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Simplex Linear Codes Over the Ring $F_2 + vF_2$

التراميز الخطية المبسطة على الحلقة $F_2 + vF_2$

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Abstract

In this paper, we construct simplex linear codes over the ring $F_2 + vF_2$ of types α and β, where $v^2 = v$ and $F_2 = \{0,1\}$. We also determine some of their properties. These codes are extension and generalization of simplex codes over the rings Z_4, Z_6 and $F_2 + uF_2$ where $u^2 = 0$.

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ملخص

في هذا البحث قمنا بتعريف التراميز الخطية المبسطة على الحلقة $F_2 + vF_2$ من نوع ألفا وبيتا حيث $F_2 = \{0,1\}$ و $v^2 = v$ وتعرفنا على بعض خصائصها. هذه التراميز هي توسيع وتعميم للتراميز المبسطة على الحلقات Z_4, Z_6, $F_2 + uF_2$ حيث $u^2 = 0$.

1. Introduction

There are various binary linear codes such as the Hamming codes, the first order Reed Muller codes and the simplex codes. Any nonzero codeword of the simplex code has many of the properties that we would expect from a sequence obtained by tossing a fair coin $2^m - 1$ times. This randomness makes these codewords very useful in number of applications such as range-finding, synchronizing, modulation scrambling etc. Hamming code is the dual of the simplex code. All these codes have been generalized to codes over the Galois fields $GF(q)$. Recently there has been much interest in codes over finite rings, especially the rings Z_{2^s} where Z_{2^s} denotes the ring of integers modulo 2^s. In particular, codes over $\mathbb{Z}_4$ and $\mathbb{F}_2 + u\mathbb{F}_2$ have been widely studied. See (Bonnecaze & Udaya, 1999, p. 1250-1254), (Dougherty, et al., 1999, p.32-45), (Dougherty, et al., 1999, p.2345-2360), (Gupta, 2000, p. 1-98), (Rains & Sloane, 1998, p. 1-140) and (EL-Atrash & AL-Ashker, 2003, p. 53-68).

More recently Z_4-simplex codes and their Gray images have been investigated by Bhandari, Lal and Gupta in (Bhandrri, et al., 1999, p. 170-180). Good binary linear and non-linear codes can be obtained from codes over Z_4 via the Gray map. In (Gupta, et al., 2001, p. 112-121) Gupta, Clyun and Gulliver studied senary simplex codes of type α and two versions of types (β and γ), self-orthogonality, torsion codes weight distribution and weight hierarchy properties were investigated. They gave a new construction of senary codes via their binary and ternary counter part and show that types α and β simplex codes can be constructed by this method. In (AL-Ashker, 2005, p. 277-285) and (AL-Ashker, 2005, p. 221-233) respectively simplex codes of types α and β over the rings $\mathbb{F}_2 + u\mathbb{F}_2$ where $u^2 = 0$ and the ring $\sum_{n=0}^{n=s} u^n \mathbb{F}_2$ were given as generalizations and extensions of simplex codes over Z_4 and Z_{2^s}. In this paper we describe linear simplex codes and their properties over the ring $R = F_2 + vF_2$ where $v^2 = v$ and $F_2 = \{0,1\}$.

2. Definitions and preliminaries

The commutative ring $R = F_2 + vF_2 = \{0,1, v, 1+v\}$, where $v^2 = v$ and $F_2 = \{0,1\}$, was introduced in (Bachoc, 1997, p. 92-119) to construct lattices. In (Dougherty, et al., 1999, p.2345-2360) it was shown that this ring is isomorphic to the ring $F_2 \times F_2$. Addition and multiplication operations over R are given in the following tables:

+	0	1	v	1+v
0	0	1	v	1+v
1	1	0	1+v	v
v	v	1+v	0	1
1+v	1+v	v	1	0

.	0	1	v	1+v
0	0	0	0	0
1	0	1	v	1+v
v	0	v	v	0
1+v	0	1+v	0	1+v

The above table shows that v and $1+v$ are orthogonal idempotents and their sum is equal to 1. Following (Dougherty, et al., 1999, p.2345-2360), this ring is a semi-local ring with two maximal ideals; (v) and $(1+v)$. Observe that $R/(v)$ and $R/(1+v)$ are isomorphic to F_2. The Chinese Remainder Theorem (CRT) (Dougherty, et al., 1999, p. 253-283) tells us that $R = (v) \oplus (1+v)$.

