An-Najah National University
Faculty of Graduate studies

RADIUS OF CONVERGENCE FOR SOME (MULTIVARIABLE) HYPERGEOMETRIC FUNCTIONS

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This Thesis is submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics, Faculty of Graduate Studies, An-Najah National University, Nablus, Palestine

2024

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Dedication

To my loving parents, my dad Yahya Kashou and my mother Tahani Kashou, whose unwavering support and encouragement have been my driving force. Their belief in me has been a constant source of strength, and I am forever grateful for their endless love and sacrifices.

To my siblings, Lara Kashou, Ladin Kashou, Lujayn Kashou and Leen kashou, who have always stood by me, cheering me on through every challenge, your support has meant the world to me.

To the homeland of the martyrs, and the blood spilling over the breadth and length of this country, to the steadfast Gaza, its people and its resistance.

Acknowledgement

Foremost, my grateful thanks to Allah, the almighty, for granting me the blessings and giving me the power and energy to conduct this humble work.

I would like to express my deepest gratitude to the discussion committee members, starting with my supervisor Dr. Muath Karki for his guidance, encouragement and the internal and external examiners Dr. Yahay Jaafra, Dr. Mohammad Marabeh for their time and generosity in reading this humble work.

I also would like to express my sincere gratitude and thanks to Dr. Ahmed Basalat for his encouragement, unlimited care and giving me the opportunity to train in France for two months to learn more about the subject, and to Prof. Samuel Friot for supervising me during that time.

I want to express my heartfelt gratitude to all those who have imparted knowledge to me, from the dedicated teachers of my school days to the esteemed professors and doctors at my university An-Najah National University. Your dedication has had a huge impact on me.

Last but not least, I am grateful to everyone who has supported in completing this dissertation.

Declaration

I, the undersigned, declare that I submitted the thesis entitled:

RADIUS OF CONVERGENCE FOR SOME (MULTIVARIABLE) HYPERGEOMETRIC FUNCTIONS

I declare that the work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's Name: Layth Yahya Abdulrahim Kashou

Signature:

Date: 15/07/2024

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RADIUS OF CONVERGENCE FOR SOME (MULTIVARIABLE) HYPERGEOMETRIC FUNCTIONS

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Abstract

Background: Hypergeometric functions are a class of special functions in mathematics that play a crucial role in various branches of science and engineering. Its importance lies in their versatility and their ability to represent a vast array of mathematical and physical phenomena. The key aspects that underscore their significance and applications include solutions to differential equations, solving Schrödinger's equation for various physical systems, studying complex integrals and contour integrals, solving problems involving electromagnetic fields and wave propagation in different media and they have applications in celestial mechanics for predicting the positions and orbits of celestial bodies.

Aims: We have two main objectives. The first one is deriving new transformation formulas for the Kampé de Fériet function taking into account the radius of convergence of each transformation. While the other is developing alternative methods for determining the radius of convergence for well-known multivariable (double and triple) hypergeometric series.

Methods: In this thesis, we will use Miller-Paris transformation formulas for generalized hypergeometric functions ${}_{r+1}F_{r+1}(z)$, ${}_{r+2}F_{r+1}(z)$ and its radii of convergence to derive new transformation formulas for the Kampé de Fériet function. Also, we will use Mathematica to develop an alternative method to calculate the radii of convergence for some well-known multivariable hypergeometric series.

Results: While testing the Kampé de Fériet transformations we derived using various parameter values and variables within the radius of convergence on Mathematica, we found that the left-hand side and the right-hand side are equal for all tested cases. Also, when we try to calculate the radii of convergence for the selected well-known multivariable hypergeometric series by plotting them on Mathematica the results indicate that our findings

are identical with the ones presented in Srivastava's book.

Keywords: Hypergeometric functions, Kampé de Fériet function, Radius of convergence, Miller-Paris transformations, Srivastava-Daoust series, Horn series, Appell hypergeometric functions, Gaussian Hypergeometric function, Mellin-Barnes integral.

