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On Continued Fractions and its Applications

By

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Dedication

To my parents, the most wonderful people in my life who taught me that there is no such thing called impossible.

To my beloved brothers "Mohammad and Khalid",

To my lovely sisters "Walaa and Ruba",

To all my family

With my Respect and Love

****

Acknowledgment

After a long period of hard working, writing this note of thanks is the finishing touch on my thesis.
First of all, I'd like to thank God for making this thesis possible and helping me to complete it.

I would like to express my sincere gratitude to my supervisor Dr. "Mohammad Othman" Omran for the valuable guidance, continuous support and encouragement throughout my work on this thesis.

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Last but not least, I wish to thank my family and all my friends who helped and supported me.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان

On Continued Fractions and its Applications

أقر بأن ما شملت عليه الرسالة هو نتاج جهدي الخاص, باستثناء ما تمت الإشارة إليه حيثما ورد, وأن هذه الرسالة ككل أو أي جزء منها لم يقدم من قبل لنيل أي درجة أو لقب علمي أو بحثي لدى أي مؤسسة علمية أو بحثية.

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On Continued Fractions and its Applications

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Abstract

In this thesis, we study finite simple continued fractions, convergents, their properties and some examples on them. We use convergents and some related theorems to solve linear Diophantine equations. We also study infinite simple continued fractions, their convergents and their properties. Then, solving Pell’s equation using continued fractions is discussed. Moreover, we study the expansion of quadratic irrational numbers as periodic continued fractions and discuss some theorems. Finally, the relation between convergents and best approximations is studied and we apply continued fractions in calendar construction and piano tuning.

Introduction

Continued fraction is a different way of looking at numbers. It is one of the most powerful and revealing representations of numbers that is ignored in mathematics that we've learnt during our study stages.

A continued fraction is a way of representing any real number by a finite (or infinite) sum of successive divisions of numbers.

Continued fractions have been used in different areas. They've provided us with a way of constructing rational approximations to irrational numbers. Some computer algorithms used continued fractions to do such approximations. Continued fractions are also used in solving the Diophantine and Pell's equations. Moreover, there is a connection between continued fractions and chaos theory as Robert M. Corless wrote in his paper in 1992.

The use of continued fractions is also important in mathematical treatment to problems arising in certain applications, such as calendar construction, astronomy, music and others.

History of Continued Fractions

Mathematics is constantly built upon past discoveries. In doing so, one is able to build upon past accomplishments rather than repeating them. So, in order to understand and to make contributions to continued fractions, it is necessary to study its history.

The history of continued fractions can be traced back to an algorithm of Euclid for computing the greatest common divisor. This algorithm generates a continued fraction as a by-product. $\lfloor 34 \rfloor$

For more than a thousand years, using continued fractions was limited to specific examples. The Indian mathematician Aryabhata used continued fractions to solve a linear indeterminate equation. Moreover, we can find specific examples and traces of continued fractions throughout Greek and Arab writings. $\lfloor 34 \rfloor$

From the city of Bologna, Italy, two men, named Rafael Bombelli and Pietro Cataldi also contributed to this branch of mathematics. Bombelli was the first mathematician to make use of the concept of continued fractions in his book **L'Algebra** that was published in 1572. His approximation method of the square root of 13 produced what we now interpret as a continued fraction. Cataldi did the same for the square root of 18. He represented $\sqrt{18}$ as $4.\ \&\frac{2}{8.}\&\frac{2}{8.}\&\frac{2}{8.}$ with the dots indicate that the following fraction is added to the denominator. It seems that he was the first to develop a symbolism for continued fractions in his essay **Trattato del modo brevissimo Di trouare la Radici quadra delli numeri** in

1613 . Besides these examples, however, both of them failed to examine closely the properties of continued fractions. $\lfloor 34,35,36,37 \rfloor$

In 1625, Daniel Schwenter was the first mathematician who made a material contribution towards determining the convergents of the continued fractions. His main interest was to reduce fractions involving large numbers. He determined the rules we use now for calculating successive convergents. $\lfloor 35 \rfloor$

Continued fractions first became an object of study in their own in the work which was completed in 1655 by Viscount William Brouncker and published by his friend John Wallis in his **Arithmetica infinitorum** written in 1656.

Wallis represented the identity $\frac{4}{\pi}=\frac{1\times3\times3\times5\times5\times7\times...}{2\times2\times4\times4\times6\times6\times...}$ and Brouncker converted it to the form $\frac{4}{\pi}=1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\ddots}}}}$.

In his book **Opera Mathematica** (1695), Wallis explained how to compute the n^{th} convergent and discovered some of the properties of convergents. On the other hand, Brouncker found a method to solve the Diophantine Equation $x^2 - Ny^2 = 1$. $\lfloor 34,36 \rfloor$

The Dutch mathematician Huygens was the first to use continued fractions in a practical application in 1687. His desire to build an accurate mechanical planetarium motivated him to use convergents of a continued fraction to find the best rational approximations for gear ratios. $\lfloor 38 \rfloor$

Later, the theory of continued fractions grew with the work of Leonard Euler, Johan Heinrich Lambert and Joseph Louis Lagrange. Euler laid down much of the modern theory in his work **De Fractionlous Continious** (1737). He represented irrational and transcendental quantities by infinite series in which the terms were related by continuing division. He called such series fractions continuae, perhaps echoing the use of the similar term fractions continuae fractae (continually broken fractions) by John Wallis in the **Arithmetica Infinitorum**. He also found an expression for e in continued fraction form and used it to show that e and e^2 are irrationals. He showed that every rational can be expressed as a finite simple continued fraction and used continued fractions to distinguish between rationals and irrationals. Euler then gave the nowadays standard algorithm used for converting a simple fraction into a continued fraction. Moreover, he calculated a continued fraction expansion of $\sqrt{2}$ and gave a simple method to calculate the exact value of any periodic continued fraction and proved a theorem that every such continued fraction is the root of a quadratic equation. $\lfloor 34,36 \rfloor$

In 1761, Lambert proved the irrationality of π using a continued fraction of *tan x*. He also generalized Euler work on e to show that both e^x and *tan x* are irrationals if x is nonzero rational. $\lfloor 34,38,40 \rfloor$

Lagrange used continued fractions to construct the general solution of Pell's Equation. He proved the converse of Euler's Theorem, *i.e.*, if x is a quadratic irrational (a solution of a quadratic equation), then the regular continued fraction expansion of x is periodic. In 1776, Lagrange used continued fractions in integral calculus where he developed a general method for obtaining the continued fraction expansion of the solution of a differential equation in one variable. $\lfloor 41,42,43 \rfloor$

In the nineteenth century, the subject of continued fractions was known to every mathematician and the theory concerning convergents was developed. In 1813, Carl Friedrich Gauss derived a very general complex - valued continued fraction by a clever identity involving the hypergeometric function. Henri Pade defined Pade approximant in 1892. In fact, this century can probably be described as the golden age of continued fractions. Jacobi, Perron, Hermite, Cauchy, Stieljes and many other mathematicians made contributions to this field. $\lfloor 34,39 \rfloor$

During the 20th century, continued fractions appeared in other fields. In 1992, for instance, the connection between continued fractions and chaos theory was studied in a paper written by Rob Corless.

Chapter One

Definitions and Basic Concepts

Definition 1.1: $\lfloor 15 \rfloor$

Let p and q be two integers where at least one of them is not zero. The **greatest common divisor** of p and q, denoted by gcd(p, q), is the positive integer d satisfying:

1) d divides both p and q.
2) If c divides both p and q, then $c \leq d$.

Definition 1.2: $\lfloor 15 \rfloor$

Two given integers p and q are called **relatively prime** if gcd(p, q) = 1.

Theorem 1.1: $\lfloor 15, p.4 \rfloor$

Let p, q & s be integers. If p divides both q and s, then p divides $qx + sy$ for every $x \,\&\, y \in Z$.

Theorem 1.2: $\lfloor 15, p.7 \rfloor$ **(The Division Algorithm)**

Given integers p and q, with $q > 0$, there exists unique integers m and r such that $p = q.m + r$, with $0 \leq r < q$. p is called the dividend, q the divisor, m the quotient and r is the remainder.

Lemma 1.1: $\lfloor 15, p.30 \rfloor$

Let p and q be two integers. If $p = q.m+r$, then gcd(p, q) = gcd(q, r).

The Euclidean Algorithm:

Euclidean algorithm is a method of finding the greatest common divisor of two given integers. It consists of repeated divisions. In this algorithm we apply the Division Algorithm repeatedly until we obtain a zero remainder. Since the gcd(p, q) = gcd($\pm p$, $\pm q$), we may assume that both p and q are positive integers with $p > q$.

Theorem 1.3: $\lfloor 15, p.29 \rfloor$ **(Euclidean algorithm)**

Let p and q be two positive integers, where $p > q$ and consider the following sequence of repeated divisions:

$$
\begin{aligned}
&p = q.a_1 + r_1, \quad 0 < r_1 < q \\
&b = r_1.a_2 + r_2, \quad 0 < r_2 < r_1 \\
&r_1 = r_2.a_3 + r_3, \quad 0 < r_3 < r_2 \\
&r_2 = r_3.a_4 + r_4, 0 < r_4 < r_3 \\
&\qquad . \\
&\qquad . \\
&\qquad . \\
&r_{n-2} = r_{n-1}.a_n + r_n, 0 < r_n < r_{n-1} \\
&r_{n-1} = r_n.a_{n+1} + 0
\end{aligned}
$$

Then gcd(p, q) = r_n, the last non-zero remainder of the division process.

Proof:

We need to prove that the greatest common divisor of p and q is r_n.

Using Lemma 1.1 repeatedly, we get the following:

gcd(p, q)=gcd(q, r_1)=gcd(r_1, r_2)=gcd(r_2, r_3) =…=gcd(r_{n-1}, r_n)=gcd(r_n, 0) =r_n.

Hence, the greatest common divisor of p and q is r_n.

Theorem 1.4: $\lfloor 15, p.13 \rfloor$

Given two integers p and q not both zero. Then the greatest common divisor of p and q is a linear combination of them. i.e. there exist two integers m and n such that gcd(p, q)=mp+nq.

Theorem 1.5: $\lfloor 15, p.16 \rfloor$

Given two integers p and q, then $\frac{p}{\gcd(p,q)}$ & $\frac{q}{\gcd(p,q)}$ are relatively prime.

Theorem 1.6: $\lfloor 15, p.18 \rfloor$ **(Euclid's Lemma)**

If p and q are relatively prime and p divides qs then p divides s.

Definition 1.3: $\lfloor 17 \rfloor$**(Algebraic and Transcendental Numbers)**

A complex number y is said to be algebraic if y is a root of a non-zero polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$ with integer coefficients a_0, a_1, …, a_n . The number which is not algebraic is transcendental.

Binomial Theorem:

For any positive integer n, the expansion of $(x+y)^n$ is given by:

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + ... + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ is called the binomial coefficient.

Chapter Two

Finite Simple Continued Fractions

Section 2.1: What is a Continued Fraction? $\lfloor 1,3,4,6 \rfloor$

Definition 2.1:

A **continued fraction (c.f.)** is an expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \ddots}}}}$$

where a_0, a_1, a_2, …, b_0, b_1, b_2, … can be either real or complex numbers.

Definition 2.2:

A **simple (regular)** continued fraction is a continued fraction of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}$$

where a_i is an integer for all i with $a_1, a_2, a_3 \ldots . > 0$.

The numbers a_i , $i = 0, 1, 2, \ldots.$ are called **partial quotients** of the **c.f.**

A simple continued fraction can have either a **finite** or **infinite** representation.

Definition 2.3:

A **finite** simple continued fraction is a simple continued fraction with a finite number of terms. In symbols:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

It is called an n^{th}-order continued fraction and has (n+1) elements (partial quotients).

It is also common to express the finite simple continued fraction as $a_0 + \frac{1}{a_1 +}\frac{1}{a_2 +}\frac{1}{a_3 +}....\frac{1}{a_n}$ or simply as $[a_0, a_1, a_2, ..., a_n]$.

Definition 2.4:

An **infinite** simple continued fraction is a simple continued fraction with an infinite number of terms. In symbols:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}$$

It can be also expressed as $a_0 + \frac{1}{a_1 +}\frac{1}{a_2 +}\frac{1}{a_3 +}....$ or simply as $[a_0, a_1, a_2, ...]$.

Example 2.1:

a) $6 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{5 + ...}}}}$ and $3 + \frac{1}{7 +}\frac{1}{15 +}\frac{1}{1 +}\frac{1}{292 +}....$ are infinite simple continued fractions.

b) $1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6}}}}$ and $[-1,2,6]$ are finite simple continued fractions.

Definition 2.5:

A **segment** of an n^{th}-order simple continued fraction is a continued fraction of the form $[a_0, a_1, a_2, ..., a_k]$ where $0 \le k \le n$ and arbitrary $k \ge 0$ if the continued fraction is infinite.

A **remainder** of an n^{th}-order finite simple continued fraction is a continued fraction of the form $[a_r, a_{r+1}, ..., a_n]$ where $0 \le r \le n$. Similarly, $[a_r, a_{r+1}, ...]$ is a remainder of an infinite simple continued fraction for arbitrary $r \ge 0$.

Example 2.2:

a) $[0,1,2]$ is a segment of the finite simple continued fraction $[0,1,2,1,4]$ and $[2,1,4]$ is a remainder of it .

b) $[6,1,5,1]$ is a segment of the infinite simple continued fraction $[6,1,5,1,5,1,5,...]$ and $[5,1,5,1,5,...]$ is a remainder of it .

Section 2.2: Properties and Theorems

Every rational number can be expressed as a finite simple continued fraction. Before we prove it and explain the way of expansion, we will introduce the continued fractions by studying the relationship between Euclidean algorithm, the jigsaw puzzle (splitting rectangles into squares) and continued fractions. Jigsaw puzzle uses picture analogy to clarify how to convert a rational number into a continued fraction. The explanation of the puzzle's steps is through the following example. [7,8]

Example 2.3:

Suppose we are interested in finding the greatest common divisor of 64 and 17. Using Euclidean algorithm, we have:

$$64 = 3\times 17 + 13 \qquad (2.1)$$

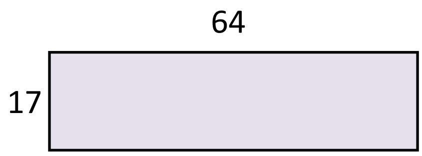
$$17 = 1\times 13 + 4 \qquad (2.2)$$

$$13 = 3\times 4 + 1 \qquad (2.3)$$

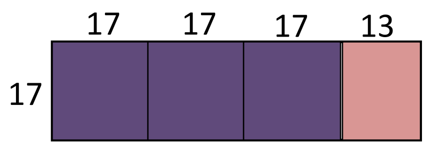
$$4 = 4\times 1 + 0 \qquad (2.4)$$

Then gcd(64, 17) = 1.

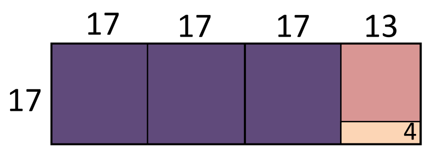
Now, consider a 64 by 17 rectangle.



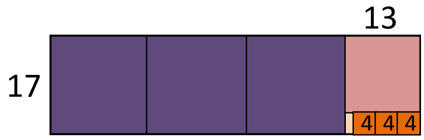
In terms of pictures, we split the rectangle into 3 squares each of side length 17 and only one 17 by 13 rectangle.



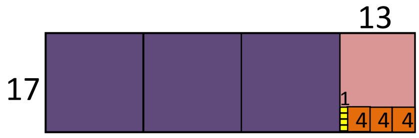
Next, it is clear that we can split the 17 by 13 rectangle into one square of side length 13 and only one 13 by 4 rectangle.



Similarly, split the 13 by 4 rectangle into 3 squares each of side length 4 and a 4 by 1 rectangle.



Finally, we can place 4 squares, each of side length 1, inside the 4 by 1 rectangle with no remaining rectangles.



We can notice that each divisor q in the Euclidean algorithm represents the length of the side of a square. For instance, the divisor 17 in equation (2.1) represents the length of the sides of the squares that we obtain from the first splitting step. Moreover, gcd(64, 17) is the length of the side of the smallest square which equals 1.

Now, divide equation (2.1) by 17 to get: $\frac{64}{17}=3+\frac{13}{17}$

Also, divide equation (2.2) by 13 to obtain: $\frac{17}{13}=1+\frac{4}{13}$

Repeat in the same way for equations (2.3) and (2.4): $\frac{13}{4}=3+\frac{1}{4}$ and $\frac{4}{1}=4$

Then, write each proper fraction in the previous equations in terms of its reciprocal as follows:

$$\frac{64}{17}=3+\frac{1}{(\frac{17}{13})} \tag{2.5}$$

$$\frac{17}{13}=1+\frac{1}{(\frac{13}{4})} \tag{2.6}$$

$$\frac{13}{4}=3+\frac{1}{(\frac{4}{1})}=3+\frac{1}{4} \tag{2.7}$$

Substitute equation (2.7) into equation (2.6) to obtain the following:

$$\frac{17}{13}=1+\cfrac{1}{3+\cfrac{1}{4}} \tag{2.8}$$

Then, substitute equation (2.8) into equation (2.5) to get:

$$\frac{64}{17}=3+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{4}}}=[3,1,3,4]$$

This is the continued fraction representation of the rational number $\frac{64}{17}$.

Note that by writing $\frac{64}{17}=[3,1,3,4]$, we do not mean an equality, but just a representation of the rational number $\frac{64}{17}$ by its continued fraction $[3,1,3,4]$.

This expression relates directly to the geometry of the rectangle as squares with the jigsaw pieces as follows:

3 squares each of side length 17, 1 square of side length 13, 3 squares each of side length 4 and 4 squares each of side length 1.

So, it's clear that the partial quotients of the continued fraction $[3,1,3,4]$ represent the number of squares that result from the splitting steps.

However, there is no need to use picture analogy each time we want to express a rational number as a continued fraction. The expansion of rational numbers into continued fractions is related to Euclidean algorithm as we've shown in the previous example. This relation will be studied closely in the proof of Theorem 2.2.

Now, to express any rational number $\frac{p}{q}$ as a continued fraction, we proceed in this manner. We split the rational number into a quotient "a_0" and a proper fraction, say $\frac{a}{b}$. If a = 1 or b = 1, stop. Otherwise, repeat the process by considering the reciprocal $\frac{1}{(\frac{b}{a})}$ of the proper fraction $\frac{a}{b}$ instead of $\frac{p}{q}$. Again, split $\frac{1}{(\frac{b}{a})}$ into a quotient "a_1" and a proper fraction, say $\frac{a}{b}$ again. Repeat this process until we get a proper fraction $\frac{1}{b}$, which is always the case for any rational number.

It is clear that if the rational number $\frac{p}{q}$ is positive and less than 1, then the continued fraction begins with zero, i.e., $a_0 = 0$. Moreover, if the rational number is negative, then the continued fraction is $[a_0, a_1, a_2, ..., a_n]$ where $a_0 < 0$ and $a_1, a_2, ...a_n > 0$.

Example 2.4:

Expand the rational number $\frac{14}{19}$ into a continued fraction.

Solution:

Since $\frac{14}{19}$ is less than 1,

then $a_0 = 0$ and $\frac{14}{19} = 0 + \frac{1}{\frac{19}{14}}$.

But $\frac{19}{14} = 1 + \frac{5}{14}$, so $\frac{14}{19} = 0 + \cfrac{1}{1 + \frac{5}{14}}$.

Also, $\frac{5}{14} = \frac{1}{\frac{14}{5}} = \cfrac{1}{2 + \frac{4}{5}}$

Therefore, $\frac{14}{19} = 0 + \cfrac{1}{1 + \cfrac{1}{2 + \frac{4}{5}}}$

Repeating the same steps for $\frac{4}{5}$, we obtain: $\frac{4}{5} = \frac{1}{\frac{5}{4}} = \cfrac{1}{1 + \frac{1}{4}}$

Thus, $\frac{14}{19} = 0 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4}}}} = [0,1,2,1,4]$

We stop here since the last proper fraction $\frac{1}{4}$ has a numerator of 1. However, looking at the last partial quotient "4" of the continued fraction, it can be written as $4 = 3 + \frac{1}{1}$. So, the continued fraction expansion $[0,1,2,1,4]$ can be also written as:

$$\frac{14}{19} = 0 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1}}}}} = [0,1,2,1,3,1]$$

As a result, the continued fraction expansion of the rational number $\frac{14}{19}$ has two forms which are obtained by changing the last quotient.

Example 2.5:

Express the rational number $\frac{59}{46}$ as a continued fraction.

Solution:

Applying the previous steps, we get:

$$\frac{59}{46} = 1 + \frac{13}{46} = 1 + \cfrac{1}{\cfrac{46}{13}} = 1 + \cfrac{1}{3 + \cfrac{7}{13}} = 1 + \cfrac{1}{3 + \cfrac{1}{\cfrac{13}{7}}} = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{6}{7}}} = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{7}{6}}}} = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6}}}}$$

$= [1,3,1,1,6]$.

Example 2.6:

Write the rational number $-\frac{7}{13}$ as a continued fraction.

Solution:

$$-\frac{7}{13}=-1+\frac{6}{13}=-1+\frac{1}{\frac{13}{6}}=-1+\frac{1}{2+\frac{1}{6}}=[-1,2,6]$$

The Continued Fraction Algorithm: $\lfloor 4,9 \rfloor$

This algorithm is a systematic approach that is used to find the continued fraction expansion of any rational number.

Let y be any non-integer rational number. To find its continued fraction expansion, we follow the next steps.