We also have

$a + vb = (a+b)v + a(v+1)$, for all $a, b \in F_2^n$.

2.1 Codes

A linear code C of length n over R is an R-submodule of R^n. An element of C is called a codeword of C. A generator matrix of C is a matrix whose rows generate C. There are three different weights for codes over R known, namely the Hamming, Lee and Bachoc weights, see (Bachoc, 1997, p. 92-119), (Betsumiya & Harada, 2004, p. 356-358) and (Betsumiya, et al., 2003, p.171-186). The Hamming weight of a codeword is the number of nonzero components. The Lee weights of the elements $0, 1, v$ and $1+v$ are $0,2,1$ and 1 respectively. The Bachoc weight is defined in (Bachoc, 1997, p. 92-119) and the weights of the elements $0,1,v$ and $1+v$ are $0,1,2$ and 2 respectively. The Lee and Bachoc weights of a codeword are the rational sums of the Lee and Bachoc weights of their components, respectively. The Lee weight for a codeword $x = (x_1, x_2, \ldots\ldots, x_n) \in R^n$ is defined by, $wt_L(x) = \sum_{i=1}^{n} wt_L(x_i)$, where

$$wt_L(x_i) = \begin{cases} 0 & \text{if } x_i = 0, \\ 1 & \text{if } x_i = v \text{ or } 1+v. \\ 2 & \text{if } x_i = 1. \end{cases}$$

The Bachoc weight is given by the relation $wt_B(x) = \sum_{i=1}^{n} wt_B(x_i)$, where

$$wt_B(x_i) = \begin{cases} 0 & \text{if } x_i = 0, \\ 1 & \text{if } x_i = 1 \\ 2 & \text{if } x_i = v \text{ or } 1+v. \end{cases} .$$

Remark 2.1 *Let $n_0(x)$ be the number of components i for which $x_i = 0$, $n_1(x)$ be the number of components i for which $x_i = 1$ and $n_2(x) = n - n_0(x) - n_1(x)$ i.e., n_2 be the number of $v's$ and $(1+v)'s$ in x.*

Then the Lee weight $wt_L(x)$ (respectively the Bachoc weight $wt_B(x)$) of $x=(x_1,x_2,.....x_n)\in R^n$ can also be obtained as follows: $wt_L(x)=n_2(x)+2n_1(x)$ and $wt_B(x)=n_1(x)+2n_2(x)$. For $x=(x_1,x_2,..........,x_n), y=(y_1,y_2,..........,y_n)\in R^n$, $d_H(x,y)=|\{i=x_i\neq y_i\}|$ is called the Hamming distance between x and y, which is equal the number of coordinates in which x and y differ.

The Lee distance between x and $y\in R^n$ is denoted by
$d_L(x,y)=wt_L(x-y)=\sum_{i=1}^{n} wt_L(x_i-y_i)$.

The Bachoc distance between x and $y\in R^n$ is denoted by
$d_B(x,y)=wt_B(x-y)=\sum_{i=1}^{n} wt_B(x_i-y_i)$.

The minimum Hamming, Lee and Bachoc weights, d_H, d_L and d_B of C are the smallest Hamming, Lee and Bachoc weights among all non-zero codewords of C, respectively. We define two inner products (x,y) and $[x,y]$ of x and $y\in R^n$. The Euclidean inner product is defined as $(x,y)=x_1y_2+x_2y_2+....+x_ny_n$ and the Hermitian inner product is defined as $[x,y]=x_1\overline{y}_2+x_2\overline{y}_2+....+x_n\overline{y}_n$, where $\overline{0}=0, \overline{1}=1, \overline{v}=v+1$ and $\overline{v+1}=v$. The dual code $C^{\perp}$ with respect to the Euclidean inner product of C is defined as, $C^{\perp}=\{x\in R^n \mid (x,y)=0 \text{ forall } y\in C\}$ and the dual code C^* with respect to the Hermitian inner product of C is defined as, $C^*=\{x\in R^n \mid [x,y]=0 \text{ forall } y\in C\}$, C is Euclidean self-dual if $C=C^{\perp}$ and C is Hermitian self dual if $C=C^*$. C is called self orthogonal if $C\subseteq C^{\perp}$ and C is called Hermitian self-orthogonal if $C\subseteq C^*$. For $R=F_2+vF_2$ we say C and C' are equivalent if either C or C' are permutation equivalent or C is permutation equivalent to the code obtained from C' by interchanging v and $1+v$ in all coordinates.

