m$$

be the input function of the dynamical system.

We note by

$$x(t_0) = x_0$$

the initial condition of the system.

A and B are constant matrices such that $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. A is called the state matrix while B is called the input matrix. A state vector of the system gives the relationship between the input and the state variables.

The second equation (1.4) is the output equation for a linear dynamical system (2) where y is a column vector of the output variables, and represents the response of the system. C and D are constant matrices with dimensions $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. C is called the output map and describes the interacts between the system and the outside world. D is the feed through matrix that describe the weight of the system input. Under the assumption that there is no feed through matrix, i.e. D=0. In the case when $m = p = 1$ the system with single input and single output is called a SISO system. While the system with two input terminals or more and output terminals or more is called multi-input, multi-output (MIMO) [39].

Definition 1.1 [2]:

A linear system in internal or state space description is a quadruple of linear maps (matrices)

$$\Sigma_q = \left(\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \end{array}\right) \tag{1.5}$$

The dimension of the system is defined as the dimension of related state space, that is:

$$Dim(\Sigma_q) = n \tag{1.6}$$

When $D = 0$, the system denoted by:

$$\Sigma_q = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right) \tag{1.7}$$

1.2 Stability of continuous time system

The stability of the linear dynamical invariant (LDI) system is described by the eigenvalues of the state matrix.

In this work, the stable system will be our primary concern.

Definition 1.2 [2]:

A matrix M is called stable if all eigenvalues of M have strictly negative real parts.

The system $\sum$ is bounded input, bounded output (BIBO) stable if any bounded input results in bounded output.

Definition 1.3 [10]:

The system

$$\Sigma_q = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right) \tag{1.8}$$

is called asymptotically stable if $Re\{\lambda_i(A)\} < 0$, and it is called stable if $Re\{\lambda_i(A)\} \leq 0$ such that $Re\{\lambda_i(A)\}$ denote the real part of eigenvalues of matrix A. otherwise the system is called unstable system.

There are many methods for determining stability of the system, two of these common classical methods are: the matrix equation approach and the characteristic polynomial approach [10, 12].

1.3 Controllability and Observability

In this section we will consider two of the basic properties related to continuous LTI dynamical system. These are controllability and observability.

The first property deals with a state-equation that is controllable from the input, whereas the second property deals with the initial state that is observable from the output.

Consider the state equation with dimension n and input p:

$$\dot{x} = Ax + Bu \tag{1.9}$$

where A and B are constant matrices with dimensions $n \times n$ and $n \times p$, respectively.

Since the input affects on controllability, so we can ignore the output equation.

Definition 1.4 [10, 40]:

The state equation (1.9) or the pair (A, B) is said to be controllable, if for any initial state $x(0) = x_0$ and the final state x_1, there exist a continuous input $u(t)$ which transfers x_0 to x_1 in a finite time, such that $x(t_1) = x_1$ where $t_1 > 0$. Otherwise (1.9) or the pair (A, B) is said to be uncontrollable.

Definition 1.5 [12, 40]:

The n dimenstional pair (A, B) is controllable if and only if the matrix

$$C(A, B) = (B \quad AB \quad A^2B \quad A^3B \ldots \quad A^{n-1}B) \tag{1.10}$$

has rank n, where n is a positive integer, or the $n \times n$ matrix CC^T is nonsingular.

The observability and controllability are dual concepts, the first one tests the possibly of surveillance the state from the input, while the second tests the possibly of estimating the state from the output.

Definition 1.6 [2, 40]:

The dynamical system

$$\Sigma_q = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

or the pair (C, A) is said to be observable if, for any $t_1 > 0$, the initial state $x(0)$ can be uniquely specified from the time history of the input $u(t)$ and the output $y(t)$ for all t in the interval $[0, t_1]$.

Definition 1.7 [12]:

The matrix

$$O(C,A) = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{n-1} \end{pmatrix} \quad (1.11)$$

is called the observability matrix if and only if it has rank n, where n is a positive interger.

Now, we will define two primary matrices for linear dynamical systems; these are controllability and the observability Gramains. In addition we will discuss some theorems related to these matrices.

Definition 1.8 [26]:

The matrix

$$W_c = \int_0^\infty e^{At} B B^T e^{A^T t} dt \quad (1.12)$$

is called the controllability Gramian while the matrix

$$W_0 = \int_0^\infty e^{A^T t} C^T C e^{At} dt \quad (1.13)$$

is called the observability Gramian.

This controllability Gramian's property holds for continuous time dynamical system.

$$W_c(t) = W_c^T(t) \geq 0, \forall t > 0 \quad (1.14)$$

Theorem 1.9 (Controllability Conditions) [40]:

The following statements are equivalent:

1. The pair $(A, B), A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, is controllable.

2. The controllability Gramian W_c is positive definite, $W_c(t) > 0$, for some $t > 0$.

3. The row rank of controllable matrix is full (i:e rank $C(A, B) = n$).

Theorem 1.10 (Observability Conditions) [2]:

These statements are equivalent.

1. The pair $(C, A), C \in \mathbb{R}^{p\times n}, A \in \mathbb{R}^{n\times n}$, is observable.

2. The rank of observable matrix is full (i:e rank $O(C, A) = n$).

3. The observablility Gramian W_0 is positive definite, $W_0(t) > 0$, for some $t > 0$.

Theorem 1.11 (Theorem of duality) [13]:

The pair (A, B) is controllable if and only if the pair (A^T, B^T) is observable.

Proof. The pair (A, B) is controllable if and only if the controllability Gramian W_c is nonsingular for any t. On the other hand the pair (A^T, B^T) is observable if and only if it satisfies the observability Gramian W_0, $W_0 = \int_0^\infty e^{A^T t} B^T B e^{At} dt$, by replacing A by A^T and B by B^T , W_0 becomes:

$$W_0 = \int_0^\infty e^{At} B B^T e^{A^T t} dt \tag{1.15}$$

is nonsingular for any t, the two conditions are thus identical.

The duality between controllability and observability means that we can test the observability of a pair (A, C)) by using the controllability tests that we already know on the pair (A^T, C^T).

1.4 The Laplace transform

In this section we will discuss the importance and applications of Laplace transform to linear time – invariant dynamical systems.

Definition 1.12 [12]:

Let $g(x)$ be a real-valued function defined for $x \geq 0$. Suppose that $g(x)$is multiplied by e^{-sx}, and then the result is integrated with respect to x from 0 to ∞. If the intergral converges, then it is a function of s. That is

$$G(s) = \int_0^\infty g(x)e^{-sx}dx \tag{1.16}$$

This is called the Laplace transform of $g(x)$ and written as:

$$G(s) = L[g(x)] = \int_0^\infty g(x)e^{-sx}dx = \lim_{A\to\infty} \int_0^A g\,(x)e^{-sx}dx \tag{1.17}$$

where $s = \sigma + i\omega, \sigma$ and ω are real variables.

The inverse Laplace transformation of a function $G(s)$ is the unique function $g(x)$ that is continuous on $[0, \infty)$ and satisfies:

$$L^{-1}[G(s)] = g(x) \tag{1.18}$$

Now we will list some important properties of the Laplace transformation:

Let α and β be constants and $G(s) = L[g(x)],\ F(s) = L[f(x)]$ then

1. $L[\alpha g(x) + \beta f(x)] = \alpha G(s) + \beta F(s)$. (Linearity property)

2. $L[e^{\alpha x} g(x)] = G(s - \alpha)$. (shifting property)

3. Let $g'(x)$ be the first derivative of $g(x)$, then $L[g'(x)] = sG(s) - g(0)$

Note that: this property (transform of derivatives) can be generalized to the n^{th}derivative:

$$L[g^n(x)] = s^n G(s) - s^{n-1} g'(0) \dots - sg^{(n-2)}(0) - g^{(n-1)}(0)$$

4. $L\left[\int_0^t g(r)dr\right] = \frac{G(s)}{s}$. (Transform of intergrals)

5.$L[x^n g(x)] = (-1)^n \frac{\partial^n}{\partial s^n} G(s), n = 1,2, \dots$

6. (Convolution)

$$L[g(x) \star \gamma(x)] = L[g]L[\gamma(x)]$$
$$= G(s)\Gamma(s)$$

where the convolution operation is defined as:

$$(g \star \gamma)(t) = \int_0^t g(\tau)\gamma(t - \tau)dr$$
$$= \int_0^t g(t - \tau)\gamma(\tau)dr$$

1.5 Lyapunov equations

We present in this section a set of important equations emerges in many branches of control theory, such as stability analysis and optimal control.

Definition 1.13 [**2, 18**]:

The matrix equations

$$MA + A^T M = -F \tag{1.19}$$

and

$$AM + MA^T = -F \tag{1.20}$$

are called the Lyapunov equations.

where

$$F \in \mathbb{R}^{n\times n}$$

Theorem 1.14 [2, 10] (Lyapunov stability theorem) :

The system

$$\Sigma_q = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

is asymptotically stable if and only if for any symmetric positive definite matrix F, there exists a unique symmetric positive definite matrix M satisfying the equations:

$$MA + A^T M = -F$$

and

$$AM + MA^T = -F$$

Proof $(\Rightarrow)$ lets define the matrix M by:

$$M = \int_0^\infty e^{A^T t} F e^{At} dt$$

Next, we show that if the system is asymptotic stable then M is symmetric positive definite and a unique solution of equation (1.19). Using M in equation (1.19) we have

$$MA + A^T M = \int_0^\infty e^{A^T t} \mathrm{F} e^{At} A\, dt + \int_0^\infty A^T e^{A^T t}\, \mathrm{F} e^{At} dt$$

$$= \int_0^\infty \frac{d}{dt}\left(e^{A^T t}\mathrm{F} e^{At}\right) dt = \left[e^{A^T t}\mathrm{F} e^{At}\right]_0^\infty$$

Since A is stable, then $e^{A^T t} \to 0$ when $t \to \infty$. Thus $MA + A^T M = -F$ so, M satisfies equation (1.19).

We can say that M is positive definite after proving that $u^T M u > 0$ for any nonzero vector u

$$u^T M u = \int_0^\infty u^T e^{A^T t} F e^{At} u dt$$

Since $e^{A^T t}$ and e^{At} are both nonsingular and F is positive definite, we conclude that $u^t M u > 0$.

Finally we must show that, M is unique. Suppose that M_1 and M_2 are solutions for equation (1.20) then:

$$A^T(M_1 - M_2) + (M_1 - M_2)A = 0$$

which implies that

$$e^{A^T t}\left(A^T\left(M_1 \; - M_2 \;\right) + \left(M_1 \; - M_2 \;\right)A\right)e^{At} = 0$$

or

$$\frac{d}{dt}\left[e^{A^T \, t}\left(M_1 \; - M_2 \;\right)e^{At}\right] = 0$$

and hence $e^{A^T \, t}\left(M_1 \; - M_2 \;\right)e^{At}$ is a constant matrix for all t. At $t = 0$ and $t = \infty$, calculated value of $\left(M_1 \; - M_2 \;\right) = 0$, so M is unique solution.

(⇐) Conversely, if M is symmetric positive definite solution of (1.19), then A is stable.

Let (λ, x) be an eigenpair of A. then multiply (1.19) by x^* from left and by x from right, we got this:

$$x^* \, MAx + x^* \, A^T \, Mx = \lambda x^* \, Mx + \bar{\lambda} x^* \, Mx = \left(\lambda + \bar{\lambda}\right)x^* \, Mx = -x^* \, \mathrm{F}x$$

Since F and M are both symmetric positive definite, we have

$\lambda + \bar{\lambda} < 0$ or $Re(\lambda) < 0$

Because λ was arbitrary, then A is stable.

The two matrices W_c and W_0 are both solutions of the Lyapunov equations, so we have:

$$AW_c + W_c A^T + BB^T = 0 \quad (1.21)$$

$$W_0 A + A^T W_0 + C^T C = 0 \quad (1.22)$$

Proposition 1.15 [40]:

Let

$$\Sigma_q = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

be a stable continuous time system and let W_c and W_0 be the controllability and observability Gramians of Σ_q , then W_c and W_0 satisfy the continuous time Lyapunov equations:

$$AW_c + W_c A^T + BB^T = 0$$

$$W_0 A + A^T W_0 + C^T C = 0$$

Proof. Suppose Σ_q is stable, then

$$AW_c + W_c A^T = \int_0^\infty \left(Ae^{At}BB^T e^{A^T t} + e^{At}BB^T e^{A^T t}A^T\right)dt$$

$$= \int_0^\infty \frac{d}{dt}\left(e^{At}BB^T e^{A^T t}\right)dt$$

$$= \left[e^{A}BB^T e^{A^T t}\right]_0^\infty$$

$$= BB^T$$

$AW_c + W_c AR^T + BB^T = 0$

$$W_0 \; A + A^T \, W_0 \; = \int_0^\infty \left(e^{A^T\,t} C^T C e^{At} A + A^T \, e^{A^T\,t} C^T C e^{At} \right) dt$$

$$= \int_0^\infty \frac{d}{dt} \left(e^{A^T\,t} C^T C e^{AT} \right) dt$$

$$= \left[e^{A^T\,t} C^T C e^{At} \right]_0^\infty$$

$$= -C^T C$$

$$W_0 \; A + A^T \, W_0 \; + C^T C = 0$$

Let A be a stable matrix and let F be symmetric, positive definite or semidefinite then:

1. The unique solution M of Lyampunov equation:

$$MA + A^T \, M = -F$$

is given by:

$$M = \int_0^\infty e^{A^T\,t} F e^{At} dt \tag{1.23}$$

2. The unique solution M of Lyapunov equation:

$$AM + MA^T \; = -F$$

is given by:

$$M = \int_0^\infty e^{At} F e^{A^T\,t} dt \tag{1.24}$$

Chapter Two

Model Order Reduction Using Balanced Truncation

Model order reduction is one of the most important methods to obtain low order controller. A central concept in system theory with application to model reduction is that of balanced representation of a system Z.