Step 1: Set $y=y_0$. The first partial quotient of the continued fraction is the greatest integer less than or equal y_0. (i.e., $a_0=[[y_0]]$), where [[.]] is the greatest integer function.

Step 2: Define $y_1=\frac{1}{y_0-[[y_0]]}$ and set $a_1=[[y_1]]$.

As long as y_j is non-integer, continue in this manner:

$$y_2=\frac{1}{y_1-[[y_1]]},\ a_2=[[y_2]],$$

.

.

$$y_k=\frac{1}{y_{k-1}-[[y_{k-1}]]},\ a_k=[[y_k]], \text{ where } y_k-[[y_k]]=0.$$

Step 3: Stop when we find a value $y_k \in N$.

Note 2.1:

This algorithm is also true for any real number. In this case, the process may continue indefinitely. This idea will be illustrated in Chapter Three.

Example 2.7:

Calculate the continued fraction expansion of $\frac{315}{201}$ using the continued fraction algorithm.

Solution:

Let $y_0 = \frac{315}{201} \approx 1.567164179$. Then $a_0 = [[y_0]] = [[\frac{315}{201}]] = 1$.

$$y_1 = \frac{1}{y_0 - [[y_0]]} = \frac{1}{\frac{315}{201} - [[\frac{315}{201}]]} = \frac{1}{\frac{315}{201} - 1} = \frac{201}{114} \approx 1.763157895, a_1 = [[y_1]] = 1$$

$$y_2 = \frac{1}{y_1 - [[y_1]]} = \frac{1}{\frac{201}{114} - [[\frac{201}{114}]]} = \frac{1}{\frac{201}{114} - 1} = \frac{114}{87} \approx 1.310344828, \ a_2 = [[y_2]] = 1$$

$$y_3 = \frac{1}{y_2 - [[y_2]]} = \frac{1}{\frac{114}{87} - [[\frac{114}{87}]]} = \frac{1}{\frac{114}{87} - 1} = \frac{87}{27} \approx 3.222222222, \ a_3 = [[y_3]] = 3$$

$$y_4 = \frac{1}{y_3 - [[y_3]]} = \frac{1}{\frac{87}{27} - [[\frac{87}{27}]]} = \frac{1}{\frac{87}{27} - 3} = \frac{27}{6} = 4.5, \ a_4 = [[y_4]] = 4$$

$$y_5 = \frac{1}{y_4 - [[y_4]]} = \frac{1}{\frac{27}{6} - [[\frac{27}{6}]]} = \frac{1}{\frac{27}{6} - 4} = \frac{6}{3} = 2, \ a_5 = [[y_5]] = 2$$

We stop here since $y_5 = 2 \in N$. Thus, $[1,1,1,3,4,2]$ is the continued fraction representation of $\frac{315}{201}$.

What about the converse? $\lfloor 8 \rfloor$

Given a continued fraction representation of a number *y*, we find *y* by using the following relationship repeatedly:

$$[a_0, a_1, a_2, ..., a_{n-1}, a_n] = [a_0, a_1, a_2, ..., a_{n-1} + \frac{1}{a_n}]$$

Example 2.8:

Find the rational number who has the continued fraction representation $[2,2,1,2,1]$.

Solution:

$$[2,2,1,2,1] = [2,2,1,2+\frac{1}{1}] = [2,2,1,3] = [2,2,1+\frac{1}{3}] = [2,2,\frac{4}{3}] = [2,2+\frac{1}{(\frac{4}{3})}]$$

$$= [2,2+\frac{3}{4}] = [2,\frac{11}{4}] = [2+\frac{1}{(\frac{11}{4})}] = [2+\frac{4}{11}] = \frac{26}{11}$$

Theorem 2.1: $\lfloor 2,\ p.553 \rfloor$

Every finite simple continued fraction represents a rational number.

Proof:

Let $[a_0, a_1, a_2, ..., a_n]$ be a given n^{th}- order finite simple continued fraction. We show that this continued fraction represents a rational number using induction on the number of partial quotients.

If n = 1, then $[a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}$.

Since a_0 and a_1 are integers, then $\frac{a_0 a_1 + 1}{a_1}$ is a rational number.

Now assume any finite simple continued fraction with $k < n$ partial quotients represents a rational number. Then:

$$[a_0, a_1, a_2, ..., a_k] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{... + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}}}} = a_0 + \frac{1}{Y}\ ,$$

Where

$$Y = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\dots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}}} = [a_1, a_2, \dots, a_k].$$

Since $[a_1, a_2, \dots, a_k]$ is a finite simple continued fraction with k partial quotients, it represents a rational number, say $\frac{d}{f}$. So, $Y = \frac{d}{f}, f \neq 0.$

Thus, $[a_0, a_1, a_2, \dots, a_k] = a_0 + \frac{1}{Y} = a_0 + \frac{1}{\frac{d}{f}} = a_0 + \frac{f}{d} = \frac{a_0 d + f}{d}$ which is a rational number since a_0, d and f are integers.

So, any finite simple continued fraction $[a_0, a_1, a_2, \dots, a_n]$ represents a rational number for any $n \in N$.

Theorem 2.2: $\lfloor 2, p.553 \rfloor \& \lfloor 3, p.10 \rfloor$

Every rational number can be represented as a finite simple continued fraction in which the last term can be modified so as to make the number of terms in the expansion either even or odd.

Proof:

Let $\frac{p}{q}$, $q > 0$ be any rational number. By the Euclidean algorithm

$$p = q.a_1 + r_1, 0 < r_1 < q \tag{2.9}$$

$$q = r_1.a_2 + r_2, 0 < r_2 < r_1 \tag{2.10}$$

$$r_1 = r_2.a_3 + r_3, 0 < r_3 < r_2$$

$$r_2 = r_3.a_4 + r_4, 0 < r_4 < r_3$$

$$.$$

$$.$$

$$r_{n-3} = r_{n-2}.a_{n-1} + r_{n-1}, 0 < r_{n-1} < r_{n-2}$$

$$r_{n-2} = a_n.r_{n-1} + 0$$

The quotients $a_2, a_3, a_4, ...a_n$ and the remainders $r_1, r_2, r_3, ..., r_{n-1}$ are positive integers, while a_1 can be a positive integer, negative integer or zero.

Now, dividing equation (2.9) by q and then taking the reciprocal of the proper fraction we get: $\frac{p}{q} = a_1 + \frac{r_1}{q} = a_1 + \frac{1}{\frac{q}{r_1}}, 0 < r_1 < q$

Also divide equation (2.10) by r_1 and take the reciprocal of the proper fraction to get:

$$\frac{q}{r_1} = a_2 + \frac{r_2}{r_1} = a_2 + \frac{1}{\frac{r_1}{r_2}}, 0 < r_2 < r_1 \qquad (2.11)$$

Repeating the same process to each equation in the above Euclidean algorithm, we have:

$$\frac{r_1}{r_2} = a_3 + \frac{r_3}{r_2} = a_3 + \frac{1}{\frac{r_2}{r_3}}, 0 < r_3 < r_2 \qquad (2.12)$$

$$\frac{r_2}{r_3} = a_4 + \frac{r_4}{r_3} = a_4 + \frac{1}{\frac{r_3}{r_4}}, 0 < r_4 < r_3 \qquad (2.13)$$

.

.

$$\frac{r_{n-3}}{r_{n-2}} = a_{n-1} + \frac{r_{n-1}}{r_{n-2}} = a_{n-1} + \frac{1}{\frac{r_{n-2}}{r_{n-1}}}, 0 < r_{n-1} < r_{n-2} \qquad (2.14)$$

$$\frac{r_{n-2}}{r_{n-1}} = a_n$$

Now, substituting $\frac{q}{r_1}$ and each of $\frac{r_{i-1}}{r_i}$ back into equations (2.11) through (2.14) yields:

$$\frac{p}{q}=a_1+\cfrac{1}{a_2+\cfrac{1}{r_1}}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{(\frac{r_2}{r_3})}}}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cfrac{1}{(\frac{r_3}{r_4})}}}}$$

Continue in the same manner to get:

$$\frac{p}{q}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cfrac{1}{\dots+\cfrac{1}{a_{n-1}+\cfrac{1}{(\frac{r_{n-2}}{r_{n-1}})}}}}}}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cfrac{1}{\dots+\cfrac{1}{a_{n-1}+\cfrac{1}{a_n}}}}}}$$

$= [a_1, a_2, ..., a_n]$.

Thus, every rational number can be represented as a finite simple continued fraction.

In fact, we can always modify the last partial quotient a_n of this representation so that the number of terms is either even or odd.

If $a_n = 1$, then $\cfrac{1}{a_{n-1}+\cfrac{1}{a_n}}=\cfrac{1}{a_{n-1}+\cfrac{1}{1}}=\cfrac{1}{a_{n-1}+1}$

and $\frac{p}{q}=[a_1,a_2,...,a_{n-1},a_n]=[a_1,a_2,...,a_{n-1}+1]$.

Else, if $a_n > 1$, then $\cfrac{1}{a_{n-1}+\cfrac{1}{a_n}}=\cfrac{1}{a_{n-1}+\cfrac{1}{(a_n-1)+1}}=\cfrac{1}{a_{n-1}+\cfrac{1}{(a_n-1)+\cfrac{1}{1}}}$

and $\frac{p}{q}=[a_1,a_2,...,a_{n-1},a_n]=[a_1,a_2,...,a_{n-1},a_n-1,1]$.

Theorem 2.3: $\lfloor 3, p.12 \rfloor$

Let p and q be two integers such that $p > q > 0$. Then $[a_0, a_1, a_2, ..., a_{n-1}, a_n]$ is a continued fraction representation of $\frac{p}{q}$ if and only if $\frac{q}{p}$ has $[0, a_0, a_1, a_2, ..., a_{n-1}, a_n]$ as its continued fraction representation.

Proof:

Since $p > q > 0$, $\frac{p}{q} > 1$ and equals $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{... + \cfrac{1}{a_n}}}}}$ where a_0

is the greatest integer less than $\frac{p}{q} = [[\frac{p}{q}]] > 0$.

The reciprocal of $\frac{p}{q}$ is

$$\frac{q}{p} = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{... + \cfrac{1}{a_n}}}}}} = 0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{... + \cfrac{1}{a_n}}}}}} = [0, a_0, a_1, a_2, ..., a_n]$$

Conversely, since $p > q > 0$, $0 < \frac{q}{p} < 1$ and equals

$$\frac{q}{p} = 0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{... + \cfrac{1}{a_n}}}}}} = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{... + \cfrac{1}{a_n}}}}}}$$

The reciprocal of $\frac{q}{p}$ is

$$\frac{p}{q}=\cfrac{1}{\cfrac{1}{a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{\dots+\cfrac{1}{a_n}}}}}}}=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{\dots+\cfrac{1}{a_n}}}}}=[a_0,a_1,a_2,\dots,a_n].$$

Theorem 2.4: $\lfloor 5,\ pp.82-83 \rfloor$

The continued fraction $[a_0,a_1,a_2,\dots,a_{n-1},a_n]$ and its reversal $[a_n,a_{n-1},a_{n-2},\dots,a_1,a_0]$ with $a_0>0$ have the same numerators.

Proof:

This theorem is proved by Euler. See $\lfloor 5 \rfloor$.

For example, the continued fractions [5,3,2,4] and [4,2,3,5] have the same numerator “164”.

Section 2.3: Convergents

In order to have a thorough understanding of continued fractions, we must study some of their properties in details.

Consider the continued fraction representation $[2,2,7]$ of the rational number $\frac{37}{15}$. The segments of this continued fraction are:

$$[2]=2,\ [2,2]=2+\frac{1}{2},\ [2,2,7]=2+\cfrac{1}{2+\cfrac{1}{7}}$$

Since each segment is a finite simple continued fraction, it represents a rational number. These segments are called convergents of the continued fraction $[2,2,7]$.

Definition 2.6: $\lfloor 4,9 \rfloor$

Let $[a_0, a_1, \ldots, a_n]$ be a finite simple continued fraction representation of a rational number $\frac{p}{q}$. Its segments:

$$c_0=[a_0]=a_0,\ c_1=[a_0,a_1]=a_0+\frac{1}{a_1},\ c_2=[a_0,a_1,a_2]=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2}},$$

$$\ldots,\ c_n=[a_0,a_1,a_2,\ldots,a_n]=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ldots+\cfrac{1}{a_n}}}}$$

are all called **convergents** of the continued fraction with c_k is the $\boldsymbol{k}^{\textbf{th}}$ **convergent**, $k = 0, 1, \ldots, n$.

Note that we have $n+1$ convergents and each convergent c_k represents a rational number of the form $c_k = \frac{p_k}{q_k}$, where p_k and q_k are integers with $c_n = \frac{p}{q}$.

We shall use the representation of a convergent $c_k = [a_0, a_1, ..., a_k]$ and $\frac{p_k}{q_k}$ interchangeably to mean the same thing.

Example 2.9:

Find all of the convergents for the continued fraction $[3,5,1,7]$.

Solution:

$$c_0 = [3] = 3$$

$$c_1 = [3,5] = 3 + \frac{1}{5} = \frac{16}{5}$$

$$c_2 = [3,5,1] = 3 + \cfrac{1}{5 + \cfrac{1}{1}} = 3 + \frac{1}{6} = \frac{19}{6}$$

$$c_3 = [3,5,1,7] = 3 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{7}}} = 3 + \cfrac{1}{5 + \cfrac{1}{\frac{8}{7}}} = 3 + \cfrac{1}{5 + \cfrac{7}{8}} = 3 + \cfrac{1}{\frac{47}{8}} = 3 + \frac{8}{47} = \frac{149}{47}$$

Note that the 3rd convergent $c_3 = \frac{149}{47}$ represents the fraction itself.

The following theorem gives a recursion formula to calculate the convergents of a continued fraction.

Theorem 2.5: $\lfloor 1, p.21 \rfloor \& \lfloor 4, p.7 \rfloor$ **(Continued Fraction Recursion Formula)**

Consider the continued fraction $[a_0, a_1, ..., a_n]$ of a given rational number. Define $\begin{matrix} p_{-1} = 1, p_{-2} = 0 \\ q_{-1} = 0, q_{-2} = 1 \end{matrix}$. Then $\begin{matrix} p_k = a_k p_{k-1} + p_{k-2} \\ q_k = a_k q_{k-1} + q_{k-2} \end{matrix}$, for k = 0, 1, 2,

…, n, where $p_0, p_1, p_2, \ldots, p_n$ are the numerators of the convergents of the given continued fraction and $q_0, q_1, q_2, \ldots, q_n$ are their denominators.

Proof:

We prove this theorem using induction on k.

For $k = 0$, we have:

$$c_0 = \frac{p_0}{q_0} = a_0 = \frac{a_0}{1} = \frac{a_0.1+0}{a_0.0+1} = \frac{a_0.p_{-1} + p_{-2}}{a_0.q_{-1} + q_{-2}}$$

Therefore, $p_0 = a_0 p_{-1} + p_{-2}$ and $q_0 = a_0.0 + 1$

For $k = 1$, we have:

$$c_1 = \frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{a_1.a_0 + 1}{a_1.1 + 0} = \frac{a_1.p_0 + p_{-1}}{a_1.q_0 + q_{-1}}$$

Then, $p_1 = a_1.p_0 + p_{-1}$ and $q_1 = a_1.q_0 + q_{-1}$

Thus, the formula $\begin{matrix} p_k = a_k p_{k-1} + p_{k-2} \\ q_k = a_k q_{k-1} + q_{k-2} \end{matrix}$ is true for $k = 0, 1$.

Assume the theorem is true for $k = 2, 3, \ldots, j$, where $j < n$.

i.e. $c_k = \dfrac{p_k}{q_k} = \dfrac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}$, for $k = 2, 3, \ldots, j$ (2.15)

So, $p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$

Now, we prove that the formula is true for the next integer $j+1$.

$$c_{j+1}=[a_0,a_1,...,a_j,a_{j+1}]=a_0+\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{a_j+\cfrac{1}{a_{j+1}}}}}=a_0+\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{(a_j+\cfrac{1}{a_{j+1}})}}}$$

$$=[a_0,a_1,...,a_j+\frac{1}{a_{j+1}}].$$

This suggests that we can calculate c_{j+1} from the formula of c_j obtained from equation (2.13) after replacing k by j. Before we continue, we must make sure that the values of $p_{j-1},p_{j-2},q_{j-1},q_{j-2}$ won't change if a_j in equation (2.13) is replaced by another number. To do this, first replace k in the equation by j-1, and then by j-2, j-3 to get:

$$c_{j-1}=\frac{p_{j-1}}{q_{j-1}}=\frac{a_{j-1}.p_{j-2}+p_{j-3}}{a_{j-1}.q_{j-2}+q_{j-3}},\qquad c_{j-2}=\frac{p_{j-2}}{q_{j-2}}=\frac{a_{j-2}.p_{j-3}+p_{j-4}}{a_{j-2}.q_{j-3}+q_{j-4}}$$

$$c_{j-3}=\frac{p_{j-3}}{q_{j-3}}=\frac{a_{j-3}.p_{j-4}+p_{j-5}}{a_{j-3}.q_{j-4}+q_{j-5}}$$

We notice that p_{j-1} and q_{j-1} depend only on a_{j-1} while the numbers $p_{j-2},p_{j-3},q_{j-2},q_{j-3}$ depend upon the preceding a's, p's and q's. Thus, the numbers $p_{j-1},p_{j-2},q_{j-1},q_{j-2}$ depend only on $a_0,a_1,...,a_{j-1}$ and not on a_j. This implies that they will remain the same when we replace a_j by $a_j+\frac{1}{a_{j+1}}$.

Back to equation (2.13), replace a_j by $a_j+\frac{1}{a_{j+1}}$ to get:

$$c_{j+1}=\frac{(a_j+\frac{1}{a_{j+1}}).p_{j-1}+p_{j-2}}{(a_j+\frac{1}{a_{j+1}}).q_{j-1}+q_{j-2}}=\frac{(\frac{a_ja_{j+1}+1}{a_{j+1}}).p_{j-1}+p_{j-2}}{(\frac{a_ja_{j+1}+1}{a_{j+1}}).q_{j-1}+q_{j-2}} \tag{2.16}$$

Multiply the numerator and denominator of equation (2.14) by a_{j+1} and rearrange the terms to obtain:

$$c_{j+1}=\frac{(a_j a_{j+1}+1).p_{j-1}+a_{j+1}p_{j-2}}{(a_j a_{j+1}+1).q_{j-1}+a_{j+1}q_{j-2}}=\frac{a_{j+1}(a_j p_{j-1}+p_{j-2})+p_{j-1}}{a_{j+1}(a_j q_{j-1}+q_{j-2})+q_{j-1}}$$

But from our assumption, $a_j p_{j-1}+p_{j-2}=p_j$ and $a_j q_{j-1}+q_{j-2}=q_j$.

Then, $c_{j+1}=\dfrac{a_{j+1}p_j+p_{j-1}}{a_{j+1}q_j+q_{j-1}}$.

Thus, the formula is true for $k = j+1$. So, by induction, the theorem is true for $0 \le k \le n$.

Note 2.2:

1) $\dfrac{p_{-1}}{q_{-1}}$ and $\dfrac{p_{-2}}{q_{-2}}$ are not convergents. p_{-1}, p_{-2}, q_{-1} and q_{-2} are just initial values used to calculate c_0 and c_1.
2) $q_k > 0$, $k = 0, 1, \ldots, n$.
3) Since $a_k > 0$ for $1 \le k \le n$ and $q_k > 0$ for $0 \le k \le n$, it follows that $q_k > q_{k-1}$, $k = 2, \ldots, n$.

Example 2.10:

Find the convergents of the continued fraction representation of the rational number $\dfrac{320}{171}$ using Continued Fraction Recursion Formula.

Solution:

First of all, the continued fraction representation of $\dfrac{320}{171}$ is $[1,1,6,1,3,2,2]$ and we have $a_0=1, a_1=1, a_2=6, a_3=1, a_4=3, a_5=2, a_6=2.$

With $\begin{matrix} p_{-1}=1, p_{-2}=0 \\ q_{-1}=0, q_{-2}=1 \end{matrix}$, calculate p_k and q_k using the recursion formula.

$$\begin{matrix} p_k=a_k p_{k-1}+p_{k-2} \\ q_k=a_k q_{k-1}+q_{k-2} \end{matrix}, \text{ for } k = 0, 1, 2, \ldots, 6.$$

For k = 0:

$$p_0 = a_0 p_{-1} + p_{-2} = 1\times 1 + 0 = 1$$
$$q_0 = a_0 q_{-1} + q_{-2} = 1\times 0 + 1 = 1$$

For k = 1:

$$p_1 = a_1 p_0 + p_{-1} = 1\times 1 + 1 = 2$$
$$q_1 = a_1 q_0 + q_{-1} = 1\times 1 + 0 = 1$$

For k = 2:

$$p_2 = a_2 p_1 + p_0 = 6\times 2 + 1 = 13$$
$$q_2 = a_2 q_1 + q_0 = 6\times 1 + 1 = 7$$

For k = 3:

$$p_3 = a_3 p_2 + p_1 = 1\times 13 + 2 = 15$$
$$q_3 = a_3 q_2 + q_1 = 1\times 7 + 1 = 8$$

For k = 4:

$$p_4 = a_4 p_3 + p_2 = 3\times 15 + 13 = 58$$
$$q_4 = a_4 q_3 + q_2 = 3\times 8 + 7 = 31$$

For k = 5:

$$p_5 = a_5 p_4 + p_3 = 2\times 58 + 15 = 131$$
$$q_5 = a_5 q_4 + q_3 = 2\times 31 + 8 = 70$$

For k = 6:

$$p_6 = a_6 p_5 + p_4 = 2\times 131 + 58 = 320$$
$$q_6 = a_6 q_5 + q_4 = 2\times 70 + 31 = 171$$

Thus, $c_0 = \frac{p_0}{q_0} = \frac{1}{1} = 1$, $c_1 = \frac{p_1}{q_1} = \frac{2}{1} = 2$, $c_2 = \frac{p_2}{q_2} = \frac{13}{7}$, $c_3 = \frac{p_3}{q_3} = \frac{15}{8}$,
$c_4 = \frac{p_4}{q_4} = \frac{58}{31}$, $c_5 = \frac{p_5}{q_5} = \frac{131}{70}$, $c_6 = \frac{p_6}{q_6} = \frac{320}{171}$.

