A study on projective Modules and some weak forms of projectivity.

By

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Dedication

To my family members for the

support and encouragement.

A knowlegment:

First of all. I thank my God for all the blessings, he bestowed on me and continues to bestow on me.

I wish to extend my gratitude and appreciation to my supervisor, Dr. Ali Abdel-Mohsin for his help and encouragement during the preparation of this theses. I also wish to thank, Dr. Mohammed. Saleh and, Dr. Jaber Abu-Joukha for accepting to be members of the defense committee and for their valuable remarks.

Warmest thanks go to my family for their support and encouragement. Special thanks go to my Instructors and my friends for their help and encouragement. Finally I would like to thank the family of shua'a' center for printing this thesis.

Contents

Introduction

In this thesis we consider projective modules and some weak forms of projectivity and we try to study the most important known results concerning these modules.

In chapter one we summarize some of the essential and basic concepts in rings and module theory. This chapter consists of three sections, section 1 presents definitions and basic properties of modules. In section 2, we study the radical of modules and rings which plays an important role in our study. In section 3 we study simple and semisimple modules and rings.

In chapter Two, which is the main body of our thesis, we study the main characterizations and properties of projective modules. Morover we study radicals and endomorphism rings of projective modules. Finally, " projective covers " was studied in this chapter.

In chapter three we study semi-perfect and perfect rings as an a pplications to projective modules, those over which all finitely generated modules and, respectively, all modules have projective covers.

In chapter four, we study some weak forms of projective modules such as, weakly projective modules, ideal projectivity , Jacobson radical projectivity, and simple projectivity.

Chapter one

Modules

In this chapter we summarize some of the essential and basic concepts in ring and module theory needed in the sequal chapters. We note here that all rings used in this thesis are rings with unity.

Section (1.1): Definitions and Basic Properties

Let M be an abelian group,an endomorphism of M is just a group homomorphism $f:M\rightarrow M$, writing our functions on the left this means that $f(x+y) = f(x) + f(y)$, $\forall x,y \in M$, then the set of all such endomrphisms of M forms a ring with respect to addition $(f,g) \rightarrow f+g$ defined by $(f+g)(x) = f(x) + g(x)$, $x \in M$ and the composition of functions defined by $(fg)(x) = f(g(x))$.

This ring is denoted by $End^l(M)$, and called the ring of left endomorphisms of M, now the right endomorphism ring is defined simillarly and is denoted by $End^r(M)$.

Let R be a ring and M an abelian group and let λ be a map from R to $End^l(M)$, then we define the left R-module as follows.

Definition (1.1.1)

Let M be an abelian group and $\lambda: R \to End^l(M)$ be a map, then the pair (M,λ) is called a left R-module if λ is a ring homomorphism.

This means that for each $a \in R$, there is a mapping $\lambda(a): M \to M$ such that for all $a,b \in R$ and all $x,y \in M$ we have:

1- $\lambda(a)(x+y) = \lambda(a)(x) + \lambda(a)(y)$

2- $\lambda(a+b)(x) = \lambda(a)(x) + \lambda(b)(x)$

3- $\lambda(ab)(x) = \lambda(a)(\lambda(b)(x))$

4- $\lambda(1)(x) = x$

Now we may think of λ as defining a left scalar multiplication so we can write $\lambda : R \times M \to M$ such that $(a,x) \mapsto ax$. Using this notation (M,λ) is a left R-module if the following hold for every $a,b \in R$ and $x,y \in M$

1- $a(x+y) = ax + ay$

2- $(a+b)x = ax + bx$

3- $(ab)x = a(bx)$

4- $1x = x$

We simply say that M is a left R-module denoted by ${}_RM$ rather than (M,λ).

By a right R-module we mean an abelian group M and a ring homomorphism $\rho:R \rightarrow End^r(M)$, which means that there is a right scalar multiplication $\rho:M \times R \rightarrow M$ such that $(x,a) \mapsto xa,\ x \in M,\ a \in R$ satisfying for all $a,b \in R$ and $x,y \in M$

1- $(x+y)\,a = xa + ya$

2- $x\,(a+b) = xa + xb$

3- $x\,(ab) = (xa)\,b$

4- $x\,1 = x$

we note that if R is a commutative ring, then the left and right R-modules are the same. In this thesis we deal with left R-modules.

Example1: Every abelian group is a left Z-module.

Proof

Let A be an abelian group and Z the ring of integers, then there is a natural action of Z on A given by the mapping $Z \times A \rightarrow A$ defined by $(m,a) \rightarrow ma$ where

$$ma = \begin{cases} a + a + \ldots a & ,m\,times \quad if\ m > 0 \\ 0 & , \qquad\qquad if\ m = 0 \\ -a + -a + \ldots. + -a, & -m\,times \quad if\ m < 0 \end{cases}$$

This mapping satisfies the following properties for $m,n \in Z$ and $a,b \in A$

1- $(m+n)\,a = ma + na$

2- $m\,(a+b) = ma + mb$

3- (mn) a = m (na)

4- 1a = a

Hence this map is a ring homomorphism and so A is a left Z-module.

Example (2):

Given an abelian group M and a ring R, then it may not be possible to endow M with an R-module structure.

Proof

Suppose M is a left Z_n – module, where n is a positive integer, $n \geq 2$, then for the coset $m + (n) = \bar{m}$ and $a \in M$ we have

$0_M = 0_{z_n} a = \bar{n} a = (\bar{1} + \bar{1} + ... \bar{1}) a = \bar{1}a + ... \bar{1}a = a + a + ... + a = na$ Thus, every element of M has order dividing n, so for an a belian group M to be a left Z_n – module we must have $O(a) \mid n \ \forall a \in M$.

Definition (1.1.2):

Let M be a left R-module, then we say that N is a submodule of $_RM$, denoted $_RN \subseteq {_RM}$, if

1- N is a subgroup of M and

2- For $r \in R$, $n \in N$, $rn \in N$

Remark (1):

For a ring R we may regard $_RR$ or R_R, then a submodule of $_RR$ is just a left ideal of R and a submodule of R_R is just a right ideal of R.

Definition (1.1.3):

Let $f: {}_RM \to {}_RN$ be a group homomorphism and let $m \in M$ and $r \in R$ then, f is called an R-homomorphism provided $f(rm) = rf(m)$ for all $m \in M$, $r \in R$.

Now a one-one R-homomorphism is called an R-monomorphism and an onto R-homomorphism is called an R-epimorphism. An R-homomorphism which is both an R-monomorphism and an R-epimorphism is called an R-isomorphism.

Definition (1.1.4):

Let ${}_RN$ be a submodule of ${}_RM$, then $M/N = \{x + N : x \in M\}$ is called the left R-factor module of M modulo N.

It is easy to see that M/N is a left R-module under the operations

$(x + N) + (y + N) = (x + y) + N$ and

$r(x + N) = rx + N$, *where* $x,y \in M$, $r \in R$.

Theorem (1.1.5): (The Factor Theorem)

Let M, $\bar{M}$ and N be left R-modules and let $f: M \to N$ be an R-homomrphism. If $g: M \to \bar{M}$ is an epimorphism with $\ker g \subseteq \ker f$ then there exists a unique homomorphism $h: \bar{M} \to N$ such that $f=hg$. Moreover, $\ker h = g(\ker f)$ and $Im(h) = Im(f)$.

Proof

Suppose $g: M \to \bar{M}$ is an epimorphism, then for each $\bar{m} \in \bar{M}$ there is at least one $m \in M$ such that $g(m) = \bar{m}$.

If also $l \in M$ with $g(l) = m$, then $g(m-l)=0$ and so $m-l \in \ker(g)$, but since $\ker g \subseteq \ker f$, then $f(m-l) = 0$ which implies that $f(m) = f(l)$. Thus, there is a well defined function $h: \bar{M} \to N$ such that $f=hg$. To see that h is an R-homomorphism, let $\bar{x}, \bar{y} \in \bar{M}$ and let $x,y \in M$ with $g(x) = \bar{x}$ and $g(y) = \bar{y}$, then for each $a,b \in R$ $g(ax+by) = a\bar{x} + b\bar{y}$, thus

$h(g(ax+by)) = h(a\bar{x} + b\bar{y}) = f(ax+by) = af(x) + bf(y) = ah(g(x)) + bh(g(y)) = ah(\bar{x}) + bh(\bar{y})$.

Now for the uniqueness of h, suppose there is $\bar{h}: \bar{M} \to N$ an R-homomorphism such that $f = \bar{h}g$, then $hg = \bar{h}g$, thus for $\bar{x} \in \bar{M}$, there is $x \in M$ such that $g(x) = \bar{x}$ which implies that $hg(x) = \bar{h}g(x)$ and so $h(\bar{x}) = \bar{h}(\bar{x})$ which completes the proof.

Definition (1.1.6):

A pair of homomorphisms $M' \xrightarrow{f} M \xrightarrow{g} M''$ is said to be exact at M if, $Im(f) = \ker(g)$.

We can generalize this definition on a finite or infinite sequence of homomorphisms. We say that a sequence

$$\ldots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \to \ldots.$$

is exact in case it is exact at each M_i, that means for each successive pair f_n, f_{n+1} we have $Im\,(f_n) = ker\,(f_{n+1})$

Proposition (1.1.7):

Given modules M and N and a homomorphism $f{:}M \to N$, then the sequence

a) $0 \to M \xrightarrow{f} N$ is exact if and only if f is one - to - one

b) $M \xrightarrow{f} N \to 0$ is exact if and only if f is onto

c) $0 \to M \xrightarrow{f} N \to 0$ is exact if and only if f is an isomorphism

Proof

a) $0 \to M \xrightarrow{f} N$ is exact iff $Im\,(0 \to m) = ker\,f$ iff $0 = ker\,f$ iff f is 1-1.

b) $M \xrightarrow{f} N \to 0$ is exact iff $Im\,(f) = ker\,(N \to 0)$ iff

$Im\,(f) = N$ iff f is onto.

c) By using (a) and (b)

Example (3):

The sequence of canonical homomorphisms

$$0 \to L \xrightarrow{i} M \xrightarrow{\zeta} M/L \to 0$$

where i is the inclusion map and ζ is the natural epimorphism is an exact sequence.

Proof

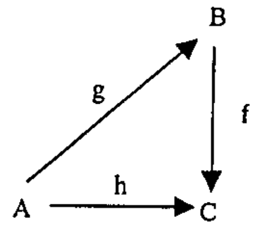
we note that $Im\ (0 \to L) = 0 = ker\ (i)$ and $Im\ (i) = L = ker\ (\zeta)$. So the sequence is exact.

Definition (1.1.8):

The exact sequence $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ is called a short exact sequence.

Definition (1.1.9):

The following diagram is said to be commutative if $h = fg$ and the same terminology is also used in more complicated diagrams.



Lemma (1.1.10):

In a short exact sequence $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$,f must be a monomorphism and g is an epimorphism.

Proof

since $Im\ (0 \to K) = 0 = ker\ (f)$, then f is monomorphism and since $Im\ (g) = Ker\ (N \to 0) = N$. thus g is an epimorphism.

We now turn to a related concept to module homomorphisms.

Definition (1.1.11):

An R-module M is the (internal) direct sum of its submodules M_1, M_2 if and only if $M=M_1+M_2$ and $M_1 \cap M_2 = 0$, and we write $M=M_1 \oplus M_2$

Thus $M = M_1 \oplus M_2$ if and only if for each $x \in M$, there exist unique elements $x_1 \in M_1$ and $x_2 \in M_2$ such that $x = x_1 + x_2$.

From the definition we note that not every submodule of a module M need to apear in such a direct foctorization of M, so the submodules that do have important a pplications.

Definition (1.1.12):

A submodule M_1 of M is called a direct summand of M if there is a submodule M_2 of M such that $M_1 \oplus M_2 = M$.

We note that M_2 is also a direct summand of M, and we *call* M_1, M_2 as a complemantary direct summands or direct complements of each other.

The following lemma shows how direct summands can be used in the study of homomorphisms.

Lemma (1.1.13):

Let $f: M \to N$ and $\bar{f}: N \to M$ be homomorphisms such that $f\bar{f} = I_N$. Then f is an epimorphism, $\bar{f}$ is a monomorphism and $M = \ker f \oplus Im\ \bar{f}$.

Proof

Let $y \in N$, then $f\bar{f}(y) = I_N(y) = y$, so $f(\bar{f}(y)) = y$, let

$\bar{f}(y) = x \in M$, then $f(x) = y$ and so f is an epimorphism, next let $y_1, y_2 \in N$ such that $\bar{f}(y_1) = \bar{f}(y_2)$, if we a pply f we get $f\bar{f}(y_1) = f\bar{f}(y_2)$ which implies that $y_1 = y_2$ and therefore f is monomorphism.

Now if $x = \bar{f}(y) \in ker\, f \cap Im\, \bar{f}$, then $0 = f(x) = f\bar{f}(y) = y$ and so $x = \bar{f}(y) = \bar{f}(0) = 0$, therefore $ker\, f \cap Im\, \bar{f} = 0$, let $x \in M$, then $f(x - \bar{f}f(x)) = f(x) - f\bar{f}(f(x)) = f(x) - f(x) = 0$ but $x = (x - \bar{f}f(x)) + \bar{f}f(x) \in kerf + Im\, \bar{f}$, which implies that $M = ker\, f \oplus Im\, \bar{f}$.

Definition (1.1.14):

Let $\bar{f}:M \to N$ and $\bar{f}:N \to M$ be homomorphisms with $f\bar{f} = I_N$ then we say that f is a split epimorphism and $\bar{f}$ is a split monomorphism.

Consider the short exact sequence $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ then by lemma (1.1.10) we see that f is a monomorphism and g is an epimorphism. Thus we can define the splitness of this sequence as the following.

Definition (1.1.15):

A short exact sequence $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ is split if and only if f is a split monomorphism and g is a split epimorphism.

The following proposition is an important charactorization of the above definition.

Proposition (1.1.16):

The following statements a bout a short exact sequence

$0 \rightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \rightarrow 0$ are equivalent.

a) The sequence is split.

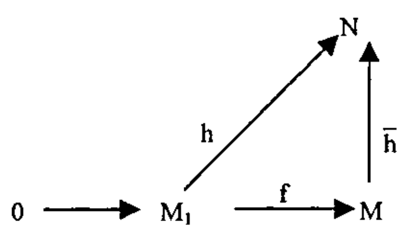
b) The monomorphism $f{:}M_1 \rightarrow M$ is split.

c) The epimorphism $g{:}M \rightarrow M_2$ is split.

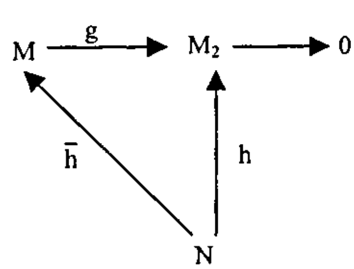
d) $Imf = ker\ g$ is a direct summand of M.

e) Every homomorphism $h{:}M_1 \rightarrow N$ factors through f (see diagram (1) below).

f) Every homormophism $h{:}N \rightarrow M_2$ factors through g (see diagram (2) below).



(1)



(2)

Proof

(a) implies (b) and (a) implies (c) by the definition, and (b) implies (d) and (c) implies (d) by lemma (1.1.13).

Now since (b) and (c) together give (a), then it is sufficaint to prove that (d) implies (e) implies (b) and (d) implies (f) implies (c).

(d) implies (e): Suppose $M = Im\, f \oplus K$, and $h{:}M_1 \to N$, then since f is monomorphism then for each $m \in M$, there is a unique $m_1 \in M_1$ and $k \in K$ such that $m = f(m_1) + k$. Define $\bar{h}: M \to N$ by $\bar{h}: m = f(m_1) + k \mapsto h(m_1)$, then $\bar{h}$ is an R-homomorphism and $\bar{h}f(m_1) = \bar{h}(f(m_1)) =$

$\bar{h}(f(m_1)+k) = h(m_1)$, so $\bar{h}f = h$.

(e) implies (b): let $h = I_N$, then $M_1 = N$ and so $\bar{h}f = I_N$ thus by the definition, f is split monomorphism.

(d) implies (f): suppose $M = kerg \oplus K$ and $h{:}N \to M_2$, then since $K \cap kerg = 0$ and $g(M) = g(K)$, we see that $(g/K){:}K \to M_2$ is an isomorphism. Let $\bar{g}: M_2 \to K$ be the inverse of g, then $\bar{h} = \bar{g}h: N \to M$ is an R-homomorphism such that $g\,\bar{h} = g\,\bar{g}h = I_{M_2}h = h$, thus $g\,\bar{h} = h$.

(f) implies (c): Let $h = I_N$, so $N = M_2$ and therefore g $\bar{h} = I_N$ which implies that g is a split epimorphism.

Example (4):

The sequence $0 \to Z_2 \xrightarrow{i} Z_6 \xrightarrow{\xi} Z_3 \to 0$ is a split exact sequence.

Proof

Since $Z_6 = Z_2 \oplus Z_3$, then by the a bove proposition the sequence is split.

Remark (2):

We note that splitting is defined only for exact sequences.

Proposition (1.1.17): Let N be an R-module and $\{L_i\}_{i\in I}$ an indexed family of submodules of N. Then their intersection $\bigcap_{i\in I} L_i$ is also a submodule of N.

Proof

Let $L = \bigcap_{i\in I} L_i$, then L is not empty since $0\in L$, and if $x,y \in L$ and $r\in R$ then $x + y \in L$ and $rx\in L$.

Now Let U be a subset of an R-module M, then by the above proposition the intersection L of all submodules of M which contains U is also a sumbodule and in fact is the smallest submodule of M containing U.

Definition (1.1.18):

Le U be a subset of $_RM$, and $\{L_i\}_{i\in I}$ an indexed family of submodules of $_RM$ containing U, then the submodule $L=\bigcap_{i\in I} L_i$ is called the submodule generated by U. and U is called a generating set for L.

If it happens that the submodule generated by U is M itself then we say that U is a system of generators for M.

Proposition (1.1.19):

Let U be a subset of an R-module M, and let $L=\bigcap_{i\in I} L_i$, where $\{L_i\}_{i\in I}$ is a family of submodules of M containing U, then $x\in L$ if and only if $x=r_1u_1 +r_2u_2 +\ldots+ r_nu_n$, where $r_i\in R$ and $u_i\in U$.

Proof

Let $\bar{L}$ be the set of all elements which can be written in the form $r_1 u_1 + r_2 u_2 + \ldots + r_n u_n$ with $r_i \in R$ and $u_i \in U$.

We want to show that $\bar{L}=L$. Now since $u_i \in U\ \forall i$ then $u_i \in \bigcap_{i \in I} L_i = L$, so $\sum_{i=1}^{n} r_i u_i \in L$, thus $\bar{L} \subseteq L$. Also if $u \in U$. then $u=1.u \in \bar{L}$, and hence $U \subseteq \bar{L}$, but $\bar{L}$ is a submodule of M. To see this, let $x,y \in \bar{L}$ and $r \in R$, then by the definition of L, we can write $x=r_1 u_1 + \ldots + r_n u_n$,

$y = r_1^* u_1^* + \cdots + r_t^* u_t^*$ where $r_i, r_i^* \in R$, $u_i, u_i^* \in U$, then we have

$x - y = (r_1 u_1 + \ldots + r_n u_n) - (r_1^* u_1^* + \ldots + r_t^* u_t^*)$ and $rx = (rr_1) u_1 + \ldots + (rr_n) u_n$ but this shows that $x + y$, rx both belong to L, and since L is the smallest submodule containing U we shall have $L \subseteq \bar{L}$.