2.1 The energy of controlling and observing state

One of the important properties of dynamical system will be discussed in this section, namely, construction of model reduction and its application to states in terms of controllability and observablility.

The controllability function basically measures the minimal amount of input $u(t)$ energy desired to approach a specific x_0 from the zero $(t = -\infty)$.

Definition 2.1[6, 40]:

The controllability function is defined as

$$L_c\left(x_0\right) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 \, dt \tag{2.1}$$

subject to the stable, controllable and observable dynamical system

$$\Sigma_q = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right), x(0) = x_0$$

The observability function measures the maximum output energy generated when the initial state of the system is x_0 and the control input $u(t)$ is equal zero.

Definition 2.2[6, 31]:

The observability function is defined as

$$L_0\left(x_0\right) = \frac{1}{2}\int_0^{\infty} \|y(t)\|^2 \, dt \qquad (2.2)$$

such that $x(0) = x_0 \,, u(t) = 0, 0 \leq t < \infty$

We define the controllability and observability Gramains respectively by:

$$W_c = \int_0^{\infty} e^{At} B B^T e^{A^T t} dt$$

and

$$W_0 = \int_0^{\infty} e^{A^T t} C^T e^{At} dt$$

and we prove that unique positive definite solutions of Lyapunov equations, this make us redefine the controllability and observability functions in term of W_c and W_0 .

Theorem 2.3[40]:

Let W_c and W_0 are controllability and observability Gramains respectively. Moreover, W_c and W_0 are unique positive definite solutions of the Lyapunov equations. Then we define the controllability and observability functions as:

$$L_c\left(x_0\right) = \frac{1}{2} x_0^T W_C^{-1} x_0 \qquad (2.3)$$

and

$$L_0\left(x_0\right) = \frac{1}{2} x_0^T W_0 \; x_0 \qquad (2.4)$$

2.2 Realization of Transfer function

In this section, we will study the concept of transfer function of dynamical linear system, and the state-space realizations of transfer functions $Q(s)$ for continuous time-invariant systems.

Let

$$\Sigma_q = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right) \tag{2.5}$$

be a linear continuous time dynamical system. Use Laplace transformation for the state and output equations of the system (1.3) and (1.4), we obtain:

$$L[\dot{x}] = L[Ax] + L[Bu]$$

$$sX(s) - x(0) = AX\ (s) + BU\ (s)$$

$$sX(s) - AX(s) = BU(s) + x(0)$$

$$(sI - A)X(s) = BU(s) + x(0)$$

$$X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0) \tag{2.6}$$

The matrix $(sI - A)^{-1}$ is called a function matrix or transition matrix.

$$L[y] = L[Cx] + L[Du]$$

$$Y(s) = CX(s) + DU(s) \tag{2.7}$$

In our study $\mathrm{D} = 0$, so equation (2.7) becomes:

$$Y(s) = CX(s) \tag{2.8}$$

From equations (2.6) and (2.8),we get

$$Y(s) = C(sI - A)^{-1}BU(s) + C(sI - A)^{-1}x(0) \tag{2.9}$$

In our case, the initial condition is zero, that means $x(0) = 0$, then equation (2.9) becomes:

$$Y(s) = C(sI - A)^{-1}BU(s) \tag{2.10}$$

Definition 2.4 [2, 40]:

Let $(A, B, C, 0)$ be a state space realization then the function matrix or transition matrix $Q(s)$ from u to y with zero initial condition is presented by:

$$Q(s) = C(sI - A)^{-1}B \tag{2.11}$$

thus equation (2.10) becomes:

$$Y(s) = Q(s)U(s) \tag{2.12}$$

$Q(s)$ can be defined as:

$$Q(s) = \frac{Y(s)}{U(s)} \tag{2.13}$$

Definition 2.5 [12]:

A state space model (A, B, C) is a realization of a transfer function $Q(s)$ if

$$Q(s) = C(sI - A)^{-1}B \tag{2.14}$$

A transfer function $Q(s)$ is called realizable if there exists a state space model (A, B, C) with transfer function $Q(s)$.

Note: A rational transfer function matrix is proper if it is bounded at infinity.

Lemma 2.6 [12, 28]:

If $Q(s)$ is a realizable transfer function then $Q(s)$ is a real-rational and proper.

Definition 2.7 [2]:

A state space realization (A, B, C) of $Q(s)$ is said to be minimal realization of $Q(s)$ if A has no other realization of $Q(s)$ of smaller dimension.

The next theorem gives us a characterization of the minimal realization:

Theorem 2.8 [2, 12]:

The following statements are equivalent:

1. A state space realization $(A, B, C, 0)$ of $Q(s)$ is minimal.

2. (A, B) is controllable, and (C, A) is observable.

The property of minimal realization is verified by the following theorem:

Theorem 2.9 [40]:

Let $(\mathrm{k}_1, \mathrm{L}_1, \mathrm{N}_1, 0)$ and $(\mathrm{K}_2, \mathrm{L}_2, \mathrm{N}_2, 0)$ are two minimal realizations of a real-rational transfer function $Q(s)$, and let $\mathrm{C}_1, \mathrm{C}_2, \mathrm{O}_1$, and O_2 be the corresponding controllability and observability matrices respectively, then there exists a unique non-singular T such that:

$$K_2 = TK_1 T^{-1}, L_2 = TL_1, N_2 = N_1 T^{-1}$$

Furthermore, T can be shown as:

$$T = \left(O_2^T O_2\right)^{-1} O_2^T O_1, or\ T^{-1} = C_1 C_2^T \left(C_2 C_2^T\right)^{-1}$$

2.3 Balanced truncation method

In this section, we will focus on one of the most important model reduction schema that is well established in theory and most commonly used is the so-called balanced truncation. It was first introduced by Mullis and Roberts (1976) and later in systems and control literature by Moore (1981). Consider the (LTI) continuous system:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

with $x(0) = x_0$

The Balanced Truncation method depends on the controllability and observability gramians W_c and W_0 which are symmetric positive definite solutions of the Lyapunov equations[6], that is,

$$AW_C + W_c A^T + BB^T = 0$$

$$A^T W_0 + W_0 A + C^T C = 0$$

To get a reduced order model, we balance the system then we delete the states that are difficult to control (i.e need large amount of control energy) and difficult to observe (i.e yield small amount of energy). These states are unimportant and have no effect on the transfer function [17, 30].

Now we introduce the Hankel singular Values (HSVs) of the dynamical system.

Definition 2.10 [2]:

Let $\Sigma_q = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$ be an n-dimensional controllable, observable and stable continuous time system, the Hankel Singular Values (HSVs)

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n \geq 0$$

of Σ_q defined as the square roots of the eigenvalues of the product W_c and W_0 and denoted by

$$\sigma_i \left(\Sigma_q\right) = \sqrt{\lambda_i \left(W_c W_0\right)} \qquad (2.15)$$

The diagonal matrix of the (HSVs) is denoted by:

$$\Sigma_q = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \qquad (2.16)$$

Where

$$\sigma_1 \ \geq \sigma_2 \ \geq \sigma_3 \ \geq \cdots \geq \sigma_n \ \geq 0$$

Definition 2.11 [2, 40]:

The controllable, observable and stable system

$\Sigma_q = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$ is called balanced if:

$$W_c = W_0 = \Sigma = diag(\sigma_1 \ \geq \sigma_2 \ \geq \sigma_3 \ \ldots \geq \sigma_n \ )$$

In the following theorem we show the technique of balancing which used to find a coordinate transformation Υ such that

$$\bar{x} = \Upsilon^{-1} x \qquad (2.17)$$

in which the controllability and observability Gramians become diagonal and equal.

Theorem 2.12[12]:

There exists a state space transformation $\bar{x} = \Upsilon^{-1} x$ for the system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

such that the transformed system

$$\dot{x} = \bar{A}\bar{x} + \bar{B}u$$

$$\bar{y} = \bar{C}\bar{x} \qquad (2.18)$$

is balanced and $\bar{A} = \Upsilon A \Upsilon^{-1}$, $\bar{B} = \Upsilon B$ and $\bar{C} = C\Upsilon^{-1}$.

For more details see [12].

If we let $\bar{Q}$ be the transfer function of the transformed system (2.1), then:

$$\bar{Q} = \left(\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \end{array}\right) = \left(\begin{array}{c|c} \Upsilon A \Upsilon^{-1} & \Upsilon B \\ \hline C\Upsilon^{-1} & \end{array}\right) \qquad (2.19)$$

Let $\bar{W}_c$ and $\bar{W}_0$ be the controllability and observability Gramians of the balance system (2.18) we have:

$$\bar{W}_c \ = \mathrm{Y}^{-1} W_c \ \mathrm{Y}^{-1^T} \tag{2.20}$$

and

$$\bar{W}_0 \ = \mathrm{Y}^T W_0 \ \mathrm{Y} \tag{2.21}$$

Since the two Gramians are equal, then:**.**

$$\bar{W}_c \ = \bar{W}_0 \ = \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \tag{2.22}$$

such that

$$\sigma_1 \ \geq \sigma_2 \ \geq \sigma_3 \ \dots \ \geq \sigma_n$$

The controllability and observability Gramians $\left(\bar{W}_c \ \ and \ \bar{W}_0 \ \right)$ in equation (2.22) are solutions of Lyapunov equations:

$$\bar{A}\Sigma + \Sigma\bar{A}^T \ + \bar{B}\bar{B}^T \ = 0$$

$$\bar{A}^T \ \Sigma + \Sigma\bar{A} + \bar{C}^T \ \bar{C} = 0$$

Lemma 2.13 (Balancing transformation) [2]:

Given the controllable, observable and stable system $\left(\begin{array}{c|c} A & B \\ C & \end{array}\right)$ with their corresponding Gramians W_c and W_0 , let U and L are the upper and the lower triangle matrices, then a balancing transformation is given as:

$$S = UY\Sigma^{-\frac{1}{2}} \tag{2.23}$$

$$S^{-1} = \Sigma^{\frac{-1}{2}} X^T L^{-1} \tag{2.24}$$

Where $\sum$ is the diagonal matrix, X and Y is the orthogonal matrices.

Definition 2.14 [40]:

The controllability and observability funtions of the transformed system (2.18) are defined as:

$$\bar{L}_c\left(\bar{x}_0\right) = \frac{1}{2}\bar{x}_0^T \Sigma^{-1} \bar{x}_0 \tag{2.25}$$

$$\bar{L}_0\left(\bar{x}_0\right)=\frac{1}{2}\bar{x}_0^T\Sigma\bar{x}_0 \qquad (2.26)$$

If $\sigma_i \gg \sigma_{i+1}$ for $i=1,2,\ldots,n$ then the amount of control energy to reach the state $\bar{x}$ is large for small values of σ_i , and the output energy at $\bar{x}$ is small for large values of σ_i .

To decrease the number of state components of the system, we omit the state components from x_{i+1} to x_n for $\sigma_i \gg \sigma_{i+1}$ [40].

Now, by the following procedure we can obtain balanced realization for $Q=\left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$ when it is a minimal realization :

1. Compute the controllability and observability Gramians $\left(W_c \ \ and\ W_0\right)$ for the system.

2. Find a matrix U such that $W_c = U^T U$.

3. Diagonalize $U^T W_0 \ U$ to obtain

$$W_c = U^T W_0 \ U = L\Sigma^2 L^T$$

4. Let

$$\mathrm{S}^{-1} = U^T L\Sigma^{-\frac{1}{2}} \mathrm{S} W_c \ \mathrm{S}^T$$

$$= \mathrm{S}^{-1^T} W_0 \ \mathrm{S}^{-1}$$

$$= \Sigma$$

and

$$\left(\begin{array}{c|c} SAS^{-1} & SB \\ \hline CS^{-1} & \end{array}\right)$$

is balanced.

2.4 Error bounds for Linear Dynamical System using balance truncation

To investigate the priori error bounds for linear dynamical system using balance truncation, we reconsider the (LTI) continuous homogeneous system:

$$\dot{x} = Ax + Bu$$
$$y = Cx \qquad (2.27)$$
$$x(0) = x_0$$

Also the transfer function:

$$Q(s) = C(sI - A)^{-1}B$$

Assumption 2.15 [35]:

The system (2.27) is asymptotically stable and minimal realization of $Q(s)$, consequently, (A, B) is controllable and (C, A) is observable.

