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عن ورقة البحث:

"Examples in Cone Metric Spaces: A Survey" *Middle East Journal of Scientific Research, 11(12):1636-1640, 2014, M. Asadi, H. Soleimani "*

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Abstract

The paper "*Examples in Cone Metric Spaces: A Survey*" had overlooked the fact that ℓ^p-spaces are Banach spaces only for $p \geq 1$. Here, we show that, for $0 \prec p \prec 1$, ℓ^p is not even a normed space .We also pointed out that the domain of the function ϕ of Example (1.17) of the paper "*Examples in Cone Metric Spaces: A Survey*" does not allow the cone metric to be defined, so we made two possible alternatives for that to make sense. Furthermore, we give a few remarks on cone metrics and cone norms which were not fully dealt with in the literature.

Keywords: Cone Metric Space, Cone Normed Space.

ملخص

لقد أغفلت الورقه "Examples in Cone Metric Spaces: A Survey" حقيقة أن فضاءات ℓ^p هي فضاءات بناخيه فقط عندما تكون $p \geq 1$. هنا نثبت أنه عندما تكون

فإن الفضاء $0 < p < 1$ ℓ^p لا يكون فضاءً معياريا. كما أن مجال الاقتران ϕ في المثال رقم (1.17) من الورقة لا يسمح بتعريف القياس. وهنا أعطينا خيارين لتصويب التعريف. وختاما كان لنا على القياسات المخروطية والمعيارات المخروطية ملاحظات لم يتم تناولها بشكل واضح في الماضي.

الكلمات المفتاحية: فضاء قياسي مخروطي، فضاء معياري مخروطي.

1. Introduction and Preliminaries

Cone metric spaces were introduced in (Huang, & Hang, 2007. 1468-1476) by means of partially ordering real-Banach spaces by specified cones. In (Abdeljawad, Turkoglo, & Abuloha, 2010. 739-753) and (Turkoglo, Abuloha, & Abdeljawad, 2012), the notion of cone – normed spaces was introduced. Cone – metric spaces, and hence, cone-normed spaces were shown to be first countable topological spaces. The reader may consult (Turkoglo, & Abuloha, 2010. 489-496) for this development.

In (Asadi, Vaezpour, & Soleimani, 2011. 1102. 2353), it was shown that, in a sense, cone –metric spaces are not, really, generalizations of metric spaces. This was the motive to do further investigations. Now we put things in order.

Definition1.1: from (Huang, & Hang, (2007). 1468-1476): Let (E , $\|.\|$) be a real Banach space and P a subset of E. Then P is called a *cone* if:

(a) P is closed , convex, nonempty , and $P \neq \{0\}$

(b) $a,b \in R; a,b \geq 0; x,y \in P \Rightarrow ax+by \in P$

(c) $x \in P and -x \in P \Rightarrow x = 0$

Example 1.2: from (Rezapour, 2007. 85-88): Let $E = \ell^1$, the absolutely summable real sequences. Then the set $P = \{x \in E : x_n \geq 0 \forall n\}$ is a cone in E .

For a cone $P \subset E$, we define (on E) a *partial order* with respect to P as:

$x \leq y$ if $y - x \in P$. We write x<y to indicate that $x \leq y$ but $x \neq y$, and $x << y$ for $y - x \in P^{\circ}$ (the interior of P). The cone P is called *normal* if there is a positive number K such that: For $x, y \in E$, if $0 \leq x \leq y$ then $\|x\| \leq k\|y\|$. The smallest such k is called the *normal constant* of P. P is called *strongly minihedral* if every subset of E which is bounded above has a supremum. Throughout, we will assume that P is a strongly minihedral normal cone with respect to a real Banach space $(E, \|.\|)$. It therefore follows that every subset of P has an infimum (Abdeljawad, Turkoglo, & Abuloha, 2010. 739-753).

2. Cone Normed Spaces

Definition 2.1: Suppose that P is a cone in a normed space $(E, \|.\|)$, and let X be a nonempty set. The pair $(X, \|.\|_C)$ is called a *cone-normed space* relative to the cone P if $\|.\|_c : X \to E$ is a function that satisfies:

(a) $0 \leq \|x\|_c \; \forall x \in X$, and equality holds if and only if $x = 0$.

(b) $\|ax\|_c = |a| \|x\|_c \; \forall a \in R$ and $x \in X$.

(c) $\|x + y\|_c \leq \|x\|_c + \|y\|_c \; \forall x$ and $y \in Y$.

It should be noted that: letting $D(x,y) = \|x - y\|_c$ defines a cone metric on the set X , but not conversely.

For a rigorous development of cone metric spaces, we refer the reader to (Huang, & Hang, 2007. 1468-1476). We construct the following example to show that cone metrics do not necessarily produce cone norms.

Example 2.2: Let X = ℓ^1, $P = [0,\infty)$, and let E =R. For $x, y \in X$, define

d (x,y) = $\sum_{n=1}^{\infty} |x_n - y_n|$, then let D (x,y) = $\frac{d(x,y)}{1+d(x,y)}$.