The last convergent, c_6 in this example, must be equal to the rational number the continued fraction represents.

However, a convergent table can be used to save time in calculating p_k and q_k. Table 2.1 explains the manner.

Table 2.1

k	-2	-1	0	1	2	…	…	…	n
a_k			a_0	a_1	a_2	…	…	…	a_n
p_k	$p_{-2}=0$	$p_{-1}=1$	p_0	p_1	p_2	…	…	…	p_n
q_k	$q_{-2}=1$	$q_{-1}=0$	q_0	q_1	q_2	…	…	…	q_n
c_k			c_0	c_1	c_2	…	…	…	c_n

The first row of the table is filled with the values of k that always range from -2 to n. In the second row, we write the partial quotients of the given continued fraction. Now, to fill the 3rd and 4th rows, we write the values $p_{-2} = 0$, $q_{-2} = 1$, $p_{-1} = 1$, $q_{-1} = 0$ under $k = -2$, $k = -1$, respectively. Then we compute the values of p_k's and q_k's using the recursion formula. For example, to find p_1 and q_1, we follow the arrows, (look at the table):

$$p_{-1} \xleftarrow{+} p_0 \xleftarrow{\times} a_1 \qquad\qquad q_{-1} \xleftarrow{+} q_0 \xleftarrow{\times} a_1$$

This manner gives us the following equations which we obtain when we set $k = 1$ in the recursion formula:

$$p_1 = a_1 p_0 + p_{-1}$$
$$q_1 = a_1 q_0 + q_{-1}$$

In the same process we find p_k and q_k for each value of k.

The last row contains the convergents c_k's, where $c_k = \dfrac{p_k}{q_k}, 0 \le k \le n$.

Back to our example, the table is filled in the same manner and the result is:

Table 2.2

k	-2	-1	0	1	2	3	4	5	6
a_k			1	1	6	1	3	2	2
p_k	0	1 ←	1	2	13	15	58 ←	131	320
q_k	1	0	1	1 ←	7	8	31	70	171
c_k			$\frac{1}{1}=1$	$\frac{2}{1}=2$	$\frac{13}{7}$	$\frac{15}{8}$	$\frac{58}{31}$	$\frac{131}{70}$	$\frac{320}{171}$

Theorem 2.6: $\lfloor 10,\ p.358 \rfloor$ **(Difference of Successive Convergents Theorem)**

$$c_k - c_{k-1} = \frac{(-1)^{k-1}}{q_k q_{k-1}}, 1 \le k \le n$$

To prove this theorem we need the following lemma.

Lemma 2.1: $\lfloor 4,\ p.7 \rfloor$

Let $\frac{p_k}{q_k}$ be the k^{th} convergent of the continued fraction $[a_0, a_1, \ldots, a_n]$, where p_k and q_k are defined as in Theorem 2.5. Then:

$$p_{k-1}q_k - p_k q_{k-1} = (-1)^k, -1 \le k \le n.$$

Proof:

This lemma will be proved by induction on k and using the formula that we've proved in the previous theorem. Direct calculations show the theorem is true for k = -1, 0 and 1.

For k = -1: $p_{-2}q_{-1} - p_{-1}q_{-2} = 0.0 - 1.1 = -1 = (-1)^{-1}$

For k = 0: $p_{-1}q_0 - p_0 q_{-1} = 1.1 - a_0.0 = 1 = (-1)^0$

For k = 1: $p_0q_1 - p_1q_0 = a_0a_1 - (a_0a_1 + 1).1 = a_0a_1 - a_0a_1 - 1 = -1 = (-1)^1$

Assume the lemma is true for some integer $s < n$, i.e. $p_{s-1}q_s - p_sq_{s-1} = (-1)^s$.

Now, for $k = s+1$, we have:

$$p_sq_{s+1} - p_{s+1}q_s = p_s(a_{s+1}q_s + q_{s-1}) - (a_{s+1}p_s + p_{s-1})q_s$$

$$= p_sa_{s+1}q_s + p_sq_{s-1} - a_{s+1}p_sq_s - p_{s-1}q_s = p_sq_{s-1} - p_{s-1}q_s = -1.(p_{s-1}q_s - p_sq_{s-1})$$

$$= -1.(-1)^s = (-1)^{s+1}.$$

Therefore, the formula is true for $k = s+1$ and so by induction the lemma is true for $-1 \le k \le n$.

Proof of Theorem 2.6:

For $1 \le k \le n$:

$$c_k - c_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{p_kq_{k-1} - p_{k-1}q_k}{q_kq_{k-1}} = -\frac{p_{k-1}q_k - p_kq_{k-1}}{q_kq_{k-1}}$$

Using Lemma 2.1, $c_k - c_{k-1} = -\dfrac{(-1)^k}{q_kq_{k-1}} = \dfrac{(-1)^{k+1}}{q_kq_{k-1}} = \dfrac{(-1)^{k-1}}{q_kq_{k-1}}$.

Example 2.11:

Verify Lemma 2.1 using the convergents of the continued fraction $[1,1,6,1,3,2,2]$.

Solution:

Using the values of p_k's $=\{1, 2, 13, 15, 58, 131, 320\}$ and q_k's $=\{1, 1, 7, 8, 31, 70, 171\}$ obtained in Example 2.10, we get:

For k = -1: $p_{-2}q_{-1} - p_{-1}q_{-2} = 0\times 0 - 1\times 1 = -1 = (-1)^{-1}$

For k = 0: $p_{-1}q_0 - p_0q_{-1} = 1\times 1 - 1\times 0 = 1 = (-1)^0$

For k = 1: $p_0q_1 - p_1q_0 = 1\times 1 - 2\times 1 = -1 = (-1)^1$

For k = 2: $p_1q_2 - p_2q_1 = 2\times 7 - 13\times 1 = 1 = (-1)^2$

For k = 3: $p_2q_3 - p_3q_2 = 13\times 8 - 15\times 7 = -1 = (-1)^3$

For k = 4: $p_3q_4 - p_4q_3 = 15\times 31 - 58\times 8 = 1 = (-1)^4$

For k = 5: $p_4q_5 - p_5q_4 = 58\times 70 - 131\times 31 = (-1)^5$

For k = 6: $p_5q_6 - p_6q_5 = 131\times 171 - 320\times 70 = (-1)^6$

Thus, $p_{k-1}q_k - p_kq_{k-1} = (-1)^k$ for -1 $\le k \le$ 6.

Corollary 2.1: $\lfloor 10, p.358 \rfloor$

$c_k - c_{k-2} = \dfrac{(-1)^k a_k}{q_kq_{k-2}}$, $2 \le k \le n.$

Proof:

By Theorem 2.6, $c_k - c_{k-1} = \dfrac{(-1)^{k-1}}{q_kq_{k-1}}$ and $c_{k-1} - c_{k-2} = \dfrac{(-1)^{k-2}}{q_{k-1}q_{k-2}}$

Adding these two equations, we get:

$$c_k - c_{k-2} = \frac{(-1)^{k-1}}{q_kq_{k-1}} + \frac{(-1)^{k-2}}{q_{k-1}q_{k-2}} = \frac{(-1)^{k-1}q_{k-2} + (-1)^{k-2}q_k}{q_kq_{k-1}q_{k-2}} = \frac{(-1)^{k-2}(q_k - q_{k-2})}{q_kq_{k-1}q_{k-2}}$$

But from the continued fraction recursion formula, $q_k - q_{k-2} = a_kq_{k-1}$

Thus, $c_k - c_{k-2} = \dfrac{(-1)^{k-2}(a_kq_{k-1})}{q_kq_{k-1}q_{k-2}} = \dfrac{(-1)^{k-2}(a_k)}{q_kq_{k-2}} = \dfrac{(-1)^k(a_k)}{q_kq_{k-2}}$.

Corollary 2.2: $\lfloor 2, p.561 \rfloor$

For $1 \le k \le n$, p_k and q_k are relatively prime.

Proof:

Let d = gcd(p_k, q_k). Then d divides $p_{k-1}q_k - p_kq_{k-1} = (-1)^k, 1 \le k \le n$.

Hence, d=1=gcd(p_k, q_k). So, p_k and q_k are relatively prime for all $1 \le k \le n$.

To illustrate this property, consider the convergents of the continued fraction in Example 2.10. We find that

gcd(p_1, q_1) = gcd(2, 1) = 1; gcd(p_2, q_2) = gcd(13, 7) = 1, gcd(p_3, q_3) = gcd(15, 8) = 1; gcd(p_4, q_4) = gcd(58, 31) = 1,

gcd(p_5, q_5) = gcd(131, 70) = 1; gcd(p_6, q_6) = gcd(320, 171) = 1.

Thus, p_k *and* q_k are relatively prime for each value of k, where $1 \leq k \leq 6$.

Example 2.12:

Given $[1,1,1,3,1,2]$ is the continued fraction representation of the rational number $\frac{39}{25}$, find the convergents.

Solution:

Applying Theorem 2.5, we find the convergents of $[1,1,1,3,1,2]$:

$c_0 = 1$, $c_1 = 2$, $c_2 = \frac{3}{2}$, $c_3 = \frac{11}{7}$, $c_4 = \frac{14}{9}$, $c_5 = \frac{39}{25}$.

Notice that:

1) The even convergents 1, $\frac{3}{2}$, $\frac{14}{9}$ form an increasing sequence and approach the actual value $\frac{39}{25}$ from below, i.e. $c_0 < c_2 < c_4$.
2) The odd convergents 2, $\frac{11}{7}$, $\frac{39}{25}$ form a decreasing sequence and approach the actual value $\frac{39}{25}$ from above, i.e. $c_1 > c_3 > c_5$.
3) The convergents c_k approach the actual value $\frac{39}{25}$ as k increases, where $0 \leq k \leq 5$. Moreover, they are alternatively less than and greater than $\frac{39}{25}$ except the last convergent c_5. Therefore, we conclude that $c_0 < c_2 < c_4 < \frac{39}{25} = c_5 < c_3 < c_1$. Figure 2.1 illustrates these notes.

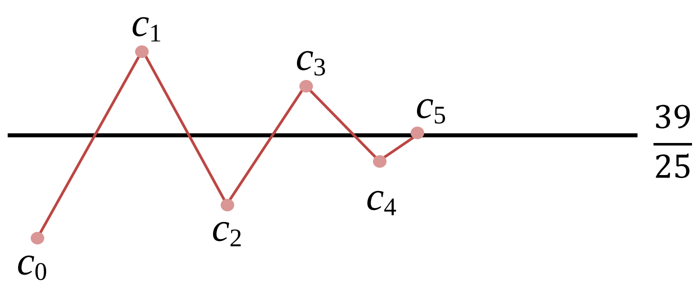


Figure 2.1

These notes lead to the following theorem.

Theorem 2.7: $\lfloor 2, p.562 \rfloor$

Let c_0, c_1, ..., c_n be the convergents of the continued fraction $[a_0, a_1, ..., a_n]$. Then even–numbered convergents form an increasing sequence and odd-numbered convergents form a decreasing sequence. Moreover every odd-numbered convergent is greater in value than every even-numbered convergent. In other words:

$c_{2m} < c_{2m+2}$, $c_{2m+3} < c_{2m+1}$ and $c_{2j} < c_{2r+1}$, $m, j, r \geq 0$.

Proof:

By Corollary 2.1, $c_{2k} - c_{2k-2} = \dfrac{(-1)^{2k} a_{2k}}{q_{2k} q_{2k-2}}$, $k \geq 1$. (2.17)

Since a_k, q_k, $q_{k-2} > 0$, then $c_{2k} - c_{2k-2} \geq 0$. Hence,

$$c_{2k} \geq c_{2k-2} \tag{2.18}$$

Thus, the even–numbered convergents form an increasing sequence $c_0 < c_2 < c_4 < \dots$.

Similarly, by Corollary 2.1, $c_{2k+1} - c_{2k-1} = \dfrac{(-1)^{2k+1} a_{2k+1}}{q_{2k+1} q_{2k-1}}$, $k \geq 1$

and so $c_{2k-1} > c_{2k+1}$ (2.19)

Thus, the odd–numbered convergents form a decreasing sequence $c_1 > c_3 > c_5 > \dots$.

Finally, put $k = 2s + 1$, $s \geq 0$ in Theorem 2.6, we obtain

$c_{2s+1} - c_{2s} = \dfrac{(-1)^{2s}}{q_{2s+1} q_{2s}} > 0$. With $q_{2s+1}, q_{2s}, (-1)^{2s} > 0$, we get

$$c_{2s} < c_{2s+1} \tag{2.20}$$

From (2.18), (2.19) & (2.20):

$$c_0 < c_2 < c_4 < \dots < c_{2k} < c_{2k+1} < c_{2k-1} < \dots < c_3 < c_1, \text{ if } n = 2k+1$$

and

$$c_0 < c_2 < c_4 < \dots < c_{2k} < c_{2k-1} < c_{2k-3} < \dots < c_3 < c_1, \text{ if } n = 2k$$

Section 2.4: Solving Linear Diophantine Equations

Many puzzles, enigmas and trick questions lead to mathematical equations whose solutions are required to be integers. Such equations are called Diophantine equations, named after the Greek mathematician Diophantus who wrote a book about them.

Definition 2.7: $\lfloor 1,2,12 \rfloor$

Diophantine Equation is an algebraic equation in one or more unknowns with integral coefficients such that only integral solutions are sought. This type of equations may have no solution, a finite number or an infinite number of solutions.

Example 2.13:

The following equations are Diophantine equations, where integral solutions are required for *x*, *y* and *z*.

$3x+5y=7$, $x^2+y^2=1$, $x^2+y^2=z^2$, $x^2-3y^2=1$.

Definition 2.8: $\lfloor 2 \rfloor$

Linear Diophantine Equation "LDE" in two variables x and y is the simplest case of Diophantine equations and has the form $ax+by=c$ where *a*, *b* and *c* are integers.

Example 2.14:

$3x+5y=1$, $6x-4y=2$, $-5x+5y=8$ are linear Diophantine equations in two variables.

In this section, we are interested in solving linear Diophantine equations in two variables. i.e., finding integral solutions of $ax+by=c$. If a and b are both zeros, then the equation is either trivially true when $c = 0$ or trivially false when $c \neq 0$. Moreover, if one of a or b equals zero, then the case is also trivial. So we omit these two cases and assume that both a and b are nonzero integers.

Geometrically, this equation represents a line in the Cartesian plane that is not parallel to either axis. Solutions of the equation $ax+by=c$ are the points on the line with integral coordinates. Points with integral coordinates are called **lattice points.**

However, does every linear Diophantine equation $ax+by=c$ have an integral solution? If not, what are the conditions necessary for a LDE to have a solution? The following theorem answers these questions.

Theorem 2.8: $\lfloor 14, p.12 \rfloor$

Let a, b & c be integers with $ab \neq 0$. The linear Diophantine equation $ax+by=c$ is solvable if and only if gcd(a, b) divides c. If (x_0, y_0) is a particular solution of the LDE, then all its solutions are given by:

$(x, y) = (x_0 + \frac{b}{\gcd(a,b)}t, y_0 - \frac{a}{\gcd(a,b)}t)$, where t is an arbitrary integer.

Proof:

First, we show that if the LDE $ax+by=c$ is solvable, then gcd(a, b) divides c.

Suppose (x_1, y_1) is a solution of $ax+by=c$. Then, $ax_1+by_1=c$.

But gcd(a, b) divides both a and b, then , by Theorem 1.1, gcd(a, b) divides $ax_1 + by_1$. i.e. gcd(a, b) divides c.

Next, we want to prove that if gcd(a, b) divides c, then the LDE $ax + by = c$ is solvable.

Suppose that gcd(a, b) divides c. Then $c = k$. gcd(a, b) for some integer k.

Now, by Theorem 1.4, there exists two integers m and n such that $ma + nb = \gcd(a,b)$.

Multiply both sides of this equation by k to get: $kma + knb = k \gcd(a,b) = \mathrm{c}$.

Thus $x_0 = km$, $y_0 = kn$ is a solution of the LDE $ax + by = c$. Therefore, the LDE is solvable.

Now assume that (x_0, y_0) is a particular solution of $ax + by = c$, then $x = x_0 + \frac{b}{\gcd(a,b)}t$ and $y = y_0 - \frac{a}{\gcd(a,b)}t, t \in Z$ also satisfy the LDE:

$$\Rightarrow ax + by = a(x_0 + \frac{b}{\gcd(a,b)}t) + b(y_0 - \frac{a}{\gcd(a,b)}t) = ax_0 + \frac{ab}{\gcd(a,b)}t + by_0 - \frac{ab}{\gcd(a,b)}t$$

$$= ax_0 + by_0 = c.$$

Thus, $(x_0 + \frac{b}{\gcd(a,b)}t, y_0 - \frac{a}{\gcd(a,b)}t)$ is a solution for any integer t.

Finally, we want to prove that any solution (x', y') of the LDE $ax + by = c$ is of the form $(x_0 + \frac{b}{\gcd(a,b)}t, y_0 - \frac{a}{\gcd(a,b)}t)$ for some integer t.

Since (x_0, y_0) and (x', y') are solutions of $ax + by = c$, then:

$ax_0 + by_0 = c$ and $ax' + by' = c$. That is $ax_0 + by_0 = ax' + by'$.

Hence, $a(x' - x_0) = b(y_0 - y')$ (2.21)

Dividing both sides of this equation by $\gcd(a,b)$, we have:

$$\frac{a}{\gcd(a,b)}(x' - x_0) = \frac{b}{\gcd(a,b)}(y_0 - y')$$

Note that $\frac{a}{\gcd(a,b)} = a_1$ & $\frac{b}{\gcd(a,b)} = b_1 \in Z$ are relatively prime by Theorem 1.5. So, we obtain $a_1(x' - x_0) = b_1(y_0 - y')$

This shows that b_1 divides $a_1(x' - x_0)$. But, since $\gcd(a_1, b_1) = 1$, then by Theorem 1.6, b_1 divides $(x' - x_0)$.

Hence, $x' - x_0 = b_1 t = \frac{b}{\gcd(a,b)} t, t \in Z$. (2.22)

That is $x' = x_0 + \frac{b}{\gcd(a,b)} t.$

Similarly, is $y' = y_0 - \frac{a}{\gcd(a,b)} t.$

Thus, every solution $(x_0 + \frac{b}{\gcd(a,b)} t,\ y_0 - \frac{a}{\gcd(a,b)} t), t \in Z$ of the linear Diophantine equation is of the desired form.

Note 2.3:

We conclude from this theorem that every solvable linear Diophantine equation $ax + by = c$ has infinitely many solutions. They are given by the general solution:

$x = x_0 + \frac{b}{\gcd(a,b)} t$ and $y = y_0 - \frac{a}{\gcd(a,b)} t,$ where t is an arbitrary integer.

By giving different values to t, we can find any number of particular solutions.

Corollary 2.3: $\lfloor 14, p.13 \rfloor$

Suppose that gcd(a,b) = 1. Then the LDE $ax + by = c$ is solvable for all integers c. Moreover, if (x_0, y_0) is a particular solution, then the general solution is $x = x_0 + bt$, $y = y_0 - at, t \in Z$.

Example 2.15:

Determine whether the following LDE's are solvable.

a) $6x+18y=30$

b) $2x+3y=7$

c) $6x+8y=15$

d) $59x-29y=-5$

Solution:

a) gcd(6,18) = 6 which divides 30, then the LDE $6x+18y=30$ is solvable.

b) gcd(2,3) = 1, so $2x+3y=7$ is solvable.

c) gcd(6,8) = 2, but 2 does not divide 15, then $6x+8y=15$ is not solvable.

d) gcd(59,29) = 1, so $59x-29y=-5$ is solvable.

How to find a particular solution to the LDE $ax+by=c$?

It is not difficult to find a particular solution. One of the methods that are used is the Euclidean Algorithm method. $\lfloor 2,16 \rfloor$

To find a particular solution to a solvable LDE $ax+by=c$, we follow these steps.

1) Step 1: Write (a,b) as a linear combination of *a* and *b*. That is:

   $ar_0+bs_0=\gcd(a,b)$, r_0 and s_0 are integers.

2) Step 2: multiply both sides of this equation by *c* and then divide it by gcd(*a*, *b*): $a(\frac{r_0\times c}{\gcd(a,b)})+b(\frac{s_0\times c}{\gcd(a,b)})=c$.

3) Step 3: we obtain ($x_0=\frac{r_0\times c}{\gcd(a,b)}, y_0=\frac{s_0\times c}{\gcd(a,b)}$) as a particular solution of the linear Diophantine equation.

LDE's were known in ancient China and India as applications to astronomy and puzzles. The following puzzle is due to the Indian mathematician Mahavira (ca. A.D. 850).

Example 2.16:

Twenty-three weary travelers entered the outskirts of a lush and beautiful forest. They found 63 equal heaps of plantains and seven single fruits, and divided them equally. Find the number of fruits in each heap and the number of fruits received by each traveller.

