An-Najah National University

Faculty of Graduate Studies

A COMPARISON OF NUMERICAL SOLUTIONS FOR FREDHOLM INTEGRAL EQUATIONS SYSTEM OF THE SECOND KIND

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Dedication

First and foremost, I am deeply grateful to Almighty God for His endless blessings, guidance, and strength throughout my academic journey. His grace has been my source of perseverance and success.

I dedicate this work to my beloved parents, whose unwavering love, sacrifices, and encouragement have shaped the person I am today. Their endless support and prayers have been my greatest motivation.

To my dear wife, whose patience, understanding, and belief in me have been a pillar of strength. To my wonderful children, who inspire me every day with their smiles and joy—you are my greatest blessings.

I also extend my heartfelt gratitude to my family, friends, mentors, and everyone who has supported and guided me throughout this journey especially my supervisor Prof. Naji Qatanani**.**

Your encouragement and kindness have been invaluable in achieving this milestone.

This thesis is dedicated to all of you, with my deepest appreciation and love.

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To everyone who has walked beside me in this academic endeavor—thank you for your kindness, patience, and belief in me.

Declaration

I I, the undersigned, declare that I submitted the thesis entitled:

A COMPARISON OF NUMERICAL SOLUTIONS FOR FREDHOLM INTEGRAL EQUATIONS SYSTEM OF THE SECOND KIND

I declare that the work provided in this thesis, unless otherwise referenced, is the researcher’s own work, and has not been submitted elsewhere for any other degree or qualification.

Student's Name: **Mohammed Jawad Badrieh**

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Date: 06/02/2025

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A COMPARISON OF NUMERICAL SOLUTIONS FOR FREDHOLM INTEGRAL EQUATIONS SYSTEM OF THE SECOND KIND

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Abstract

The main goal of this work is to propose various numerical techniques for approximating the solution of a system of Fredholm integral equations of the second kind. The methods proposed involve the Chebyshev collocation method, the Haar wavelet method and the reconstruction of the variational iteration method.

After reviewing the fundamental concepts of Fredholm integral equations and addressing the mathematical framework of these numerical methods, we provide some illustrative numerical examples with known exact solution to illustrate the effectiveness and the efficiency of these methods.

Numerical results show clearly that Chebyshev collocation method is one of the most efficient method for solving system of Fredholm integral equations in comparison with its counterparts.

Keywords: Fredholm integral equations**;** Chebyshev collocation method; Haar wavelet method; Reconstruction of variational iteration method; and numerical methods

Chapter One

Introduction

Mathematical models of FIE exists in different fields such as physics, engineering, economics, and biology. This equation arises in problems where the unknown function is defined in terms of its integral, which usually involves a kernel function. Although they provide elegant methods for describing details, solving them analytically is often difficult or even impossible due to non-linear or singular nature

To overcome these difficulties, numerical methods have emerged as indispensable tools for solving the Fredholm integral equations. These methods provide an efficient and accurate framework for computing approximate solutions that satisfy integral equations at a desired level of accuracy Among the many numerical methods available, numerical methods for system solving of the FIE stand out as versatile methods capable of handling complex systems with many equations and unknown

In this context, this paper explores three numerical methods tailored for the solution of systems of FIE, these are Reconstruction of variational iteration method Haar –wavelets method, Chebyshev collocation method. We discuss their theoretical foundations, computational implementations, strengths, and limitations. Through illustrative examples, we aim to gain a deeper understanding of these numerical techniques and their application to solving various problems

An integral equation is a mathematical expression where the unknown function is found inside the integral sign. These equations are classified into various forms, including FIE of types I and II, VIE of types I and II, and integro-differential equations

The Fredholm equation system, in addition to the Volterra or singular integral equations, often arises in various practical applications. They are central to our understanding of various physical phenomena such as desert wind waves, nano-hydrodynamics, population growth models, crystal formation behavior, oceanography, scattering quantum mechanics and water wave dynamics.

FIE of the second and the first kind were extensively researched. Ivar Fredholm's establishment of FIE in the beginning of the 20^{th} century [1] was supplemented by studies that extended Fredholm's work to Banach and Hilbert spaces [2] [3]. Kernel functions occupy a key role in such equations and their properties influence the solvability of Fredholm systems [4].

Analytical solutions of FIE systems are infeasible in most cases, leading to the development of numerical methods. Quadrature methods such as Nystrom's method [5] and Galerkin methods [6] have been applied widely for linear FIE. Collocation methods, including polynomial and splined-based methods have also been used to approximate solutions with efficiency [7]. Successive approximations and projections methods have been tried for solving high dimensional problems [8] [9]. Deep learning methods have been proposed in recent research for the solutions of complex integral equations [10].

Among numerical techniques, Haar wavelet methods have been shown to e very efficient and accurate in the solution of integral equations by transforming them into systems of algebraic equations [11]. Similarly, Chebyshev collocation methods apply Chebyshev polynomials as basis functions, Reconstruction of Variational Iteration method (RVIM) has improved convergence by reconstructing the Lagrange multiplier [12].

Advanced techniques include the Singular value Decomposition (SVD) technique, which effectively solves FIE by stabilizing the numerical solution [13]. Machine Learning techniques, such as Least Squares Support Vector Regression have been employed to approximate solutions in an effective way [14]. Sinc projection techniques, particularly in solving weakly singular integral equation [15].

The Haar wavelet method has emerged as a significant numerical approach in recent years. It was first introduced by Alfred Haar, who pioneered its development. Building on his foundation, Ulo Lepik and Enn Tamme utilized the Haar wavelet method [16] to solve integral equations. Subsequently, Reihani and Abadi [17] proposed a variant of the method to address linear Fredholm and Volterra integral equations of the second kind. Further advancements came in [18], where E. Babolian and A. Shahsavaran developed a numerical method for solving nonlinear Fredholm and Volterra integral equations of the second kind. This approach combined Haar wavelets with the collocation method. Similarly, Mingxu Yi and Yiming Chen, in reference [19], introduced a Haar wavelet operational matrix designed for solving fractional partial differential equations.

Chebyshev polynomials was discovered by the Russian mathematician Chebyshev. Their important for practical computation was rediscovered by C. lanczos [20] and then has been extended by C.W. Clenshaw [21] to differential equations and then by Sezar and Kaynak to Chebyshev-Matrix methods [22]. Y. H. Youssri [23] used Chebyshev collocation to solve Volterra–Fredholm integral equation. Dolapci İhsan [24] used Chebyshev collocation method to solve linear differential equations, Abd-Elhameed and Youssri [25] used numerical solutions for Volterra–Fredholm Hammerstein integral equations via second kind Chebyshev quadrature collocation algorithm. Also, Abd-Elhameed, and Youssri [26] used Fifth-kind orthonormal Chebyshev polynomial solutions for fractional differential equations.

The Reconstruction of the Variational Iteration Method (RVIM) integrates the Laplace transform into the original Variational Iteration Method (VIM). A key advantage of RVIM is its ability to generate rapidly converging successive approximations of the exact solution. It is also widely recognized for its computational simplicity, requiring no restrictive assumptions.

Chapter Two

Mathematical Preliminaries

2.1 Definition

An integral equation is an equation where the unknown function is found within one or more integral expressions. The most general form of an integral equation is represented as follows:

$$h(x)u(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,s)\, u(s) ds \qquad (2.1)$$

In this context, $u(x)$ represents an unknown function referred to as the solution of the integral equation. The limits of integration, $a(x)$ and $b(x)$, can be constants, variables, or a combination of both. The parameter $\lambda \neq 0$, while $h(x)$ is a function that defines the homogeneity of the equation. The functions $h(x), f(x)$, and $k(x,s)$ are known, where $k(x,t)$ is specifically called the kernel of the integral equation. [27]

2.2 Classification of integral equations

2.2.1 Classification depends on the limit of the integration

1. Volterra Integral equation

If one or both limits of integration is variable then equation (2.1) is called Volterra Integral equation.

$$h(x)u(x) = f(x) + \lambda \int_{a}^{x} k(x,s)\, u(s) ds \qquad (2.2)$$

2. Fredholm Integral equation

If limits of integration of equation (2.1) are constants then the equation is called Fredholm Integral equation

$$h(x)u(x) = f(x) + \lambda \int_{a}^{b} k(x,s)\, u(s) ds \qquad (2.3)$$

3. Singular Integral equation

If any of the limits of integrations is infinity, or both or if the kernel becomes unbounded at any point in the interval of integration then the equation (2.1) is called Singular Integral equation.

4. Volterra-Fredholm integral equation

The Volterra-Fredholm integral equations appear in two forms

$$h(x)u(x) = f(x) + \lambda_1 \int_a^x k_1(x,s)\, u(s)ds + \lambda_2 \int_a^b k_2(x,s)\, u(s)dt \qquad (2.4)$$

and

$$h(x,s)u(x,s) = f(x) + \lambda_1 \int_0^t \int_\Omega F\big(x,s,\varepsilon,\tau,u(\varepsilon,\tau)\big)\ d\varepsilon d\tau \qquad (2.5)$$

Ω is a closed subset of R^n

2.2.2 Classification depends on the known functions $\boldsymbol{h(x)}, \boldsymbol{f(x)}$

1. Integral equation of the first kind

If $h(x) = 0$

Then equation (2.3) is called FIE of first kind

$$f(x) = \lambda \int_a^b k(x,s)\, u(s)ds \qquad (2.6)$$

and equation (2.2) is called VIE of first kind

$$f(x) = \lambda \int_{a(x)}^{b(x)} k(x,s)\, u(s)ds \qquad (2.7)$$

2. Integral equation of the second kind

If $h(x) = 1$

Then equation (2.3) is called FIE of second kind

$$u(x) = f(x) + \lambda \int_a^b k(x,s)\, u(s)ds \qquad (2.8)$$

and equation (2.2) is called VIE of second kind

$$u(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,s)\, u(s)ds \qquad (2.9)$$

3. Homogeneous integral equation

If $h(x) = 1\,, f(x) = 0$

Then equation (2.3) is called Fredholm homogeneous IE

$$u(x) = \lambda \int_a^b k(x,s)\, u(s) ds \tag{2.10}$$

and equation (2.2) is called Volterra homogenous IE

$$u(x) = \lambda \int_{a(x)}^{b(x)} k(x,s)\, u(s) ds \tag{2.11}$$

2.2.3 Other types of integral equations

1. Integro- differential equation

The integro-differential equation contains one of the $u(x)$ derivatives outside the integral sign

$$h(x)\, u(x)^{(k)} = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,s)\, u(s) ds, \qquad u(x)^{(k)} = \frac{d^k\, u}{d\, x^k} \tag{2.12}$$

And can be classified into Fredholm and Volterra – integro differential equations.

2. Two-dimensional integral equations

$$h(x,y)\, u(x,y) = f(x,y) + \lambda \int_a^b \int_c^d k(x,y,s,w)\, u(s,w) ds\, dw \tag{2.13}$$

And also, can be classified into Fredholm and Volterra or mixed integral equations.

2.3 Linearity concept of integral equations

If the unknown function $u(x)$ has exponents other than one or is contained in a nonlinear expression, the integral equation is said to be nonlinear.

2.4 System of integral equations

A system of integral equations consists of two or more integral equations, all equations of the system being of the same kind [28].

The system of integral equations general form may be written as

$$H(x)U(x) = F(x) + \lambda \int_{a(x)}^{b(x)} K(x,s)\, U(s) ds \tag{2.14}$$

$H(x) = [h_i(x)]_{n\times n}$

$U(x) = [u_i(x)]_{n\times 1}$

$F(x) = [f_i(x)]_{n\times 1}$

$U(s) = [u_i(s)]_{n\times 1}$

$\lambda K(x) = \left[\lambda_{ij} k_{ij}(x,s)\right]_{n\times n}$

$i = 1, 2, 3, \dots., n \qquad j = 1, 2, 3, \dots., n$

$a(x)$ and $b(x)$are limits of integration which can be constants or variables or mixed, λ is a nonzero constant

2.4.1 Classification of System of Integral Equations

1. System of Fredholm integral equations

The system has a standard form

$$H(x)U(x) = F(x) + \lambda \int_a^b K(x,s)\, U(s)ds \qquad (2.15)$$

$H(x) = [h_i(x)]_{n\times n}$

$U(x) = [u_i(x)]_{n\times 1}$

$F(x) = [f_i(x)]_{n\times 1}$

$U(s) = [u_i(s)]_{n\times 1}$

$\lambda K(x) = \left[\lambda_{ij} k_{ij}(x,s)\right]_{n\times n}$

$i = 1, 2, 3, \dots., n \qquad j = 1, 2, 3, \dots., n$

$a, b \in R \quad , \quad \lambda \neq 0$

The system can be classified into first kind and second kind as in 2.2.2

2. System of Volterra integral equations

The system has a standard form

$$H(x)U(x) = F(x) + \lambda \int_a^x K(x,s)\, U(s)ds \qquad (2.16)$$

$H(x) = [h_i(x)]_{n\times n}$

$U(x) = [u_i(x)]_{n\times 1}$

$F(x) = [f_i(x)]_{n\times 1}$

$U(s) = [u_i(s)]_{n\times 1}$

$\lambda K(x) = \left[\lambda_{ij} k_{ij}(x,s)\right]_{n\times n}$

$i = 1, 2, 3, \dots., n \qquad j = 1, 2, 3, \dots., n$

$a \in R\,, \qquad \lambda \neq 0$

The system can also be classified into first kind and second kind as in 2.2.2

3. System of integro-differential equations

$$H(x)U^{(n)}(x) = F(x) + \lambda \int_{a(x)}^{b(x)} K(x,s)\, U(s)ds \tag{2.17}$$

$H(x) = [h_i(x)]_{n\times n}$

$U(x) = [u_i(x)]_{n\times 1}$

$F(x) = [f_i(x)]_{n\times 1}$

$U(s) = [u_i(s)]_{n\times 1}$

$\lambda K(x) = \left[\lambda_{ij} k_{ij}(x,s)\right]_{n\times n}$

$i = 1, 2, 3, \dots., n \qquad j = 1, 2, 3, \dots., n$

$U^{(n)}(x)$ is the derivative of $U(x)$ of order n

$a(x)$ and $b(x)$ are limits of integration which can be constants or variables or mixed λ is a nonzero constant

Also, this system can be classified to Volterra integro-differential equation and Fredholm integro-differential equation depends on the limits of the integral.

2.5 Linearity concept of systems of integral equations

System (2.14) is considered linear if the exponents of the unknown functions under the integral sign are equal to one, and none of the equations include nonlinear functions of $U(x)$. otherwise, the system is classified as nonlinear.

2.6 Homogeneity concept of systems of integral equations

System (2.14) of IE is considered homogeneous if $F(x)$ is identically zero. If not, it is referred to as nonhomogeneous.

2.7 Existence and uniqueness

This section focuses on examining the existence and uniqueness of solutions to the system of FIE within a Banach space, utilizing fixed-point theory as the foundational approach.

We will start with some definitions and results from the theory of normed vector spaces which will be needed.

2.7.1 Definition

A metric on a nonempty set V is a function $d: V \times V \to [0, \infty)$ such that $\forall\, x, y, z \in V$:

- $d(x, y) = 0$ for all $x \in V$.
- If $x, y \in V$ and $d(x, y) = 0$ then $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in V$.
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in V$.

A metric space is a pair (V, d), where V is a nonempty set and d is a metric on V. [29]

2.7.2 Definition Limit

Suppose (V, d) is a metric space. $x_1, x_2, x_3, \ldots$ is a sequence in V, and $x \in V$

Then $\lim\limits_{n\to\infty} x_n \leftrightarrow \lim\limits_{n\to\infty} x_n \; d(x_n, x) = 0$

Which means that a sequence $x_1, x_2, x_3, \ldots$ converges in V to $x \in V$ If for every $\epsilon > 0$

There exist $n \in Z^+$ such that $d(x_n, x) < \epsilon$ for all integers $k \geq n$. [29]

2.7.3 Definition continuity

Suppose $(V, d_v)\ and\ (W, d_w)$ are metric spaces and $f: V \ \rightarrow W$ is a function.

The Function F is continuous at $x \in V$ if for every $\in > 0$, there exist $\delta > 0$

Such that $d_w\big(f(x),\ f(y)\big) < \epsilon$ for all $y \in V$ with $d_v(x, y) < \delta$.