Definition 2.1 *Consider the map* $\phi : F_2^{\ n} + vF_2^{\ n} \rightarrow F_2^{\ n} \times F_2^{\ n}$ defined as $\phi(x+vy) = (x, x+y)$ for all $x, y \in \mathbb{F}_2^n$. ϕ is called Gray map and it can be shown that ϕ is an isomorphism, see (Betsumiya & Harada, 2004, p. 356-358), (Dougherty, et al., 1999, p. 253-283). This map can be extended naturally from $(\mathbb{F}_2 + v\mathbb{F}_2)^n$ to $\mathbb{F}_2^{2n}$. The Lee weight of $x+vy$ is the Hamming weight of its gray image. In (Betsumiya & Harada, 2004, p. 356-358) it was shown that if C is a code over R, then there are binary codes C_1 and C_2 such that $C = \phi^{-1}(C_1, C_2)$.

Proposition 2.1 (Betsumiya & Harada, 2004, p. 356-358) *Let* d_H *and* d_L *be the minimum Hamming and Lee weights of* $C = \phi^{-1}(C_1, C_2)$, *respectively. Then* $d_H = d_L = \min\{d(C_1), d(C_2)\}$, *where* $d(C_i)$ *denotes the minimum weight of a binary code* C_i.

Definition 2.2 *A self-dual code for the Euclidean dot product is doubly even (Type II) if the Lee weight of all its words is divisible by* 4 *and singly even otherwise.*

Theorem 2.2 (Bachoc, 1997, p.92-119) *If* $C \subseteq R^n$ *is a self-dual Hermitian code, then* $d_B \leq 2(1 + \lfloor \frac{n}{3} \rfloor)$.

Codes meeting that bound with equality are called extremal.

Definition 2.3 *We say that a self-dual code with the highest minimum Bachoc weight among all self-dual codes of that length is optimal.*

2.2 The Macwilliams relations (Dougherty, et al., 1999, p. 2345-2360)

The Hamming weight enumerator for a code over R is defined by:

$$W_C(x,y)=\sum_{u\in C}x^{n-wt(u)}y^{wt(u)}=\sum_{i=0}^{n}A_i x^{n-i}y^i.$$

Where $A_i = A(C)$ is the number of codewords of weight i in the codes C.

The complete weight enumerator for a code over R is defined by:

$$cwe_C(x_0,x_1,x_v,x_{1+v})=\sum_{c\in C}cwt(c),$$

where $cwt(c)=\prod a^{n_0(c)}b^{n_1(c)}c^{n_v(c)}d^{n_{1+v}(c)}$ and n_α is the number of times α appears in the codeword c.

Now define the Lee composition of x say $L_i(x)=0,1,2$ as the number of entries in x of Lee weight i. The symmetrized weight enumerator (swe) is defined by:

$$swe_C(a,b,c)=\sum_{x\in C}a^{L_0(x)}b^{L_1(x)}c^{L_2(x)}$$

and is given by

$$swe_C(a,b,c)=cwe(a,c,b,b).$$

2.3 Binary structure of codes over R

Following (Dougherty, et al., 1999, p. 2345-2360), any code over R is permutation equivalent to a code generated by the following matrix:

$$\begin{pmatrix} I_{k_1} & vB_1 & (1+v)A_1 & (1+v)A_2+vB_2 & (1+v)A_3+vB_3 \\ 0 & (1+v)I_{k_2} & 0 & (1+v)A_4 & 0 \\ 0 & 0 & vI_{k_3} & 0 & vB_4 \end{pmatrix},$$

where A_i and B_j are binary matrices. Such a code is said to have rank $\{2^{k_1}, 2^{k_2}, 2^{k_3}\}$.