It is easy to check that D is a cone metric relative to the cone P which is not compatible with any cone norm.

3. ℓ^p - SPACES

Recall that for $p \succ 0$, the space ℓ^p is defined as the set of all p-summable sequences. For $p \geq 1$, the space ℓ^p is a Banach space under the norm $\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$.

Example 3.1: (for $0 \prec p \prec 1$, $\|x\|_p$ is not a norm):

Take $x = (1,0,0,0,...)$ and $y = (0,1,0,0,...)$. Then $\|x+y\|_p = (1^p + 1^p)^{1/p} \succ 2$, while $\|x\|_p + \|y\|_p = 1+1=2$.

So, the triangle inequality for norms fails to hold.

Thus, Example (1.18) of (Asadi, Hossein, 2012. 1636-1640), should state:

Example 3.2: Let $p \geq 1, E = \ell^P$, and let P = $\{x : x_n \geq 0\}$. P is a normal cone with normal constant 1, (A sadi, & Hossein, 2012. 1636-1640). For any metric space (X,p), define d: $X \times X \to E$ as:

d(x,y) = $\left((\frac{p(x,y)}{2^n})^{\frac{1}{p}} \right)_{n=1}^{\infty}$. Then (X, d) is a cone metric space, (A sadi, & Hossein, 2012. 1636-1640).

4. EXAMPLE *(1.16) of* (A sadi, & Hossein, 2012. 1636-1640)

In this quick section we make a generalization of the example.

To be specific, the following is found in (Asadi, Hossein, 2012. 1636-1640).:

Example 4.1: Let

$E = R^n, P = \{(x_1, x_2, ..., x_n) : x_i \geq 0 \forall i = 1,2,...,n\}, X = R,$

and let $d : X \times X \to E$ be defined as:

$d(x, y) = (|x_1 - y_1|, a_1|x_2 - y_2|, a_2[x_3 - y_3], ..., a_{n-1}|x_n - y_n|$,

where $a_i \geq 0 \forall 1 \leq i \leq n-1$. Then (X, d) is a *cone metric space*.

It is a routine check to see that the following example is a generalization to the foregoing:

Example 4.2:

Let $E = R^n, P = \{(x_1, x_2, ..., x_n) : x_i \geq 0, \forall i = 1,2,...\}$.

Let X=R, and let d: $X \times X \to E$ be defined as:

$d(x, y) = (a_1|x_1 - y_1|, a_2|x_2 - y_2|, ..., a_n|x_n - y_n|)$, where

$a_i \succ 0, \forall i \;\; 1 \leq i \leq n$. Then (X,d) is a *cone metric space*.

5. EXAMPLE (1.17) of (A sadi, & Hossein, 2012. 1636-1640)

Remark 5.1: In Example, (1, 17) of (A sadi, & Hossein, 2012. 1636-1640), The assumption that $\phi : [0,1] \to [0,\infty)$makes

d(x, y) undefined on the interval (1,∞) .

So, a modified statement will now be:

Example 5.2: Let E = C_R $[0,\infty)$, and P = $\{f : f(t) \geq 0\}$

Let ϕ: $[0,\infty) \to [0,\infty)$ be continuous, and (X,p) be a metric space. Now define

d: $X \times X \to E$ as: D(x,y) = p(x,y) ϕ.

Then (X,d) is a cone metric space, (A sadi, & Hossein, 2012. 1636-1640).

The following modification also does another choice for modification as well.

***Example* 5.3**: Take E=$C_R[0,1]$ and the continuous function

$\phi : [0,1] \to [0,\infty)$. Let (X, ρ) be a metric space

Now define d: $X \times X \to E$ as: d(x,y) = $\rho(x,y)\phi$

Then (X, d) is a cone metric space, (A sadi, & Hossein, 2012. 1636-1640).

Proposition 5.4:

Consider the cone metric space (R^n , d,P) , where d and P are as in Example (4.2).

Let x = (x_1, x_2 ,…., x_n) $\in R^n$. Then a sequence $(x^k)_{k=1}^{\infty}$ of elements of R^n converges to x if and only if for each i=1,2,…,n , $\lim_{k\to\infty} x_i^k = x_i$.

Proof:

Let $\in$ > 0 be arbitrary. Choose $k_\circ \in N$ such that: $k \ge k_\circ \Rightarrow d(x^k, x) << (a_1 \in, a_2 \in, ..., a_n \in)$

Thus, for $k \ge k_\circ$, and for i = 1,2,…n , One has : $a_i \in -a_i |x_i^k - x_i| > 0,$ or equivalently , $|x_i^k - x_i| < \in$. Therefore,

$\forall i = 1,2,....,n, \ \lim_{k\to\infty} x_i^k = x_i$.

Conversely, suppose that $\forall i = 1,2,....,n, \lim_{k\to\infty} x_i^k = x_i$

Let c = (c_1, c_2, …, c_n)>> 0 be arbitrary.