The function f is continuous if f is continuous at every $x \in V$. [29]

2.7.4 Definition Cauchy sequence

A sequence $f_1, f_2, f_3, \ldots$ in a metric space (V, d) is a Cauchy sequence if for every $\epsilon > 0$, there exist $n \in Z^+$ such that $d\big(f_i, f_j\big) < \in$ for all integers $i \ \geq n$

And $j \ \geq n$ [29]

2.7.5 Definition

A metric space V is complete if every Cauchy sequence in V converges to some element of V. [29]

2.7.6 Definition vector space

A vector space over the field of real numbers R is a set V along with addition and scalar multiplication on V such the following properties hold:

- Commutativity

$x + y = y + x$ For all $x, y \ \in V$.

- Associativity

$(x + y) + z = x + (y + z)$ For all $x, y, z \in V$.

- Scalar multiplication

$(\alpha\beta)x = \ \alpha(\beta x)$ Where $\alpha, \beta \ \in R\ and\ x \in V$.

- Additive identity

There exists an element $0 \in V$ such that $x + 0 = x$ for all $x \in V$. [29]

- Additive inverse

For every $x \in V$ there exist $y \in V$ such that $x + y = 0$.

- Multiplication identity

$1. x = x$ For all $x \in$ V.

- Distributive property

$\alpha(x + y) = \alpha x + \alpha y$ And $(\alpha + \beta)x = \alpha x + \beta x$ for all $\alpha, \beta \in R \quad , x \in V.$

2.7.7 Definition Subspace

A Subset U of V is called subspace of V if U is also a vector space using the same addition and scalar multiplication as on V [29]

2.7.8 Definition norm; normed vector space

A norm on a vector space V over R is a function $\|.\| : V \leftrightarrow (0, \infty]$ such that

- Positive definite

$\|x\| = 0 \; if \; and \; only \; if \; x = 0.$

- Homogeneity

$\|\alpha x\| = |\alpha|. \|x\|$ For all $\alpha \in R \; and \; x \in V.$

- Triangle inequality

$\|x + y\| \leq \|x\| + \|y\|$ For all $x, y \in V.$

A normed vector space is a pair $(V, \|.\|)$, where V is a vector space and $\|.\|$ is a norm on V. [29]

2.7.9 Theorem normed vector spaces are metric spaces

Suppose $(V, \|.\|)$ Is a normed vector space.

Define $d: V \leftrightarrow (0, \infty]$ by $d(x, y) = \|x - y\|$ then d is a metric on V. [29]

2.7.10 Definition Banach space

Banach space is a complete normed vector space. [29]

2.7.11 Definition Linear map (operator)

Suppose V and W are vector spaces. A function $T: V \to W$ is called linear if

- $T(x+y) = T(x) + T(y)$ For all $x, y \in V$.
- $T(\alpha x) = \alpha T(x)$, $\alpha \in R, x \in V$. [29]

2.7.12 Definition Bounded linear map

Suppose V and W are normed vector spaces and $T: V \to W$ is a linear map.

- The norm of T ,denoted $\|T\| = sup\{\|Tx\| : x \in V \; and \; \|x\| \leq 1\}$
- T is called bounded if $\|T\| < \infty$. [29]

2.7.13 Definition Contraction

Suppose (V, d) Is a Metric space and $T: V \to V$ is a linear operator.

T Is said to be contraction operator if there exist $0 < \alpha < 1$ such that

$d(Tf - Tg) \leq \alpha \, d(f - g)$ For all $f, g \in V$. [30]

2.7.14 Theorem Banach fixed point

Suppose (V, d) Is a complete metric space with a contraction mapping $T: V \to V$ then there exist a unique fixed point $x \in V$

i.e., $T(x) = x$. [30]

2.7.15 Definition The $L^p -$ norm of vector

For a vector $x = [x_1, x_2, x_3, ...]^T$ and for $0 < p < \infty$. The $p -$ norm are defined as

$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$. [29]

Example: The $1-$ norm

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

Example: The $2-$ norm

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

2.7.16 Definition The $\infty-$ norm

$$\|x\|_\infty = \lim_{p\to\infty} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

$$\|x\|_\infty = \max_{1\le i\le n}\{x_1, x_2, x_3, \dots x_i \dots x_n\}\ .\ [31]$$

Now for the Existence and Uniqueness of the solution of System of FIE on Banach space the fixed-point theorem gives a way to solve the system of IE by starting with initial approximation and continue in the iterative scheme.

Consider a system of FIE of the second kind

$$U(x) = F(x) + \lambda \int_a^b K(x,s)\, U(s) ds$$

Define the operator T as $T\big(U(x)\big) = U(x)$

$$T\big(U(x)\big) = F(x) + \lambda \int_a^b K(x,s)\, U(s) ds, \quad T = [T_i(x)]_{n\times 1}$$

Choose $u_i^0(x) \in [a,b]$ as an initial approximation and follow the fixed-point iteration

$$T_i(U^{k-1}) = u_i{}^k = f_i(x) + \sum_{j=1}^{n} \lambda_{ij} \int_a^b k_{ij}(x,s)\, u_j{}^{k-1} ds \qquad\qquad k \ge 1$$

After applying multiple approximations, the sequence $\{u_i^n\}$ converges to the fixed point which is the solution $U(x)$ as $n \rightarrow \infty$

2.7.17 Theorem Existence and Uniqueness

For system of FIE of the second kind in Banach space $(C[a,b], \|.\|_\infty)$, if

- $T: C[a,b] \rightarrow C[a,b]$ is a bounded linear map
- $f_i(x): [a,b] \rightarrow R$ and $k_{ij}: [a,b] \times [a,b] \rightarrow R$ are continuous
- The system satisfies the following condition (contractive mapping)

$$\lambda_i = \max_{j=1,2,\dots,n} |\lambda_{ij}| < \frac{1}{n(b-a)^2 M_i}$$

$$|K_{ij}(x,s)| \leq M_{ij} \quad and \quad M_i = \max_{j=1,2,\dots,n} M_{ij}$$

Then

- T_i has a unique fixed point
- For any initial approximation, the sequence of iterations defined by $T_i(U^{k-1}) = U^k$ converges to the fixed point.

2.8 Orthogonality

2.8.1 Theorem

Suppose x and y are elements of an inner product space. Then

$$\|\langle x, y\rangle\| \leq \|x\| \, \|y\|.$$

2.8.2 Definition

The inner product of two functions$f(x)$ and $g(x)$ on an interval $[a,b]$ is the number

$\langle f, g\rangle = \int_a^b f(x)g(x)\, dx.$ [32]

2.8.2 Definition

A set of real valued functions $\{f_1(x), f_2(x), f_3(x), \dots\}$ is said to be orthogonal on an interval $[a, b]$ if

$$\langle f_i, f_j \rangle = \int_a^b f_i(x) f_j(x)\, dx = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad [32]$$

2.8.3 Definition

A set of real valued functions $\{f_1(x), f_2(x), f_3(x), \dots\}$ is said to be orthogonal with respect to a weight function $w(x)$ on an interval $[a, b]$ if

$$\langle f_i, f_j \rangle = \int_a^b w(x) f_i(x) f_j(x)\, dx = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad [32]$$

Chapter Three

Numerical Techniques for Solving Linear System of Fredholm Integral Equations of the Second Kind

Various methods exist for solving systems of IE. These include direct approaches such as the Adomian Decomposition Method, the Laplace Transform Method, and Variational Iteration Methods. [33] or like converting the system to different type of systems, whether represented as linear equations or differential equations, often involve scenarios where direct methods are not applicable. Consequently, the significance of finding approximate solutions to such systems of integral equations becomes undeniable [28].

In this chapter we will discuss three numerical methods for solving system of FIE: Reconstruction of variational iteration method, Chebyshev collocation method, The Haar wavelet method.

3.1 Variational Iterations Method

A system of FIE can be considered in general form as

$$\begin{cases} u_1(x) = f_1(x) + \int_a^b (k_{11}(x,s)u_1 + k_{12}(x,s)u_2 + k_{13}(x,s)u_3 + \cdots + k_{1n}(x,s)u_n)\,ds \\ u_2(x) = f_2(x) + \int_a^b (k_{21}(x,s)u_1 + k_{22}(x,s)u_2 + k_{23}(x,s)u_3 + \cdots + k_{2n}(x,s)u_n)\,ds \\ \vdots \\ u_n(x) = f_n(x) + \int_a^b (k_{n1}(x,s)u_1 + k_{n2}(x,s)u_2 + k_{n3}(x,s)u_3 + \cdots + k_{nn}(x,s)u_n)ds \end{cases} \quad (3.1)$$

In applying the VIM, the following non-homogeneous system of differential equations is considered [34]

$$\begin{cases} M_1 u_1(s) + N_1\big(u_1(s), u_2(s), \dots, u_n(s)\big) = g_1(s) \\ M_2 u_2(s) + N_2\big(u_1(s), u_2(s), \dots, u_n(s)\big) = g_2(s) \\ \vdots \\ M_n u_n(s) + N_n\big(u_1(s), u_2(s), \dots, u_n(s)\big) = g_n(s) \end{cases} \quad (3.2)$$

In the above system of equations (2.2), $M_1, M_2, \dots, M_n$ Linear differential operators and $N_1, N_2, \dots, N_n$ are nonlinear operators and $g_1(s), g_2(s), \dots, g_n(s)$ are some given functions.

Based on the VIM, the correction functional for the system can be written as follows

$$\begin{cases} u_{1,(m+1)}(x) = u_{1,m}(x) + \int_a^b \lambda_1(\tau)\left(M_1\, u_{1,m} + N_1(\tilde{u}_{1,m}\,, \tilde{u}_{2,m}, \dots, \tilde{u}_{n,m}\ ) - g_1(\tau)\right) d\tau \\ u_{2,(m+1)}(x) = u_{2,m}(x) + \int_a^b \lambda_2(\tau)\left(M_2\, u_{2,m} + N_2(\tilde{u}_{1,m}\,, \tilde{u}_{2,m}, \dots, \tilde{u}_{n,m}\ ) - g_2(\tau)\right) d\tau \\ \vdots \\ u_{n,(m+1)}(x) = u_{n,m}(x) + \int_a^b \lambda_n(\tau)\left(M_1\, u_{n,m} + N_n(\tilde{u}_{1,m}\,, \tilde{u}_{2,m}, \dots, \tilde{u}_{n,m}\ ) - g_n(\tau)\right) d\tau \end{cases} \tag{3.3}$$

Where $\lambda_1, \lambda_2, \dots, \lambda_n$ represent the general Lagrange multipliers, which can be determined optimally using variational theory.

$\tilde{u}_i$ is a restricted variation, i.e., $\delta\tilde{u}_i = 0$ for $i = 1,2, \dots, n$

To apply VIM to system (3.1), the method works effectively if the kernel k_{ij} is separable i.e., $k(x,s) = g(x)h(s)$

This implies that we need to differentiate both sides of each equation in order to transform the system into a Fredholm integro-differential equation.

$u_{i,0}(x)$ can be any selective function, but preferably for fast convergence to be selected using Taylor series

$u_{i,0}(x) = u_i(0)$ For first order $\grave{u}_i$

$u_{i,0}(x) = u_i(0)\ + x\acute{u}_i(0)$ For second order $\acute{\acute{u}}_i$

$u_{i,0}(x) = u_i(0)\ + x\grave{u}_i(0) + \frac{1}{2!}x^2\acute{\acute{u}}_i(0)$ For third order $\acute{\acute{\acute{u}}}_i$

.
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.

The iteration system (3.3) will give several approximations of $(u_1, u_2, \dots, u_n)$ and consequently the solution

$$u_i(x) = \lim_{m\to\infty} u_{i,m}(x) \quad \text{for } i = 1,2, \dots, n$$

And the determination of the Lagrange multipliers is considered by

If $u_i^{(m)} + f\left(u_i(\tau), \acute{u}_i(\tau), \dots, u_i^{(m)}(\tau)\right) = 0$

$$\lambda_i(x) = (-1)^n \frac{1}{(m-1)!}(\tau - x)^{m-1} \ \text{ for } i = 1,2, \dots, n \tag{3.4}$$

3.1.1 Reconstruction of Variational Iterations Method

The reconstruction of VIM is an alternate technique that uses the Laplace transform to determine the optimal value of the Lagrange multiplier.

First, we introduce the necessary definitions and theorems needed for RVIM

3.1.1.1 Definition

Let f be a function defined for$x \geq 0$. Then the integral

$$L\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)dx$$

is said to be Laplace transform of f, provided that the integral converges. [32]

3.1.1.2 Definition

If $F(s)$ represents the Laplace transform of a function $f(x)$, then we say that $f(x)$ is the inverse Laplace transform of $F(s)$ and we write $f(x) = L^{-1}\{F(s)\}$ [32]

Linear Transform Property

Laplace transform satisfy the linear transform property

$$L\{\alpha f(x) + \beta g(x)\} = \alpha L\{f(x)\} + \beta L\{g(x)\}$$

and the inverse Laplace transform is also a linear transform; that

$$L^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha L^{-1}\{F(s)\} + \beta L^{-1}\{G(s)\}$$

where F and G are the transforms of some functions f and g

$\alpha, \beta \in R$

3.1.1.3 Theorem

If $f, f', f'', \ldots f^{(n-1)}$ are continuous on $[0, \infty)$ and $f^{(n)}(x)$ is piecewise continuous on $[0, \infty)$ then

$$L\{f^{(n)}(x)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

where $F(s) = L\{f(x)\}$ [32]

3.1.1.4 Definition

If functions f and g are piecewise continuous on the interval $[0, \infty)$, then $f * g$ is called the convolution of f and g defined by the integral

$(f * g)(x) = \int_0^x f(\tau) g(t - \tau)\, d\tau$ [32]

3.1.1.5 Theorem

If functions $f(x)$ and $g(x)$ are piecewise continuous on the interval $[0, \infty)$, then

$L\{f * g\} = L\{f(x)\}\, L\{g(x)\} = F(s)G(s)$ [32]

Method of solution

Back to the correction functional (2.3), $N_i\left(\tilde{u}_{1,m}, \tilde{u}_{2,m}, \dots, \tilde{u}_{n,m}\right)$ is a restricted variation

i.e. $N_i\left(\tilde{u}_{1,m}, \tilde{u}_{2,m}, \dots, \tilde{u}_{n,m}\right) = 0$, the general form of Lagrange multiplier is found to be of the form $\lambda_i = \overline{\lambda}_i(x - \tau) \qquad i = 1,2, \dots, n$

We now proceed to take the Laplace transform on both sides of each integral equation within the system (3.3)

$$L\left[u_{i,m+1}(x)\right] = L\left[u_{i,m}(x)\right] + L\left[\int_a^b \bar{\lambda}_i(x - \tau)\left[M\left(u_{i,m}\right) + N\left(\tilde{u}_{1,m}, \tilde{u}_{2,m}, \dots, \tilde{u}_{n,m}\right) - g_i(\tau)\right] d\tau\right]$$

By the convolution theory, therefore

$$L\left[u_{i,m+1}(x)\right] = L\left[u_{i,m}(x)\right] + L\left[\bar{\lambda}_i(x) * \left(M\left(u_{i,m}(x)\right) + N\left(\tilde{u}_{1,m}, \tilde{u}_{2,m}, \dots, \tilde{u}_{n,m}\right) - g_i(\tau)\right)\right]$$

$$L\left[u_{i,m+1}(x)\right] = L\left[u_{i,m}(x)\right] + L\left[\bar{\lambda}_i(x)\right] . L\left[M\left(u_{i,m}(x)\right) + N\left(\tilde{u}_{1,m}, \tilde{u}_{2,m}, \dots, \tilde{u}_{n,m}\right) - g_i(\tau)\right] \tag{3.5}$$

To determine the optimal value of $\bar{\lambda}(x - \tau)$, we compute the variation of expression (3.5) with respect to $u_m(x)$

$$\begin{aligned}&\frac{\delta}{\delta u_{i,m}} L\left[u_{i,m+1}(x)\right]\\ &= \frac{\delta}{\delta u_{i,m}} L\left[u_{i,m}(x)\right]\\ &+ \frac{\delta}{\delta u_{i,m}} \left(L\left[\bar{\lambda}_i(x)\right].L\left[M\left(u_m(x)\right) + N\left(\tilde{u}_{1,m}, \tilde{u}_{2,m}, \dots, \tilde{u}_{n,m}\right) - g_i(\tau)\right]\right) \quad (3.6)\end{aligned}$$

Hence

$$\frac{\delta}{\delta u_{i,m}} L\left[u_{i,m+1}(x)\right] = \frac{\delta}{\delta u_{i,m}} L\left[u_{i,m}(x)\right] + \frac{\delta}{\delta u_{i,m}} L\left[\bar{\lambda}_i(x)\right].L\left[M\left(u_{i,m}(x)\right)\right]$$

$$\delta\, L\left[u_{i,m+1}(x)\right] = \delta\, L\left[u_{i,m}(x)\right] + \delta\, L\left[\bar{\lambda}_i(x)\right].L\left[M\left(u_{i,m}(x)\right)\right] \qquad (3.7)$$

It is assumed that M represents a linear differential operator with constant coefficients, defined as follows.

$$Mu(x) = a_n u^{(n)} + a_{n-1} u^{(n-1)} + a_{n-2} u^{(n-2)} + \cdots + a_2 \acute{\acute{u}} + a_1 \acute{u} + a_0 u \qquad (3.8)$$

The Laplace transform to (3.8) of both sides

$$L[Mu(x)] = L\left[a_n u^{(n)} + a_{n-1} u^{(n-1)} + a_{n-2} u^{(n-2)} + \cdots + a_2 \acute{\acute{u}} + a_1 \acute{u} + a_0 u\right]$$

$$= a_n L\left[u^{(n)}\right] + a_{n-1} L\left[u^{(n-1)}\right] + a_{n-2} L\left[u^{(n-2)}\right] + \cdots + a_0 L[u] \qquad (3.9)$$

$$a_n L\left[u^{(n)}\right] = a_n s^n L[u] - a_n \sum_{k=1}^{n} s^{k-1}\, u^{(n-k)}(0)$$

$$a_{n-1} L\left[u^{(n-1)}\right] = a_{n-1} s^{n-1} L[u] - a_{n-1} \sum_{k=1}^{n-1} s^{k-1}\, u^{(n-1-k)}(0)$$

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.