The controllability and obsrevability Gramians W_c , W_0 are positive semi-definite and solutions of the Lyapunov equations.

By theorem (2.3.3) we obtain the next balanced system:

$$\dot{x} = \bar{A}\bar{x} + \bar{B}u$$
$$\bar{y} = \bar{C}\bar{x} \qquad (2.28)$$

such that $\bar{A} = \Upsilon A \Upsilon^{-1}, \bar{B} = \Upsilon$ and $\bar{C} = C\Upsilon^{-1}$.

We will partition the balanced system (A, B, C) and the Gramian Σ as:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$$

where A_{11} and A_{22} are matrices of the dimension $r \times r$ and $n - r \times n - r$ respectively, and the other matrices have dimension corresponds to the original system.

Suppose that the corresponding Hankel Singular Values satisfy

$\sigma_r > \sigma_{r+1}$, then the reduced order model obtained from the Balance Truncation method is given by:

$$\dot{x}_r = A_{11}x_r + B_1 u \qquad (2.29)$$

$$y_r = C_1 x_r$$

and the transfer function of reduced system is:

$$Q_r(s) = C_1 \left(sI - A_{11}\right)^{-1} B_1 \qquad (2.30)$$

The subsystem $\left(A_{11}, B_1, C_1\right)$ is a good approximation of the balanced system $(\bar{A}, \bar{B}, \bar{C})$.

Lemma 2.16 [22, 40]:

The subsystems $\left(A_{ii}, B_i, C_i\right)(i = 1,2)$ are internally balanced with Gramian Σ_1 and Σ_2 .

Lemma 2.17 [22, 40]:

The matrix A_{ii} $(i = 1,2)$ is asymptotically stable $\begin{pmatrix} i.e\ Re\left(\lambda_k\left(A_{ii}\right) < 0\right) \\ i = 1,2, \forall k \end{pmatrix}$

if Σ_1 and Σ_2 have no diagonal entries in common. Furthermore, the system $\left(A_{11}, B_1, C_1\right)$ is controllable and observable.

Now, we discuss a very important notion in control theory, namely, we compute the H_∞ norm of the transfer function of the model. Moreover we compare the difference with the norm of the transfer function of our reduced order model using the balanced truncation.

Definition 2.18 [7-9]:

The H_∞ norm is defined as:

$$\|Q(j\omega)\|_\infty = \underset{\omega \in \mathbb{R}}{sup} \sigma\{Q(j\omega)\} \tag{2.31}$$

Let $Q(s)$ be the transfer function of the balanced system $(\bar{A}, \bar{B}, \bar{C})$ and $Q_r\ (s)$ be the transfer function of the reduced system $(A_{11}, B_1\ , C_1\ )$, then the upper bound for the approximation error is given as follows:

Lemma 2.19 [7-9]:

$$\|Q - Q_r\ \|_\infty \leq 2(\sigma_{r+1} + \sigma_{r+2} + \cdots + \sigma_n\ ) \tag{2.32}$$

Where σ_{r+1} is the first neglected Hankel Singular Value of $Q(s)$.

Chapter Three

Singular Perturbation Approximation

In previous chapter we used the balanced truncation scheme to reduce the dimension of the original system, and acquired an error bound. Then we introduce the properties of reciprocal system and extend the error bound to reduce reciprocal system.

3.1 Reciprocal system of a linear dynamical system

Some properties of a reciprocal system of the balanced realization for the infinite dimensional system will be discussed in this section.

Consider a linear time-invariant continuous system represented by:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Assume the system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is balanced with gramian Σ , then we have:

$$A\Sigma + \Sigma A^T \ + BB^T \ = 0$$

$$A^T \, \Sigma + \Sigma A + C^T \, C = 0$$

Let

$$Q(s) = C(sI - A)^{-1}B + D$$

be the transfer function of the balanced system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$.

The reciprocal system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of the system (A, B, C, D) is defined as [13]:

$$\begin{aligned} \hat{A} &= A^{-1} \\ \hat{B} &= A^{-1}B \\ \hat{C} &= -CA^{-1} \\ \hat{D} &= D - CA^{-1}B \end{aligned} \qquad (3.1)$$

Remark 3.1 [22, 29]:

If we compute $Q(0)$ we have:

$$Q(0) = -CA^{-1}B + D = \hat{D}$$

Remark 3.2 [22, 29]:

If the matrix A is given as:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

then

$$A^{-1} = \begin{pmatrix} \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} & -\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1}A_{12}A_{11}^{-1} \\ -\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}A_{21}A_{11}^{-1} & \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \end{pmatrix}$$

Also we have

$$A^{-1} = \begin{pmatrix} \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} & -A_{11}^{1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \\ -A_{11}^{-1}A_{21}\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} & \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \end{pmatrix}$$

The transfer function of the reciprocal system $\left(\hat{A}, \hat{B}, \hat{C}, \widehat{D}\right)$ is:

$$\hat{Q}(s) = \hat{C}\left(sI - \hat{A}\right)^{-1}\hat{B} + \widehat{D} \tag{3.2}$$

The relation between $Q(s)$ and $\hat{Q}(s)$ is given as [29]:

$$Q(s) = C(sI - A)^{-1}B + D$$

$$= C(sI - A)^{-1}AA^{-1}B + D$$

$$= C\frac{I}{s}(A^{-1} - \frac{I}{s})^{-1}\,B + D$$

$$= -C\left(\frac{I}{s} - A^{-1} + A^{-1}\right)\left(\frac{I}{s} - A^{-1}\right)^{-1}A^{-1}B + D$$

$$= -CA^{-1}B - CA^{-1}\left(\frac{I}{s} - A^{-1}\right)^{-1}A^{-1}B + D \tag{3.3}$$

$$= -CA^{-1}(\frac{I}{s} - A^{-1})^{-1}A^{-1}B + D - CA^{-1}B$$

$$= \hat{C}\left(\frac{I}{s} - \hat{A}\right)^{-1}\hat{B} + \widehat{D}$$

$$= \hat{Q}\left(\frac{1}{s}\right)$$

Lemma 3.3 [22, 29]:

Let $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ be the minimal and balanced realization of the linear time-invariant system with gramian Σ. Then the reciprocal system $\left(\hat{A}, \hat{B}, \hat{C}, \widehat{D}\right)$is balanced with the same gramian Σ.

Proof. Since Σ satisfies the Lyapunove equations:

$$A\Sigma + \Sigma A^T + BB^T = 0$$

$$A^T\Sigma + \Sigma A + C^TC = 0$$

then multiply the first equation by A^{-1} from the left and multiply it by ${A^{-1}}^T$ from the right to get:

$$A^{-1}(A\Sigma){A^{-1}}^T + A^{-1}\left(\Sigma A^T\right){A^{-1}}^T + A^{-1}\left(BB^T\right){A^{-1}}^T = 0$$

so

$$\Rightarrow \Sigma {A^{-1}}^T + A^{-1}\Sigma + \left(A^{-1}B\right)\left(A^{-1}B\right)^T = 0$$

Substituting the values in equation (3.1) we have:

$$\hat{A}\Sigma + \Sigma\hat{A}^T + \hat{B}\hat{B}^T = 0$$

Multiply the second Lyapunov equation by ${A^{-1}}^T$ from the left and by A^{-1} from the right , we get:

$${A^{-1}}^T \left(A^T \Sigma\right)A^{-1} + {A^{-1}}^T (\Sigma A)A^{-1} + {A^{-1}}^T \left(C^T C\right)A^{-1} = 0$$

so

$$\Rightarrow \Sigma A^{-1} + {A^{-1}}^T \Sigma + \left(CA^{-1}\right)^T \left(CA^{-1}\right) = 0$$

Using equation (3.1), we have:

$$\hat{A}^T\Sigma + \Sigma\hat{A} + \hat{C}^T\hat{C} = 0$$

This means that the reciprocal system $\left(\hat{A}, \hat{B}, \hat{C}, \hat{D}\right)$ is balanced with the same gramian Σ.

Let us partition the reciprocal system $\left(\hat{A}, \hat{B}, \hat{C}, \hat{D}\right)$ and the gramian Σ as:

$$\hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix}, \hat{B} = \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \end{pmatrix}, \hat{C} = \begin{pmatrix} \hat{C}_1 & \hat{C}_2 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \qquad (3.4)$$

Lemma 3.4 [22]:

Assume that the hypothesis of lemma (3.3) hold and the reciprocal system $\left(\hat{A}, \hat{B}, \hat{C}, \hat{D}\right)$ is partitioned as in equation (3.4), then the subsystem $\left(\hat{A}_{ii}, \hat{B}_i, \hat{C}_i, \hat{D}\right)(i = 1,2)$ is also internally balanced with gramian Σ_i $(i = 1,2)$.

Lemma 3.5 [22]:

Let the hypothesis of lemma (3.4) hold. Then the subsystem matrix $\hat{A}_{ii}$ $(i = 1,2)$ is a asymptotically stable if Σ_1 and Σ_2 have no common diagonal

element. Furthermore, the subsystem
$(\hat{A}_{ii}, \hat{B}_i \ , \hat{C}_i \ , \hat{D})$ $(I = 1,2)$ is controllable and observable.

Before applying the balanced truncation method on the reciprocal system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, we assume that the Hankel singular Values $\sigma_j \ , j = 1,2, \dots, r$ are distinct and such that $\sigma_1 \ > \sigma_2 \ > \cdots > \sigma_n \ > 0$, from this condition we have $\Sigma_1 \ > 0$.

Then we get a balanced $\mathrm{r} \times \mathrm{r}$ reduced system $(\hat{\mathrm{A}}_{\mathrm{ii}}, \hat{\mathrm{B}}_{\mathrm{i}} \ , \hat{\mathrm{C}}_{\mathrm{i}} \ , \hat{\mathrm{D}})$ represented by the following state space equation:

$$\hat{\mathrm{x}} = \hat{\mathrm{A}}_{11}\hat{\mathrm{x}} + \hat{\mathrm{B}}_1 \ \hat{\mathrm{x}} \qquad (3.5)$$

$$\hat{\mathrm{y}} = \hat{\mathrm{C}}_1 \ \hat{\mathrm{x}} + \hat{\mathrm{D}}\mathrm{u}$$

By equation (3.1) and remark (3.2) we can find the values of $\hat{\mathrm{A}}_{11}$,
$\hat{\mathrm{B}}_1$, $\hat{\mathrm{C}}_1$ and D as

$$\hat{\mathrm{A}}_{11} = \left(\mathrm{A}_{11} - \mathrm{A}_{12}\mathrm{A}_{22}^{-1}\mathrm{A}_{21}\right)^{-1}$$

$$\hat{\mathrm{B}}_1 \ = \left(\mathrm{A}_{11} - \mathrm{A}_{12}\mathrm{A}_{22}^{-1}\mathrm{A}_{21}\right)^{-1}\left(\mathrm{B}_1 \ - \mathrm{A}_{12}\mathrm{A}_{22}^{-1}\mathrm{B}_2 \ \right) \qquad (3.6)$$

$$\hat{\mathrm{C}}_1 \ = \left(\mathrm{C}_1 \ - \mathrm{C}_2 \ \mathrm{A}_{22}^{-1}\mathrm{A}_{21}\right)\left(\mathrm{A}_{11} - \mathrm{A}_{12}\mathrm{A}_{22}^{-1}\mathrm{A}_{21}\right)^{-1}$$

$$\hat{\mathrm{D}} = \mathrm{D} - \mathrm{CA}^{-1}\mathrm{B}$$

For the reduced system $(\hat{\mathrm{A}}_{\mathrm{ii}}, \hat{\mathrm{B}}_{\mathrm{i}} \ , \hat{\mathrm{C}}_{\mathrm{i}} \ , \hat{\mathrm{D}})$ the transfer function is given as

$$\hat{\mathrm{G}}_{\mathrm{r}} \ (\mathrm{s}) = \hat{\mathrm{C}}_1 \ \left(\mathrm{sI} - \hat{\mathrm{A}}_{11}\right)^{-1}\hat{\mathrm{B}}_1 \ + \hat{\mathrm{D}} \qquad (3.7)$$

Now, the error bound is given in the following lemma.

Lemma 3.6 [7-9]: We have:

$$\left\|\hat{G} - \hat{G}_r \ \right\|_\infty \leq 2\left(\sigma_{r+1} + \sigma_{r+2} + \cdots + \sigma_n \ \right). \qquad (3.8)$$

3.2 Singular perturbation approximation method

In previous sections we studied a balanced truncation scheme to reduce the dimension of the system. In this section we study another method used to reduce the order of the system called singular perturbation approximation method (SPAM).

The two methods gives the same error bound. For (BTM) the error bound is small at high frequencies and large at low frequencies, but for (SPAM) the error bound is large at high frequencies and small at low frequencies.