Chapter One

Gaussian Hypergeometric Function ${}_2F_1$ and its Well-Known Transformations using Euler Integral and Mellin-Barnes integral

1.1 Introduction

Within this research, we embark on an exploration of certain aspects of the transformation theory of (multivariable) hypergeometric functions, as well as the convergence properties of their series representations. Our initial chapter One, delves into introducing the Gaussian Hypergeometric function ${}_2F_1(z)$, outlining its definition, its Euler integral representation and some of its well-known transformations. However, these transformations do not make it possible to extend the range of z. Consequently, we will see how the Gaussian hypergeometric series can be extended by analytic extension for large values of z using the integral representation of Mellin-Barnes.

In chapter Two, we will introduce the Miller-Paris transformation formulas for the generalised hypergeometric functions ${}_{r+1}F_{r+1}(z)$ and ${}_{r+2}F_{r+1}(z)$. But as we will see, these transformations face certain limitations, especially when the free bottom parameter is slightly greater than the free top parameter by a positive integer. However, in [1], this challenge has been addressed by Karp and Prilepkina, who have ingeniously solved this problem. They achieved this by computing the limit cases of these transformations, even in situations that were previously considered impossible. This leads us to new transformations for the generalised hypergeometric functions ${}_{r+1}F_{r+1}(z)$ and ${}_{r+2}F_{r+1}(z)$ as we will see in section 2.2.

As we advance further in our thesis and proceed to chapter Three, we will build upon the previously introduced Miller-Paris transformations for the generalised hypergeometric functions ${}_{r+1}F_{r+1}(z)$ and ${}_{r+2}F_{r+1}(z)$. Utilizing these transformations, we will proceed to derive new transformations for the Kampé de Fériet function. Additionally, we will use the radius of convergence for Miller-Paris transformations and Kampé de Fériet function to calculate the radius of convergence for each transformation, while also testing each transformation (for various parameter values and variable z within the radius of convergence of each series) using Mathematica.

Moving on to the last chapter Four, we will shift our focus to other well-known double hypergeometric series, specifically the Appell hypergeometric functions denoted as F_1, F_2, F_3 and F_4, and the Horn series encompassing G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_5, H_6 and H_7. In this section, we will strive to calculate the radius of convergence for each of these series by rewriting each one of them as the Srivastava-Daoust series. As we progress, we will compare our results with those obtained in ([2], p.59- p.60), to validate our calculations and ensure the accuracy of our conclusions.

Actually, this analysis bears importance, as Srivastava's book ([2], p.59- p.65) does not explicitly how to establish some formulas, which are an essential component for applying the Horns theorem to calculate the radius of convergence for double hypergeometric series. By delving into this crucial part, we aim to shed light on this intriguing aspect, contributing to the study of hypergeometric series and their convergence properties.

Furthermore, we will delve into the study of Srivastava's triple hypergeometric series H_c, along with other triple hypergeometric series introduced in ([2], p70- p.107). Our main objective is to determine their respective radii of convergence by reformulating them as Srivastava-Daoust series. By utilizing this approach, we aim to compare our results with the findings presented in ([2], p.70- p.107) ensuring the reliability and accuracy of our analysis as we said above.

1.2 The Gaussian hypergeometric function ${}_2F_1$

The Gaussian hypergeometric function ${}_2F_1$ is defined as one of the six solutions to a certain type of differential equation known as the Gauss's hypergeometric differential equation ([3], p.242-p.243)

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0 \tag{1.1}$$

This function is a special function represented by the hypergeometric series

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, \quad |x|<1 \tag{1.2}$$

The notation $(\omega)_n$ refers to Pochhammer symbols and is defined as $(\omega)_n = \Gamma(\omega+n)/\Gamma(\omega)$ where n is natural number and $\omega \neq 0$. Using the ratio test we can easily find that the above series convergence for all x such that $|x| < 1$. Outside of this domain, Gauss's hypergeometric function ${}_2F_1$ can be extended by analytic extension as we will explain in the next sections. In this first part of our thesis, we will study the function ${}_2F_1$: its different representations, transformations and analytic extensions.