$L[a_1 \acute{u} + a_0 u] = a_1 s\, L[u] - a_1 u(0) + a_0 L[u]$

So, the variation with respect to u is

$\delta\, L\left[a_n u^{(n)}\right] = a_n s^n\, L[\delta u]$

$\delta\, L\left[a_{n-1} u^{(n-1)}\right] = a_{n-1} s^{n-1}\, L[\delta u]$

$\delta\, L[a_1 \acute{u}] = a_1 s\, L[\delta u]$

$\delta\, L[a_0 u] = a_0\, L[\delta u]$

Then, for (3.9) we have

$$L[Mu(x)] = L[\delta u] \sum_{k=0}^{n} a_k\, s^k \tag{3.10}$$

Substituting (3.10) in (3.7)

$$L[\delta\, u_{m+1}(x)] = L[\delta\, u_m(x)] + L\left[\,\bar{\lambda}(x)\right]. L[\delta u_m(x)] \left(\sum_{k=0}^{n} a_k\, s^k\right)$$

$$L[\delta\, u_{m+1}(x)] = L[\delta\, u_m(x)] \left[1 + L\left[\bar{\lambda}(x)\right] \left(\sum_{k=0}^{n} a_k\, s^k\right)\right] \tag{3.11}$$

The extreme condition of $u_{m+1}(x)$ requires that $\delta\, u_{m+1}(x) = 0$

$$L\left[\bar{\lambda}(x)\right] = \frac{-1}{\sum_{k=0}^{n} a_k s^k} \tag{3.12}$$

Taking the Laplace inverse for (3.12) gives the optimal value of $\bar{\lambda}$ and thus we have the iteration formula

$$L\left[u_{i,m+1}(x)\right] = L\left[u_{i,m}(x)\right]$$
$$+ L\left[\int_a^b \bar{\lambda}_i(x-\tau)\left[M\, u_{im} + N\left(\tilde{u}_{1,m}\,, \tilde{u}_{2,m}, \dots, \tilde{u}_{n,m}\right) - g_i(\tau)\right] d\tau\right] \tag{3.13}$$

3.2 Chebyshev Collocation Method

The Chebyshev collocation method has been introduced as an approach to solve systems of IE using Chebyshev polynomials. This technique converts the integral system into a matrix equation by utilizing Chebyshev collocation points, where the unknown in this equation corresponds to a Chebyshev coefficient matrix. Initially formulated for solving systems of integro-differential equations, the Chebyshev collocation method has been adapted for systems of differential equations and subsequently applied to integral systems. [35]

The truncated Chebyshev series is defined by

$$u_i(x) = \sum_{j=0}^{n} a_{ij}\, T_j\,(x) \qquad \begin{matrix} i = 1\,, 2\,, 3\,, \dots n \\ -1 \le x \le 1 \end{matrix} \tag{3.14}$$

Where $T_j(x)$ denote the Chebyshev polynomials of the first kind, a_{ij} are the unknown coefficients to be determined and n is chosen any positive integer.

The Chebyshev polynomials $\{T_j(x)\}$ are orthogonal on $(-1\,,1)$ with respect to the weight function

$w(x) = \frac{1}{\sqrt{1-x^2}}$ and $T_j(x)$ has exactly j zeros within the interval $(-1\,,1)$.

For $x \in [-1,1]$, define

$$T_n(x) = cos\left(ncos^{-1}(x)\right) \qquad for\; n \geq 0$$

$$T_0(x) = 1 \;\;,\;\; T_1(x) = x$$

Then a recursive relation is derived

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad for\; n \geq 1$$

The orthogonality property of this polynomials is given as

$$\int_{-1}^{1} T_n(x)\, T_m(x) w(x) dx = \begin{cases} \frac{\pi}{2}, & n = m \neq 0 \\ 0\,, & n \neq m \\ \pi, & n = m = 0 \end{cases}$$

Method of solution

The solutions of system (3.1) are assumed to be representable as a truncated Chebyshev series.

$$u_i(x) = T(x)A_i$$

$$T(x) = \begin{bmatrix} T_0(x) & T_1(x) & T_2(x) & \dots & T_n(x) \end{bmatrix}$$

$$A_i = \begin{bmatrix} a_{i0} & a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}^T$$

The matrix $U(x)$ is defined as a column matrix of unknown functions which can be expressed by

$U(x) = T(x)A$, so that,

$$\begin{bmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \\ . \\ \vdots \\ u_n(x) \end{bmatrix} = \begin{bmatrix} T(x) & 0 & \dots & 0 \\ 0 & T(x) & \dots & 0 \\ 0 & . & \dots & . \\ \vdots & \vdots & \dots & . \\ & & \dots & . \\ 0 & 0 & \dots & T(x) \end{bmatrix} . \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ . \\ \vdots \\ A_n \end{bmatrix}$$

Substituting into the system of FIE

$$\sum_{j=0}^{n} a_{ij}\, T_j\,(x) \;= f_i(x) + \int_a^b \left(\sum_{j=1}^{n} k_{ij}(x,s) \sum_{j=0}^{n} a_{ij}\, T_j\,(x)\right) ds \qquad (3.15)$$

$i = 1\,,2\,,3\,, \dots .\, m$

Chebyshev collocation points defined as

$$x_k = \cos\frac{(2k+1)\pi}{2n} \qquad , \qquad k = 0,1,2, \dots , n-1$$

Substituting this collocation points in system (3.15)

$$\sum_{j=0}^{n} a_{ij}\, T_j\,(x_k) \;= f_i(x_k) + \int_a^b \left(\sum_{j=1}^{n} k_{ij}(x_k,s) \sum_{j=0}^{n} a_{ij}\, T_j\,(x_k)\right) ds$$

$$\sum_{j=0}^{n} a_{ij}\, T_j\,(x_k) - \int_a^b \left(\sum_{j=1}^{n} k_{ij}(x_k,s) \sum_{j=0}^{n} a_{ij}\, T_j\,(x_k)\right) ds = f_i(x_k) \qquad (3.16)$$

$i = 1,2,3, \dots , m$

Write (3.16) in matrix form

$$T(x_k)A - \int_a^b (K\, .\; T(x_k)A)dt = F$$

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ & \vdots & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

$$\left(T(x_k) - \int_a^b \left(K\,.\,T(x_k)\right)ds\right).A = F$$

$$A = \left(T(x_k) - \int_a^b \left(K\,.\,T(x_k)\right)ds\right)^{-1}\,.\,F \tag{3.17}$$

This corresponds to a system of linear algebraic equations involving the unknown Chebyshev coefficients. By solving these equations, the coefficients a_{ij} can be determined, allowing us to obtain the solution to the Fredholm integral system expressed in the truncated Chebyshev series.

3.2.1 Shifted Chebyshev polynomials

To use Chebyshev polynomials over the interval $[0\,,L]$, we define the shifted Chebyshev polynomials as follows

$$T_i^*(x) = T_i\left(\frac{2}{L}x - 1\right), \quad i = 1\,,2\,,3\,,\dots n \tag{3.18}$$

Then, we have

$T_0^*(x) = 1\ ,\ T_1^*(x) = \frac{2}{L}x - 1$

$T_{n+1}^*(x) = 2\left(\frac{2}{L}x - 1\right)T_n^*(x) - T_{n-1}^*(x) \qquad for\ n \geq 1$

They are orthogonal on the interval $[0,1]$ with the weight function $w(x) = \frac{1}{\sqrt{x-x^2}}$ and the zeros of $T_n^*(x)$ are $x_i = \frac{1}{2} + \frac{1}{2}cos\left(\frac{2i+1}{2n}\pi\right)$, $\qquad i = 0,1,2,\dots,n-1$

After collocating at the distinct $N+1$ roots of the shifted Chebyshev polynomials

$$\sum_{j=0}^{n} a_{ij}\,T_j^*\,(x_k) - \int_0^b \left(k_{ij}(x_k,s)\sum_{j=0}^{n} a_{ij}\,T_j^*\,(x_k)\right)ds = f_i(x_k) \tag{3.19}$$

the solution will be as (3.16)

$$A = \left(T^*(x_k) - \int_0^b \left(K\,.\,T^*(x_k)\right)ds\right)^{-1}\,.F \tag{3.20}$$

3.3 The Haar Wavelet Method

Let’s consider the interval $x \in [A, B]$, we divide the interval $[A, B]$ into $2M$ subintervals of equal length; each interval has a length $\Delta x = \frac{B-A}{2M}$, the j^{th} Haar wavelet family is defined as follows

$$h_j(x) = \begin{cases} 1\,, & \tau_1 \le x < \tau_2 \\ -1\,, & \tau_2 \le x < \tau_3 \\ 0\,, & elsewhere \end{cases} \tag{3.21}$$

$\tau_1 = A + 2c\mu\Delta x$

$\tau_2 = A + (2c + 1)\mu\Delta x$

$\tau_3 = A + 2(c + 1)\mu\Delta x$

$$\mu = \frac{M}{m}$$

The integer $m = 2^i$ where $i = 0, 1, 2, \dots, I$ indicates the level of the wavelet and as j increases the wave becomes narrower so it’s called the *dilatation parameter* and $M = 2^I$ is the maximal level of resolution.

Next, the parameter $c = 0, 1, 2, \dots, m - 1$ localize the position of the wavelet so it’s called the *translation parameter*. The wavelet number identified as $j = m + c + 1$

$(m > c \; and \; m \neq c)$ [36] [37]

The equations are valid for $j \geq 2$, for $j = 1$ the function $h_1(x)$ is defined as

$$h_1(x) = \begin{cases} 1\,, & 0 \le x < 1 \\ 0\,, & elsewhere \end{cases} \tag{3.22}$$

is called the scaling function

Table 3.3

Index computation for Haar wavelet functions

i	0	1	1	2	2	2	…
c	0	0	1	0	1	2	…
j	2	3	4	5	6	7	…

Simple calculations show that when $j = 2$, $x \in [0,1[$ the function $h_2(x)$ is defined as

$$h_2(x) = \begin{cases} 1\,, & 0 \le x < \frac{1}{2} \\ -1\,, & \frac{1}{2} \le x < 1 \\ 0\,, & elsewhere \end{cases} \tag{3.23}$$

When $j = 3$, $x \in [0,1[$ the function $h_3(x)$ is defined as

$$h_3(x) = \begin{cases} 1\,, & 0 \le x < \frac{1}{4} \\ -1\,, & \frac{1}{4} \le x < \frac{2}{4} \\ 0\,, & elsewhere \end{cases} \tag{3.24}$$

When $j = 4$, $x \in [0,1[$ the function $h_4(x)$ is defined as

$$h_4(x) = \begin{cases} 1\,, & \frac{2}{4} \le x < \frac{3}{4} \\ -1\,, & \frac{3}{4} \le x < \frac{4}{4} \\ 0\,, & elsewhere \end{cases} \tag{3.25}$$

We see that the Haar wavelets are orthogonal to each other

$$\int_A^B h_i(x)h_j(x)dx = \begin{cases} 0\,, & i \neq j \\ 1\,, & i = j \end{cases}$$

The Haar wavelet series expansion for a given function $g(x)$ is

$$g(x) = \sum_{j=1}^{2M} b_j h_j(x) \tag{3.26}$$

where $x \in [A, B]$, $b_{j's}$ the wavelet coefficients are to be determined. In order to use the Haar wavelets for the numerical solutions we must put them into a discrete form, so we will use the collocations method.

The collocation points are defined as $x_k = A + (k - 0.5)\Delta x$ [37]

and the discrete form of $g(x)$ is

$$g(x_k) = \sum_{j=1}^{2M} b_j h_j(x_k) \tag{3.27}$$

$k = 1\,, 2\,, 3\,, \ldots, 2M$

Method of solution

Let us consider a system of FIE

$$u_i(x) = f_i(x) + \int_a^b \left(\sum_{j=1}^{N} k_{ij}(x,s) u_j(s) \right) ds\,, \qquad 1 \leq i \leq N$$

It’s expedient to write the system in the matrix form

$$U(x) = F(x) + \int_a^b (\sum K(x,s)U(s))ds \tag{3.28}$$

$where$

$$U(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \\ \cdot \\ \cdot \\ u_N(x) \end{bmatrix}, F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ \cdot \\ \cdot \\ f_N(x) \end{bmatrix}, K(x,s) =$$

$$\begin{bmatrix} k_{11}(x,s) & k_{12}(x,s) & k_{13}(x,s) & \ldots & k_{1N}(x,s) \\ k_{21}(x,s) & k_{22}(x,s) & k_{21}(x,s) & \ldots & k_{2N}(x,s) \\ k_{31}(x,s) & k_{32}(x,s) & k_{33}(x,s) & \ldots & k_{3N}(x,s) \\ \vdots & \vdots & \cdot & & \cdot \\ k_{N1}(x,s) & k_{N2}(x,s) & k_{N3}(x,s) & \cdots & k_{NN}(x,s) \end{bmatrix}$$

Approximate the unknown functions $u_i(x)$ using Haar wavelet series expansion to get the following system

$$\sum_{j=1}^{2M} b_{i,j}\, h_j(x_k) = f_i(x_k) + \int_0^{x_k} \left(\sum_{j=1}^{N} k_{ij}(x_k,s) \sum_{j=1}^{2M} b_{i,j}\, h_j(s) \right) ds \tag{3.29}$$

$$\begin{bmatrix} \sum_{j=1}^{2M} b_{1,j}\, h_j(x_k) \\ \sum_{j=1}^{2M} b_{2,j}\ h_j(x_k) \\ \sum_{j=1}^{2M} b_{3,j}\ h_j(x_k) \\ . \\ . \\ \sum_{j=1}^{2M} b_{N,j}\ h_j(x_k) \end{bmatrix} = \begin{bmatrix} f_1(x_k) \\ f_2(x_k) \\ f_3(x_k) \\ . \\ . \\ f_N(x_k) \end{bmatrix} + \int_0^{x_k} \begin{bmatrix} k_{11}(x_k,s) & k_{12}(x_k,s) & k_{13}(x_k,s) & \dots & k_{1N}(x_k,s) \\ k_{21}(x_k,s) & k_{22}(x_k,s) & k_{21}(x_k,s) & \dots & k_{2N}(x_k,s) \\ k_{31}(x_k,s) & k_{32}(x_k,s) & k_{33}(x_k,s) & \dots & k_{3N}(x_k,s) \\ \vdots & \vdots & \ddots & & \ddots \\ k_{N1}(x_k,s) & k_{N2}(x_k,s) & k_{N3}(x_k,s) & \cdots & k_{NN}(x_k,s) \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{2M} b_{1,j}\ h_j(s) \\ \sum_{j=1}^{2M} b_{2,j}\ h_j(s) \\ \sum_{j=1}^{2M} b_{3,j}\ h_j(s) \\ . \\ . \\ \sum_{j=1}^{2M} b_{N,j}\ h_j(s) \end{bmatrix} ds$$

$$f_i(x_k) = \sum_{j=1}^{2M} b_{i,j}\ h_j(x_k) - \int_0^{x_k} \left(\sum_{j=1}^{N} k_{ij}(x_k,s) \sum_{j=1}^{2M} b_{i,j}\ h_j(s) \right) ds$$

$$f_i(x_k) = \sum_{j=1}^{2M} b_{i,j} \left[h_j(x_k) - \int_0^{x_k} \left(\sum_{j=1}^{N} k_{ij}(x_k,s)\ h_j(s) \right) ds \right]$$

Assume that

$$v_{i,j} = \int_0^{x_k} \left(\sum_{j=1}^{N} k_{ij}(x_k,s)\ h_j(s) \right) ds$$

$$f_i(x_k) = \sum_{j=1}^{2M} b_{i,j} \left[h_j(x_k) - v_{i,j} \right] \tag{3.30}$$

$$\sum_{j=1}^{2M} b_{i,j} = f_i(x_k) \left[h_j(x_k) - v_{i,j} \right]^{-1} \tag{3.31}$$

The Wavelet coefficients $b_{i's}$, $i = 1,2,3,\dots,2M$ are obtained by solving the 2M system of equations in (3.31)

Then substituting these coefficients in (3.26) to obtain the Haar – Wavelet solution at the collocation points

Chapter Four

Numerical Examples and Results

This chapter introduces several numerical examples to evaluate the accuracy and convergence of the methods discussed in Chapter Two. Furthermore, a comparison between the exact solutions and their approximations will be presented both in tabular form and through graphical representations.