If H is a code over R, let H^+(respectively H^-) be the binary code such that $(1+v)H^+$(respectively vH^-) is read $H \bmod v$ (respectively $H \bmod (1+v)$).

We have

$H = (1+v)H^+ \oplus vH^-$.

With,

$H^+ = \{s \mid \exists t \in F_2^n \mid (1+v)s + vt \in H\}$;

$H^- = \{t \mid \exists s \in F_2^n \mid (1+v)s + vt \in H\}$.

The code H^+ is permutation equivalent to a code with generator matrix of the form

$$\begin{pmatrix} I_{k_1} & 0 & A_1 & A_2 & A_3 \\ 0 & I_{k_2} & 0 & A_4 & 0 \\ & & & & \end{pmatrix},$$

where A_i are binary matrices.

And the binary code H^- is permutation equivalent to a code with generator matrix of the form:

$$\begin{pmatrix} I_{k_1} & B_1 & 0 & B_2 & B_3 \\ 0 & 0 & I_{k_3} & 0 & B_4 \\ & & & & \end{pmatrix},$$

where B_i are binary matrices. The preceding statements show that any code

H over R is completely characterized by its associated codes H^+ and H^- and conversely.

3. R-Simplex codes of type α

Following (Bhandrri, et al., 1999, p.170-180), (Gupta, et al., 2001, p.112-121) and (Gupta, 2000, p. 1-98), we construct simplex codes over the ring R of type α in the following way.

For convenience we set $w = 1 + v$. Let G_k be a $k \times 2^{2k}$ matrix over R defined inductively by:

$$\left(\begin{array}{c|c|c|c} 00 \cdots 0 & 11 \cdots 1 & vv \cdots v & ww \cdots w \\ G_{k-1} & G_{k-1} & G_{k-1} & G_{k-1} \end{array}\right) \tag{3.1}$$

where $G_1 = (01vw)$.

The columns of G_k consist of all distinct $k-$ tuples over R. The code S_k^α generated by G_k has length 2^{2k}.

The following observations are useful to obtain Hamming, Lee, Bachoc and distribution weights of S_k^α.

Remark 3.1 *If A_{k-1} denotes the $(4^{k-1} \times 4^{k-1})$ array consisting of all codewords in S_{k-1}^α and $i = (i, i, ..., i)$, then the $(4^k \times 4^k)$ array of codewords of S_k^α is given by*

$$\begin{bmatrix} A_{k-1} & A_{k-1} & A_{k-1} & A_{k-1} \\ A_{k-1} & 1+A_{k-1} & v+A_{k-1} & w+A_{k-1} \\ A_{k-1} & v+A_{k-1} & v+A_{k-1} & A_{k-1} \\ A_{k-1} & w+A_{k-1} & A_{k-1} & w+A_{k-1} \\ & & & \end{bmatrix}.$$

Remark 3.2 *If $R_1, R_2, ..., R_k$ denote the rows of the matrix G_k^α, then*

- $wt_H(R_i) = 3 \cdot 2^{2(k-1)}$, $wt_H(vR_i) = wt_H(wR_i) = 2^{2k-1}$.
- $wt_L(R_i) = 2^{2k}$, $wt_L(vR_i) = wt_L(wR_i) = 2^{2k-1}$.
- $wt_B(R_i) = 5.2^{2(k-1)}$, $wt_B(vR_i) = wt_B(wR_i) = 2^{2k}$.

It may be observed that each element of R occurs equally often in every row of G_k^α.

Let $c = (c_1, c_2, ..., c_n) \in C$. For each $j \in R$, let $w_j(c) = |\{i \mid c_i = j\}|$, we have the following lemma.