1.2.1 Express hypergeometric function ${}_2F_1$ using Euler integral

The Euler integral representation should not be confused with the Euler integral of the first kind, called the Beta function (1.4), nor with the Euler integral of the second kind, called the Gamma function (1.3) (See [4],p.116-p117 and p.134-p135)

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \ , \ \Re(x) > 0 \tag{1.3}$$

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \ , \ \Re(x) > 0, \Re(y) > 0 \tag{1.4}$$

we call the following integral(1.5), Euler integral

$$\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-ux)^{-a}du \tag{1.5}$$

Let us try to calculate the following integral

$$f(x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-ux)^{-a}du \tag{1.6}$$

we can use binomial theorem to express the term $(1-ux)^{-a}$ as the following series

$$\begin{aligned}(1-ux)^{-a} &= 1 + axu + a(a+1)\frac{(xu)^2}{2!} + \ \ldots \ + \ a(a+1)\ldots(a+n-1)\frac{(xu)^n}{n!} + \ldots \\ &= \sum_{n=0}^{\infty}(a)_n\frac{(xu)^n}{n!} \end{aligned}\tag{1.7}$$

By using term-by-term integration theorem, definition of the Euler integral of the first kind(beta function) and Pochhammer's symbol we will obtain:

$$f(x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n \int_0^1 u^{b+n-1}(1-u)^{c-b-1} du = {}_2F_1(a,b;c;x) \quad (1.8)$$

1.2.2 Transformations of ${}_2F_1$ by changes of variables

In order to obtain linear transformations for the hypergeometric function ${}_2F_1$ by changes of variables, we will take the integral mentioned above (1.6). For simplicity, we will take h to be the following function:

$$h(x) = \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-ux)^{-a} du \quad (1.9)$$

1. **<u>First Transformation</u>**

   If we consider the following change of variable:

$$\begin{cases} u = 1 - v \\ du = -dv \end{cases} \quad (1.10)$$

   we will get the following integral:

$$\begin{aligned} h(x) &= \int_0^1 v^{c-b-1}(1-v)^{b-1}(1-x+vx)^{-a} dv \\ &= (1-x)^{-a} \int_0^1 v^{c-b-1}(1-v)^{b-1} \left(1 - \frac{x}{x-1} v\right)^{-a} dv \\ &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} (1-x)^{-a} \, {}_2F_1\left(a, c-b, c; \frac{x}{x-1}\right) \end{aligned} \quad (1.11)$$

   As a result, the following transformation will obtained:

$${}_2F_1(a,b,c;x) = (1-x)^{-a} \; {}_2F_1\left(a, c-b, c; \frac{x}{x-1}\right) \quad (1.12)$$

   The above expression will be convergence by ratio test for $|\frac{x}{x-1}| < 1$, which implies $|x| < |x-1|$. This means that above transformation (1.12) convergence for all x such that $|x| < 1$, $\Re(x) < 1/2$ [1].

[1] $\Re(x)$ means real part of x.

2. **Second Transformation**

If we consider the following change of variable:

$$\begin{cases} u = \dfrac{v}{1 - x + vx} \\ du = \dfrac{(1 - x + vx) - vx}{(1 - x + vx)^2} dv = \dfrac{1 - x}{(1 - x + vx)^2} dv \end{cases} \tag{1.13}$$

we will get the following integral:

$$\begin{aligned} h(x) &= \int_0^1 \frac{v^{b-1}(1-v)^{c-b-1}(1-x)^{c-b-a}}{(1-x+vx)^{c-a}} dv \\ &= (1-x)^{-b} \int_0^1 v^{b-1}(1-v)^{c-b-1} \left(1 - \frac{x}{x-1} v\right)^{a-c} dv \\ &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} (1-x)^{-b} \ {}_2F_1 \left(c - a, b, c; \frac{x}{x-1}\right) \end{aligned} \tag{1.14}$$

As a result, the following transformation will obtained:

$${}_2F_1(a, b, c; x) = (1-x)^{-b} \ {}_2F_1 \left(c - a, b, c; \frac{x}{x-1}\right) \tag{1.15}$$

Similarly, the above expression will be convergence for all x such that $|x| < 1$, $\Re(x) < 1/2$.