4.1 Example

$$\begin{cases} u(x) = x^2 - \frac{11}{12}x + \int_0^1 xsu(s)\,ds + \int_0^1 xs^2v(s)\,ds + \int_0^1 xs^3w(s)\,ds \\ v(x) = 1 - \frac{11}{12}x + \int_0^1 xsu(s)\,ds + \int_0^1 xs^2v(s)\,ds + \int_0^1 xs^3w(s)\,ds \\ w(x) = x^3 + \frac{1}{12}x + \int_0^1 xsu(s)\,ds + \int_0^1 xs^2v(s)\,ds + \int_0^1 xs^3w(s)\,ds \end{cases} \tag{4.1}$$

System (4.1) has the exact solution $u(x) = x^2,\ v(x) = x + \ x^3\ and\ \ w(x) = 1$

We will find the approximate solutions of system (4.1) by the following methods

4.1.1 Reconstruction of Variational Iterations Method

Differentiating both sides of system (4.1) with respect to x using Leibniz rule gives

$$\begin{cases} u'(x) = 2x - \frac{11}{12} + \int_0^1 su(s)\,ds + \int_0^1 s^2v(s)\,ds + \int_0^1 s^3w(s)\,ds \\ v'(x) = \frac{-11}{12} + \int_0^1 su(s)\,ds + \int_0^1 s^2v(s)\,ds + \int_0^1 s^3w(s)\,ds \\ w'(x) = 3x^2 + \frac{1}{12} + \int_0^1 su(s)\,ds + \int_0^1 s^2v(s)\,ds + \int_0^1 s^3w(s)\,ds \end{cases} \tag{4.2}$$

With the initial conditions

$(u_0, v_0, w_0) = \big(u(0), v(0), w(0)\big) = (0\,, 1\,, 0)$

By Formula (3.12) we have $\lambda_i(x) = -1$

The correction functional is given by

$$L[u_{n+1}(x)] =$$
$$L[u_n(x)] - L\left[\int_0^x \left(u_n{}'(\varepsilon) - 2\varepsilon + \frac{11}{12} - \int_0^1 su_n(s)\,ds - \int_0^1 s^2 v_n(s)\,ds - \int_0^1 s^3 w_n(s)\,ds\right) d\varepsilon\right]$$
$$L[v_{n+1}(x)] =$$
$$L[v_n(x)] - L\left[\int_0^x \left(v_n{}'(\varepsilon) - \frac{11}{12} - \int_0^1 su_n(s)\,ds - \int_0^1 s^2 v_n(s)\,ds - \int_0^1 s^3 w_n(s)\,ds\right) d\varepsilon\right] \quad (4.3)$$
$$L[w_{n+1}(x)] =$$
$$L[w_n(x)] - L\left[\int_0^x \left(w_n{}'(\varepsilon) - 3\varepsilon^2 - \frac{1}{12} - \int_0^1 su_n(s)\,ds - \int_0^1 s^2 v_n(s)\,ds - \int_0^1 s^3 w_n(s)\,ds\right) d\varepsilon\right]$$

Table (4.1) shows the first 8 iterations of correction functional for system (4.1) using RVIM

Table 4.1
The first 8 iterations of correction functional for system (4.1) using RVIM

n	$u_n(x)$	$v_n(x)$	$w_n(x)$
1	$x^2 - 0.6666666666x$	$x^3 + 0.3333333333x$	$1 - 0.6666666666x$
2	$x^2 - 0.5222222222x$	$x^3 + 0.4777777777x$	$1 - 0.5222222222x$
3	$x^2 - 0.4090740740x$	$x^3 + 0.5909259259x$	$1 - 0.4090740740x$
4	$x^2 - 0.3204413580x$	$x^3 + 0.679558641x$	$1 - 0.3204413580x$
5	$x^2 - 0.2510123971x$	$x^3 + 0.7489876028x$	$1 - 0.2510123971x$
6	$x^2 - 0.1966263777x$	$x^3 + 0.8033736222x$	$1 - 0.1966263777x$
7	$x^2 - 0.1540239958\text{x}$	$x^3 + 0.8459760041x$	$1 - 0.1540239958x$
8	$x^2 - 0.1206521301\text{x}$	$x^3 + 0.8793478698x$	$1 - 0.1206521301x$

It’s obvious that the following successive iterations gives the exact solutions of system (4.1)

$$u(x) = \lim_{n\to\infty} u_n(x) = x^2$$

$$v(x) = \lim_{n\to\infty} v_n(x) = x^3 + x$$

$$w(x) = \lim_{n\to\infty} w_n(x) = 1$$

A comparison of functions $u(x)$ and $v(x)$ exact and approximate solution with $n = 50$ for system (4.1) using RVIM with $n = 50$ is shown in figure (4.1)

Figure 4.1

Exact and estimated solution comparison of functions $u(x)$ *and* $v(x)$ *with* $n = 50$ *for system (4.1) using RVIM*

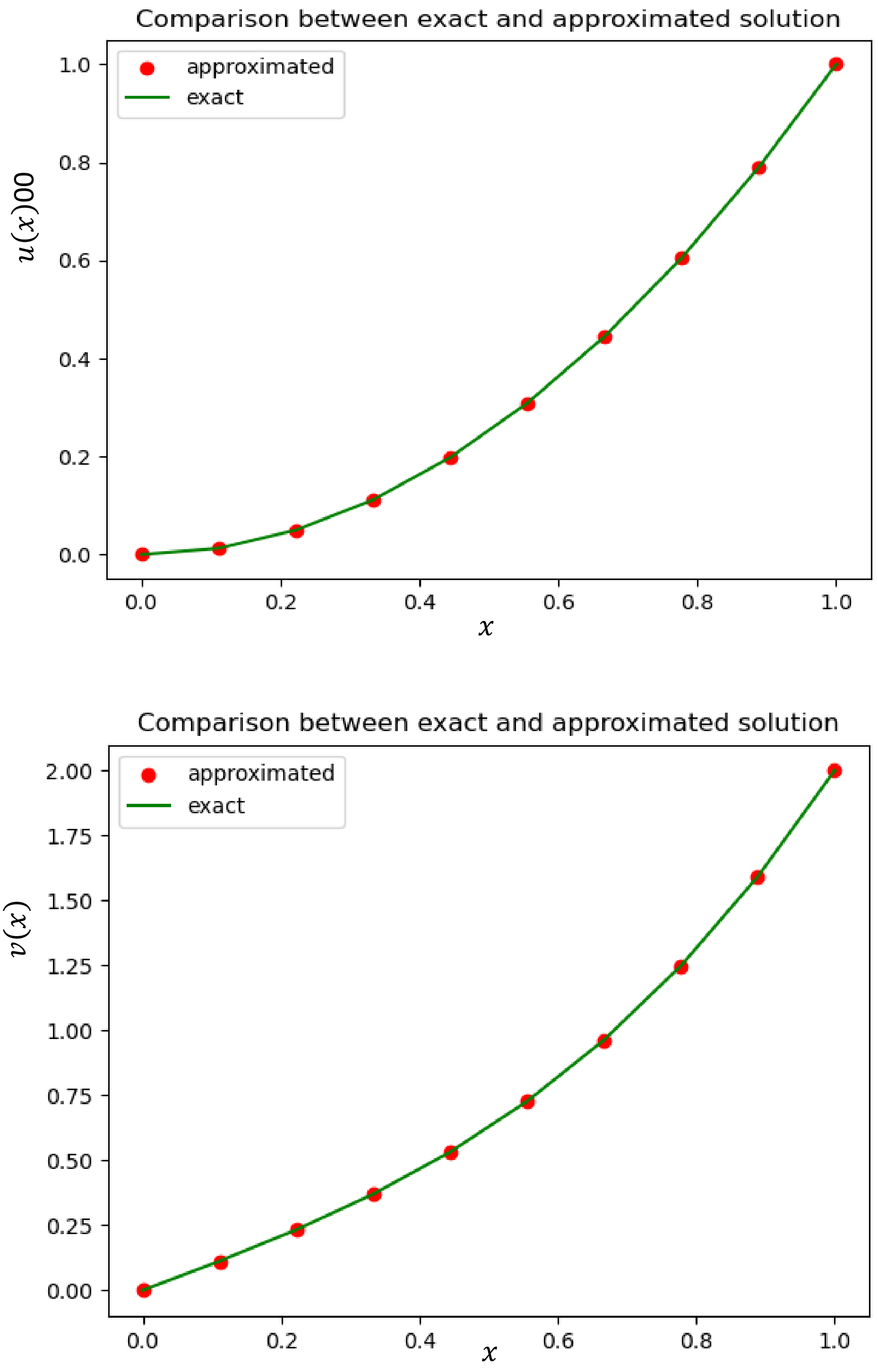


Table (4.2) shows $u(x)$ exact and approximate solution comparison and absolute error with $n = 50$ for system (4.1) using RVIM

Table (4.3) shows $v(x)$ exact and approximate solution comparison and absolute error with $n = 50$ for system (4.1) using RVIM

Table (4.4) shows $w(x)$ exact and approximate solution comparison and absolute error with $n = 50$ for system (4.1) using RVIM as shown in appendix A.

Table (4.5) shows maximum absolute error of each function of system (4.1) with $n = 50$ for system (4.1) using RVIM

Table 4.5

Maximum absolute error of each function of system (4.1) with $n = 50$ *using RVIM*

Function	Maximum error
$u(x) = x^2$	5.41×10^{-6}
$v(x) = x^3 + x$	5.41×10^{-6}
$w(x) = 1$	5.41×10^{-6}

4.1.2 Chebyshev Collocation Method

With $n = 3$,

$$f_i(x) = \sum_{k=0}^{3} a_{ik} T_k(x), \qquad i = 1,2,3$$

So, we have

$$T(x) = \left[1, 2x - 1,\ 8\left(x - \tfrac{1}{2}\right)^2 - 1, -6x + 32\left(x - \tfrac{1}{2}\right)^3 + 3\right]$$

Calculating collocation points which are the zeros of $T_4(x)$ and substituting in (3.18), gives the coefficients

$$A_1 = [0.3750 \quad 0.5 \quad 0.1249 \quad 4.813034903046741 \times 10^{-17}]^T$$

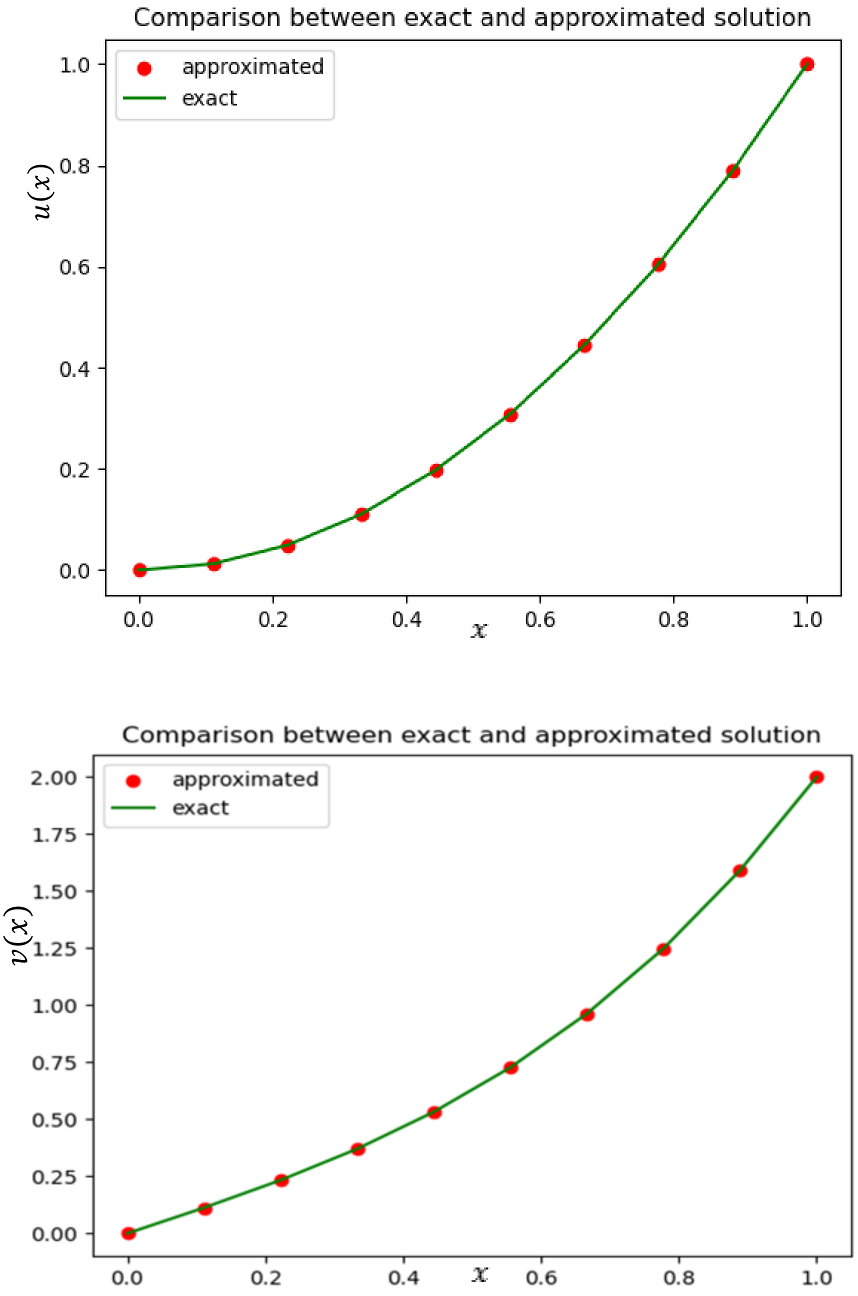
$$A_2 = [0.8125 \quad 0.96875 \quad 0.18749 \quad 0.03125]^T$$

$$A_3 = [1 \quad 2.294694835036 \times 10^{-16} \quad -7.85046229341 \times 10^{-17} \quad 4.248639577543 \times 10^{-17}]^T$$