We want to find the error bound for the reduced order model using (SPAM), to get this error bound, we show the relationship between the reduced model of reciprocal system and the reduced model obtained by (SPAM)

Consider the linear time invariant (LTI) continuous system:

$$\dot{x} = Ax + Bu \tag{3.9}$$

$$y = Cx$$

The controllability and observability gramians W_c and W_0 respectively are positive semi-definite and can be expressed as [14]:

$$W_c = UU^T$$

$$W_0 = LL^T$$

The balanced gramian Σ is partitioned as:

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$$

where

$$\Sigma_1 = diag\,(\sigma_1, \sigma_2, \ldots \sigma_r)$$

and

$$\Sigma_2 = diag\left(\sigma_{r+1}, \sigma_{r+2}, \dots \sigma_n\right)$$

This partition shows which singular values are important and the ones we can delete [28, 30]. Also, the balanced transformation S satisy this equation:

$$\mathrm{S} = UY\Sigma^{-\frac{1}{2}}$$

$$\mathrm{S}^{-1} = \Sigma^{-\frac{1}{2}} X^T L^T$$

Suppose $\sigma_{r+1} \ll \sigma_r$, and Hankel Singular Values(HSVs) are coordinate invariant, since $\sigma_{r+1} > \sigma_{r+2} > \cdots > \sigma_n > 0$, then a reduced order system with small parameter can be obtained [17].

By replacing Σ_2 by $\epsilon\Sigma_2$ we have:

$$\left(\sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_n\right) \to \epsilon\left(\sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_n\right), \epsilon > 0$$

We change the coordinates using balanced transformation $\mathrm{Y}(\epsilon)$ such that :

$$x \to \mathrm{S}(\epsilon)x$$

By letting $\mathrm{S}^{-1}(\epsilon) = \varrho(\epsilon)$, we obtain the new balanced partitioned matrices:

$$\mathrm{S}(\epsilon) = \begin{pmatrix} \mathrm{S}_{11} & \frac{1}{\sqrt{\epsilon}}\mathrm{S}_{12} \\ \mathrm{S}_{21} & \frac{1}{\sqrt{\epsilon}}\mathrm{S}_{22} \end{pmatrix} \tag{3.10}$$

and the inverse

$$\varrho(\epsilon) = \begin{pmatrix} \varrho_{11} & \varrho_{12} \\ \frac{1}{\sqrt{\epsilon}}\varrho_{21} & \frac{1}{\sqrt{\epsilon}}\varrho_{22} \end{pmatrix} \tag{3.11}$$

which gives rise to the balanced coefficients written as:

$$\tilde{A}(\epsilon) = \varrho(\epsilon)AS(\epsilon)$$

$$= \begin{pmatrix} \varrho_{11} & \varrho_{12} \\ \frac{1}{\sqrt{\epsilon}}\varrho_{21} & \frac{1}{\sqrt{\epsilon}}\varrho_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \mathrm{S}_{11} & \frac{1}{\sqrt{\epsilon}}\mathrm{S}_{12} \\ \mathrm{S}_{21} & \frac{1}{\sqrt{\epsilon}}\mathrm{S}_{22} \end{pmatrix} \tag{3.12}$$

$$= \begin{pmatrix} \tilde{A}_{11} & \frac{1}{\sqrt{\epsilon}}\tilde{A}_{12} \\ \frac{1}{\sqrt{\epsilon}}\tilde{A}_{21} & \frac{1}{\epsilon}\tilde{A}_{22} \end{pmatrix}$$

$$\tilde{B}(\epsilon) = \varrho(\epsilon)B$$

$$= \begin{pmatrix} \varrho_{11} & \varrho_{12} \\ \frac{1}{\sqrt{\epsilon}}\varrho_{21} & \frac{1}{\sqrt{\epsilon}}\varrho_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \tag{3.13}$$

$$= \begin{pmatrix} \tilde{B}_1 \\ \frac{1}{\sqrt{\epsilon}}\tilde{B}_2 \end{pmatrix}$$

and

$$\tilde{C}(\epsilon) = C\mathrm{S}(\epsilon)$$

$$= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} \mathrm{S}_{11} & \frac{1}{\sqrt{\epsilon}}\mathrm{S}_{23} \\ \mathrm{S}_{21} & \frac{1}{\sqrt{\epsilon}}\mathrm{S}_{22} \end{pmatrix} \tag{3.14}$$

$$= \begin{pmatrix} \tilde{C}_1 & \frac{1}{\sqrt{\epsilon}}\tilde{C}_2 \end{pmatrix}$$

In equation (3.12), $\tilde{A} = \varrho(1)A\mathrm{S}(1)$ denotes the balanced matrix A for $\epsilon = 1$.

The balancing transformation can be reformulated as:

$$\mathrm{S}(\epsilon) = \Gamma(\epsilon)\mathrm{S}(1)$$

and

$$\varrho(\epsilon) = \varrho(1)\Gamma(\epsilon)$$

where $\Gamma(\epsilon)$ is the diagonal scaling matrix:

$$\Gamma(\epsilon) = \begin{pmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{\epsilon}} I \end{pmatrix}$$

In what follows we delete the tilde from the balanced matrices to get matrices:

$$A = \begin{pmatrix} A_{11} & \frac{1}{\sqrt{\epsilon}} A_{12} \\ \frac{1}{\sqrt{\epsilon}} A_{22} & \frac{1}{\epsilon} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ \frac{1}{\sqrt{\epsilon}} B_2 \end{pmatrix}, C = \begin{pmatrix} C_1 & \frac{1}{\sqrt{\epsilon}} C_2 \end{pmatrix}$$

We define the new variable $w = (w_1 \ , w_2 \ )$ which can be balanced using the balance transformation $\varrho(\epsilon)$ and w written in the balance form as:

$$w = \varrho(\epsilon) x$$

In this moment, our dynamical linear system in equation (3.9) turns in to the singular perturbed system of equation:

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & \frac{1}{\sqrt{\epsilon}} A_{12} \\ \frac{1}{\sqrt{\epsilon}} A_{21} & \frac{1}{\epsilon} A_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} B_2 \\ \frac{1}{\sqrt{\epsilon}} B_2 \end{pmatrix} u \tag{3.15}$$

$$y = (C_1 \quad \frac{1}{\sqrt{\epsilon}} C_2 \ ) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Equation (3.15) can written as:

$$\begin{aligned} \dot{w}_1 \ &= A_{11} w_1 \ + \tfrac{1}{\sqrt{\epsilon}} A_{12} w_2 \ + B_1 \ u \\ \dot{w}_2 \ &= \tfrac{1}{\sqrt{\epsilon}} A_{21} w_1 \ + \tfrac{1}{\epsilon} A_{22} w_2 \ + B_2 \ u \\ y &= C_1 \ w_1 \ + \tfrac{1}{\sqrt{\epsilon}} C_2 \ w_2 \end{aligned} \tag{3.16}$$

We scale w_2 as:

$$w_2 \ \rightarrow \sqrt{\epsilon} w_2$$

Then equation (3.16) becomes:

$$\begin{aligned} \dot{w}_1 \ &= A_{11} w_1 \ + \frac{1}{A_{12}} w_2 \ + B_1 \ u \\ \epsilon \dot{w}_2 \ &= A_{21} w_1 \ + A_{22} w_2 \ + B_2 \ u \end{aligned} \tag{3.17}$$

$$y = C_1\ w_1\ + \frac{1}{\sqrt{\epsilon}} C_2\ w_2$$

We can write this system in matrix form as:

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon} A_{21} & \frac{1}{\epsilon} A_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{1}{\epsilon} B_2 \end{pmatrix} u \qquad (3.18)$$

$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

where the submatrices $A_{11}, A_{12}, \ldots$ are in balanced form, and scalar ϵ represents all the small parameters to be neglected [17].

In order to reduce the order of the system to gain a reduced order model, set the singular perturbation $\epsilon = 0$. Singular perturbation cause a multi-time scale behavior of dynamical systems and this yields the slow and fast variables of the system. More details about the quasi-steady-state are found in [20].

Assumption 3.7 : The block matrix A_{22} is invertable and stable, i.e

$$\mathbb{R}\{\lambda(A_{22})\} < 0$$

Assumption 3.8 : The equation:

$$\epsilon \dot{w}_2\ = A_{21} w_1\ + A_{22} w_2\ + B_2\ u \qquad (3.19)$$

has distinct roots when $\epsilon = 0$.

w_1 and w_2 which appeared in the dynamical system expressed by equation (3.17) are the slow and the fast variables respectively [12].

According to the assumptions (3.7) - (3.8) and equation (3.18), then setting $\epsilon = 0$ makes the roots of equation (3.19) denoted by $\overline{w}_2$ look like:

$$\overline{w}_2\ = A_{22}^{-1} A_{21} \overline{w}_1\ - A_{22}^{-1} B_2\ u \qquad (3.20)$$

In the first part of equation (3.17), if we substitute the value of $\bar{w}_2$ we have the reduced order model given by the state equations:

$$\dot{\bar{w}}_1 = \bar{A}\bar{w}_1 + \bar{B}u$$
$$\bar{y} = \bar{C}\bar{w}_1 + \bar{D}u \qquad (3.21)$$
$$\bar{w}_1(0) = w_1(0)$$

such that

$$\bar{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}$$
$$\bar{B} = B_1 + A_{12}A_{22}^{-1}B_2$$
$$\bar{C} = C_1 - C_2 A_{22}^{-1}A_{21} \qquad (3.22)$$
$$\bar{D} = -C_2 A_{22}^{-1}B_2$$

Let $\bar{Q}$ be the transfer function of the reduced order model given in equation (3.21), then:

$$\bar{Q}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} \qquad (3.23)$$

In view of definition (3.5) and equations (3.6) and (3.22), we have:

$$\hat{A}_{11} = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} = (\bar{A})^{-1}$$
$$\hat{B}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}(B_1 - A_{12}A_{22}^{-1}B_2)$$
$$= (\bar{A})^{-1}\bar{B} \qquad (3.24)$$
$$\hat{C}_1 = \left(C_1 - C_2 A_{22}^{-1}A_{21}\right)\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}$$
$$= -\bar{C}(\bar{A})^{-1}$$
$$\hat{D} = \bar{D} - \bar{C}(\bar{A})^{-1}\bar{B}$$

Moreover, the following relationship between the two transfer functions $\bar{Q}(s)$ and $\bar{Q}_r(s)$ are [29]:

$\bar{Q}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} - \bar{D}$

$$= \bar{C}\left(\frac{I}{s}\right)\left(I - \frac{1}{s}\bar{A}\right)^{-1} \bar{B} + \bar{D}$$

$$= \bar{C}\left(\frac{1}{s}\right)((\bar{A})^{-1}\bar{A} - \frac{1}{s}\bar{A})^{-1}\bar{B} + \bar{D}$$

$$= \bar{C}\left(\frac{1}{s}\right)\left((\bar{A})^{-1} - \frac{I}{s}\right)^{-1} (\bar{A})^{-1}\bar{B} + \bar{D}$$

$$= \bar{C}\left(\frac{1}{s} - \bar{A}^{-1} + \bar{A}\right)(\frac{I}{s} - (\bar{A})^{-1})^{-1}(\bar{A})^{-1}\bar{B} + \bar{D} \tag{3.25}$$

$$= -\bar{C}(\bar{A})^{-1}\bar{B} - \bar{C}\left(\frac{I}{s} - (\bar{A})^{-1}\right)^{-1} \bar{A}^{-1}\bar{B} + \bar{D}$$

$$= -\bar{C}(\bar{A})^{-1}\left(\frac{I}{s} - (\bar{A})^{-1}\right)^{-1} (\bar{A})^{-1}\bar{B} + \bar{D} - \bar{C}(\bar{A})^{-1}\bar{B}$$

$$= \hat{C}\left(\frac{I}{s} - \hat{A}_{11}\right)^{-1} \hat{B}_1 + \hat{D}$$

$$= \bar{Q}_r \left(\frac{1}{s}\right)$$

Theorem 3.9 [29]:

The reduced order model by SPAM $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$is balanced with gramian Σ_1 and is asymptotically stable.

