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EXISTENCE OF PATTERNS FORMATION OF AN AGGREGATION-DIFFUSION SYSTEM IN 2-D

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Dedication

I would want to start by dedicating this to my beloved country of Palestine and hoping that it will help with its growth and prosperity. To my parents, first and foremost, I must thank my parents for their endless support throughout my life, for from them I learned patience and perseverance, no matter the difficulties.

To my husband for his love and support, which have helped me stay motivated and persevered through difficult moments.

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I dedicate you all to my master’s thesis in pure mathematics.

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Declaration

I, the undersigned, declare that I submitted the thesis entitled:

EXISTENCE OF PATTERNS FORMATION OF AN AGGREGATION-DIFFUSION SYSTEM IN 2-D

I declare that the work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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Abstract

In this thesis, the researcher introduced the aggregation-diffusion equation in two-dimensional space and investigated the existence of steady-states, which contain several states, some of which are trivial and some of which are not. The categories studied numerically using the finite-volume method, then followed by the use of the third-order SSP-RK method, and finally used MATLAB to clarify the state of stability in each case. The categories are divided into three main categories depending on the value of theta, one of which diffusion prevails, while aggregation dominates, and the last one, in which the balance between diffusion and aggregation wins.

Chapter One

Mathematical Preliminaries

1.1 Introduction

The field of partial differential equations (PDEs) is a wide field of diversity and size, and it is one of the most important points of attraction in modern mathematics, especially in engineering [1]and analysis [2], and it has many uses and applications, such as finance [3], insurance[4], physics [1], chemistry [5], biology [6], economic [7] and social [8], etc. You can notice, that most of the challenges facing the world are related in some way to social and economic origins. To solve these challenges and understand their complexities and implications, we need practical descriptive tools to understand, analyze and study them.

The most important application of the partial differential equations in the present and the future is mathematical modeling, which is an important step to move from maths training to application-oriented maths experience, which makes the student or researcher suitable to master the challenges of our modern technological culture. The model helps to explain and study the effects of different components and make predictions about behavior. It can be defined as translating problems from applications into mathematical formulas that can be analyzed theoretically and numerically. Time management is the biggest risk facing the development of a real model.

In the last quarter-century, due to the development of new techniques for collecting biological data, a resort has been made to integrate mathematical and theoretical thinking into biology. Here comes the role of mathematical modeling, which is not only used to validate hypotheses but to design and test new models that lead to testable experimental predictions. It provided new insight into biological systems.

The most important and most famous example in biology is population growth over time [10]. This model was presented using mathews, who used the following law

$$\frac{dP}{dt} = rP, \tag{1.1}$$

which shows how the population P changes over time t, assuming that r is a variable whose change depends on the species. The predator-prey model [11] is a well-known example of a mathematical model that was developed from the Lotka-Volterra model, which was created to analyze the population of fish in the Mediterranean during and after World War II. The model combines biological and ecological systems, and it is a non-linear application of partial differential equations in a simple ecosystem, where it describes the competition between two species living in the same conditions and in the same ecosystem, assuming that the system is closed, that is, it does not allow migration out of the system or the entry of new elements into the system.

Bacterial swarms are an example of a microbiology mathematical model in which cells self-assemble [12]. Swarming occurs when bacteria interact with surfaces and media in their environment, they reprogramme their behavior, motility, and growth while also responding to changes in their new environment by coordinating their existence with other cells. In the colony, where everything does not happen at random but is driven by several factors,

About fifteen years ago, a type of equation spread widely and attracted great mathematical interest, which is the aggregation-diffusion equation, which is considered a type of partial differential equation. This kind of equation explains how systems with diffusing and aggregating particles interact. This equation helped them in explaining many phenomena in many fields such as biology [31, 32], models of granular media [43], social sciences [48], opinion formation modeling [40], and chemotaxis [49], which are all largely driven by the long-range attractive force.

The Keller-Segel model of chemical concentration is included in the aggregation-diffusion equation [13], which describes the mass movement of cells (bacteria or mold) attracted by a self-released chemical (for one or a group of agglomerations). The model is mathematically expressed using the following set of equations

$$\rho_t = \Delta\rho - \nabla \cdot (\rho \nabla c), \tag{1.2}$$

$$\varepsilon c_1 = \Delta c - \alpha c + \rho, \tag{1.3}$$

$\rho(x, t)$ is unknown refers to the population density of the cell colony of a two-dimensional surface subject to Brownian motion and t is time and because of the tendency of these cells to move towards higher concentrations chemically, we have an additional drift velocity

$c(x,t)$.

The Smoluchowski equation [14], which is widely used in physics, chemistry, and materials science, is also another example of an aggregation-diffusion equation. It describes the coagulation or aggregation of tiny particles to form bigger clusters. The formula for the equation is:

$$\frac{\partial C}{\partial t} = D\nabla^{2C} - K\int C(x)C(x-y)dy, \tag{1.4}$$

C(x) denotes the particle concentration at position x, D is the diffusion coefficient, K the coagulation rate constant, and the integral term denote the rate of coagulation between particles separated by distance y. The formula predicts how the concentration of the particles will change over time as they diffuse and aggregate.

What we want to study in our thesis is the steady-state of aggregation-diffusion equations in two-dimensional space, which represents the state of equilibrium, which means that the partial derivative with respect to time is zero for all future time, in other words, that the solution of the process or system at a certain point will stop changing over time.

This thesis is organized as follows: it is divided into four basic chapters, In the first chapter, a simple introduction to partial differential equations was presented, and the system was also presented numerically in a one-dimensional space with little about the background of the study, while the second chapter dealt with the system in a two-dimensional space, and it was analyzed numerically, using the Finite Volume method, and then Matlab in finding the steady-state, as for the third chapter, we dealt with some examples of different situations that may result in the steady-state, and finally, the fourth chapter, we dealt with the difficulties of the study, the conclusions we got, and some suggestions for future work.

1.2 Aggregation-Diffusion equations in one-dimensional space

In this section, we introduce an equation with aggregation and diffusion terms in one-dimensional space which models a type of competition with long-range attractive effects with short-repulsive ones which occur simultaneously. This equation is very used in many fields such as population dynamics [15], chemical reactions [16], biological systems [17], traffic flow [18], phase transition [19], chemotaxis [20], swarming whether it is animals [21] or bacteria [22], material sciences [23], etc. This section is introduced and devoted to presenting the main properties and results of this equation such as the existence and uniqueness of solutions, and the existence of steady-states with their properties. We introduce analytical results as well as a numerical simulation showing the shape of steady-state.

The equation we consider in one-dimensional space [15] is defined as:

$$\partial_t \rho = \partial_x(\rho \partial_x(\theta\rho - S * \rho)), \tag{1.5}$$

where the unknown $\rho = \rho\,(x\,t)$ is non-negative L^1 densities with fixed masses defined on $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, t is the time, θ is a positive constant modelling the diffusion, S is an interaction kernel modelling non-local attractive effects. The symbol $S * \rho$ denotes the convolution of two functions S and ρ, a mathematical technique for combining two functions to create a third function that acts as a sort of combination of the two original functions and is inserted as part of the product of two functions over a given period. And mathematically it is expressed as follows

$$(S * \rho)\,(x) = \int_{\mathbb{R}} S\,(x-y)\,\rho\,(y)\,dy \tag{1.6}$$

where $S : \mathbb{R} \to [0, +\infty)$. The equation (1.5) consists of two terms, the first term is

$$D := \frac{\theta}{2}\partial_{xx}\rho^2, \tag{1.7}$$

called the quadratic diffusion term which models the local repulsive effects with diffusion constant $\theta > 0$, this term is used to model situations where the diffusion rate is not constant. While the second term is

$$A := \partial_x(\rho S' * \rho), \tag{1.8}$$

a nonlocal attractive term, it represents an interaction or an all-pervasive, non-local force that attracts things together (acting at a distance, not just between nearby or neighboring objects), the phenomenon of long-range interactions between particles is modeled using this term.

The aggregation-diffusion equation in one-dimensional space does not have a generic solution that is known but the solution of equation (1.5) can take two ways, the first is that the solution approaches the steady state, or follows the other, that it decays to zero when the time is very large. In other words, more accurately, if the energy E is finite, then the solution decays to zero in L^2 and almost everywhere if θ is greater than 1 and x in when the time approaches infinity. While if the support for the steady-state is in the region where the kernel S is convex and assuming the support for the initial datum is very close to the steady-state, then for any time the solution is close to the steady-state.

The existence and uniqueness of solutions depend on the existence and uniqueness of the entropy solutions of the equation and this is proven in several ways, the most important of which is developed by Ambrosio, which is the theory of graduated flows in the Wasserstein scale, for the proof and more details we refer the reader to [], provided that the existence and uniqueness of the solutions are in the bounded convex field.

The solution of the aggregation-diffusion equation in one-dimensional space will also exist and be unique in the long-range if the initial conditions and limits are suitable, such as being smooth, finite, and with suitable decay.

1.2.1 Some analytical results about the solution of equation(1.5)

We'd like to present a quick rundown of the analytical conclusions of the aggregation diffusion equations, which describe the conditions that ensure the presence, existence, or nonexistence of steady-states, and long-range behavior of solutions. Consider the stationary version of the equation (1.5) as follows to achieve all of the above

$$0 = (\rho(\theta\rho - S * \rho)_x)_x, \tag{1.9}$$

this equation is a second-order ODE. And in a simpler expression

$$0 = \frac{\theta}{2}\partial_{xx}\rho^2 - \partial_x(\rho S' * \rho), \tag{1.10}$$

We'll first make a few assumptions that will make it simpler to find steady-state. One of these assumptions is non-negativity. Because system applications demanded it, this assumption was made, which means that $\rho \geq 0$.

The distribution of particles or species across the entire research region being equal is one of the most important assumptions we wish to accept, and this is the case when t=0.

The second assumption is that the total mass is conserved in time; that is, $\int \rho(x,t)dx$ remains constant during the time evolution. As a result, equilibria can be considered as having a fixed mass, which is determined by the initial configuration.

Proof. To prove that the total mass is conserved in time in one-dimensional space for the aggregation-diffusion equation, we need to show that the total mass remains constant as time evolves, i.e., the integral of the density field $\rho(x,t)$ over the entire space remains constant. Suppose that

$$\frac{\partial \rho}{\partial t} = \partial_x(\rho \partial_x(\theta\rho - S * \rho)), \tag{1.11}$$

where θ is the diffusion constant and $S * \rho$ is the convolution of the density ρ with the aggregation kernel S.

We multiply both sides of the equation by x and integrate over the entire domain:

$$\int_{-\infty}^{\infty} x\frac{\partial \rho}{\partial t}dx = \int_{-\infty}^{\infty} x\partial_x(\rho\partial_x(\theta\rho - S * \rho))dx,$$

We apply integration by parts to the right-hand side:

$$\int_{-\infty}^{\infty} x\frac{\partial \rho}{\partial t}dx = [x\rho\partial_x(\theta\rho - S * \rho)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(\theta\rho - S * \rho)\partial_x(x\rho)dx,$$

The first term on the right-hand side evaluates to zero because ρ and its derivative are assumed to approach zero at infinity. We are left with:

$$\int_{-\infty}^{\infty} x\frac{\partial \rho}{\partial t}dx = \int_{-\infty}^{\infty}(\theta\rho - S * \rho)\rho dx,$$

Taking the time derivative of the left-hand side and substituting in the original equation, we obtain:

$$\frac{d}{dt}\int_{-\infty}^{\infty} x\rho dx = \int_{-\infty}^{\infty}(\theta\rho - S * \rho)\rho dx,$$

This shows that the time derivative of the total mass of the system is equal to the net production of mass due to diffusion and aggregation. Since the right-hand side is zero in the steady state, the total mass is conserved in time.

Therefore, the total mass is conserved in time in the aggregation-diffusion equation, regardless of the diffusion constant θ and the aggregation kernel S. □

Finally, for the sake of simplicity, we'll assume Smooth for the solution to $\rho(x)$, which means $\rho(x)$ can be differentiated anywhere.
All of the previous assumptions were made in order for the equation to yield a correct and precise result. Let's move on to the assumptions for S. Let S interaction potential which is a mathematical function that describes the interactions between individuals within a population or species and a well-known example of the Lennard-Jones potential is used to describe the interactions between atoms and molecules, $S : \mathbb{R}^d \to [0, +\infty)$ satisfying the following assumptions:

- $S \geq 0$,and **supp**$(S) = \mathbb{R}$.
- $S \in L^1(\mathbb{R}) \bigcap L^\infty(\mathbb{R}) \bigcap C^2(\mathbb{R})$.
- $S(x) = S(-x)$ for all $x \in \mathbb{R}$.
- $S''(x) < -c < 0$ on $[-\lambda, \lambda]$ for some $\lambda, c > 0$.
- $S'(x) < 0$ for all $x > 0$.

It's worth noticing that each of the previous assumptions is essential in something which is explained as follows: The first assumption suggests that the term non-local drift in (1.5) is attractive, as it leads to a reduction in the time of all moments. This is a crucial assumption that leads to the conclusion that the support of a steady-state is connected. The second assumption is about the properties of regularities on S that are required to pass derivatives inside the integral operator $S * \rho$, assumption 3 is about symmetry (even function), and assumptions 4 and 5 are about the kernel S being perfectly attractive and interactions decaying at infinity.
For simplicity we will assume that

$$\|S\|_{L^1(\mathbb{R})} = 1 \tag{1.12}$$

this is not a restrictive assumption since the Kernel S can always be normalized by adjusting the diffusion constant and time scale as follows

$$\partial_\tau \rho = \rho(\acute{\theta}\rho - \tilde{S} * \rho)_x, \quad \tau = \|S\|_{L^1(\mathbb{R})}\,, \tag{1.13}$$

and

$$\tilde{S} = S/\left\|S\right\|_{L^1(\mathbb{R})}, \quad \acute{\theta} = \theta/\left\|S\right\|_{L^1(\mathbb{R})}, \tag{1.14}$$

Example 1.2.1. A good example of aggregation kernel fulfilling the above assumptions is

$$S(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$$

it's called the Gaussian Kernel. If we stick to the previous assumptions, we'll see that S is always positive. $\mathbb{R}$ is the smallest closed set with all non-zero points. As a result, the support for S is $\mathbb{R}$, confirming the first assumption.
We can see that S is an even function($S(x) = S(-x)$), confirming the symmetry assumption. S' is the derivative of S which $S')(x) = \frac{-2x}{\sqrt{\pi}}e^{-x^2}$. When we examine its sign, we can see that it is always negative. This leads us to the last condition, which is that the function is decreasing. See Appendix A: Figure 1.

Example 1.2.2. let us take the interaction potential kernel as

$$S(x) = \frac{1}{2}e^{-|x|},$$

if we check the assumptions of S we can see that this example does not fulfill the assumptions for S so we can't consider it as an aggregation kernel. See Appendix A: Figure 2.

Before talking about the steady-state of equation (1.5) we want to define a tool. This tool is one of the most important tools in studying our equation. It is called energy function also known as entropy defined as

$$E(\rho) = \frac{\theta}{2}\int_{\mathbb{R}} \rho^2 dx - \frac{1}{2}\int_{\mathbb{R}} \rho S * \rho dx, \tag{1.15}$$

The diffusion of particles is represented by the first component of the energy function, while the aggregation and attraction of particles are represented by the second part. It is a measurement of the spatial expression of particle concentration and aggregation force.
And another thing we want to define is the next space, which we will use in many places when defining the steady-state

$$\mathcal{P} = \{\rho \in L^1_+(\mathbb{R}) : \int_{\mathbb{R}} \rho(x)dx = 1\}, \tag{1.16}$$

In terms of the steady-state, the first step in finding it is to look for evidence of its existence, which is crucial and necessary. Since the basic idea is to find a non-local minimizer, we will use the standard method, which is the direct method for calculating variances. If we take the minimizing sequence and trace it, this shows that it converges until later and establishes that half of the continuity of the function decreases along this sequence. Depending on the value of the integral for S, the non-local minimizer's existence or non-existence is determined. Specifically, a non-local minimizer exists if and only if the integral of S is finite, for any mass.

The steady-state of the aggregation-diffusion equation in one-dimensional space is a stable solution in which the concentration of particles is constant in time.

In the following we have three different situations for the existence of steady-states depending on the value of θ and S where S is non-negative and integrable, We want to mention each case separately with the theories and the proofs that follow it.