A comparison of functions $u(x)$ and $v(x)$ exact and approximate solution with $n = 3$ for system (4.1) using Chebyshev Collocation Method is shown in figure (4.2)

Figure 4.2

Exact and estimated solution comparison of functions $u(x)$ *and* $v(x)$ *with* $n = 3$ *for system (4.1) using Chebyshev Collocation Method*



A comparison of $w(x)$ exact and approximate solution with n = 3 for system (4.1) using Chebyshev Collocation Method is shown in figure (4.3)

Figure 4.3

Exact and estimated solution comparison of $w(x)$ with n = 3 for system (4.1) using Chebyshev Collocation Method

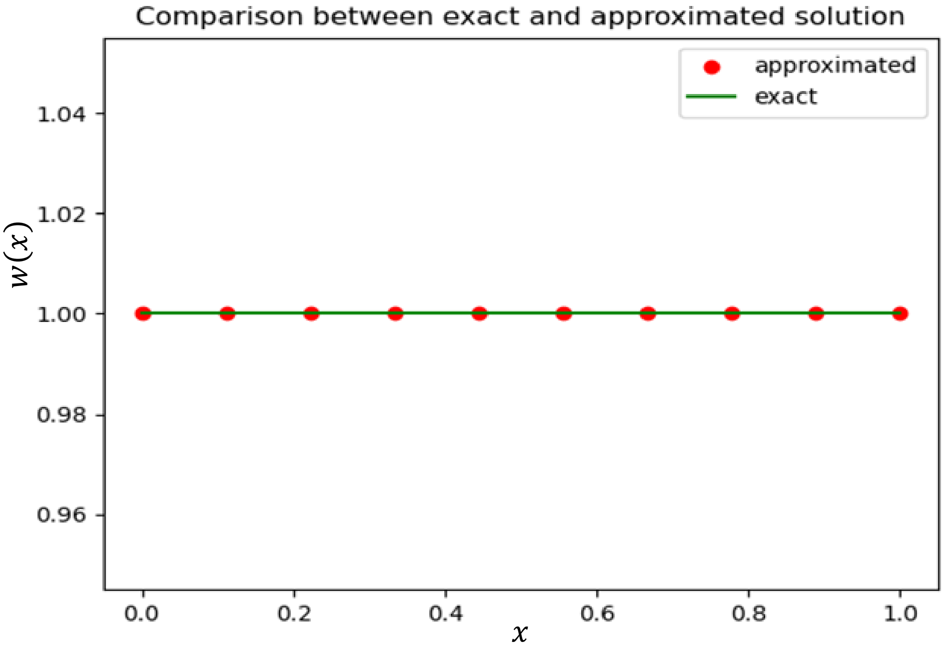


Table (4.6) shows $u(x)$ exact and approximate solution comparison and absolute error with $n = 3$ for system (4.1) using Chebyshev Collocation Method

Table (4.7) shows $v(x)$ exact and approximate solution comparison and absolute error with $n = 3$ for system (4.1) using Chebyshev Collocation Method

Table (4.8) shows $w(x)$ exact and approximate solution comparison and absolute error with $n = 3$ for system (4.1) using Chebyshev Collocation Method As shown in appendix A.

Table (4.9) shows maximum absolute error of each function of system (4.1) with $n = 3$ for system (4.1) using Chebyshev Collocation Method

Table 4.9

Maximum absolute error of each function of system (4.1) with $\mathbf{n} = \mathbf{3}$ *using Chebyshev Collocation Method*

Function	Maximum error
$u(x) = x^2$	$4.44089209850063\times 10^{-16}$
$v(x) = x^3 + x$	$2.22044604925031\times 10^{-16}$
$w(x) = 1$	$6.66133814775094\times 10^{-16}$

4.1.3 Haar Wavelet Method

Set the dilation parameter I = 3

M = 8

j = 0, 1, 2… 16

$$f_i(x) = \sum_{k=0}^{16} b_{ik} h_k(x)\,, \qquad\qquad i = 1,2,3 \tag{4.4}$$

Substituting in (3.31) and solving 16×16 system of equations gives the coefficients of the functions

A comparison of functions $u(x)$ and $v(x)$ exact and approximate solution with $I = 3$ for system (4.1) using Haar Wavelet Method is shown in Figure (4.4)

Figure 4.4

Exact and estimated solution comparison of functions $u(x)$ *and* $v(x)$ *with* $I = 3$ *for system (4.1) using Haar Wavelet Method*

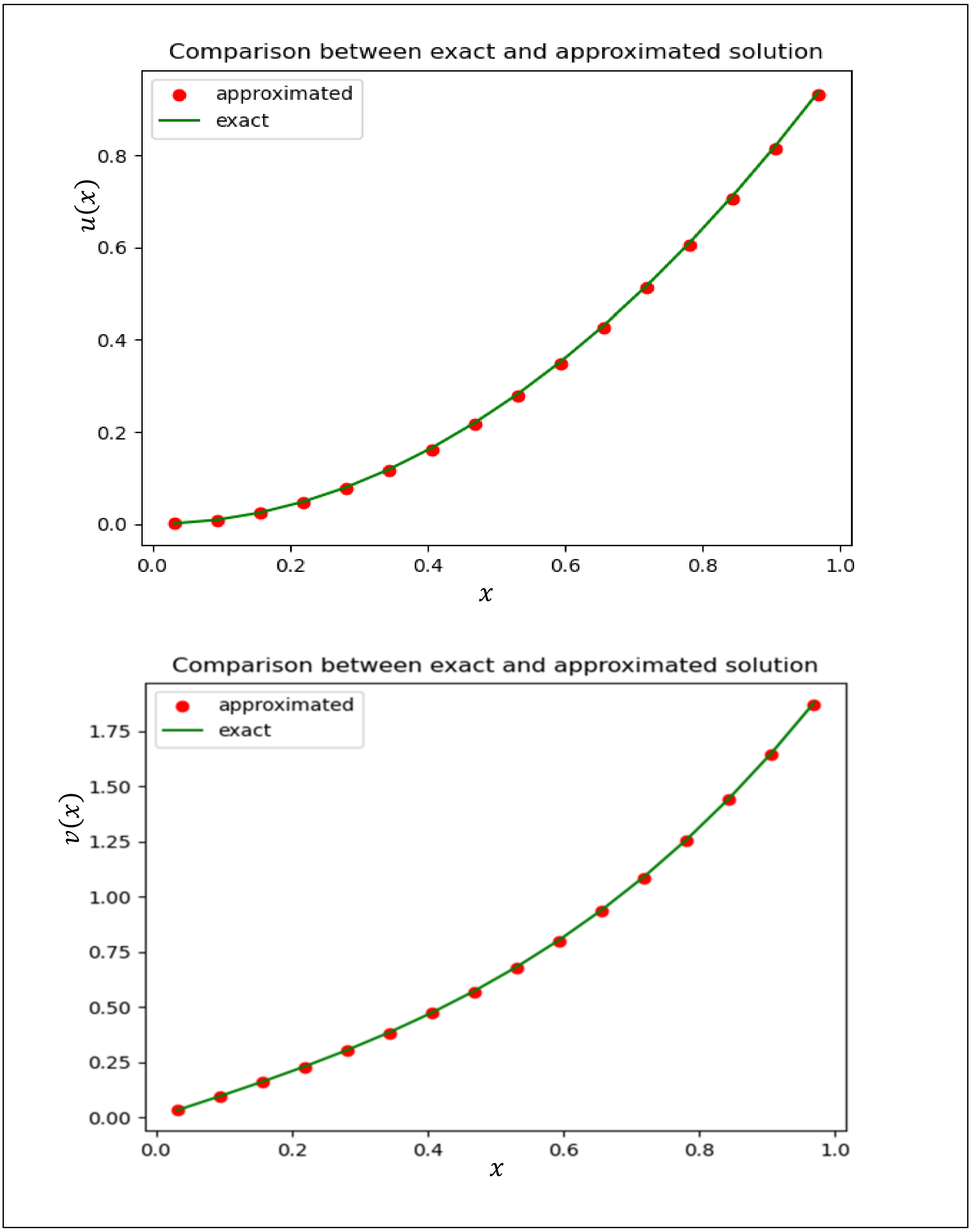


Table (4.10) shows $u(x)$ exact and approximate solution comparison and absolute error at each collocation point with $I = 3$ for system (4.1) using Haar Wavelet Method

Table (4.11) shows $v(x)$ exact and approximate solution comparison and absolute error at each collocation point with $I = 3$ for system (4.1) using Haar Wavelet Method

Table (4.12) shows $w(x)$ exact and approximate solution comparison and absolute error at each collocation point with $I = 3$ for system (4.1) using Haar Wavelet Method As shown in appendix A.

Table (4.13) shows maximum absolute error of each function of system (4.1) with $I = 3$ using Haar Wavelet Method

Table 4.13

Maximum absolute error of each function of system (4.1) with $I = 3$ using Haar Wavelet Method

Function	Maximum error
$u(x) = x^2$	$6.87814396052833\times 10^{-3}$
$v(x) = x^3 + x$	$6.87814396052877\times 10^{-3}$
$w(x) = 1$	$6.87814396052833\times 10^{-3}$

In example (4.1) we approximated the solution of a linear nonhomogeneous system of 3 equations with polynomial kernels and as a result, RVIM and Haar wavelet method provided good approximations but it is evident that the Chebyshev collocation method provides the most precise results.

4.2 Example

$$\begin{cases} u(x) = f_1(x) + \int_0^1 2xs^2u(s)\,ds + \int_0^1 x^2sv(s)\,ds \\ v(x) = f_2(x) + \int_0^1 -xsu(s)\,ds + \int_0^1 x^2(s+1)v(s)\,ds \end{cases} \tag{4.5}$$

where,

$$f_1(x) = x^2\left(\frac{2}{e} - 1\right) + x(4 - 2e) + e^x$$

$$f_2(x) = x^2\left(\frac{3}{e} - 2\right) + x + e^{-x}$$

System (4.5) has the exact solution $u = e^x \ and \ \ v = e^{-x}$

We will find the approximate solutions of system (4.5) by the aforementioned methods

4.2.1 Reconstruction of Variational Iterations Method

With initial condition

$(u_0, v_0) = \left(u(0), v(0)\right) = (1\,,1\,)$

By Formula (3.12) we have $\lambda_i(x) = -1$

The correction functional is given by

$$L[u_{n+1}(x)] = L[u_n(x)] - L\left[\int_0^x \left(u_n{}'(\varepsilon) - f_1'(x) - x\int_0^1 s^2 u_n(s)\,ds - 2x\int_0^1 s v_n(s)\,ds\right) d\varepsilon\right]$$

$$L[v_{n+1}(x)] = L[v_n(x)] - L\left[\int_0^x \left(v_n{}'(\varepsilon) - f_2'(x) + \int_0^1 s u_n(s)\,ds - 2x\int_0^1 (s+1) v_n(s)\,ds\right) d\varepsilon\right] \tag{4.6}$$

$f_1'(x) = 2x\left(\frac{2}{e} - 1\right) + (4 - 2e) + e^x$

$f_2'(x) = 2x\left(\frac{3}{e} - 2\right) + 1 + e^{-x}$

A comparison of functions $u(x)$ and $v(x)$ exact and approximate solution with $n = 50$ for system (4.5) using RVIM is shown in Figure (4.5)

Figure 4.5

Exact and estimated solution comparison of functions $u(x)$ *and* $v(x)$ *with* $n = 50$ *for system (4.5) using RVIM*

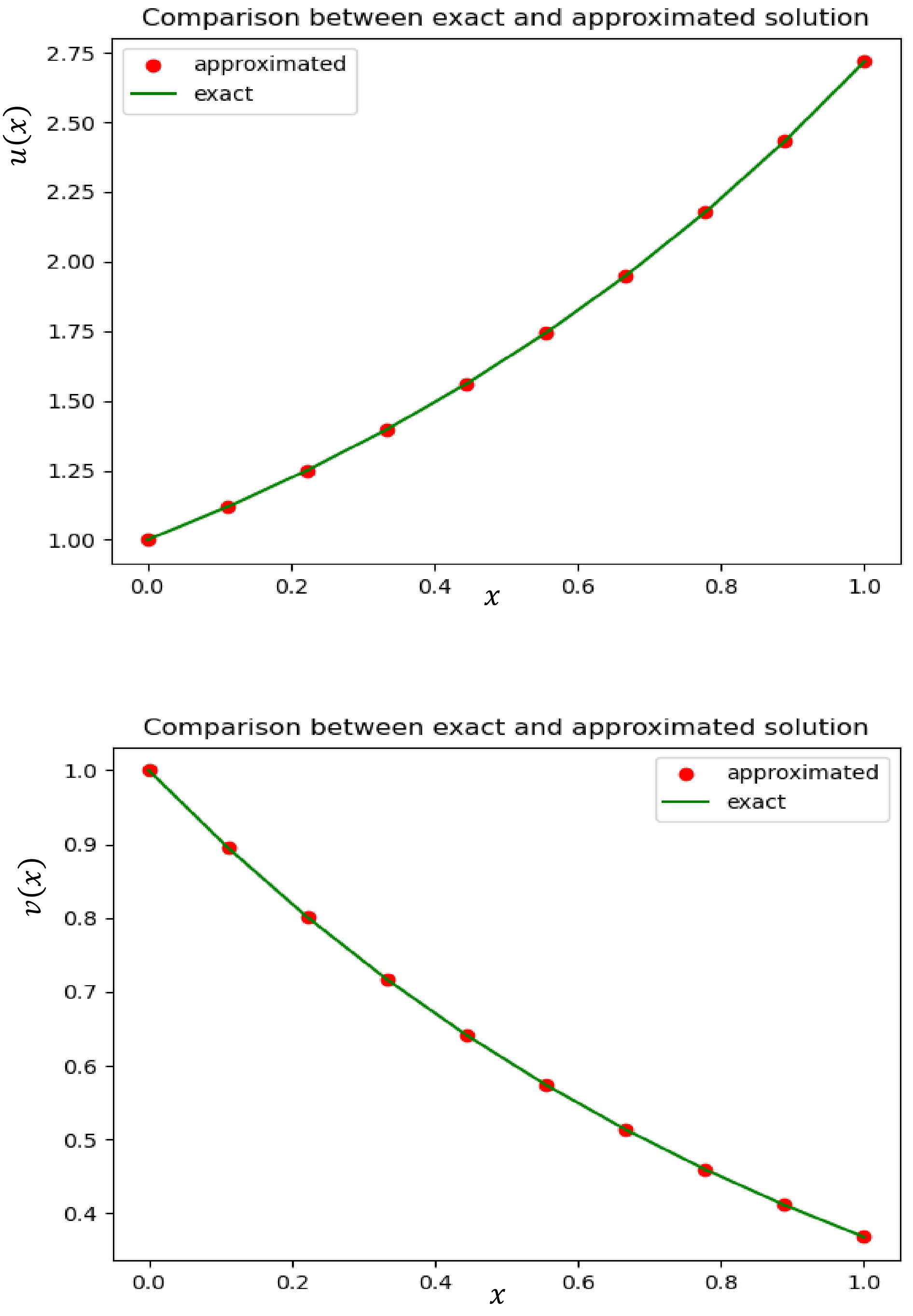


Table (4.14) shows $u(x)$ exact and approximate solution comparison and absolute error with $n = 50$ for system (4.5) using RVIM

Table (4.15) shows $v(x)$ exact and approximate solution comparison and absolute error with $n = 50$ for system (4.5) using RVIM As shown in appendix A.