Suppose that the elements $x_1, x_2, \ldots, x_n$ in a left R-module M, then we say that $x_1, x_2, \ldots, x_n$ generate the R-module M if and only if every element of the module M can be expressed in the form $r_1 x_1 + r_2 x_2 + \ldots + r_n x_n$ where $r_i \in R$.

Definition (1.1.20):

An R-module which can be generated by a finite number of elements is said to be a finitely generated module. If an R-module can be generated by one element a lone then it is said to be singly generated or cyclic.

This means that an R-module M is finitely generated if and only if for each $x \in M$, $x = r_1 x_1 + \dots + r_n x_n$ where $x_i \in M$, $r_i \in R$.

Example 5:

Let R be a ring, then the module $_R R$ is finitely generated by its unity, thus it is cyclic.

Proposition (1.1.21):

Let $f: M \to N$ be a module homomorphism and U a generating set in M. Then.

i) $f(U)$ is a generating set of $Im(f)$.

ii) If M is finitely generated (cyclic), then $Im(f)$ is finitely generated (cyclic).

iii) If $g: M \to N$ is another homomorphism, then $g=f$ if and only if $g(u) = f(u)$ for every $u \in U$.

Proof

The proof is immediate from the definition of the generating set, finitely generated module and homomorphism.

Definition (1.1.22): The family $\{x_i\}_{i \in I}$ is said to constitute a basis for M if each element $x \in M$ has a unique representation in the form

$x = \sum_{i \in I} r_i x_i$, where $x_i \in M$, $r_i \in R$ and $r_i = 0$ for almost all i in I.

Corollary (1.1.23):

If 0 has a unique representation, then so is every element in M.

Proof

Let $x \in M$, if $x = \sum_{i \in I} r_i x_i = \sum_{i \in I} r_i' x_i$, then $\sum_{i \in I} r_i x_i - \sum_{i \in I} r_i' x_i = 0$ which implies that $\sum_{i \in I} (r_i - r_i') x_i = 0$, but 0 has a unique representation, therefore $r_i - r_i' = 0$ implies $r_i = r_i'$

Definition (1.1.24):

An R-module which possesses a basis is said to be free.

Example (6):

1) The zero module is a free module with the empty set forming a base.

2) ${}_R R$ is free, since the identity element of R is a base.

Proposition (1.1.25):

A left R-module M is free if and only if $M \cong R^{(I)}$ for some I, where $R^{(I)} = \bigoplus_{i \in I} R$

Proof

The module $R^{(I)}$ is free with a basis $\{e_{ij}\}$, where

$$e_{ij} = \begin{cases} 1 & , \text{if } i = j \\ 0 & , \text{if } i \neq j \end{cases}$$

but $M \cong R^{(I)}$ which implies that M is free, conversly, if M is free with basis $\{x_i\}_I$, then we define $\alpha : R^{(I)} \to M$ by $(r_i)_I \mapsto \sum_{i \in I} r_i x_i$ which is an isomorphism by the definition of a basis.

Lemma (1.1.26):

Every module is a factor module of a free module.

Proof

Let $(x_i)_I$ be a generating set of the module M, $(x_i)_I$ exist since we can take $(x_i)_I$ as M itself. Then define $\alpha : R^{(I)} \to M$ as follows.

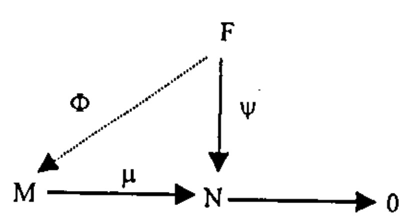
$\alpha ((r_i)_I) = \Sigma_I \, r_i \, x_i$ then α is an epimorphism, since for each $x \in M$, we have $x = \Sigma_I \, r_i \, x_i$, so there is $(r_i)_I$ such that $\alpha ((r_i)_I) = \sum_{i \in I} r_i x_i$.

By the first isomorphism theorem $R^{(I)}/ker\alpha \cong Im\,\alpha$ but α is onto, so $R^{(I)}/ker\alpha \cong M$ and by the a bove proposition (1.1.25), $R^{(I)}$ is free. Thus M is a factor module of a free module.

The following theorem represent a fundamantal property of a free module which we'll concentrate on later.

Theorem (1.1.27):

Let F be a fee R-Module, $\psi : F \to N$ an R-homomorphism and $\mu : M \to N$ an epimorphism of R-modules. Then it is possible to find an R-homomorphism $\Phi : F \to M$ such that the following diagram is commutative. That is $\mu\phi = \psi$.



Proof

Let $(u_i)_{i\in I}$ be a base for F, then $\Psi(u_i)$ is an element of N for each i, now since μ is an epimorphism, we can find an element $v_i \in M$ such that $\mu\ (v_i)= \Psi(u_i)$, but F is free, hence there is a homomorphism $\Phi:F\rightarrow M$ such *that* $\Phi(u_i)=v_i$ and then we have $\mu\Phi(u_i)= \mu\ (v_i)= \Psi\ (u_i)$.

Thus $\mu\ \Phi= \Psi$.

Section (1.2): Radical of modules and rings.

We begin this section by the following definition

Definition (1.2.1):

A submodule K of M is superfluous (small) in M a bbreviated K ⟨⟨M in case for every submodule L of M such that $K+L=M$, then $L=M$.

Example (1): The only small submodule of $_zZ$ is 0.

To show this. If N is a submodule of $_zZ$, then $N=nZ$ for some $n\in Z$, then there is a prime number p such that p dosen't divides n, hence $GCD\ (p,n)=1$ which implies that $pZ+nZ = (1)= Z$, but $pZ \neq Z$,so N is not small in Z and thus 0 is the only small sumbodule of $_ZZ$.

Now we'll represent the concept of nilpotency which is needed later on.

Definition (1.2.2):

An element x of a ring R is nilpotent in case there is a natural number n such that $x^n=0$, the least such n is called the niplotency index of the element.

The elementwise concept of nilpotence can be extended. Thus, a subset A of a ring R is nilpotent in case there is an integer $n>0$ such that $x_1x_2...x_n=0$ for every sequence $x_1,x_2,...,x_n$ in A.

Definition (1.2.3):

A subset A of a ring R is nil if each of its elements is nilpotent.

Remark (1): Every nilpotent subset of R is nil.

Example (2): Any nil (lef t) ideal I in a ring R is small as a left module.

Proof

suppose $R=I+J$ for some left ideal J in R, then $1=i+j$ for some suitable $i\in I$ and $j\in J$, now since I is nil then i is nilpotent element, so there is $k\in N$ such that $i^k=0$, but $i=1-j$ so we get $0=i^k=(1-j)^k=1+\bar{j}$ for some $\bar{j}\in J$, therefore $1\in J$ and this implies that $J=R$.

Proposition (1.2.4): Let ${}_RM$ be a left R-module, then

1- If A,B are small in M, then $A+B$ is small in M.

2- If $f: {}_RM\to{}_RN$ is a homomorphism and A is small in M then $f(A)$ is small in N.

Proof

1) Let K be a submodule of M such that $(A+B)+K=M$ hence $A+(B+K)=M$. As $A \langle\langle M$. then $B+K=M$ and also since $B \langle\langle M$ we get $K=M$ and hence $A+B \langle\langle M$.

2) Let B be a submodule of N and $f(A)+B=N$. Let $m \in M$, then $f(m) \in N$ and hence we can write f(m) as $f(m)=f(a)+b$ where $a \in A$ and $b \in B$ and this implies than $f(m-a)=b$ which implies that

$(m-a) \in f^{-1}(B)$

hence $m = a+f^{-1}(B)$ and then $M=A+f^{-1}(B)$. But $A \langle\langle M$, so $f^{-1}(B)=M$ which implies that $f(M) \subseteq B$ and so $f(A) \subseteq f(M) \subseteq B$. Therefore

$N=f(A)+B=B$. which completes the proof.

We see from the previous proposition that the sum of small submodules is small and a homomorphic image of a small submodule is also samll.

Now we can give the definition of the radical of a left R-module.

Definition (1.2.5):

Let $\{M_\alpha\}_{\alpha \in A}$ be infinite family of submodules of M, a sum is defined by:

$$\sum_{\alpha \in A} M_\alpha = \{ \sum_{k=1}^{r} m_{\alpha_k} : r \in N, \alpha_k \in A, m_{\alpha_k} \in M_{\alpha_k} \}.$$

lemma (1.2.6):

$\sum_{\alpha \in A} M_\alpha$ is a submodule of M, and moreover it is the smallest submodule containing all M_α.

Proof

To see that $S = \sum_{\alpha\in A} M_\alpha$ is a submodule of M, let $a, b \in S$, then $a = \sum_{k=1}^{s} m_{\alpha_k}$, $b = \sum_{k=1}^{t} m_{\alpha_k}$ for some $s, t \in N$, so $a + b = \sum_{k=1}^{r} m_{\alpha_k} \in S$. Let $r \in R$, then $ra = \sum_{k=1}^{s} rm_{\alpha_k} \in S$. Moreover if N is any submodule of M containing all M_α, then for $x \in S$, $x = \sum_{k=1}^{r} m_{\alpha_k}$, and since N contains each M_α then $m_{\alpha_k} \in N$ for each $k = 1, \ldots, r$. Thus $x \in N$.

Definition (1.2.7): let ${}_RM$ be a left R- module, then the radical of ${}_RM$ is denoted $Rad(M)$ or $J(M)$ is defined by:

$J(M) = \Sigma\{L: L \text{ is a small submodule of } M\}$.

Corollary (1.2.8): From the previous definition we note that the radical of a module M is a submodule of M.

Definition (1.2.9):

A submodule ${}_RB$ of ${}_RM$ is a maximal submodule provided $B \neq M$ and if C is a submodule of M with $B \subseteq C \subseteq M$, then $C=B$ or $C=M$.

Theorem (1.2.10):

Let ${}_RM$ be a left R-module. then $J(M) = \bigcap_{B\in\Omega} B$ where $\Omega = \{B : B$ is a maximal submodule of ${}_RM\}$. Moreover if $\Omega = \varnothing$, then $J(M)=M$.

Proof

Let A be any small submodule of M, and B any maximal submodule of M. Now since $A+B$ is a submodule of M and A is small in M, then $A+B \neq M$. And also we have $B \subseteq A+B$, but B is maximal in M so $A+B \subseteq B$, hence $B=A+B$ which implies that $A \subseteq B$. And this shows that every small submodule is contained in all maximal submodules thus $J(M) \subseteq \bigcap_{B\in\Omega} B$. Conversly, let $x \in M$. If N is a submodule of M with $Rx + N = M$, then either $N=M$ or there is a maximal submodule K of M with $N \subseteq K$ and $x \notin K$. Now if

$x \in \bigcap_{B\in\Omega} B$, then $x \in K$, so the latter cannot occur, thus $Rx \ll M$ which implies that $Rx \subseteq J(M)$. Thus $x \in J(M)$.

Now, Let us return to small submodules for more investigation.

Lemma (1.2.11):

Let S, $\bar{M}$, M be left R-modules such that $S \subseteq \bar{M} \subseteq M$ and S is small in $\bar{M}$, then S is small in M.

Proof

Let $S+N=M$, where N is a submodule of M. Then $S+(N\cap \bar{M})=\bar{M}$, but S is small in $\bar{M}$, so $N\cap M=\bar{M}$ which implies than $N\supseteq \bar{M}\supseteq S$, thus $N=M$.

Lemma (1.2.12):

If S is a small submodule in M, then every submodule of S is small in M and if S_i $(1\leq i\leq n)$ are small in M, then $\sum_{i=1}^{n} S_i$ is also small in M.

Proof

Let K be any submodule of S, and $S \ll M$ and let $K+N=M$, for some submodule N of M, then since $K\subseteq S$, $K+N\subseteq S+N$ which implies that $M=S+N$, thus $N=M$. Now if each S_i is small in M and $\sum_{i=1}^{n} S_i+\bar{M}=M$ for some submodule $\bar{M}$ of M, then we can use induction to get $\bar{M}=M$.

Proposition (1.2.13):

Let M,N be R-modules, and let $f:M\rightarrow N$ be a homomorphism, then $f(J(M))\subset J(N)$.

Proof

By proposition (1.2.4) and the definition of the radical.

Proposition (1.2.14):

Let $\bar{M}\subseteq M$ be R-modules then $J(\bar{M})\subseteq J(M)$.

Proof

Since $J(\overline{M})$ is the sum of all small submodules S of $\overline{M}$ we have by lemma (1.2.11) that each small S of $\overline{M}$ is small in M, so we get that each small submodule in $\overline{M}$ is also small in M, thus $J(\overline{M}) \subseteq J(M)$.

Theorem (1.2.15):

$$\textit{If } M = \bigoplus_{i \in I} M_i \textit{ then } J(M) = \bigoplus_{i \in I} J(M_i)$$

Proof: by the above proposition (1.2.14).

Definition (1.2.16):

The radical of ${}_R R$ is called the Jacobson radical of R,

i.e $Jac(R) = Rad({}_R R) = J(R)$.

Now by the previous characterizations of the radicals we see that $J(R)$ is the intersection of all maximal left ideals in R. And also the sum of all small left ideals in R.

Definition (1.2.17):

An element r in a ring R is called left (right) quasi – regular if and only if $(1-r)$ has a left (right) inverse.Equivalentely r is a left (right) quasi-regular if there is $a \in R$ s.t, $a + r + ar = 0 (a + r + ra = 0)$.

Definition (1.2.18):

A left (right) ideal I of R is left (right) quasi-reguler provided $(1-a)$ has a left (right) inverse for each $a \in I$.

An element $r \in R$ is called quasi-regular if it is left and right quasi-regular. Also an ideal I is called quasi-regular if it is left and right quasi-regular.

Lemma (1.2.19):

If I is an ideal of R, then I is left quasi-regular if and only if I is right quasi-regular.

Proof

Suppose I is an ideal of R and I is left quasi-regular. If $a \in I$, then $(1-a)$ has a left inverse say b, such that $b\ (1-a)=1$,so $b-ba=1$ or $b=1-(-ba)$, since $-baI$ then $b=1-(-ba)$ has a left inverse. But b has a right inverse namely $1-a$, thus b is invertable and $b^{-1} = 1-a$, so $(1-a)$ has a right inverse. The converse is simillar.

Lemma (1.2.20):

A left (right) ideal I of R is left (right) quasi-regular if and only if Ra (aR) is small in $_RR$ (R_R) for each $a \in I$.

Proof

Suppose I is a left quasi-regular ideal of R. If $a \in I$ and B is a left ideal of R such that $Ra+B=R$, then $ra+b=1$, for some $r \in R$, $b \in B$, so $b=1-ra$ has a left inverse $\bar{b}$ such that $\bar{b}b=1 \in B$, thus $B=R$.

Conversly suppose I is a left ideal of R and Ra is small in ${}_RR$, then for each $a \in I$, $Ra+R\ (1-a)=R$, but $Ra \langle\langle R$ So $R(1-a)=R$, hence there is $r \in R$ such that $r(1-a)=1$, therefore $1-a$ has aleft inverse.

The proof is simillar in the case that I is right quasi-regular.

Corollary (1.2.21):

$J(R)$ is a quasi-regular ideal in R.

Proof

Let $x \in J(R)$, then $Rx \subseteq J(R)$. Thus by the definition of $J(R)$, Rx is small in R, hence by the above lemma's (1.2.19), (1.2.20), $J(R)$ is a quasi-regular ideal.

Corollary (1.2.22):

$J(R)$ is a small ideal in R.

Proof

Let K be an ideal of R, with $J(R)+K=R$, then $j+k=1$ for some $k \in K$, $j \in J(R)$. since $k=1-j$ and $J(R)$ is quasi-regular then k is invertible, therefore $1 \in K$ which implies that $K=R$.

Proposition (1.2.23):

For a module ${}_RM$, $J(R)M \subseteq J(M)$.

Proof

For $x \in M$, we define $f_x: {}_RR \to {}_RM$ by $f_x: r \to rx$ is an R-homomorphism. Now $f_x(J(R)) = J(R)x \subseteq J(M)$ by proposition (1.2.13). Hence $J(R)M \subseteq J(M)$.

Lemma (1.2.24):

If R is a ring, then $J(R)$ contains no non-zero idempotents.

Proof

If $e \in R$ is an idempotent in $J(R)$, then $Re + R(1-e) = R$ but $Re \subset J(R)$, hence it is small in R and therefore $R(1-e) = R$ which implies that $R - Re = R$ and hence $e = 0$.

Theorem (1.2.25):

Every non-zero finitely generated module M contains a maximal submodule.

Proof

Let μ be the set of all proper submodules of M, partially ordered under inclusion, let T be a totally ordered subset of M, and let $\bar{L}$ be the sum of all $L \in T$. Now if we show that $\bar{L} \neq M$, then L will be an upper bound for T in μ, and Zorn's Lemma can be applied to give a maximal proper submodule of M. If $\bar{L} = M$. then $\bar{L}$ contains a finite generating set

$x_1, \ldots, x_n$ for M and each x_i lies in a module belonging to T, so $\{x_1, \ldots, x_n\}$ $\subset L$ for some $L \in T$ which implies that $L=M$, which is a contradiction.

The above theorem establishes the existence of maximal ideals in R, because each ring R is finitely generated.

Corollary (1.2.26):

If ${}_RM \neq 0$ is a finitely generated module then $Rad\ (M) \neq M$.

Proof

By the above theorem ${}_RM$ has a proper maximal submodule. And since $Rad\ (M)$ is the intersection of all maximal submodules of M. Thus

$Rad\ (M) \neq M$.

Now by using the above corollary, we have the following very useful characterization of $J(R)$. It is often called Nakayama's lemma.

Proposition (1.2.27): (Nakayama's Lemma)

For a left ideal I of a ring R, the following are equivalent:

a) $I \subset J\ (R)$

b) For any finitely generated left R-module M,if

$IM=M$, *then* $M=0$.

c) For any finitely generated left R-module M, $IM \langle\langle M$.

Proof

(a) implies (b).

Suppose $M \neq 0$ is a finitely generated module, then by theorem (1.2.25), M has a maximal submodule K, hence by proposition (1.2.23), $J(R)M \subseteq J(M) \subseteq K$. Now if $IM=M$ then $IM=M \subset J(R)M \subseteq J(M) \subseteq K$. Which implies that $M=0$

(b) implies (c): Let N be a submodule of M such that $IM+N=M$, then $I(M/N) = (IM+N)/N=M/N$, so if M is finitely generated, then M/N is also finitely generated and so we can apply (b) here to get $M/N=0$ and so $M=N$. therefore $IM \langle\langle M$.

(c) implies (a).

Assume (c), then since ${}_RR$ is finitely generated, we have $IR \langle\langle R$, thus $I \subset IR \subset J(R)$.