Now, we want to start with the most important case that is useful to us in our work, which is $0 < \theta < \|S\|_{L^1(\mathbb{R})}$. In this case, there exists a non-trivial $L^2 \bigcap \mathcal{P}$ steady-state this means that the functional E has a non-local minimizer in $L^1(\mathbb{R}^d)$ under the constraint $\int \rho dx = M > 0$.

The following theorem is the first theorem that we will start with, which is related to the above case, which is $\theta < \|S\|_{L^1(\mathbb{R})}$.

Theorem 1.2.1. Let $\theta < \|S\|_{L^1(\mathbb{R})}$. Then, there exists a radially symmetric non-increasing minimizer $\rho \in L^2 \bigcap \mathcal{P}$ for the entropy functional E restricted to $\mathcal{P}$ with $\rho \neq 0$.[29]

The second case that we will mention is $\theta \geq \|S\|_{L^1(\mathbb{R})}$ this case does not have any non-trivial steady-state but it has a trivial steady-state $\rho \equiv 0$. In other words, there is no existence of a non-local minimizer in $L^1(\mathbb{R}^d)$ under the constraint $\int \rho dx = M > 0$, In this case, intermolecular attraction is weak and diffusion dominates the system.

The following lemma is specific to a part of the above case, which is $\theta > \|S\|_{L^1(\mathbb{R})}$

Lemma 1.2.1. Let $\theta > \|S\|_{L^1(\mathbb{R})}$. Then, there exists no steady-state in the space $L^2 \bigcap \mathcal{P}$.[29]

Proof. first, we will prove that there is no minimizer for $E(\rho)$ under the mass constraint $\int \rho = 1$ and $\rho \geq 0$, by using the young inequality we get

$$E(\rho) = \frac{\theta}{2}\int \rho^2 dx - \frac{1}{2}\int \rho S{*}\rho dx \geq \frac{\theta}{2}\int \rho^2 dx - \frac{\|S\|_{L^1}}{2}\int \rho^2 dx = \frac{\theta-1}{2}\int \rho^2 dx, \quad (1.17)$$

since $\theta > \|S\|_{L^1(\mathbb{R})}$ and $\|S\|_{L^1(\mathbb{R})} = 1$ we conclude $\theta - 1 > 0$ $E(\rho) \leq \mathrm{C}\,\|\rho\|^2_{L^2}$. Let us take a family of functions $\rho_\lambda(x) \geq 0$ such that

$$\int \rho_\lambda(x)dx = 1 \quad \text{and} \quad \int \rho^2_\lambda(x)dx \to 0 \quad \text{and} \quad \lambda \to +\infty,$$

Let us take a fixed $L^2(\mathbb{R})$ function $\rho \not\equiv 0$ to construct a family, then rescale it by

$$\rho_\lambda(x) = \lambda^{-d}\rho(\lambda^{-1}x)$$

For such a family we therefore have

$$E(\rho_\lambda) \to 0 \qquad \text{as} \qquad \lambda \to +\infty.$$

Therefore, it is impossible to have a minimizer ρ_∞ for $E(\rho)$ in the set $\{\rho \in L^1_+ : \int \rho = 1\}$ because (1.17) imply that $E(\rho_\infty) > 0$ and we would necessarily have $0 < E(\rho_\lambda) < E(\rho_\infty)$ for λ large enough.
Now we will prove in this case there is no steady-state. Then, ρ is a steady point for E. Referring to Lemma [3.2] in paper[] implies that E is strong convex which leads to a single fixed point with a non-local minimizer. which is a contradiction.

□

The following theorem is concerned with the other part of the above case and it is as follows:

Theorem 1.2.2. (Non-existence of nontrivial steady states for $\theta = 1$). There exists no solution to (1.9) in $L^2 \bigcap \mathcal{P}$.[29]

We do not want here to mention the complete proof of the theorem, but we will mention the sketch of the proof, in the first we use the Cauchy-Schwarz inequality, and by applying it to our equation, we get

$$\int_{\mathbb{R}} \rho S * \rho dx \leq \|\rho\|_{L^2(\mathbb{R}^d)} \|S * \rho\|_{L^2(\mathbb{R})} \quad (1.18)$$

then use the previous lemma [1.2.1] .

And the last case we want to mention is that $\theta = 0$ this case leads us to the aggregation equation and the solution for this equation is a Dirac delta solution centered at the initial center of mass as $t \to +\infty$, As a result, the diffusion rate is negligible in respect to the aggregation rate, and the diffusion coefficient is nonexistent.

The non-trivial steady-state which exists in the first case above ($0 < \theta < \|S\|_{L^1(\mathbb{R})}$) for(1.5)in one-dimensional space has general properties, including:

The uniqueness of the steady state of the equation is its most important property; it is dependent on the system characteristics and initial conditions.

In other words, there exists a unique weak steady state with unite mass and zero centers of mass for the equation (1.5) in $L^2 \bigcap \mathcal{P}$.

Theorem 1.2.3. Let ρ_0 have a connected compact support and be smooth positive. Assume S satisfies the assumptions. Then there exists a unique weak solution ρ to (1.1) with

1. $E[\rho(t)] < +\infty$ for all $t \geq 0$.
2. $\sqrt{\rho}\nabla(\theta\rho - S * \rho) \in L^2([0,T] \times \mathbb{R}^2)$ for all $T > 0$.

such that the following energy identity is satisfied [29]

$$E[(t)] + \int_0^T \int_{\mathbb{R}} \rho\nabla \left|\theta\rho - S * \rho\right|^2 dxdt = E[\rho_0].$$

The steady-state for the equation (1.5) in one-dimensional space has a regularizing effect, this means that the steady-state can approximate to a regular series function over time to infinity in $L^2 \bigcap \mathcal{P}$ this means that the steady stats satisfies

$$\int \rho \left|\nabla\rho\right|^2 dx < +\infty \tag{1.19}$$

for every $t > 0$, this lead us to $\rho \in \mathrm{C}^2$on supp $[\rho]$,

The organizing effect results from the interaction of the terms diffusion, which tends to calm the solution, and aggregation, which tends to form compact groups.

One-dimensional space steady-states have connected support (i.e., supp(ρ) is a connected set), which means we cannot divide supp(ρ) into two nonempty subsets such that each subset has no points in common with the set closure of the other. In other words, the

solution exists and is non-zero for the spatial variable over a continuous time interval. To obtain the connected support, the equation's diffusion and aggregation terms must be in equilibrium.
The steady-states in one-dimensional space are symmetric this indicates that the solution has not changed as a result of reflection or reflection with regard to the center of the spatial field, this implies mathematically that

$$E(\tilde{\rho}) = E(\rho), \tag{1.20}$$

that means $\tilde{\rho}$ and ρ have the same energy.
Concavity of ρ for small θ is also a property of steady-state in one-dimensional space this means that the steady-state for all θ is concave on the whole interval for $\theta \in (0, \theta_0)$ where $\theta_0 \in (0, 1)$, in a more straightforward manner, all points in the spatial domain have negative values for the second derivative of the function ρ. The concavity of the ρ is caused by an equilibrium between the diffusion and aggregation conditions in the equation.
ρ has a multiplicity of behavior for a fixed θ. That means if we set all the values for all variables except θ, we can get completely different solutions.
The last property we want to indicate that is the support of ρ is compact. This property shows us that the support can only be closed and bounded intervals. This indicates that the range of possible aggregation patterns can be restricted by the compact support because they are defined over a certain period of time and are closed in a one-dimensional space. If the kernel and the initial state meet specific criteria, compact support can be used to prove the existence and uniqueness of a steady-state.
It is very important to be careful when choosing the numerical method to be used in solving the aggregation-diffusion equations due to its sensitivity. There are many numerical methods, including the finite-volume method, as well as the finite element method and other methods, and given that the solutions are available as long as the numerical method used is suitable, accurate, and stable.
In the next section, we want to use the finite-volume method on the aggregation-diffusion equations in a one-dimensional space, and then we want to use MATLAB to apply it numerically to our equation for the steady-state observation.

1.2.2 Finite Volume Method

In this section, we will study numerically the diffusion-collection equations in one - dimensional space, and here we will adopt the finite-volume method (FVM) [24] [25] to study the diffusion-collection equations which are a kind of partial differential equations, especially the equations preheating This method can be applied to a very wide range describing conservation laws, which include aggregation and diffusion equations.

The integrated conservation law is the basis of the finite-volume method, and the idea begins with dividing the field into a limited number of separate cells and then approximating the integrated conservation law on separated cells separately.

The use of this method has many advantages, including the ability to use it to solve a variety of equations, which means that it is flexible, including the ability to use it in time-dependent steady-state problems, and can also be used for all structural and unstructured networks. Since it also contains a set of algebraic equations, one of its benefits is that it is simple to implement. This technique gives us exceptional accuracy and is simple to adapt to complex geometric shapes. In addition, it may be used in multidimensional space.

Despite the many advantages of the finite-volume method, it has some disadvantages, One of these is that it depends on the network, making it sensitive to the network's selection and the size of the control units. One of its drawbacks is that it could take a while, especially for complicated geometric patterns, and that occasionally accuracy might be compromised by the numerical spread.

It nevertheless remains a widely used method in spite of its disadvantages because of its flexibility, strength, and simplicity of use. We need to follow a series of steps in order to apply it to the aggregation-diffusion equation in a one-dimensional space.

In the first step of this method, the domain is partitioned into cells of finite volume as follows: $U_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ of a uniform size Δx with $x_i = i\Delta x$ where $i \in \{-M, ..., M\}$.
Let

$$\tilde{\rho}_i(t) = \frac{1}{\Delta x}\int_{U_i} \rho(x,t)dx, \tag{1.21}$$

be the averages solution ρ computed at each cell U_i(Want to assume it is known at $t \geq 0$). A semi-discrete finite volume scheme is obtained by integrating equation 3 over each cell U_i and is given by the following system of ODEs for $\tilde{\rho}_i$

$$\frac{d\tilde{\rho}_i(t)}{dt} = -\frac{F_{i+\frac{1}{2}}(t) - F_{i-\frac{1}{2}}(t)}{\Delta x}, \tag{1.22}$$

where the numerical flux $F_{i+\frac{1}{2}}$ and $F_{i-\frac{1}{2}}$ which are a major component of the finite-volume method which are considered as an approximation for the continuous fluxes $-\rho(\theta\rho - S * \rho)_x$ at a cell interface $x_{i+\frac{1}{2}}$, where $F_{i+\frac{1}{2}}(t)$ and $F_{i-\frac{1}{2}}(t)$ are computed as a following

$$F_{i+\frac{1}{2}}(t) = \max(u_{i+1}, 0)\left[\tilde{\rho}_i + \frac{\Delta x}{2}(\rho_x)_i\right] + \min(u_{i+1}, 0)\left[\tilde{\rho}_i - \frac{\Delta x}{2}(\rho_x)_i\right], \tag{1.23}$$

and

$$F_{i-\frac{1}{2}}(t) = \max(u_{i-1}, 0)\left[\tilde{\rho}_i + \frac{\Delta x}{2}(\rho_x)_i\right] + \min(u_{i-1}, 0)\left[\tilde{\rho}_i - \frac{\Delta x}{2}(\rho_x)_i\right], \tag{1.24}$$

where

$$u_{i+1} = \sum_j \tilde{\rho}_j(S(x_{i+1} - x_j) - S(x_i - x_j)) - \frac{\theta}{\Delta x}(\tilde{\rho}_{i+1} - \rho_i), \tag{1.25}$$

and

$$u_{i-1} = \sum_j \tilde{\rho}_j(S(x_i - x_j) - S(x_{i-1} - x_j)) - \frac{\theta}{\Delta x}(\tilde{\rho}_i - \rho_{i-1}), \tag{1.26}$$

where u_{i+1} and u_{i-1} are the discrete values of the velocities and $(\rho_x)_i$ represented the corresponding slope and we want to recalculate it by using the minmod, it is defined as follows

$$(\rho_x)_i = \text{minmod}(2\frac{\tilde{\rho}_{i+1} - \tilde{\rho}_i}{\Delta x}, \frac{\tilde{\rho}_{i+1} - \tilde{\rho}_{i-1}}{2\Delta x}, 2\frac{\tilde{\rho}_i - \tilde{\rho}_{i-1}}{\Delta x}), \tag{1.27}$$

where the definition of minmod limiter [26] is as follows

$$\text{minmod}\,(a_1, a_2, a_3) = \begin{cases} \min\,(a_1, a_2, a_3) & \text{, if } a_i > 0\ \forall i \\ \max\,(a_1, a_2, a_3) & \text{, if } a_i < 0\ \forall i \\ 0 & \text{, otherwise.} \end{cases} \tag{1.28}$$

The Minimod Limiter is a type of regression determinant of the finite-volume method. It is usually used to solve partial differential equations.
The final step is to integrate the semi-discrete scheme in equation (1.2) numerically by using the third-order preserving Runge-Kutta (SSP-RK) ODE solver used in [27].
As this method allows for longer time steps than the explicit traditional methods while keeping stability, the goal of employing it is to develop a stable and effective numerical method. The method is given by the following equations

$$K_1 = \rho_0^n(x) + \Delta t F(\rho_0), \tag{1.29}$$

$$K_2 = \frac{3}{4}\rho_0^n(x) + \frac{1}{4}K_1 + \frac{1}{4}\Delta t F(K_1), \tag{1.30}$$

$$K_3 = \frac{1}{3}\rho_0^n(x) + \frac{2}{3}K_2 + \frac{2}{3}\Delta t F(K_2), \tag{1.31}$$

where ρ_0 is the initial condition which we want to start from it and F is the main function in the numerical method where we approximate the continuous fluxes $-\rho(\theta\rho - S * \rho)_x$ that given in (1.22), this equation system operates with respect to time, where (n) is the system's time step.
The following are a few instances where the aggregation-diffusion equations in a one-dimensional space can achieve steady-states, some of which are non-trivial and some of which are trivial, depending on the values of the variables in them.

Example 1.2.3. This is the first example that we want to take on the steady state of the aggregation-diffusion equation in a one-dimensional space, consider the initial condition as a constant function as

$$\rho_0(x) = 1,$$

we must tread carefully when choosing the initial condition because we first assumed that the area under the curve is equal to one. The limits of the initial condition in this example are -0.5 and -0.5. and consider the interaction potential kernel is the Gaussian kernel which is defined as

$$S(x) = \frac{1}{\sqrt{\pi}} e^{-x^2},$$

defined on the interval $x \in [-0.2, 0.2]$ with diffusion constant $\theta = .002$ and the number of particles $N = 71$. see figure 8 at top teft to see the finite volume steps and the steady-state for this example.

First, you can notice that the steady-state in this example is non-trivial because the diffusion coefficient is equal to 0.1, which belongs to period $]0, 1[$. This case describes an equilibrium between aggregation and diffusion. The shape of the steady state is similar to a symmetrical bell. Also, the steady state is symmetrical around $x = 0$.

Example 1.2.4. Let us in this example take

$$\rho_0(x) = 1,$$

and we want to take the Gaussian kernel

$$S(x) = \frac{1}{\sqrt{\pi}} e^{-x^2},$$

on the interval $x \in [-1, 1]$ and for $\theta = 0$ This case belong to case where the steady-state is trivial, you can see that when the time increasing the steady-state converges to Dirac-Delta solution which is centered at zero. see 8 at top right.

Example 1.2.5. In this example, we want to take all the same values as in the previous example while we want to change the value of the initial condition. let us take the initial as the following

$$\rho_0(x) = \frac{21}{8}(1 - \frac{49}{9}x^2),$$

and we want to take the Gaussian kernel

$$S(x) = \frac{1}{\sqrt{\pi}} e^{-x^2},$$

defined on the interval $x \in [-0.4, 0.4]$ and for the same diffusion constant $\theta = .1$ with same number of particles $N = 71$. see 8 at mid left.

Example 1.2.6. In this example, we take the initial condition as a quadratic function

$$\rho_0(x) = \frac{93}{8}(1 - \frac{961}{4}x^2),$$

and the interaction Kernel as

$$S(x) = \frac{1}{\sqrt{\pi}}e^{-x^2},$$

a Gaussian kernel which defied on $[-0.35, 0.35]$ and for the diffusion consant $\theta = 0.001$ with the number of particles $N = 71$. 8 at mid right. In this example, you can see that the steady-state gradually descends to the bottom while conserving the mass.