Table (4.16) shows maximum absolute error of each function for system (4.5) using RVIM

Table 4.16
Maximum absolute error of each function for system (4.5) with n = 50 using RVIM

Function	Maximum error
$u(x) = e^{x}$	$2.21230589403376\times 10^{-11}$
$v(x) = e^{-x}$	$1.06194497639933\times 10^{-11}$

4.2.2 Chebyshev Collocation Method

Taking $n = 5$,

$$f_i(x) = \sum_{k=0}^{5} a_{ik} T_k(x) , \qquad i = 1,2 \tag{4.7}$$

A comparison of functions $u(x)$ and $v(x)$ exact and approximate solution with $n = 5$ for system (4.5) using Chebyshev Collocation Method is shown in Figure (4.6)

Figure 4.6

Exact and estimated solution comparison of functions $u(x)$ *and* $v(x)$ *with* $n = 5$ *for system (4.5) using Chebyshev Collocation Method*

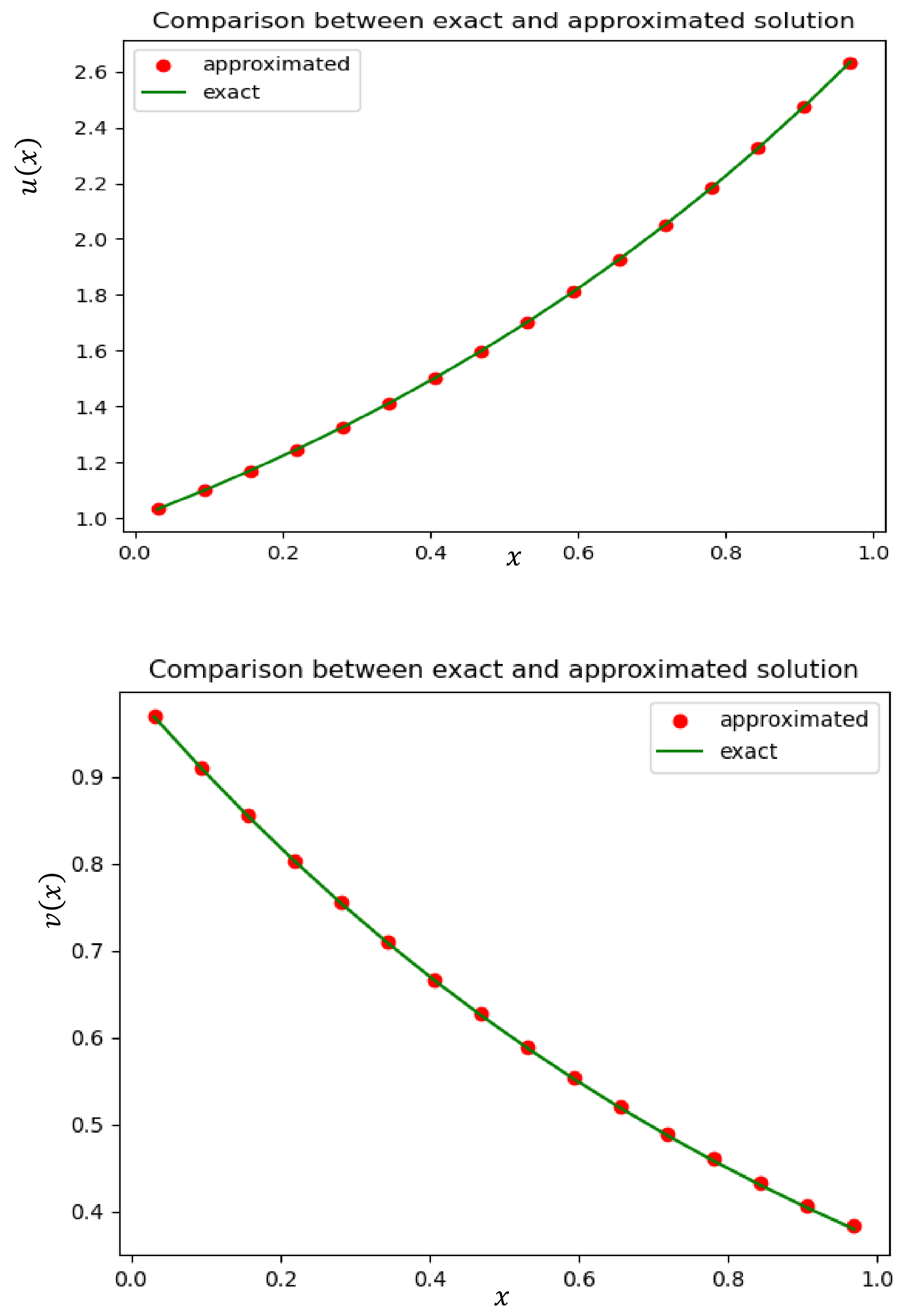


Table (4.17) shows $u(x)$ exact and approximate solution comparison and absolute error with $n = 5$ for system (4.5) using Chebyshev Collocation Method

Table (4.18) shows $v(x)$ exact and approximate solution comparison and absolute error with $n = 5$ for system (4.5) using Chebyshev Collocation Method As shown in appendix A.

Table (4.19) shows maximum absolute error of each function at different degree of Chebyshev polynomials for system (4.5)

Table 4.19
Maximum absolute error of each function at different degree of Chebyshev polynomials for system (4.5)

n	Maximum error $u(x) = e^x$	Maximum error $v(x) = e^{-x}$
5	$1.15928541832844\times 10^{-6}$	$4.52938180850460\times 10^{-7}$
6	$4.23796793214137\times 10^{-8}$	$1.57563855296416\times 10^{-8}$
10	$1.99840144432528\times 10^{-14}$	$7.99360577730113\times 10^{-15}$
15	$4.44089209850063\times 10^{-16}$	$3.33066907387547\times 10^{-16}$

4.2.3 Haar Wavelet Method

Set the dilation parameter I = 3

M = 8

j = 0, 1, 2… 16

$$f_i(x) = \sum_{k=0}^{16} b_{ik} h_k(x), \qquad i = 1,2 \qquad (4.8)$$

A comparison of functions $u(x)$ and $v(x)$ exact and approximate solution with $I = 3$ for system (4.5) using Haar Wavelet Method is shown in Figure (4.7)

Figure 4.7

Exact and estimated solution comparison of functions $u(x)$ *and* $v(x)$ *with* $I = 3$ *for system (4.5) using Haar Wavelet Method*

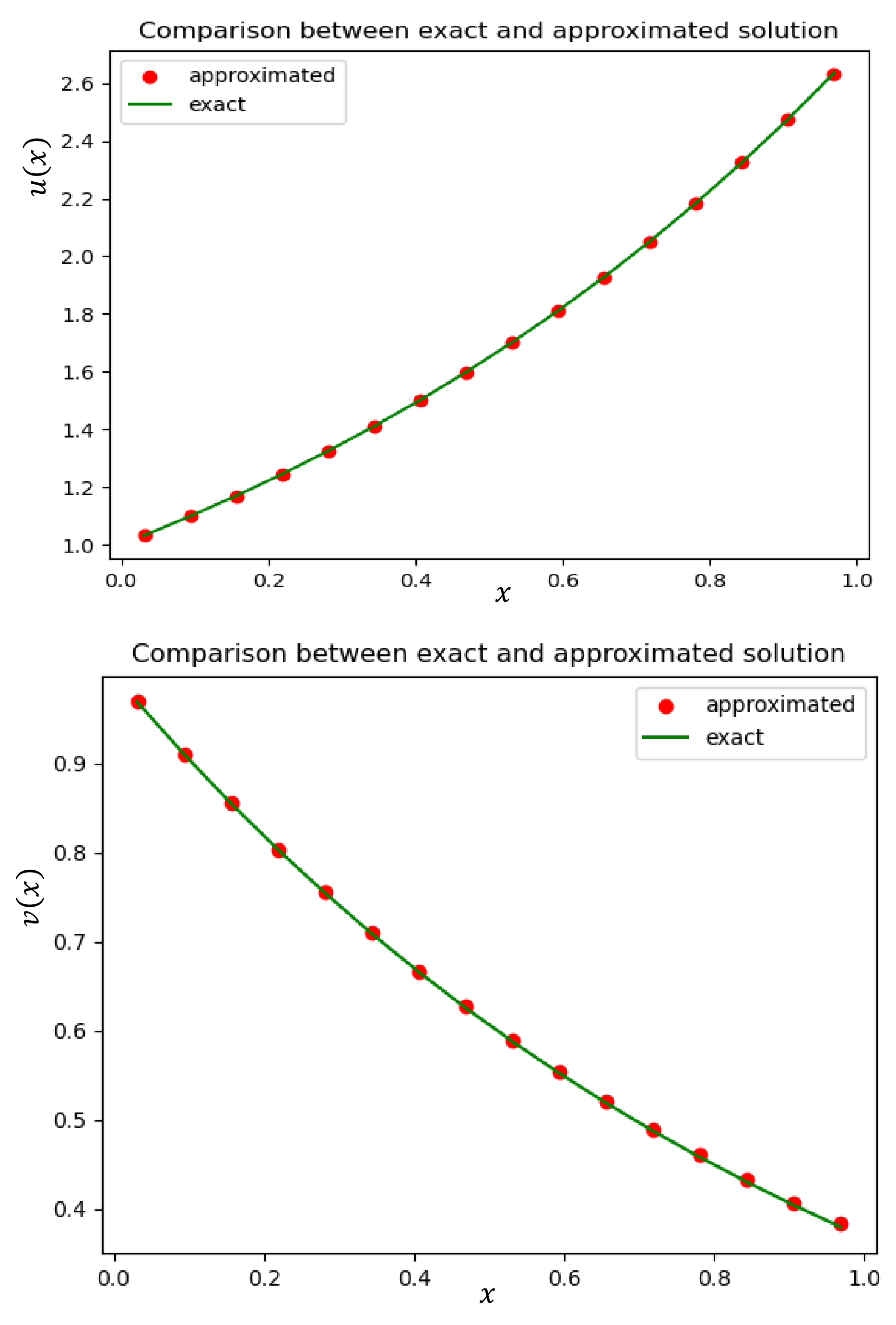


Table (4.20) shows $u(x)$ exact and approximate solution comparison and absolute error at each collocation point with $I = 3$ for system (4.5) using Haar Wavelet Method

Table (4.21) shows $v(x)$ exact and approximate solution comparison and absolute error at each collocation point with $I = 3$ for system (4.5) using Haar Wavelet Method As shown in appendix A.

Table (4.22) shows maximum absolute error of each function at different values of dilation parameter for system (4.5) using Haar Wavelet Method

Table 4.22

Maximum absolute error of each function at different values of dilation parameter for system (4.5) using Haar Wavelet Method

I	2M	Maximum error of $u(x) = e^x$	Maximum error of $v(x) = e^{-x}$
2	8	$4.025524187788\times 10^{-3}$	$1282729880596\times 10^{-2}$
3	16	$9.985804122250\times 10^{-4}$	$3.3997008709756\times 10^{-3}$
4	32	$2.\ 493580293987\times 10^{-4}$	$8.\ 738185054210\times 10^{-4}$
5	64	$6.231541316426\times 10^{-5}$	$2.214223490232\times 10^{-4}$

In example (4.2) we approximated the solution of a nonhomogeneous linear system of 2 equations with exponential functions kernels and as a result, Haar wavelet method gave a good approximation at $I = 4$ also RVIM provided a very good approximation but Chebyshev collocation method provides the most accurate results.

4.3 Example

$$\begin{cases} u(x) = f_1(x) + \int_0^1 s.sin(x)u(s)\,ds + \int_0^1 \mathrm{s}.\sin(x)v(s)\,ds \\ v(x) = f_2(x) + \int_0^1 s^2cos(x)u(s)\,ds + \int_0^1 s^2cos(x)v(s)\,ds \end{cases} \tag{4.9}$$

where,

$f_1(x) = sin(x)(x - 2\cos(1) - 3\sin(1) + 3)$

$f_2(x) = \cos(x)(4sin(1) - 7cos(1) + 1)$

System (4.9) has the exact solution $u(x) = xsin(x)\ and\ \ v(x) = cos(x)$

We will find the approximate solutions of system (4.9) by the aforementioned methods

4.3.1 Reconstruction of Variational Iterations Method

With initial condition

$(u_0, v_0) = \big(u(0), v(0)\big) = (0\,, 0.8756517\ )$

By Formula (3.12) we have $\lambda_i(x) = -1$

The correction functional is given by

$$\begin{aligned} L[u_{n+1}(x)] &= L\big(u_n(x)\big) \\ &- L\left[\int_0^x \left(u_n{}'(\varepsilon) - f_1'(x) - cos(x)\int_0^1 s u_n(s)\,ds - cos(x)\int_0^1 s v_n(s)\,ds \right) d\varepsilon \right] \\ L[v_{n+1}(x)] &= L\big(v_n(x)\big) \\ &- L\left[\int_0^x \left(v_n{}'(\varepsilon) - f_1'(x) + sin(x)\int_0^1 s^2 u_n(s)\,ds - sin(x)\int_0^1 s^2 v_n(s)\,ds \right) d\varepsilon \right] \end{aligned} \tag{4.10}$$

$f_1'(x) = \ sin(x) + cos(x)(x - 2\cos(1) - 3\sin(1) + 3)$

$f_2'(x) = -sin(x)(4sin(1) - 7cos(1) + 1)$

A comparison of functions $u(x)$ and $v(x)$ exact and approximate solution with $n = 25$ for system (4.9) using RVIM is shown in Figure (4.8)

Figure 4.8

Exact and estimated solution comparison of functions $u(x)$ *and* $v(x)$ *with* $n = 25$ *for system (4.9) using RVIM*

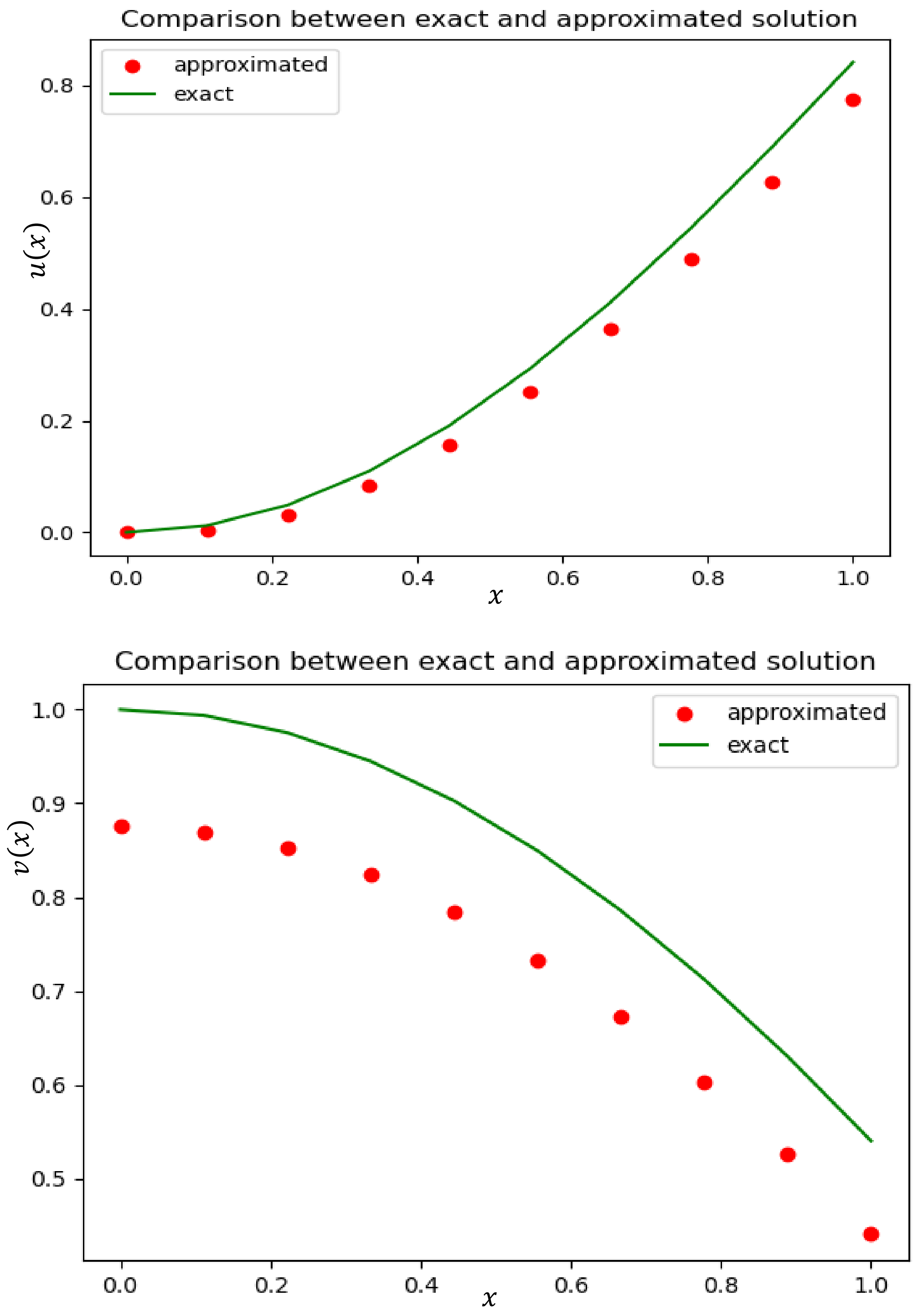


Table (4.23) shows $u(x)$ exact and approximate solution comparison and absolute error with n = 25 for system (4.9) using RVIM

Table (4.24) shows $v(x)$ exact and approximate solution comparison and absolute error with n = 25 for system (4.9) using RVIM As shown in appendix A.