3. **Third Transformation**

If we consider the following change of variable:

$$\begin{cases} u = \dfrac{1 - v}{1 - vx} \\ du = \dfrac{-(1 - vx) + (1 - v)x}{(1 - vx)^2} = \dfrac{x - 1}{(1 - vx)^2} dv \end{cases} \tag{1.16}$$

we will get the following integral:

$$\begin{aligned} h(x) &= \int_0^1 v^{c-b-1}(1-v)^{b-1}(1-x)^{c-b-a}(1-vx)^{a-c} dv \\ &= (1-x)^{c-b-a} \int_0^1 v^{c-b-1}(1-v)^{b-1}(1-vx)^{a-c} dv \\ &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} (1-x)^{c-b-a} \ {}_2F_1(c-a, c-b, c; x) \end{aligned} \tag{1.17}$$

As a result, the following transformation will obtained:

$$ {}_2F_1(a,b,c;x) = (1-x)^{c-b-a}\ {}_2F_1(c-a,c-b,c;x) \tag{1.18} $$

Also, the above expression will be convergence for all x such that $|x| < 1$.

As we see above, we have succeeded in obtaining new transformations for the hypergeometric function ${}_2F_1$ using Euler integral. But these transformations do not make it possible to extend the range of x. Therefore, another representation is necessary. In the next section (1.3), we will prove that Gaussian hypergeometric series can be extended by analytic extension for large values of x.

1.3 Integral representation of Mellin-Barnes

Any integral on the following form is called Mellin-Barnes integral ([5], p.49)

$$ \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} x^{-s}\frac{\prod_{j=1}^{m}\Gamma(a_j s+b_j)}{\prod_{k=1}^{n}\Gamma(c_k s+d_k)}ds \tag{1.19} $$

In order to study the hypergeometric function ${}_2F_1$, we will define the function g to be the following MB-integral ([6], p.292)

$$ g(x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}(-x)^s\,\Gamma(-s)\frac{\Gamma(b+s)\Gamma(a+s)}{\Gamma(c+s)}ds \tag{1.20} $$

where $-a < -b < 0$, $a-b,\ b-c$, and $a-c$ not a natural numbers (we will see below why we put these conditions on a, b, c).
The calculation of (1.20) goes through the residue theorem ([7], p.231-p.252). For simplicity, we will take f to be as follows:

$$ f(s) = (-x)^s\,\Gamma(-s)\frac{\Gamma(b+s)\Gamma(a+s)}{\Gamma(c+s)} \tag{1.21} $$

The pols of f will be as follows:

1. For the term $(-x)^s$, f have pole when $\Re(s) < 0$ and $x = 0$. But when $x = 0$, $f(x) = 0$ so we will assume that $x \neq 0$.

2. To find the poles of $\Gamma(-s)$, we will consider the following definition for Gamma function:

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{z}}{1+\frac{z}{n}} \tag{1.22}$$

Its clear that $\Gamma(z)$ has poles at $\{z = 0$ and $1 + \frac{z}{n} = 0\}$ which equivelent to $\{z = 0$ and $z = -n\}$. Therefore, $\Gamma(-s)$ has poles at $\{z_n = n \mid n \in \mathbb{N} \cup \{0\}\} = \mathcal{N}$.

3. Using the same definition for Gamma, we can find the poles for $\Gamma(a + s)$. Therefore, the poles will be at $\{z_k = -(k + a) \,|k \in \mathbb{N} \cup \{0\}\} = \mathcal{K}$.

4. Similarly, the poles for $\Gamma(b + s)$ will be at $\{z_j = -(j + b) \,|j \in \mathbb{N} \cup \{0\}\ \} = \mathcal{J}$.

5. Also, from the same definition of Gamma it's clear that $\forall z \in \mathbb{C}$, $\frac{1}{\Gamma(z)} \neq 0$, which means $\frac{1}{\Gamma(z)}$ has no poles. Therefore, $\frac{1}{\Gamma(s+c)}$ has no poles.

It's important to note that, no pole can disappear or pass from order 1 to order 2, 3 or more when we product the Gamma functions because $a - b,\ b - c,\ a - c$ not a natural number. Therefore, the set of points $\mathcal{N} \cup \mathcal{K} \cup \mathcal{J} = \mathcal{F}$ is denote the all poles of f with degree 1.