Example 1.2.7. The following example shows a case where we have a trivial solution. let us take the initial condition as a quadratic polynomial which is

$$\rho_0(x) = x^2,$$

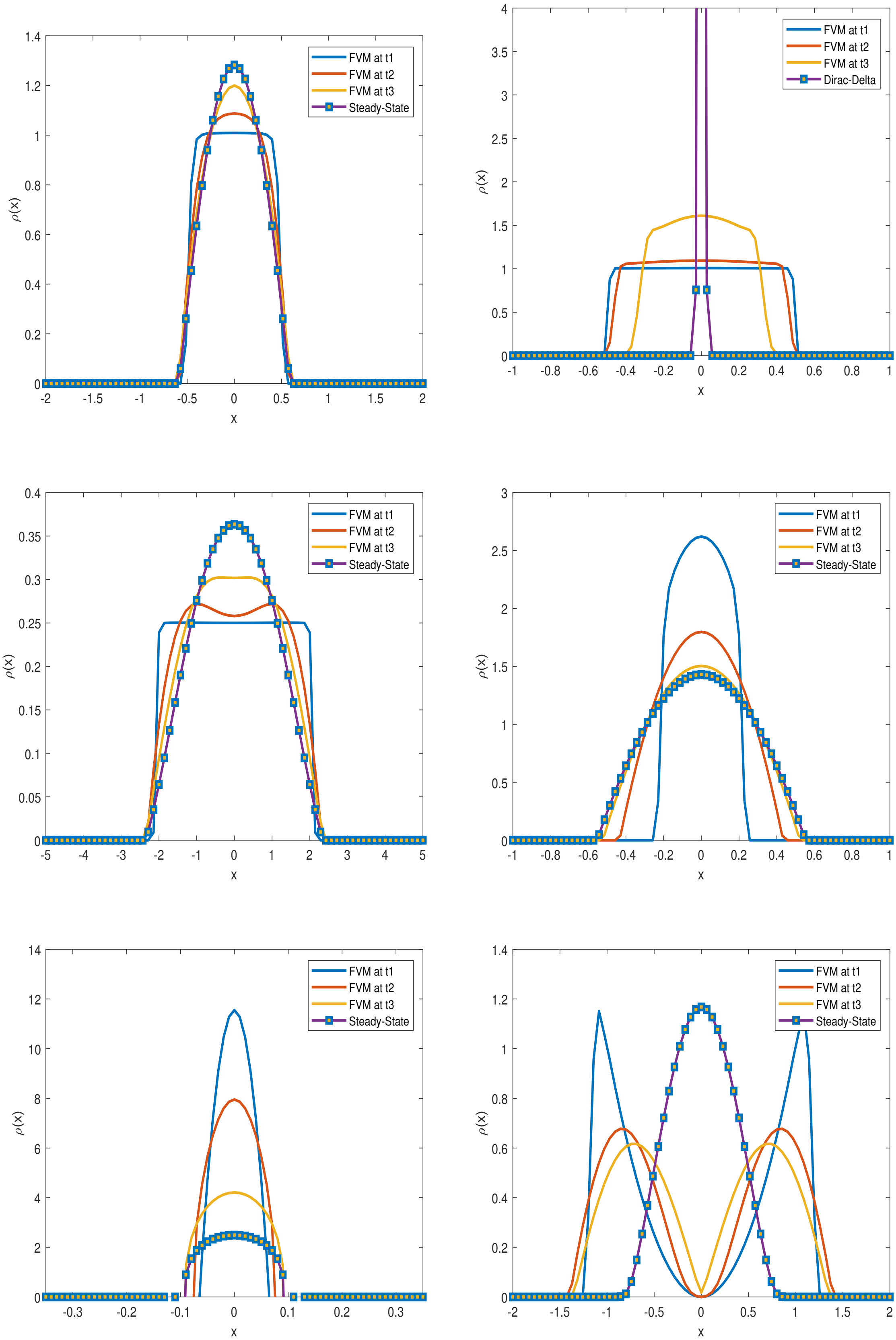
and we want to take the Gaussian kernel

$$S(x) = \frac{1}{\sqrt{\pi}}e^{-x^2},$$

on the interval $x \in [-2, 2]$ and for $\theta = 0.1$ with number of particles $N = 71$
In this example, the initial state was concave up and after passing time the equation reached to steady-state and the steady-state is appear like a bell. see 8 end one.

Figure 1

Various steady-state examples of aggregation-diffusion equations in one-dimension



Chapter Two

Numerical simulations of the steady-states in two-dimensional space

2.1 Steady-States Solution in two-dimensional space

The aggregation-diffusion equation in two dimensions describes the concentration of a scalar quantity in a two-dimensional field under the impact of the combined processes of diffusion and aggregation. The coordinated movement of robots and drones in a two-dimensional environment, the spread of illness and pollution, and many other disciplines use the aggregation-diffusion equations. These disciplines include ecology, computer science, engineering, materials science, epidemiology, and robotics. This kind of equation considers the mechanisms of aggregation and dissemination as well as the spatial and temporal changes in a population or quantity. It can be applied to population behavior or future projection.

The main purpose of this chapter is to study and analyze the two-dimensional aggregation-diffusion equations numerically using the finite volume method, which was previously utilized to solve the one-dimensional aggregation-diffusion equations. Our strategy is to investigate the existence of a steady-state equation in two-dimensional space using MATLAB code. Then we want to compare the results with the ones obtained in one-dimensional space.

The following is the result of rewriting the equation in two-dimensional space

$$\partial_t \rho = \partial_x(\rho \partial_x(\theta \rho - S * \rho)) + \partial_y(\rho \partial_y(\theta \rho - S * \rho)), \tag{2.1}$$

where $\rho = \rho(x_1, x_2, t)$ is unknown non-negative L^1 densities with fixed masses defined on $(x, y, t) \in^2 \times^+$, θ positive constant modeling the diffusion, S is the interaction kernel modeling non-local attractive and non-local repulsive effects. The notation $S * \rho$ is the convolution of the two functions S and ρ in two-dimensional space

$$(S * \rho)(x, y) = \int_{\mathbb{R}} S(x - x_1, y - y_1)\rho(x_1, y_1, t)dx_1 dy_1, \tag{2.2}$$

where $S : \times\mathbb{R} \to [0, +\infty) \times [0, +\infty)$.There are two portions to the equation: one for describing motion in the x-dimension $\partial_x(\rho\partial_x(\theta\rho - S * \rho))$ and the other for describing motion in the y-dimension $\partial_y(\rho\partial_y(\theta\rho - S * \rho))$. The first portion of each of these two parts represents a local force of attraction effect, whereas the second part describes a local repulsive effect.

In the aggregation-diffusion equation in a two-dimensional space, the energy function known as (entropy) is similar to that in a one-dimensional space but with the addition of the second spatial dimension, and it may be expressed mathematically as follows

$$E(\rho) = \frac{\theta}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho^2 dxdy - \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho S * \rho dxdy, \tag{2.3}$$

The energy function is used to study the dynamics of a group of particles that move and aggregate based on the balance between diffusion and gravity. Understanding the system's characteristics, such as its stability and equilibrium states, is achievable. By determining the optimal geographic distribution of particles, developing methods for managing them, and predicting their behavior, it can be used as a tool for system optimization.

To get the equation's steady-state in two-dimensional space we must solve and study the following equation

$$0 = (\rho(\theta\rho - S * \rho)_x)_x + (\rho(\theta\rho - S * \rho)_y)_y, \tag{2.4}$$

As in the case of the equation in one-dimensional space, we want to set some conditions and assumptions for the equation in two-dimensional space, in order to simplify finding a solution to the equation.

One of the basic assumptions of this equation is non-negativity (positivity), which states that the population density at any time or place cannot be negative. And because the solution is financially important and that it clarifies the reality, it is necessary to ensure that the positive equation is achieved. Mathematically, this means that

$$\rho(x, y, t) \geq 0. \tag{2.5}$$

One of the important assumptions also in the aggregation-diffusion equation in a two-dimensional space is the conservation of the total mass. In other words, the total mass of the population, which represents the integration of density over all space, is preserved over time. It can be mathematically expressed as follows

$$\frac{\partial}{\partial t}\int \rho(x,y,t) = 0, \tag{2.6}$$

where $\rho(x, y, t)$ denotes the density at time t and position x and y . This assumption is important because it ensures that the total area of the population remains constant over time. The aggregation-diffusion equation can lead to nonphysical solutions where population density expands infinitely or disappears over time if the conservatism of the total mass assumption is not met.

Making this assumption for solving (2.9) is essential, because the diffusion term in the aggregation-diffusion equation requires the presence of second derivatives of the density, and this is valid only in the assumption of smoothness. Therefore, we assume that the density function $\rho(x, y)$ is a smooth function of space. In other words, the first and second derivatives are continuous in both directions x and y. The aggregation-diffusion equation could not have well-defined solutions or might not accurately depict how the population density changes over time if this smoothness assumption is not satisfied.

Finally, the initial density function $\rho(x, y)$ is often assumed to be a bounded and continuous function in the aggregation diffusion equation. These assumptions of boundedness and continuity are important for several reasons. First, a bounded density ensures that the total mass of the density is finite and can be conserved over time. Second, the continuity of the density function is necessary for the well-posedness of the diffusion equation. The diffusion term in the equation requires the existence of continuous first and second derivatives of the density function. Without this continuity assumption, the diffusion term may not be well-defined over time.

For simplicity and to ensure integrability we want to assume that the volume under the initial condition is equal to 1 as follows

$$\int_{-a}^{a}\int_{-a}^{a} \rho(x,y)dxdy = 1. \tag{2.7}$$

In contrast, the current assumptions are for the Interaction Potential Kernel S in two dimensions. Let $S(x, y)$ be the Interaction Potential Kernel and $S : \mathbb{R} \times \mathbb{R} \to [0, +\infty) \times [0, +\infty)$. The assumptions are as follows:

- $S(x, y) \geq 0$,and $\mathbf{supp}(S) = \mathbb{R} \times \mathbb{R}$.
- $S(x, y) \in L^1(\mathbb{R} \times \mathbb{R}) \bigcap L^\infty(\mathbb{R} \times \mathbb{R}) \bigcap C^2(\mathbb{R} \times \mathbb{R})$.

- $S(x, y) = S(-x, -y)$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$.

- $S''(x, y) < -c < 0$ on $[-\lambda, \lambda] \times [-\lambda, \lambda]$ for some $\lambda, c > 0$.

- $S'(x, y) < 0$ for all $x, y > 0$.

Each of these assumptions has some usefulness in finding steady-states for aggregation-diffusion equations in two-dimensional space. For example, the first assumption claims that the non-local drift term in (2.1) is attractive because it results in the time reduction of all moments. This is a critical assumption that leads to the conclusion that steady-state support is connected. while the second assumption is about the regularities of $S(x, y)$ required to pass derivatives inside the convolution. Regarding the third presumption, it shows that the Interaction Potential Kernel $S(x, y)$ is symmetric. Finally, assumptions 4 and 5 state that interactions decay at infinity and that the kernel $S(x, y)$ is perfectly attractive.

For simplicity we will assume that

$$\|S\|_{L^1(\mathbb{R}\times\mathbb{R})} = \int\int S(x, y)dxdy = 1. \tag{2.8}$$

Example 2.1.1. An example of an interaction potential kernel in a two-dimensional space is

$$S(x, y) = \frac{1}{\sqrt{(2\pi)^2}} e^{\frac{-x^2-y^2}{2}},$$

we observe that S satisfies all of the assumptions that we establish for it by keeping track of them in a two-dimensional space.See Appendix A: Figure 3.

Example 2.1.2. This example of an interaction potential kernel in a two-dimensional space that does not satisfy all of the assumptions of $S(x, y)$ see Appendix A: Figure 4.

$$S(x, y) = \frac{1}{\sqrt{(2\pi)^2}} e^{\frac{x-y^2}{100}}.$$

In two-dimensional space, we must solve the following equation to get the equation's steady-state.

$$0 = (\rho(\theta\rho - S * \rho)_x)_x + (\rho(\theta\rho - S * \rho)_y)_y. \tag{2.9}$$

The steady-state of equation (2.9) belongs to one of the following three categories, depending mostly on the initial conditionρ_0, the value of θ, and the function of the interaction potential kernel $S\ (x, y)$. Where $S\ (x, y)$ non-negative integrable function and satisfies all the assumptions, and ρ_0 satisfies the smoothness, non-negativity, and has connected support.

With the understanding of steady-states and their properties in a multi-dimensional space, we now want to discuss each instance independently and mention the characteristics of each one.

We begin with the first trivial situation, where theta is bigger than $\|S\|_{L^1(\mathbb{R}\times\mathbb{R})} = 1$. The attraction between the molecules is modest and gets weaker as the value of theta rises and as the time approaches infinity, while diffusion dominates the equation.

Lemma 2.1.1. Let $\theta > \|S\|_{L^1(\mathbb{R}\times\mathbb{R})}$. Then, there exists no steady-state to in the space $L^2 \bigcap \mathcal{P}$ where $\mathcal{P} = \{\rho \in L^1_+(\mathbb{R}^2) : \int_{\mathbb{R}^2} \rho(x, y, t)dxdy = 1\}$. [29]

In the second case, which we shall discuss, equilibrium dominates the system and the θ is between zero and one. This is the most significant case. In this case, the aggregation-diffusion equation's non-trivial steady-state in a two-dimensional space is obtained.

Theorem 2.1.1. Let $\theta < \|S\|_{L^1(\mathbb{R}\times\mathbb{R})}$. Then, there exists a radially symmetric non-increasing minimizer $\rho \in L^2 \bigcap \mathcal{P}$ for the entropy functional E restricted to $\mathcal{P}$ with $\rho \neq 0$ where $\mathcal{P} = \{\rho \in L^1_+(\mathbb{R}^2) : \int_{\mathbb{R}^2} \rho(x, y, t)dxdy = 1\}$.[29]

The final situation, where the steady-state is present, is also trivial. θ being equal to zero is the situation. In this situation, the diffusion component of the equation is completely absent, remaining only the aggregation component. When time becomes close to infinity. In other words, we have in this case what is called Dirac Delta, which is defined such that it is zero everywhere except at the origin, where it is infinitely large.

As in the second chapter, finding the stead-state in a two-dimensional space depends on the value of θ and S, where S is non-negative and integrable. The stead-state can be included in three main categories.

The aggregation-diffusion equation can be solved numerically in a two-dimensional space using a variety of techniques, including finite volume, finite elements, and boundary elements, all of which involve identifying the domain and approximating the solution at separate points. The final volume method will be used in the following subsection, and the results will be noted.

2.1.1 Finite Volume Method

In this part of the chapter, we will study in depth the mechanism of the finite-volume method in the aggregation-diffusion equations in two-dimensional space. The method includes many steps, including dividing the continuous field into a large number of separate and small control sizes, these sizes have several shapes, usually they may be rectangles or they may be triangles. The solution is solved by solving equations on the entire field and it is in the form of matrix equations, which can then be solved using a variety of techniques, such as Gaussian-Seidel, gradient-flow, or Range-Kutta (SSP-RK).