Solution:

Let x denote the number of fruits in a heap and y denote the number of fruits received from each traveller.

Then we get the linear Diophantine equation:

$$63x + 7 = 23y$$

i.e. $$63x - 23y = -7$$

x and y must be positive, so we are looking for positive integral solutions of the LDE.

Since gcd(63, 23) = 1, then, by Corollary 2.3, the LDE is solvable.

To find a particular solution, we apply the Euclidean Algorithm:

$$63 = 2 \times 23 + 17 \tag{2.23}$$

$$23 = 1 \times 17 + 6 \tag{2.24}$$

$$17 = 2 \times 6 + 5 \tag{2.25}$$

$$6 = 1 \times 5 + 1 \tag{2.26}$$

$$5 = 5 \times 1$$

Now, use equations (2.21), (2.22), (2.23) and (2.24) in reverse order to get:

$$\begin{aligned} 1 &= 6-1\times 5 \\ &= 6-1(17-2\times 6) \\ &= 3\times 6-1\times 17 \\ &= 3\times(23-1\times 17)-1\times 17 \\ &= 3\times 23-4\times 17 \\ &= 3\times 23-4\times(63-2\times 23) \\ &= 11\times 23-4\times 63 \end{aligned}$$

Thus, $63(-4)-23(-11)=1$. Multiplying both sides of this equation by -7, we have: $63(-4\times -7)-23(-11\times -7)=-7$.

That is: $63(28)-(23)(77)=-7$.

Therefore, (28, 77) is a particular solution of $63x-23y=-7$.

By Corollary 2.3, the general solution of the LDE is:

$(x, y)=(28-23t, 77-63t)$, t is arbitrary integer.

Finally, since $x > 0$ and $y > 0$, then:

$$\begin{aligned} 28-23t &> 0 \quad \text{and} \quad 77-63t > 0 \\ t &< \frac{28}{23} \approx 1.217 \quad \text{and} \quad t < \frac{77}{63} \approx 1.222 \end{aligned}$$

So, $(x, y)=(28-23t, 77-63t)$, where t is an integer less than or equal 1, is a positive integral solution of the LDE $63x+7=23y$.

Continued Fractions and Linear Diophantine Equations $\lfloor 1,2,13 \rfloor$

Another way to find a particular solution to a solvable LDE $ax+by=c$ is the continued fraction method. Our approach to explain this method will be a step-by-step process until we'll be able to find integral solutions to any solvable LDE of the form $ax+by=c$. This method depends on the formula stated in Lemma 2.1.

> **Solving the LDE** $ax+by=1$**;** a & b are positive relatively prime integers.

To solve this LDE, we express $\frac{a}{b}$ as a finite simple continued fraction.

$$\frac{a}{b}=[a_0,a_1,...,a_{n-1},a_n]$$

Then we calculate the convergents c_0, c_1, c_2, …, c_{n-1} ,c_n. The last two convergents $c_{n-1}=\frac{p_{n-1}}{q_{n-1}}$ and $c_n=\frac{p_n}{q_n}$ with the relation stated in Lemma 2.1 are the key to the solution: $p_{n-1}q_n-p_nq_{n-1}=(-1)^n$

With $p_n=a$ and $q_n=b$ we have: $bp_{n-1}-aq_{n-1}=(-1)^n$

Or

$$a(-1)^{n-1}q_{n-1}+b(-1)^n p_{n-1}=1$$

Comparing this equation with the LDE $ax+by=1$, we conclude that: $(x_0=(-1)^{n-1}q_{n-1},y_0=(-1)^n p_{n-1})$ is a particular solution of $ax+by=1$**.**

Therefore, if n is even, then $(x_0,y_0)=(-q_{n-1},p_{n-1})$ and if n is odd, then $(x_0,y_0)=(q_{n-1},-p_{n-1})$.

We have four cases $\pm ax\pm by=1$ according to the sign of both a and b:

Case 1: $a>0\&b>0$
Equation: $ax+by=1$
Solution: $(x_0,y_0)=((-1)^{n-1}q_{n-1},(-1)^n p_{n-1})$

Case 2: $a>0\&b<0$
Equation: $ax-by=1$
Solution: $(x_0,y_0)=((-1)^{n-1}q_{n-1},(-1)^{n-1} p_{n-1})$

Case 3: $a<0\&b>0$
Equation: $-ax+by=1$
Solution: $(x_0,y_0)=((-1)^n q_{n-1},(-1)^n p_{n-1})$

Case 4: $a<0 \& b<0$

Equation: $-ax-by=1$

Solution: $(x_0, y_0)=((-1)^n q_{n-1}, (-1)^{n-1} p_{n-1})$

Example 2.17:

Solve the LDE $204x-91y=1$ using continued fraction method.

Solution:

First of all, gcd(204 , 91) = 1, then the LDE is solvable.

To find a particular solution, we represent $\frac{204}{91}$ as a finite simple continued fraction.

$$\frac{204}{91}=[2,4,7,3]$$

Then we construct the convergent table as shown in Table 2.3. From this table: $n = 3$, $p_{n-1} = p_2 = 65$ and $q_{n-1} = q_2 = 29$.

Table 2.3

k	-2	-1	0	1	2	3
a_k			2	4	7	3
p_k	0	1	2	9	65	204
q_k	1	0	1	4	29	91
c_k			$\frac{2}{1}=2$	$\frac{9}{4}$	$\frac{65}{29}$	$\frac{204}{91}$

Thus, a particular solution to the LDE $204x-91y=1$ is:

$$x_0=(-1)^2\times 29=29$$
$$y_0=(-1)^2\times 65=65$$

Finally, by Corollary 2.3, the general solution is:

$$\begin{aligned} x &= 29+(-91)t = 29-91t \\ y &= 65-204t \end{aligned}$$, t is an arbitrary integer.

Now, what if we replace the number 1 in any LDE in the cases above by another integer "*c*"? In other words, what is the particular solution of the LDE $ax+by=c$**,** $\gcd(a,b)=1$**?**

> Solving the LDE $ax+by=c$**, where *a, b* and *c* are *integers,*** $\gcd(a,b)=1$**.**

The first step in solving this LDE is to find a particular solution (x_0, y_0) of the LDE $ax+by=1$ using the formulas we've studied and derived according to the case we have.

From $ax_0+by_0=1$, we have: $a(cx_0)+b(cy_0)=c$

Thus, (cx_0, cy_0) is a particular solution of the LDE $ax+by=c$.

> Solving the LDE $Ax+By=C$**, where *A, B* and *C* are *integers,*** $\gcd(A,B)\neq 1$**.**

As we have proved in Theorem 2.8, the LDE $Ax+By=C$ is solvable if and only if gcd(*A*, *B*) divides *C*. If so, divide both sides of the LDE by gcd(*A*, *B*) to reduce it to the equation of the form:

$$ax+by=c, \tag{2.27}$$

where *a*, *b* and *c* are integers, gcd(*a*, *b*)=1.

The solution of equation (2.27) has been discussed and is easy to solve. Finally, any solution of this equation is automatically a solution of the original equation $Ax+By=C$.

Example 2.18:

Solve the LDE $65x-182y=299$ using continued fraction method.

Solution:

gcd(65,182) = 13, and 13 divides 299. So, the LDE $65x-182y=299$ is solvable.

Divide both sides of the equation $65x-182y=299$ by 13 to get the LDE $5x-14y=23$.

Now, we find a particular solution to the LDE $5x-14y=1$.

$\frac{5}{14}=[0,2,1,4]$. The following table is the convergent table.

Table 2.4

k	-2	-1	0	1	2	3
a_k			0	2	1	4
p_k	0	1	0	1	1	5
q_k	1	0	1	2	3	14
c_k			0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{14}$

From this table: $n = 3$, $p_2 = 1$ and $q_2 = 3$.
Thus, a particular solution to the LDE $5x-14y=1$ is:

$$x_0=(-1)^2\times 3=3$$
$$y_0=(-1)^2\times 1=1$$

So, $(23x_0, 23y_0) = (69, 23)$ is a particular solution to the LDE $5x-14y=23$

Finally, the general solution of $5x-14y=23$ is:

$$\begin{aligned} x &= 69+-14t=69-14t \\ y &= 23-5t \end{aligned}$$, t is an arbitrary integer.

Note 2.4:

The continued fraction method for finding a particular solution for a solvable LDE is equivalent to the Euclidean algorithm method. This is due to the fact that the continued fraction of $\frac{a}{b}$ is derived from the Euclidean algorithm as we have already studied in Chapter Two. However, generating the convergents using the recurrence relations to solve a LDE is quicker than to find Euclidean algorithm equations and then use them in reverse order.

Chapter Three

Infinite Simple Continued Fractions

Section 3.1: Properties and Theorems

Irrationals are numbers that cannot be written as a ratio of two integers. Some irrationals are of the form $\frac{A \pm \sqrt{B}}{C}$, where *A* and *C* are integers, B is a positive no-perfect square integer. Irrationals of this form are the roots of the quadratic equation $C^2X^2 - 2ACX + (A^2 - B) = 0$, so they are called **quadratic irrationals or quadratic surds**. However, there are irrational numbers which are not quadratic surds such as π, e, cube roots, fifth roots, etc. Our discussion will concentrate on the continued fraction expansions of quadratic irrationals.

The numbers π and e are examples of transcendental numbers. The expansion of transcendental numbers into continued fractions is not easy, but using decimal approximations to them, such as $\pi = 3.141592\ldots..$ and $e = 2.7182818\ldots.$, we can find some of the first terms of their continued fraction expansions:

$e = [2,1,2,1,1,4,1,1,6,1,1,8,1,.....]$ and $\pi = [3,7,15,1,292,1,1,1,2,1,.....]$

As we can see, e has apparent pattern occurs in its expansion, but the expansion of the irrational number π does not appear to follow any pattern. However, mathematicians found the expansions of π and e using methods which are beyond the scope of this thesis.

In Chapter Two, we've defined the infinite simple continued fraction that has the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}$$

But, in that chapter, our study of continued fractions has been limited to the expansion of rational numbers. In this chapter, we study the continued fraction expansion of irrational numbers, state their properties and some related theorems.

The Continued Fraction Algorithm: $\lfloor 4, p.3 \rfloor$

For the continued fraction expansion of irrational numbers, we'll use the same algorithm as in the continued fraction expansion of rational numbers. Let y be an irrational number.

Set $y = y_0$ and let $a_0 = [[y_0]]$;

$$y_1 = \frac{1}{y_0 - [[y_0]]},\ a_1 = [[y_1]];$$

$$y_2 = \frac{1}{y_1 - [[y_1]]},\ a_2 = [[y_2]];$$

.

.

$$y_k = \frac{1}{y_{k-1} - [[y_{k-1}]]};\ a_k = [[y_k]],$$

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We continue in this manner. Here, the process will continue indefinitely but the expansion exhibit nice periodic behavior for quadratic irrationals. We'll prove this algorithm in Theorem 3.5.

Example 3.1:

Find the continued fraction expansion of $\sqrt{2}$.

Solution:

Let $y_0 = \sqrt{2}$, $1 < \sqrt{2} < 2$, $a_0 = [[y_0]] = [[\sqrt{2}]] = 1$;

$y_1 = \dfrac{1}{\sqrt{2}-1}$. Rationalize the denominator of y_1:

$$y_1 = \frac{1.(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{\sqrt{2}+1}{1} = \sqrt{2}+1,\ a_1 = [[\sqrt{2}+1]] = 2\,;$$

$$y_2 = \frac{1}{\sqrt{2}+1-2} = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 = y_1,\ a_2 = [[\sqrt{2}+1]] = 2\,.$$

Since $y_1 = y_2$, it is clear that $y_2 = y_3 = y_4 = y_5 = ... = \sqrt{2}+1$ and $a_2 = a_3 = a_4 = a_5 = ... = 2.$

Hence,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}} = [1,2,2,2,...] = [1,\overline{2}]$$, where the bar over 2 indicates that the number 2 is repeated over and over.

Example 3.2:

Using the continued fraction algorithm, find the first 6 terms of the infinite continued fraction expansion of e.

Solution:

Let $y_0 = e = 2.7182818285...,\ a_0 = [[y_0]] = [[e]] = 2\,;$

$$y_1 = \frac{1}{2.7182818285...-2} = 1.3922111911...,\ a_1 = [[y_1]] = 1;$$

$$y_2 = \frac{1}{1.3922111911...-1} = 2.5496467788...,\ a_2 = [[y_2]] = 2\,;$$

$$y_3 = \frac{1}{2.5496467788\ldots - 2} = 1.8193502419\ldots,\ a_3 = [[y_3]] = 1;$$

$$y_4 = \frac{1}{1.8193502419\ldots - 1} = 1.2204792881\ldots,\ a_4 = [[y_4]] = 1;$$

$$y_5 = \frac{1}{1.2204792881\ldots - 1} = 4.5355734255\ldots,\ a_5 = [[y_3]] = 4\,.$$

Thus, $e = [2,1,2,1,1,4,\ldots]$.

Convergents: $\lfloor 1,2,6,9 \rfloor$

The corresponding convergents to any infinite continued fraction form an infinite sequence:

$$c_0 = \frac{p_0}{q_0}, c_1 = \frac{p_1}{q_1}, \ldots, c_i = \frac{p_i}{q_i}, \ldots.$$

These convergents are evaluated in the same way as convergents of finite simple continued fractions since each convergent is finite and represents a rational number. So we calculate them using the formula:

$$\begin{matrix} p_k = a_k p_{k-1} + p_{k-2} \\ q_k = a_k q_{k-1} + q_{k-2} \end{matrix},\ k \geq 0 \text{ and } \begin{matrix} p_{-1} = 1, p_{-2} = 0 \\ q_{-1} = 0, q_{-2} = 1 \end{matrix}.$$

They also have the same properties of convergents of finite simple continued fractions, and we can summarize them as follows:

* $p_{k-1}q_k - p_k q_{k-1} = (-1)^k,\ k \geq -1$

* $c_k - c_{k-1} = \dfrac{(-1)^{k-1}}{q_k q_{k-1}},\ k \geq 1.$

* $c_k - c_{k-2} = \dfrac{(-1)^k a_k}{q_k q_{k-2}},\ k \geq 2.$

* For $k \geq 1$, p_k and q_k are relatively prime.

* $c_0 < c_2 < c_4 < \ldots < c_{2k} < \ldots < c_{2k+1} < \ldots < c_3 < c_1$.

The proofs of these properties are the same as before since the proofs given there were independent of whether the continued fraction is finite or infinite.

Moreover, it is important to note here that since a_s and q_k, are positive integers for $s \geq 1$ & $k \geq 0$, it follows from the equation $q_k = a_k q_{k-1} + q_{k-2}$ that $\{q_k, k = 0, 1, 2, ...\}$ is an increasing unbounded sequence.

Theorem 3.1: $\lfloor 4, p.10 \rfloor$

a) $\dfrac{p_k}{p_{k-1}} = [a_k, a_{k-1}, ..., a_1, a_0],\ k \geq 0,\ a_0 > 0$.

b) $\dfrac{q_k}{q_{k-1}} = [a_k, a_{k-1}, ..., a_1],\ k \geq 1$.

Proof:

We only prove a). The proof of b) is similar.

Using induction on *k*:

For $k = 0$: $$\frac{p_0}{p_{-1}} = \frac{a_0}{p_{-1}} = a_0 = [a_0]$$

For $k = 1$: $$\frac{p_1}{p_0} = \frac{a_1 p_0 + p_{-1}}{p_0} = a_1 + \frac{p_{-1}}{p_0} = a_1 + \frac{1}{a_0} = [a_1, a_0]$$

Suppose the statement is true for $k = n > 1$. That is,
$$\frac{p_n}{p_{n-1}} = [a_n, a_{n-1}, ..., a_1, a_0]$$

Now, $p_{n+1} = a_{n+1} p_n + p_{n-1}$. Divide both sides by p_n to get:

$$\frac{p_{n+1}}{p_n} = a_{n+1} + \frac{p_{n-1}}{p_n} = a_{n+1} + \cfrac{1}{\cfrac{p_n}{p_{n-1}}}$$

$$= a_{n+1} + \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_0}}}} = [a_{n+1}, a_n, \ldots, a_1, a_0]$$

Thus, the statement is true for $k \geq 0$.

Example 3.3:

Find the first seven convergents of the continued fraction expansion of π.

Solution:

We can find some of the first terms of the infinite continued fraction for π:

$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, \ldots]$.

We want to find c_i for each $0 \leq i \leq 6$, so we construct the following convergent table:

Table 3.1

k	-2	-1	0	1	2	3	4	5	6
a_k			3	7	15	1	292	1	1
p_k	0	1	3	22	333	355	103993	104348	208341
q_k	1	0	1	7	106	113	33102	33215	66317
c_k			3	$\frac{22}{7}$	$\frac{333}{106}$	$\frac{355}{113}$	$\frac{103993}{33102}$	$\frac{104348}{33215}$	$\frac{208341}{66317}$

Now, $c_0 = 3$

$$c_1 = \frac{22}{7} \approx 3.1428571428571429$$
$$c_2 = \frac{333}{106} \approx 3.1415094339622642$$
$$c_3 = \frac{355}{113} \approx 3.1415929203539823$$
$$c_4 = \frac{103993}{33102} \approx 3.1415926530119026$$
$$c_5 = \frac{104348}{33215} \approx 3.141592653921421$$
$$c_6 = \frac{208341}{66317} \approx 3.1415926534674367$$

and π = 3.1415926535897932….

Notice that the convergents $c_0, c_1, ..., c_6$ are good approximations for π to 0, 2, 4, 6, 9, 9, 9 decimal places, respectively. Hence, they give successively better approximations to π.

Now, from the property $c_0 < c_2 < c_4 < \ldots < c_{2m} < \ldots < c_{2m+1} < . \; . \; . < c_3 < c_1$, the sequence of even convergents $\{c_{2m}\}$ is an increasing sequence that is bounded above by c_1, so it is a convergent sequence. Moreover, the sequence of odd convergents $\{c_{2m+1}\}$ is a decreasing sequence that is bounded below by c_0, so it is also a convergent sequence. Hence as m approaches ∞, the sequence $\{c_{2m}\}$ approaches a limit M_e and the sequence $\{c_{2m+1}\}$ approaches a limit M_o. That is, $\lim_{m\to\infty} c_{2m} = M_e$ and $\lim_{m\to\infty} c_{2m+1} = M_o$.

Since even convergents are less than all odd convergents, then the limit M_e is less than all odd convergents. Similarly, the limit M_o is greater than any even convergent.

These two limits are equal according to the following theorem:

Theorem 3.2: $\lfloor 1, pp.68-70 \rfloor \& \lfloor 2, p.568 \rfloor$

Let $[a_0, a_1, a_2, ...]$ be an infinite simple continued fraction expansion of a number y and let $c_i = [a_0, a_1, ..., a_i]$ denotes the i^{th} convergent of the continued fraction, then:

1) $\lim_{m\to\infty} c_{2m} = \lim_{m\to\infty} c_{2m+1}$.

2) $\lim_{m\to\infty} c_m = y$.

Proof:

1) Using the property $c_k - c_{k-1} = \dfrac{(-1)^{k-1}}{q_k q_{k-1}}$, $k \geq 1$, we obtain:

$$c_{2m+1} - c_{2m} = \frac{(-1)^{2m}}{q_{2m+1}q_{2m}} = \frac{1}{q_{2m+1}q_{2m}}, \; m \geq 0.$$

But we know that $q_{2m+1} > q_{2m}$, then $\dfrac{1}{q_{2m+1}q_{2m}} < \dfrac{1}{q_{2m}^2}$.

Now, as m increases, q_{2m} and q_{2m}^2 both increase and so $\{\dfrac{1}{q_{2m}^2}\}$ is a bounded decreasing positive sequence, hence it is convergent to 0.

So, $\lim_{m\to\infty}(c_{2m+1} - c_{2m}) = 0$ and hence, $\lim_{m\to\infty} c_{2m+1} = \lim\limits_{m\to\infty} c_{2m}$.

2) Now, $\{c_{2m}\}$ and $\{c_{2m+1}\}$ are subsequences of the sequence $\{c_m\}$ and they both have a common limit, say M. Then $\lim_{m\to\infty} c_m = M$.

Hence, we can say that every infinite simple continued fraction converges to a limit M. This limit is greater than all even convergents and less than all odd convergents. We prove that the limit M equals the number y.

Given $y = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n + \cfrac{1}{a_{n+1} + \ddots}}}}}}$,

define $y_n = a_n + \cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2} + \ddots}}$, $y_{n+1} = a_{n+1} + \cfrac{1}{a_{n+2} + \cfrac{1}{a_{n+3} + \ddots}}$, … and so on.

Then, we can write $y = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{y_n}}}}}$ and $y_n = a_n + \dfrac{1}{y_{n+1}}$.

It is clear that $y_{n+1} > 0$. So, $y_n > a_n$ for all $n \geq 0$.

Hence, $a_n < y_n < a_n + \dfrac{1}{a_{n+1}}$. (3.1)

Now, comparing the three following expressions:

$$c_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}} \tag{3.2}$$

$$y = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{y_n}}}}} \tag{3.3}$$

and

$$c_{n+1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n + \cfrac{1}{a_{n+1}}}}}}} \tag{3.4}$$

We can see that they have the term $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1}}}}}$ in common and differ in the terms $\frac{1}{a_n}$, $\frac{1}{y_n}$, $\cfrac{1}{a_n + \cfrac{1}{a_{n+1}}}$, respectively.