As a result, the integral can be calculated from two choices of distinct closed contours. They are represented below (see Figure (1)) by contours $\mathcal{C}_{\gamma 1}$ and $\mathcal{C}_{\gamma 2}$. It can be seen graphically that each of the contours encloses a certain subset of residuals. Thus, we obtain the following two cases:

1. along first contour $\mathcal{C}_{\gamma 1}$

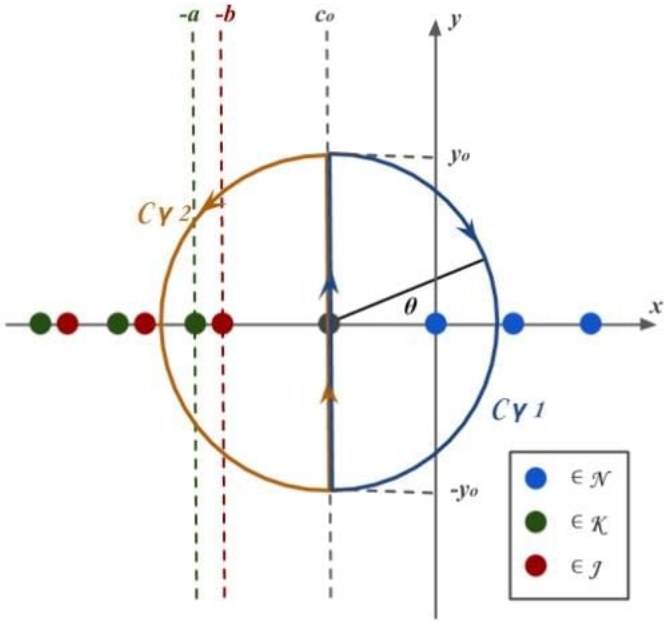
$$\lim_{y_0\to\infty}\oint_{\mathcal{C}_{\gamma 1}(y_0)} f(z)dz = 2\pi i\sum_{n=0}^{\infty}(-1).res(f;n)$$

2. along second contour $\mathcal{C}_{\gamma 2}$

$$\lim_{y_0\to\infty}\oint_{\mathcal{C}_{\gamma 2}(y_0)} f(z)dz = 2\pi i\left[\sum_{k=0}^{\infty}res(f;-(k+a)) + \sum_{j=0}^{\infty}res(f;-(j+b))\right]$$

Figure 1

Graphical representation of contours and residuals



1.3.1 Calculation the integral via first contour $\mathcal{C}_{\gamma 1}$

In our case, we have residues of order 1 which are simply expressed by:

$$res(f;n) = \lim_{z \to n}(z-n)f(z) = \lim_{z \to n}(z-n)(-x)^z \Gamma(-z) \frac{\Gamma(a+z)\Gamma(b+z)}{\Gamma(c+z)} \tag{1.23}$$

Take m_n to be natural number such that $m_n > n$, then using Pochhammer's symbol we have:

$$\Gamma(-z) = \frac{\Gamma(-z+m_n)}{(-z)_{m_n}} = \frac{\Gamma(-z+m_n)}{(-z+m_n-1)\dots(-z+n)\dots(-z)} \tag{1.24}$$

Therefore, equation (1.23) can be written as

$$\begin{aligned} res(f;n) &= \lim_{z \to n}(z-n)(-x)^z \frac{\Gamma(-z+m_n)\Gamma(a+z)\Gamma(b+z)}{(-z+m_n-1)\dots(-z+n)\dots(-z)\Gamma(c+z)} \\ &= \lim_{z \to n} \frac{(-1)(-x)^z \Gamma(-z+m_n)\Gamma(a+z)\Gamma(b+z)}{(-z+m_n-1)\dots(-z+n+1)(-z+n-1)\dots(-z)\Gamma(c+z)} \\ &= (-1)(-x)^n \frac{\Gamma(a+n)\Gamma(b+n)}{(-1)\dots(-n)\Gamma(c+n)} \\ &= -\frac{(x)^n}{n!} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \end{aligned} \tag{1.25}$$

Thus,

$$\begin{aligned}\lim_{y_0\to\infty}\oint_{\mathcal{C}_{\gamma 1}(y_0)} f(z)dz &= 2\pi i\sum_{n=0}^{\infty}\frac{(x)^n}{n!}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \\ &= 2\pi i\,\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{(x)^n}{n!} \\ &= 2\pi i\,\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\;{}_2F_1(a,b,c;x)\end{aligned} \tag{1.26}$$