Corollary (1.2.28):

If ${}_RM$ is a finitely generated module then $J(M)$ is small in M.

Proof

Suppose ${}_RM$ is finitely generated, then by corollary (1.2.26) $J(M) \neq M$. Now Let L be any submodule of ${}_RM$ such that $J(M)+L=M$ and $L \neq M$. by theorem (1.2.25), L must be contained in a maximal proper submodule K, so $M=J(M)+L \subseteq J(M)+K$, but $J(M) \subseteq K$, therefore $M=K$ which contradicts that K is a proper maximal submodule. Thus $L=M$ and hence $J(M)$ is small in M.

Proposition (1.2.29):

Let M be an R-module, and $f:M\rightarrow N$ be a homomorphism then

1) $f(RadM) = Rad(f(M))$, if $Ker f \subset Rad M$

2) $Rad(M/Rad M) = 0$

Proof

(1) $f(RadM) \subseteq Rad(f(M))$ is true in general by proposition (1.2.13). Conversly, since $ker f \subset Rad M$, then $Ker f$ is contained in each maximal submodule of M and since $f:M\rightarrow f(M)$ is an epimorphism, then by the correspondence theorem there is a one-one correspondence between the maximal submodules K of M that contains $Kerf$ and the submodules of $f(M)$ given by $K\rightarrow f(K)$. Thus we have $Rad(f(M))=f(RadM)$.

(2) Let $\zeta:M\rightarrow M/RadM$ be the natural epimorphism, since $ker\zeta \subset Rad M$, then by (1), $Rad(\zeta(M))=Rad(M/RadM)=\zeta(Rad M)=0$

Definition (1.2.30):

A ring R is called local if the set of non-invertible elements of R is closed under a ddition.

Now by using the jacobson radical of a ring R, we have the following characterization of this class of rings.

Proposition (1.2.31):

For a ring R the following statements are equivalent

a) R is a local ring

b) R has a unique maximal left ideal

c) $J(R)$ is a maximal left ideal.

d) The set of elements of R without left inverses is closed under addition.

e) $J(R) = \{x \in R \mid Rx \neq R\}$

f) $R/J(R)$ is a division ring

g) $J(R) = \{x \in R : x \text{ is not invertible}\}$.

h) If $x \in R$ then either x or $1-x$ is invertible.

Proof

(b) iff (c): It is clear since $J(R)$ is the intersection of all maximal ideals in R.

(c) implies (d): suppose that $J(R)$ is a maximal left ideal, then $J(R)$ is unique. Let $x, y \in R$ be non-left invertible, then since every proper left ideal is contained in a maximal one say $Rx, Ry \subset J(R)$ thus, $x+y \in J(R)$, so $x+y$ is not invertible.

(d) implies (e): Assume (d), since $J(R)$ is a proper left ideal, then we only show that if $x \in R$ with $Rx \neq R$, then $x \in J(R)$. Now for each $r \in R$, rx doesn't have a left inverse and $1 = rx + (1-rx)$, so by (d) $1-rx$ has a left inverse implies that rx is a left quasi-regular element, so $rx \in J(R)$ which implies that $x \in J(R)$.

(e) implies (f): Let $J(R) = \{x \in R: Rx \neq R\}$, it follows that every non-zero element of $R/J(R)$ has a left inverse, thus $R/J(R)$ is a division ring.

(f) implies (b): Since a division ring has no non-trivial ideals then if $R/J(R)$ is a division ring, then $J(R)$ is a maximal left ideal.

(h) implies (g): Assume (h), and let $x \in R$ be non-invertable, say x has no left inverse, then no rx is invertable, so by (h) each rx is quasi-reqular, thus $x \in J(R)$.

(f) implies (g): Let $R/J(R)$ be a division ring, and let $x \in R$ and $x \notin J(R)$, then $x+J(R)$ is invertible, that is $Rx+J(R)=R$ *and* $xR+J(R)=R$, *but* $J(R)$ is small in R by corollary (1.2.28), thus we get $Rx=R$ *and* $xR=R$, so x is invertable.

Finally (g) implies (f) and (g) implies (a) implies (h) are clear.

Section (1.3): Simple and semisimple modules and rings.

In this section we'll study an important class of modules and rings, namely simple and semisimple modules and rings.

We'll present the basic properties and the most well known characterizations of these modules. The importance of these classes of rings comes from its relation with projective modules.

Definition (1.3.1):

Let R be a ring, and $_RM$ be a module then M is called a simple R-module if M has no R–submodules other than (0) and M.

Definition (1.3.2):

The left R-module M is called semisimple if it is the direct sum of simple submodules of M.

Remark(1.3.3): If $_RM$ is simple, then it is also semisimple. To see this suppose M is a simple module, then M has only two submodules M and (0), so $M=M\oplus 0$.

Lemma (1.3.4):

Every simple R-module is a cyclic module

Proof

Let x be a non-zero element in M, then Rx is a submodule of M and since M is simple then $Rx=M$. Thus M is cyclic.

Theorem (1.3.5):

Let $_RM$ be a simple module, then $_RM \cong R/B$ where B is a maximal ideal of R.

Proof

Let $0\neq x\in M$, then by the above lemma (1.3.4) $Rx=M$, hence $f_x: {}_RR\rightarrow {}_RM = Rx$ is an R-epimorphism and $ker\ (f_x) = \{r\in R \mid rx=0\}$ is a left ideal of

R, and by the corrospondence theorem $ker\ f_x$ is a maximal left ideal of R. Thus by the fundamental R-homomorphism theorem $_RM \cong R/ker f_x$.

Lemma (1.3.6):

Let $(T_\alpha)_{\alpha\in A}$ be an indexed set of simple submodules of the left R-module M. If $M=\sum_{\alpha\in A} T_\alpha$, then for each submodule K of M there is a subset B of A such that $\left(K \cap \left(\sum_{\beta\in B} T_\beta\right) = 0\right)$ and $M=K\oplus \left(\oplus_B T_\beta\right)$.

Proof

Let K be a submodule of M, then by the maximum principle there is a subset B of A which is maximal with respect to the condition that

$K \cap \left(\sum_{\beta\in B} T_\beta\right) = 0$, then the sum

$N = K + \sum_{\beta\in B} T_\beta$ is direct. Now we claim that $N=M$, let $\alpha \in A$, since T_α is simple, then either $T_\alpha \cap N = T_\alpha$ or $T_\alpha \cap N=0$, but $T_\alpha \cap N=0$ contradicts the maximality of B. Thus $T_\alpha \cap N= T_\alpha$ and so T_α is a submodule of N for each $\alpha \in A$, so $M=N$.

The above lemma verifies that for a module M,M is semisimple if every submodule of M is a direct summand of M.

Corollary (1.3.7):

If a module M is generated by an indexed set $\{T_\alpha\}_{\alpha\in A}$ of simple submodules, then for some $B\subseteq A$, $M = \oplus_{\beta\in B} T_\beta$, that is M is semisimple.

Proof

Use the above lemma with $K=0$.

Proposition (1.3.8):

Let M be a semisimple left R-module with semisimple decomposition $M = \bigoplus_{\alpha \in A} T_\alpha$ .If $0 \to K \to M \to N \to 0$ is an exact sequence of R-modules, then the sequence splits and both K and N are semisimple. Indeed there is a subset $B \subseteq A$ and isomorphisms $N \cong \bigoplus_{B} T_\beta$ *and* $K \cong \bigoplus_{A-B} T_\alpha$

Proof

Since $Im\ (f)$ is a submodule of M, then by lemma (1.3.6) there is a subset $B \subseteq A$ such that $M = Imf \oplus (\bigoplus_{B} T_\beta)$. Thus by proposition (1.1.16) the sequence splits and by the fundamental theorem of isomorphism, $N \cong M/kerg = M/Imf \cong \bigoplus_{B} T_\beta$, But $M = (\bigoplus_{A-B} T_\alpha) \oplus (\bigoplus_{B} T_\beta)$, so $K = \bigoplus_{A-B} T_\alpha$.

Remark: since the sequence $0 \to K \xrightarrow{i} M \xrightarrow{\xi} M/K \to 0$ is an exact sequence, where i is the inclusion map and ξ is the natural epimorphism.Then by the above proposition if M is a semisimple module, then K and M/K are also semisimple modules. So every submodule and every factor module of a semisimple module is semisimple.

Corollary (1.3.9):

Let $(T_\alpha)_{\alpha\in A}$ be an indexed set of simple submodules of M. if T is a simple submodule of M such that $T\cap(\sum_{\alpha\in A} T_\alpha)\neq 0$, then there is an $\alpha\in A$ such that $T\cong T_\alpha$.

Proof

If T is simple and $T\cap(\sum_{\alpha\in A} T_\alpha)\neq 0$, then $T=T\cap(\sum_{\alpha\in A} T_\alpha)$ so $T\subset \sum_{\alpha\in A} T_\alpha$. Then we may assume that $M=\sum_{\alpha\in A} T_\alpha$, thus by corollary (1.3.7), M is semisimple and $M=\oplus_B T_\beta$ for some $B\subseteq A$, Now applying proposition (1.3.8) we have $T\cong T_\alpha$ for some $\alpha\in A$.

Now we have the following fundamental characterization of semisimple modules.

Theorem (1.3.10):

For a left R-module TFAE:

a) M is semisimple

b) M is generated by simple submodules.

c) M is the sum of some set of simple submodules

d) M is the sum of its simple submodules.

e) Every submodule of M is a direct summand.

f) Every short exact sequence $0\to K\to M\to N\to 0$ of left R-modules splits.

Proof

(a) implies (f). By proposition (1.3.8) and (f) implies (e) by proposition (1.1.16) and (b) implies (a) by proposition (1.3.7), also (b) iff (c) iff (d) are all trivial.

We want to show that (e) implies (d). Suppose that M satisfies (e), we claim that every non-zero submodule of M has a simple submodule, to see this let $0 \neq x \in M$, then by theorem (1.2.25), Rx has a maximal submodule say H, so by (e) we have $M = H \oplus \bar{H}$ for some $\bar{H}$ submodule of M, thus $Rx = R \cap M = H \oplus (Rx \cap H)$. Therefore $Rx/H \cong Rx \cap \bar{H}$ is simple, so Rx has a simple submodule. Let N be the sum of all simple submodules of M then by using (e), $M = N \oplus \bar{N}$ for some submodule $\bar{N}$ of M. since $N \cap \bar{N} = 0$, N has no simple submodules which means that $\bar{N} = o$, so $M = N$.

Definition (1.3.11):

The ring R is said to be left semisimple if and only if ${}_RR$ is semisimple.

Definition (1.3.12): The ring R is simple if and only if ${}_RR$ is simple .

Now we close this section by the following theorem :

Theorem (1.3.13) :

Let R be a ring and let M be a left R - module . Then the following statements are equivalent.

a) All left R-modules are semisimple.

b) R is a semisimple ring.

Proof

(b) implies (a). Let M be any left R-module, where R is a semisimple ring, then any cyclic submodule Rm of M is semisimple. Since $M = \sum_{m \in M} R.m$, then by theorem (1.3.10.c), M is semisimple. The converse is trivial.

Chapter Two

Projective Modules

This chapter is the main body of our thesis, in it we give the definition of " projective modules ", and discuss the main characterizations and properties of these modules.

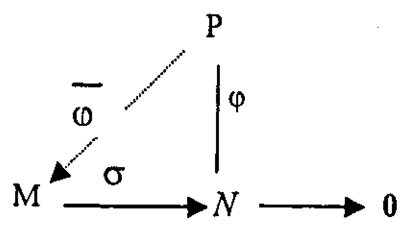
Section (2.1): Definition and characterizations of projective modules

The notion of projective module is a generalization of the concept of a free module which have the fundamental property mentioned in theorem (1.1.27) where we saw that if F is a free R-module, $\Psi : F \to N$ is an R-homomorphism and $\lambda : M \to N$ is an epimorphism of R-modules, then there is a homomorphism $\phi : F \to M$ such that $\lambda \phi = \Psi$.

This property of free modules defines the class of projective modules.

Definition (2.1.1):

Let P, M, N be left R-modules , then P is projective relative to M (or P is M-Projective) if and only if for each epimorphism $\sigma : M \to N$ and each homomorphism $\varphi : P \to N$, there is an R-homomorphism $\bar{\varphi} : P \to M$ such that the following diagram commutes, that is $\sigma \bar{\varphi} = \varphi$.



Defintion (2.1.2): A module ${}_RP$ is said to be projective if and only if it is projective relative to every module ${}_RM$.

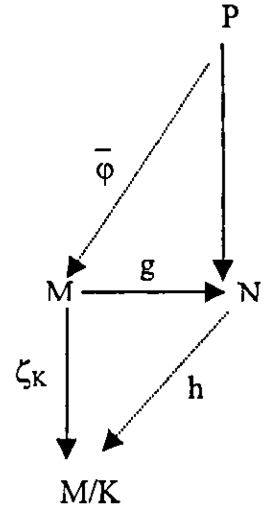
Theorem (2.1.3): Let P, M be left R-modules , then P is M-projective if and only if for each submodule K of M, every R-homomorphism $f:P\to M/K$ factors through the natural epimorphism $\zeta_K: M\to M/K$.

Proof

Let $g:M\to N$ be an epimorphism of R-modules with $K= ker g$. Since $Ker g = K \subseteq ker\ \zeta_K = K$, then by the factor theorem (1.1.5) there is a unique homomorphism $h:N \to M/K$ such that $hg = \zeta_K$,

with $ker\ h= g\ (ker\ \zeta_K) = g\ (K) = g\ (ker\ g) = 0$ and $Im\ (h) = Im\ (\zeta_K) = M/K$, hence h is an isomorphism .

Now If $\varphi : P \to N$ is a homomorphism then $h\ \varphi : P \to M/K$ factors through ζ_K, that is there is $\bar{\varphi} : P \to M$ such that the following diagram commutes . That is $\zeta_K\ \bar{\varphi} = h\ \varphi$



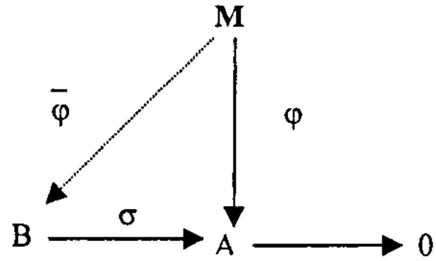
Thus we have, $hg\ \bar{\varphi} = \zeta_K\ \bar{\varphi} = h\ \varphi$, but since h is an isomorphism , then ,$g\bar{\varphi} = \varphi$ and this implies that P is M-projective. The converse is trivial.

Proposition (2.1.4): Every free module is projective.

Proof

Let ${}_RM$ be free with basis $\{m_i : i \in I\}$, and let $\varphi : M \to A$ be a homomorphism . Also let $\sigma : B \to A$ be an epimorphism, then since σ is onto for each $i \in I$ there exists $b_i \in B$ such that $\varphi(m_i) = \sigma(b_i)$. Now for any $m \in M$, $m = \sum_{i \in I} r_i m_i$ we define $\bar{\varphi} : M \to B$ such that $\bar{\varphi}(m) = \sum_{i \in I} r_i b_i$

where $r_i = 0$ for all $i \in I$ except for a finite number. Clearly $\bar{\varphi}$ is a homomorphism and $\sigma\bar{\varphi}\left(\sum_I r_i m_i\right) = \sigma\left(\sum_I r_i b_i\right) = \varphi\left(\sum_I r_i m_i\right)$ hence $\sigma\bar{\varphi} = \varphi$

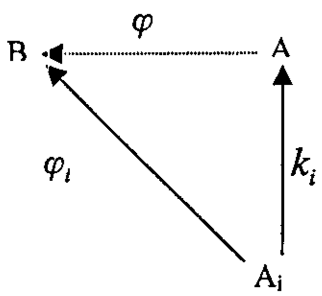


Corollary (2.1.5): ${}_RR$ is projective .

Proof:

Since ${}_RR$ is free, then by the a bove proposition it is projective.

Proposition (2.1.6): If A is the direct sum of a family of modules $\{A_i : i \in I\}$ with inclusion mappings $k_i : A_i \to A$,then for every module B and for every family of homomorphisms $\varphi_i : A_i \to B$,there exists a unique homomorphism $\varphi : A \to B$ such that $\varphi k_i = \varphi_i$.



Proof:

Define φ:$A \to$B by $\varphi(a) = \sum_{i \in I} \varphi_i(a_i)$, it is clear that φ is a homomorphism and that

$$\varphi k_j(a_j) = \varphi(k_j(a_j)) = \sum \varphi_i((k_j a_j)(i))$$

$= \varphi_j(a_j)$ .Thus $\varphi k_j = \varphi_j$. If also $\Psi k_j = \varphi_j$, then

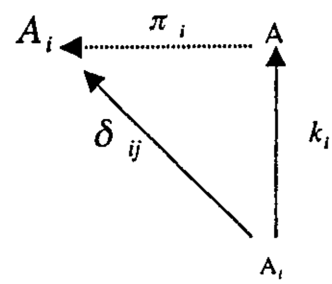
$\Psi(a) = \sum_{i \in I} \psi(k_i(a_i)) = \sum_{i \in I} \varphi_i(a_i) = \varphi(a)$, hence $\Psi = \varphi$.

Corollary (2.1.7): If A is isomorphic to the direct sum of modules A_i with inclusion mappings $k_i : A_i \rightarrow A$, then there exist a projection mappings $\pi_i : A \rightarrow A_i$ such that $\pi_i k_i = 1$ and $\pi_i k_j = 0$ when $i \neq j$.

Proof:

For fixed i consider the mapping $\delta_{ij} : A_j \rightarrow A_i$, where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$. Then by the above proposition there exists $\pi_i : A \rightarrow A_i$ such that, for all j $\pi_i k_j = \delta_{ij}$ (take $\varphi_j = \delta_{ij}$ and $\varphi = \pi_i$).

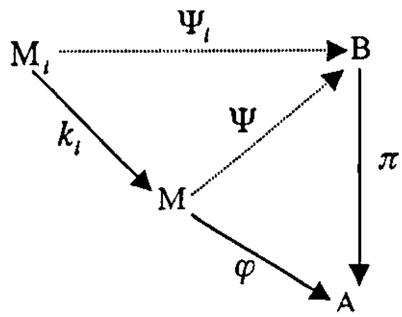
Which implies that $\pi_i k_i = 1$ and $\pi_i k_j = 0$ when $i \neq j$.



Proposition (2.1.8): If M is the direct sum of a family of modules $\{M_i : i \in I\}$, then M is projective if and only if each M_i is projective.

Proof:

Suppose that each M_i is projective, and let $\varphi : M \rightarrow A$ be a homomorphism, $\pi : B \rightarrow A$ be an epimorphism .Then by the projectivity of M_i, $\varphi k_i : M_i \rightarrow A$ can be lifted to $\Psi_i : M_i \rightarrow B$ such that $\pi \Psi_i = \varphi k_i$ where $k_i : M_i \rightarrow M$ is the canonical mapping. Now, by proposition (2.1.6), there exists a unique $\Psi : M \rightarrow B$ such that $\Psi k_i = \Psi_i$ for all $i \in I$. Since $\varphi k_i = \pi\Psi_i$ and $\pi\Psi k_i = \pi\Psi_i$ then by the uniqueness of φ we have $\pi\Psi = \varphi$. Thus M is projective.