Lemma 3.1 *Let $c \in S_k^\alpha, c \neq 0$. Then*

- If for at least one i, c_i is a unit, then $\forall j \in R, \omega_j = 4^{k-1}$.
- If $\forall\, i, c_i \in \{0, v\}$, then $\forall j \in \{0, v\}, \omega_j = 2^{2k-1}$ *in c.*
- If $\forall\, i, c_i \in \{0, w\}$, then $\forall j \in \{0, w\}, \omega_j = 2^{2k-1}$ *in c.*

Proof. By Remark (3.1) any $x \in S_{k-1}^\alpha$ gives rise to the following four codewords of S_k^α.

- $y_1 = (x \mid x \mid x \mid x)$.

- $y_2 = (x \mid 1+x \mid v+x \mid w+x)$.
- $y_3 = (x \mid v+x \mid v+x \mid x)$.
- $y_4 = (x \mid w+x \mid w+x \mid x)$.

The assertion follows by induction.

Now we will give some facts about binary simplex codes.

Let $G(S_k)$ (columns consisting of all nonzero binary k-tuples) be a generator matrix for an $[n,k]$ binary simplex code S_k. Then the extended binary simplex code $\hat{S}_k$ generated by the matrix.

$G(\hat{S}_k) = [0 \mid G(S_k)]$.

Inductively,

$$G(\hat{S}_k) = \left[\begin{array}{c|c} 00\cdots 0 & 11\cdots 1 \\ G(\hat{S}_{k-1}) & G(\hat{S}_{k-1}) \\ & \end{array}\right], \text{ with } G(\hat{S}_1) = [0\ 1] \qquad (3.2).$$

Lemma 3.2 *The H^+(or H^-) binary codes of S_k^α are equivalent to the 2^k copies of $\hat{S}_k$.*

Proof. First we will prove the H^+ case by induction on k. Observe that the binary H^+ code of S_k^α is the set of codewords obtained by replacing w by 1 in all $w-$ linear combination of the rows of the matrix wG_k (where G_k is defined in 3.1). For $k = 2$ the result holds and.

$$G_2 = \left[\begin{array}{c|c|c|c} 0000 & 1111 & vvvv & wwww \\ 01vw & 01vw & 01vw & 01vw \\ & & & \end{array}\right]$$

$$H^+ = \left[\begin{array}{c|c|c|c} 0000 & 1111 & 0000 & 1111 \\ 0101 & 0101 & 0101 & 0101 \\ & & & \end{array}\right]$$

If wG_{k-1} is permutation equivalent to 2^{k-1} copies of $wG(\hat{S}_{k-1})$, then the matrix wG_k takes the form:

$$\left[\begin{array}{c|c|c|c} 00\cdots 0 & ww\cdots w & 00\cdots 0 & ww\cdots w \\ \hline wG(\hat{S}_{k-1})\,|\cdots|\,wG(\hat{S}_{k-1}) & wG(\hat{S}_{k-1})\,|\cdots|\,wG(\hat{S}_{k-1}) & wG(\hat{S}_{k-1})\,|\cdots|\,wG(\hat{S}_{k-1}) & wG(\hat{S}_{k-1})\,|\cdots|\,wG(\hat{S}_{k-1}) \\ & & & \end{array}\right].$$

Now regrouping the columns according to (3.2) gives the desired result. The proof for the H^- case is similar to the above case.

Definition 3.1 *For each*
$1 \le i \le n$, let $A_H(i)$ ($A_L(i)$ or $A_B(i)$) be the number of code words of Hamming, Lee or Bachoc weight i in the code C,

Then
$\{A_H(0), A_H(1), ...A_H(n)\}, (\{A_L(0), A_L(1), ...A_L(n)\})$ or $(\{A_B(0), A_B(1), ...A_B(n)\})$ is called the Hamming (Lee) or (Bachoc) weight distribution of C.

The Hamming, Lee and Bachoc weight distributions of S_k^{α} are given in the following theorem.

Theorem 3.3 *Hamming, Lee and Bachoc weight distributions of S_k^{α} are:*

- $A_H(0)=1$, $A_H(2^{2k-1})=2(2^k-1)$ and $A_H(3.2^{2(k-1)})=(2^k-1)(2^k-1)$.
- $A_L(0)=1$, $A_L(2^{2k-1})=2.(2^k-1)$ and $A_L(4^k)=(2^k-1)(2^k-1)$.
- $A_B(0)=1$, $A_B(4^k)=2\cdot(2^k-1)$, $A_B(5.2^{2(k-1)})=(2^k-1)(2^k-1)$.