Proof. By lemma (3.4) the reduced reciprocal system $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D})$ is balanced with gramain Σ_1 that satisfy Lyapunov equations:

$$\hat{A}_{11}\Sigma_1 + \Sigma_1 \hat{A}_{11}^T + \hat{B}_1 \hat{B}_1^T = 0$$

$$\hat{A}_{11}^T\Sigma_1 + \Sigma_1 \hat{A}_{11} + \hat{C}_1^T\hat{C}_1 = 0$$

premultiplying the first equation by $\hat{A}_{11}^{-1}$ and postmultiplying by $(\hat{A}_{11}^{-1})^T$, and multiplying the second equation from the left by $(\hat{A}_{11}^{-1})^T$ and from the right by $\hat{A}_{11}^{-1}$, yields:

$$\hat{A}_{11}^{-1}(\hat{A}_{11}\Sigma_1)(\hat{A}_{11}^{-1})^T + \hat{A}_{11}^{-1}(\Sigma_1 \hat{A}_{11}^T)(\hat{A}_{11}^{-1})^T + \hat{A}_{11}^{-1}(\hat{B}_1 \hat{B}_1^T)(\hat{A}_{11}^{-1})^T = 0$$

$$\Rightarrow \Sigma_1 (\hat{A}_{11}^{-A})^T + \hat{A}_{11}^{-1}\Sigma_1 + (\hat{A}_{11}^{-1}\hat{B}_1)(\hat{A}_{11}^{-1}\hat{B}_1)^T = 0$$

Using equation (3.24), we have

$$\bar{A}\Sigma_1 + \Sigma_1 \bar{A}^T + B\bar{B}^T = 0$$

and

$$\left(\hat{A}_{11}^{-1}\right)^T \left(\hat{A}_{11}^T\Sigma_1\right)\hat{A}_{11}^{-1}) + \left(\hat{A}_{11}^{-1}\right)^T \left(\Sigma_1 \hat{A}_{11}\right)\hat{A}_{11}^{-1} + \left(\hat{A}_{11}^{-1}\right)^T \left(\hat{C}_1^T\hat{C}_1\right)\hat{A}_{11}^{-1} = 0$$

$$\Rightarrow \Sigma_1 \hat{A}_{11}^{-1} + (\bar{A}_{11}^{-1})^T \Sigma_1 + \left(\hat{C}_1 \hat{A}_{11}^{-1}\right)^T) \left(\bar{C}_1 \bar{A}_{11}^{-1}\right) = 0$$

In the same way, using equation (3.24), we obtain:

$$\bar{A}^T \Sigma_1 + \Sigma_1 \bar{A} + \bar{C}^T \bar{C} = 0$$

this implies that $(\bar{A}, \bar{B}, \bar{C}, \bar{C})$ is balanced with gramian Σ_1 . Hence the eigenvalues of $\bar{A}$ are $\lambda_i = \frac{1}{\lambda}$ due to $\bar{A} = \bar{A}_{11}^{-1}$ and $\hat{A}_{11}$ is also asymptotically stable, (i.e $\mathbb{R}\{\lambda(\hat{A}_{11})\} < 0$), with λ be the eigenvalues of $\hat{A}_{11}$.

We conclude that $\mathbb{R}\{\lambda_i (\hat{A})\} < 0$, which means the reduced order model by SPA method is asymptotically stable.

The next theorem gives the error bound of reduced order model using SPA method in the form of H_∞ norm

Theorem 3.10 [7-9]:

Suppose Q and $\bar{Q}_r$ are the transfer functions of the main system and r-th order model ingenious by SPAM $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ of Q, then we have:

$$\|Q(s) - \bar{Q}(s)\|_\infty \leq 2\sum_{i=r+1}^{\infty} \sigma_i \tag{3.26}$$

Proof. From equations (3.3)-(3.25) and lemma (3.1.6), and using the triangle inequality we obtain:

$$\|Q(s)-\bar{Q}(s)\|_\infty = \left\|Q(s)-\bar{Q}\left(\frac{1}{s}\right)+\bar{Q}\left(\frac{1}{s}\right)-\bar{Q}_r\ \left(\frac{1}{s}\right)+\hat{Q}_r\ \left(\frac{1}{s}\right)-\bar{Q}(s)\right\|_\infty$$

$$\leq \left\|Q(s)-\hat{Q}\left(\frac{1}{s}\right)\right\|_\infty + \left\|\hat{Q}\left(\frac{1}{s}\right)-\hat{Q}_r\ \left(\frac{1}{s}\right)\right\|_\infty + \left\|\hat{Q}_r\ \left(\frac{1}{s}\right)-\hat{Q}_r\ \left(\frac{1}{s}\right)\right\|_\infty$$

$$\leq \left\|\hat{Q}\left(\frac{1}{s}\right)-\hat{Q}_r\ \left(\frac{1}{s}\right)\right\|_\infty$$

$$\leq 2\sum_{i=r+1}^{\infty}\sigma_i$$

Chapter Four

Numerical Examples and Results

To illustrate the effectiveness of the Balanced Truncation (BT) and the Singular Perturbation (SP) approximation for constructing a reduced order model for stable finite dimensional linear time-invariant (LTI) dynamical system with zero initial condition we consider the following numerical examples:

Example 4.1: Mass – spring damping system

For convenience, we will start with three masses and apply Newton's Second Law of motion to them. Suppose that m_1 ,m_2 and m_3 are the masses described in figure (4.1)

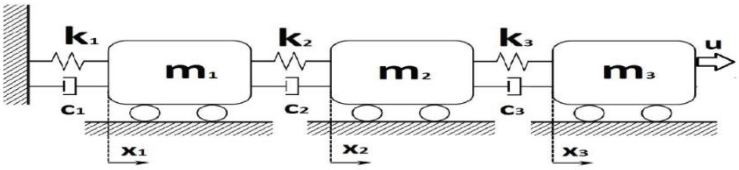

Figure 4.1: Three Mass-Spring Damping

where x_1 , x_2 and x_3 are the positions of the masses m_1 , m_2 and m_3 respectively and k_1 , k_2 , k_3 and c_1 , c_2 , c_3 are constants that represent the stiffness and the damping of the springs with u is the force acting on the mass m_3 . For mass m_1 in figure (4.2) and apply Newton's Second Law we get the following differerntial equation:

$$m_1 \ddot{x}_1 + c_1 \dot{x}_1 + k_1 x_1 + c_2 (\dot{x}_2 - \dot{x}_1) + k_2 (x_2 - x_1) = 0$$

$$m_1 \ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + c_2 \dot{x}_2 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \qquad (4.1)$$

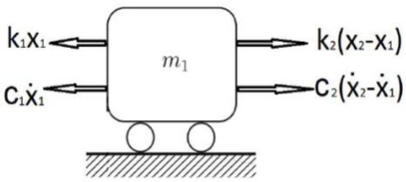

Figure 4.2: Mass 1

Likewise for the mass m_2 in figure (4.3), we obtain:

$$m_2 \ddot{x}_2 + c_2 \ddot{x}_2 + k_2 x_2 + c_3 (\dot{x}_3 - \dot{x}_2) + k_3 (x_3 - x_2) = 0$$

$$m_2 \ddot{x}_2 + (c_2 + c_3)\ddot{x}_2 + c_2 \dot{x}_3 + (k_2 + k_3)x_2 - k_3 x_3 = 0 \qquad (4.2)$$

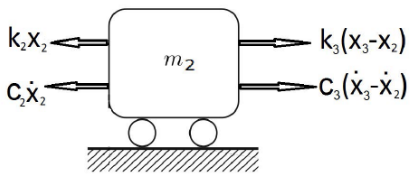

Figure 4.3: Mass 2

and the differential equation for the mass m_3 in figure (4.4) is:

$$m_3\,\ddot{x}_3 \;+ c_1\,(\ddot{x}_3 \;- \dot{x}_2\,) + k_3\,(x_3 \;- x_2\,) = u$$

$$m_3\,\ddot{x}_3 \;- c_3\,\dot{x}_2 \;+ c_3\,\dot{x}_3 \;+ k_3\,x_3 \;- k_3\,x_2 \;= u \tag{4.3}$$

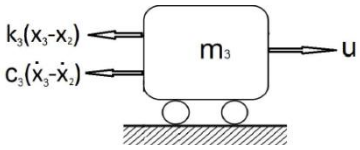

Figure 4.4: Mass 3

We can write these set of differential equations (4.1), (4.2) and (4.3) in matrix form as:

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{pmatrix} c_1 \;+ c_2 & -c_2 & 0 \\ -c_2 & c_2 \;+ c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}$$
$$+ \begin{pmatrix} k_1 \;+ k_2 & -k_2 & 0 \\ -k_2 & k_2 \;+ k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \tag{4.4}$$

The differential equation that represents this system has the form:

$$M\ddot{x} + C\dot{x} + Kx = Lu \tag{4.5}$$

where M is the mass matrix, C is the damping matrix, K is the stiffness and L is an 3×1 column vector.

$$L = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

To find the state space equation for the previous linear continuous dynamical system we assume that:

$$\dot{x} = z \tag{4.6}$$

Furthermore

$$\ddot{x} = \dot{z} \tag{4.7}$$

Assume that M^{-1} exists, then insert equations (4.6) and (4.7) into equation (4.5), we obtain:

$$\dot{x} = z$$

$$\dot{x} = -M^{-1}Kx - M^{-1}Cz + M^{-1}Lu \tag{4.8}$$

and in matrix form we have

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ -M^{-1}L \end{pmatrix} u \tag{4.9}$$

Let $A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix}$ be of size (6×6) and $B = \begin{pmatrix} 0 \\ -M^{-1}L \end{pmatrix}$ of size (6×1), then the state space equation for this system is:

$$\dot{X} = AX + Bu \tag{4.10}$$

where $X = \begin{pmatrix} x \\ z \end{pmatrix}$ is the state vector of the linear dynamical system of size (6×1).

Now, we want to find the state space representation for any n mass-spring damping continuous system.

Using same approach, we can derive the state space equation for n masses and apply Newton's Second Law on the mass m_i to get the following differential equation

$$m_i \ddot{x}_i - c_i \dot{x}_{i-1} + (c_i + c_{i+1})\dot{x}_i - c_{i+1}\dot{x}_{i+1} - k_i x_{i-1}$$

$$+k_{i+1}(x_{i+1} - x_i) = bu \tag{4.11}$$

where $i = 1,2,3, \dots, n$

The value of B is zero when $i \neq n$ and one when $i = n$, but $x_0 = 0$ for $i = 1$ and $x_{n+1} = k_{n+1} = d_{n+1} = 0$ for $i = n$

The matrix representation for the differential equation described by equation (4.11) can be written as:

$$M\ddot{x} + C\dot{x} + kx = Lu \tag{4.12}$$

where

$$M = \begin{pmatrix} m_1 & 0 & 0 & \ldots & \ldots & 0 \\ 0 & m_2 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & \ddots & 0 & \ldots & 0 \\ \vdots & \vdots & 0 & m_i & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & m_n \end{pmatrix}_{n\times n}$$

is called the mass matrix of the system and

$$C = \begin{pmatrix} c_1 + c_2 & -c_2 & 0 & \ldots & \ldots & 0 \\ -c_2 & c_2 + c_3 & -c_3 & 0 & \ldots & 0 \\ 0 & -c_3 & c_3 + c_4 & & \ldots & 0 \\ 0 & 0 & & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -c_n \\ 0 & 0 & 0 & \ldots & -c_n & c_n \end{pmatrix}_{n\times n}$$

is the damping matrix

and

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & 0 & \ldots & \ldots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 & \ldots & 0 \\ 0 & -k_3 & k_3 + k_4 & \ddots & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -k_n \\ 0 & 0 & 0 & \ldots & -k_n & k_m \end{pmatrix}_{n\times n}$$

is the Stiffnes matrix. Finally, the vector

$$L = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}_{n\times 1}$$

represents the number of controllers acting on masses.

To obtain the state space equation for the linear continuous system, we let:

$$\dot{x} = z$$

and

$$\ddot{x} = \dot{z}$$

Sitting these equations into equation (4.12), we get the following system:

$$\dot{x} = z$$

$$\dot{z} = -M^{-1}Kx - M^{-1}Cz + M^{-1}Lu \tag{4.13}$$

and in matrix form, we have:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ -M^{-1}L \end{pmatrix} u \tag{4.14}$$

If we let

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}$$

of size $(2n \times 2n)$

and

$$B = \begin{pmatrix} 0 \\ -M^{-1}L \end{pmatrix}$$

of size $(2n \times 1)$, then the state space equation for this system is:

$$\dot{X} = AX + Bu \tag{4.15}$$

where $X = \binom{x}{z}$ is the state vector of size $(2n \times 1)$

Tables (4.1) and (4.2) show the value of $\|G - G_r\|_\infty$ and $\| \quad \|_{L_2}$ bound of the approximation error between the output y and yr of the original and reduced systems and the error bound for the SPA and BT methods.

Figure (4.5) represents the Hankel singular values of damping's mass-spring system for both SPA and BTM.

Table 4.1: the L^2 norm of $y - y_r$ of the damping mass- spring

r_x	$\|G - G_r\|_\infty$ by SPA	$\|y_0 - y_r\|_{L_2}$ SPA	$2\sum_{i=r+1}^{10} \sigma_i$
1	3.5016	6.5955e-06	3.6523
2	0.0055	2.1036e-10	0.1450
3	0.0593	1.4866e-08	0.0849
4	0.0018	1.8108e-11	0.0259
5	0.0097	1.2979e-10	0.0159
6	5.9742e-04	8.6645e-13	0.0063
7	0.0013	1.3055e-10	0.0051
8	5.1416e-04	1.0127e-12	0.0039
9	8.1211e-04	6.5479e-12	0.0030
10	4.7972e-04	4.9314e-13	0.0021

Now, by applying the balanced truncation method for this system with zero initial condition obtained a reduce order system, and compute $\|.\|_\infty$ bound of the approximation error given in section (2.4).