Conversely, let M be projective, and let $\varphi_i : M_i \to A$ and $\pi_i : M \to M_i$ canonically. Then $\varphi_i\pi_i : M \to A$ can be lifted to $\Psi : M \to B$ such that $\pi\Psi = \varphi_i\pi_i$. But $\pi_i k_i = 1$, hence $\pi\Psi k_i = \varphi_i\pi_i k_i = \varphi_i$, and so φ_i has been lifted to Ψk_i. Therefore M_i is projective.

Corollary (2.1.9): Every direct summand of a projective module is again R-projective .

Proof:

Let P be a projective module and $P = M \oplus N$, then by the above proposition M is projective.

Lemma (2.1.10):

Every module is isomorphic to a factor module of a projective module.

Proof:

By lemma (1.1.26) we see that every module is a factor module of a free module. And since free modules are projective, then every module is isomorphic to a factor module of a projective module.

Theorem (2.1.11): Every module is an epimorphic image of a projective module.

Proof:

Let ${}_RF$ be a free module, then by proposition (1.1.25) F is isomorphic to a direct sum of copies of $R, F \cong \underset{I}{\oplus} R$. Now let A be any module and take $I=A$ be the indexing set, then there is a map $\varphi : F \to A$ defined by $(r_\alpha)_{\alpha \in A} \to \sum_{\alpha \in A} r_\alpha \alpha$ .This map is an epimorphism by definition. Since F is free then it is projective and so A is an epimorphic image of F.

This fact is yet another important reason to study projective modules. Thus we can study any module through projective modules.

Now we`ll study the basic characterizations of projective modules.

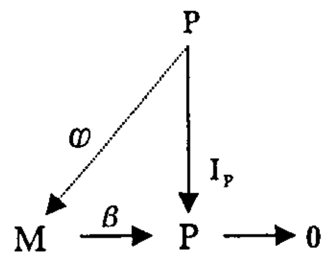
Theorem (2.1.12): The following properties of a module P are equivalent.

(a) P is a projective module

(b) P is a direct summand of a free module.

(c) Every exact sequence $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} P \to 0$ splits

Proof: (a) implies (c): Suppose P is projective. Since $\beta: M \rightarrow P$ is an epimorphism, then the identity map $I_P: P \rightarrow P$ can be lifted to $\varphi: P \rightarrow M$ such that $\beta\varphi = I_P$. Thus β splits, hence the sequence also splits by proposition (1.1.16).



(c) implies (b): Suppose that every exact sequence $0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} P \rightarrow 0$ splits. By lemma (1.1.26), there is a free module F and a submodule K of F such that $P \cong F/K$. Now the sequence $0 \rightarrow K \xrightarrow{i} F \xrightarrow{\zeta} P \rightarrow 0$, where i is the inclusion map and ζ is the natural epimorphism, is a split sequence. So by proposition (1.1.16) . $F \cong P \oplus K$. Thus P is a direct summand of a free module.

(b) implies (a): Since P is a direct summand of a free module, $P \oplus K \cong F$ for some free module F, but F is projective, hence P is a direct summand of projective module .Thus P is projective.

Corollary (2.1.13): A finitely generated R-module P is projective if and only if for some module ${}_R\overline{P}$ and some integer $n > 0$, $P \oplus \overline{P} \cong R^{(n)}$.

Proof:

A module ${}_RP$ is finitely generated if and only if for some natural number n, there is an epimorphism $R^{(n)} \to P \to 0$. By theorem (2.1.12), this epimorphism splits if and only if P is projective, So, there exists a module $\bar{P}$ such that $P \oplus \bar{P} \cong R^{(n)}$.

Corollary (2.1.14): If P is a projective R-module, then there exists a free module F such that $P \oplus F \cong F$.

Proof

Suppose P is projective,then by theorem(2.1.12) there exists a free R-module $\bar{F}$ such that $\bar{F} \cong \bar{P} \oplus P$ for some R-module $\bar{P}$ let

$F \cong \bar{P} \oplus P \oplus \bar{P} \oplus P \oplus \ldots$, where there are countably many summands.Thus F is free and since it is the direct sum of copies of the free module $\bar{F}$,then

$$\begin{aligned} P \oplus F &\cong P \oplus \bar{P} \oplus P \oplus \bar{P} \oplus \ldots \\ &\cong \bar{P} \oplus P \oplus \bar{P} \oplus P \oplus \ldots \\ &\cong F \end{aligned}$$

Now, we`ll give and example of a projective module which is not free.

Example 1: We have $2Z + 3Z = Z$ and $2Z \cap 3Z = 6Z$, hence, if $R = Z/6Z, A = 2Z/6Z$ and $B = 3Z/6Z$, then $R = A \oplus B$, Since ${}_RR$ has its identity element as a basis, then it is free , thus A is projective. But R contains six elements. Consequently a free R-module contains either an infinite number of elements or a finite number ***n*** of elements, where ***n*** is a multiple of six. But the number of elements in A lies between zero and six. Thus A is not free.

Proposition (2.1.15): Let $M = \sum_{i\in I} M_i$ be a direct sum of submodules M_i. Assume that M_i, and all submodules of each M_i are projective. Then any submodule N of M is isomorphic to a direct sum $\sum_{i\in I} N_i$, where $N_i \subseteq M_i$, for every $i \in I$.

Proof

We may take I to be the set of all ordinal numbers $< r$. Now for any ordinal $k \le r$ put $M^k = \sum_{i<k} M_i$. The projection mapping $\pi_i : M^k + M_k \to M_k$ sends $N \cap M^{k+1}$ onto some submodule N_k of M_k. but N_k is projective hence it is isomorphic to a direct summand of $N \cap M^{k+1}$,in fact $N \cap M^{k+1} = (N \cap M^k) + \overline{N_k}$, where $N \cap M^k \cap \overline{N}_k = 0$ and $\overline{N}_k \cong N_k$.

We'll prove by transfinite induction that for all $i \le r$ $N \cap M^i = \sum_{j<i} \overline{N}_j$ as a direct sum of submodules. This is trivial for $i = o$ and holds for $i = k+1$ if it holds for $i = k$ *by* $N \cap M^{k+1} = (N \cap M^k) + \overline{N}_k = \sum_{j<k} \overline{N}_j + \overline{N}_k = \sum_{j<k+1} \overline{N}_j$.

Now let k be any limit ordinal and a ssume the result for all $i<k$, then $N \cap M^k = N \cap \bigcup_{i<k} M^i = \bigcup_{i<k} (N \cap M^i) = \bigcup_{i<k} \sum \overline{N}_j = \sum_{j<k} \overline{N}_j$ and the sum $\sum_{j<k} \overline{N}_j$ is

direct, thus our statement holds for all $i \leq r$, in particular for $i = r$, we have $N = \sum_{i \in I} \overline{N_i}$.

Corollary (2.1.16): If R is a *PID*, then every submodule of a free R-module is free.

Proof

Let M be a free module , then M is a direct sum of submdules $M_i \cong_R R$. Now any non – zero ideal N_i of R is isomorphic to R as a left R-module, hence for all nonzero N_l, $N_i \cong_R R$. If N is a submodule of M, then by the above proposition, N is also free.

Lemma (2.1.17): An abelian group is free if and only if it is projective.

Proof: Let Z be the ring of integers, then abelian groups are just Z-modules Now an abelian group will be projective if and only if it is a direct summand of a free a belian group, but by the above corollary since Z is an integral domain then, this direct summand is free.

Theorem (2.1.18): (Dual Basis Theorem)

The module ${}_RP$ is projective if and only if there exist families $\{\alpha_i\}_{i \in I}$ of element of P and $\{f_i\}_{i \in I}$ of elements of *Hom (P,R)* such that

(a) for each $\alpha \in P$, $f_i(\alpha) = o$ for almost all i , and (b) $\alpha = \sum f_i(\alpha)\alpha_i$ for every $\alpha \in P$. The family $\{\alpha_i\}$ may be chosen to be any generating system of P.

Proof

Assume that $\Psi : F \to P$ is an epimorphism , where F is a free R-module with a base $(e_i)_{i\in I}$. By theorem (2.1.12) P is projective if and only if $\Psi\phi$ is an identity map for some R-homomorphism $\phi : P \to F$.

Now suppose first that P is projective , and choose such an F,Ψ,ϕ , and put $\alpha_i = \Psi(e_i)$. If $\alpha \in P$, then we can write $\phi(\alpha) = \sum_{i\in I} r_i\, e_i$. Define f_i as

follows $f_i : P \xrightarrow{\phi} F \xrightarrow{\pi_i} R$, where $\pi_i : F = \bigoplus_{i\in I} Re_i \to R$ given by

$\pi_i\left(\sum_{i\in I} r_i e_i\right) = r_i$,then for $\alpha \in P$, $f_i(\alpha) = \pi_i\left(\sum_{i\in I} r_i e_i\right) = r_i$,thus $\phi(\alpha) = \sum_{i\in I} f_i(\alpha)\, e_i$,

hence $\Psi\phi\,(\alpha) = \Psi\left(\sum_{i\in I} f_i(\alpha)\, e_i\right) = \sum_{i\in I} f_i(\alpha)\Psi(e_i)$ which implies that

$\alpha = \sum_{i\in I} f_i\,(\alpha)\alpha_i$.

Also it is clear by the definition of f_i that $f_i(\alpha) = o$ for almost all i ,and by the choice of F and Ψ we can take $\{\alpha_i\}_{i\in I}$ to be any generating set of P.

Conversely suppose that we have families $\{\alpha_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ satisfying conditions (a) and (b), then we can construct a homomorphism $\Psi : F \to P$ in which F is free with a basis $\{e_i\}_{i\in I}$ and $\Psi(e_i) = \alpha_i$. Define $\phi : P \to F$ such that $\phi(\alpha) = \sum f_i(\alpha)\, e_i$. Then $\Psi\phi(\alpha) = \sum_{i\in I} f_i(\alpha)\alpha_i = \alpha$. Thus $\Psi\phi$ is an identity map and so Ψ is split epimorphism . which implies that P is projective.

Now since $(\alpha_i)_{i\in I}$ may be taken to be any generating system of P then if P is finitely generated projective module,so I may be taken to be finite.

Now as an application on a Dual Basis theorem we have the following.

Lemma (2.1.19): Let M be any R-module, and let

$I=\{\sum f(M) : f\in Hom(M,R)\}$ then I is an ideal of R.

Proof

Let $r_1,r_2 \in I$, then $r_1\in f(M)$ and $r_2\in g(M)$, for some $g,f \in Hom(M,R)$. Thus there is $m_1, m_2 \in M$ such that $f(m_1)=r_1$ and $g(m_2)=r_2$. Now $r_1+r_2=f(m_1)+g(m_2) \in f(M)+g(M)\subseteq I$. Next for any $r\in R$ we have $rI=r\left(\sum_{f:M\to R} f(M)\right)=\sum_{f:M\to R} rf(M)$, but $rf\in Hom(M,R)$.

Thus $rI\subseteq I$, therefore I is a left ideal of R and similarly I is a right ideal , so it is an ideal of R.

Definition (2.1.20): The ideal I in the above lemma is called the trace ideal of M.

Corollary (2.1.21): Let P be a projective module and I the trace ideal P then.

(a) $PI=P$

(b) $I^2=I$

Proof

(a): Since P is a projective module then , by the dual basis lemma it has a dual basis say $(\{f_i\}_J, \{x_i\}_J)$.

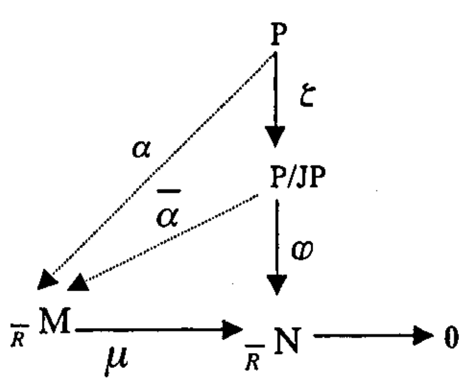
Now for $x \in P$, $x = \sum_J f_i(x)\, x_i \in PI$. Thus $P \subseteq PI$. Also since I is an ideal we have $PI \subseteq P$, so $P = PI$.

(b) $I = \sum_{f:P\to R} f(P) = \sum_{f:P\to R} f(PI) = \left(\sum_{f:P\to R} f(P)\right) I = I^2$.

Lemma (2.1.22): Let R be a ring and $\overline{R} = R/J$, where J is an ideal of R. If P is a projective R-module, then P/JP is a projective R/J – module.

Proof

Let $\mu : {}_{\overline{R}}M \to {}_{\overline{R}}N$ be an epimorphism of $\overline{R}$- modules and let $\varphi : P/JP \to N$ be any homomorphism. Also



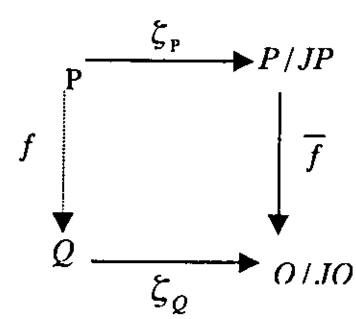
let $\zeta : P \to P/JP$ be the natural epimorphism, then by the projectivity of P there is $\alpha : P \to M$, such that $\mu\alpha = \varphi\zeta$ .Now since $ker\,\zeta \subseteq ker\,\alpha$, then by the factor theorem (1.1.5) there exist a unique homomorphism $\overline{\alpha}$ such that $\alpha = \overline{\alpha}\,\zeta$. Thus $\mu\,\overline{\alpha}\zeta = \varphi\zeta$. Therefore $\mu\overline{\alpha} = \varphi$.

Let us return to local rings. To determine the structure of finitely generated projective modules over a local ring we have the following lemma.

Lemma (2.1.23): Let R be a ring and $\overline{R} = R/J$, where J is an ideal of R contaid in Rad(R) . Let P , Q be finitely generated projective left R-modules. Then P $\cong Q$ as R-modules iff $P/JP \cong Q/JQ$ as $\overline{R}$ -modules.

Proof

Consider the following diagram, where $\overline{f}$ is a given isomorphism from *P/JP* to *Q/JQ*:



Since P is a projective R-module, then there exists an R- homomorphism $f : P \to Q$ which makes the diagram commutative such that $\zeta_Q\, f = \overline{f}\,\zeta_P$.

The surjectivity of $\overline{f}$ implies that $\mathrm{Im}(f) + JQ = Q$. Since Q is finitely generated we have by Nakayama`s lemma, $Im(f) = Q$, thus f is onto . But then by the projectivity of Q, there exists a decomposition $P = P' + Q'$ where $P' = Ker(f)$ and $f': Q' \to Q$ is an isomorphism. Reducing modulo J, we get $P/JP \cong P'/JP' \oplus Q'/JQ'$. Since $\overline{f}$ is an isomorphism then $P'/JP' = 0$. Thus $P' = JP'$. However, being a direct summand of P, P' is also finitely generated as R-module, by applying Nakayama's lemma again, we see that $P' = 0$. This means that f is 1-1, hence $f: P \to Q$ is an isomorphism.

This lemma leads quickly to the following well-known result.

Theorem (2.1.24): Let R be any local ring. Then any finitely generated projective R-module P is free.

Proof

Let $J=Rad(R)$, and reducing modulo J. Then by lemma (2.1.22) and proposition (1.1.21, P/JP is also a finitely generated projective module over R/J. By proposition (1.2.31). R/J is a division ring . Therefore $P/JP \cong (R/JR)^n$ for some integer n. By the above lemma, we conclude that $P \cong R^n$. Thus P is free.

We close this section with a theorem which illustrates how looking at the modules over a ring can give us information about the ring's internal structure.

Theorem (2.1.25): The following conditions on a ring R are equivalent

1. R is semisimple

2. All left R-modules are projective

3. All finitely generated left R-modules are projective

4. All cyclic left R-modules are projective

Proof

(1) iff (2) follows from (1.3.10,f) and (1.3.12), and (2) implies (3) implies (4) are trivial.

We finish by showing that (4) $\Rightarrow$(1). Consider any left ideal $I \subseteq R$ .By (4) as the left R-module R/I is cyclic. Then it is projective, so by theorem (2.1.12) the short exact sequence $0 \to I \to R \to R/I \to 0$ splits. This implies that I is an R-module direct summand of ${}_RR$, therefore by theorem (1.3.10,e) R is semisimple.

Section 2.2: Radicals and Endomorphism rings of projective Modules.

Radicals and Endomorphisms of projective modules are special and they play an important role in studying projective modules.

Lemma (2.2.1): Let R be a ring with radical $J(R)=J$. If P is a projective left R-module , then $Rad\ (P) = JP$.

Proof

By theorem (2.1.12), P is a direct summand of a free module such that $P + \overline{P} = F$, for some free module F.

Let $F \cong R^I \cong \underset{I}{\oplus} R$ for some I,then by theorem (1.2.15),$Rad\ (P)\ \oplus Rad\ (\overline{P}) =$

$Rad\ (R^I)\ = (Rad\ R\ )^I = (J\ (R))^I = (J.R)^I = J.\ \overset{I}{R}\ = JP \oplus J\overline{P}$. However by proposition (1.2.23) $JP \subseteq Rad\ (P)$ and $J(\overline{P}) \subseteq Rad\ (\overline{P})$ .Thus $Rad(P) = JP$.

Now before stating the next theorem , let us recall that for $s,t \in End\ (P)$, we have the following facts.

1. $Im\ (s \mp t) \subseteq \mathrm{Im}\ (s) \mp \mathrm{Im}\ (t)$
2. $Im\ (t\ s) \subseteq Im\ (s)$
3. $Im\ (s\ t) \subseteq (Im\ (s)\ )\ t$

Lemma (2.2.2): Let M be a left R-module and $S = End(M)$, then $\mathrm{N} = \{s \in S : \mathrm{Im}(s)\ \textit{is small in } M\}$ is an ideal of S.

Proof

Let $s\ ,\ t \in N$, then $Im\ (s + t) \subseteq Im\ (s) + Im\ (t)$, but $Im\ (s)$ and $Im(t)$ are small submodules of M. Thus $Im\ (s) + Im\ (t)$ is also small, therefore by lemma (1.2.12). $Im\ (s + t)$ is small in M.

Let $v \in S$, then $Im(vs) \subseteq Im(s)$ .since $Im(s)$ is small in M, then $Im\ (v\ s)$ is small in M. also $Im(sv) \subseteq v(Im(s))$ but $Im(s)$ is small in M, then by proposition (1.2.4) , $v(Ims)$ is small in M, thus by lemma (1.2.12), $Im\ (sv)$ is small in M. hence N is an ideal of S.

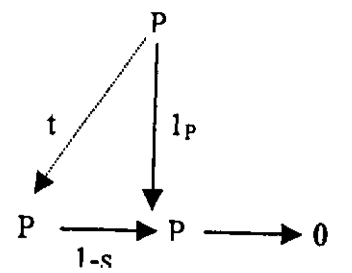
Theorem (2.2.3): If $_RP$ is a projective module and $S = End\ (P)$.