Table (4.25) shows maximum absolute error of each function of system (4.9) with $n = 25$ using RVIM

Table 4.25

Maximum absolute error of each function of system (4.9) with $n = 25$ using RVIM

Function	Maximum error
$u(x) = xsinx(x)$	0.0671540244377784
$v(x) = cos(x)$	0.124348302768088

4.3.2 Chebyshev Collocation Method

Taking $n = 6$,

$$f_i(x) = \sum_{k=0}^{6} a_{ik} T_k(x), \qquad i = 1,2 \tag{4.11}$$

A comparison of functions $u(x)$ and $v(x)$ exact and approximate solution with $n = 6$ for system (4.9) using Chebyshev Collocation Method is shown in Figure (4.9)

Figure 4.9

Exact and estimated solution comparison of functions $u(x)$ *and* $v(x)$ *with* $n = 6$ *for system (4.9) using Chebyshev Collocation Method*

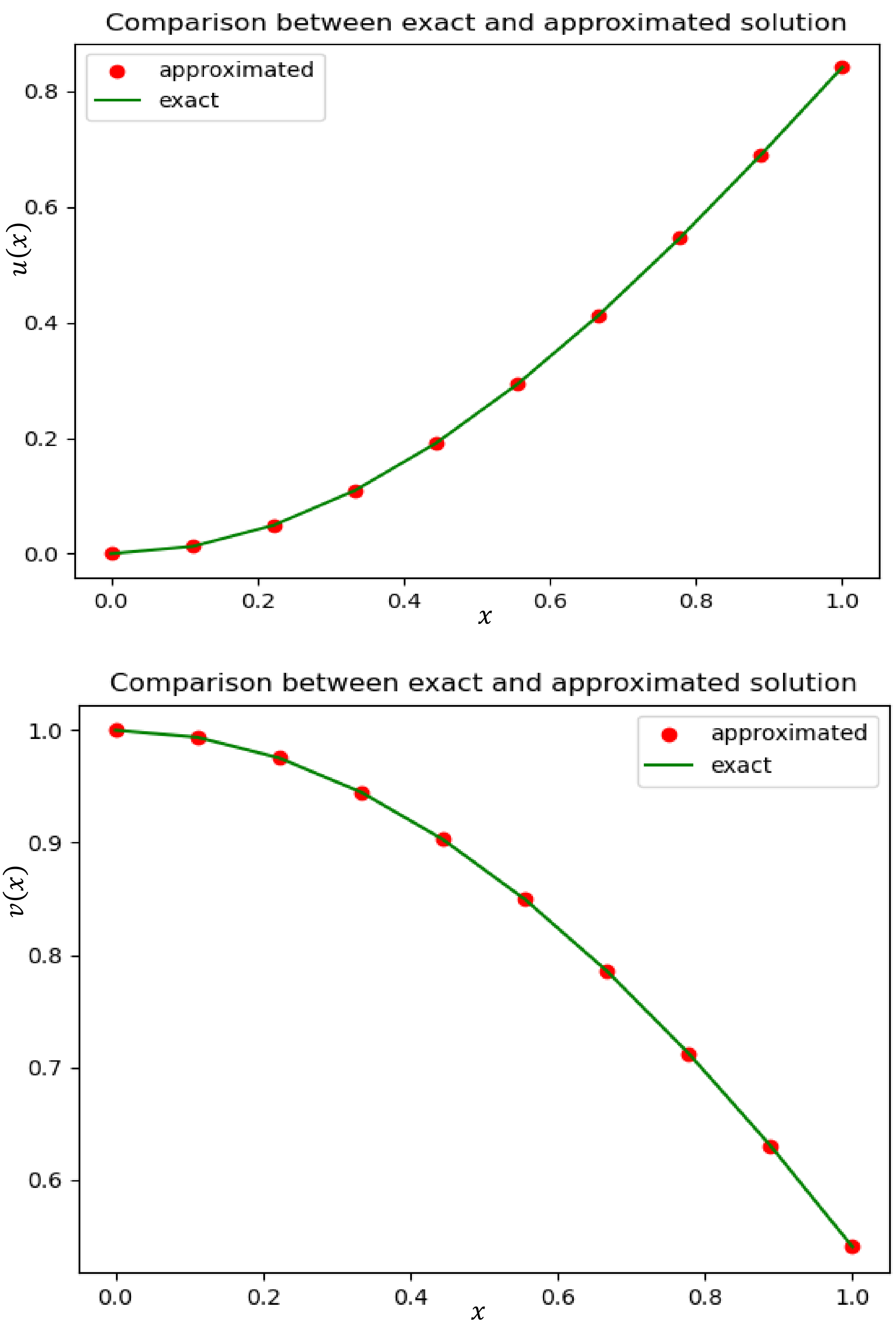


Table (4.26) shows $u(x)$ exact and approximate solution comparison and absolute error with n = 6 for system (4.9) using Chebyshev Collocation Method

Table (4.27) shows $v(x)$ exact and approximate solution comparison and absolute error with n = 6 for system (4.9) using Chebyshev Collocation Method As shown in appendix A.

Table (4.28) shows maximum absolute error of each function at different degree of Chebyshev polynomials for system (4.9) using Chebyshev Collocation Method

Table 4.28
Maximum absolute error of each function at different degree of Chebyshev polynomials for system (4.9) using Chebyshev Collocation Method

n	Maximum error of $u(x)$	Maximum error of $v(x)$
6	$9.88687255398801\times 10^{-8}$	$1.39194389348063\times 10^{-8}$
8	$1.06607611627396\times 10^{-10}$	$1.16274767592017\times 10^{-11}$
10	$7.26085858104852\times 10^{-14}$	$6.55031584528842\times 10^{-15}$
12	$3.33066907387547\times 10^{-16}$	$3.33066907387547\times 10^{-16}$

4.3.3 Haar Wavelet Method

Set the dilation parameter I = 3

M = 8

j = 0, 1, 2… 16

$$f_i(x) = \sum_{k=0}^{16} b_{ik} h_k(x)\,, \qquad i = 1,2 \qquad (4.12)$$

A comparison of functions $u(x)$ and $v(x)$ exact and approximate solution with $I = 3$ for system (4.9) using Haar Wavelet Method is shown in Figure (4.10)

Figure 4.10

Exact and estimated solution comparison of functions $u(x)$ *and* $v(x)$ *with* $i = 3$ *for system (4.9) using Haar Wavelet Method*

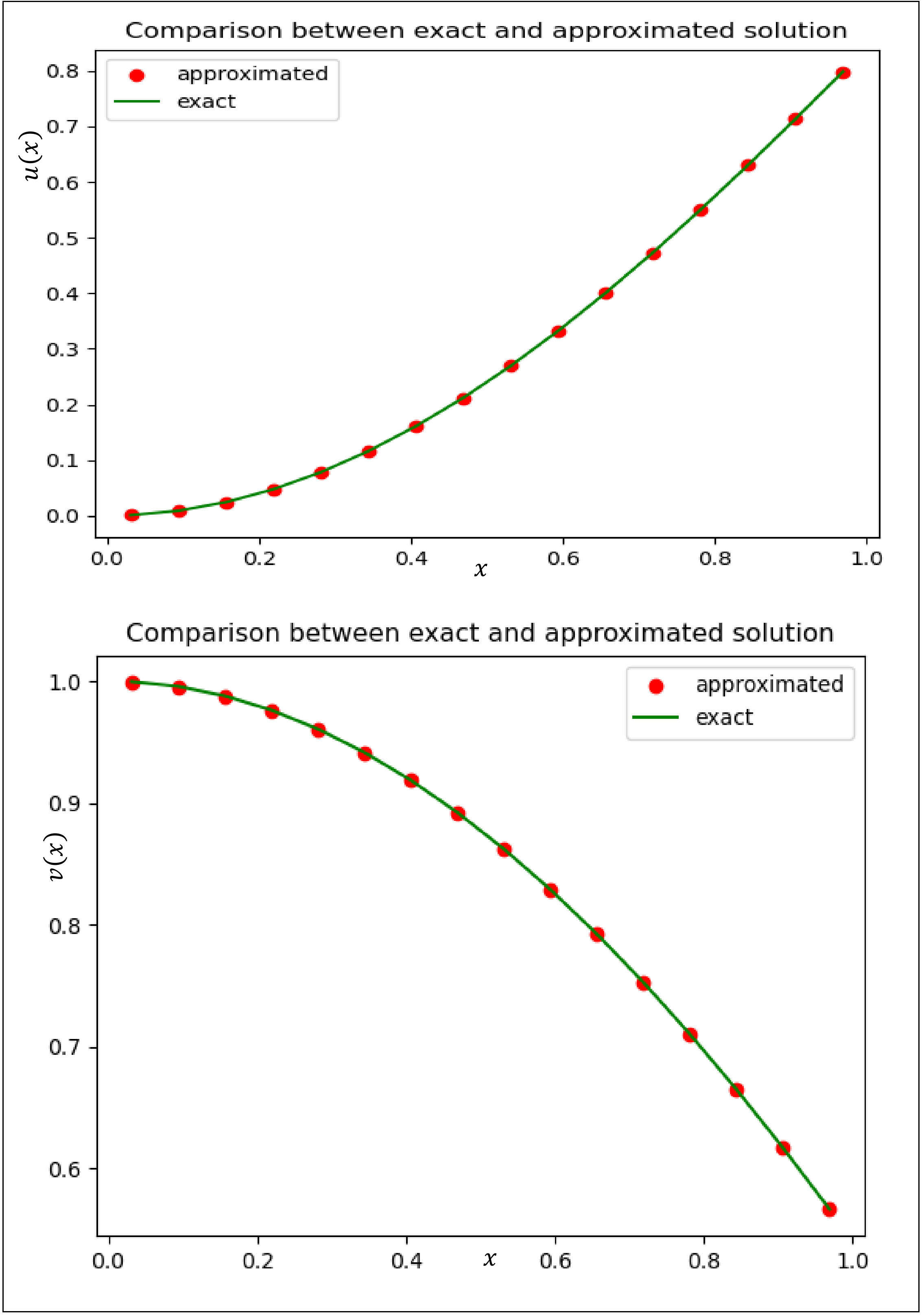


Table (4.29) shows $u(x)$ exact and the approximate solution comparison and absolute error at each collocation point with $I = 3$ for system (4.9) using Haar Wavelet Method

Table (4.30) shows $v(x)$ exact and approximate solution comparison and absolute error at each collocation point with $I = 3$ for system (4.9) using Haar Wavelet Method

Table (4.31) shows maximum absolute error of each function at different values of dilation parameter for system (4.9) using Haar Wavelet Method As shown in appendix A.

In example (4.3) we approximated the solution of a nonhomogeneous linear system of 2 equations with sinusoid function kernels and as a result, Haar wavelet method gave a good approximation at $I = 4$ but Chebyshev collocation method provides the most accurate results.

4.4 Conclusion

The application of numerical methods for solving systems of FIE has been found to be highly effective.

Throughout this thesis, we have seen that the choice of method depends largely on the specific characteristic of the integral equations to converge to the solution such as kernel.

In this thesis, we employed several numerical methods to solve three systems of FIE of the second kind: the Haar wavelet method, reconstruction of the variational iteration method, and the Chebyshev collocation method. Numerical results demonstrate that the convergence and accuracy of these methods closely approximate the known analytical solutions. Furthermore, the Chebyshev collocation method proved to be the most efficient among the three methods compared.

In summary, solving systems of Fredholm integral equations remains a challenging task, but the use of numerical methods has provided us with valuable tools to approach these problems effectively.

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Appendices

Appendix A

Tables

Table 4.2

A comparison between the exact and the approximate solution of $u(x)$ *and absolute error with* $n = 50$ *for system (4.1) using RVIM*

x	Approximate Solution u_{app}	Exact solution $u(x) = x^2$	Absolute error $\lvert u(x) - u_{app} \rvert$
0	0	0	0
0.1	0.012345	0.012346	6.01×10^{-7}
0.2	0.049382	0.049383	1.20×10^{-6}
0.3	0.111109	0.111111	1.80×10^{-6}
0.4	0.197528	0.197531	2.4×10^{-6}
0.5	0.308639	0.308642	3.01×10^{-6}
0.6	0.444441	0.444444	3.61×10^{-6}
0.7	0.604934	0.604938	4.21×10^{-6}
0.8	0.790119	0.790123	4.81×10^{-6}
1	0.999995	1	5.41×10^{-6}

Table 4.3

A comparison between the exact and the approximate solution of $v(x)$ *and absolute error with* $n = 50$ *for system (4.1) using RVIM*

x	Approximate Solution v_{app}	Exact solution $v(x) = x^3 + x$	Absolute error $\lvert v(x) - v_{app} \rvert$
0.1	0.112482	0.112483	6.01×10^{-7}
0.2	0.233195	0.233196	1.20×10^{-6}
0.3	0.370369	0.37037	1.80×10^{-6}
0.4	0.532234	0.532236	2.4×10^{-6}
0.5	0.72702	0.727023	3.01×10^{-6}
0.6	0.962959	0.962963	3.61×10^{-6}
0.7	1.248281	1.248285	4.21×10^{-6}
0.8	1.591216	1.591221	4.81×10^{-6}
1	1.999995	2	5.41×10^{-6}

Table 4.4

A comparison between the exact and the approximate solution of $w(x)$ *and absolute error of* $w(x)$ *at* $n = 50$ *for system (4.1) using RVIM*

x	Approximate Solution w_{app}	Exact solution $w(x) = 1$	Absolute error $\lvert w(x) - w_{app} \rvert$
0.1	0.999999	1	6.01×10^{-7}
0.2	0.999999	1	1.20×10^{-6}
0.3	0.999998	1	1.80×10^{-6}
0.4	0.999998	1	2.4×10^{-6}
0.5	0.999997	1	3.01×10^{-6}
0.6	0.999996	1	3.61×10^{-6}
0.7	0.999996	1	4.21×10^{-6}
1	0.999995	1	5.41×10^{-6}

Table 4.6

A comparison between the exact and the approximate solution of $u(x)$ *and absolute error with* $n = 3$ *for system (4.1) using Chebyshev Collocation Method*

x	Approximate Solution u_{app}	Exact solution $u(x) = x^2$	Absolute error $\left\|u(x) - u_{app}\right\|$
0	-1.11×10^{-16}	0	1.11×10^{-16}
0.1	0.012346	0.012346	8.33×10^{-17}
0.2	0.049383	0.049383	1.94×10^{-16}
0.3	0.111111	0.111111	2.22×10^{-16}
0.4	0.197531	0.197531	3.05×10^{-16}
0.5	0.308642	0.308642	3.33×10^{-16}
0.6	0.444444	0.444444	2.78×10^{-16}
0.7	0.604938	0.604938	4.44×10^{-16}
0.8	0.790123	0.790123	4.44×10^{-16}
1	1	1	4.44×10^{-16}

Table 4.7

A comparison between the exact and the approximate solution of $v(x)$ *and absolute error with* $n = 3$ *for system (4.1) using Chebyshev Collocation Method*

x	Approximate Solution v_{app}	Exact solution $v(x) = x^3 + x$	Absolute error $\left\|v(x) - v_{app}\right\|$
0	-2.22×10^{-16}	0	2.22×10^{-16}
0.1	0.112483	0.112483	6.94×10^{-17}
0.2	0.233196	0.233196	5.55×10^{-17}
0.3	0.37037	0.37037	5.55×10^{-17}
0.4	0.532236	0.532236	0
0.5	0.727023	0.727023	1.11×10^{-16}
0.6	0.962963	0.962963	0
0.7	1.248285	1.248285	2.22×10^{-16}
1	2	2	0