Proof. Note that
$A_H(0)=A_L(0)=A_B(0)=1$, $A_H(2^{2k-1})=A_L(2^{2k-1})=A_B(4^k)=2(2^k-1)$ and
$A_H(3\cdot 2^{2(k-1)})=A_L(4^k)=A_B(5\cdot 2^{2(k-1)})=(2^k-1)(2^k-1)$. By remark (3.2)
each nonzero codeword of S_k^{α} has Hamming weight either $3\cdot 2^{2(k-1)}$ or 2^{2k-1},
Lee weight is either 4^k or 2^{2k-1} and Bachoc weight is either $5\cdot 2^{2(k-1)}$ or 4^k.
And by

Lemma (3.2), the dimension of H^+ code of S_k^{α} is k, thus the number of codewords is 4^k and there will be $(2^k-1)(2^k-1)$ codewords of Hamming weight $3\cdot 2^{2(k-1)}$. Therefore the number of codewords having Hamming weight 2^{2k-1} is

$$4^k-[(2^k-1)(2^k-1)+1]=4^k-[2^{2k}-2\cdot 2^k+1+1]=4^k-4^k+2\cdot 2^k-2=2\cdot 2^k-2=2(2^k-1).$$

Similar arguments hold for the other weights.

The symmetrized weight enumerator (swe) of S_k^{α} is given by the following formula,

$$swe(x,y,z)=x^n+3^{2(k-1)}x^{4^{k-1}}y^{4^{k-1}}z^{2^{2k-1}}+2\cdot 3^{k-1}x^{2^{2k-1}}z^{2^{2k-1}}$$

Remark 3.3

- the Simplex code S_k^{α} is not equidistant with respect to Hamming, Lee and Bachoc distances.

- The minimum weights of S_k^α are: $d_H = 2^{2k-1}, d_L = 2^{2k-1}$ and $d_B = 2^{2k}$.

4. Simplex codes of type β

The length of S_k^α is large and increases fast, so we can omit some columns from G_k^α to obtain good codes over R of smaller length and we will call the simplex codes of type β.

Let λ_k be the $k \times 2^k(2^k - 1)$ matrix defined inductively by $\lambda_1 = [1v]$ and

$$\lambda_k = \left[\begin{array}{c|c|c|c} 00\cdots0 & 11\cdots1 & vv\cdots v & ww\cdots w \\ \lambda_{k-1} & G_{k-1}^\alpha & G_{k-1}^\alpha & \lambda_{k-1} \\ & & & \end{array}\right].$$

for $k \ge 2$ and let δ_k be the $k \times 2^k(2^k - 1)$ matrix defined inductively by $\delta_1 = [1w]$ and

$$\delta_k = \left[\begin{array}{c|c|c|c} 00\cdots0 & 11\cdots1 & vv\cdots v & ww\cdots w \\ \delta_{k-1} & G_{k-1}^\alpha & \delta_{k-1} & G_{k-1}^\alpha \\ & & & \end{array}\right].$$

For $k \ge 2$ where G_{k-1}^α is the generator matrix of S_{k-1}^α.

Now let G_k^β be the $k \times [(2^k - 1)(2^k - 1)]$ matrix defined inductively by

$$G_2^\beta = \left[\begin{array}{c|c|c|c} 1111 & 0 & vv & ww \\ 01vw & 1 & 1w & 1v \\ & & & \end{array}\right]$$

and for $k > 2$.

$$G_k^{\beta} = \left[\begin{array}{c|c|c|c} 11\cdots 1 & 00\cdots 0 & vv\cdots v & ww\cdots w \\ G_{k-1}^{\alpha} & G_{k-1}^{\beta} & \delta_{k-1} & \lambda_{k-1} \\ & & & \end{array}\right] \qquad (4.1).$$

Note that the generator matrix G_k^{β} is obtained by deleting $2^{k+1}-1$ columns of the generator matrix G_k^{α}. By induction it is easy to verify that no two columns of G_k^{β} are multiple of each other.