Then $J(S) \cong \{s \in S\ :\ \mathrm{Im}(s)\ \textit{is small in } P\}$.

Proof

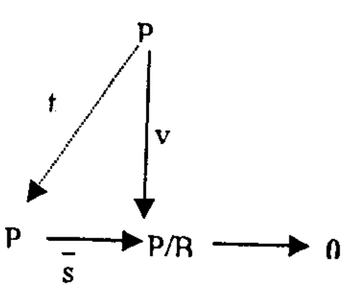
Let $N = \{s \in S : \text{Im}(s) \text{ is small in } P\}$. We want to show that $N = J(S)$. By above lemma we see that N is an ideal of S.

Let $s \in N$, then $Im(1\text{-}s + s) = Im(1) = P = Im(s) + Im(1\text{-}s)$. But $Im(s)$ is small in P, so $P = Im(1\text{-}s)$, hence $1\text{-}s$ is an epimorphism. So by the projectivity of P, there exists $t \in S$ such that $t(1\text{-}s) = 1$.



Thus N is a left quasi – regular ideal of S. So by corollary (1.2.21), $N \subseteq J(S)$.

Conversely, if $s \in J(S)$ and $Im(s) + B = P$ for some submodule B in P. We'll show that $Im(s)$ is small in P i-e $B=P$. Let $\bar{s} : P \to P/B$ and $v : P \to P/B$ defined where for $x \in P$ $x\bar{s} = xS + B$ and $xv = x + B$. Now since $\bar{s}$ is an epimorphism, then by the projectivity of P there exists $t \in S$ such that $t\bar{s} = v$



Now for $x \in P$, we have

$x(t\bar{s}) = xts + B = xv = x + B$, *hence* $x\,ts - x \in B$ *so* $x(ts - 1) \in B$ which implies that $Im\ (1-ts) \subseteq B$. but since $s \in J(S)$, *then* $ts \in J(S)$ implies that ts is a quasi-regular element. So *1-ts* is invertible , hence *1- ts* is an isomorphism of P. Therefore $P = Im\ (1-ts) \subseteq B$, so $P = B$ and consequently *Im (s)* is small in P which implies that $N = J(S)$.

Corollary (2.2.4): Let $J = J(R)$. If P is a projective left R-module such that $JP \langle\langle P$, then $J\ (End\ (P)) = Hom\ (P, JP)$ and

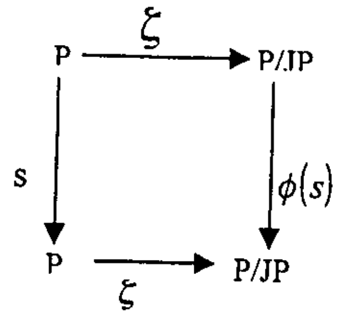
$End(P)/J(End(P)) \cong End(P/JP)$.

Proof

By Lemma (2.2.1) we have *Rad (P)=JP*, the hypothesis $JP \langle\langle P$ insures that a submodule of P is small iff it is contained in *JP*. In particular by theorem (2.2.3) , an endomorphism s of P belongs to

J(End(P) iff $Im(s) \langle\langle P \subset JP$ .Thus $J(End(P)) = Hom\ (P,JP)$.

Now observe that, since *JP* is stable under endomorphisms of P $\phi(s):(x+JP) \to xs+JP$,where $x \in P, s \in End(P)$ defines a ring homomorphism $\phi: End(P) \to End\ (P/JP)$. Now since P is projective, then the diagram below commutes.



So ϕ is subjective . But clearly we have

$$ker\ \phi = \{s : \mathrm{P} \to \mathrm{P} : s \in End(\mathrm{P})\ and\ \phi(s)(x+JP) = xs + JP = JP\}$$

$= \{s \in End(P) : \phi(s) \in JP\} = Hom(P, JP)$

Thus by the fundamental theorem of homomorphisms we have

$End(P)/\ker\phi \cong End(P/JP)$.so $End(P)/J(End(P)) \cong End(P/JP)$

Theorem (2.2.5): If ${}_RP$ is a projective module and *J(P)=P* then *P =0* .

Which means that every projective module has maximal submodules.

Proof

Suppose ${}_RP$ is projective and *J(P)=P*. By theorem (2.1.12) *P* is a direct summand of a free module *F* such that $F=P \oplus Q$.

Let $\{x_i\}_I$ be a basis for *F*, if $x \in P$, then $x = \sum_{i=1}^{n} r_i \; x_i$, where $\{x_i\}_{i=1}^{n} \subseteq \{x_i\}_I$ .Let $x_i = P_i + q_i$ where $P_i \in P$, $q_i \in Q$ then we have $x = \sum_{i=1}^{n} r_i x_i = \sum_{i=1}^{n} r_i P_i + \sum_{i=1}^{n} r_i q_i$ or

$x - \sum_{i=1}^{n} r_i P_i = \sum_{i=1}^{n} r_i q_i$.*Thus* $\sum_{i=1}^{n} r_i q_i \in Q \cap P = 0$, so

$x = \sum_{i=1}^{n} r_i P_i$. Next we define $\alpha : F \to F$ by

$$x_i \alpha = \begin{cases} P_i & for \quad i = 1,\dots, n \\ 0 & for \quad i \notin \{1,\dots, n\} \end{cases}$$

So $Im(\alpha) = R P_1 + R P_2 + ... + R P_n$. Thus $Im(\alpha) \subset P = J(P) \subset J(F)$ which implies that $Im(\alpha)$ is a finitely generated submodule of $J(F)$, hence $Im(\alpha)$ is a small . Submodule of F. Now it is clear that

$Im\alpha + Im(1\text{-}\alpha) = F$. Thus $Im\ (1\text{-}\ \alpha)=F$, so $1\text{-}\alpha$ is an isomorphism of F.

Now for

$$x \in P\ , x(1-\alpha) = x - x\alpha = x - \left(\sum_{i=1}^{n} r_i x_i\right)\alpha = x - \sum_{i=1}^{n} r_i p_i = x - x = 0$$

, but since $1\text{-}\alpha$ is injective, then $x = 0$, hence $P = 0$.

Corollary (2.2.6): If P is a non-zero projective module such that

$P = P_1 \oplus P_2$ *with* $P_2 \subset Rad\ (P)$, *then* $P_2 = 0$.

Proof

Let $\pi_2 : P \rightarrow P_2$ denote the projection on to P_2, then by proposition (1.2.13), $P_2 = \pi_2(P_2) \subset \pi_2(Rad\ (P)) \subset Rad\ (P_2)$.

Thus $P_2 = Rad\, P_2$ .And by the a bove theorem,$P_2 = 0$.

Section 2.3: Projective covers.

Projective covers plays an important role for studying projective modules and rings .Thus we`ll see that modules which have projective covers play a main role in our study.

Definition (2.3.1): An epimorphism $f{:}M \rightarrow N$ is called superfluous (minimal) if and only if $ker\ \ f \ll M$.

Ramark: Let K be a submodule of M then, $K \langle\langle M$ if and only if the canonical projection $\zeta : M \rightarrow M/K$ is a superfluous epimorphism. This is true since $ker\ \zeta = K$.

The following lemma is an important characterization of superfluous epimorphism.

Lemma (2.3.2): An epimorphism $f: M \rightarrow N$ is superfluous if and only if for every homomorphism $h: L \rightarrow M$ with fh epimorphism is an epimorphism.

Proof

If fh is an epimorphism and $m \in M$, then there exists $x \in L$ with $f(m) = fh(x)$. Which means that $m = h(x) + (m - h(x)) \in Im(h) + ker(f)$. Thus $M = Im(h) + ker(f)$.

As $ker(f) \langle\langle M$, then $M = Im(h)$, so h is an epimorphism. Conversly, suppose $L \subset M$ such that $L + ker\ f = M$. Let $i: L \rightarrow M$ be the inclusion map, then $fi(L) = f(i(L)) = f(L) = f(L + ker\ f) = f(M) = N$. So f is an epimorphism. Hence by the assumption i has to be an epimorphism. Thus $f(L) = L = M$ and therefore $ker(f) \langle\langle M$.

Now we need the following lemma on small submodules.

Lemma (2.3.3): Let A, B, M be R- modules. If $A \subseteq B \subseteq M$, and $B \langle\langle M$, then $A \langle\langle M$.

Proof

Assume B is small in M, and Let $K \subseteq M$ such that $A+K = M$. Since $A \subset B$, then $A+K \subseteq B+K$ i-e $M \subseteq B+K$. Hence $M = B + K$. As $B \langle\langle M$, then $K=M$ which implies that $A \langle\langle M$.

An important property of superfluous epimorphisms is given in the next theorem.

Theorem (2.3.4): Let L, M and N be R-modules, and $f: M \rightarrow N, g: N \rightarrow L$ be two epimorphisms , then, gf is superfluous if and only if f and g are superfluous.

Proof

Suppose f and g are superfluous epimorphisms and let $h: M' \rightarrow M$ is a homomorphism with gfh is an epimorphism, then fh is an epimorphism. Thus h is an epimorphism and so gf is superfluous.

Conversly , suppose gf is superfluous epimorphism , then $ker\ (gf) \langle\langle M$. As $ker\ (f) \subseteq ker\ (gf)$, then by lemma (2.3.3) $ker f \langle\langle M$. Thus f is superfluous.On the other hand since $ker(gf) \langle\langle M$ then by proposition(1.2.4), $f(ker\ gf)$ is small in N . Now we claim that $ker\ g \subseteq f\ (ker\ gf)$.If this is true then by lemma (2.3.3) , we have $ker\ g \langle\langle N$.

To see this let $x \in ker\ g$, then $x \in N$. As f is onto there is $y \in M$ such that $f(y)=x$. Now $gf\ (y) = g\ (f(y)) = g(x)=0$. so $y \in ker\ gf$.

Therefore $x = f(y) \in f(\ker gf)$. Which implies that g is superfluous.

Corollary (2.3.5): Let K, L and M be R-modules. If $K \subset L \subset M$ then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.

Proof

By using the above theorem with the canonical mappings $f: M \to M/K$ and $g: M/K \to M/L$. As $\ker gf = L$, $\ker f = K$ and $\ker g = L/K$.

Corollary (2.3.6): If $K \subset L \subset M$ are R-modules, and L is a direct summand in M, then $K \ll M$ if and only if $K \ll L$.

Proof

Let $M = L \oplus N$ for submodule N of M, and let $\zeta : L \to M$, $\pi : M \to L$ be the cononical mappings. Now if $K \ll L$, then $\zeta(K) = K \ll M$ and if $K \ll M$ then $\pi(K) = K \ll L$.

We saw in theorem (2.1.11) that every module is an epimorphic image of a projective module. Now for some module M an even stronger assertion is possible in case there is a projective module P and a superfluaus epimorphism $f: P \to M$.

Definition (2.3.7): A pair (P, μ) is a projective cover of the module ${}_RM$ in case P is a projective R-module and $P \xrightarrow{\mu} M \to 0$ is a superfluous epimorphism.

We'll simply call P it self a projective cover of M. Before giving an examples on projective covers we have the following.

Lemma (2.3.8): If ${}_RM$ is a finitely generated module, then $M/Rad\ (M)$ is finitely generated and the natural epimorphism $M \xrightarrow{\xi} M/Rad\,M \to 0$ is superfluous.

Proof

By proposition (1.1.21), $\xi(M)=M/Rad\,M$ is finitely generated . And by corollary (1.2.28) $J(M) \ll M$. Thus ξ is superfluous .

Lemma (2.3.9): If $e=e^2$ is an idempotent in a ring R , then

(1) $R\,e$ is a projective as an R-module.

(2) $Rad\ (Re)=J(R)\ e = Je$.

Proof

(1) Since $Re \oplus R(1-e)=R$, then $R\ e$ is a direct summand of a free module R. Thus by theorem (2.1.12), Re is a projective R-module.

(2) By lemma (2.2.1) since Re is projective , then $Rad\ (Re) = J(R)\ Re = J(R)e = Je$.

Now we'll see the following examples on projective covers.

Example 1: If e is an idempotent in R. Then by lemma (2.3.8) and lemma (2.3.9), $Rad\ (Re)=Je \ll Re$ and Re is a projective.

Thus the natural map $\zeta : Re \rightarrow Re/Je$ is superfluous epimorphism, since $ker\ \zeta \subseteq Je \ll Re$, hence $ker\ \zeta \ll Re$. Which implies that *Re* is projective cover of *Re/Je.*

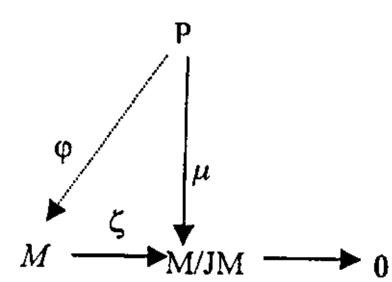
Now we give an example on a module which doesn't have a projective cover.

Example 2: Let Z be the ring of integers. Then the pair (Z, r_2) where $r_2 : Z \rightarrow Z/2Z$ is the natural epimorphism is not a projective cover since $2Z$ is not small in Z which have 0 as the only small submodule.

Theorem (2.3.10): Every finitely generated module over a local ring has projective cover.

Proof

Let R be a local ring and ${}_RM$ finitely generated, then R/J is a division ring and M/JM is a finite dimensional vector space over R/J, say M/JM is k-dimensional, and set $P = R^{(k)}$, then there is an epimorphism $\mu : P \rightarrow M/JM$ where $\ker \mu = JP$. Since P projective and the natural map $\zeta : M \rightarrow M/JM$ is an epimorphism , then there is a homomorphism $\varphi : P \rightarrow M$ such that $\zeta \varphi = \mu$.



Now by Nakayama's lemma , $JM \ll M$. So ζ is a superfluous epimorphism . hence by lemma (2.3.2) , φ is an epimorphism and since $ker\varphi \subseteq ker\ \mu = JP \ll P$, then P is a projective cover of M.

Now we'll prove the fundamental lemma for projective covers . One its consequences is that if a module does have a projective cover, then it has (essentially) only one .

Lemma (2.3.11): Suppose $_RM$ has a projective cover $\alpha: P \to M$. If $_RQ$ is projective and $\beta: Q \to M$ is an epimorphism, then Q has a decomposition $Q = P' \oplus P''$ such that

(1) $P' \cong P$

(2) $P'' \subseteq \ker\ \beta$

(3) $(\beta / P'): P' \to M$ is a projective cover for M. Moreover if $f: M_1 \to M_2$ is an isomorphism and if $\phi: P_1 \to M_1$ and $\psi: P_2 \to M_2$ are projective covers , then there is an is omorphism $\bar{f}: P_1 \to P_2$ such that $\psi \bar{f} = f\phi$.

Proof

By the projectivity of Q there is a commutative

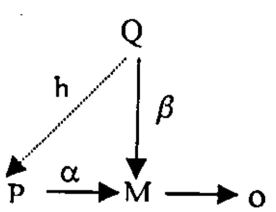


diagram with exact row and column. Since α is a superfluous epimorphism and $\alpha\ h = \beta$ is an epimorphism, then by lemma(2.3.2), h is also an epimorphism. But P is projective , so $h:Q \to P$ splits, so there is a monomorphism $g:P \to Q$ such that $h\ g = I_P$, and hence by lemma (1.1.13), $Q = Im\ g \oplus$ her h.

Now,setting $P' = Im\,g$, $P'' = \ker h$ .We see that (1) holds since $Im(g) = P'$ and g is monomorphism, so $P' \cong P$. Also (2) holds because $\alpha h = \beta$, we have $ker(h) = P'' \subset ker\ \beta$. To prove (3).We have $\beta(P') = \beta(P' + P'') = \beta(Q) = M$ and since $\beta\ g = \alpha\, h\, g = \alpha I_P = \alpha$, then $ker\ (\beta / P') = g\ (\ker \alpha)$ is small submodule of $g\ (P) = P'$. Thus $(\beta / P') : P' \to$ M is a projective cover for M.

Moreover, to prove the last statement, let $\alpha = \psi$, $\beta = f\ \phi$ and $\overline{f} = h$, *then* $\psi\ \overline{f} = \alpha h = \beta = f\ \phi$. *Also* $\overline{f} = h$ is an epimorphism and $\ker \overline{f} = \ker \phi$ is small direct summand of P_1. Thus $\overline{f}$ is an isomorphism.

Corollary (2.3.12): Let e and f be idempotents in a ring R and let $J{=}J(R)$, the TFAE.

(a) $Re \cong Rf$

(b) $Re/Je \cong Rf/Jf$

Proof

Clearly (a) implies (b). To show the converse let h:Re/$Je \to Rf/Jf$ be an isomorphism, since the natural maps Re $\to$ Re/Je *and* $Rf \to Rf/Jf$ are projective covers, It follows from lemma (2.3.11) that there is an isomorphism $\bar{h} : Re \to Rf$.

Now we'll study projective covers of simple modules. We need now the following lemma.

Lemma (2.3.13): (schur's Lemma)

If ${}_RS$ is a simple left R-module, then *End(S)* is a division ring.

Proof

Let $0 \neq f \in End(S)$. Then $Im(f) \neq 0$ and $ker(f) \neq S$.

Since *Im (f)* and *ker (f)* are both submodules of S, and S is simple, it follows that $Im(f) = S$ and $ker(f) = 0$. Thus f is invertible in *End (S)*. So, *End (S)* is a division ring.

Proposition (2.3.14): Let R be a ring with radical $J = J(R)$, then the following statements a bout a projective left R-module P are equivalent:

(a) P is the projective cover of a simple left R-module

(b) JP is a small maximal submodule of P.

(c) *End (P)* is a local ring.

Proof

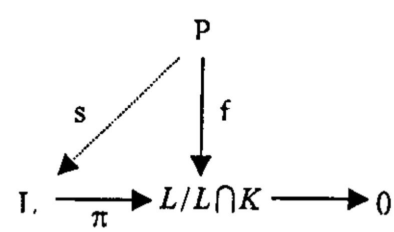
(a) implies (b) : let (μ, P) be a projective cover of a simple module S such that $\mu : P \to S \to 0$, then $ker\,\mu$ P and since $P/ker\,\mu \cong S$, so

$ker\,\mu$ is a small maximal submodule of P. Now JP is contained in every maximal submodule ,and JP contains every small submodule of P. So $JP = ker\ \mu$ is small maximal submodule of P.

(b) implies (c) : If JP is a small maximal submodule of P , then by schur`s lemma and corollary (2.2.4) .We have, $End\ (P)/J(End(P)) \cong End\ (P/JP)$ is a division ring . Thus by proposition (1.2.31). *End (P)* is a local ring.

(c) implies (a) : Suppose that *End (P)* is a local ring , then $P \neq 0$ so by theorem (2.2.5) there is a maximal submodule K of P .