As we knew previously, the first, basic, and most important stage is to divide the domain into separate cells of finite size. which is $U_{j,k} = \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right] \times \left[y_{k-\frac{1}{2}}, y_{k+\frac{1}{2}}\right]$ of a uniform size $\Delta x \Delta y$ that is $\Delta x = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \forall j$ and $\Delta y = y_{k+\frac{1}{2}} - y_{k-\frac{1}{2}}, \forall k$.

$$\bar{\rho}_{j,k}(t) = \frac{1}{\Delta x \Delta y} \iint_{U_{j,k}} \rho(x,y,t) dxdy, \tag{2.10}$$

is the average of the solutions ρ computed at each cell $U_{j,k}$, then compute the integral for the ODE equation on each $U_{j,k}$

$$\frac{\mathrm{d}\bar{\rho}_{j,k}}{\mathrm{d}t} = -\frac{F^x_{j+\frac{1}{2},K} - F^x_{j-\frac{1}{2},K}}{\Delta x} - \frac{F^y_{j,K+\frac{1}{2}} - F^y_{j,K-\frac{1}{2}}}{\Delta y}, \tag{2.11}$$

where the numerical fluxes $F^x_{j+\frac{1}{2},K}$,$F^x_{j-\frac{1}{2},K}$,$F^y_{j,K+\frac{1}{2}}$ and $F^y_{j,K-\frac{1}{2}}$ which are the main elements of the finite volume method which is considered as an approximation for the continuous fluxes $-\rho(\theta\rho - S * \rho)_x - \rho(\theta\rho - S * \rho)_y$ respectively. which is defined by

$$F^x_{j+\frac{1}{2},K} = \max(u^x_{j+\frac{1}{2},k}0) \left[\bar{\rho}_{j,k} + \frac{\Delta x}{2}(\rho_x)_{j,k}\right] + \min(u^x_{j+\frac{1}{2},k}, 0) \left[\bar{\rho}_{j+1,k} - \frac{\Delta x}{2}(\rho_{x+1})_{j,k}\right], \tag{2.12}$$

and

$$F^x_{j-\frac{1}{2},K} = \max(u^x_{j-\frac{1}{2},k}, 0) \left[\bar{\rho}_{j-1,k} + \frac{\Delta x}{2}(\rho_x)_{j-1,k}\right] + \min(u^x_{j-\frac{1}{2},k}, 0) \left[\bar{\rho}_{j,k} - \frac{\Delta x}{2}(\rho_x)_{j,k}\right], \tag{2.13}$$

and

$$F^y_{j,K+\frac{1}{2}} = \max(u^y_{j,k+\frac{1}{2}}, 0)\left[\bar{\rho}_{j,k} + \frac{\Delta y}{2}(\rho_y)_{j,k}\right] + \min(u^y_{j,k+\frac{1}{2}}, 0)\left[\bar{\rho}_{j,k+1} - \frac{\Delta y}{2}(\rho_y)_{j,k+1}\right], \tag{2.14}$$

and

$$F^y_{j,K-\frac{1}{2}} = \max(u^y_{j,k-\frac{1}{2}}, 0)\left[\bar{\rho}_{j,k-1} + \frac{\Delta y}{2}(\rho_y)_{j,k-1}\right] + \min(u^y_{j,k-\frac{1}{2}}, 0)\left[\bar{\rho}_{j,k} - \frac{\Delta y}{2}(\rho_y)_{j,k}\right], \tag{2.15}$$

where the discrete values of the velocities $u^x_{j+\frac{1}{2},k}$, $u^x_{j-\frac{1}{2},k}$, $u^y_{j,k+\frac{1}{2}}$ and $u^y_{j,k-\frac{1}{2}}$ are defined as the following

$$u^x_{j+\frac{1}{2},k} = \Delta y \sum_l \sum_i \bar{\rho}_{i,l}(S(x_{j+1} - x_i, y_k - y_l) - S(x_j - x_i, y_k - y_l)) - \frac{\theta}{\Delta x}(\bar{\rho}_{j+1,k} - \bar{\rho}_{j,k}), \tag{2.16}$$

and

$$u^x_{j-\frac{1}{2},k} = \Delta y \sum_l \sum_i \bar{\rho}_{i,l}(S(x_j - x_i, y_k - y_l) - S(x_{j-1} - x_i, y_k - y_l)) - \frac{\theta}{\Delta x}(\bar{\rho}_{j,k} - \bar{\rho}_{j-1,k}), \tag{2.17}$$

and

$$u^y_{j,k+\frac{1}{2}} = \Delta x \sum_l \sum_i \bar{\rho}_{i,l}(S(x_j - x_i, y_{k+1} - y_l) - S(x_j - x_i, y_k - y_l)) - \frac{\theta}{\Delta y}(\bar{\rho}_{j,k+1} - \bar{\rho}_{j,k}), \tag{2.18}$$

and

$$u^y_{j,k-\frac{1}{2}} = \Delta x \sum_l \sum_i \bar{\rho}_{i,l}(S(x_j - x_i, y_k - y_l) - S(x_j - x_i, y_{k-1} - y_l)) - \frac{\theta}{\Delta y}(\bar{\rho}_{j,k} - \bar{\rho}_{j,k-1}), \tag{2.19}$$

and $(\rho_x)_{j,k}$ and $(\rho_y)_{j,k}$ represented the corresponding slops for x and y dimensions

$$(\rho_x)_{j,k} = \textbf{minmod}(2\frac{\bar{\rho}_{j,k} - \bar{\rho}_{j-1,k}}{\Delta x}, \frac{\bar{\rho}_{j+1,k} - \bar{\rho}_{j-1,k}}{2\Delta x}, 2\frac{\bar{\rho}_{j+1,k} - \bar{\rho}_{j,k}}{\Delta x}), \tag{2.20}$$

and

$$(\rho_y)_{j,k} = \textbf{minmod}(2\frac{\bar{\rho}_{j,k} - \bar{\rho}_{j,k-1}}{\Delta y}, \frac{\bar{\rho}_{j,k+1} - \bar{\rho}_{j,k-1}}{2\Delta y}, 2\frac{\bar{\rho}_{j,k+1} - \bar{\rho}_{j,k}}{\Delta y}), \tag{2.21}$$

where the following is the definition of a minmod limiter:

$$\text{minmod}(a_1, a_2, a_3) = \begin{cases} \min(a_1, a_2, a_3) & \text{, if } a_i > 0 \; \forall i \\ \max(a_1, a_2, a_3) & \text{, if } a_i < 0 \; \forall i \\ 0 & \text{, otherwise.} \end{cases} \tag{2.22}$$

The final step involves numerically integrating the Semi-Discreet Scheme into the equation using the nonlinear Range-Kutta (SSP-RK) method of the strong stability of the third-order, which is a numerical method for solving nonlinear ordinary differential equations. Multiple time steps, each using the previous time step's solution as the beginning condition for the subsequent time step, can be utilized with the SSP-RK method to supply the solution in time. The outcome found after a predetermined number of time steps is regarded as the numerical outcome of the initial PDEs.

We obtain the following equations by rewriting the third-order Range-kutta (SSP-RK) equations over the aggregation-diffusion equations in a two-dimensional space

$$K_1 = \rho_0^n(x, y) + \Delta t F(\rho_0(x, y)), \tag{2.23}$$

$$K_2 = \frac{3}{4}\rho_0^n(x, y) + \frac{1}{4}K_1 + \frac{1}{4}\Delta t F(K_1), \tag{2.24}$$

$$K_3 = \frac{1}{3}\rho_0^n(x, y) + \frac{2}{3}K_2 + \frac{2}{3}\Delta t F(K_2), \tag{2.25}$$

where F is the main function in the numerical method where we approximate the continuous fluxes $-\rho(\theta\rho - S * \rho)_x - \rho(\theta\rho - S * \rho)_y$ that given in (2.11), this equation system operates with respect to time, where (n) is the system's time step.

Chapter Three

Numerical Examples and Results

Here, in this chapter, we present many examples of the fixed solutions of the aggregation-diffusion equation, using the finite volume method that we described in the second chapter, using the MATLAB program, also we want here to present the simulation in a two-dimensional space, knowing that this method can extend to many dimensions. We will then go into detail about the results we found, explain why they appeared, and determine whether or not they match our initial expectations. Results in one-dimensional space and those in two-dimensional space will be compared.

3.1 Examples of situations when aggregation dominates and diffusion fails

This section offers a variety of examples, all of which have in common that the diffusion coefficient (θ) is zero, and when $\theta = 0$, the diffusion term disappears from the equation, and we are left with:

$$0 = (\rho(S * \rho)_x)_x + (\rho(S * \rho)_y)_y, \tag{3.1}$$

which is a pure aggregation equation. This equation describes the evolution of the density over time in the absence of diffusion.

Example 3.1.1. In our first example of the equation in a two-dimensional space, we want to assume that the initial dynamics of the solutions are as follows. First, the initial datum is

$$\rho_0(x, y) = 1,$$

the interaction potential S is the Gaussian Kernel which

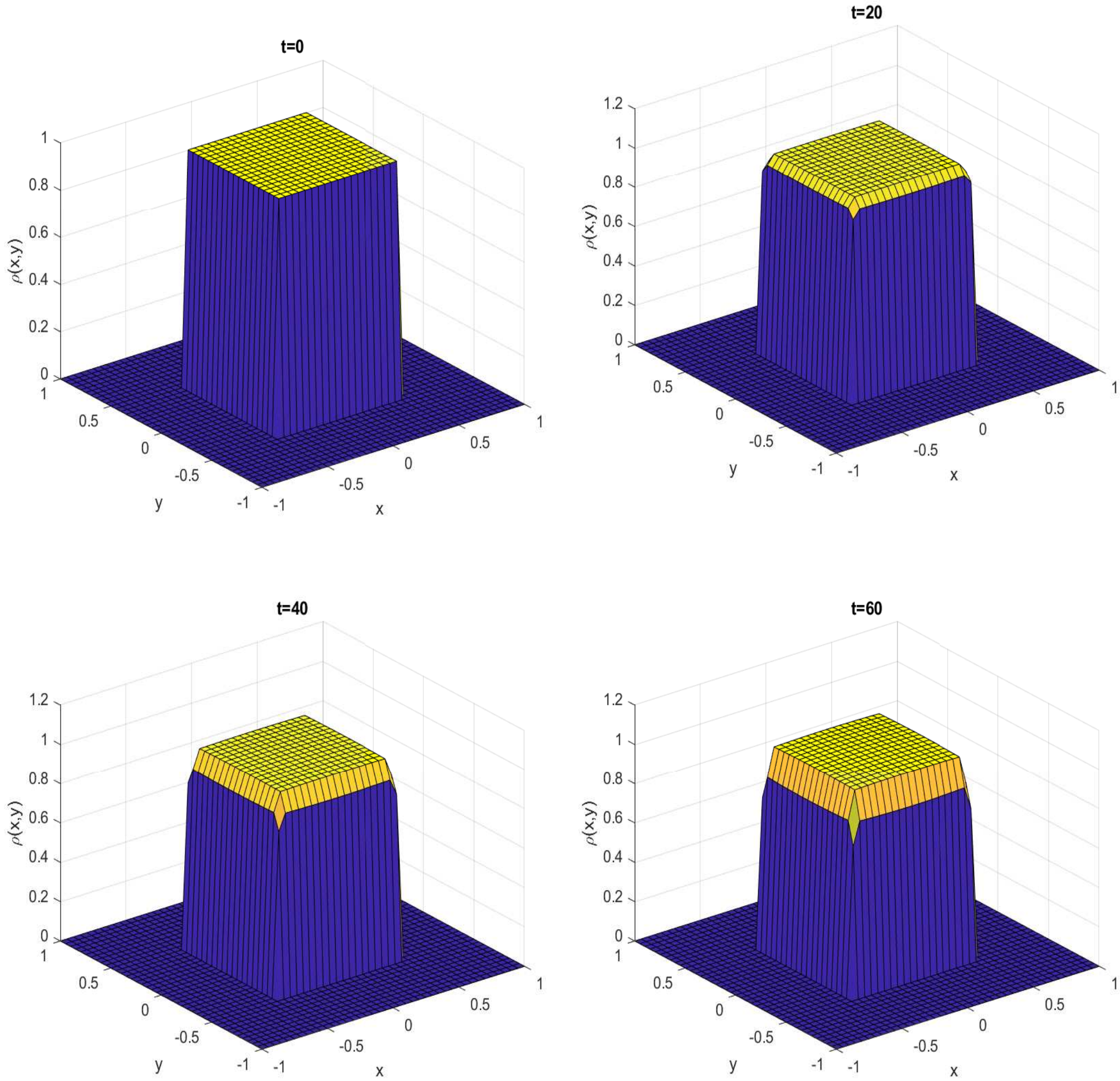
$$S = \frac{1}{\sqrt{(2\pi)^2}} e^{\frac{-x^2-y^2}{2}},$$

the diffusion constant $\theta = 0$ which is a trivial case, defined on $x \times y = [-2, 2] \times [-2, 2]$ and the particles number $N = 40$.

Here, ensure that $\rho_0(x, y) = 1$ satisfies the requirements for positive, continuity, boundness, smoothness, and Connected Compact support.
For simplicity and to ensure integrability

$$\int_{-0.5}^{0.5} \int_{-0.5}^{0.5} \rho(x, y) dx dy = \int_{-0.5}^{0.5} \int_{-0.5}^{0.5} 1 dx dy = 1.$$

Figure 2
Steady-state for aggregation-diffusion equation for $\rho_0(x, y) = 1$*,* $N = 40$*.*



Since θ in our example was equal to 0, you can see that it belongs to an interval that has a trivial steady-state. In this illustration, we can see how the steady-state slowly aggregated, approaching nearer zero. As time gets closer to infinity, we have what is known as a Dirac Delta Solution, which was considered to be trivial. This means that the mass of the system remains constant(conserved), but it becomes more and more concentrated at a single point as time goes to infinity. This solution is considered trivial because it does not exhibit any interesting patterns or behaviors.

Example 3.1.2. This is another trivial steady-state situation, where aggregation triumphs and diffusion fails. This example supports the assertion in the previous example that if theta were 0, the steady state would be the weight of a column with zero at its center as time increases and approaches infinity. That is, it will approach the Dirac Delta Solution To observe this let us call

$$\begin{cases} 0 & \text{if } x \leq -0.425 \text{ or } 0.425 \geq x \text{ or } y \leq -0.425 \text{ or } 0.425 \geq y \\ 1 & \text{if } x \leq -0.075 \text{ or } 0.075 \geq x \text{ or } y \leq -0.075 \text{ or } 0.075 \geq y \\ 0 & \text{if otherwise} \end{cases},$$

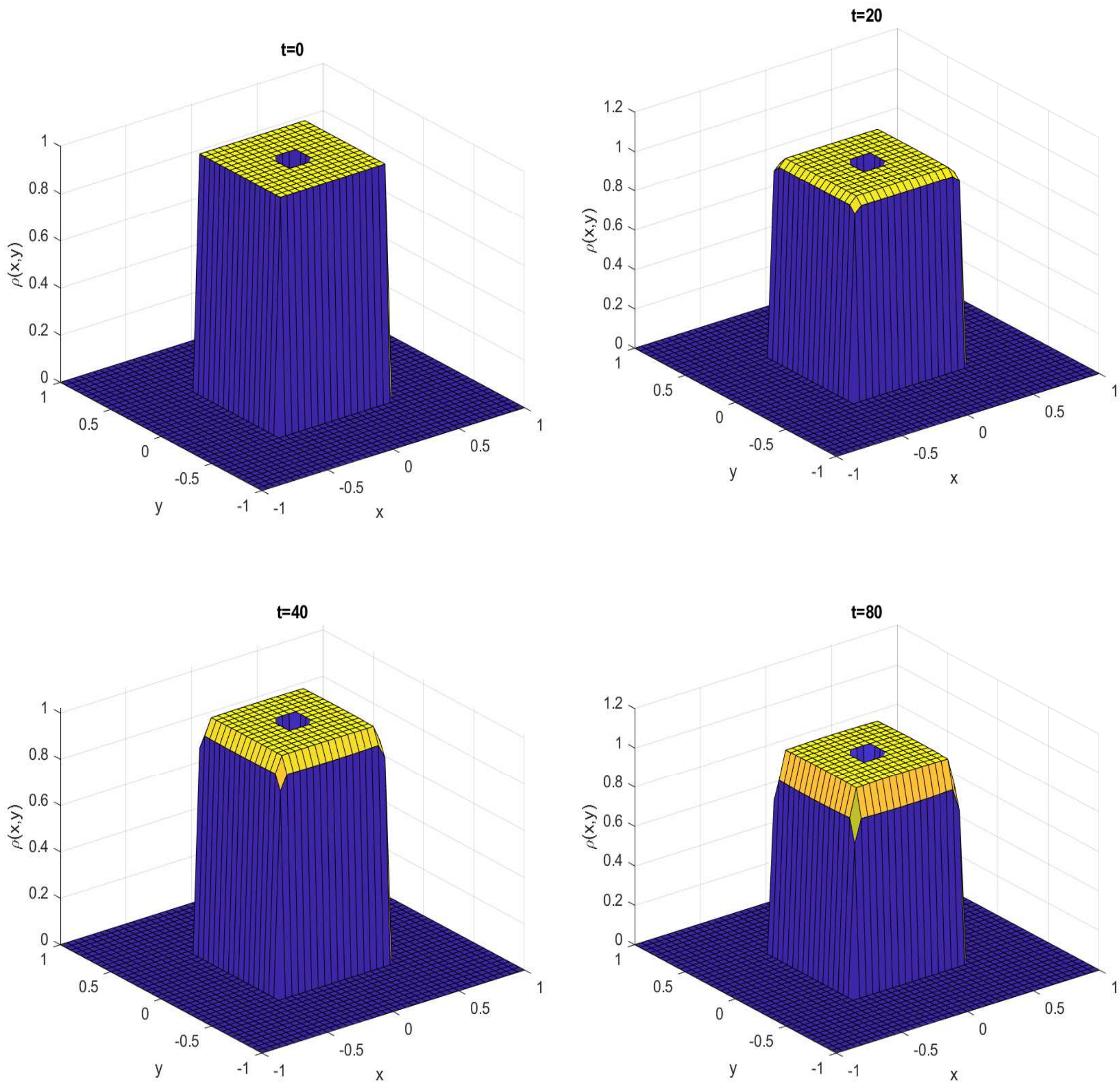
with the potential Kernel

$$S = \frac{1}{\sqrt{(2\pi)^2}} e^{\frac{-x^2-y^2}{2}},$$

and the number of particles $N = 40$ on the interval $x \times y = [-1,1] \times [-1,1]$ by specifying $\theta = 0$.