We take a system of size N = 20 and reduce its order to obtain reduced order system of size r = 2.

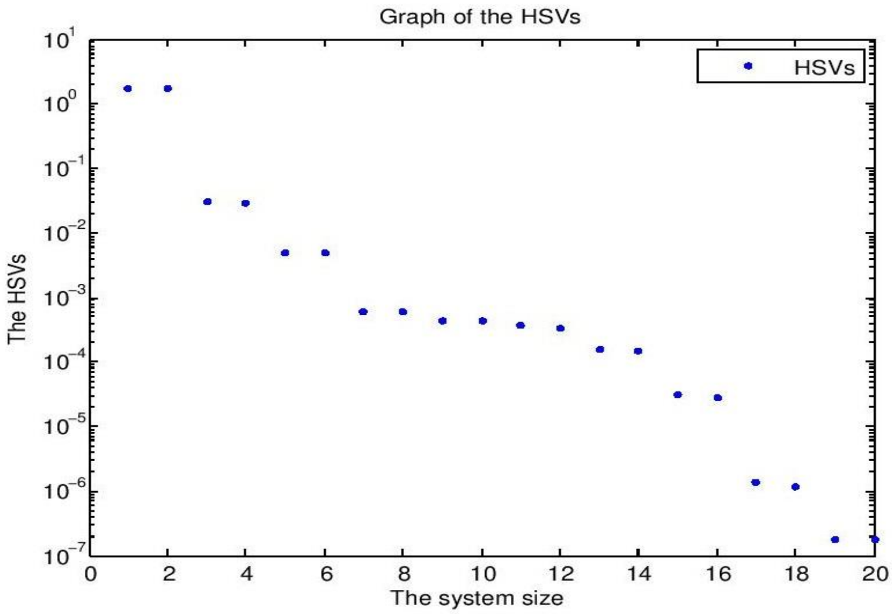

Figure 4.5: HSVs of the mass-spring damping

Table 4.2: the L^2 norm of $y - y_r$ of the damping mass- spring

r_x	$\|G - G_r\|_\infty$ by BT	$\|y_0 - y_r\|_{L_2}$ BT	$2\sum_{i=r+1}^{10} \sigma_i$
1	3.5129	6.5973e-06	3.6523
2	0.0072	2.1064e-10	0.1450
3	0.0607	1.4896e-08	0.0849
4	0.0023	1.8201e-11	0.0259
5	0.0104	1.3042e-10	0.0159
6	5.9823e-04	8.6729e-13	0.0063
7	0.0019	1.3131e-10	0.0051
8	5.1485e-04	1.0216e-12	0.0039
9	8.1276e-04	6.5547e-12	0.0030
10	4.8018e-04	4.9424e-13	0.0021

Figure (4.6) and (4.8) represents the maximum singular value decomposition (MSVD) of (G – Gr) and the error bound $2\sum_{i=r+1}^{10}\sigma_i$ for SPA and BTM. We see clearly that the balanced truncation method yields a reduced order model with smaller error at high frequencies and larger error at low frequencies.

Next, figures (4.7) and (4.9), show the plot of the outputs y and yr of the original and reduced systems respectively and their differences $(y - yr)$ and error bound by applying singular perturbation and balanced truncation method on the numerical example respectively.

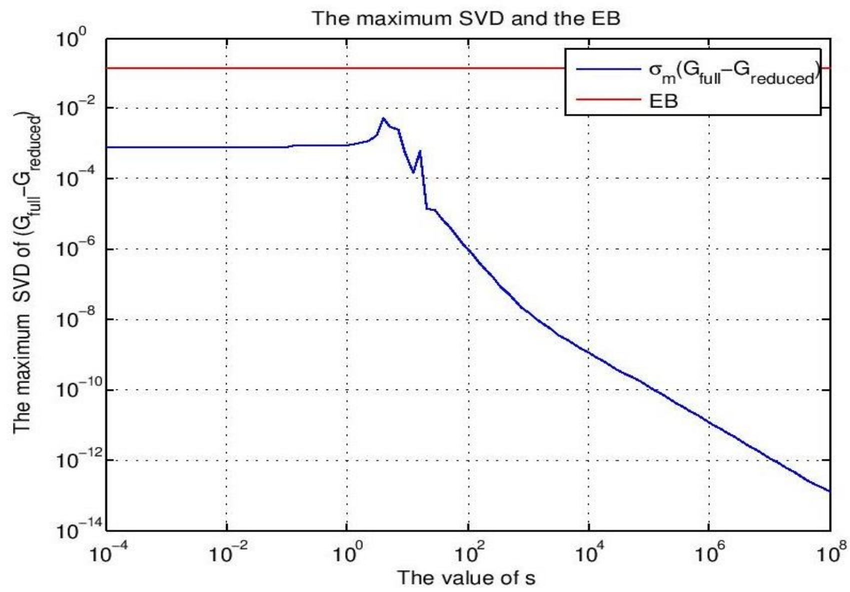

Figure 4.6: The MSVD and the error bound for the damping mass-spring

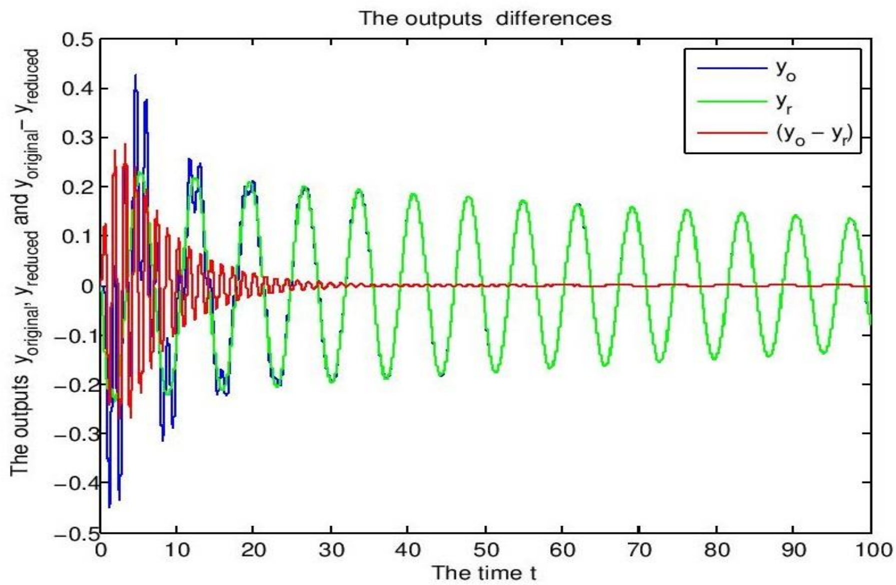

Figure 4.7: The outputs of the mass-spring damping

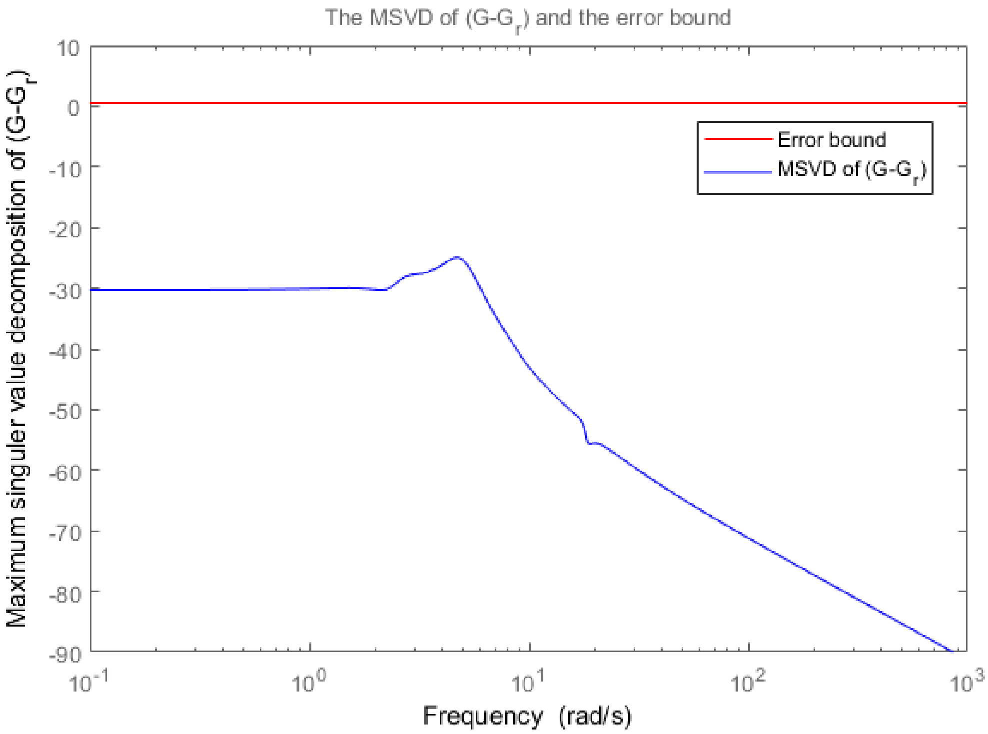

Figure 4.8: The MSVD and the error bound for the damping mass-spring

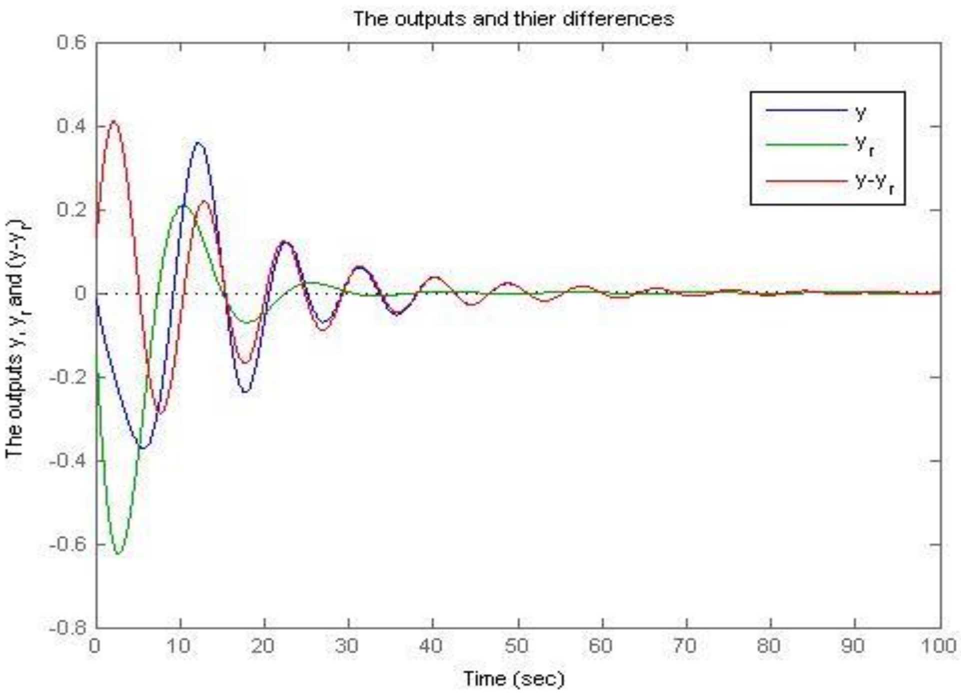

Figure 4.9: The outputs of the mass-spring damping

Example 4.2: RC-circuit:

We will study an RC-filter, RC-network and resistor–capacitor circuit (RC-circuit). The RC-circuit shown in figure (4.10) is a collection of resistors and capacitors.

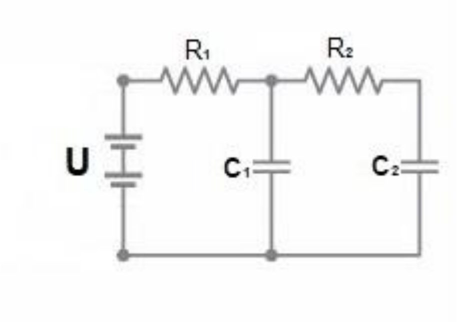

Figure 4.10: Simple RC-circuit

driven by a voltage or current exporter. RC-circuits may be used to filter a signal by blocking confirmed frequencies and passing others.

Suppose we denote by i_ρ and v_ρ the current and voltage through the capacitor C_ρ, let U denote the voltage source in the circuit and I (the output of the system) is the current across a resistor $n, \rho = 1,2, \cdots, n$.