Using inequality (3.1), we get:

$$\cfrac{1}{a_n + \cfrac{1}{a_{n+1}}} < \frac{1}{y_n} < \frac{1}{a_n} \tag{3.5}$$

Thus, from equations (3.2), (3.3), (3.4) and inequality (3.5), we conclude that y must lie between two consecutive convergents c_n and c_{n+1}. That is:

$$c_n < y < c_{n+1} \qquad \text{or} \qquad c_{n+1} < y < c_n$$

But we know that even convergents are less than odd convergents. Thus, we conclude that

$$c_{2m} < y < c_{2m+1}, \qquad m = 0, 1, 2, \ldots$$

and in expanded form, we write:

$$c_0 < c_2 < c_4 < \ldots < c_{2m} < \ldots < y < \ldots < c_{2m+1} < \ldots < c_5 < c_3 < c_1$$

Thus, as m increases, even convergents approach y from the left and odd convergents approach y from the right. Hence, the limit M obtained in the previous proof is the same as the number y. In symbols, $\lim_{m\to\infty} c_m = y$.

Definition 3.1: $\lfloor 20 \rfloor$

Given an infinite simple continued fraction $[a_0, a_1, a_2, a_3, ...]$, the term $y_n = [a_n, a_{n+1}, a_{n+2}, ...]$ is called the $(n+1)$-*st* complete quotient of the continued fraction.

Theorem 3.3: $\lfloor 2, p.569 \rfloor$

Any infinite simple continued fraction represents an irrational number.

Proof: (by contradiction)

Let $y = [a_0, a_1, a_2, ...]$ be an infinite simple continued fraction. Then, by Theorem 3.2, $y = \lim_{m\to\infty} c_m$ and $c_{2m} < y < c_{2m+1}$.

Thus, $0 < y - c_{2m} < c_{2m+1} - c_{2m}$. But, $c_{2m+1} - c_{2m} = \dfrac{1}{q_{2m+1}q_{2m}}$.

So, $0 < y - \dfrac{p_{2m}}{q_{2m}} < \dfrac{1}{q_{2m+1}q_{2m}}$

Now, suppose by contradiction that y is a rational number, say $y = \dfrac{l}{s}$, where $s > 0$.

Then,

$$0 < \frac{l}{s} - \frac{p_{2m}}{q_{2m}} < \frac{1}{q_{2m+1}q_{2m}}.$$

That is,

$$0 < lq_{2m} - sp_{2m} < \frac{s}{q_{2m+1}}.$$

So, $lq_{2m} - sp_{2m}$ is a positive integer less than $\dfrac{s}{q_{2m+1}}$.

As we studied before, as m increases, q_{2m+1} also increases. Thus, there is an integer i such that $q_{2i+1} > s$.

Then, $\frac{s}{q_{2i+1}} < 1$. This implies that $0 < lq_{2i} - sp_{2i} < 1$. Hence $lq_{2i} - sp_{2i}$ is a positive **integer** less than one, which is a contradiction. So, y is an irrational number.

Theorem 3.4: $\lfloor 4, p.7 \rfloor$

Let $\{\frac{p_k}{q_k}, k = 0,1,2,...\}$ be the sequence of convergents of an irrational number y. Define y_k as in the continued fraction algorithm,

i.e., $y_k = \frac{1}{y_{k-1} - [[y_{k-1}]]}$. Then: $y = \frac{y_k p_{k-1} + p_{k-2}}{y_k q_{k-1} + q_{k-2}}, k \geq 0.$

Proof:

We prove this theorem using induction on k. First, remember that $p_{-1} = 1, p_{-2} = 0$, $q_{-1} = 0, q_{-2} = 1$, $p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$.

For k = 0:

$$\frac{y_0 p_{-1} + p_{-2}}{y_0 q_{-1} + q_{-2}} = \frac{y_0.1 + 0}{y_0.0 + 1} = y_0 = y$$

For k = 1:

$$\frac{y_1 p_0 + p_{-1}}{y_1 q_0 + q_{-1}} = \frac{y_1.a_0 + 1}{y_1.1 + 0} = \frac{(\frac{1}{y - a_0}).a_0 + 1}{(\frac{1}{y - a_0})} = y$$

So, the statement is true for k = 0 and k = 1.

Assume that the statement holds for an arbitrary number $j \geq 2$. That is,

$$\frac{y_j p_{j-1} + p_{j-2}}{y_j q_{j-1} + q_{j-2}} = y.$$

Now,

$$\frac{y_{j+1}p_j + p_{j-1}}{y_{j+1}q_j + q_{j-1}} = \frac{(\frac{1}{y_j - a_j})p_j + p_{j-1}}{(\frac{1}{y_j - a_j})q_j + q_{j-1}} = \frac{p_j + p_{j-1}(y_j - a_j)}{q_j + q_{j-1}(y_j - a_j)} = \frac{a_j p_{j-1} + p_{j-2} + p_{j-1}(y_j - a_j)}{a_j q_{j-1} + q_{j-2} + q_{j-1}(y_j - a_j)}$$

$$= \frac{y_j p_{j-1} + p_{j-2}}{y_j q_{j-1} + q_{j-2}} = y.$$

So, the statement is true for $j+1$. Thus, the theorem is true for $k \geq 0$.

Note 3.1: $\lfloor 3,4 \rfloor$

The property considered in Theorem 3.4 is also true for rational numbers. That is $[a_0, a_1, ..., a_n] = \dfrac{y_k p_{k-1} + p_{k-2}}{y_k q_{k-1} + q_{k-2}}, 0 \leq k \leq n.$

The following theorem shows that any irrational number can be written as an infinite simple continued fraction.

Theorem 3.5: $\lfloor 2, p.570 \rfloor$

Let $y = y_0$ be an irrational number. Define the sequence $\{a_k\}_{k=0}^{\infty}$ of integers a_k recursively as follows:

$$a_k = [[y_k]] \quad , \quad y_{k+1} = \frac{1}{y_k - a_k} \quad , \quad k \geq 0$$

Then $y = [a_0, a_1, a_2, a_3, ...]$.

Proof:

It is clear that, for any $k \geq 0$, a_k is an integer.

By induction, we prove that y_k is an irrational number for every $k \geq 0$.

Note that y_0 is an irrational number and $a_0 = [[y_0]] \neq y_0$. Then, $y_0 - a_0$ is irrational and so $y_1 = \dfrac{1}{y_0 - a_0}$ is irrational.

Assume that y_k is an irrational number for an arbitrary integer $k \geq 0$. This implies that $y_k - a_k$ and $\frac{1}{y_k - a_k}$ are also irrationals, which means that y_{k+1} is irrational. So, by induction, y_k is an irrational number for every $k \geq 0$.

Next, we show that $a_k \geq 1$ for every $k \geq 1$. y_k is an irrational number and $a_k = [[y_k]]$ is an integer, then $a_k \neq y_k$ and $y_k - a_k > 0$.

But, $y_k - a_k = y_k - [[y_k]] < 1$. Then, $0 < y_k - a_k < 1$ and so $y_{k+1} = \frac{1}{y_k - a_k} > 1$. Therefore, $a_{k+1} = [[y_{k+1}]] \geq 1$. It means that $a_1, a_2, a_3, \ldots$ are all positive integers.

Finally, we prove that $y = [a_0, a_1, a_2, \ldots]$.

Using the recursive formula:

$$y_{k+1} = \frac{1}{y_k - a_k},$$

we find that

$$y_k = a_k + \frac{1}{y_{k+1}}, \quad k \geq 0.$$

Now, $y_0 = a_0 + \frac{1}{y_1}$. Successively substituting for $y_1, y_2, y_3, \ldots$ yields:

$$y_0 = a_0 + \frac{1}{y_1} = [a_0, y_1]$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{y_2}} = [a_0, a_1, y_2]$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{y_3}}} = [a_0, a_1, a_2, y_3]$$

$$\vdots$$

$$\vdots$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_m + \cfrac{1}{y_{m+1}}}}}}} = [a_0, a_1, a_2, a_3, \ldots, a_m, y_{m+1}] \ , \ \ m \geq 0.$$

Using Theorem 3.4, we get:

$$y_0 = \frac{y_{m+1} p_m + p_{m-1}}{y_{m+1} q_m + q_{m-1}}$$

Let $c_k = \dfrac{p_k}{q_k}$ be the k^{th} convergent of the continued fraction $[a_0, a_1, a_2, a_3, \ldots]$.

Then

$$y_0 - c_m = \frac{y_{m+1} p_m + p_{m-1}}{y_{m+1} q_m + q_{m-1}} - \frac{p_m}{q_m}$$

$$= \frac{p_{m-1} q_m - p_m q_{m-1}}{(y_{m+1} q_m + q_{m-1}) q_m}$$

But, $p_{m-1} q_m - p_m q_{m-1} = (-1)^m$. Then,

$$y_0 - c_m = \frac{(-1)^m}{(y_{m+1} q_m + q_{m-1}) q_m}.$$

So, $|y_0 - c_m| = \dfrac{1}{(y_{m+1} q_m + q_{m-1}) q_m}$.

But, $y_{m+1} > a_{m+1}$, so $|y_0 - c_m| < \dfrac{1}{(a_{m+1} q_m + q_{m-1}) q_m} = \dfrac{1}{q_{m+1} q_m}$.

As m approaches ∞, q_m gets larger and larger, and so, $\dfrac{1}{q_{m+1} q_m}$ approaches zero. It means that $c_m \to y_0$ as $m \to \infty$.

Hence, $y = y_0 = \lim_{m\to\infty} c_m = [a_0, a_1, a_2, ...]$.

We conclude that we can approximate any irrational number by a rational number.

Theorem 3.6: $\lfloor 20, p.253 \rfloor$

The infinite simple continued fraction expansion of an irrational number is unique.

Proof:

Let *x* be an irrational number. Suppose that there are two infinite simple continued fractions representing the irrational number *x*.

$$x = [d_0, d_1, d_2, ...] = [h_0, h_1, h_2, ...]$$

It is clear that $d_0 = [[x]]$ and $h_0 = [[x]]$. So, $d_0 = h_0$.

Next, using the continued fraction algorithm, we find that $d_1 = [[\frac{1}{x - [[x]]}]]$ and $h_1 = [[\frac{1}{x - [[x]]}]]$. So, $d_1 = h_1$.

Suppose that $d_k = h_k$ for all $k < n$. We'll prove that $d_n = h_n$.

Using the complete quotient, we can write:

$$x = [d_0, d_1, d_2, ...d_{n-1}, x_n] = [d_0, d_1, d_2, ...d_{n-1}, x'_n]$$

Now

$$x = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}} = \frac{x'_n p_{n-1} + p_{n-2}}{x'_n q_{n-1} + q_{n-2}}.$$

So

$$(x_n p_{n-1} + p_{n-2})(x'_n q_{n-1} + q_{n-2}) = (x_n q_{n-1} + q_{n-2})(x'_n p_{n-1} + p_{n-2}).$$

This implies that

$$x'_n p_{n-2} q_{n-1} + x_n p_{n-1} q_{n-2} = x_n q_{n-1} p_{n-2} + x'_n p_{n-1} q_{n-2}$$

Then

$$(x_n' - x_n) p_{n-2} q_{n-1} - (x_n' - x_n) p_{n-1} q_{n-2} = 0$$

$$(x_n' - x_n)[p_{n-2} q_{n-1} - p_{n-1} q_{n-2}] = 0$$

But

$$p_{n-2} q_{n-1} - p_{n-1} q_{n-2} = (-1)^{n-1}$$

Thus

$$x_n' - x_n = 0 .$$

As a result, $x_n' = x_n$. So, $d_n = [[x_n]] = [[x_n']] = h_n$.

Thus, we deduce that $d_n = h_n$.

Therefore, we've proved by induction that the simple continued fraction representation of an irrational number is unique.

Section 3.2: Periodic Continued Fractions

In the previous section, we studied the representation of irrational numbers as infinite simple continued fractions. We also discussed their convergents and some related theorems. In this section, we study quadratic irrationals in details, i.e., irrationals of the form $\frac{A \pm \sqrt{B}}{C}$, where A and C are integers, B is a positive non-perfect square integer.

Definition 3.2: $\lfloor 3,6,9,18 \rfloor$

The infinite continued fraction $[a_0, a_1, a_2, a_3, ...]$ is **periodic with period *d*** if there exists a smallest positive integer d and a nonnegative integer f such that $a_{n+d} = a_n$ for all $n \geq f$. It can be represented as $[a_0, a_1, ..., a_{f-1}, \overline{a_f, ..., a_{f+d-1}}]$.

The quotients $a_0, a_1, ..., a_{f-1}$ are called non-repeating quotients and the quotients $a_f, a_{f+1}, ..., a_{f+d-1}$ are called the repeating quotients of the fraction.

A continued fraction is called **purely periodic with period *d*** if it is periodic with f = 0, that is if there is no non-repeating quotients. It can be represented as $[\overline{a_0, a_1, ..., a_{d-1}}]$.

Example 3.4:

Find the continued fraction expansion of $\frac{1+\sqrt{35}}{2}$.

Solution:

Let $y_0 = \frac{1+\sqrt{35}}{2}, a_0 = [[y_0]] = 3$

$$y_1 = \frac{1}{y_0 - a_0} = \frac{1}{\frac{1+\sqrt{35}}{2} - 3} = \frac{2}{\sqrt{35}-5} = \frac{2(\sqrt{35}+5)}{35-25} = \frac{\sqrt{35}+5}{5}, a_1 = 2.$$

$$y_2 = \frac{1}{y_1 - a_1} = \frac{1}{\frac{\sqrt{35}+5}{5} - 2} = \frac{5}{\sqrt{35}-5} = \frac{5(\sqrt{35}+5)}{35-25} = \frac{\sqrt{35}+5}{2}, a_2 = 5.$$

$$y_3 = \frac{1}{\frac{\sqrt{35}+5}{2} - 5} = \frac{2}{\sqrt{35}-5} = \frac{\sqrt{35}+5}{5} = y_1, a_3 = a_1 = 2.$$

Since $y_3 = y_1$, it is clear that $y_4 = y_2$, $y_5 = y_1$, …, $y_{2k} = y_2$, $y_{2k+1} = y_1$, and the corresponding partial quotients alternate between 2 and 5 indefinitely.

Hence, $\frac{1+\sqrt{35}}{2} = [3,2,5,2,5,2,...] = [3,\overline{2,5}]$.

The continued fraction expansions of the irrational numbers $\sqrt{2}$ (in Example 3.1) and $\frac{1+\sqrt{35}}{2}$ are periodic but not pure.

Example 3.5:

Find the continued fraction expansion of $3+\sqrt{11}$.

Solution:

Let $y_0 = \sqrt{11}+3, a_0 = [[y_0]] = 6$

$$y_1 = \frac{1}{y_0 - a_0} = \frac{1}{3+\sqrt{11}-6} = \frac{1}{\sqrt{11}-3} = \frac{1(\sqrt{11}+3)}{(\sqrt{11}-3)(\sqrt{11}+3)} = \frac{\sqrt{11}+3}{2} \quad , \quad a_1 = 3$$

$$y_2 = \frac{1}{y_1 - a_1} = \frac{1}{\frac{\sqrt{11}+3}{2} - 3} = \frac{2}{\sqrt{11}-3} = \frac{2(\sqrt{11}+3)}{11-9} = \sqrt{11}+3 \quad , \quad a_2 = 6$$

Since $y_2 = y_0$, it is clear that $y_3 = y_1$, $y_4 = y_0$, …, $y_{2k} = y_0$, $y_{2k+1} = y_1$, and the corresponding partial quotients alternate between 6 and 3 indefinitely.

Therefore, $\sqrt{11}+3 = [6,3,6,3,6,...] = [\overline{6,3}]$ is purely periodic with period 2.

Now, if we want to convert a periodic continued fraction to a quadratic irrational, what shall we do? $\lfloor 1,11 \rfloor$

We'll explain the method by the following example.

Example 3.6:

Convert the continued fraction $[2,3,\overline{1,2,1}]$ to the form $\frac{A \pm \sqrt{B}}{C}$.

Solution:

Let $x = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}$ and y represents the repeating quotients of the continued fraction.

That is $y = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \ddots}}}}}$.

Now, $y = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{y}}} = 1 + \cfrac{1}{2 + \cfrac{y}{y+1}} = 1 + \frac{y+1}{3y+2} = \frac{4y+3}{3y+2}$.

And so, $3y^2 - 2y - 3 = 0$. Solving this quadratic equation, we get:

$$y = \frac{1 \pm \sqrt{10}}{3}.$$

Since y is positive, then $y = \frac{1 + \sqrt{10}}{3}$.

But, $x = 2 + \cfrac{1}{3 + \cfrac{1}{y}}$. Substitute the value of y in this equation to find the value of x:

$$x = 2 + \cfrac{1}{3 + \cfrac{1}{\cfrac{1+\sqrt{10}}{3}}} = 2 + \cfrac{1}{\cfrac{3\sqrt{10}+6}{1+\sqrt{10}}} = \frac{7\sqrt{10}+13}{3\sqrt{10}+6}.$$

Rationalize the denominator to get:

$x = \dfrac{132 - 3\sqrt{10}}{54} = \dfrac{44 - \sqrt{10}}{18}$. Hence, $[2,3,\overline{1,2,1}] = \dfrac{44 - \sqrt{10}}{18}$.

Definition 3.3: $\lfloor 3,21 \rfloor$

An irrational number is called a **quadratic irrational** if it is a root of a quadratic equation $ay^2 + by + c = 0$ where a, b, c are integers, $a \neq 0$ and its discriminant $b^2 - 4ac$ is a positive non-perfect square integer.

Lemma 3.1: $\lfloor 22, p.281 \rfloor$

Let x be a quadratic irrational and let $y = \dfrac{ax+b}{cx+d}$, where a, b, c and d are integers, c and d are not both zeros. Then x is a quadratic irrational if and only if $ad - bc \neq 0$.

Proof: see $\lfloor 22 \rfloor$.

Theorem 3.7: $\lfloor 3, p.20 \rfloor$

If the continued fraction expansion of y is purely periodic, then y is a quadratic irrational.

Proof:

Let y be represented by a purely periodic continued fraction. That is $y = [\overline{a_0, a_1, ..., a_{d-1}}]$.

Then

$$y = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{d-1} + \cfrac{1}{y}}}}} = [a_0, a_1, ..., a_{d-1}, y]$$

and $y = y_0 = y_d = y_{2d} = \cdots$.

Using Theorem3.4, $y = \dfrac{yp_{d-1} + p_{d-2}}{yq_{d-1} + q_{d-2}}$.

So, $q_{d-1}y^2 + (q_{d-2} - p_{d-1})y - p_{d-2} = 0.$

Hence y is a root of a quadratic equation. If y is rational, then it has a finite simple continued fraction representation, not an infinite pure periodic continued fraction presentation. Thus y is a quadratic irrational.

Corollary 3.1: $\lfloor 21, p.168 \rfloor \& \lfloor 22, p.281 \rfloor$

A periodic continued fraction represents a quadratic irrational.

Proof:

Let y be a real number represented by a periodic continued fraction. That is $y = [a_0, a_1, \ldots, a_{f-1}, \overline{a_f, a_{f+1}, \ldots, a_{f+d-1}}]$.

Let x be represented by the periodic part of y. That is

$x = [\overline{a_f, a_{f+1}, \ldots, a_{f+d-1}}]$. Then, by Theorem 3.7, x is a quadratic irrational.

So,

$$y = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{f-1} + \cfrac{1}{a_f + \cfrac{1}{a_{f+1} + \cfrac{1}{\ddots + \cfrac{1}{a_{f+d-1} + \cfrac{1}{\ddots}}}}}}}}}$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{f-1} + \cfrac{1}{x}}}}} = [a_0, a_1, \ldots, a_{f-1}, x]$$

Using Theorem 3.4, $y=\frac{xp_{f-1}+p_{f-2}}{xq_{f-1}+q_{f-2}}$. Then, by Lemma 3.1 , y is a quadratic irrational since $p_{f-1}q_{f-2}+p_{f-2}q_{f-1}=(-1)^f \neq 0$.

The quadratic equation $ay^2+by+c=0$, where $a \neq 0$, b and c are integers with a positive non-perfect square discriminant has two roots.

The first one is $\omega=\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{A+\sqrt{B}}{C}$ and the second is $\omega'=\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{A-\sqrt{B}}{C}$, where $A=-b,\ B=b^2-4ac,\ C=2a$ are integers. ω and ω' are conjugates. Notice that $\omega+\omega'=\frac{2A}{C}=\frac{-b}{a}$ and $\omega\omega'=\frac{A^2-B}{C^2}=\frac{c}{a}$.

For instance, $-\sqrt{5}$ is the conjugate of $\sqrt{5}$ and $14+\sqrt{7}$ is the conjugate of $14-\sqrt{7}$.

Definition 3.4: $\lfloor 1,23 \rfloor$

Let ω be a quadratic irrational satisfying the quadratic equation $ay^2+by+c=0$, where a, b and c are integers. Then ω is called a **reduced** quadratic irrational if $\omega>1$ and $-1<\omega'<0$.

Example 3.7:

The quadratic irrational $3+\sqrt{10}$ is greater than 1 and satisfies the quadratic equation $(x-3)^2=10$, that is $x^2-6x-1=0$. also, its conjugate $3-\sqrt{10}$ lies between -1 and 0. Thus, $3+\sqrt{10}$ is a reduced quadratic irrational.

In general, we know that if B is a positive non-perfect square integer, then $0<\sqrt{B}-[[\sqrt{B}]]<1$ and so, $-1<[[\sqrt{B}]]-\sqrt{B}<0$. Thus, $\sqrt{B}+[[\sqrt{B}]]>1$ is a reduced quadratic irrational.