Figure 3

A trivial steady-state for Piecewise Function, $N = 40$*.*



Example 3.1.3. This is another example of this kind of trivial situation. In this example, we take the initial datum as

$$\rho_0(x,y) = x^2y^2,$$

and the potential kernel as

$$S = \frac{1}{\sqrt{4\pi^2}} e^{\frac{-x^2-y^2}{2}},$$

where the other variable is as $x \times y = [-1,1] \times [-1,1]$, the particles number $N = 40$ and $\theta = 0$

You can see that the four vertices in this example gradually converge and finally merge to create the trivial solution, the Dirac delta.

Example 3.1.4. You can see in this example how the steady-state for a large time converges to a case of the Dirac-Delta solution. suppose that

$$\rho_0(x,y) = 10 - 4x^2 - 2y^2,$$

and

$$S = \frac{1}{\sqrt{\pi}} e^{\frac{-x^2-y^2}{2}},$$

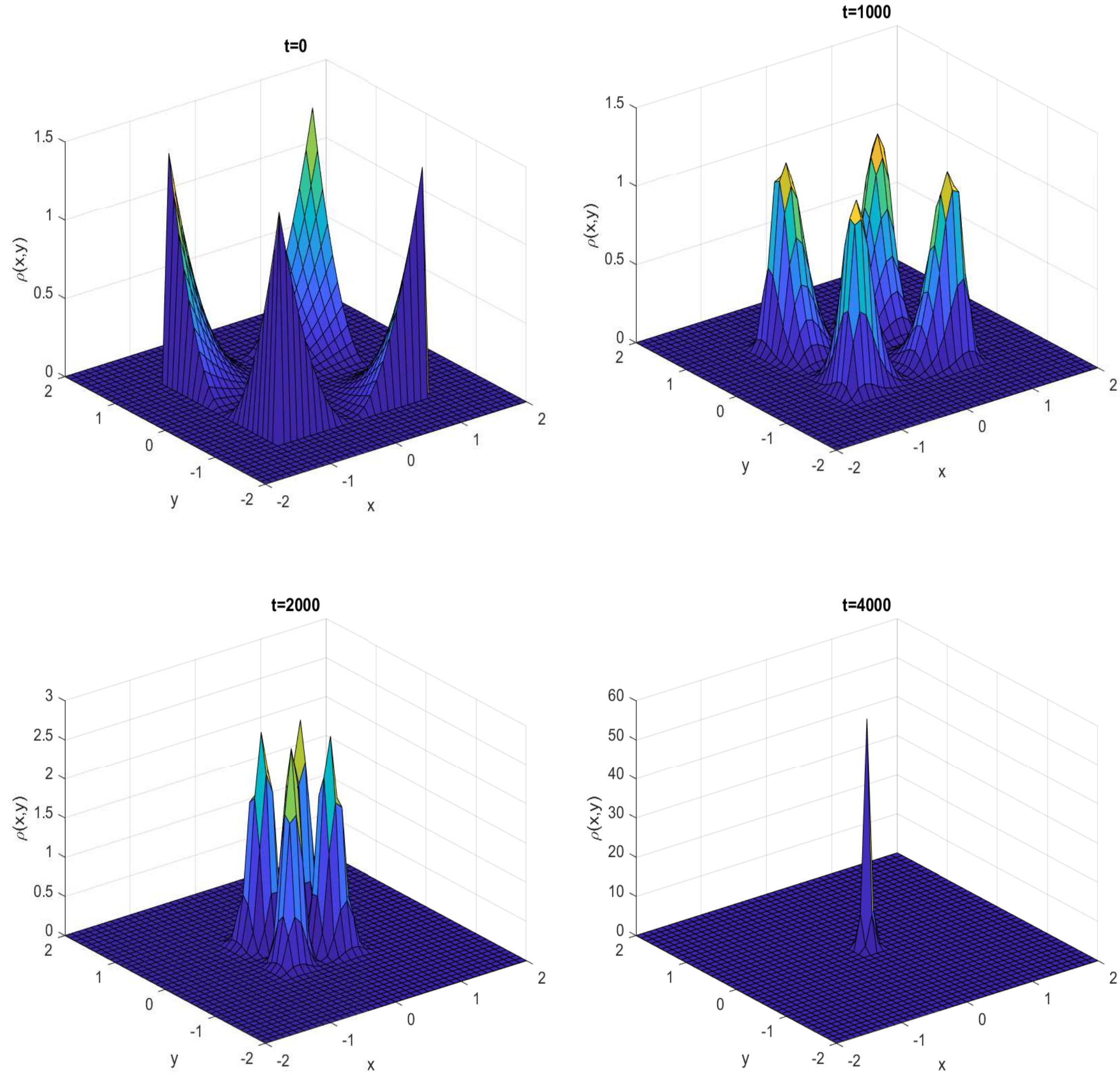
which is defied on $x \times y = [-1,1] \times [-1,1]$ and the particles number $N = 40$ and with diffusion constant $\theta = 0$.

By examining the previous three examples in this section, we can see that they all share a number of characteristics, the most significant of which is that the value of θ is equal to zero. In this case, the solution is trivial, as expected theoretically, and we have demonstrated this numerically. The solution in all of these examples will be Dirac Delta as time approaches infinity. See Appendix B.

3.2 A few instances where diffusion defeats aggregation

For this section, we present a set of examples in which diffusion is the dominant mechanism, and they have a common characteristic in that the diffusion coefficient is greater

Figure 4
Trivial steady-state $\rho_0(x, y) = x^2y^2$, $N = 40$



than 1 in all of them, in another way it means that the rate of diffusion is faster than the rate of aggregation.

Example 3.2.1. This is the first example of how diffusion triumphs and aggregation collapse over time. This is another type of trivial solution. This happened when we took θ greater than 1 since we took it equal to 2 and took the other variables as

$$S = \frac{1}{\sqrt{(2\pi)^2}} e^{\frac{-x^2-y^2}{2}},$$

on the interval $x \times y = [-1, 1] \times [-1, 1]$ and

$$\rho_0(x, y) = 1,$$

with the number of particles $N = 40$. You can see that there is a high particle diffusion in this example, which is what we expected theoretically and what this numerical example made extremely clear. This solution is considered trivial because θ falls outside the interval of non-trivial solutions that are greater than 1.

Example 3.2.2. In this example, we'll go over another situation in which steady-states are trivial since the datum's are diffusive to infinity. Let's take the value of the constant $\theta = 2$, the potential kernel

$$S = \frac{1}{\sqrt{(2\pi)^2}} e^{\frac{-x^2-y^2}{2}}$$

on the interval $x \times y = [-1, 1] \times [-1, 1]$ and

$$\rho_0(x, y) = \begin{cases} 0 & \text{if } x \leq -0.425 \text{ or } 0.425 \geq x \text{ or } y \leq -0.425 \text{ or } 0.425 \geq y \\ 1 & \text{if } x \leq -0.075 \text{ or } 0.075 \geq x \text{ or } y \leq -0.075 \text{ or } 0.075 \geq y \\ 0 & \text{if otherwise} \end{cases} \cdot$$

with the number of particles $N = 40$ The cases of constancy over time will converge to infinity, that is, they will fully apply to the surface, and this is what happens when θ converges to infinity. Diffusion triumphs over aggregation in this situation.

Figure 5

A trivial steady-state for $\rho_0(x, y) = 1$, $N = 40$, $\theta = 2$

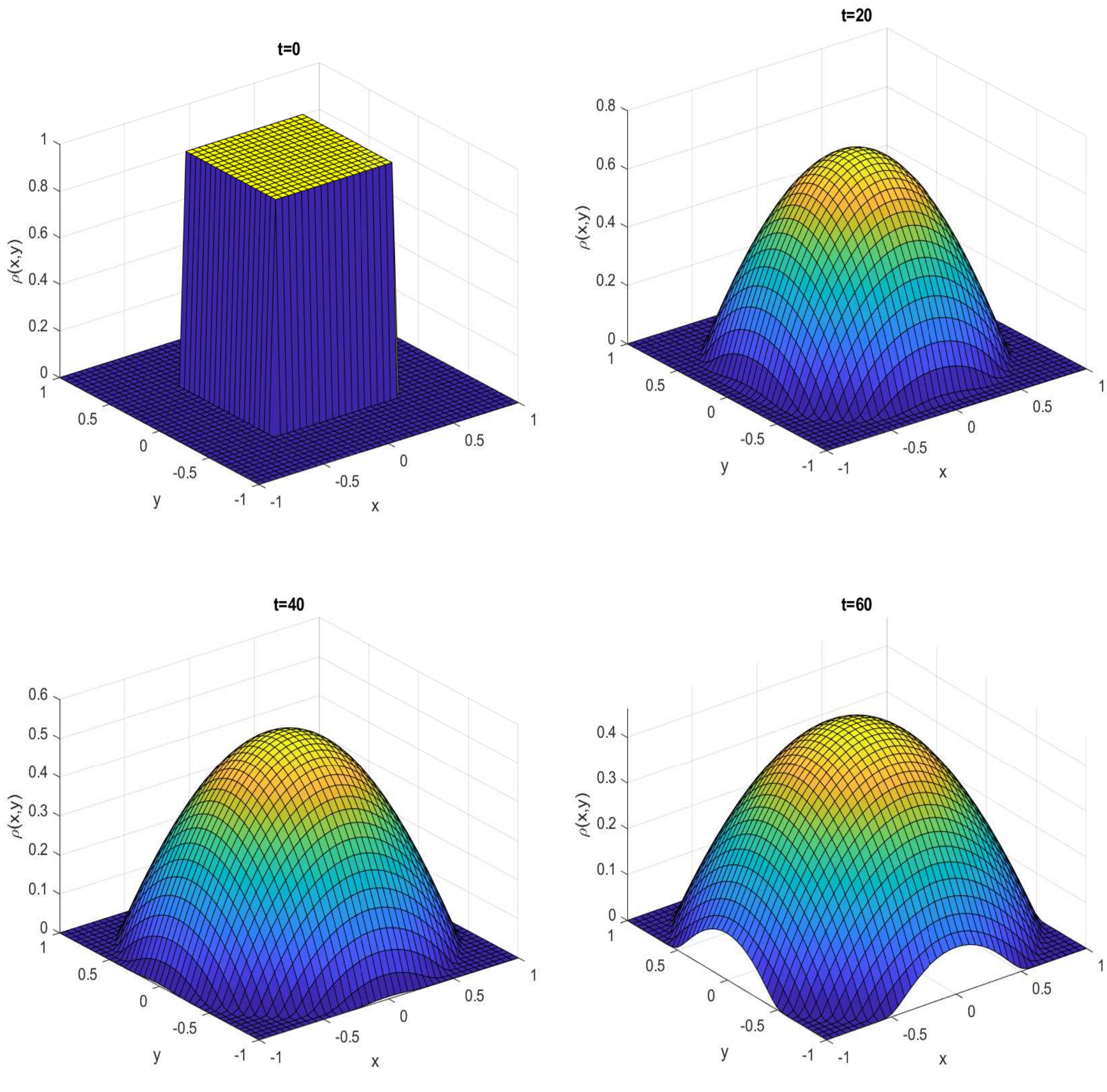
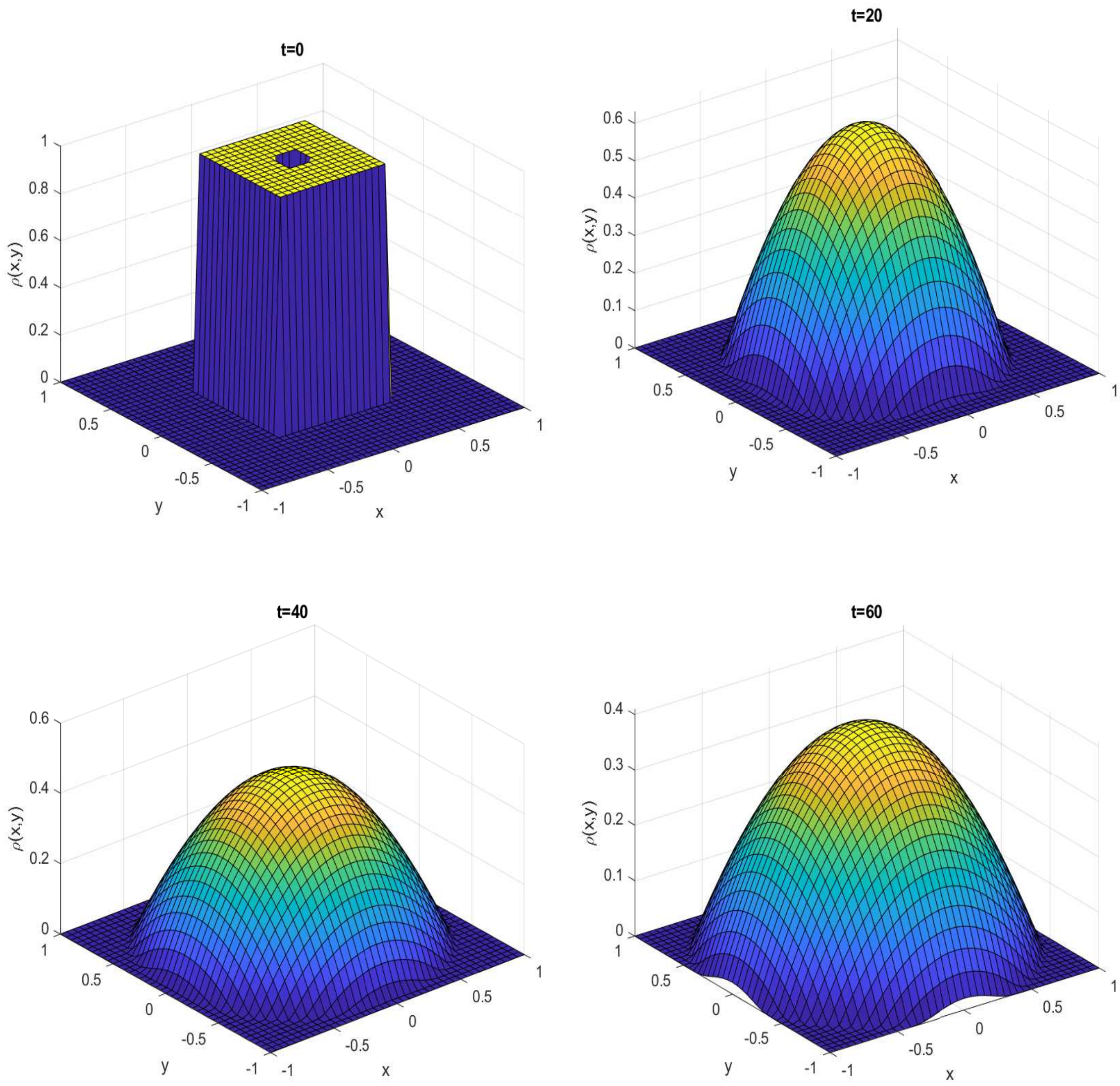


Figure 6
Diffusion defeats aggregation for Piecewise Function, $N = 40$.