We claim that the natural epimorphism $P \xrightarrow{\zeta} P/K \to 0$ is a projective cover . For in that case suppose $K+L=P$ for some submodule L of P. Then $P/K \cong (L+K)/K \cong L/L \cap K$. So there is a non-zero homomorphism $f: P \to L/L \cap K$. By the projectivity of P there is an endomorphism $s: P \to L \subseteq P$ such that the following diagram commutes



Since $0 \neq f = s\pi$, $Im\ s \subsetneq K$. It follows that *Im (s)* is not small in P . So by theorem (2.2.3) , $s \notin J(End\ (P))$ and hence by proposition (1.2.31), s is an invertible endomorphism of P. Therefore $L=P$ and hence $K \ll P$ which implies that ζ is a projective cover of the simple module P/K .

The following corollary determine the structure of projective module ${}_RP$ when *End (P)* is local .

Corollary (2.3.15): If ${}_RP$ is a projective module and *End (P)* is a local ring, then $P=Re$ for some idempotent $e \in R$.

Proof

For any $x \in P$, if $P=J(P)$ then it is trivial. If $J(P) \not\subset P$,then there is $x \in P$, $x \notin J(P)$, so $J(P) + Rx = P$, by proposition (2.3.14) $JP \langle\langle P$, and so $P = Rx$. Let $f_x : R \to Rx = P$, then the exact sequence.

$0 \to \ker f_x \xrightarrow{\subseteq} R \xrightarrow{f_x} Rx \to 0$ splits. Thus $R = \ker f_x \oplus Rx$,so $P = Rx$ is a direct summand of R, and hence $P = Re$ for some $e = e^2 \in R$.

Lemma(2.3.16): If e is a non-zero idempotent in a ring R then $\lambda : eRe \to End(Re)$ defined by $\lambda(x) = f(x)$, $f \in End(Re)$ is a ring isomorphism.

Proof

It is clear that eRe is a ring with identity e and λ is a ring homomorphism. Now we'll show that λ is onto.

Let $f \in End (Re)$ and let $x=ef$, then $ex = e.ef = e^2 f = ef = x$. Since $x \in Re$, then $xe = x$,and $x=ef = (ee)f = e(ef) = ex$. So $x = ex = e(xe) = exe \in eRe$. Thus λ is onto . Now for $y \in Re$, $ye = y$, so $yf = (ye)f = y(ef) = yx = yfx$. hence $f = f(x)$. Suppose $\lambda(x) = 0$, then $f(x) = 0$, so

$efx = ex = 0$. Thus we have $eRe \cong End\,(Re)$.

Corollary(2.3.17): Let $J=J(R)$ and $e = e^2 \in R$, then the following are equivalent

(a) Re/Je is simple

(b) Je is the unique maximal submodule of Re.

(c) eRe is a local ring.

Proof

(a)implies (b): Let Re/Je be a simple module , then we have $Re \to Re/Je$ as a projective cover of Re/Je . Thus by proposition (2.3.14), $J(Re)=Je$ is the unique maximal submodule of Re .

(b)implies (c): By lemma (2.3.16), $\lambda : eRe \to End\,(Re)$ is an isomorphism , then $End\,(Re)$ is local , and so eRe is local .

(c)implies (a): since $End\,(Re)$ is local , then Re is a projective cover of a simple module . But $Re \to Re/Je$ is a projective cover . So Re/Je is simple.

Lemma(2.3.18): Suppose that $K_1 \subset M_1 \subset M$, $K_2 \subset M_2 \subset M$ and $M=M_1 \oplus M_2$, *then* $K_1 \oplus K_2 \langle\langle M_1 \oplus M_2$ *iff* $K_1 \langle\langle M_1$ *and* $K_2 \langle\langle M_2$.

Proof

Suppose $K_1 \oplus K_2 \langle\langle M_1 \oplus M_2$ and let $P_i : M \to M_i$ denote the projection of M on M_i along M_j $(i \neq j)$. Then by proposition (1.2.4) $P_1(K_1 + K_2) = K_1 \langle\langle M_1$

and $P_2\ (K_1+K_2) = K_2 \ll M_2$. Conversely if $K_1 \ll M_1$ and $K_2 \ll M_2$, then by proposition (1.2.4), $K_1+K_2 = K_1 \oplus K_2 \ll M$.

Remark: We can use induction to show that if $K_i \subset M_i \subset M$ for $i=1,\ldots,n$, *then* $\overset{n}{\underset{i=1}{\oplus}} K_i \ll \overset{n}{\underset{i=1}{\oplus}} M_i$ iff $K_i \ll M_i$.

Definition (2.3.19): Let $(M_\alpha)_{\alpha\in A}$ be an indexed set of R-modules, and $(f_\alpha)_{\alpha\in A}$ is an indexed set of homomorphisms such that $f_\alpha : M_\alpha \to N$, where N is a left R-module , $\alpha \in A$, then we define a homomorphism $f : \underset{A}{\oplus} M_\alpha \to N$ such that for each $x = (x_\alpha)_{\alpha\in A} \in \underset{A}{\oplus} M_\alpha$, $f(x) = \sum_A f_\alpha(x_\alpha)$. We call f the direct sum of $(f_\alpha)_{\alpha\in A}$ and denote it by $f = \underset{A}{\oplus} f_\alpha$.

Lemma (2.3.20): If $\pi_1 : P_1 \to N_1$, $\pi_2 : P_2 \to N_2$ are projective covers of N_1, N_2, then $\pi_1 \oplus \pi_2 : P_1 \oplus P_2 \to N_1 \oplus N_2$ is a projective cover of $N_1 \oplus N_2$.

Proof

Since P_1, P_2 are projective , then $P_1 \oplus P_2$ is also projective . and also $ker(\pi_1 \oplus \pi_2) = \ker\pi_1 \oplus \ker\pi_2 \ll P_1 \oplus P_2$ by lemma (2.3.18).

By induction we can show that if $\pi_i : P_i \to N_i$ is a projective cover N_i for $i = 1, \ldots, n$, then $\oplus\pi_i : \oplus P_i \to \oplus N_i$ is also projective cover.

Definition (2.3.21): A ring R is called Jacobson semisimple (or J-semisimple if and only if $Rad\,(R) = 0$.

Lemma (2.3.22) : Let R be J-semisimple ring , then a module ${}_RM$ has a projective cover if and only if M is already projective.

Proof

Suppose $\mu : P \to M$ is a projective cover . Then $ker\ \mu \ll P$, thus we have that $ker\ \mu \subseteq J(P)$, but P is projective so $J(P) = JP$. Since R is J-semisimple then $J = J(R) = 0$. Therefore $ker\ \mu \subseteq J(P) = JP = 0$. So μ is an isomorphism and M is projective.

It follows, for instance , that over Z , the only modules a dmitting projective covers are the free a belian groups.

Lemma (2.3.23): Let I be an ideal in R and let $\bar{R} = R/I$. Let M be a left $\bar{R}$- module which is, therefore , also a left R-module. If ${}_RM$ has a projective cover over R then ${}_{\bar{R}}M$ also has a projective cover over $\bar{R}$.

Proof

Let $\mu : P \to {}_RM$ be a projective cover of M over R , we claim that $\bar{\mu} : P/IP \to {}_{\bar{R}}M$ is a projective cover of M over $\bar{R}$. To prove the claim we have $ker\ \bar{\mu} = \ker \mu / IP$. Now we want to show that this is small in P/IP .

Assume that $N/IP + \ker\mu/IP = P/IP$, where N is some R-submodule of P, but this implies that $N+ \ker\mu = P$, and since $\ker\mu \ll P$, then $N=P$, So $N/IP = P/IP$.

Theorem (2.3.24): Let $f: M \to N$ be a superfluous epimorphism and let $\varphi : P \to M$ be an R-homomorphism. Then $\varphi : P \to M$ is a projective cover if and only if $f\varphi : P \to N$ is a projective cover.

Proof

Suppose $\varphi : P \to M$ is a projective cover, so it is superfluous epimorphism. Since f is an epimorphism then $f\varphi$ is an epimorphism, to see that $f\varphi$ is superfluous.

we use lemma (2.3.2): If h is a homomorphism with $f\varphi h$ epimorphism, φh is an epimorphism, then h is an epimorphism. Thus $f\varphi$ is superfluous.

Conversely, if $f\varphi$ is a projective cover, then it is a superfluous epimorphism and so by lemma (2.3.2) we have φ is an epimorphism, and it is superfluous, as $\ker\varphi \subseteq \varphi(\ker f) \ll P$.

Lemma (2.3.25): Let ${}_RM \neq 0$, and let $\mu : P \to M$ be a projective cover, then μ gives a one – one correspondence between the maximal submodules of P and those of M. In particular, $\mu(\text{rad } P) = \text{rad } M$.

Proof

This is because any maximal submodule of P contains $ker\ \mu$ which is small in P. Thus by proposition (1.2.29) since $ker\, f \subseteq Rad\ P$,then

μ (radP) = rad M.

Corollary (2.3.26): Any nonzero module M with $Rad\ M=M$ cannot a dmit a projective cover.

Proof

Suppose $\theta : P \rightarrow M$ is a projective cover, then by lemma (2.3.24) $\theta\ (Rad\ P) = Rad\ (M)$ and since $Rad\ P \subsetneq P$, by theorem (2.2.5), we have $Rad\ (M) \subsetneq M$. Thus If $Rad\ (M) = M$ then M cannot a dmit a projective cover.

Chapter Three

Applications on projective modules

In this chapter we'll study semi-perfect and left perfect rings as an application on projective modules, those over which all finitely generated modules and, respectively, all modules have projective covers.

Section (3.1): Lifting Idempotents

The notion of lifting idempotents can be used to study a semi-perfect ring.

The idempotents in a ring R represent idempotents in every factor ring of R, but idempotent cosets in a factor ring of R need not have idempotent representatives in R, to see this we give an example.

Example 1: Let $R = Z$ which has two idempotents namely 0 and 1, while $Z/6Z$ has four idempotents, namely 0,1,3 and 4 .

Definition (3.1.1): Let I be an ideal in a ring R and let $g + I$ be an idempotent element of R/I. We say that this idempotent can be lifted modulo I in case there is an idempotent $e \in R$ such that $g + I = e + I$.

We say that idempotents lift modulo I in case every idempotent in R/I can be lifted to an idempotent in R.

Theorem (3.1.2): If I is a nil ideal in a ring R, then idempotents lift modulo I.

Proof

Let I be a nil ideal in R and $g \in R$ such that $g+I=g^2+I$, let n be the nilpotency index of $g\text{-}g^2$, then we can use the binomial formula as follows

$$0=(g-g^2)^n=\sum_{k=0}^{n}\binom{n}{k}g^{n-k}\left(-g^2\right)^k=\sum_{k=0}^{n}(-1)^k\binom{n}{k}g^{n+k}=(-1)^0\binom{k}{0}g^n$$

$$-g^{n+1}\left(\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}g^{k-1}\right)=g^n-g^{n+1}\left(\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}g^{k-1}\right).$$

Now let $t=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}g^{k-1} \in R$, then we have

$g^n=g^{n+1}t$, so $g^n=g^n\,gt$ which implies that $gt=e=tg$, thus

$e=g^n t^n=\left(g^{n+1}t\right)t^n=g^{n+1}t^{n+1}=\left(g^{n+2}t\right)t^{n+1}=g^{n+2}t^{n+2}=\dots=g^{2n}t^{2n}=\left(g^n t^n\right)^2=e^2$, So $e=g^n t^n$ is an idempotent in R, also

$$g+I=g^n+I=g^{n+1}t+I=\left(g^{n+1}+I\right)(t+I)=(g+I)(t+I)=gt+I=e+I.$$

Lemma (3.1.3): Every nil left ideal of a ring R is contained in $J(R)$.

Proof

Let I be a nil left ideal and let $a \in I$, then $a^n=0$ for some integer n, so we have $a+(-a+a^2-a^3+\dots+(-1)^{n-1}a^{n-1})+(-a+a^2-a^3+\dots+(-1)^{n-1}a^{n-1})a=0$.

Now let $a'=-a+a^2-a^3+\dots+(-1)^{n-1}a^{n-1}$, then $a+a'+a'a=0$.

Thus $(1-a)(1-a')=1$,which implies that a is a left quasi-regular element.

So, $a \in J(R)$.

The following two lemma`s on projective covers will be needed.

Lemma (3.1.4): Let ${}_RM$ have a decomposition $M=M_1\oplus....\oplus M_n$ such that each term M_i has a projective cover. Then an R- homomorphism $\phi : P \to M$ is a projective cover if and only if P has a decomposition $P=P_1\oplus...\oplus P_n$ such that for each $i=1,...,n$ $(\phi|P_i) : P_i \to M_i$ is a projective cover .

Proof

Suppose that each M_i has a projective cover and $\phi : P \to M$ is a projective cover .Let $q_i : Q_i \to M_i$ $(i = 1,...,n)$ be projective covers, then it follows from lemma (2.3.20) that $\bigoplus_{i=1}^{n} q_i : \bigoplus_{i=1}^{n} Q_i \to \bigoplus_{i=1}^{n} M_i$ is a projective cover. By letting $q_i= (\phi|P_i)$ we have by uniqueness of projective covers that $P = P_1\oplus ... \oplus P_n$.

Conversely. If $P = P_1 \oplus ... \oplus P_n$ is such that for each $i=1,...,n$ $(\phi|P_i) : P_i \to M_i$ is a projective cover, then we have by lemma (2.3.20) that $\bigoplus_{i=1}^{n}(\phi|P_i) : \bigoplus_{i=1}^{n} P_i \to \bigoplus_{i=1}^{n} M_i$ is a projective cover , which implies that $\phi : P \to M$ is a projective cover.

Lemma (3.1.5): A cyclic module $_RM$ has a projective cover if and only if $M \cong Re/Ie$ for some idempotent $e \in R$ and some left ideal $I \subseteq J(R)$. For e and I satisfying this condition the natural map $Re \to Re/Ie \to 0$ is a projective cover.

Proof

Suppose $_RM$ is cyclic with projective cover $\phi : P \to M$. As M is cyclic, then there is an epimorphism $f: R \to Rx = M$. So by lemma (2.3.11) R has a decomposition $R = P \oplus P'$ such that $\phi = (f|P)$. Thus for some idempotent $e \in R$, $P = Re$ and $Ie = \ker \phi \ll Re$, we have $Ie \subseteq J(R)e \subseteq J(R)$ and so $M \cong Re/Ie$. Conversely the natural map $Re \to Re/Ie$ has kernel Ie, so if $I \subseteq J(R)$, then $Ie \subseteq J(R)e \ll Re$. Thus $_RM$ has a projective cover.

Definition (3.1.6): A pair of idempotents e_1 and e_2 in a ring R are said to be orthogonal if $e_1 e_2 = e_2 e_1 = 0$

Definition (3.1.7): A set of idempotents $(e_\alpha)_{\alpha \in A}$ in a ring R is said to be orthogonal if and only if the set is pairwise orthogonal.

Definition (3.1.8): A finite orthogonal set of idempotents $e_1, \ldots, e_n$ in a ring R is said to be complete in case $e_1 + \ldots + e_n = 1 \in R$.

Proposition (3.1.9): Let $I_1, \ldots, I_n$ be left ideals of the ring R, then the following statements are equivalent about the left R- module $_RR$.

(i) $R = I_1 \oplus \ldots \oplus I_n$

(ii)Each element $r \in R$ has a unique expression

$r = r_1 + \ldots + r_n$, with $r_i \in I_i$ $(i = 1,\ldots,n)$

(iii) There exist a (necessarily unique) complete set $e_1,\ldots,e_n$ of pairwise orthogonal idempotents in R with $I_i = Re_i$.

Now the following proposition is a characterization of lifting idempotents.

Proposition (3.1.10): Let R be a ring and I be an ideal of R with $I \subseteq J(R)$.Then the following are equivalent .

(i) Idempotents lift modulo I

(ii) Every direct summand of the left R-module R/I has a projective cover.

(iii) Every (complete) finite orthogonal set of idempotents in R/I lifts to a (complete) orthogonal set of idempotents in R.

Proof

(i) implies (ii): Suppose that idempotents lift modulo I, and let ${}_R(M/I)$ be a direct summand of ${}_R(R/I)$, where $M \subseteq R$, then ${}_R(M/I)$ is also R/I – module and so it is generated by an idempotent of R/I hence by assumption we can lift any such idempotent , so we want such idempotent , so we want to show that if $e \in R$ is idempotent , then $(Re+I)/I$ has a projective cover.

But $(Re+I)/I \cong Re/I \cap Re \cong Re/Ie$ and by lemma (3.1.5) $(Re+I)/I$ has a projective cover.

(ii) implies (iii) : Let $g_1, \dots, g_n \in R$ be a complete orthogonal set of idempotents in R/I.Since $I \subseteq J(R) \ll R$,then the natural map $\zeta_I : R \to R/I$ is a projective cover , hence by hypothesis each term in $R/I = (R/I)(g_1+I) \oplus \dots \oplus (R/I)(g_n+I)$ has a projective cover. So by lemma (3.1.4) and proposition (3.1.10), there is a complete orthogonal set of idempotents $e_1, \dots, e_n \in R$ such that $(R/I)(e_i+I) = \zeta_I(Re_i) = (R/I)(g_i+I)$ $(i=1,\dots,n)$ and by the uniqueness part of proposition (3.1.10) , we have $e_i + I = g_i + I$ $(i=1,\dots,n)$

(iii) implies (i). This is clear by the definition of lifting idempotents.

Section (3.2): T- nilpotency

The concept of T- nilpotency serve us in introducing the notion of left perfect rings.

Definition (3.2.1): A subset A of a ring R is called left (right) T-nilpotent if, for any sequence of elements $\{a_1, a_2, a_3, \dots\} \subseteq A$, there exists an integer $n \geq 1$ such that $a_1 a_2 \dots a_n = 0$ $(a_n \dots a_2 a_1 = 0)$.

We shall use this notion of T-nilpotency mainly for ideals . So we have.

Remark 1: Every left T-nilpotent ideal I is nil .

Proof

If I is a left T-nilpotent ideal and $a \in I$, then by applying the definitien to $\{a, a, \ldots\}$, we see that each $a \in I$ is nilpotent and so I is nil.

Remark2: Every nilpotent ideal is T-nilpotent.

Proof

Let I be a nilpotent ideal, and n be the index of nilpotency of I such that $I^n = 0$, which means for any $\{a_1, a_2, \ldots\} \subseteq I$, $a_1 a_2 \ldots a_n = 0$. Therefore I is left T-nilpotent.

Remark3: By remark1 and remark2 we see that every nilpotent ideal I is nil.

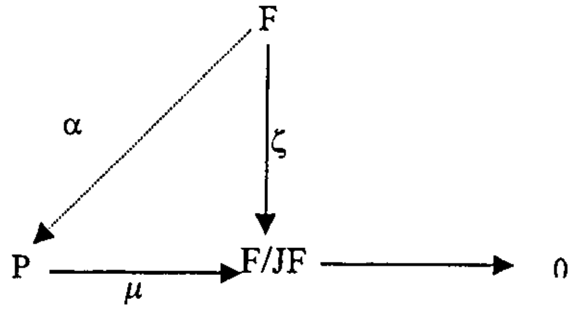
Now we'll give a characterization for the left (right) T-nilpotency, but firstly we need the following lemma which is related to projective covers.