Theorem 3.8: $\lfloor 24, p.405 \rfloor$

If y has a purely periodic continued fraction expansion, then y is a reduced quadratic irrational.

Proof:

Let $y = [\overline{a_0, a_1, ..., a_{d-1}}]$ be the value of a purely periodic continued fraction. In Theorem 3.7, we've proved that y is a root of the quadratic equation:

$$q_{d-1}y^2 + (q_{d-2} - p_{d-1})y - p_{d-2} = 0,$$

hence y is a quadratic irrational. Now, we shall prove that y is reduced.

Since a_k's are positive integers for $k \geq 1$ and y is purely periodic, $a_0 = a_d = a_{2d} = ... \geq 1$ and so $y > 1$. Also, notice that p_k's and q_k's are positive for all k.

y and its conjugate y' are the roots of the quadratic polynomial:

$$g(x) = q_{d-1}x^2 + (q_{d-2} - p_{d-1})x - p_{d-2}$$

Now,

$g(-1) = (q_{d-1} - q_{d-2}) + (p_{d-1} - p_{d-2}) > 0$ since $p_{d-1} > p_{d-2}$ and $q_{d-1} > q_{d-2}$

and $g(0) = -p_{d-2} < 0$.

By the Intermediate Value Theorem, there is a root of $g(x)$ between –1 and 0. But y is greater than 1, so the root is its conjugate y'. Thus, $-1 < y' < 0$.

As a result, y is a reduced quadratic irrational.

Theorem 3.9: $\lfloor 1, p.93 \rfloor \& \lfloor 23, p.169 \rfloor$

Let $w = [\overline{a_0, a_1, ..., a_n}]$ be a purely periodic continued fraction. Then:

$\frac{-1}{w'} = [\overline{a_n, a_{n-1}, ..., a_1, a_0}]$, where w' is the conjugate of w.

Proof:

w represents the purely periodic continued fraction $[\overline{a_0, a_1, ..., a_n}]$. Then, it can be written as

$$w = [a_0, a_1, ..., a_n, w]$$

Using Theorem 3.4,

$$w = \frac{wp_n + p_{n-1}}{wq_n + q_{n-1}}$$

where $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{p_{n-1}}$ are the n^{th} and $(n\text{-}1)^{st}$ convergents of the continued fraction $[\overline{a_0, a_1, ..., a_n}]$, respectively.

This implies that (by Theorem 3.7),

$$q_n w^2 + (q_{n-1} - p_n)w - p_{n-1} = 0 \qquad (3.6)$$

Next, let v be the purely continued fraction representation of of w but in reverse order,

i.e.,
$$\begin{aligned} v &= [\overline{a_n, a_{n-1}, ..., a_1, a_0}] \\ &= [a_n, a_{n-1}, ..., a_1, a_0, v] \end{aligned}$$

Again, using Theorem 3.4,

$$v = \frac{vr_n + r_{n-1}}{vs_n + s_{n-1}} \qquad (3.7)$$

where $\frac{r_n}{s_n}$ and $\frac{r_{n-1}}{s_{n-1}}$ are the n^{th} and $(n\text{-}1)^{st}$ convergents of the continued fraction $[\overline{a_n, a_{n-1}, ..., a_1, a_0}]$, respectively.

We have:

$$\frac{p_n}{p_{n-1}} = [a_n, a_{n-1}, ..., a_0] = \frac{r_n}{s_n}$$

and
$$\frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, ..., a_1] = \frac{r_{n-1}}{s_{n-1}}$$

Then

$$p_n = r_n\,, \qquad p_{n-1} = s_n$$
$$q_n = r_{n-1}\,, \qquad q_{n-1} = s_{n-1}$$

since convergents are in their lowest terms. Substituting these results in equation (3.7), we get:

$$v = \frac{vp_n + q_n}{vp_{n-1} + q_{n-1}}$$

So, $$p_{n-1}v^2 + (q_{n-1} - p_n)v - q_n = 0$$

Dividing both sides of this equation by $-v^2$ ($v > 1$)

$$q_n(\frac{-1}{v})^2 + (q_{n-1} - p_n)(\frac{-1}{v}) - p_{n-1} = 0$$

Thus, $\frac{-1}{v}$ is a root of equation (3.6) and since v and w are positive, $\frac{-1}{v}$ is negative and different from w. so, $\frac{-1}{v}$ must be the conjugate w' of w.

Hence, $w' = \frac{-1}{v}$ and so $v = \frac{-1}{w'} = [\overline{a_n, a_{n-1}, \ldots, a_0}]$.

Theorem 3.10: $\lfloor 23, p.169 \rfloor$

Let w be a quadratic irrational and w' be its conjugate. If d and l are rational numbers, then $d + lw'$ is the conjugate of $d + lw$. Moreover, $d + \frac{l}{w'}$ is the conjugate of $d + \frac{l}{w}$.

Proof: see $\lfloor 23 \rfloor$.

Theorem 3.11: $\lfloor 3, p.21 \rfloor$ **(Lagrange's theorem)**

If w is a quadratic irrational number, then it has a periodic continued fraction expansion.

Proof: see $\lfloor 3 \rfloor$.

Theorem 3.12: $\lfloor 24, p.405 \rfloor \& \lfloor 25, p.45 \rfloor$

If w is a reduced quadratic irrational number, then it has a purely periodic continued fraction expansion.

Proof:

Let $w = w_0$ be a reduced quadratic irrational. So, $w > 1$ and its conjugate w' lies between -1 and 0.

First, we prove that each complete quotient $w_k, k \geq 1$ is a reduced quadratic irrational.

$w = a_0 + \dfrac{1}{w_1}$, $a_0 = [[w]]$. So, $w_1 = \dfrac{1}{w - a_0} > 1$. By Theorem 3.10, $w' = a_0 + \dfrac{1}{w_1'}$ and so $w_1' = \dfrac{1}{w' - a_0}$.

w_1' lies between -1 and 0 since $-1 < w' < 0$. So, w_1 is a reduced quadratic irrational.

Suppose w_n is reduced. That is, $w_n > 1$ and $-1 < w_n' < 0$.

Now, $w_n = a_n + \dfrac{1}{w_{n+1}}$, $a_n = [[w_n]]$. So, we have $w_{n+1} = \dfrac{1}{w_n - a_n} > 1$ and

$w_{n+1}' = \dfrac{1}{w_n' - a_n}$.

$a_n = [[w_n]] \geq 1$ since $w_n > 1$. So, $w_n' - a_n < -a_n \leq -1$ and $-1 < w_{n+1}' < 0$.

Hence, w_{n+1} is a reduced quadratic irrational and by induction, w_k is reduced for $k \geq 0$.

Next, we complete the proof using contradiction. Suppose that the continued fraction expansion of w is not purely periodic and has the form $[a_0, a_1, ..., a_{n-1}, \overline{a_n, a_{n+1}, ..., a_{n+d-1}}]$, where a_n is the first repeating quotient.

$w_{n-1} - w_{n+d-1} = (a_{n-1} + \dfrac{1}{w_n}) - (a_{n+d-1} + \dfrac{1}{w_{n+d}})$.

But $w_n = w_{n+d}$. So, $w_{n-1} - w_{n+d-1} = a_{n-1} - a_{n+d-1}$ is a non-zero integer since otherwise a_{n-1} and a_{n+d-1} would be equal and hence the period would begin one position sooner.

This implies that $w'_{n-1} - w'_{n+d-1}$ is also a non-zero integer. But, $-1 < w'_{n-1} < 0$ and $-1 < w'_{n+d-1} < 0$. So, $-1 < w'_{n-1} - w'_{n+d-1} < 1$, a contradiction.

As a result, the continued fraction expansion of a reduced quadratic irrational is purely periodic.

Theorem 3.13: $\lfloor 1, p.112 \rfloor \& \lfloor 25, p.47 \rfloor$ **(The continued fraction expansion for** $\sqrt{T}$ **)**

Let T be a positive integer, not a perfect square. Then

$$\sqrt{T} = [a_0, \overline{a_1, a_2, a_3, \ldots, a_{n-1}, a_n, 2a_0}]$$

where $a_{n+1-j} = a_j,\ j = 1, 2, \ldots, n.$

i.e., $\sqrt{T} = [a_0, \overline{a_1, a_2, a_3, \ldots, a_2, a_1, 2a_0}]$

Proof:

At first, notice that $\sqrt{T} > 1$ and so $-\sqrt{T} < -1$. Thus, $\sqrt{T}$ is not a reduced quadratic irrational.

Let $$\sqrt{T} = [a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} \tag{3.8}$$

Since $a_0 = [[\sqrt{T}]]$, $a_0 + \sqrt{T} > 1$ and its conjugate $-1 < a_0 - \sqrt{T} < 0$. Then, $a_0 + \sqrt{T}$ is a reduced quadratic irrational and has a purely periodic continued fraction.

Add a_0 to both sides in equation (3.8) to get:

$$a_0+\sqrt{T}=2a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\ddots}}$$

But the expansion of a reduced quadratic irrational is purely periodic. Then,

$$a_0+\sqrt{T}=2a_0+\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{a_n+\cfrac{1}{2a_0+\cfrac{1}{a_1+\ddots}}}}}=[\overline{2a_0,a_1,a_2,\ldots,a_n}]$$

So,

$$\sqrt{T}=a_0+\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{a_n+\cfrac{1}{2a_0+\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{a_n+\cfrac{1}{2a_0+\ddots}}}}}}}}=[a_0,\overline{a_1,a_2,\ldots,a_n,2a_0}] \qquad (3.9)$$

Now, using Theorem 3.9,

$$\frac{-1}{a_0-\sqrt{T}}=\frac{1}{\sqrt{T}-a_0}=[\overline{a_n,a_{n-1},\ldots,a_1,2a_0}] \qquad (3.10)$$

Moreover, using equation (3.9), we can find $\frac{1}{\sqrt{T}-a_0}$. Subtract a_0 from both sides to get:

$$\sqrt{T}-a_0 = 0+\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{a_n+\cfrac{1}{2a_0+\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{a_n+\cfrac{1}{2a_0+\cfrac{1}{\ddots}}}}}}}}} = [0,\overline{a_1,a_2,\dots,a_n,2a_0}]$$

Thus, by Theorem 2.3,

$$\frac{1}{\sqrt{T}-a_0} = a_1+\cfrac{1}{\ddots+\cfrac{1}{a_n+\cfrac{1}{2a_0+\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{a_n+\cfrac{1}{2a_0+\cfrac{1}{\ddots}}}}}}}} = [\overline{a_1,a_2,\dots,a_n,2a_0}] \qquad (3.11)$$

However, since the continued fraction expansion is unique and comparing both equations (3.10) & (3.11), we find:

$$a_n = a_1, a_{n-1} = a_2, \dots, a_2 = a_{n-1}, a_1 = a_n$$

So, $$\sqrt{T} = [a_0,\overline{a_1,a_2,\dots,a_2,a_1,2a_0}].$$

In other words, the periodic part is symmetrical except for the term $2a_0$. It may or may not have a central term.

For instance, the symmetrical part of the periodic expansion for $\sqrt{29} = [5,\overline{2,1,1,2,10}]$ has no central term. But, in the periodic expansion for $\sqrt{31} = [5,\overline{1,1,3,5,3,1,1,10}]$, 5 is the central term.

Example 3.8:

Find the continued fraction expansion of $\sqrt{11}$.

Solution:

Let $y_0 = \sqrt{11}, a_0 = [[\sqrt{11}]] = 3.$

$$y_1 = \frac{1}{\sqrt{11}-3} = \frac{\sqrt{11}+3}{2},\ \ a_1 = 3$$

$$y_2 = \frac{1}{\frac{\sqrt{11}+3}{2}-3} = \frac{2}{\sqrt{11}-3} = \sqrt{11}+3,\ \ a_2 = 6$$

$$y_3 = \frac{1}{\sqrt{11}+3-6} = \frac{\sqrt{11}+3}{2} = y_1,\ \ a_3 = a_1 = 3$$

Since $y_3 = y_1$, it is clear that $y_1 = y_3 = y_5 = \ldots$ and $a_1 = a_3 = a_5 = \ldots = 3$. So,

$y_2 = y_4 = y_6 = \ldots$ and $a_2 = a_4 = a_6 = \ldots = 6$

Hence, $\sqrt{11} = 3 + \cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{\ddots}}}}} = [3,3,6,3,6,...] = [3,\overline{3,6}]$. Notice that

$6 = 2(3) = 2a_0$.

For more examples, look at Table 3.2.

Table 3.2

T	**The continued fraction expansion for $\sqrt{T}$**
2	$[1,\overline{2}]$
3	$[1,\overline{1,2}]$
5	$[2,\overline{4}]$
6	$[2,\overline{2,4}]$
7	$[2,\overline{1,1,1,4}]$
8	$[2,\overline{1,4}]$
10	$[3,\overline{6}]$
11	$[3,\overline{3,6}]$
12	$[3,\overline{2,6}]$
13	$[3,\overline{1,1,1,1,6}]$
14	$[3,\overline{1,2,1,6}]$
15	$[3,\overline{1,6}]$
17	$[4,\overline{8}]$
18	$[4,\overline{4,8}]$
19	$[4,\overline{2,1,3,1,2,8}]$
20	$[4,\overline{2,8}]$
21	$[4,\overline{1,1,2,1,1,8}]$
22	$[4,\overline{1,2,4,2,1,8}]$
23	$[4,\overline{1,3,1,8}]$
24	$[4,\overline{1,8}]$
26	$[5,\overline{10}]$
27	$[5,\overline{5,10}]$
28	$[5,\overline{3,2,3,10}]$
29	$[5,\overline{2,1,1,2,10}]$
30	$[5,\overline{2,10}]$

Section 3.3: Solving Pell's Equation

In this section, we study the solution of one type of Diophantine equations, called Pell's equation, using continued fractions method. The continued fraction expansion of $\sqrt{T}$ plays an important role in our discussion.

Definition 3.5: $\lfloor 5,9,12 \rfloor$

Pell's equation is a Diophantine equation of the form $x^2 - Ty^2 = \pm M$, where *T* is a positive non-perfect square integer and *M* is a fixed natural number.

Indian mathematicians Brahmagupta and Bhaskara are the first to study Pell's equation. This equation appears in problems in mathematics. One of these problems is "The Cattle Problem" of Archimedes. In this problem, there are eight unknowns represent the number of cattle in different kinds. After many steps, one can reduce the problem to $x^2 - 4729494y^2 = 1$.

In this section, we are interested in solving the Pell's equation $x^2 - Ty^2 = \pm 1$, where *T* is not a perfect square since if *T* is a square natural number, i.e. $T = s^2$ for some natural number *s*, then we get a linear system of equations and so the case is trivial.

For the case of $x^2 - Ty^2 = 1$:

$$x^2 - Ty^2 = 1$$

$$x^2 - s^2y^2 = 1$$

$$(x - sy)(x + sy) = 1$$

Thus, $$\begin{matrix} x - sy = 1 \\ x + sy = 1 \end{matrix} \quad \text{or} \quad \begin{matrix} x - sy = -1 \\ x + sy = -1 \end{matrix}$$

Solving these two linear systems, we get $(x, y) = (1, 0)$ or $(x, y) = (-1, 0)$, respectively.

And for the case of $x^2 - Ty^2 = -1$:

$$x^2 - Ty^2 = -1$$

$$x^2 - s^2y^2 = -1$$

$$(x - sy)(x + sy) = -1$$

$$\begin{matrix} x - sy = 1 \\ x + sy = -1 \end{matrix} \quad \text{or} \quad \begin{matrix} x - sy = -1 \\ x + sy = 1 \end{matrix}$$

Solving these two linear systems, we get $(x, y) = (0, \frac{-1}{s})$ or $(x, y) = (0, \frac{1}{s})$, respectively. So, we have the trivial solutions $(x, y) = (0,-1)$ or $(x, y) = (0,1)$ if s = 1.

Another note about Pell's equation is that if (m,n) is a solution, then there are three other solutions located at the vertices of the rectangle centered at the origin and having (m,n) as one of its vertices, i.e., the other three solutions are $(m,-n),(-m,n)$ &$(-m,-n)$. Thus, it is enough to consider positive solutions only, i.e., *m & n* are positive integers.

Remark 3.1:

If (a,b) is a solution of the equation $x^2 - Ty^2 = \pm 1$, then gcd(a,b) =1. Otherwise if gcd(a,b) = $c \neq 1$ then $c^2(r^2 - Ts^2) = \pm 1$. But c, r, s and T are all integers. So c = 1.

Theorem 3.14: $\lfloor 14, p.88 \rfloor \& \lfloor 26, p.332 \rfloor$

If (a,b) is a positive solution of the equation $x^2 - Ty^2 = \pm 1$, then $\frac{a}{b}$ is a convergent of the continued fraction expansion of $\sqrt{T}$.

Before we write the proof of this theorem, we need the following lemma:

Lemma 3.2: $\lfloor 26, p.326 \rfloor$

Let y be an irrational number. If $\frac{u}{v}, v \geq 1$ and $\gcd(u,v)=1$ satisfies $\left|y-\frac{u}{v}\right| < \frac{1}{2v^2}$, then $\frac{u}{v}$ is one of the convergents of the continued fraction expansion of y.

Proof: see $\lfloor 26 \rfloor$.

Proof of Theorem 3.14:

First, if (a,b) is a positive solution of $x^2 - Ty^2 = 1$, then $a^2 - Tb^2 = 1$ and $a > b\sqrt{T}$.

Now,

$$a^2 - Tb^2 = 1$$

$$(a - b\sqrt{T})(a + b\sqrt{T}) = 1$$

So, $$0 < \frac{a}{b} - \sqrt{T} = \frac{1}{b(a + b\sqrt{T})} < \frac{1}{b(b\sqrt{T} + b\sqrt{T})} = \frac{1}{2b^2\sqrt{T}} < \frac{1}{2b^2}$$

By Lemma 3.2, $\frac{a}{b}$ is a convergent of the continued fraction expansion of $\sqrt{T}$.

Second, if (a,b) is a positive solution of $x^2 - Ty^2 = -1$, then $a^2 - Tb^2 = -1$.

Rewrite the equation as $b^2 - \frac{1}{T}a^2 = \frac{1}{T}$.

Now, $(b - \frac{1}{\sqrt{T}}a)(b + \frac{1}{\sqrt{T}}a) = \frac{1}{T}$.

Notice that $\frac{1}{T} > 0$ and so $\frac{b}{a} > \frac{1}{\sqrt{T}}$.

Therefore, $$0 < \frac{b}{a} - \frac{1}{\sqrt{T}} < \frac{1}{Ta(b + \frac{1}{\sqrt{T}}a)} < \frac{1}{Ta(\frac{1}{\sqrt{T}}a + \frac{1}{\sqrt{T}}a)} = \frac{1}{2a^2\sqrt{T}} < \frac{1}{2a^2}.$$

This implies that

$$\left|\frac{1}{\sqrt{T}} - \frac{b}{a}\right| < \frac{1}{2a^2}$$

and thus, by Lemma 3.2, $\frac{b}{a}$ is a convergent of $\frac{1}{\sqrt{T}}$.

Let $\sqrt{T}=[a_0,a_1,a_2,...]$. Then, by Theorem 2.3, $\frac{1}{\sqrt{T}}=[0,a_0,a_1,a_2,...]$. Since $\frac{b}{a}$ is a convergent of $\frac{1}{\sqrt{T}}$, then $\frac{b}{a}=[0,a_0,a_1,a_2,...,a_n]$ for some *n*.

But, $\frac{a}{b}=[a_0,a_1,a_2,...,a_n]$. Therefore, $\frac{a}{b}$ is a convergent of $\sqrt{T}$.

Definition 3.6: $\lfloor 14 \rfloor$

The positive solution (x_0, y_0) to Pell's equation $x^2-Ty^2=\pm1$ is called the **least positive solution** or the **fundamental solution** if $x_0<f$ and $y_0<g$ for every other positive solution (f,g).

It is easy to find the solution of Pell's equation $x^2-Ty^2=1$ using the continued fraction expansion of $\sqrt{T}=[a_0,\overline{a_1,a_2,a_3,...,a_{n-1},a_n,2a_0}]$.

Now, $\frac{p_n}{q_n}=[a_0,a_1,...,a_{n-1},a_n]$ and $\frac{p_{n-1}}{q_{n-1}}=[a_0,a_1,...,a_{n-1}]$ are the n^{th} and (*n*-1)$^{\text{st}}$ convergents of the continued fraction expansion of $\sqrt{T}$.

Using Theorem 3.4, $\sqrt{T}=\frac{t_{n+1}p_n+p_{n-1}}{t_{n+1}q_n+q_{n-1}}$,

where $t_{n+1}=[\overline{2a_0,a_1,...,a_n}]=a_0+\sqrt{T}$.

Thus, $\sqrt{T}=\frac{(a_0+\sqrt{T})p_n+p_{n-1}}{(a_0+\sqrt{T})q_n+q_{n-1}}$

If we multiply and rearrange the terms we obtain:

$$\sqrt{T}(a_0q_n+q_{n-1})+Tq_n=a_0p_n+p_{n-1}+p_n\sqrt{T}$$

But $\sqrt{T}$ is an irrational number and $a_0,q_{n-1},q_n,p_{n-1},p_n,T$ are all integers.