Example 3.2.3. Another example is when diffusion dominates, let us take the initial datum ρ_0 as

$$\rho_0 = e^{x+y},$$

and the interaction Kernal as

$$S = \frac{1}{\sqrt{\pi}} e^{\frac{-x^2-y^2}{2}},$$

with these function defined on $x \times y = [-1, 1] \times [-1, 1]$ and the particles number $N = 40$.

Example 3.2.4. finally, the last example will take in this case is this, we take the initial condition as

$$\rho_0(x, y) = x^2 y^2,$$

and the Kernel as

$$S = \frac{1}{\sqrt{\pi}} e^{\frac{-x^2-y^2}{2}},$$

with the diffusion constant $\theta = 3$, defined on $x \times y = [-2, 2] \times [-2, 2]$ and the particles number $N = 40$.
The observation that the four heads began to gather gradually with time, smelled after that began to spread rapidly, and this is what exists theoretically. See Appendix C.

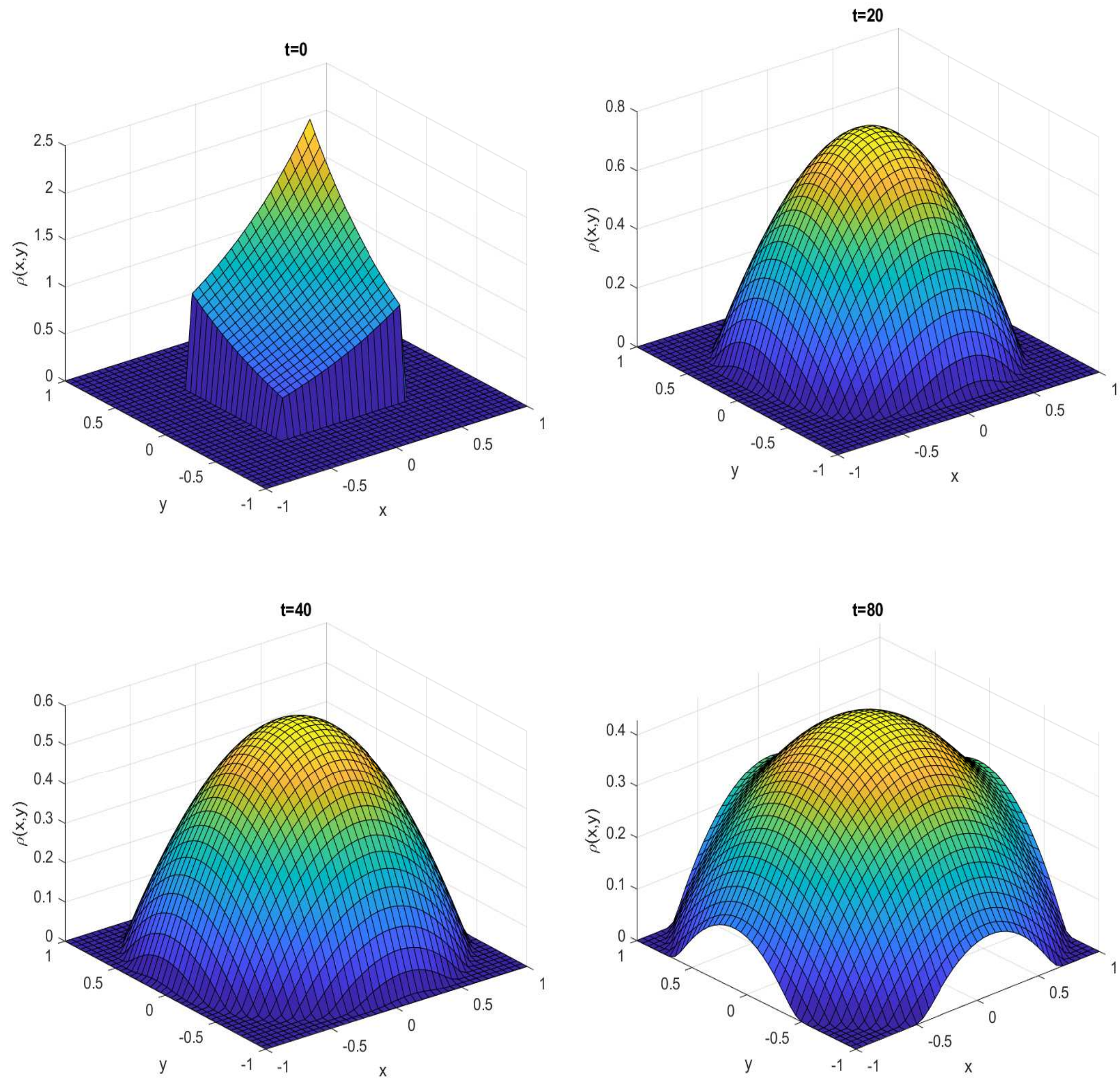
In the set of instances in this section, which all share the characteristic that the diffusion coefficient is very high, larger than 1, and it is a trivial case, we note that with time the particles spread gradually, applying to the surface, that is, approaching infinity.

3.3 Examples of situations where aggregation and diffusion are balanced

This section contains the most significant collection of examples where θ is constrained between 0 and 1 and the solution is non-trivial, as it is in most applications. In this case, the diffusion term and the aggregation term have roughly equal effects on the density.

Figure 7

A trivial steady-state for $\rho_0(x, y) = e^{x+y}$, $N = 40$.



This means mathematically that $\frac{\partial \rho}{\partial t}$ after a long time equal 0.

Example 3.3.1. Now we want to start with a set of examples in which the solution is not trivial. We start with the following example, let us take the interaction potential kernel as previous examples which is a Gaussian Kernel as

$$S = \frac{1}{\sqrt{(2\pi)^2}} e^{\frac{-x^2-y^2}{2}},$$

and the initial condition as

$$\rho_0(x, y) = 1,$$

with diffusion coefficient $\theta = 0.2$, defined on $y \times x = [-1, 1] \times [-1, 1]$ and the particles number $N = 40$.

It becomes extremely simple for you to confirm that the initial value achieves positivity and smoothness and has Connected Compact Sport.

Seeing the finite volume's steps makes it clearly evident that the solution approaches the steady-state, which resembles a bell, with increasing time.

Another important note is that the steady state is symmetric about $x = 0$ and $y = 0$

From the steady-state curve, if we checked a little, we would find that it does not contain sudden interruptions, which means that it is connected.

Also, from the figure, we can check and conclude that the Support is Connected and also Compact.

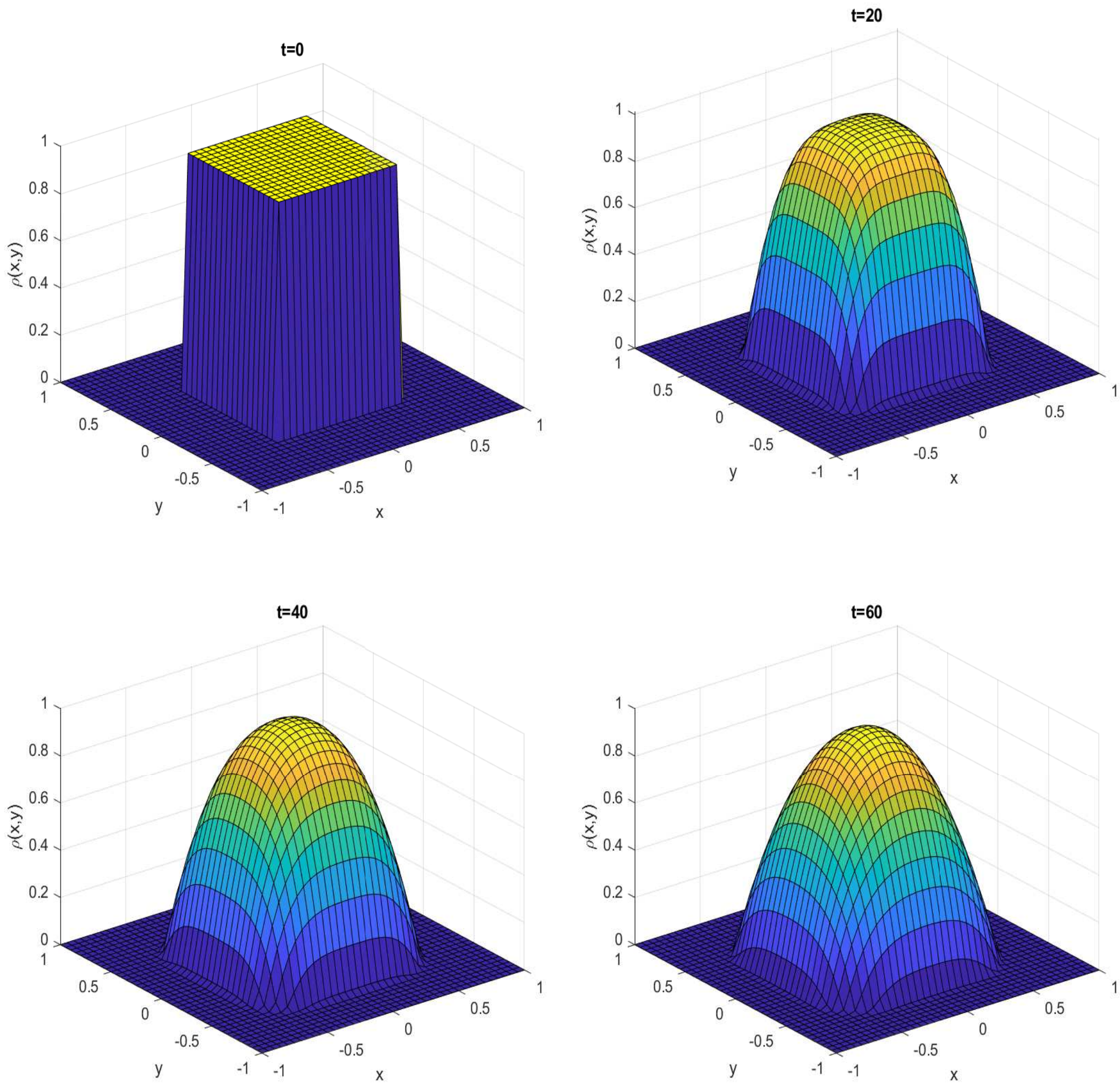
And because the circumference of the steady-state does not contain any interruptions or sudden angles, this means that it is smooth.

In this illustration, you can see how the solution eventually tends to aggregate at the top, approaching to the stable solution or steady-state and attaining what was theoretically predicted. see figure 3.9 .

Example 3.3.2. In the following numerical example, we choose the initial data to be an exponential function given by

Figure 8

Balanced steady-state for $\rho_0(x, y) = 1$, $N = 40$



$$\rho_0(x,y) = \begin{cases} 0 & \text{if } x \leq -0.425 \text{ or } 0.425 \geq x \text{ or } y \leq -0.425 \text{ or } 0.425 \geq y \\ 1 & \text{if } x \leq -0.075 \text{ or } 0.075 \geq x \text{ or } y \leq -0.075 \text{ or } 0.075 \geq y \\ 0 & \text{if otherwise} \end{cases},$$

and the potential Kernel as the following

$$S = \frac{1}{\sqrt{(2\pi)^2}} e^{\frac{-x^2-y^2}{2}},$$

with the constant $\theta = 0.2$ and the particles number $N = 40$ on the interval $x \times y = [-1, 1] \times [-1, 1]$.

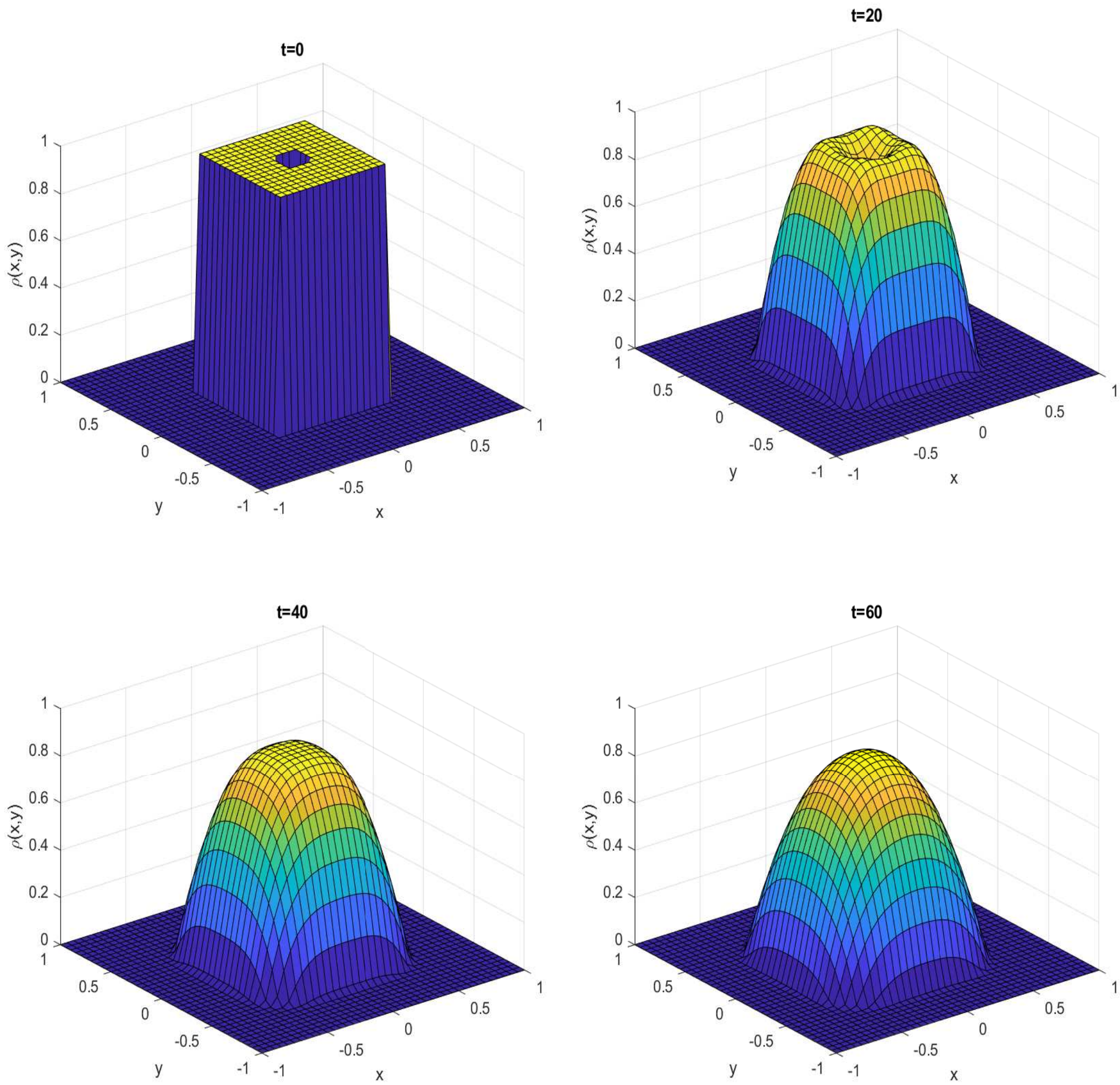
In this example, we notice that in the initial state of the particles, there was a gap in the middle, but it disappeared as the solution approached the steady-state. As it takes the form of a symmetrical bell at the end.

You can also notice that the stationary state at the end has become continuous, smooth and regular, regardless of its initial state.

Looking at Figure 3.10 you can see that there is almost no difference between the time step at 40 and the steady state at time 60, and this is all due to the value of theta.

Figure 9

Balanced steady-state for Piecewise Function, $N = 40$



Example 3.3.3. In this example, we'll talk about a situation where the first data is a quadratic function. As a result, we intend to use

$$\rho_0(x, y) = 10 - 4x^2 - 2y^2,$$

and want to use

$$S = \frac{1}{\sqrt{(2\pi)^2}} e^{\frac{-x^2-y^2}{2}},$$

as in the previous example, while we want to take $\theta = 0.4$ with the particles number $N = 40$ on the interval $x \times y = [-1, 1] \times [-1, 1]$.

In this example, you can notice that by following the steps of the final volume, it decreases to reach a steady-state, while maintaining the shape of the bell for the steady-state. see Figure 3.11.

As in the previous cases, the steady state is positive, continuous, smooth, and has Connected support and Compact support.