As a result, a second integral representation of the Gaussian hypergeometric series ${}_2F_1$ will obtained:

$${}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}(-x)^s\,\Gamma(-s)\frac{\Gamma(b+s)\Gamma(a+s)}{\Gamma(c+s)}ds \tag{1.27}$$

1.3.2 Calculation the integral via second contour $\mathcal{C}_{\gamma 2}$

Also, in this case we have residues of order 1 which are simply expressed by:

$$\begin{aligned}res(f;-(k+a)) &= \lim_{z\to-(k+a)}(z+k+a)f(z) \\ &= \lim_{z\to-(k+a)}(z+k+a)(-x)^z\Gamma(a+z)\frac{\Gamma(-z)\Gamma(b+z)}{\Gamma(c+z)}\end{aligned} \tag{1.28}$$

Take m_k to be natural number such that $m_k > k$, then using Pochhammer's symbol we have:

$$\Gamma(z+a) = \frac{\Gamma(z+a+m_k)}{(z+a)_{m_k}} = \frac{\Gamma(z+a+m_k)}{(z+a+m_k-1)\dots(z+a+k)\dots(z+a)} \tag{1.29}$$

Therefore, equation (1.28) can be written as:

$$\begin{aligned}res(f;-(k+a)) &= \lim_{z\to-(k+a)}(z+k+a)f(z) \\ &= \lim_{z\to-(k+a)}\frac{(z+k+a)(-x)^z\Gamma(z+a+m_k)\Gamma(-z)\Gamma(b+z)}{(z+a+m_k-1)\dots(z+a+k)\dots(z+a)\Gamma(c+z)} \\ &= (-x)^{-(k+a)}\frac{\Gamma(k+a)\Gamma(b-a-k)}{(-1)\dots(-k)\Gamma(c-a-k)} \\ &= (-x)^{-a}\frac{(x)^{-k}}{k!}\;\frac{\Gamma(k+a)\Gamma(b-a-k)}{\Gamma(c-a-k)}\end{aligned} \tag{1.30}$$

By applying the formula

$$\Gamma(\alpha - n) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)(-1)^n}{\Gamma(n+1-\alpha)}\ , \quad \forall \alpha \notin -\mathbb{N} \cup \{0, 1, 2, \ldots, n+1\} \tag{1.31}$$

we will obtain:

$$\begin{aligned}\frac{\Gamma(k+a)\Gamma(b-a-k)}{\Gamma(c-a-k)} &= \frac{\Gamma(b-a)\Gamma(1-b+a)}{\Gamma(1-b+a+k)}\ \frac{\Gamma(1-c+a+k)}{\Gamma(c-a)\Gamma(1-c+a)}\ \Gamma(a)(a)_k \\ &= \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(c-a)} \frac{(a)_k(1-c+a)_k}{(1-b+a)_k}\end{aligned} \tag{1.32}$$

Therefore, equation (1.30) can be simplified as follows:

$$\begin{aligned}res(f; -(k+a)) &= (-x)^{-a} \frac{(x)^{-k}}{k!}\ \frac{\Gamma(k+a)\Gamma(b-a-k)}{\Gamma(c-a-k)} \\ &= (-x)^{-a} \frac{(x)^{-k}}{k!}\ \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(c-a)}\ \frac{(a)_k(1-c+a)_k}{(1-b+a)_k}\end{aligned} \tag{1.33}$$

As a result,

$$\begin{aligned}2\pi i \sum_{k=0}^{\infty} res(f; -(k+a)) &= 2\pi i\ (-x)^{-a} \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(c-a)} \sum_{k=0}^{\infty} \frac{(\frac{1}{x})^k}{k!}\ \frac{(a)_k(1-c+a)_k}{(1-b+a)_k} \\ &= 2\pi i\ (-x)^{-a} \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(c-a)}\ {}_2F_1\left(a, 1-c+a, 1-b+a; \frac{1}{x}\right)\end{aligned} \tag{1.34}$$