Let $i_\rho = C_\rho \frac{d}{dt} v_\rho(t)$

$$e_1: \frac{U - v_1}{R_1} + i_1 + \frac{v_2 - v_1}{R_2} = 0 \ \rightarrow \ \frac{d}{dt} v_1(t) = \frac{v_1 - U}{C_1 R_1} + \frac{v_1 - v_2}{C_1 R_2}$$

$$e_2: \frac{v_2 - v_1}{R_2} + i_2 + \frac{v_3 - v_2}{R_3} = 0 \ \rightarrow \ \frac{d}{dt} v_2(t) = \frac{v_1 - v_2}{C_2 R_2} + \frac{v_2 - v_3}{C_2 R_3}$$

$$\vdots$$

$$e_\rho: \frac{v_\rho - v_{\rho-1}}{R_\rho} + i_\rho + \frac{v_{\rho+1} - v_\rho}{R_{\rho+1}} = 0 \ \rightarrow \ \frac{d}{dt} v_\rho(t)$$

$$= \frac{v_{\rho-1} - v_\rho}{C_\rho R_\rho} + \frac{v_\rho - v_{\rho+1}}{C_\rho R_{\rho+1}}$$

$$\vdots$$

$$e_n: \frac{v_n - v_{n-1}}{R_n} + i_n = 0 \ \rightarrow \ \frac{d}{dt} v_n(t) = \frac{v_{n-1} - v_n}{C_n R_n}$$

$I = \frac{v_n - v_{n-1}}{R_n}$.

This system can be written in state space representation as:

$$\dot{V} = AV + BU$$
$$I = CV,$$

where:

$$A = \begin{pmatrix} \frac{1}{C_1R_2}+\frac{1}{C_1R_1} & -\frac{1}{C_1R_2} & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{1}{C_2R_2} & -\frac{1}{C_2R_2}+\frac{1}{C_2R_3} & -\frac{1}{C_2R_3} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \frac{1}{C_3R_3} & -\frac{1}{C_3R_3}+\frac{1}{C_3R_4} & -\frac{1}{C_3R_4} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \frac{1}{C_4R_4} & \ddots & \ddots & 0 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{C_nR_n} & -\frac{1}{C_nR_n} \end{pmatrix}_{n\times n}$$

$$B = \begin{pmatrix} \frac{-1}{C_1R_1} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}_{n\times 1}$$

and

$$C = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & \frac{-1}{R} & \frac{1}{R} \end{pmatrix}$$

We choose the size of the system n=20, and the reduced system r=4. the following matrices represent the linear dynamical system:

A = matrix of size 20x20

$$A = \begin{pmatrix} A1 & A2 \\ A3 & A4 \end{pmatrix}$$

$$A1=\begin{pmatrix}
-0.0016 & 0.8865 & 0.0021 & -0.0021 & -0.0014 & 0.0016 & -0.0010 & -0.0001 & -0.0007 & 0.0003\\
-0.8865 & -0.0016 & -0.0021 & 0.0022 & 0.0014 & -0.0016 & 0.0010 & 0.0001 & 0.0007 & -0.0003\\
-0.0021 & -0.0021 & -0.0397 & 4.4801 & 0.0630 & -0.0514 & 0.0399 & 0.0030 & 0.0243 & -0.0110\\
-0.0021 & -0.0022 & -4.4801 & -0.0433 & -0.0474 & 0.0752 & -0.0403 & -0.0033 & -0.0264 & 0.0112\\
-0.0014 & -0.0014 & -0.0630 & -0.0474 & -0.1050 & 7.6950 & -0.1390 & -0.0140 & -0.1078 & 0.0398\\
-0.0016 & -0.0016 & -0.0514 & -0.0752 & -7.6950 & -0.1395 & 0.2091 & 0.0128 & 0.1057 & -0.0556\\
-0.0010 & -0.0010 & -0.0399 & -0.0403 & -0.1390 & -0.2091 & -0.4736 & -14.5428 & -2.1095 & 0.1543\\
0.0001 & 0.0001 & 0.0030 & 0.0033 & 0.0140 & 0.0128 & 14.5428 & -0.0030 & -0.0275 & 0.0704\\
0.0007 & 0.0007 & 0.0243 & 0.0264 & 0.1078 & 0.1057 & 2.1095 & -0.0275 & -0.2629 & 10.6777\\
0.0003 & 0.0003 & 0.0110 & 0.0112 & 0.0398 & 0.0556 & 0.1543 & -0.0704 & -10.6777 & -0.0519
\end{pmatrix}$$

$$A2=\begin{pmatrix}
-0.0011 & -0.0007 & -0.0007 & 0.0005 & -0.0004 & -0.0003 & -0.0001 & -0.0001 & 0.0000 & 0.0000\\
0.0011 & 0.0008 & 0.0007 & -0.0005 & 0.0004 & 0.0003 & 0.0001 & 0.0001 & 0.0000 & 0.0000\\
0.0402 & 0.0286 & 0.0255 & -0.0187 & 0.0133 & 0.0096 & 0.0031 & 0.0028 & 0.0010 & -0.0001\\
-0.0434 & -0.0295 & -0.0266 & 0.0199 & -0.0141 & -0.0100 & -0.0033 & -0.0029 & -0.0011 & 0.0001\\
-0.1753 & -0.1060 & -0.0982 & 0.0772 & -0.0535 & -0.0379 & -0.0126 & -0.0111 & -0.0041 & 0.0004\\
0.1769 & 0.1424 & 0.1226 & -0.0852 & 0.0619 & 0.0449 & 0.0147 & 0.0130 & 0.0049 & -0.0005\\
-2.4564 & -0.4429 & -0.4888 & 0.5996 & -0.3399 & -0.2213 & -0.0762 & -0.0677 & -0.0251 & 0.0025\\
-0.0487 & -0.1217 & -0.0657 & 0.0290 & -0.0243 & -0.0192 & -0.0061 & -0.0053 & -0.0020 & 0.0002\\
-0.4730 & -2.6044 & -0.8452 & 0.2999 & -0.2658 & -0.2174 & -0.0674 & -0.0593 & -0.0221 & 0.0022\\
2.6587 & 0.1520 & 0.1776 & -0.2739 & 0.1374 & 0.0858 & 0.0301 & 0.0268 & 0.0099 & -0.0010
\end{pmatrix}$$

$$A3=\begin{pmatrix}
0.0011 & 0.0011 & 0.0402 & 0.0434 & 0.1753 & 0.1769 & 2.4564 & -0.0487 & -0.4730 & -2.6587\\
-0.0007 & -0.0008 & -0.0286 & -0.0295 & -0.1060 & -0.1424 & -0.4429 & 0.1217 & 2.6044 & 0.1520\\
-0.0007 & -0.0007 & -0.0255 & -0.0266 & -0.0982 & -0.1226 & -0.4888 & 0.0657 & 0.8452 & 0.1776\\
-0.0005 & -0.0005 & -0.0187 & -0.0199 & -0.0772 & -0.0852 & -0.5996 & 0.0290 & 0.2999 & 0.2739\\
0.0004 & 0.0004 & 0.0133 & 0.0141 & 0.0535 & 0.0619 & 0.3399 & -0.0243 & -0.2658 & -0.1374\\
-0.0003 & -0.0003 & -0.0096 & -0.0100 & -0.0379 & -0.0449 & -0.2213 & 0.0192 & 0.2174 & 0.0858\\
-0.0001 & -0.0001 & -0.0031 & -0.0033 & -0.0126 & -0.0147 & -0.0762 & 0.0061 & 0.0674 & 0.0301\\
0.0001 & 0.0001 & 0.0028 & 0.0029 & 0.0111 & 0.0130 & 0.0677 & -0.0053 & -0.0593 & -0.0268\\
0.0000 & 0.0000 & -0.0010 & -0.0011 & -0.0041 & -0.0049 & -0.0251 & 0.0020 & 0.0221 & 0.0099\\
0.0000 & 0.0000 & -0.0001 & -0.0001 & -0.0004 & -0.0005 & -0.0025 & 0.0002 & 0.0022 & 0.0010
\end{pmatrix}$$

$$A4=\begin{pmatrix}
-0.8580 & -14.8917 & -1.8800 & 0.5648 & -0.5188 & -0.4357 & -0.1333 & -0.1171 & -0.0438 & 0.0043\\
14.8917 & -0.4522 & -0.5514 & 1.0657 & -0.4676 & -0.2814 & -0.1002 & -0.0893 & -0.0331 & 0.0032\\
1.8800 & -0.5514 & -0.7737 & 17.2204 & -1.0070 & -0.4947 & -0.1905 & -0.1713 & -0.0632 & 0.0062\\
0.5648 & -1.0657 & -17.2204 & -0.4543 & 0.5343 & 0.5697 & 0.1538 & 0.1337 & 0.0503 & -0.0049\\
-0.5188 & 0.4676 & 1.0070 & 0.5343 & -1.1073 & -19.1988 & -0.5478 & -0.4456 & -0.1735 & 0.0168\\
0.4357 & -0.2814 & -0.4947 & -0.5697 & 19.1988 & -0.6193 & -0.3899 & -0.3778 & -0.1338 & 0.0133\\
0.1333 & -0.1002 & -0.1905 & -0.1538 & 0.5478 & -0.3899 & -1.3950 & -21.7036 & -0.8116 & 0.1036\\
-0.1171 & 0.0893 & 0.1713 & 0.1337 & -0.4456 & 0.3778 & 21.7036 & -1.2316 & -1.0784 & 0.0783\\
0.0438 & -0.0331 & -0.0632 & -0.0503 & 0.1735 & -0.1338 & -0.8116 & 1.0784 & -1.1434 & 20.3942\\
0.0043 & -0.0032 & -0.0062 & -0.0049 & 0.0168 & -0.0133 & -0.1036 & 0.0783 & -20.3942 & -0.0111
\end{pmatrix}$$

B = vector of size 20×1

$$B=\begin{pmatrix} B1\\ B2 \end{pmatrix}$$

$$B1=\begin{pmatrix} -0.0756\\ -0.0757\\ -0.0489\\ -0.0506\\ -0.0323\\ -0.0367\\ -0.0240\\ 0.0019\\ 0.0152\\ 0.0067 \end{pmatrix}\quad,\quad B2=\begin{pmatrix} 0.0250\\ -0.0174\\ -0.0156\\ -0.0115\\ 0.0082\\ -0.0059\\ -0.0019\\ 0.0017\\ -0.0006\\ -0.0001 \end{pmatrix}$$

C Vector of size 1×20

$$C = \begin{pmatrix} C1 & C2 \end{pmatrix}$$

C1(−0.0756 0.0757 0.0489 −0.0506 −0.0323 0.0367 −0.0240 −0.0019 −0.0152 0.0067)

$C2$(−0.0250 −0.0174 −0.0156 0.0115 −0.0082 −0.0059 −0.0019 −0.0011 −0.0007 −0.0001)

Table 4.3: the L^2 norm of $y - y_r$ of RC-circuit example

r_x	$\|G - G_r\|_\infty$ by SPA	$\|y_0 - y_r\|_{L_2}$ SPA	$2\sum_{i=r+1}^{10} \sigma_i$
1	3.7667	3.5426e-06	3.8608
2	0.0059	1.1235e-12	0.1533
3	0.0666	6.5791e-13	0.0898
4	0.0020	1.0095e-14	0.0274
5	0.0114	5.3814e-15	0.0169
6	5.6448e-04	1.1438e-16	0.0067
7	0.0018	1.1536e-16	0.0054
8	5.2135e-04	6.2286e-17	0.0041
9	9.3891e-04	5.2701e-17	0.0032
10	5.0001e-04	1.8344e-17	0.0023

Table 4.4: the L^2 norm of $y - y_r$ of RC-circuit example

r_x	$\|G - G_r\|_\infty$ by BT	$\|y_0 - y_r\|_{L_2}$ BT	$2\sum_{i=r+1}^{10} \sigma_i$
1	3.7714	3.5512e-06	3.8608
2	0.0068	1.1308e-12	0.1533
3	0.0692	6.5834e-13	0.0898
4	0.0028	1.0128e-14	0.0274
5	0.0122	5.3936e-15	0.0169
6	5.6508e-04	1.1517e-16	0.0067
7	0.0022	1.1649e-16	0.0054
8	5.2214e-04	6.2347e-17	0.0041
9	9.3932e-04	5.2856e-17	0.0032
10	5.0018e-04	1.8518e-17	0.0023

Tables (4.3) and (4.4) show the value of $\|G - G_r\|_\infty$ and $\| \quad \|_{L_2}$ bound of the approximation error between the output y and yr of the original and reduced systems and the error bound for SPA and BTM.