Table 4.8

A comparison between the exact and the approximate solution of $w(x)$ *and absolute error with* $n = 3$ *for system (4.1) using Chebyshev Collocation Method*

x	Approximate Solution w_{app}	Exact solution $w(x) = 1$	Absolute error $\left\|w(x) - w_{app}\right\|$
0	1.000	1.000	0
0.1	1.000	1.000	4.44×10^{-16}
0.2	1.000	1.000	4.44×10^{-16}
0.3	1.000	1.000	4.44×10^{-16}
0.4	1.000	1.000	4.44×10^{-16}
0.5	1.000	1.000	6.66×10^{-16}
0.6	1.000	1.000	6.66×10^{-16}
0.7	1.000	1.000	6.66×10^{-16}
0.8	1.000	1.000	6.66×10^{-16}
1	1.000	1.000	6.66×10^{-16}

Table 4.10

A comparison between the exact and the approximate solution of $u(x)$ and absolute error at each collocation point with $I = 3$ for system (4.1) using Haar Wavelet Method

Collocation point	Approximate Solution u_{app}	Exact solution $u(x) = x^2$	Absolute error $\lvert u(x) - u_{app}\rvert$
0.03125	0.000755	0.000977	0.000222
0.09375	0.008123	0.008789	0.000666
0.15625	0.023305	0.024414	0.001109
0.21875	0.046298	0.047852	0.001553
0.28125	0.077105	0.079102	0.001997
0.34375	0.115723	0.118164	0.002441
0.40625	0.162155	0.165039	0.002884
0.46875	0.216398	0.219727	0.003328
0.53125	0.278455	0.282227	0.003772
0.59375	0.348323	0.352539	0.004216
0.65625	0.426005	0.430664	0.004659
0.71875	0.511498	0.516602	0.005103
0.78125	0.604805	0.610352	0.005547
0.84375	0.705923	0.711914	0.005991
0.90625	0.814855	0.821289	0.006434
0.96875	0.931598	0.938477	0.006878

Table 4.11

A comparison between the exact and the approximate solution of $v(x)$ and absolute error at each collocation point with $I = 3$ for system (4.1) using Haar Wavelet Method

Collocation Point	Approximate Solution v_{app}	Exact solution $v(x) = x^3 + x$	Absolute error $\lvert v(x) - v_{app}\rvert$
0.03125	0.031059	0.031281	0.000222
0.09375	0.093908	0.094574	0.000666
0.15625	0.158955	0.160065	0.001109
0.21875	0.227664	0.229218	0.001553
0.28125	0.3015	0.303497	0.001997
0.34375	0.381928	0.384369	0.002441
0.40625	0.470413	0.473297	0.002884
0.46875	0.568419	0.571747	0.003328
0.53125	0.677411	0.681183	0.003772
0.59375	0.798854	0.80307	0.004216
0.65625	0.934214	0.938873	0.004659
0.71875	1.084954	1.090057	0.005103
0.78125	1.25254	1.258087	0.005547
0.84375	1.438437	1.444427	0.005991
0.90625	1.644109	1.650543	0.006434
0.96875	1.871021	1.877899	0.006878

Table 4.12

A comparison between the exact and the approximate solution of $w(x)$ and absolute error at each collocation point with $I = 3$ for system (4.1) using Haar Wavelet Method

Collocation Point	Approximate Solution w_{app}	Exact solution $w(x) = 1$	Absolute error $\lvert w(x) - w_{app} \rvert$
0.03125	0.999778	1	0.000222
0.09375	0.999334	1	0.000666
0.15625	0.998891	1	0.001109
0.21875	0.998447	1	0.001553
0.28125	0.998003	1	0.001997
0.34375	0.997559	1	0.002441
0.40625	0.997116	1	0.002884
0.46875	0.996672	1	0.003328
0.53125	0.996228	1	0.003772
0.59375	0.995784	1	0.004216
0.65625	0.995341	1	0.004659
0.71875	0.994897	1	0.005103
0.78125	0.994453	1	0.005547
0.84375	0.994009	1	0.005991
0.90625	0.993566	1	0.006434
0.96875	0.993122	1	0.006878

Table 4.14

A comparison between the exact and the approximate solution of $u(x)$ and absolute error with $n = 50$ for system (4.5) using RVIM

x	Approximate Solution u_{app}	Exact solution $u(x) = e^{x}$	Absolute error $\lvert u(x) - u_{app} \rvert$
0	1	1	0
0.1	1.117519	1.117519	1.62×10−11
0.2	1.248849	1.248849	3.45×10−11
0.3	1.395612	1.395612	5.49×10−11
0.4	1.559623	1.559623	7.74×10−11
0.5	1.742909	1.742909	1.02×10−10
0.6	1.947734	1.947734	1.29×10−10
0.7	2.17663	2.17663	1.57×10−10
1	2.718282	2.718282	2.21×10−10

Table 4.15

A comparison between the exact and the approximate solution of $v(x)$ and absolute error with $n = 50$ for system (4.5) using RVIM

x	Approximate Solution v_{app}	Exact solution $v(x) = e^{-x}$	Absolute error $\lvert v(x) - v_{app} \rvert$
0	1	1	0
0.1	0.894839	0.894839	7.35E×10−12
0.2	0.800737	0.800737	9.92×10−12
0.3	0.716531	0.716531	7.70×10−12
0.4	0.64118	0.64118	6.85×10−13
0.5	0.573753	0.573753	1.11×10−11
0.6	0.513417	0.513417	2.77×10−11
0.7	0.459426	0.459426	4.91×10−11
1	0.367879	0.367879	1.06×10−10

Table 4.17

A comparison between the exact and the approximate solution of $u(x)$ *and absolute error with* $n = 5$ *for system (4.5) using Chebyshev Collocation Method*

x	Approximate Solution u_{app}	Exact solution $u(x) = e^{x}$	Absolute error $\left\|u(x) - u_{app}\right\|$
0	0.999999	1	1.05×10−6
0.1	1.11752	1.117519	6.39×10−7
0.2	1.248848	1.248849	9.86×10−7
0.3	1.395612	1.395612	4.76×10−7
0.4	1.559624	1.559623	9.03×10−7
0.5	1.74291	1.742909	9.23×10−7
0.6	1.947734	1.947734	4.83×10−7
0.7	2.176629	2.17663	1.04×10−6
1	2.718281	2.718282	1.16×10−6

Table 4.18

A comparison between the exact and the approximate solution of $v(x)$ *and absolute error with* $n = 5$ *for system (4.5) using Chebyshev Collocation Method*

x	Approximate Solution v_{app}	Exact solution $v(x) = e^{-x}$	Absolute error $\left\|v(x) - v_{app}\right\|$
0	1	1	4.46×10−7
0.1	0.89484	0.894839	2.55×10−7
0.2	0.800737	0.800737	4.08×10−7
0.3	0.716531	0.716531	2.07×10−7
0.4	0.641181	0.64118	3.06×10−7
0.5	0.573754	0.573753	2.93×10−7
0.6	0.513417	0.513417	2.20×10−7
0.7	0.459425	0.459426	4.14×10−7
1	0.367879	0.367879	4.53×10−7

Table 4.20

A comparison between the exact and the approximate solution of $u(x)$ *and absolute error at each collocation point with* $I = 3$ *for system (4.5) using Haar Wavelet Method*

Collocation point	Approximate Solution u_{app}	Exact solution $u(x) = e^{x}$	Absolute error $\left\|u(x) - u_{app}\right\|$
0.03125	1.031677	1.031743	6.60×10−5
0.09375	1.098094	1.098285	0.000191
0.15625	1.168811	1.169118	0.000307
0.21875	1.244106	1.24452	0.000414
0.28125	1.324272	1.324785	0.000513
0.34375	1.409624	1.410226	0.000602
0.40625	1.500496	1.501178	0.000682
0.46875	1.597242	1.597995	0.000753
0.53125	1.700242	1.701057	0.000815
0.59375	1.809898	1.810766	0.000868
0.65625	1.926638	1.92755	0.000912
0.71875	2.050919	2.051867	0.000947
0.78125	2.183227	2.184201	0.000973
0.84375	2.324079	2.32507	0.000991
0.90625	2.474025	2.475024	0.000999
0.96875	2.633651	2.634649	0.000998

Table 4.21

A comparison between the exact and the approximate solution of $v(x)$ and absolute error at each collocation point with $I = 3$ for system (4.5) using Haar Wavelet Method

Collocation Point	Approximate Solution v_{app}	Exact solution $v(x) = e^{-x}$	Absolute error $\lvert v(x) - v_{app}\rvert$
0.03125	0.969272	0.969233	3.83×10−5
0.09375	0.91064	0.91051	0.000129
0.15625	0.855584	0.855345	0.000239
0.21875	0.803891	0.803523	0.000368
0.28125	0.755356	0.75484	0.000516
0.34375	0.709789	0.709106	0.000683
0.40625	0.667013	0.666144	0.000869
0.46875	0.626858	0.625784	0.001074
0.53125	0.589168	0.58787	0.001298
0.59375	0.553794	0.552252	0.001541
0.65625	0.520597	0.518793	0.001803
0.71875	0.489446	0.487361	0.002085
0.78125	0.460218	0.457833	0.002385
0.84375	0.432799	0.430095	0.002704
0.90625	0.407079	0.404037	0.003042
0.96875	0.382957	0.379557	0.0034

Table 4.23

A comparison between the exact and the approximate solution of $u(x)$ and absolute error with n = 25 for system (4.9) using RVIM

x	Approximate Solution u_{app}	Exact solution $u(x) = xsinx$	Absolute error $\lvert u(x) - u_{app}\rvert$
0	0.003471	0.01232	0.008849
0.1	0.031388	0.048977	0.017589
0.2	0.082953	0.109065	0.026112
0.3	0.156779	0.191092	0.034313
0.4	0.250918	0.293009	0.042091
0.5	0.362897	0.412247	0.049349
0.6	0.489766	0.545765	0.055999
0.7	0.62815	0.690108	0.061959
0.8	0.774317	0.841471	0.067154
1	0.003471	0.01232	0.008849

Table 4.24

A comparison between the exact and the approximate solution of $v(x)$ and absolute error with n = 25 for system (4.9) using RVIM

x	Approximate Solution v_{app}	Exact solution $v(x) = cosx$	Absolute error $\lvert v(x) - v_{app}\rvert$
0	0.875652	1	0.124348
0.1	0.869819	0.993834	0.124014
0.2	0.852394	0.97541	0.123016
0.3	0.82359	0.944957	0.121367
0.4	0.783763	0.90285	0.119086
0.5	0.733405	0.849608	0.116203
0.6	0.673136	0.785887	0.112751
0.7	0.6037	0.712475	0.108775
1	0.440853	0.540302	0.09945

Table 4.26

A comparison between the exact and the approximate solution and absolute error of $u\ (x)$ with n = 6 for system (4.9) using Chebyshev Collocation Method

x	Approximate Solution u_{app}	Exact solution $u(x) = xsinx$	Absolute error $\lvert u(x) - u_{app} \rvert$
0	-8.06×10−8	0	8.06×10−8
0.1	0.01232	0.01232	4.04×10−9
0.2	0.048977	0.048977	7.16×10−8
0.3	0.109065	0.109065	5.98×10−8
0.4	0.191092	0.191092	6.22×10−8
0.5	0.293008	0.293009	6.62×10−8
0.6	0.412246	0.412247	6.69×10−8
0.7	0.545765	0.545765	7.87×10−8
0.8	0.690108	0.690108	2.53×10−9
1	0.841471	0.841471	9.89×10−8

Table 4.27

A comparison between the exact and the approximate solution of $v(x)$ and absolute error with n = 6 for system (4.9) using Chebyshev Collocation Method

x	Approximate Solution v_{app}	Exact solution $v(x) = cosx$	Absolute error $\lvert v(x) - v_{app} \rvert$
0	1	1	8.15×10−9
0.1	0.993834	0.993834	1.55×10−9
0.2	0.97541	0.97541	6.99×10−9
0.3	0.944957	0.944957	9.59×10−9
0.4	0.90285	0.90285	9.85×10−9
0.5	0.849608	0.849608	6.49×10−9
0.6	0.785887	0.785887	6.68×10−9
0.7	0.712475	0.712475	1.16×10−8
1	0.540302	0.540302	1.39×10−8

Table 4.29

A comparison between the exact and the approximate solution of $u(x)$ and absolute error at each collocation point with $I = 3$ for system (4.9) using Haar Wavelet Method

Collocation point	Approximate Solution u_{app}	Exact solution $u(x) = xsinx$	Absolute error $\lvert u(x) - u_{app} \rvert$
0.03125	0.000964	0.000976	1.24×10−5
0.09375	0.008739	0.008776	3.72×10−5
0.15625	0.024253	0.024315	6.19×10−5
0.21875	0.047385	0.047471	8.63×10−5
0.28125	0.077952	0.078063	0.00011
0.34375	0.115717	0.115851	0.000134
0.40625	0.16038	0.160537	0.000157
0.46875	0.211588	0.211768	0.00018
0.53125	0.268936	0.269137	0.000201
0.59375	0.331965	0.332187	0.000223
0.65625	0.400168	0.400411	0.000243
0.71875	0.472995	0.473257	0.000262
0.78125	0.549851	0.550131	0.00028
0.84375	0.630103	0.6304	0.000297
0.90625	0.713084	0.713397	0.000313
0.96875	0.798095	0.798423	0.000328

Table 4.30

A comparison between the exact and the approximate solution of $v(x)$ *and absolute error at each collocation point with* $I = 3$ *for system (4.9) using Haar Wavelet Method*

Collocation point	Approximate Solution v_{app}	Exact solution $v(x) = cosx$	Absolute error $\lvert v(x) - v_{app} \rvert$
0.03125	0.999177	0.999512	0.000334
0.09375	0.995276	0.995609	0.000333
0.15625	0.987487	0.987818	0.000331
0.21875	0.975843	0.976169	0.000327
0.28125	0.960388	0.960709	0.000321
0.34375	0.941182	0.941497	0.000315
0.40625	0.918302	0.918609	0.000307
0.46875	0.891835	0.892134	0.000299
0.53125	0.861886	0.862174	0.000289
0.59375	0.828571	0.828848	0.000277
0.65625	0.792021	0.792286	0.000265
0.71875	0.752378	0.752629	0.000252
0.78125	0.709796	0.710034	0.000238
0.84375	0.664443	0.664666	0.000222
0.90625	0.616496	0.616702	0.000206
0.96875	0.566141	0.56633	0.00019

Table 4.31

Maximum absolute error of each function at different values of dilation parameter for system (4.9) using Haar Wavelet Method

I	$2M$	Maximum error $u(x) = xsinx$	Maximum error $v(x) = cosx$
2	8	0.00128417170160999	0.00133713494293064
3	16	0.000327772774280199	0.000334459856355718
4	32	8.27826737478388×10−5	8.36257476017899×10−5
5	64	2.08005533458566×10−5	2.09071073725697×10−5

جامعة النجاح الوطنية

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قدمت هذه الرسالة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات، من كلية الدراسات العليا، في جامعة النجاح الوطنية، نابلس–فلسطين.

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الملخص

الهدف الرئيسي من هذه الرسالة اقتراح طرق عددية متنوعة لإيجاد حلول تقريبية لنظام من معادلات فريدهولم التكاملية من النوع الثاني. تشمل الطرق المقترحة طريقة التجميع باستخدام كثيرات الحدود تشيبيشيف، وطريقة مويجات هار، وطريقة إعادة بناء التكرار التبايني

بعد استعراض المفاهيم الأساسية لمعادلات فريدهولم التكاملية ومعالجة الإطار الرياضي لهذه الطرق العددية، نقدم بعض الأمثلة العددية التوضيحية ذات الحلول المعروفة لإظهار فعالية وكفاءة هذه الطرق

تُظهر النتائج العددية بوضوح أن طريقة التجميع باستخدام كثيرات الحدود تشيبيشيف تُعد من أكثر الطرق كفاءةً في حل نظام معادلات فريدهولم التكاملية مقارنةً بنظيراتها.

الكلمات المفتاحية: معادلات فريدهولم التكاملية، كثيرات حدود تشيبيشيف، طريقة مويجات هار، طريقة إعادة بناء التكرار التبايني، طرق عددية.