Now let S_k^{β} be the code generated by G_k^{β}, to determine the weight distribution of S_k^{β} we first make the following observations.

Remark 4.1 *Each row of G_k^{β} has Hamming weight $2^{k-2}[3(2^k-1)-1]$, Lee weight $2^k(2^k-1)$ and Bachoc weight $2^k[2(2^{k-1}-1)+2^{k-2}]$.*

Proposition 4.1 *Each row of G_k^{β} contains $2^{2(k-1)}$ units and* $\omega_v = \omega_w = 2^{2(k-1)} - 2^{k-1} = 2^{k-1}(2^{k-1}-1)$.

Proof. The result can be easily verified for $k=2$. Assume that the result holds for each row of G_{k-1}^{β}. Then the number of units in each row of G_{k-1}^{β} is equal to $2^{2(k-2)}$. By Lemma (3.1), the number of units in any row of G_{k-1}^{α} is 2^{2k-3}. Hence the total number of units in any row of G_k^{β} will be $2^{2k-3} + 2\cdot 2^{2(k-2)} = 2^{2(k-1)} = 4^{k-1}$. A similar argument holds for the number of $v's$ and $w's$.

Theorem 4.2 *The Hamming, Lee and Bachoc weight distributions of S_k^{β} are:*

- $A_H(0)=1, A_H(2^{k-2}(3(2^k-1)-1))=(2^k-1)(2^k-1)$ and $A_H(2^{k-1}(2^k-1))=2(2^k-1)$.

- $A_L(0) = 1$, $A_L(2^{k-1}(2^k - 1)) = 2(2^k - 1)$ and $A_L(2^k(2^k - 1)) = (2^k - 1)(2^k - 1)$
- $A_B(0) = 1$, $A_B(2^k[2(2^{k-1} - 1) + 2^{k-2}]) = (2^k - 1)(3 + 2^{k-1})$ and $A_B(2^k(2^k - 1)) = 2 \cdot 3^{k-3}(2^k - 1)$.

Proof. Similar to the proof of theorem(3.3).

Remark 4.2

- The minimum Hamming weight of S_k^β is $d_H = 2^{k-1}(2^k - 1)$.
- The minimum Lee weight of S_k^β is $d_L = 2^{k-1}(2^k - 1)$.
- The minimum Bachoc weight of S_k^β is $d_B = 2^k(2(2^{k-1} - 1) + 2^{k-2})$.
- $d_H = d_L \leq \dfrac{d_B}{2}$ for S_k^β.

Now we will give the Macwilliams relations of S_k^β.

Remark 4.3

$$W_c(x, y) = x^n + q(k)x^{n-h(k)}y^{h(k)} + nx^{n-f(k)}y^{f(k)},$$

where $q(k) = 2(2^k - 1), h(k) = 2^{k-1}(2^k - 1), f(k) = 2^{k-2}(3(2^k - 1) - 1)$.

$$swe(x, y, z) = x^n + nx^{\rho(k)}y^{\delta(k)}z^{n-\rho(k)-\delta(k)} + 2(2^k - 1)x^{n-h(k)}z^{h(k)},$$

where

$n = L(k) = (2^k - 1)(2^k - 1), h(k) = 2^{k-1}(2^k - 1), \rho(k) = L(k-1) = (2^{k-1} - 1)(2^{k-1} - 1)$ and $\delta(k) = 2^{2(k-1)}$.

Remark 4.4

- $S_k^\alpha(S_k^\beta)$ are Hermitian self-orthogonal codes.
- S_k^α is self-orthogonal codes with Euclidean inner product, but S_k^β is not.
- The $S_k^\alpha(S_k^\beta)$ codes do not a chive the inequality

$$d_B \leq 2(1+\lfloor \frac{n}{3} \rfloor),$$

and so they are not Hermitian self-dual codes.

4.1 Conclusion

In this paper we have studied simplex codes of types α and β over the ring $F_2 + vF_2$. This study can be extended to study simplex codes over more rings such as $F_p + vF_p$ where p is prime integer.We hope we can find the number of errors which simplex codes will detect and correct.

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