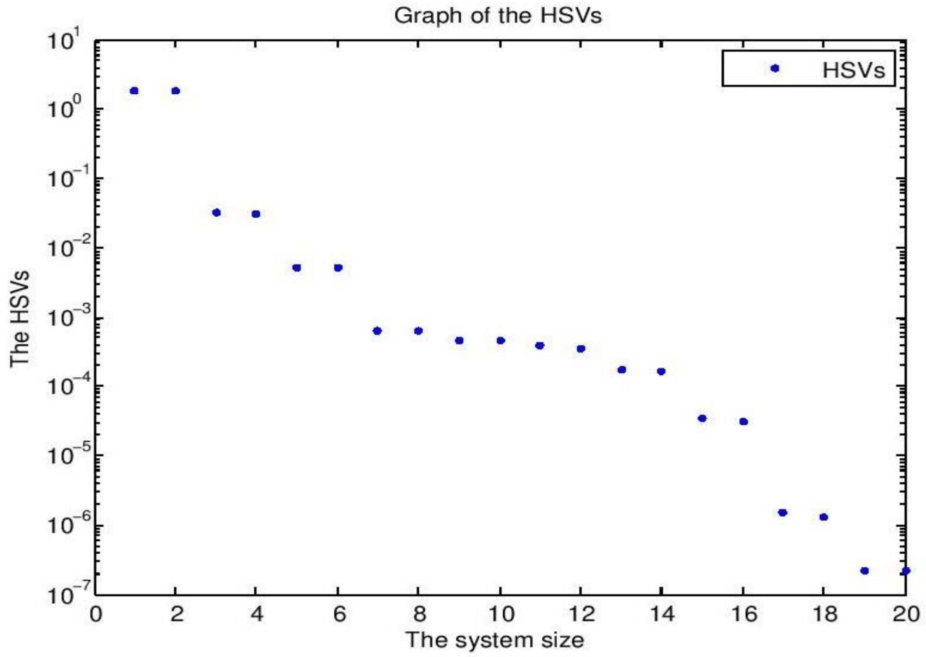

Figure 4.11: HSVs of the second example

Figure (4.11) represents the Hankel singular values of damping's mass-spring system for both SPA and BTM.

Figure (4.12) and (4.14) represents the maximum singular value decomposition (MSVD) of (G – Gr) and the error bound $2\sum_{i=r+1}^{10}\sigma_i$ for SPA and BTM. We see clearly that the balanced truncation method yields a reduced order model with smaller error at high frequencies and larger error at low frequencies.

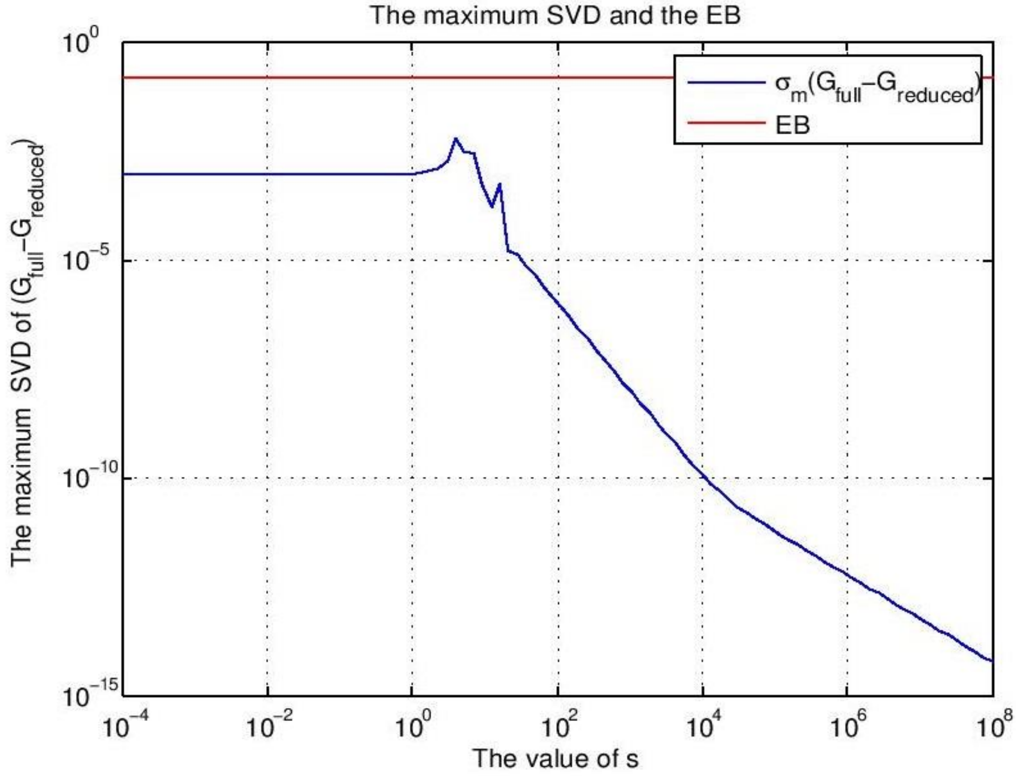

Figure 4.12: The MSVD and the error bound for the second example

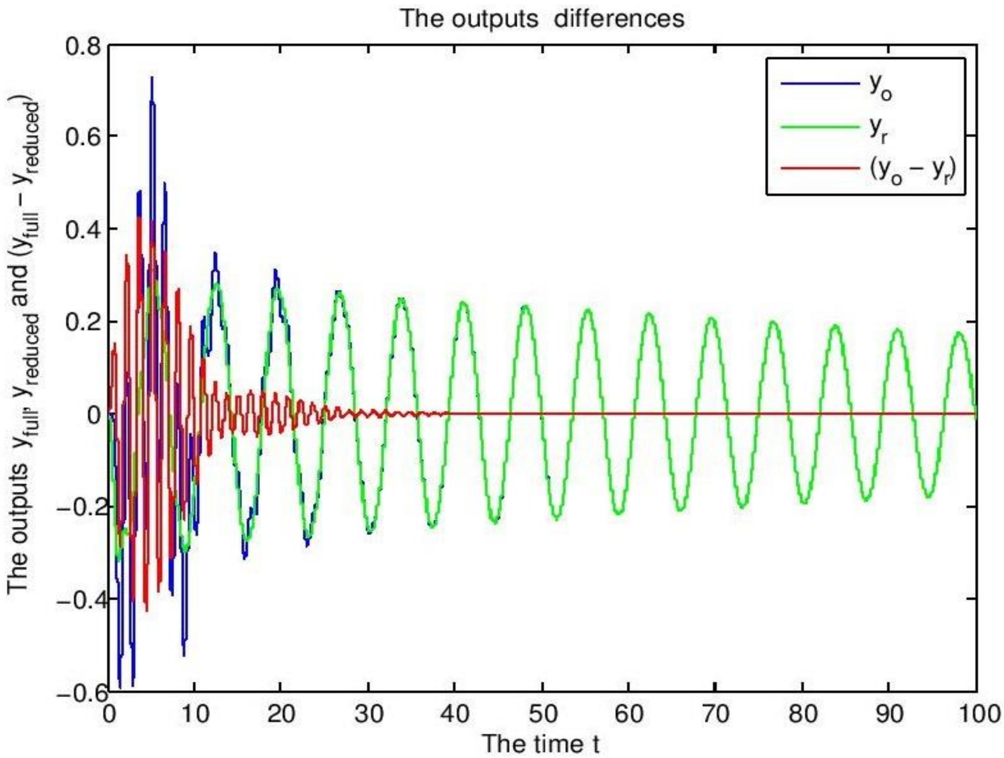

Figure 4.13: The outputs of the second example

Figures (4.13) and (4.15), show the plot of the outputs y and yr of the original and reduced systems respectively and their differences $(y - yr)$ and error bound by applying singular perturbation and balanced truncation method on the numerical example respectively.

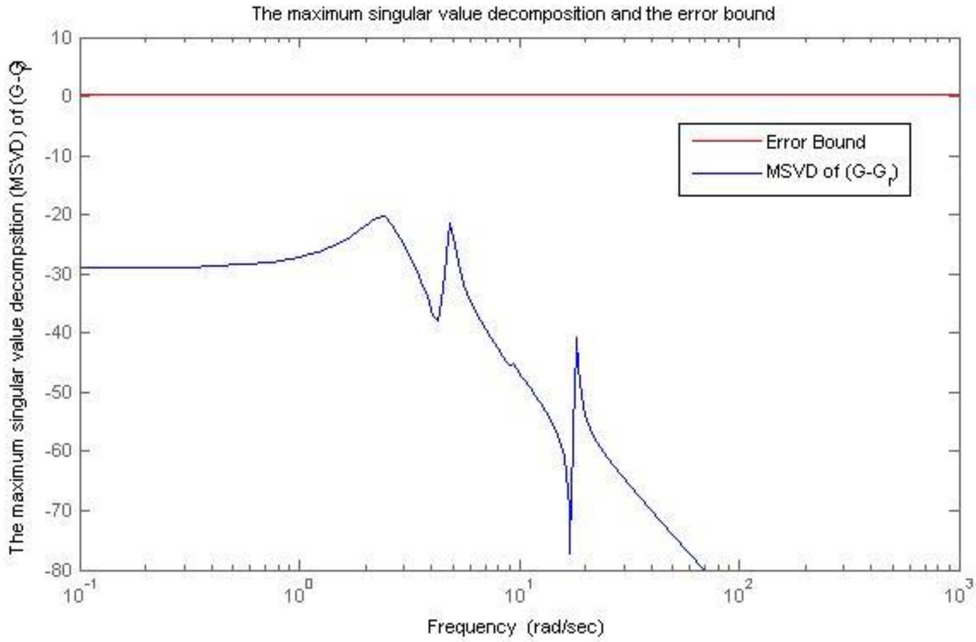

Figure 4.14: The MSVD and the error bound for the second example

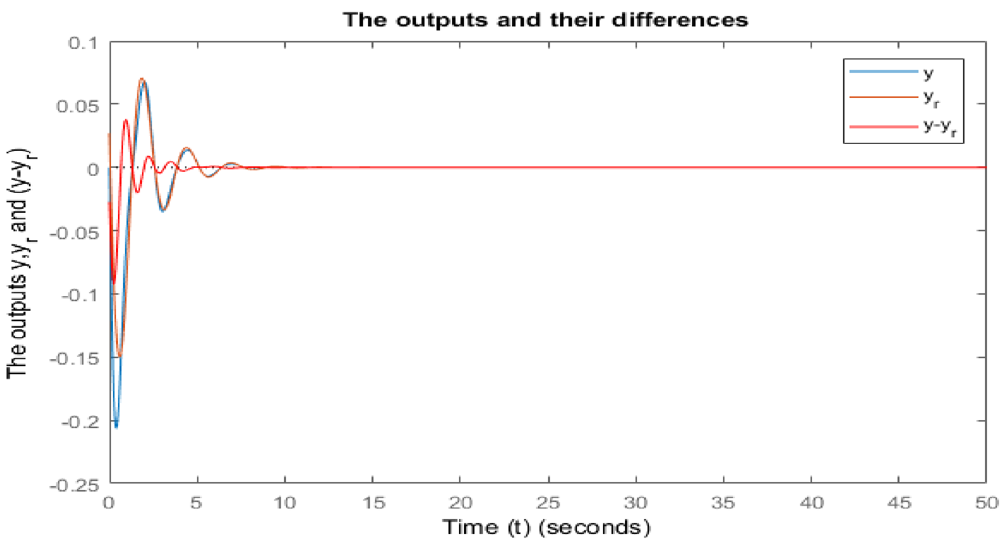

Figure 4.15: The outputs of the second example

Conclusion:

Linear large-scale systems arise in many practical applications, for instance, in circuit simulations and in control problems where the underlying physical process is modeled by partial differential equations.

Two methods namely, Balanced Truncation Method, and Singular Perturbation Approximation implemented to approximate the solution of Linear Time Invariant (LTI) system. Some illustrative examples to demonstrate the validity and applicability of these methods had solved.

The two methods gives the same error bound. For (BTM) the error bound is small at high frequencies and large at low frequencies, but for (SPAM) the error bound is large at high frequencies and small at low frequencies.

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جامعة النجاح الوطنية

كلية الدراسات العليا

طريقة تقريب اضطراب المفرد لنظام ديناميكي ذو مقاييس عالية

إعداد

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إشراف

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قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدراسات العليا، في جامعة النجاح الوطنية، نابلس – فلسطين.

2021

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د. عدنان دراغمة

الملخص

إن نظرية التحكم تعتبر من المواضيع القديمة التي ظهرت مع بداية حياة الإنسان ورغبته في إدارة ما حوله واستغلال موارد الطبيعة، إن دراسة التطبيقات الهندسية والفيزيائية والكيميائية وغيرها يقودنا إلى أنظمة من المعادلات التفاضلية العادية و الجزئية والتي تكون في الغالب ذات رتبة عالية.

لتسهيل التعامل مع هكذا أنظمة ولتشغيلها بأقل تكلفة ممكنة والحصول على أفضل المخرجات نحتاج إلى تقليل رتبتها بحيث نستثني منها المتغيرات الأقل تأثيرا للحصول على نظام جديد ذو رتبة صغيرة بأقل خطأ ممكن.

وقد ظهر في هذا المجال العديد من الأبحاث والدراسات القيمة حيث إننا في هذا البحث ركزنا على حالة النظام الديناميكي الخطي الثابت وحاولنا تخفيض رتبته والحصول على النظام الجديد بأقل خطأ.

في البداية استخدمنا طريقة الاقتطاع الثابت ثم طريقة الاضطراب المفرد لتخفيض رتبة النظام الثابت أو المستقر, ولتوضيح فعالية هذه الطرق قمنا بدراسة أمثلة عددية من واقع التطبيقات الحياتية وطبقنا عليها الطرق التي قمنا بدراستها سابقا , وقد أظهرت النتائج الفرق في نسبة الخطأ عند استخدام هذه الطرق.