If we applied the same calculations on $res(f; -(j+b))$ we will get:

$$\begin{aligned}res(f; -(j+b)) &= \lim_{z \to -(j+b)} (z+j+b) f(z) \\ &= (-x)^{-b} \frac{(x)^{-j}}{j!}\ \frac{\Gamma(b)\Gamma(a-b)}{\Gamma(c-b)}\ \frac{(b)_j(1-c+b)_j}{(1-a+b)_j}\end{aligned} \tag{1.35}$$

As a result,

$$\begin{aligned}2\pi i \sum_{j=0}^{\infty} res(f; -(j+b)) &= 2\pi i\ (-x)^{-b}\ \frac{\Gamma(b)\Gamma(a-b)}{\Gamma(c-b)} \sum_{j=0}^{\infty} \frac{(\frac{1}{x})^j}{j!}\ \frac{(b)_j(1-c+b)_j}{(1-a+b)_j} \\ &= 2\pi i\ (-x)^{-b} \frac{\Gamma(b)\Gamma(a-b)}{\Gamma(c-b)}\ {}_2F_1\left(b, 1-c+b, 1-a+b; \frac{1}{x}\right)\end{aligned} \tag{1.36}$$

Using the residue theorem the final result for the integral via the second contour $\mathcal{C}_{\gamma_2}$:

$$\begin{aligned}\lim_{y_0\to\infty}\oint_{\mathcal{C}_{\gamma 2}(y_0)} f(z)dz &= 2\pi i\left[\sum_{k=0}^{\infty} res(f;-(k+a)) + \sum_{j=0}^{\infty} res(f;-(j+b))\right] \\ &= 2\pi i\,(-x)^{-a}\frac{\Gamma(a)\Gamma(b-a)}{\Gamma(c-a)}\,{}_2F_1\left(a, 1-c+a, 1-b+a; \frac{1}{x}\right) \\ &+ 2\pi i\,(-x)^{-b}\frac{\Gamma(b)\Gamma(a-b)}{\Gamma(c-b)}\,{}_2F_1\left(b, 1-c+b, 1-a+b; \frac{1}{x}\right)\end{aligned} \tag{1.37}$$

It's clear that the above representation convergence for $|1/x| < 1$ which equivalent to $|x| > 1$. So we prove that Gaussian hypergeometric series ${}_2F_1$ can be extended by analytic extension for large values of x.

Chapter Two

Miller-Paris Transformation Formulas and their Degenerate Cases

2.1 Miller Paris transformation formulas for the generalized hypergeometric functions

The generalized hypergeometric function ${}_pF_q$ can be expressed as the following series:

$$
\begin{aligned}
{}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) &= {}_pF_q\left[\begin{matrix} a_1, & a_2, & \dots, & a_p \\ b_1, & b_2, & \dots, & b_q \end{matrix}; z\right] \\
&= \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{z^k}{k!}
\end{aligned} \tag{2.1}
$$

where the above series converges for $|z| < 1$ when $p = q + 1$, and for any z when $p \leq q$(using ratio test). In this section, we will consider the generalized hypergeometric functions ${}_{r+1}F_{r+1}(z)$, ${}_{r+2}F_{r+1}(z)$ and introduce the Miller-Paris transformations for these two generalized hypergeometric functions.

Let us define $\mathbf{m} = (m_1, m_2, \dots, m_r) \in \mathbb{N}^r$, $m = m_1 + m_2 + \cdots + m_r$, $\mathbf{f} = (f_1, f_2, \dots, f_r) \in \mathbb{C}^r$ and $m = 0$ when **m** is empty, then as we see in [8], the Miller-Paris transformations for ${}_{r+2}F_{r+1}$, ${}_{r+1}F_{r+1}$ will be as follows:

2.1.1 The first Miller-Paris transformation formula

In subsection (1.2.2), we see that the first Euler transformation for the Gauss hypergeometric function has the following formula:

$$
{}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; x\right] = (1 - x)^{-a} {}_2F_1\left[\begin{matrix} a, & c - b \\ & c \end{matrix}; \frac{x}{x - 1}\right] \tag{2.2}
$$

where $|x| < 1, \Re(x) < \frac{1}{2}$.