$\forall i = 1,2,....,n$, Choose $k_i \in N$ such that: $\forall k \geq k_i, \left|x_i^k - x_i\right| < \frac{c_i}{a_i}$

Take $k_\circ = \max\{k_i : i = 1,2,..,n\}$. It now follows that: $\forall k \geq k_\circ$,

d(x^k ,x) = $\left(a_1\left|x_i^k - x_1\right|, a_2\left|x_2^x - x_2\right|,......., a_n\left|x_n^k - x_n\right|\right)$

<<(c_1 ,c_2 , .., c_3)

=c

6. FEW REMARKS

We conclude this article with the following remarks.

Remark *6.1:* (Not every *cone metric* is induced by a *cone norm*):

Let (E,$\|\cdot\|$) be a Banach space and P a strongly minihedral cone.

On any subset X of E, define the cone metric d as: d(x,y) = inf$\{x - y, x_\circ\}$where

$x_\circ \neq 0$ is any fixed element of X. If the metric d were induced by a cone norm $\|\cdot\|_c$ then the cone normed space (X , $\|\cdot\|_c$) must be bounded. But because for $x \in X$ and $\forall a \in R, \|ax\|_c = |a|\|\cdot\|_c$, the set $\{ax_0 : a \in R\}$ is bounded, which is a contradiction, since a can be made arbitrarily large.

Remark 6.2: Cones may not be minihedral: Here is an example.

Let $E = C_R^2[0,1]$, equipped with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, and let

p = $\{f : f(t) \geq 0 \forall t \in [0,1)\}$.This cone is not minihedral because, for instance, take f(t) = sin t and g(t) = cos t. Clearly the function defined as: $h(t)=\sup_t\{f(t),g(t)\}$ is not an element in E, since $h'\left(\frac{\pi}{4}\right)$ does not exist.

Remark 6.3: Cones may be strongly minihedral but not minihedral and here is an example:

Let $E=R^2$, and $P=\{(x,0)\in R^2 : x\geq 0\}$.

Let A be a subset of R^2 which is bounded above, say by (a,b)

By the definition of P, elements of A must have the form (x,b)

Where $x\leq a$. Now, the set $X=\{x\in R:(x,b)\in A\}$ is a non – empty set of R, which is hounded above. Hence has an infimum, call it u. Now, $(u,b)=\sup A$. Thus P is strongly minihedral. However, P is not minihedral. To see this, take $x=(-1,-1)$ and $y=(-2,-2)$. If $(a,b)\geq x$ then $b=-1$, and if $(a,b)\geq y$ then $b=-2$, which is a contradiction.

Remark 6.4: This is a refinement of Example (1.21) of (Asadi, Hossein, 2012. 1636-1640). By this example, we intend to enforce the fact that norm-convergenc is different from cone metric-convergence.

Let $E=C_R^2[0,1]$, equipped with the norm $\|f\|=\|f\|_\infty+\|f'\|_\infty$, and let $P=\{f : f(t)\geq 0\forall t\in[0,1]\}$.

$\forall n\in N$, take x_n (t) = $\frac{1-\sinh(nt)}{n+2}$, so $x_n\in C_R^2[0,1]$.

Let $d:E\times E\to E$ be defined as: $d(x,y)=\begin{cases} x+y; & if\ x\neq y \\ 0; & if\ x=y \end{cases}$.

It is a direct check that d is a cone metric. To see that $x_n \xrightarrow{d} 0$,

Let $c >> 0$ be arbitrary, so $c \in P$, and as a continuous function on the compact set $[0,1]$, let $\delta_\circ = \min\{c(t) : t \in [0,1]\}$.

Since $c(t) > 0 \forall t \in [0,1], \delta_\circ > 0$

By the Archimedean Property, pick $n_\circ \in N$ such that $\frac{1}{2+n_\circ} < \delta_\circ$

Now, for all $t \in [0,1]$ and all $n \geq n_\circ$ we have :

$$c(t) - x_n(t) = c(t) - \frac{1-\sinh(nt}{n+2}$$

$$\geq \delta_\circ - \frac{1}{n+2} + \frac{\sinh(nt)}{n+2}$$

$$\geq \delta_\circ - \frac{1}{n+2}$$

$$> \delta_\circ - \frac{1}{n+2}$$

$$> 0$$.

Since t was arbitrary, it follows that: $\forall_n \geq n_\circ, c >> x_n$, Thus $x_n \xrightarrow{d} 0$.

Finally, $x_n \xrightarrow{\|\cdot\|} 0$. To see this,

$\forall n$, we have : $\|x_n\| = \left\|\frac{1-\sinh(nt)}{n+2}\right\|_\infty + \left\|\frac{n\cosh(nt)}{n+2}\right\|_\infty \geq \frac{1}{2}$

Remark 6.5: In (Turkoglo, & Abuloha, 2010. 489-496), the notion of *positive cone* was given.

For an example of a cone metric space whose *positive cone* has a non –empty interior, the authors gave L^1 form (Deilimling, K. 1985). Here we give the following example

Example 6.6: Let $E = R^2$, and take $P = \{(x,0) : x \geq 0\}$.

Of course P is a cone in E whose positive cone (which is P itself) has empty interior.

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