Example 3.3.4. This example demonstrates that the steady-state exhibits multiple behaviors, which means that the steady-state of the datum varies depending on its initial condition. Since we utilized the same potential Kernel

$$S = \frac{1}{\sqrt{(2\pi)^2}} e^{\frac{-x^2-y^2}{2}},$$

and the same θ, which was equal to 0.2 throughout the same interval $x \times y = [-1, 1] \times [-1, 1]$ and with the same number of particles $N = 40$ in both examples, but in both instances, we modified the datum's initial state, whereas in one of them

$$\rho_0(x, y) = \begin{cases} 0 & \text{if } x \leq -0.425 \text{ or } 0.425 \geq x \text{ or } y \leq -0.425 \text{ or } 0.425 \geq y \\ 1 & \text{if } x \leq -0.075 \text{ or } 0.075 \geq x \text{ or } y \leq -0.075 \text{ or } 0.075 \geq y \\ 0 & \text{if otherwise} \end{cases},$$

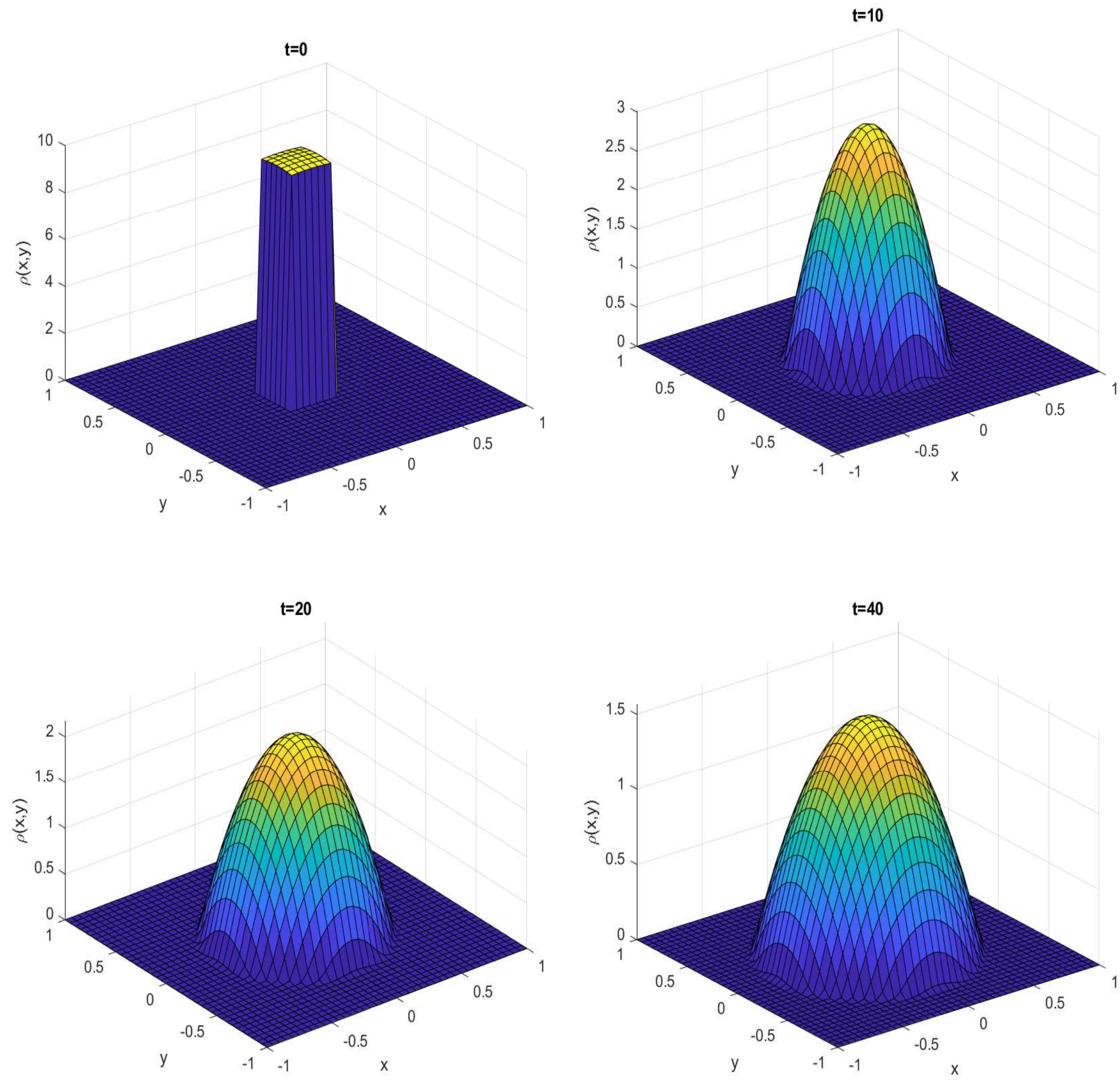
and in the other

$$\rho_0(x, y) = e^{x+y},$$

see Figures 8 and See Appendix D.

Figure 10

Balanced steady-state for $\rho_0(x, y) = 10 - 4x^2 - 2y^2$, $N = 40$



Example 3.3.5. In this example, we present a special case of the initial condition which is defined as

$$\rho_0(x,y) = x^2y^2,$$

and the

$$S = \frac{1}{\sqrt{(2\pi)^2}} e^{\frac{-x^2-y^2}{2}},$$

where the diffusion coefficient $\theta = 0.4$, defined on $x \times y = [-1,1] \times [-1,1]$ and the particles number $N = 40$.
In this example, we can notice that the initial state of the density was curved upwards, but with the steps of time it became close to the steady state and began to take the shape of a smooth symmetrical bell. See Appendix E .

You can see that the steady-state eventually takes the shape of a bell, regardless of the initial state of the particles or how they are moving.
Whatever the shape of the initial state, you can notice that by using the finite volume method, as time increases, the solution moves towards the steady-state gradually.
we can also notice the different behavior of the steady-state, depending on the initial state of the particles.
Additionally, you'll see that when you go through the finite volume method steps, you'll see that they all lead to a single, singular steady-state.
The steady-state in every example in this section is positive, symmetric, smooth, continuous, concave, has Connected support, and the support Compact.
And since the steady-state is symmetrical, continuous, and smooth in all situations, we can conclude that it is regular.
Furthermore, we can claim that the steady-state has a regularizing impact since in every case, as time approaches infinity, the solution maintains its smoothness and regularity.

Chapter Four

Conclusion and Perspectives

4.1 Difficulties

Although the aggregation-diffusion equations can be solved using the finite volume method, this method is rife with numerous numerical problems that might result in incorrect solutions. Consequently, it is crucial to be aware of these challenges and fully comprehend them in order to find a precise solution that works for complex problems.

One of these problems is that this type of problem has very high dimensions and requires a great deal of mathematical operations.

Also, among these difficulties is that this type of problem is very sensitive to small changes in the inputs, and this may lead to inaccurate results.

When the diffusion constant is equal to zero, the particles accelerate to aggregate to gather at one point, and therefore we have to be careful in running the code, and this is also considered one of the difficulties that exist in the Numerical Method.

4.2 Conclusion and perspectives

In conclusion, the aggregation-diffusion equation in two dimensions is a mathematical model of the behavior of a population of particles that are simultaneously aggregated and diffused. Depending on the initial conditions and equation parameters, a wide variety of phenomena, such as stable equilibria, traveling waves, pattern development, and chaotic dynamics, can be generated by this equation.

The aggregation-diffusion equation in two-dimensional space exhibits steady-state behavior when the density of the particle population is constant over time. This is achieved when the rates of diffusion and aggregation are balanced, resulting in a stable density distribution.

The interaction of the equation's form, reaction kinetics, diffusion coefficients, and initial conditions leads to pattern formation in aggregation-diffusion equations in two-dimensional space.

The main purpose of this thesis is to use the finite volume method to approximate solutions for a fairly wide class of aggregation and diffusion rates in two-dimensional space, which are used in many fields and application contexts.

This thesis' principal conclusion is that there is only one steady-state and that steady-state has a bell-shaped shape.
The characteristics of the steady-state in general are positivity, connectivity, smoothness, symmetry, and it has Connected support, Compact support, Regularity, and Regularizing Effect.
The datum goes to a steady-state faster the closer the value of the θ constant is to one.
Depending on the datum's initial state, the steady-state of the aggregation-diffusion equation can have multiple possible behaviors, and understanding these behaviors is important for predicting the long-term behavior of the system.
This thesis addressed a problem that can be expanded and applied to any dimensional space.

4.3 Comparison with the results in one-dimensional space

There are a number of significant differences between the behavior of the system in one-dimensional space and two-dimensional space, despite the fact that the equation's basic form is the same in both dimensions.
In two-dimensional space, the density of particles can display more complex patterns, such as clustering and spatial heterogeneity, in contrast to one-dimensional space where it typically scales as a power law with distance from the origin.
Whereas the two-dimensional space equation represents a two-dimensional system, the aggregation diffusion equation in one-dimensional space describes the behavior of particles in a linear system. When particles in a two-dimensional space system have more degrees of freedom and can interact in more complicated ways, this may have a substantial impact on how the system behaves.
The aggregation-diffusion equation can be solved numerically in one dimension, where the issue is less difficult and can be solved more quickly than in two dimensions. The computational cost can be significantly larger in two-dimensional space, and more advanced numerical methods would be needed.
Due to variations in scaling, diffusion, aggregation, and boundary conditions, the behavior of the aggregation-diffusion equation is noticeably different in one-dimensional space and two-dimensional space. The study of the aggregation-diffusion equation in both one-dimensional space and two-dimensional space is an active topic of research since these

variations can result in a wide range of behaviors.

The aggregation and diffusion equations, both in one-dimensional space and in two-dimensional space, have many common characteristics and features, although there are many differences.

One of the similarities between the two equations, such as the fact that they are both in one dimension and in two dimensions, deal with particle diffusion, or scattering.

Also, both equations at the end have a steady-state solution that eliminates the behavior of the system after a long time and in the long run. When there is a balance between diffusion and aggregation, the system after a period of time stabilizes and a steady state is reached.

In our case, we are that the density depends on the galvanic state of the system, and that is because it is not of the two equations, whether in one or two dimensions, linearly

The two equations share one dimension and two dimensions in the fact that the two equations depend on the law of conservation of mass or energy conservation, since these properties are used to describe the behavior of the system after a large time has passed.

In general, the aggregation and diffusion equations represent a more complex and more realistic model for the system because they take into account the effect of dimensions and the possibility of the existence of heterogeneous cases.

4.4 Limitations and errors of aggregation-diffusion equations in two-dimensional space

To explain a wide range of behaviors in biological, physical, and cosmic systems, aggregation-diffusion equations are used to explain, describe, and understand them. Despite their wide use, they are surrounded by a set of limitations when it comes to accurately describing the movement of particles in two-dimensional space.

The assumption that the rate of diffusion in the equation is constant and that the particles do not move randomly but move in a regular movement is considered one of the limitations of the equation, but in fact, there is a possibility of irregular movement of particles and also that the rate of diffusion may be variable and unstable and sometimes uneven.

Additionally, it presumes that the particles exist in an ideal, barrier-free environment.

One of the limitations is that the equation assumes that the particles will move toward each other at a constant rate, but in fact, the particles may move away from each other or

the rate of their approach to each other may vary.
Since they are unable to understand the complexity of the system fully, the clustering-diffusion equations can be seen as a simplification of the system. This leads to inaccurate predictions, which can be considered as one of the limitations of these equations.
Despite the limitations imposed by the equation and the claim that it is perfect, it is surrounded by errors, as these errors can be divided into three main groups: estimation errors, numerical dispersion and dispersion, and finally truncation errors.
Computing the numerical solution of a finite number of discrete volumes leads to errors in estimation while using numerical approximations to solve equations leads to numerical dispersion. And finally, when the field is divided into a certain number of volumes, truncation errors occur as a result.
The calculation of the error rate is a measure of how close the finite-volume method is to the actual solution of the equation. Once we know the error rate, we can learn, study, compare and improve the method, raise the accuracy rate and get better results with the least amount of errors.
Often, and in most cases, the exact solution is unknown, so an approximate solution is used as a reference solution, and the error between the reference solution and the finite volume solution is calculated using one of the approved error standards, and at the end of that, the error results obtained are evaluated to determine the accuracy of the finite volume method in finding the steady-state equations for aggregation - diffusion in a two-dimensional space.
Ignoring these limitations and errors, these equations are widely used in a variety of fields and applications. But in order to obtain more accurate results, we must develop these equations to include more variables and influences, which can be considered as one of the suggestions for independent work.

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Appendix A

Interaction Potential Kernel Examples

Figure 1
Interaction Potential Kernel fulfills the assumptions of S 1-D

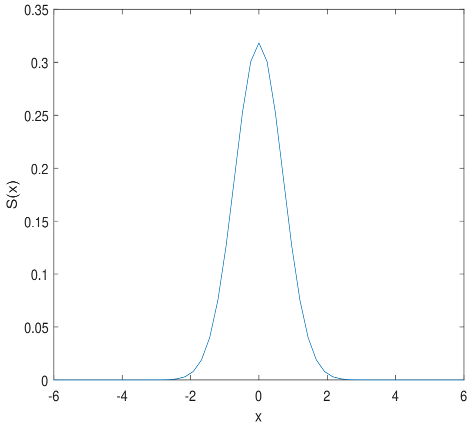


Figure 2
Interaction Potential Kernel does not fulfill the assumptions of S in 1-D

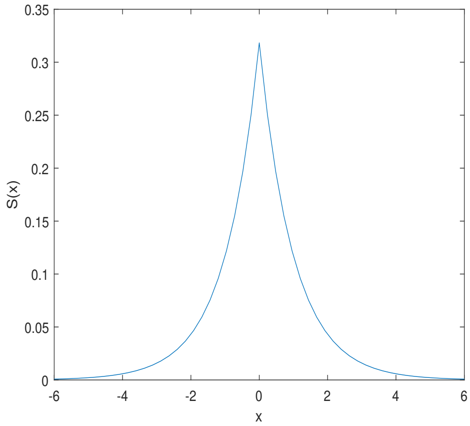


Figure 3
Interaction Potential Kernel fulfills the assumptions of S

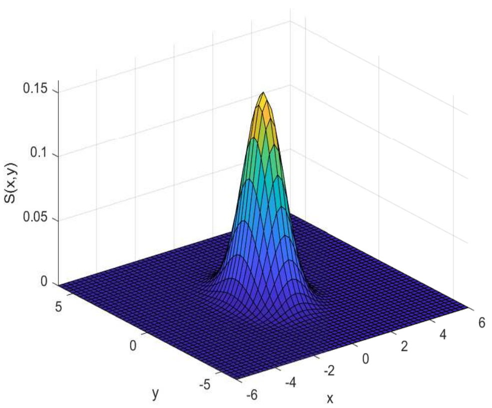
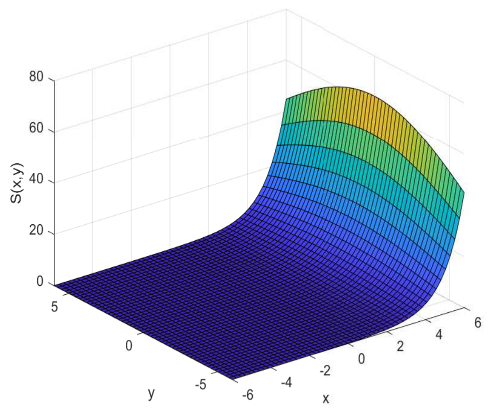