This implies that $\begin{array}{l} a_0q_n+q_{n-1}=p_n \\ Tq_n=a_0p_n+p_{n-1} \end{array}$

So

$$\begin{aligned} q_{n-1}&=p_n-a_0q_n \\ p_{n-1}&=Tq_n-a_0p_n \end{aligned} \tag{3.12}$$

Remember that

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n \qquad (3.13)$$

Substitute equations (3.12) in equation (3.13) to get:

$$(Tq_n - a_0 p_n)q_n - p_n(p_n - a_0 q_n) = (-1)^n$$

Then $$Tq_n^2 - p_n^2 = (-1)^n$$

or $$p_n^2 - Tq_n^2 = (-1)^{n-1}.$$

Therefore, (p_n, q_n) is a solution to the equation $x^2 - Ty^2 = (-1)^{n-1}$.

We have two cases. If *n* is odd, then (p_n, q_n) is a particular solution to the equation $x^2 - Ty^2 = 1$. Notice that the period length " d " of the continued fraction expansion of $\sqrt{T}$ is *n*+1. So, we can write the particular solution (p_n, q_n) as (p_{d-1}, q_{d-1}).

However, if *n* is even, then (p_n, q_n) is a particular solution to the equation $x^2 - Ty^2 = -1$. So, we try to use convergents of the second period to find a particular solution to $x^2 - Ty^2 = 1$.

Now, the term that occurs for the second time is the term a_{2n+1}. So,

$$\frac{p_{2n}}{q_{2n}} = [a_0, a_1, \ldots a_{n-1}, a_n, 2a_0, a_1, \ldots a_{n-1}]$$

and $$\frac{p_{2n+1}}{q_{2n+1}} = [a_0, a_1, \ldots a_{n-1}, a_n, 2a_0, a_1, \ldots, a_{n-1}, a_n]$$

$\sqrt{T} = \dfrac{t_{2n+2}p_{2n+1} + p_{2n}}{t_{2n+2}q_{2n+1} + q_{2n}}$, where $t_{2n+2} = [\overline{2a_0, a_1, \ldots, a_n}] = t_{n+1} = a_0 + \sqrt{T}$.

$$\sqrt{T} = \frac{(a_0 + \sqrt{T})p_{2n+1} + p_{2n}}{(a_0 + \sqrt{T})q_{2n+1} + q_{2n}}$$

Continuing as before yields:

$$\begin{aligned} q_{2n} &= p_{2n+1} - a_0 q_{2n+1} \\ p_{2n} &= Tq_{2n+1} - a_0 p_{2n+1} \end{aligned} \qquad (3.14)$$

Now, $p_{2n}q_{2n+1} - p_{2n+1}q_{2n} = (-1)^{2n+1}$ (3.15)

Substituting equation (3.14) in equation (3.15) and then dividing the resulting equation by -1 yield

$$p_{2n+1}^2 - Tq_{2n+1}^2 = (-1)^{2n} = 1$$

Thus, $(p_{2n+1}, q_{2n+1}) = (p_{2d-1}, q_{2d-1})$ is a particular solution to $x^2 - Ty^2 = 1$ where either n is even and d is odd or vice versa. Notice that whatever the case is, we can always find an integral solution to the equation $x^2 - Ty^2 = 1$.

The following theorem gives us the set of all positive solutions of $x^2 - Ty^2 = \pm 1$.

Theorem 3.15: $\lfloor 14, p.89 \rfloor$

Let T be a non-perfect square positive integer and $\dfrac{p_i}{q_i}$ is the i^{th} convergent of $\sqrt{T} = [a_0, \overline{a_1, a_2, ..., a_{d-1}, a_d}]$, where d is the period length. Then:

(a) All positive solutions of $x^2 - Ty^2 = 1$ are given by

$$(x, y) = \begin{cases} (p_{kd-1}, q_{kd-1}), k \in N & \text{if} \quad d \text{ is even} \\ (p_{2kd-1}, q_{2kd-1}), k \in N & \text{if} \quad d \text{ is odd} \end{cases}$$

(b) On the other hand, all positive solution of $x^2 - Ty^2 = -1$ are given by

$$(x, y) = \begin{cases} (p_{(2k-1)d-1}, q_{(2k-1)d-1}), k \in N & \text{if} \quad d \text{ is odd} \\ \text{no solution} & \text{if} \quad d \text{ is even} \end{cases}$$

Moreover, (p_{d-1}, q_{d-1}) is the fundamental solution of

$$\begin{cases} x^2 - Ty^2 = 1 & \text{if} \quad d \text{ is even} \\ x^2 - Ty^2 = -1 & \text{if} \quad d \text{ is odd} \end{cases}$$

and (p_{2d-1}, q_{2d-1}) is the fundamental solution of $x^2 - Ty^2 = 1$ if d is odd.

Proof: see $\lfloor 14 \rfloor$.

Example 3.9:

Find the fundamental solution to the equation:

(a) $x^2 - 19y^2 = 1$

(b) $x^2 - 13y^2 = 1$

Solution:

(a) First, $\sqrt{19} = [4, \overline{2,1,3,1,2,8}] = \left[a_0, \overline{a_1, a_2, a_3, a_4, a_5, a_6}\right]$.

The period length d = 6. So, the fundamental solution is (p_5, q_5).

The first six convergents of $\sqrt{13}$ are:

$$4, \frac{9}{2}, \frac{13}{3}, \frac{48}{11}, \frac{61}{14}, \frac{170}{39}$$

Thus, $(170,39)$ is the fundamental solution to $x^2 - 19y^2 = 1$.

In addition, the set of all positive solutions of $x^2 - 19y^2 = 1$ is

$$(x, y) = \{(p_{6k-1}, q_{6k-1}), k \in N\}$$

The following table shows solutions for k = 1, 2, …, 5.

Table 3.3

K	(p_{6k-1}, q_{6k-1})
1	$(p_5, q_5) = (170, 39)$
2	$(p_{11}, q_{11}) = (57799, 13260)$
3	$(p_{17}, q_{17}) = (19651490, 4508361)$
4	$(p_{23}, q_{23}) = (6681448801, 1532829480)$
5	$(p_{29}, q_{29}) = (2271672940850, 521157514839)$

(b) $\sqrt{13} = [3, \overline{1,1,1,1,6}] = [a_0, \overline{a_1, a_2, a_3, a_4, a_5}]$

The period length d = 5. So, the fundamental solution of $x^2 - 13y^2 = 1$ is (p_9, q_9).

The first ten convergents of $\sqrt{13}$ are:

$$3, 4, \frac{7}{2}, \frac{11}{3}, \frac{18}{5}, \frac{119}{33}, \frac{137}{38}, \frac{256}{71}, \frac{393}{109}, \frac{649}{180}$$

So, the fundamental solution is $(649, 180)$.

Notice that $(p_4, q_4) = (18, 5)$ is the fundamental solution of $x^2 - 13y^2 = -1$. Moreover, we can find all other positive solutions of Pell's equation using the fundamental solution as the following two theorems illustrate.

Theorem 3.16: $\lfloor 26, p.339 \rfloor \& \lfloor 28, p.354 \rfloor$

Let (x_0, y_0) be the fundamental solution of Pell's equation $x^2 - Ty^2 = 1$. Then, all other positive solutions (x_n, y_n) can be obtained from the equation

$$x_n + y_n\sqrt{T} = (x_0 + y_0\sqrt{T})^n, n \in N.$$

Proof: see $\lfloor 28 \rfloor$.

Theorem 3.17: $\lfloor 18, p.63 \rfloor$

Let (x_0, y_0) be the fundamental solution of the negative Pell's equation $x^2 - Ty^2 = -1$. Then, all positive solutions (x_n, y_n) of $x^2 - Ty^2 = \pm 1$ are given by $x_n + y_n\sqrt{T} = (x_0 + y_0\sqrt{T})^n, n \in N$

where odd values of n gives all positive solutions to $x^2 - Ty^2 = -1$ and even values of n gives all positive solutions to $x^2 - Ty^2 = 1$.

Proof: see $\lfloor 18 \rfloor$.

Remark 3.2:

We find the values of (x_n, y_n) by expanding

$x_n + y_n\sqrt{T} = (x_0 + y_0\sqrt{T})^n, n \in N$ by the Binomial Theorem and equating rational parts and purely irrational parts of the resulting equation. For example, for *n* = 3, we have:

$$x_3 + y_3\sqrt{T} = (x_0 + y_0\sqrt{T})^3 = \binom{3}{0}x_0^3 + \binom{3}{1}x_0^2 y_0\sqrt{T} + \binom{3}{2}x_0^1 y_0^2 T + \binom{3}{3}y_0^3 T\sqrt{T}$$
$$= x_0^3 + 3x_0^1 y_0^2 T + \sqrt{T}(3x_0^2 y_0 + y_0^3 T)$$

So, $(x_3, y_3) = (x_0^3 + 3x_0^1 y_0^2 T, 3x_0^2 y_0 + y_0^3 T)$

Example 3.10:

(1) In Example 3.9, the fundamental solution of $x^2 - 13y^2 = 1$ is (649, 180).

Set *n* = 2, we have: $x_2 + y_2\sqrt{13} = (649 + 180\sqrt{13})^2 = 842401 + 233640\sqrt{13}$

So, (842401, 233640) is the second solution of $x^2 - 13y^2 = 1$.

Set *n* = 3, we get:

$x_3 + y_3\sqrt{13} = (649 + 180\sqrt{13})^3 = 1093435849 + 303264540\sqrt{13}$

So, (1093435849, 303264540) is the third solution of $x^2 - 13y^2 = 1$.

(2) By Theorem 3.17 and using convergents in Example 3.9, we find that the fundamental solution of $x^2 - 13y^2 = -1$ is (18, 5).

To find the second solution of this equation, set *n* = 3:

$x_3 + y_3\sqrt{13} = (18 + 5\sqrt{13})^3 = 23382 + 6485\sqrt{13}$

So, (23382, 6485) is the second solution of $x^2 - 13y^2 = -1$.

If we set *n* = 2, we get a solution of $x^2 - 13y^2 = 1$:

$x_2 + y_2\sqrt{13} = (8 + 5\sqrt{13})^2 = 169 + 180\sqrt{13}$

Notice that we get the fundamental solution (169, 180) of $x^2 - 13y^2 = 1$.

Chapter Four

Best Approximation and Applications

We discuss in the first section the best approximation and its relation with convergents. In the second section we study some interesting applications of continued fractions in different fields.

Section 4.1: Continued Fractions and Best Approximation

A very important use of continued fractions is the approximation of irrational numbers by rational numbers.

The problem of approximation includes determining which of the rational numbers that have a difference "no more than a specific value" from a given irrational number has the lowest positive denominator. This way is also used to approximate rational numbers whose numerators and denominators are extremely large by a fraction with smaller numerator and denominator.

Convergents have an important role in solving the problem of best approximation of a real number since, from our study of them, they are completely determined by the number represented and closely connected with it.

Definition 4.1: $\lfloor 30 \rfloor$

A rational number $\frac{a}{b}$ is a **best approximation of a first kind** of a real number x provided that, for every rational number $\frac{c}{d} \neq \frac{a}{b}$ such that $0 < d \leq b$, we have $\left| x - \frac{a}{b} \right| < \left| x - \frac{c}{d} \right|$.

In other words, $\frac{a}{b}$ is a best approximation of the first kind if we cannot find a different rational number closer to x with denominator $\leq b$.

Definition 4.2: $\lfloor 3 \rfloor$

A rational number $\frac{a}{b}$ is a **best approximation of a second kind** of a real number x provided that, for every rational number $\frac{c}{d} \neq \frac{a}{b}$ such that $0 < d \leq b$, we have $|bx - a| < |dx - c|$

Theorem 4.1: $\lfloor 30, p.386 \rfloor$

Every best approximation of a second kind of a real number x is a best approximation of a first kind of x.

Proof:

Let $\frac{a}{b}$ be a best approximation of a second kind of x. Let $\frac{c}{d}$ be a rational number with $0 < d \leq b$. Then

$$|bx - a| < |dx - c|$$

Now, $\left|x - \frac{a}{b}\right| = \frac{|bx - a|}{b} < \frac{|dx - c|}{b} \leq \frac{|dx - c|}{d} = \left|x - \frac{c}{d}\right|$

So, $\left|x - \frac{a}{b}\right| < \left|x - \frac{c}{d}\right|$. Thus, $\frac{a}{b}$ is a best approximation of a first kind.

Remark 4.1: $\lfloor 30, p.386 \rfloor$

The converse of Theorem 4.1 is not true. The following example shows that a best approximation of a first kind may not be a best approximation of a second kind.

Example 4.1:

The rational number $\frac{13}{4}$ is a best approximation of the first kind to π since there are no rational numbers closer to π with denominator ≤ 4.

$\frac{10}{3}, \frac{7}{2}$ and $\frac{3}{1}$ are the closest **distinct** rational numbers to π with denominators 3, 2 and 1 respectively.

$$\left|\pi - \frac{13}{4}\right| = 0.108407346...$$
$$\left|\pi - \frac{10}{3}\right| = 0.191740679...$$
$$\left|\pi - \frac{7}{2}\right| = 0.358407346...$$
$$\left|\pi - \frac{3}{1}\right| = 0.141592653...$$

Thus, for every rational number $\frac{c}{d} \neq \frac{13}{4}$ with $0 < d \leq 4$, we get

$$\left|\pi - \frac{13}{4}\right| < \left|\pi - \frac{c}{d}\right|.$$

However, $\frac{13}{4}$ is not a best approximation of the second kind to π since $\frac{3}{1} \neq \frac{13}{4}$ and $0 < 1 < 4$ but $|1.\pi - 3| < |4.\pi - 13|$.

The following theorem is a generalization of the idea in Example 3.3.

Theorem 4.2: $\lfloor 3, p.31 \rfloor \& \lfloor 11, p.404 \rfloor$

For an infinite simple continued fraction representing an irrational number y, each convergent is nearer to y than the preceding convergent.

Proof:

Let $[a_0, a_1, ..., a_n, ...]$ be the infinite simple continued fraction representation of y. Then $y = [a_0, a_1, ..., a_n, y_{n+1}]$ where $y_{n+1} = [a_{n+1}, a_{n+2}, ...]$.

By Theorem 3.4,

$$y = \frac{y_{n+1}p_n + p_{n-1}}{y_{n+1}q_n + q_{n-1}}.$$

This implies that

$$y_{n+1}q_n y - y_{n+1}p_n = -q_{n-1}y + p_{n-1}$$

Thus $$y_{n+1}(q_n y - p_n) = -q_{n-1}(y - \frac{p_{n-1}}{q_{n-1}}) \qquad \text{for } n \geq 1.$$

Dividing both sides by $y_{n+1}q_n$ yields:

$$y - \frac{p_n}{q_n} = \frac{-q_{n-1}}{y_{n+1}q_n}(y - \frac{p_{n-1}}{q_{n-1}})$$

Then, take absolute value to both sides to obtain:

$$\left| y - \frac{p_n}{q_n} \right| = \left| \frac{q_{n-1}}{y_{n+1}q_n} \right| \left| y - \frac{p_{n-1}}{q_{n-1}} \right|$$

But $q_n > q_{n-1} > 0$ and $y_{n+1} > 1$ for $n \geq 1$. Therefore $0 < \left| \frac{q_{n-1}}{y_{n+1}q_n} \right| < 1$.

Thus, $\left| y - \frac{p_n}{q_n} \right| < \left| y - \frac{p_{n-1}}{q_{n-1}} \right|$ or $\left| y - c_n \right| < \left| y - c_{n-1} \right|$ for $n \geq 1$.

So, y is closer to the n^{th} convergent than to the $(n-1)^{st}$ convergent.

Theorem 4.3: $\lfloor 30, p.387 \rfloor$

Let y be a real number and let $c_n = \frac{p_n}{q_n}$ be the n^{th} convergent of the simple continued fraction representation of y. Then

$$\left| y - c_n \right| < \frac{1}{q_n q_{n+1}}.$$

Proof:

We'll prove the theorem for irrational numbers. The proof of rational numbers is in the same manner provided that c_{n+1} exists (that is $y \neq c_n$).

First, we use the inequalities:

$$c_n < c_{n+2} < y < c_{n+1}, \text{ if } n \text{ is even}$$

$$c_{n+1} < y < c_{n+2} < c_n, \text{ if } n \text{ is odd}$$

and the property $c_{n+1} - c_n = \dfrac{(-1)^n}{q_n q_{n+1}}$, $n \geq 0$.

Now, from the inequalities, we conclude:

$$|y - c_n| < |c_{n+1} - c_n|$$

and $$|c_{n+1} - c_n| = \left|\frac{(-1)^n}{q_n q_{n+1}}\right| = \frac{1}{q_n q_{n+1}}$$

Thus, $|y - c_n| < \dfrac{1}{q_n q_{n+1}}$.

Theorem 4.4: $\lfloor 25, p.20 \rfloor$

Every best approximation of the second kind of a real number x is a convergent of the simple continued fraction representation of x.

Proof:

We prove this theorem with the assumption that x is an irrational number. When $x = [a_0, a_1, ..., a_{n-1}, a_n]$ is a rational number, the proof is in the same manner but assume that the last partial quotient $a_n \neq 1$.

Let $x = [a_0, a_1, a_2, ...]$ and let $\dfrac{a}{b}$ be a best approximation of x of a second kind.

By contradiction, suppose that $\dfrac{a}{b}$ is not a convergent. We have only three cases to consider.

Case I: $\dfrac{a}{b} < \dfrac{p_0}{q_0}$

Now, $\dfrac{a}{b} < a_0 < x$ since $\dfrac{p_0}{q_0} = a_0$. This implies that $|x - a_0| < \left|x - \dfrac{a}{b}\right|$.

Then, $|1.x-a_0| \le b|x-a_0| < b\left|x-\frac{a}{b}\right| = |bx-a|, 0<1\le b$ which contradicts that $\frac{a}{b}$ is a best approximation of a second kind.

Thus, $\frac{a}{b} > \frac{p_0}{q_0}$.

Case II: $\frac{a}{b} > \frac{p_1}{q_1}$

Recall that $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots < x < \ldots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}$.

Then, $x < \frac{p_1}{q_1} < \frac{a}{b}$. Multiply this inequality by b and then subtract a from the result to get:

$$bx-a < b\frac{p_1}{q_1} - a < 0.$$

Thus,

$$0 < \frac{|bp_1 - aq_1|}{q_1} < |bx-a|.$$

Note that $bp_1 - aq_1$ is an integer and since $\frac{|bp_1-aq_1|}{q_1} > 0$ then $|bp_1 - aq_1| \ge 1$. Therefore,

$$\frac{1}{q_1} < |bx-a|.$$

But $q_1 = a_1$, so $\frac{1}{a_1} < |bx-a|$.

Remember that $x = a_0 + \frac{1}{x_1}$, where $x_1 = [a_1, a_2, a_3, \ldots]$ and $a_1 = [[x_1]] < x_1$.

So, $|x-a_0| = \frac{1}{x_1} < \frac{1}{a_1} < |bx-a|$ which contradicts the assumption that that $\frac{a}{b}$ is a best approximation of a second kind.

Case III: $\frac{a}{b}$ lies between $\frac{p_0}{q_0}$ and $\frac{p_1}{q_1}$ and is not a convergent, then

$$\left.\begin{array}{lll} & \dfrac{p_n}{q_n} < \dfrac{a}{b} < \dfrac{p_{n+2}}{q_{n+2}} < x < \dfrac{p_{n+1}}{q_{n+1}} & \text{if } \dfrac{a}{b} \text{ is to the left of } x \\ \text{or} & \dfrac{p_{n+1}}{q_{n+1}} < x < \dfrac{p_{n+2}}{q_{n+2}} < \dfrac{a}{b} < \dfrac{p_n}{q_n} & \text{if } \dfrac{a}{b} \text{ is to the right of } x \end{array}\right\} \quad (4.1)$$

Now

$$0 < \frac{|aq_n - p_n b|}{bq_n} = \left|\frac{a}{b} - \frac{p_n}{q_n}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = |c_{n+1} - c_n| = \frac{1}{q_n q_{n+1}} .$$

Then

$$\frac{1}{bq_n} < \frac{1}{q_n q_{n+1}} .$$

This is since $\dfrac{a}{b} \neq \dfrac{p_n}{q_n}$ and therefore the integer $|aq_n - p_n b| \geq 1$.

So,

$$\frac{1}{b} < \frac{1}{q_{n+1}} .$$

That is

$$q_{n+1} < b . \qquad (4.2)$$

Moreover, the inequalities (4.1) imply that

$$\frac{|aq_{n+2} - p_{n+2} b|}{bq_{n+2}} = \left|\frac{a}{b} - \frac{p_{n+2}}{q_{n+2}}\right| < \left|x - \frac{a}{b}\right|$$

But $|aq_{n+2} - p_{n+2} b| \geq 1$ since $\dfrac{a}{b} \neq \dfrac{p_{n+2}}{q_{n+2}}$.

Thus

$$\frac{1}{bq_{n+2}} < \left|x - \frac{a}{b}\right|$$

That is

$$\frac{1}{q_{n+2}} < |bx - a| \qquad (4.3)$$

Next,

$$|q_{n+1} x - p_{n+1}| = q_{n+1} \left|x - \frac{p_{n+1}}{q_{n+1}}\right|$$

By Theorem 4.3,

$$\left| x - \frac{p_{n+1}}{q_{n+1}} \right| < \frac{1}{q_{n+1}q_{n+2}}$$

So, $$\left| q_{n+1}x - p_{n+1} \right| < \frac{1}{q_{n+2}}$$

From (4.2) and (4.3), we get:

$$\left| q_{n+1}x - p_{n+1} \right| < \left| bx - a \right|, \qquad 0 < q_{n+1} < b$$

which contradicts the fact that $\frac{a}{b}$ is a best approximation of a second kind.

Hence $\frac{a}{b}$ must be a convergent of the continued fraction expansion of *x*.