Lemma (3.2.2): Let $_RF$ be a free module and $J=J(R)$. If $_R(F/JF)$ has a projective cover, then the natural epimorphism $\zeta : F \to F/JF$ is a projective cover of F/JF and consequentely JF is small in F.

Proof

Let $\mu: P \to F/JF \to 0$ be a projective cover of F/JF and $\zeta : F \to F/JF$ be the natural epimorphism.

Now since F is free so it is projective and by the projectivity of F the diagram below commutes.



So we have $\mu\ \alpha = \zeta$. Since ζ is an epimorphism then $\mu\ \alpha$ is also an epimorphism, but μ is superfluous epimorphism, thus by lemma (2.3.2). α is an epimorphism.

Since $\mu\ \alpha = \zeta$, then $ker\ \alpha \subseteq ker\ \zeta = JF$. Also by the projectivity of P the exact sequence $0 \rightarrow \ker\ \alpha \xrightarrow{\subseteq} F \xrightarrow{\alpha} P \rightarrow 0$ splits. So we have $F = ker\ \alpha \oplus P$, and so $JF = J\,ker\ \alpha \oplus JP$. Since $ker\ \alpha \subseteq JF$, then $ker\ \alpha = J\,ker\ \alpha$, but $ker\ \alpha$ is projective being a direct summand of a free module, hence by lemma (2.2.1) $J(ker\ \alpha) = J\,ker\ \alpha = ker\ \alpha$. Therefore by theorem (2.2.5) $ker\ \alpha = 0$. So α is an isomorphism, and consequently $ker\ \zeta = JF$ is small in F.

Theorem (3.2.3): Let $_RF$ be a free module with basis $\{x_n : n \in N^*\}$ and $J=J(R)$, then the following are equivalent.

(a) JF is small in F

(b) J is left T-nilpotent

(c)JM is small in ${}_RM$ for any left module ${}_RM$

Proof

(a)implies (b):Let ${}_RG = \sum_{n=1}^{N} R\left(x_n - a_n x_{n+1}\right)$,where $\left\{a_n\right\}_{n=1}^{\infty} \subset J\left(R\right)$

then we have ${}_RG + JF = F$, but JF is small in F, so ${}_RG = F$, hence

$x_1 \in G$ implies that ,

$x_1 = r_1 (x_1 - a_1 x_2) + r_2 (x_2 - a_2 x_3) + \ldots r_N (x_N - a_N x_{N+1})$

$= r_1 x_1 + (r_2 - r_1 a_1) x_2 + (r_3 - r_2 a_2) x_3 + \ldots + (r_N - r_{N-1} a_{N-1}) x_N - r_N a_N x_{N+1}$.

Since $\left\{x_n\right\}_{n=1}^{\infty}$ is a basis for F, then we get

$r_1 = 1$ and $r_2 - r_1 a_1 = r_3 - r_2 a_2 = \ldots = r_N - r_{N-1} a_{N-1} = r_N a_N = 0$.

Now by using successive substitution we get $a_1 a_2 \ldots a_N = 0$. Thus J is left T-nilpotent.

(b) implies (c). Suppose $JM + N = M$, for some proper submodule N of M, and let $X = M/N$, then $X \neq 0$ and $JX = X$, pick $a_1 \in J$, $x_1 \in X$ such that $a_1 x_1 \neq 0$, since $x_1 \in JX = X$, then x_1 is a sum of terms of the form $a\,y$, where $a \in J$ and $y \in X$, so $a_1 x_1$ is a sum of terms of the form $a_1\, a\, y$ not all of which are zero , so there is $a_2 \in J$, $x_2 \in X$ such that $a_1 a_2 x_2 \neq 0$. Now $x_2 \in X = JX$, so x_2 is a sum of terms of the form $a_1\, a_2\, a\, y$, since $a_1 a_2 x_2 \neq 0$ one of this terms is not zero , so there exist $a_3 \in J$, $x_3 \in X$ such that $a_1\ a_2\ a_3\ x_3 \neq 0$. By continue in this way we get a sequence

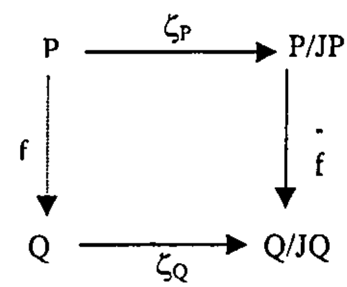
$\{a_n\}_{n=1}^{\infty} \subseteq J$ such that $a_1 a_2 \ldots a_n x_n \neq 0$, hence $a_1 a_2 \ldots a_n \neq 0$, which contradicts (b) . so $X = M/N = 0$ and hence $M = N$.

(c) implies (a): Let $M = F$, then $JM = JF$ is small in F.

Corollary (3.2.4): Let J be any left T-nilpotent left ideal of R. Then for any two projective left R-modules P and Q, $P/JP \cong Q/JQ$ implies $P \cong Q$

Proof

Consider the following diagram, where $\bar{f}$ is a given isomorphism from P/JP to Q/JQ.



Since P is projective , then there is an R-homomorphism $f: P \to Q$ which makes the diagram commutative i-e $\zeta_Q f = \bar{f} \zeta_P$.

The surjectivity of $\bar{f}$ implies that $Im(f) + JQ = Q$. Since J is left T-nilpotent , then by theorem (3.2.3), JQ is small in Q and so $Im(f) = Q$. Also by projectivity of Q, there exists a decomposition $P = P' + Q'$ where $P' = \ker f$ and $Q' \cong Q$. Now Reducing modulo J, we get $P/JP \cong P'/JP' \oplus Q'/JQ'$. Since $\bar{f}$ is an isomorphism then $P'/JP' = 0$, thus $P' = JP'$, hence by theorem (2.2.5) , $P' = \ker f = 0$. So f is an isomorphism.

Section (3.3): Semiperfect and left perfect rings and projectivity.

In this section we`ll study semiperfect and left perfect rings and their main characterizations by means of projectivity .

Definition (3.3.1): A ring R is called semiperfect if and only if $R/J(R)$ is a semisimple ring and idempotents lift modulo $J(R)$.

Now we have the following two lemma`s which are needed latter.

Lemma (3.3.2): $J(R/J(R)) = 0$

Proof

Let $\bar{P}$ be a left quasi – regular left ideal in $R/J(R)$. And let P be its inverse image in R, let $a \in P$, then $\bar{a} = a + J(R)$ and let $b \in P$ such that $\overline{a + b + ba} = \bar{a} + \bar{b} + \bar{b}\,\bar{a} = 0$, then $a + b + b\,a \in J(R)$ and so is left quasi-regular. Let c be such that $a + b + b\,a + c + c(a + b + b\,a) = 0$, then $a + (b + c + c\,b) + (b + c + c\,b)\,a = 0$ .Thus a is left quasi-regular and so P is a left quasi-regular ideal of R, hence $P \subseteq J(R)$, that is $\bar{P} = 0$. therefore $J(R/J(R)) = 0$.

Definition (3.3.3): Let M be a left R-module , then the socle of M, denoted by $soc(M)$ is defined by $Soc(M) = \sum_I M_i$, where $\{M_i\}_I$ is the set of all simple submodules of M. If $\{M_i\}_I = \phi$,then $soc(M) = 0$

Remark (3.3.4): If ${}_RM$ is semisimlpe , then $soc(M) = M$.

Lemma (3.3.5): The ring R is semisimple if and only if every simple left R-module is projective .

Proof

Suppose R is semisimple,then by theorem (2.1.25),every left R – module is projective , so every simple left R-module is projective. Conversely , it is sufficient to show that $Soc(R)=R$, suppose $Soc(R) \subsetneq R$,then $soc(R) \subset {}_RM$ for some maximal left ideal ${}_RM$ in R. Thus R/M is simple and so it is R-projective , hence the sequence $0 \rightarrow M \xrightarrow{\subseteq} R \xrightarrow{\xi} R/M \rightarrow 0$ splits . So $R= M \oplus S$,where $S \cong R/M$ is a simple left ideal of R , then we have $S \subseteq soc(R) \subset M$, which is a contradiction.

Now we have the follwing characterization of a semiperfect ring.

Proposition (3.3.6): For a ring R the following statements are equivalent.

(a) R is semiperfect

(b) R has a complete orthogonal set $e_1, ..., e_n$ of idempotents with each $e_i R e_i$ a local ring.

(c) Every simple left R-module has a projective cover .

(d) Every finitely generated left R-module has a projective cover.

Proof

Let $J = J(R)$ be the radical of R

(a)implies (b) . If R is semiperfect , then by proposition (3.1.10) we can lift the idempotents in proposition (3.1.9) for a semisimple decomposition of R/J to obtain a complete orthogonal set $e_1, \ldots, e_n$ of idempotents in R with each $Re_i / Je_i \cong (R/J)(e_i + J)$ simple. Then by corollary (2.3.17), $e_i R e_i$ is a local ring.

(b)implies (c). Assume (b), then by corollary (2.3.17) we have that each $R e_i/Je_i$ is simple , and so it has a projective cover by lemma (3.1.5). But each simple left R-module is isomorphic to a factor of $R/J \cong Re_1/Je_1 \oplus \ldots \oplus Re_n/Je_n$, and so isomorphic to one of the Re_i/Je_i so it has a projective cover .

(c)implies (d): Assume (c) and let Ω be a complete set of projective covers of simple left R-modules , then Ω generates every left R-module. Now let ${}_RM$ be finitely generated , then there is a sequence $P_1, \ldots, P_n$ in Ω and an epimorphism

$P = P_1 \oplus \ldots \oplus P_n \xrightarrow{f} M \rightarrow 0$ .And since $f(JP) = JM$, we conclude that there is an epimorphism

$P_1/JP_1 \oplus \ldots \oplus P_n/JP_n \cong P/JP \rightarrow M/JM \rightarrow 0$,but by proposition (2.3.14),each P_i/JP_i is simple , so M/JM is a finite direct sum of simple modules. So by lemma (3.1.4), M/JM has a projective cover. But $JM \ll M$. Thus $M \rightarrow M/JM$ is a superfluous epimorphism. Now a pply lemma (3.2.2) to get the result.

(d)implies (a) . Assume (d), this implies that every direct summand of R/J has a projective cover . So by proposition (3.1.10) idempotents lift modulo J .Now to see that R/J is semisimple, let $J \subseteq K \subset_R R$. Then , since the syclic R-module R/K has a projective cover , we have by lemma (3.1.5), $R/K \cong Re/Ie$ for some left ideal I such that $Ie \subseteq Je$. So $J.Re/Ie \cong J.R/K = 0$. Thus $Je = JRe \subseteq Ie$. Therefore $Ie = Je$ and so we have $R/K \cong Re/Je \cong (R/J)(e + J)$ is projective cover of R/J .Hence K/J is a direct summand of R/J and so R/J is semisimple.

Corollary (3.3.7): For any ring R, R is semiperfect if and only if every cyclic left R-module has a projective cover.

Proof

Let $_RM$ be any cyclic left R-module and R is semiperfect, then $_RM$ is finitely generated , so it has a projective cover by the a bove proposition.

Conversely , suppose that every cyclic left R-module has a projective cover , and let $_RM$ be a simple left R-module , then $_RM$ is cyclic , so it has a projective cover. Thus R is semiperfect.

Corollary (3.3.8): If R is semiperfect , then for any ideal $I \subseteq R$, the quotient ring R/I is also semiperfect.

Proof

Let M be a cyclic left R/I – module , then it is a cyclic left R-module , so it has a projective cover over R and by lemma (2.3.22) ${}_{R/I}M$ also have a projective cover over R/I . Thus R/I is semiperfect.

The following theorem is a nother charactarization of semiperfect ring.

Theorem (3.3.9): For the ring R the following are equivelent .

(a) R is a semiperfect ring.

(b) ${}_RR = P_1 \oplus \ldots \oplus P_n$ with $End\ (P_i)$ is a local ring .

For each $i=1, \ldots, n$.

Proof

For $J = J(R)$, let $\bar{R} = R/J$ and let R be a semiperfect ring , and let S be a simple left $\bar{R}$ – module , so it is a simple left R-module and so it has an R-projective cover, and by lemma (2.3.22), ${}_{\bar{R}}S$ has a projective cover, however $\bar{R}$ is semisimple , hence by lemma (3.3.5), ${}_{\bar{R}}S$ is $\bar{R}$ – projective . Now since $\bar{R}$ is semisimple ring, thus we can write

${}_R\ \bar{R} = S_1 \oplus \ldots \oplus S_n$ where S_i is a simple module for each $i = 1, \ldots, n$

furtheremore, each simple left R –module (R-module) is ispmorphic to one more than one of the ${}_RS_i$.Let $P_i \xrightarrow{\mu_i} S_i \rightarrow 0$ be a projective cover of the ${}_RS_i$, then by lemma (2.3.20), $P_1 \oplus \ldots \oplus P_n \rightarrow S_1 \oplus \ldots S_n \rightarrow 0$

is a projective cover of ${}_R R$, *but* ${}_R R \rightarrow {}_R \overline{R} \rightarrow 0$ is a projective cover of ${}_R \overline{R}$, thus we have ${}_R R \cong P_1 \oplus \dots \oplus P_n$,and by proposition (2.3.14), *End* (P_i) is a local ring for each i .

Conversely , if ${}_R S$ is a simple left R-module , then ${}_R S$ is cyclic such that $S=Rx$ for non – zero $x \in S$. Hence $Rx = P_1 x \oplus \dots \oplus P_n x$, so $S = P_i x$ for Some $i = 1, \dots, n$, and therefore we have an epimorphism $f_x : {}_R P_i \rightarrow {}_R S \rightarrow 0$. By proposition (2.3.14) this epimorphism is a projective cover. Thus ${}_R R$ is a semiperfect ring.

Now we'll give the definition of a left perfect ring

Definition (3.3.10): A ring R is a left perfect if and only if $R/J(R)$ is semisimple and $J(R)$ is left T-nilpotent.

It is clear from the definition that a left perfect ring R is semiperfect.

The following theorem is a useful characterization of a left perfect ring.

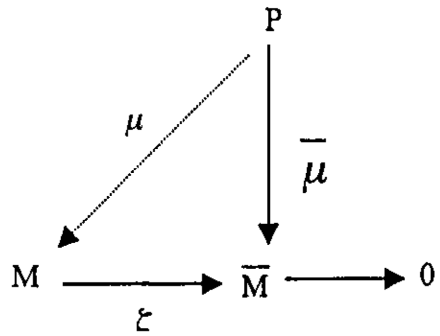
Theorem (3.3.11): For a ring R , R is left perfect if and only if every left R-module has a projective cover.

Proof

Suppose that every left R-module has a projective cover , in particular each simple left R-module has a projective cover. Hence R is semiperfect , so $R/J(R)$ is semisimple. Now let F be a free R-module, then ${}_R(F/JF)$

has a projective cover, so by lemma (3.2.2), the natural epimorhism $\zeta : F \to F / JF$ is a projective cover and $JF \langle\langle F$. Thus by theorem (3.2.3), $J(R)$ is left T-nilpotent. Hence R is left perfect.

Conversely, suppose R is a left perfect ring, then R is semiperfect and $J(R)$ is left T-nilpotent. Now for any left R-module ${}_RM$, let ${}_R\bar{M} = {}_R(M/JM)$. Since $\bar{R} = R/J(R)$ is semisimple, then by theorem (1.3.13), ${}_{\bar{R}}\bar{M}$ and also ${}_R\bar{M}$ is semisimple, thus ${}_R\bar{M}$ has a projective cover say $\bar{\mu} : P \to {}_R\bar{M}$. By the projectivity of P there is a homomorphism μ such that $\mu\zeta = \bar{\mu}$



Since $\bar{\mu}$ is an epimorphism,then $M = Im(\mu) + ker(\zeta) = Im(\mu) + JM$, but J is left T-nilpotent so JM is small in ${}_RM$ by theorem (3.2.3). Thus $M = Im(\mu)$, so μ is an epimorphism.

Since $\mu\zeta = \bar{\mu}$, then we have $ker\ \mu \subseteq ker\ \bar{\mu}$, but $ker\ \bar{\mu}$ is small in P and this implies that $ker\ \mu$ is small in P, thus μ is a projective cover of ${}_RM$.

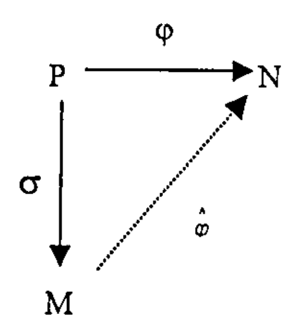
Chapter Four

Some Weak Forms of Projectivity

Section (4.1):Weakly projective modules

The object of this section is to introduce the concept of weak relative projectivity of left R-modules and to study the basic results in the subject.

Defintion (4.1.1): Let M and N be modules and assume M has a projective cover $\pi : P \to M$. We say that M is weakly N-projective if for every map $\varphi : P \to N$ there exists an epimorphism $\sigma : P \to M$ and a homomorphism $\hat{\varphi}$:M$\to N$ such that $\varphi = \hat{\varphi}\sigma$.



Definition(4.1.1*): If a module M is weakly N-projective for all finitely generated left R-modules N, we say that M is a weakly projective module.

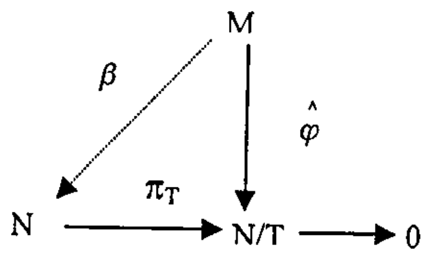
We note from the definition that weak projectivity is defined on modules which have projective covers. Also , if a left R-module M has a projective cover we get the following characterization of relative projectivity .

Theorem (4.1.2): Let M and N be left R-modules and assume M has a projective cover $\pi : P \to M$. Then M is N-projective if and only if for every homomorphism $\varphi : P \to N$, there exists a homomorphism $\varphi' : M \to N$ *such that* $\varphi = \varphi' \pi$. Equivalently, $\varphi(\ker \pi) = 0$.

Proof

Let $\varphi : P \to N$ be a homomorphism. We shall first show that $\varphi(\ker\pi) = 0$. Let $T = \varphi(\ker \pi)$ and let $\pi_T : N \to N/T$ be the natural projection. Then φ induces $\hat{\varphi} : M \to N/T$ defined by $\hat{\varphi}(m) = \pi_T \varphi(x)$, where $x \in P$ and $m = \pi(x)$. It follows that $\hat{\varphi}\pi = \pi_T \varphi$. Now since

M is N-projective, there exists a map $\beta : M \to N$ such that $\hat{\varphi} = \pi_T \beta$.



So we have $\pi_T \beta \pi = \pi_T \varphi$, which implies that $\pi_T(\varphi - \beta\pi) = 0$, hence $(\varphi - \beta\pi)P \subseteq T$. We claim that $\varphi = \beta\pi$.

Let $X = \{x \in P : \varphi(x) = \beta\pi(x)\}$. We shall show that $X = P$. Let $x \in P$, since $(\varphi - \beta\pi)(x) \in T = \varphi(\ker \pi)$, there exists $k \in \ker \pi$ such that $(\varphi - \beta\pi)(x)=\varphi(k)$. Thus , $\varphi(x\text{-}k) - \beta\pi(x\text{-}k)=0$, since $\beta\pi(k)=0$.