Figure 4
Interaction Potential Kernel does not fulfill the assumptions of S

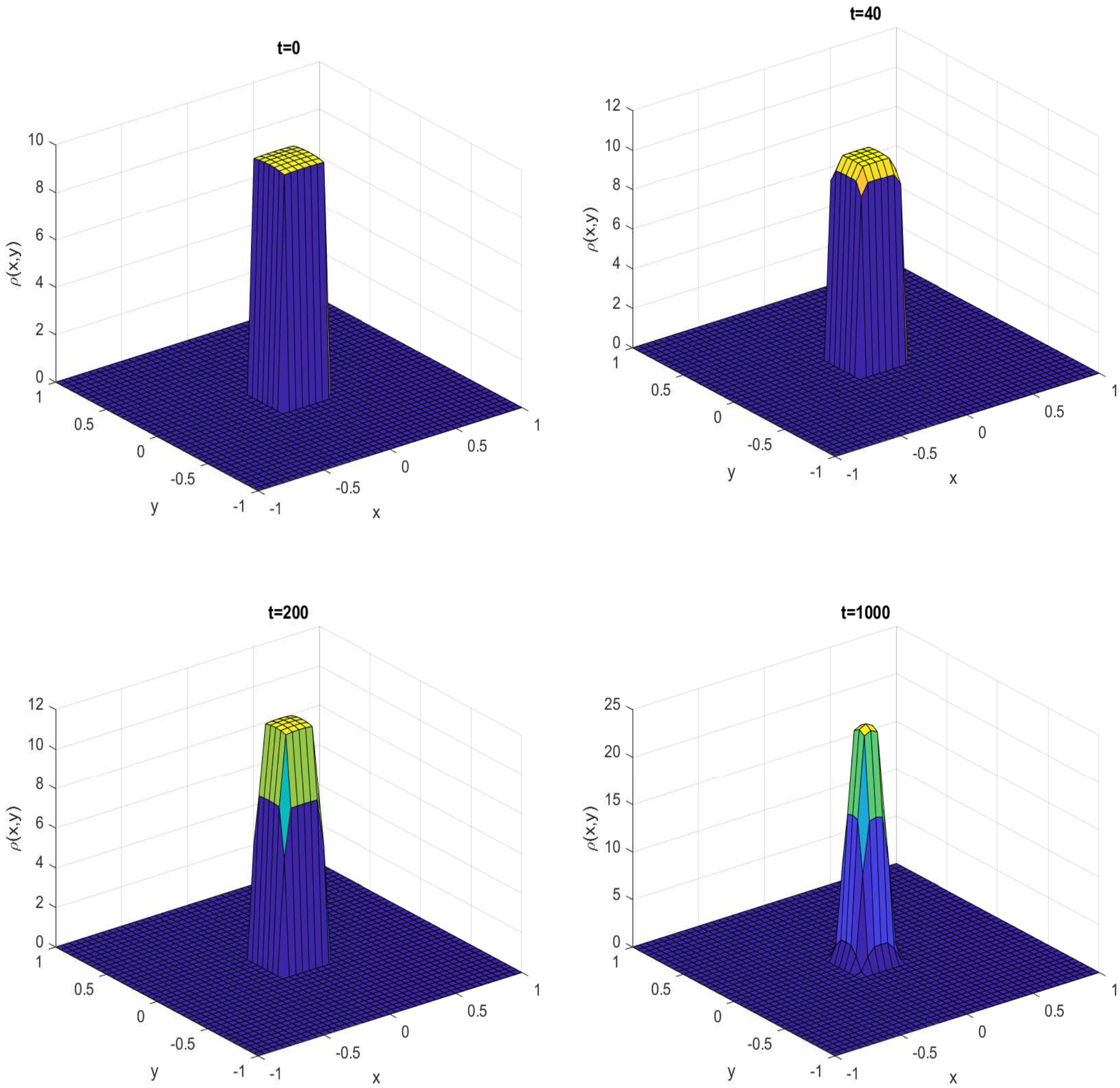


Appendix B

Example of diffusing downward converging to the steady-state

Figure 5

Diffusing downward converging to the steady-state where $\rho_0(x,y) = 10 - 4x^2 - 2y^2$*,* $N = 40$

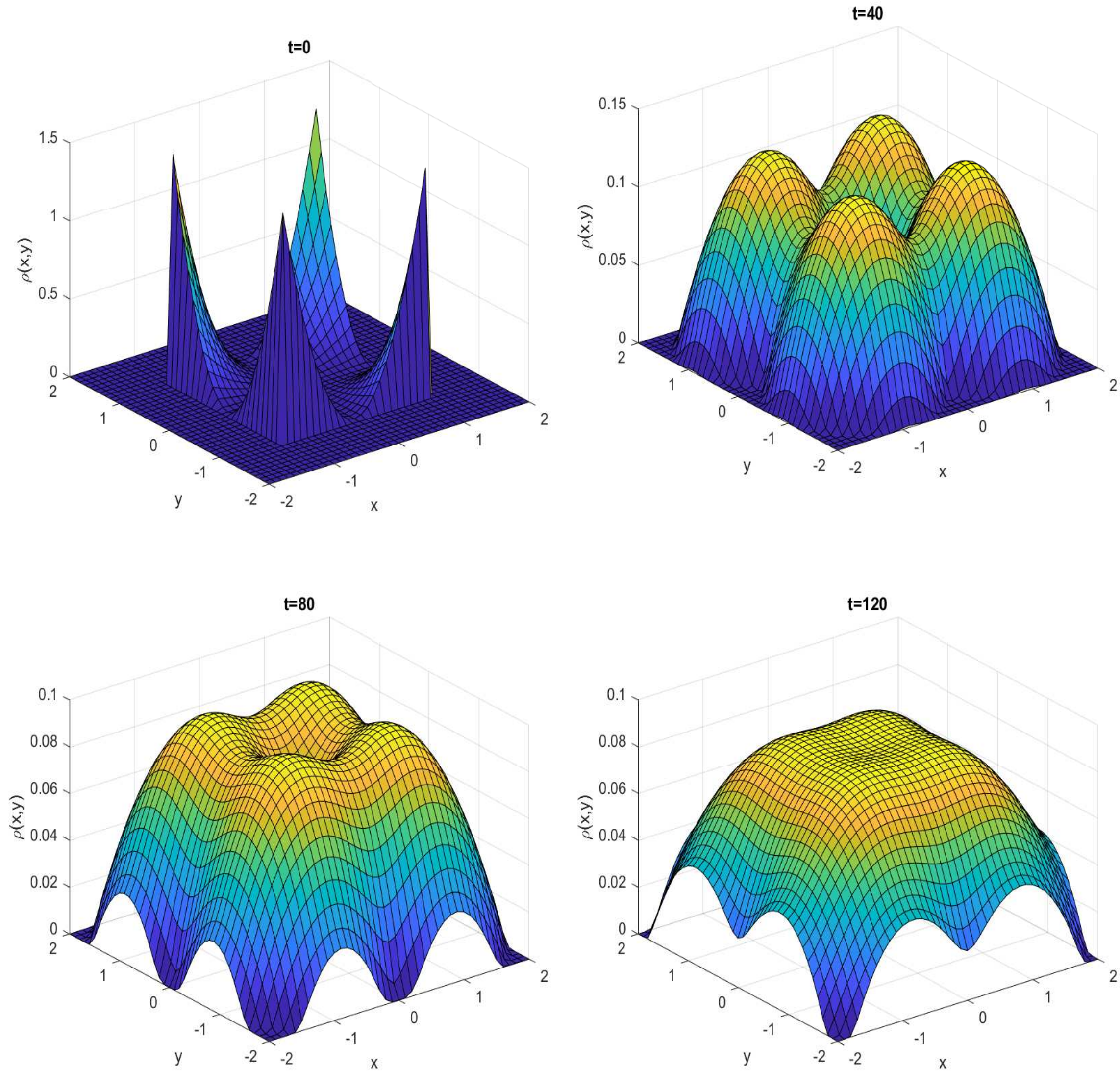


Appendix C

Example of diffusing downward converging to the steady-state

Figure 6

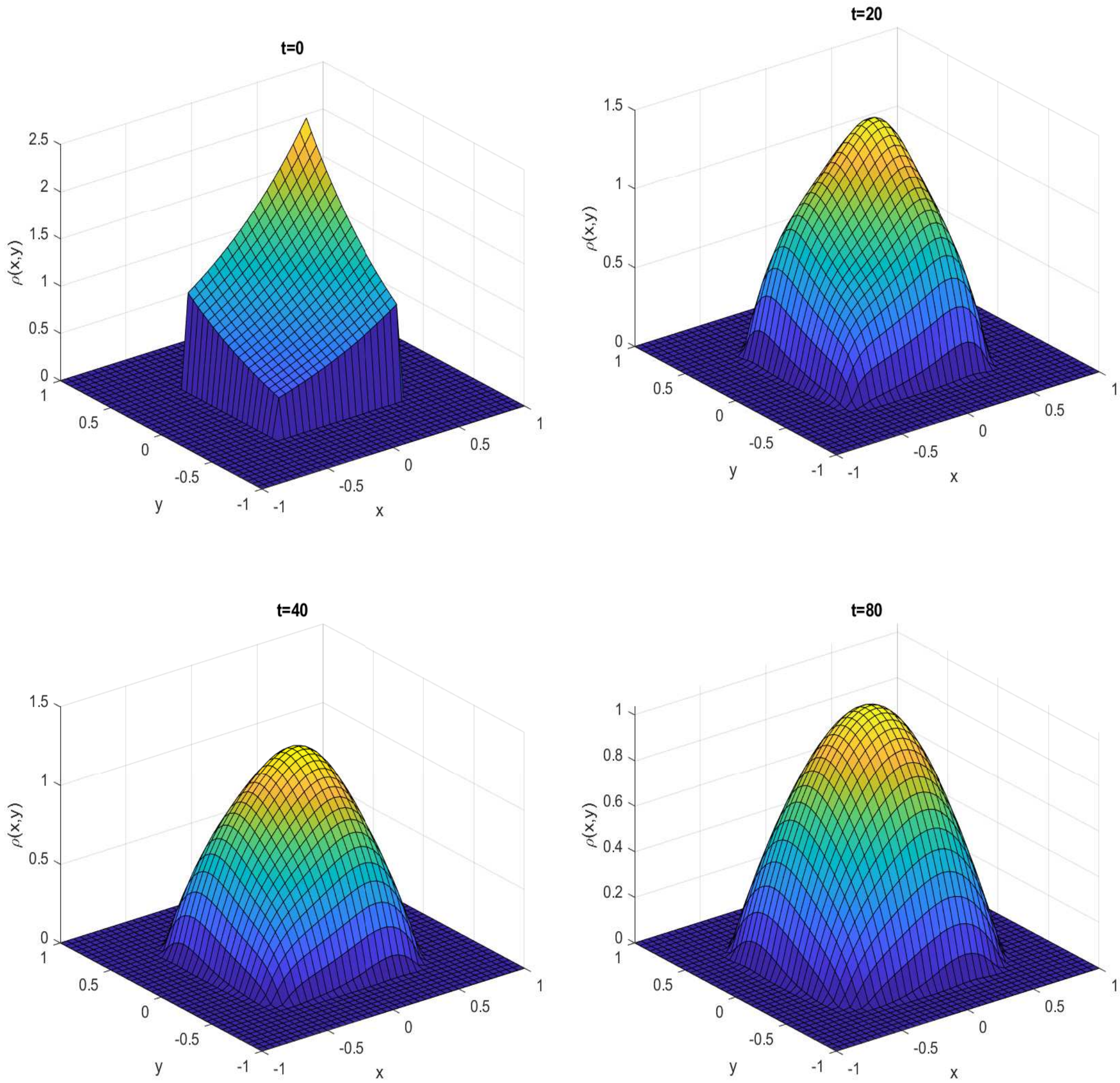
Diffusing downward converging to the steady-state for $\rho_0(x, y) = x^2y^2$, $N = 40$



Appendix D

Example of balanced steady-state

Figure 7
Balanced steady-state where $\rho_0(x, y) = e^{x+y}$, $N = 40$

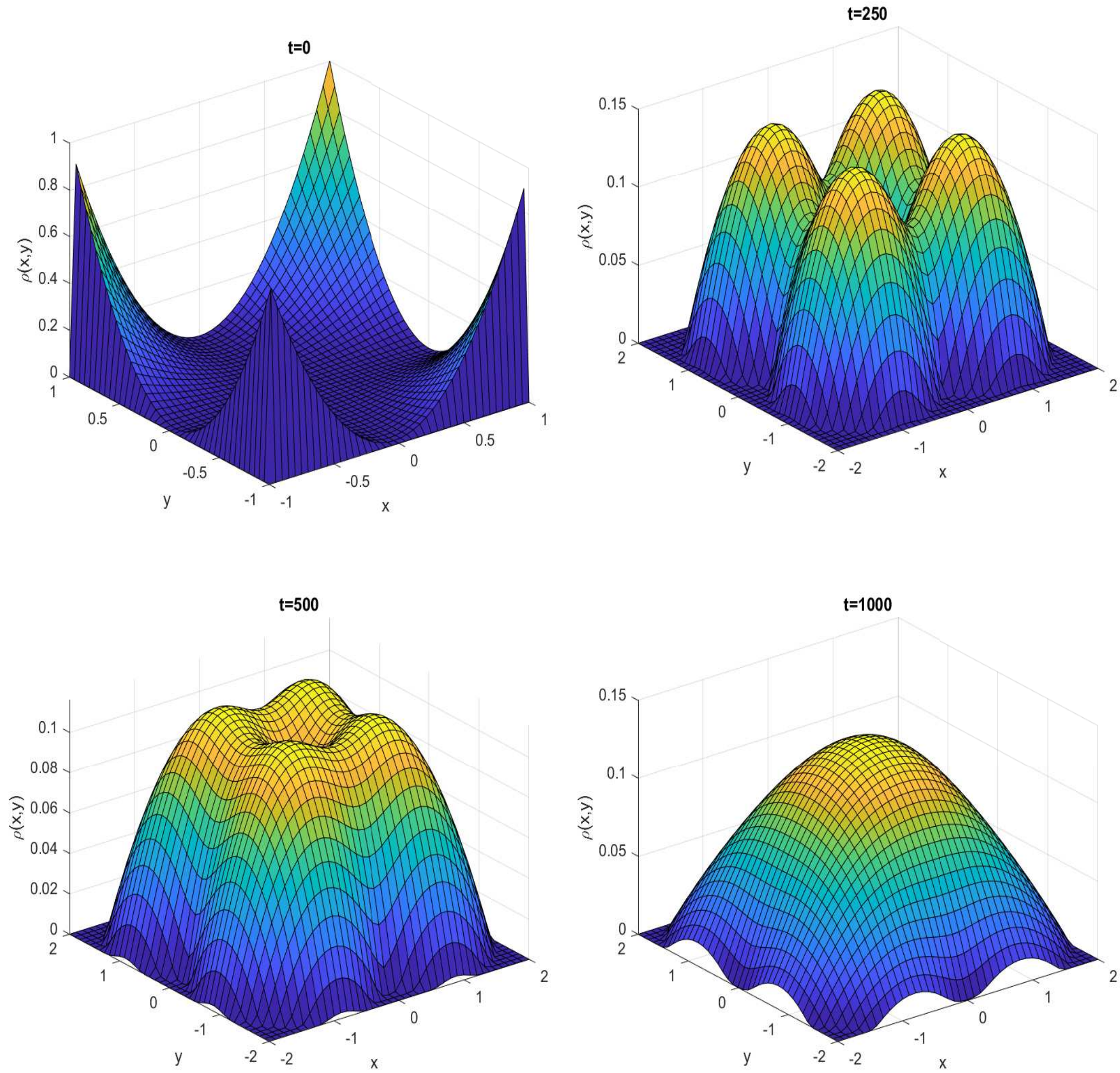


Appendix E

Example of balanced steady-state

Figure 8

Balanced steady-state where $\rho_0(x, y) = x^2y^2$, $N = 40$



جــامعـــــة النجاح الوطنية

كليـــة الدراســـــات العليـــا

وجود حلول ذات أنماط مميزة في نظام من المعادلات التفاضلية الجزئية (تجمع/انتشار) في بعدين

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إشراف

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قدمت هذه الرسالة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات، من كلية الدراسات العليا، في جامعة النجاح الوطنية، نابلس– فلسطين.

2023

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الملخص

في هذه الأطروحة قدمنا معادلة التجميع–الانتشار في فضاء ثنائي الأبعاد وبحثنا في حالات الحالة المستقرة، والتي تحتوي على عدة فئات، بعضها مباشر والبعض الآخر ليس كذلك. تمت دراسة الفئات عدديًا باستخدام طريقة الحجم المحدود، ثم تم استخدام طريقة SSP-RK من الدرجة الثالثة، وأخيرًا تم استخدام طريقة MATLABلتوضيح حالة الاستقرار في كل حالة. تنقسم الفئات إلى ثلاث فئات رئيسية اعتمادًا على قيمة θ، يسود أحدها الانتشار، بينما يهيمن التجميع في الحالة الثانية، والأخيرة يسود فيها التوازن بين الانتشار والتجميع.