Theorem 4.5: $\lfloor 25, p.21 \rfloor$

Let *x* be a real number not of the form $a_0 + \frac{1}{2}$. Then every convergent of the simple continued fraction expansion of *x* is a best approximation of a second kind to *x*.

Proof: see $\lfloor 25 \rfloor$.

Remark 4.2: $\lfloor 25 \rfloor$

If the real number *x* lies in the middle between two integers (i.e. $x = a_0 + \frac{1}{2}$, a_0 is an integer), then $\left| x - (a_0 + 1) \right| = \left| x - a_0 \right|$ since both sides equal to a half, where $\frac{a_0}{1} \neq \frac{a_0 + 1}{1}$. So, the convergent $c_0 = a_0$ is not a best approximation of a second kind.

Section 4.2: Applications

4.2.1 Calendar Construction $\lfloor 3,31,33 \rfloor$

There are many activities whose success depends on accurate planning. Some of them should be done during a certain period of a year, such as sowing and plowing. So, calendar construction is an important issue since ancient times and calendars are found in every old civilization.

By counting the days, calendars help us to determine the seasons which depend on the rotation of the Earth around the sun.

A **tropical "solar" year** is the time it takes the Earth to make one revolution around the sun = $\frac{315569259747}{864000000} = 365.24219878125$ days.

Remark 4.3:

There are 365 days, 5 hours, 48 minutes and 45.9747 seconds in a year. So, there are $(365 * 24 + 5 + 48/60) * 3600 + 45.9747 = 31556925.9747$ seconds in one year. On the other hand, there are $24 * 3600 = 86400$ seconds a day.

Babylonian Calendar "the oldest" contained 12 months with 29 and 30 days alternately. One year in this calendar had 354 days. After that, the Babylonian calendar was replaced by the Egyptian Calendar which consisted of 12 months, each month contained 30 days. One year in this calendar consisted of 360 days. Then, five days were added to adjust the calendar in Pharaonic times. This calendar was effective for more than 3000 years. However, it led to an error of quick accumulation and therefore a noticeable shift of seasons. Next, a sixth day was introduced every fourth year to give a calendar called the Alexandrian Calendar.

Our calendar comes from the ancient Roman calendar. A year in Roman calendar consisted of 365 days until an Alexandrian astronomer advised Romans to create the Julian calendar in which every year divisible by 4 was a **leap year** “consisting of 366 days” and every other year was a **common year** “consisting of 365 days”. Julian calendar was a good calendar as it accumulated a small error in a hundred years. However, over the next millennium, the discrepancy was noticed.

CALENDARIVM
GREGORIANVM
PERPETVVM.

Orbi Christiano vniuerso à Gregorio XIII. P. M. propositum. Anno M. D. LXXXII.

GREGORIVS EPISCOPVS
SERVVS SERVORVM DEI
AD PERPETVAM REI MEMORIAM.

Figure 4.1: The first page of the papal bull "Inter Gravissimas" by which Pope Gregory XIII introduced his calendar.

Finally, a new more precise calendar construction was created by Pope Gregory XIII. He decreed to omit a leap year every century except those years that are divisible by 400. The Gregorian calendar is both accurate and easy to remember.

The question now is what is the science behind this construction? In fact, continued fractions provide such a science.

The idea of constructing a modern calendar is to have a cycle of q years such that p of them are leap years. So, $q - p$ are common years. When p and q are chosen, we should take into consideration that the mean year length is very close to the tropical year. Moreover, the rule for selecting p leap years should be convenient and simple to use.

During the q-cycle with p leap years, there are $365q + p$ days. Thus, the mean year length is $365+\frac{p}{q}$.

Recall that a tropical year consists of

$$\frac{315569259747}{864000000}=365+\frac{209259747}{864000000}=365+\frac{7750361}{32000000}=365.24219878125$$

days.

Now, our purpose is to find a good approximation $\frac{p}{q}$ for $\frac{7750361}{32000000}$. The last fraction represents the error between a tropical year and a common year.

Representing this fraction as a continued fraction yields

$$\frac{7750361}{32000000}=[0,4,7,1,3,5,6,1,1,3,1,7,7,1,1,1,1,2]$$

The first 8 corresponding convergents are

$c_0 = 0$ $\qquad$ $c_1=\frac{1}{4}=0.25$

$c_2=\frac{7}{29}=0.24137931034$ $\qquad$ $c_3=\frac{8}{33}=0.24242424242$

$c_4=\frac{31}{128}=0.2421875$ $\qquad$ $c_5=\frac{163}{673}=0.24219910846$

$c_6=\frac{1009}{4166}=0.24219875180$ $\qquad$ $c_7=\frac{1172}{4839}=0.24219880141$

The Julian Calendar is realized by the first convergent c_1 which gives a 4-year cycle with one leap year in it.

The annual error considered in Julian Calendar is

$$\left|\frac{1}{4}-\frac{7750361}{32000000}\right|=0.00780121875$$

which means that the calendar accumulates about 8 extra days in 1000 years. That is a bit less than a day in 100 years.

Looking at the denominators next convergents, we realize that the numbers 29, 33, 128, 673, … provide uncomfortable lengths of a cycle. For example, the fourth convergent $\frac{31}{128}$ determines a 128-year cycle with 31 leap years in it. We can construct a corresponding calendar in which there is a leap year every fourth year in the cycle with the thirty-second leap year deleted and this construction gives an annual error

$$\left|\frac{31}{128}-\frac{7750361}{32000000}\right|=\left|-0.00001128125\right|=0.00001128125$$

which means a loss of about one day every 100000 years. This construction is more accurate than Julian Calendar but it is *uncomfortable* to use. So, no one used this calendar.

Now, we try to find a cycle several centuries long with easy and simple selection rule of leap years. Suppose that q = 100t, t is an integer lies between 1 and 9. This assumption matches with the problem of approximating the fraction $100\times\frac{7750361}{32000000}=\frac{7750361}{320000}$.

$$\frac{7750361}{320000}=[24,4,1,1,4,1,2,2,6,11,2,1,1,2]$$

The first corresponding 6 convergents are

$$c_0=24 \qquad c_1=\frac{97}{4}=24.25$$

$$c_2=\frac{121}{5}=24.2 \qquad c_3=\frac{218}{9}=24.22222222222$$

$$c_4=\frac{993}{41}=24.21951219512 \qquad c_5=\frac{1211}{50}=24.22$$

The first convergent $c_1=\frac{97}{4}$ gives a 400-cycle with 97 leap years in it. The selection rule of leap years in the cycle is that every year divisible by 4 is a leap year except 100^{th}, 200^{th}, 300^{th} years. This calendar is called Gregorian Calendar which is used nowadays in most countries.

The error results in a century from this calendar is

$$\left|\frac{97}{4}-\frac{7750361}{320000}\right|=0.030121875$$

That is an accumulation of about one extra day every 3320 years.

Another calendar could be constructed using the convergent $c_2=\frac{121}{5}$ which gives a 500-year cycle with 121 leap years in it. The selection rule of leap years in this cycle is that every year divisible by 4 is a leap year except 100^{th}, 200^{th}, 300^{th}, 400^{th} years.

The error results from this calendar in a century is

$$\left|\frac{121}{5}-\frac{7750361}{320000}\right|=\left|-0.019878125\right|=0.019878125$$

which implies that there is a loss of nearly a day every 5031 years.

Moreover, we can construct a calendar using a 900- year cycle with 218 leap years in it. The selection rule of leap years in this calendar implies 7 exceptions to the fourth year rule since $(900\div 4)-7=218$. This calendar is accurate since it accumulates only one day in about 42660 years. However, it is more complicated than the previous calendars and the 900-year cycle is long and therefore inconvenient. So, we reject this calendar and prefer the simpler ones.

A small correction can be done in future to the Gregorian Calendar to make it more accurate. Continued fractions give an easy method to carry out his correction. The idea is to find a longer cycle length q consisting of a number of 400- year cycles. Suppose $q = 400s$, where s is the number of 400 year cycles in the new longer cycle.

Represent $400 \times \frac{7750361}{32000000} = \frac{7750361}{80000}$ as a simple continued fraction to get

$$\frac{7750361}{80000} = [96, 1, 7, 3, 2, 1, 25, 2, 1, 5, 2]$$

The first corresponding four convergents are

$$c_0 = 96 \qquad c_1 = 97$$

$$c_2 = \frac{775}{8} \qquad c_3 = \frac{2422}{25}$$

The second convergent $c_2 = \frac{775}{8}$ provides us with a 3200- year cycle with 775 leap years in it.

The error of this construction is

$$\left| \frac{775}{8} - \frac{7750361}{80000} \right| = \left| -0.0045125 \right| = 0.0045125$$

which implies that there is a loss of about one day every 88643 years.

Remember that in Gregorian Calendar, there are 97 leap years in every 400- year cycle. So, we get 97×8 = 776 leap years within every 3200 years. Therefore, omitting one leap year every 3200 years will provide us with a modified Gregorian Calendar which has nearly the same construction as Gregorian Calendar but is more accurate.

4.2.2 Piano Tuning $\lfloor 3,30,32 \rfloor$

Musicians know that we cannot tune a piano perfectly. In this discussion, we study the role of continued fractions in piano tuning.

The keyboard of a piano consists of white and black keys. The standard white keys are A, B, C, D, E, F and G. A black key is called sharp and flat. It is a sharp key "#" of the white key that precedes it and a flat key "b" of the white key that follows it as Figure 4.2 shows.

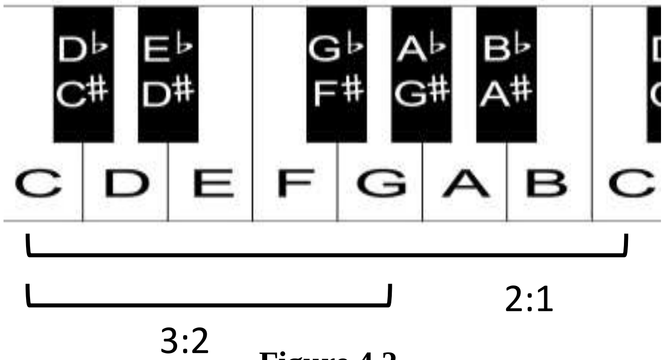

Figure 4.2

Sounds that have frequencies with small integer ratios are consonant and harmonious. Musical intervals represent the ratios of frequencies of two notes. For example, an octave, which represents the ratio 2:1, is the interval between two notes, one having double the frequency of the other. A perfect fifth is an interval represents the ratio 3:2. There are many other intervals such as perfect fourth "4:3", whole step "9:8", etc. In fact, our study here is about the first two intervals which are the most consonant intervals.

Pythagorean scale used only octaves and perfect fifths. The problem we are trying to solve comes from trying to find an integer solution to the equation $2^x = \left(\frac{3}{2}\right)^y$ in order to keep the scale finite. This equation has no integer solution except $x = y = 0$. So, we need to approximate the solution using continued fractions.

Now, $2^x = \left(\frac{3}{2}\right)^y$ implies that $2^{x/y} = \frac{3}{2}$.

So, $\frac{x}{y} = \log_2(\frac{3}{2}) = \log_2(3) - 1 \approx 0.5849625007211562$ and its continued fraction expansion is $[0,1,1,2,2,3,1,5,2,23,2,...]$

The k^{th} convergents, $2 \le k \le 6$ are: $\frac{1}{2}, \frac{3}{5}, \frac{7}{12}, \frac{24}{41}, \frac{31}{53}$

Taking the fourth convergent, we get $\frac{3}{2} \approx 2^{(7/12)} = 1.4983070768766815$.

The approximation $\frac{7}{12}$ implies that the octave consists of 12 semitones with a perfect fifth equal to 7 semitones, "a semitone is the musical interval between two adjacent notes in a 12-tone scale". In fact, in western music, they use this approximation, i.e., the octave is divided into 12 semitones.

Other approximations are inconvenient since $\frac{24}{41}$ and $\frac{31}{53}$ give 41 and 53 notes within an octave, respectively which are too many notes. Moreover, $\frac{1}{2}$ and $\frac{3}{5}$ give 2 and 5 notes within an octave, respectively which are too few notes. The percentage error results from choosing $\frac{7}{12}$ as an approximation is: $\frac{\left|(\log_2(3)-1) - \frac{7}{12}\right|}{\log_2(3)-1}.100\% = 0.278508\% < 0.3\%$.

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Appendix

Would it be possible to find convergents for a continued fraction without finding first all of the preceding convergents? It can be done using determinants.

Consider the continued fraction expansion $[a_0, a_1, a_2, ...]$ of a real number. Using the continued fraction recursion formula:

$$\begin{matrix} p_k = a_k p_{k-1} + p_{k-2} \\ q_k = a_k q_{k-1} + q_{k-2} \end{matrix}, \text{ k} \geq 0 \text{ and } \begin{matrix} p_{-1} = 1, p_{-2} = 0 \\ q_{-1} = 0, q_{-2} = 1 \end{matrix}$$

We have

$$\begin{aligned} p_0 &= a_0 \\ p_1 &= a_1 p_0 + 1 \\ p_2 &= a_2 p_1 + p_0 \\ p_3 &= a_3 p_2 + p_1 \\ &\;. \\ &\;. \\ p_{k-1} &= a_{k-1} p_{k-2} + p_{k-3} \\ p_k &= a_k p_{k-1} + p_{k-2} \\ &\;. \\ &\;. \end{aligned}$$

To compute p_k, we find the following system of linear equations in k + 1 unknowns, p_0 through p_k:

$$\begin{aligned} -p_0 \qquad\qquad\qquad\qquad\qquad &= -a_0 \\ a_1 p_0 - p_1 \qquad\qquad\qquad\qquad &= -1 \\ p_0 + a_2 p_1 - p_2 \qquad\qquad\qquad &= 0 \\ p_1 + a_3 p_2 - p_3 \qquad\qquad &= 0 \\ . \qquad\qquad & \\ . \qquad & \\ p_{k-3} + a_{k-1} p_{k-2} - p_{k-1} \qquad &= 0 \\ p_{k-2} + a_k p_{k-1} - p_k &= 0 \end{aligned}$$

To solve for p_k, we may use Cramer's Rule:

$$p_k = \frac{\begin{vmatrix} -1 & 0 & . & . & . & & 0 & -a_0 \\ a_1 & -1 & 0 & & & & . & -1 \\ 1 & a_2 & -1 & . & & & . & 0 \\ 0 & 1 & a_3 & -1 & . & & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & 0 & . \\ . & & & . & 1 & a_{k-1} & -1 & . \\ 0 & . & . & . & 0 & 1 & a_k & 0 \end{vmatrix}}{\begin{vmatrix} -1 & 0 & . & . & . & & 0 & 0 \\ a_1 & -1 & 0 & & & & . & 0 \\ 1 & a_2 & -1 & . & & & . & 0 \\ 0 & 1 & a_3 & -1 & . & & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & 0 & . \\ . & & & . & 1 & a_{k-1} & -1 & . \\ 0 & . & . & . & 0 & 1 & a_k & -1 \end{vmatrix}}$$

In the denominator we have a lower triangular matrix, so its determinant is $(-1)^{k+1}$. For the numerator, we interchange successively k^{th} columns until we get the last column in the first position and then multiply its entries by -1 to get:

$$p_k = \frac{(-1)^{k+1}\begin{vmatrix} a_0 & -1 & 0 & . & . & . & . & 0 \\ 1 & a_1 & -1 & . & & & . & . \\ 0 & 1 & a_2 & -1 & . & & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & & . & . & . & . & . & 0 \\ . & & & . & . & 1 & a_{k-1} & -1 \\ 0 & . & . & . & . & 0 & 1 & a_k \end{vmatrix}}{(-1)^{k+1}}$$

Then,

$$p_k = \begin{vmatrix} a_0 & -1 & 0 & . & . & . & . & 0 \\ 1 & a_1 & -1 & . & & & . & . \\ 0 & 1 & a_2 & -1 & . & & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & & . & . & . & . & . & 0 \\ . & & & . & . & 1 & a_{k-1} & -1 \\ 0 & . & . & . & . & 0 & 1 & a_k \end{vmatrix}$$

Again to find the value of q_k, we have:

$$\begin{aligned} -q_0 &= -1 \\ a_1q_0 - q_1 &= 0 \\ q_0 + a_2q_1 - q_2 &= 0 \\ q_1 + a_3q_2 - q_3 &= 0 \\ &. \\ &. \\ q_{k-3} + a_{k-1}q_{k-2} - q_{k-1} &= 0 \\ q_{k-2} + a_kq_{k-1} - q_k &= 0 \end{aligned}$$

Applying Cramer's Rule to find q_k and proceeding in the same way as we did for p_k we get:

$$q_k = \begin{vmatrix} 1 & -1 & 0 & . & . & . & . & 0 \\ 0 & a_1 & -1 & . & & & . & . \\ 0 & 1 & a_2 & -1 & . & & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & & . & . & . & . & . & 0 \\ . & & & . & . & 1 & a_{k-1} & -1 \\ 0 & . & . & . & . & 0 & 1 & a_k \end{vmatrix}$$

This determinant can be simplified to get:

$$q_k = \begin{vmatrix} a_1 & -1 & 0 & . & . & . & . & 0 \\ 1 & a_2 & -1 & . & & & . & . \\ 0 & 1 & a_3 & -1 & . & & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & & . & . & . & . & . & 0 \\ . & & & . & . & 1 & a_{k-1} & -1 \\ 0 & . & . & . & . & 0 & 1 & a_k \end{vmatrix}$$

Therefore,

$$c_k = \frac{\begin{vmatrix} a_0 & -1 & 0 & . & . & . & . & 0 \\ 1 & a_1 & -1 & . & & & . & . \\ 0 & 1 & a_2 & -1 & . & & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & & . & . & . & . & . & 0 \\ . & & & . & . & 1 & a_{k-1} & -1 \\ 0 & . & . & . & . & 0 & 1 & a_k \end{vmatrix}}{\begin{vmatrix} a_1 & -1 & 0 & . & . & . & . & 0 \\ 1 & a_2 & -1 & . & & & . & . \\ 0 & 1 & a_3 & -1 & . & & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & & . & . & . & . & . & 0 \\ . & & & . & . & 1 & a_{k-1} & -1 \\ 0 & . & . & . & . & 0 & 1 & a_k \end{vmatrix}}, k \geq 1$$

Example:

Using determinants find the 4th convergent of the continued fraction for each of the following:

a) $\frac{1+\sqrt{11}}{2}$

b) $\dfrac{6211}{215}$

Solution:

a) $\dfrac{1+\sqrt{11}}{2}=[2,\overline{6,3}]$

$$c_4=\frac{\begin{vmatrix}2 & -1 & 0 & 0 & 0\\ 1 & 6 & -1 & 0 & 0\\ 0 & 1 & 3 & -1 & 0\\ 0 & 0 & 1 & 6 & 0\\ 0 & 0 & 0 & 1 & 3\end{vmatrix}}{\begin{vmatrix}6 & -1 & 0 & 0\\ 1 & 3 & -1 & 0\\ 0 & 1 & 6 & -1\\ 0 & 0 & 1 & 3\end{vmatrix}}=\frac{818}{379}$$

b) $\dfrac{6211}{215}=[28,1,7,1,23]$

$$c_4=\frac{\begin{vmatrix}28 & -1 & 0 & 0 & 0\\ 1 & 1 & -1 & 0 & 0\\ 0 & 1 & 7 & -1 & 0\\ 0 & 0 & 1 & 1 & -1\\ 0 & 0 & 0 & 1 & 23\end{vmatrix}}{\begin{vmatrix}1 & -1 & 0 & 0\\ 1 & 7 & -1 & 0\\ 0 & 1 & 1 & -1\\ 0 & 0 & 1 & 23\end{vmatrix}}=\frac{6211}{215}$$

Note:

p_k and q_k are determinants of tridiagonal matrices which can be calculated inductively as follows:

Given the tridaigonal matrix $A_n, n \geq 3$:

$$A_n = \begin{bmatrix} a_1 & b_1 & 0 & . & . & . & . & 0 \\ c_1 & a_2 & b_2 & . & & & . & . \\ 0 & c_2 & a_3 & b_3 & . & & . & . \\ . & . & c_3 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & & . & . & . & . & b_{n-1} & 0 \\ . & & & . & . & c_{n-1} & a_{n-1} & b_n \\ 0 & . & . & . & . & 0 & c_n & a_n \end{bmatrix}$$

Then, $\det A_{k+1} = a_{k+1} \det A_k - b_{k+1} c_{k+1} \det A_{k-1}, k = 2, 3, ..., n-1$

جامعة النجاح الوطنية
كلية الدراسات العليا

الكسور المستمرة وتطبيقاتها

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الملخص

تم في هذه الأطروحة دراسة الكسور المستمرة البسيطة المنتهية وغير المنتهية وخصائصها وحل أمثلة عليها وإثبات بعض النظريات الهامة . تم أيضاً استخدام التقاريب وبعض النظريات الخاصة بها لحل معادلة ديوفانتاين الخطية بمتغيرين.

تم بعد ذلك دراسة بعض النظريات المتعلقة بالكسور المستمرة الدورية واستخدام الكسر المستمر الدوري للجذور الصماء والتقاريب الخاصة بها لحل حالة خاصة من معادلة بيل.

أخيراً تم دراسة العلاقة بين التقاريب الخاصة بالكسر المستمرلأي عدد حقيقي وأفضل تقريب من النوع الثاني لهذا العدد واستخدام هذه التقاريب لإيجاد أفضل تقريب للأعداد الحقيقية في بعض التطبيقات مثل بناء تقويم ميلادي دقيق وضبط البيانو.