So $x\text{-}k \in X$. Therefore $\ker\pi + X = P$, which implies that $X=P$, since

$\ker\pi \ll P$. Therefore $(\varphi - \beta\pi)P=0$, in particular, $(\varphi - \beta\pi)\ker\pi=0$.

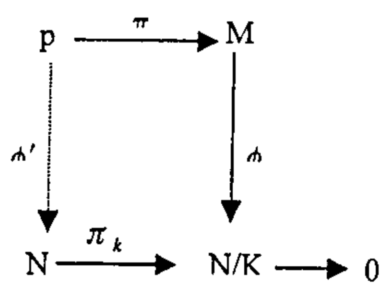
Thus $\varphi(\ker\pi)=0$. Equivalently , there exists $\varphi':M\to N$ such that

$\varphi'\pi=\varphi$.

Conversely , let $\phi:M\to N/K$ be a homomorphism.

Then by the projectivity of P there exists a homomorphism $\phi':P\to N$

such that $\phi\pi=\pi_K\phi'$.



Now by our hypothesis there exists $\phi'':M\to N$ such that $\phi''\pi=\phi'$. It follows that $\pi_K\phi''=\phi$. Thus M is N – projective.

Corollary (4.1.3): Let M and N be left R-modules and assume M has a projective cover $\pi: P\to M$. If M is N-projective, then it is weakly N-projective.

Proof

Let $\varphi: P\to N$ be any homomorphism, then by the above theorem there exists a homomorphism $\hat{\varphi}:M\to N$ such that $\varphi=\hat{\varphi}\pi$. Let $\sigma=\pi$, then we have M is weakly N-projective.

Now we have the following characterization of weak projectivity.

Theorem (4.1.4): Let M and N be modules and assume M has a projective cover $\pi : P \to M$. Then M is weakly N-projective if and only if for every map $\varphi : P \to N$ there exists a submodule $X \subset ker\ \varphi$ such that $P/X \cong M$.

Proof

Let $\varphi : P \to N$ be a homomorphism. Assume first that M is weakly N-projective and let the homomorphism $\hat{\varphi}: M \to N$ and the epimorphism $\sigma : P \to M$ be as in the definition of weak relative projectivity, such that $\varphi = \hat{\varphi}\sigma$, so, $ker\ \sigma \subset \ker\varphi$ *and*, $P/\ker\sigma \cong M$. By choosing $X = \ker\sigma$, we have $P/X \cong M$.

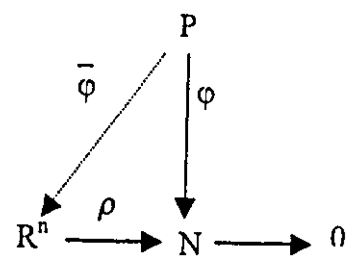
Conversely, if $X \subset P$, satisfies the condition in the statement of the theorem, then the isomorphism $P/X \cong M$, composed with the natural projection $\pi_X : P \to P/X$ is an epimorphism $\sigma : P \to M$ such that $ker\ \sigma = X \subset \ker\varphi$. Define $\hat{\varphi}: M \to N$ by $\hat{\varphi}(m) = \varphi(x), x \in P$ where $\sigma(x) = m$, then $\varphi = \hat{\varphi}\sigma$ Therefore M is weakly N-projective.

The following is another characterization of weak projectivity.

Theorem (4.1.5): A left module $_RM$ is weakly projective if and only if M is weakly R^n – projective for each $n \in Z^+$.

Proof

We only need to show that if M is weakly R^n–projective then it is weakly projective. Let N be a finitely generated left R-module and $\varphi : P \to N$ be any homomorphism. Now since N is finitely generated, there exists an epimorphism $\rho : R^n \to N$ for some $n \in Z^+$. The projectivity of P yields the existence of a homomorphism $\varphi' : P \to R^n$ such that $\rho\varphi' = \varphi$.



Since M is weakly R^n- projective,there exists $X \subseteq \ker\varphi'$ such that $P/X \cong M$. But $\ker\varphi' \subseteq \ker\varphi$. Thus $X \subseteq \ker\varphi$, so by theorem (4.1.4) ,M is weakly N-projective.

In the next proposition , we'll show that the domains of weak projectivity are closed under quotients and submodules.

Proposition (4.1.6): Let M and N be left R-modules and assume M has a projective cover $\pi : P \to M$. Then the following statements are equivalent.

(1) M is weakly N-projective

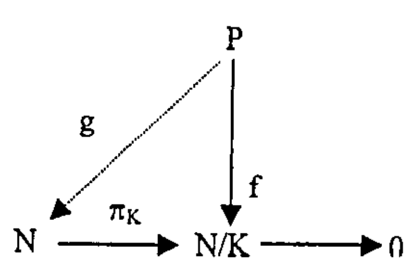
(2) For every submodule K of N, M is weakly K-projective,

(3) For every submodule K of N, M is weakly N/K-projective

Proof

(1) implies (2): suppose M is weakly N-projective and let K be a submodule of N and let $\varphi : P \to K$ be a homomorphism. Then $\Psi = i_k \varphi : P \to N$ may be expressed as a composition $\psi = \hat{\psi} \sigma$,for some homomorphism $\hat{\psi}: M \to N$ and epimorphism $\sigma : P \to M$. Since σ is onto, the range of $\hat{\psi}$ equal the range of Ψ and so it is contained in K. So , we can define $\hat{\varphi}: M \to K$ by $\hat{\varphi}(m) = \hat{\psi}(m)$, and hence $\varphi = \hat{\varphi} \sigma$. Proving M is weakly K-projective .

(1) implies (3): suppose that M is weakly N-projective and let $f: P \to N/K$ be a homomorphism . Since P is projective, there exists a map $g : P \to N$ such that $f = \pi_k g$.



Also, the weak N-projectivity of M yields an epimorphism $\sigma : P \to M$ and a homomorphism $\hat{g}: M \to N$ such that $g = \hat{g} \sigma$ .Let $\hat{f} = \pi_K \hat{g}$ then $\hat{f} \sigma = \pi_K \hat{g} \sigma = \pi_K g = f$, proving that M is weakly N/K – projective. And since either (2) or (3) clearly implies (1), then the proof is complete.

Now modules which are weakly projective relative to a fixed module are closed under finite direct sums. To see this we have the following theorem.

Theorem (4.1.7): Let M_i, $i= 1, 2, \ldots, n$ be a family of weakly N-projective modules . Then $\oplus\sum_{i=1}^{n} M_i$ is weakly N-projective.

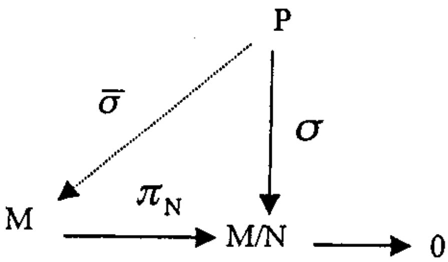
Proof

Let $\alpha_i : P_i \to M_i$ $(i=1,\ldots,n)$ be projective covers. Then by lemma (2.3.20), $\oplus\sum_{i=1}^{n}\alpha_i : \oplus\sum_{i=1}^{n} P_i \to \oplus\sum_{i=1}^{n} M_i$ is a projective cover. Let $\varphi : \oplus\sum_{i=1}^{n} P_i \to N$ and let $i_{P_i} : P_i \to \oplus\sum_{i=1}^{n} P_i$ be the inclusion map. Then by weak projectivity of M_i's, for each i, there exists an epimorphism $\sigma_i : P_i \to M_i$ and a homomorphism $\overline{\varphi}_i : M_i \to N$ such that $\overline{\varphi}_i \sigma_i = \varphi i_{P_i}$ .Now set $\overline{\varphi} = \oplus\sum_{i=1}^{n}\overline{\varphi}_i$ and $\sigma = \oplus\sum_{i=1}^{n}\sigma_i$ then it follows that $\varphi = \overline{\varphi}\sigma$, as desired.

Theorem (4.1.8): Let *M/N* be a weakly K-projective module. Then *M* is weakly K-projective when $N \ll M$.

Proof

Since $N \ll M$, then *M* and *M/N* have the same projective cover. Let $\varphi : P \to K$ and $\pi_N : M \to M/N$ be the natural epimorphism.By weak K-projectivity of M/N,there exists an epimorphism $\sigma: P \to M/N$ and a homomorphism $\bar{\varphi}: M/N \to K$ such that $\bar{\varphi}\sigma = \varphi$.Also by projectivity of P, there exists $\bar{\sigma}: P \to M$ such that $\pi_N \bar{\sigma} = \sigma$.

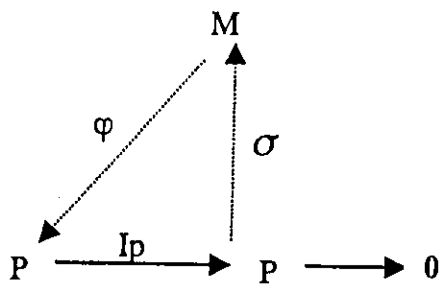


Now since $N \ll M$, it follows that $\bar{\sigma}$ is onto . and $\bar{\varphi}\pi_N \bar{\sigma} = \varphi$. Thus M is weakly K- projective. Finally we have the following result.

Theorem (4.1.9): If a module is weakly projective relative to its own projective cover, then the module is indeed projective.

Proof

Consider a module M with projective cover $\pi: P \to M$. If we assume that M is weakly P- projective, then the identity map I_P: $P \to P$ factors through M, i-e there exists an epimorphism $\sigma: P \to M$ and a homomorphism $\varphi: M \to P$ such that $\varphi\sigma = I_P$.

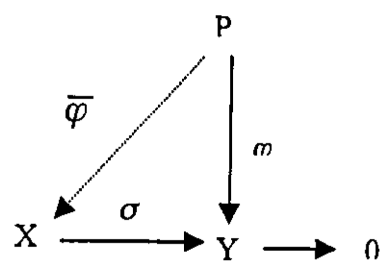


Thus $M \cong P$, so M is projective.

Section (4.2): Ideal projectivity and jacobson radical projectivity.

We begin this section by the following definition .

Definition (4.2.1): Let H be an ideal of a ring R and P a left R-module. We say that P is H-projective provided whenever $\sigma: X \rightarrow Y$ is an epimorphism of left R-modules with $HY=0$ and $\varphi: P \rightarrow Y$ an R-homomorphism, then there is an R-homomorphism $\overline{\varphi}: P \rightarrow X$ such that $\varphi = \sigma \overline{\varphi}$.



From the definition we have the following remarks.

Rrmork1: Let P be a left R-module which is H-projective, where H is an ideal of a ring R. If K is an ideal of R containing H, then P is K-projective.

Proof

Since $KY = 0$, then $HY \subseteq KY = 0$. So P is K-projective.

Remark 2: If ${}_RP$ is H- projective then (P/HP) is R/H - projective

Proof

Let σ:X$\rightarrow Y$ be an epimorphism of R-modules with $R/H.Y = H$, then $RY/H = H$. So $RY = 0$ which implies by remark (1) that $HY = 0$. So P/HP is R/H – projective.

Definition (4.2.2): If H is an ideal of R , then $N(H)$ is the set of all endomorphisms of P with image in HP for a left R-module P.

Let P be a left R-module and $P^* = Hom_R(P,R)$, then for $g_1,\dots,g_n \in P^*$ $(g_i): P \to R^n$ is an R-homomorphism given by $x\ (g_i) = (xg_1,\dots,x\ g_n)$. And if $x_1, \dots, x_n \in P$, then $(x_i): R^n \to P$ is the R-homomorphism given by $(r_1,\dots,r_n)(x_i) = \sum_{i=1}^{n} r_i x_i$. Now composition of (g_i) with (x_i) is denoted $(g_i).(x_i)$ is an endomorphism of P.

Definition (4.2.3): Let P be a left R-module and $P^* = Hom\ (P,R)$, then $P^*.P$ denotes the set of all endomorphisms of P. which factors through a finitely generated free module.

The following theorem characterizes ideal projectivity for finitely generated modules.

Theorem (4.2.4): Let P be a finitely generated left R-module , $C = End\ ({}_RP)$, H an ideal of R and $\bar{P} = P/HP$.

Then the following are equivalent

1. P is H- projective.
2. There is a homomorphism $(P \to F \to P)$ with F finitely generated free such that $(P \to F \to P \to \bar{P}) = P \to \bar{P}$.
3. There are homomrphisms $P \to F$ and $F \to \bar{P}$ with F finitely

generated free such that $(P \to F \to \bar{P}) = (P \to \bar{P})$

4. $P^* P + N(H) = C$.

Proof

(1) implies(2):

Let $y_1, \ldots, y_n$ be a set of generaters for P and F a free left R-module with basis $x_1, \ldots, x_n$. Then there is an epimorphism $F \to P$ such that $x_i \to y_i$, $i = 1, \ldots, n$, and so we have $F \to P \to \bar{P}$ is an epimorphism. But by H-projectivity of P , there is a homomorphism $P \to F$ such that $(P \to F \to P \to \bar{P}) = (P \to \bar{P})$.

(2) implies (3) : clear.

(3) implies (4): Let $P \to F$ and $F \to \bar{P}$ be as in 3, where $F = R^n$, then there is g_i in P^*, $i = 1, \ldots, n$ such that $(P \to F) = (g_i)$. Consider $(y_i) : F \to P$ be an epimorphism ,then $(g_i)\,(y_i)\, v = v$, where $v = (P \to \bar{P})$, hence for y in P, we have

$$y v = y(g_i)(y_i)\, v = \left(\sum_{i=1}^{n} (y g_i) y_i\right) v = \sum_{i=1}^{n} y(g_i\, y_i) v = y\left(\sum_{i=1}^{n} g_i \cdot y_i\right) v$$

thus $y - y\left(\sum_{i=1}^{n} g_i\, y_i\right)$ is in HP , for all $y \in P$. Thus $1 - \sum_{i=1}^{n} g_i\, y_i$ is in $N(H)$, it follows that $P^*P + N(H) = C$.

(4)implies (1): By (4), there is g_i in P^* and y_i in P , $i = 1, \ldots, n$ such that $\sum_{i=1}^{n} g_i\, y_i + t = 1$, where $Pt \subseteq HP$. Let $h: X \to Y$

be a left R-epimorphism with $HY = 0$ and $f: P \to Y$ a homomorphism. Since $HY=0$, $HP \subseteq ker\,(P \to Y)$, so there is a homomorphism $\bar{f}: \bar{P} \to Y$ such that $v\bar{f} = f$ (by the factor theorem), where $v = P \to \bar{P}$.Now for $a \in P$, $av = \left(\sum_{i=1}^{n} (a\,g_i)\, y_i\right) v = a(g_i)(y_i)\, v$, thus $(g_i)\ (y_i)\ v = v$. Let $F = R^n$, choose x_i in X such that $x_i\, h = y_i f$ and let $(x_i) : F \to X$, then for $a \in P$, we have $a((g_i)(x_i)h) = \left(\sum_{i=1}^{n} (a\,g_i)\, x_i\right) h = \sum_{i=1}^{n} (a\,g_i)(x_i\, h) = \sum_{i=1}^{n} (a\,g_i)(y_i\, f)$

$= a(g_i)(y_i) f = a(g_i)(y_i) v\bar{f} = a v \bar{f} = a f$.

As a special case of ideal projectivity , we have the **Jacobson radical projectivity (J-projectivity),** where J is the jacobson radical of a ring.

Theorem (4.2.5): Let P be a finitely generated left R-module such that every epimorphism of P to P is a monomorphism. Then the following are equivalent:

1) P is projective

2) P is J-projective.

Proof

(2) implies (1): By theorem (4.2.4), we can write $1 = c + t$, where c is in

$P^*.P$ and t in $N(J)$, so $P = Pc + Pt \subseteq Pc + JP$, but JP is small, hence $P=Pc$, thus $c= (P \to F \to P)$, with F finitely generated free, is an epimorphism of P. But by (2), c is a monomorphism of P. Therefore P is projective. The converse is trivial.

Theorem (4.2.6): Let P be a finitely generated J-projective left R-module. If P has a projective cover, then P is projective.

Proof

Let $\bar{P} = P/JP$ and $f: Q \to P$ be a projective cover of P. Now $(Q \xrightarrow{f} P \xrightarrow{g} \bar{P})$ is an epimorphism and $g: P \to \bar{P}$ is a homomorphism. So by the J-projectivity of P, there is a homomorphism $h: P \to Q$ such that $hf\,g=g$. Now since $ker(g) \subseteq JP$, which is small in P, then g is a minimal epimorphism, thus fg is a minimal epimorphism.

Now, hfg is an epimorphism and fg is minimal, this implies that h is an epimorphism, also Q is projective implies that h splits, so there is a homomorphism $k: Q \to P$ such that $hk= I_Q$ and so $P = Im(k) \oplus \ker(h)$, with $ker\ (h) \subseteq ker\ (g) \subseteq JP$, and $J(P)$ is small, thus $ker(h) =0$ and hence Q is isomorphic to P.

Section (4.3): Simple projectivity

Definition (4.3.1): A left R-module P is said to be simply projective, provided whenever $X \to Y$ is an epimorphism with Y a simple left R-module and $P \to Y$ a homomorphism , then there is a homomorphism $P \to X$ such that $(P \to X \to Y) = P \to Y$.

It is clear from the definition that a left R-module P is simply projective if and only if given a homomorphism $P \to S$ and an epimorphism $R \to S$, where S is a simple left R-module , then there is a homomorphism $P \to R$ such that $(P \to R \to S) = P \to S$.

The following property of projective modules is still true for finitely generated simply projective modules.

Proposition (4.3.2): If X is a finitely generated simply projective left R-module with trace ideal H, then

1) $HX = X$, and

2) $H^2 = H$

Proof

(1) We have $HX \subseteq X$. *If* $HX \subset X$, then there is a maximal submodule M of X such that $HX \subseteq M \subset X$.

Let $R \to X/M$ be an epimorphism , then for $X \to X/M$, and by the simple projectivity of X, there is a homomorphism $X \to R$ such that $(X \to R \to X/M) = X \to X/M$. Since $Im(X \to R) \subseteq H$ then $X \to R \to X/M$ is the zero map which is a Contradiction: To show (2), we have $H^2 = HH = H(XX^*) = (HX)X^* = XX^* = H$, where $X^* = Hom\ (X,R)$.

Appendix:

(i) **The Maximal Principle (Zorn's lemma)**

(1) Let P be a poset. An element $m \in P$ is maximal (minimal) in P in case $x \in P$ and $x \geq m$ $(x \leq m)$ implies $x = m$.

(2) A poset P is inductive in case every subchain of P has an upper bound in P, that is, for every subset C of P that is totally ordered by the partial ordering of P, there is an element of P greater than or equal to every element of C.

Now the Maximal principle states:

"Every non-empty inductive poset has at least on maximal element"

(ii) The Transfinite induction.

We use the Transfinite induction to prove a mathematical statements with infinite index set and using the steps of mathematical induction.

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