The first Miller-Paris transformation formula is a generalization for equation (2.2), and has the following expression:

$$
{}_{r+2}F_{r+1}\left[\begin{matrix} a, & b, & \mathbf{f+m} \\ & c, & \mathbf{f} \end{matrix}; x\right] = (1 - x)^{-a} {}_{m+2}F_{m+1}\left[\begin{matrix} a, & c - b - m, & \boldsymbol{\zeta} + 1 \\ & c, & \boldsymbol{\zeta} \end{matrix}; \frac{x}{x - 1}\right] \tag{2.3}
$$

where $b \neq f_j$, $j = 1, 2, \ldots, r$, $(c-b-m)_m \neq 0$, $|x| < 1, \Re(x) < \frac{1}{2}$ and $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_m)$ are the roots for the characteristic polynomial:

$$Q_m(t) = \frac{1}{(c-b-m)_m} \sum_{k=0}^{m} (b)_k C_{k,r}(t)_k (c-b-m-t)_{m-k} \tag{2.4}$$

with $C_{0,r} = 1, C_{m,r} = 1/(\mathbf{f})_{\mathbf{m}} = [(f_1)_{m_1} \ldots (f_r)_{m_r}]^{-1}$ and

$$C_{k,r}(\mathbf{f}) = \frac{(-1)^k}{k!} {}_{r+1}F_r \left[\begin{matrix} -k, \ \mathbf{f+m} \\ \mathbf{f} \end{matrix} ; 1 \right] \tag{2.5}$$

2.1.2 The second Miller-Paris transformation formula

One of the well-known transformations is Kummer's transformation for the confluent hypergeometric function, which can be expressed as:

$$ {}_1F_1 \left[\begin{matrix} b \\ c \end{matrix} ; x \right] = e^x {}_1F_1 \left[\begin{matrix} c-b \\ c \end{matrix} ; -x \right] \tag{2.6}$$

where $|x| < \infty$.

The second Miller-Paris transformation is a generalization for equation (2.6), and has the following expression:

$$ {}_{r+1}F_{r+1} \left[\begin{matrix} b, \ \mathbf{f+m} \\ c, \ \mathbf{f} \end{matrix} ; x \right] = e^x {}_{m+1}F_{m+1} \left[\begin{matrix} c-b-m, \ \boldsymbol{\zeta}+1 \\ c, \ \boldsymbol{\zeta} \end{matrix} ; -x \right] \tag{2.7}$$

where $b \neq f_j$, $j = 1, 2, \ldots, r$, $(c-b-m)_m \neq 0$ and $x \in \mathbb{C}$. Using the two formulas (2.4), (2.5) we can obtain the vector of roots $\boldsymbol{\zeta}$.

2.1.3 The third Miller-Paris transformation formula

In subsection (1.2.2), we see that the third Euler transformation for the Gauss hypergeometric function is expressed as:

$$ {}_2F_1 \left[\begin{matrix} a, \ b \\ c \end{matrix} ; x \right] = (1-x)^{c-a-b} {}_2F_1 \left[\begin{matrix} c-a, \ c-b \\ c \end{matrix} ; x \right] \tag{2.8}$$

where $|x| < 1$.

The third Miller-Paris transformation is a generalization for equation (2.8), and has the following expression:

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} a,\ b,\ \mathbf{f+m} \\ c,\ \mathbf{f} \end{matrix};x\right] = (1-x)^{c-a-b-m}{}_{m+2}F_{m+1}\left[\begin{matrix} c-a-m,\ c-b-m,\ \boldsymbol{\eta}+1 \\ c,\ \boldsymbol{\eta} \end{matrix};x\right] \tag{2.9} $$

where $(c-a-m)_m \neq 0$, $(c-b-m)_m \neq 0$,$(1+a+b-c)_m \neq 0$, $|x| < 1$, and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_m)$ are the roots for the characteristic polynomial:

$$ \hat{Q}_m(t) = \sum_{k=0}^{m} \frac{(-1)^k C_{k,r}(a)_k(b)_k(t)_k}{(c-a-m)_k(c-b-m)_k} {}_3F_2\left[\begin{matrix} -m+k,\ t+k,\ c-a-b-m \\ c-a-m+k,\ c-b-m+k \end{matrix};1\right] \tag{2.10} $$

Remark: It is clear that when **m** is empty, then equations (2.3),(2.7) and (2.9) can be reduced to equations (2.2), (2.6) and (2.8) respectively.

2.2 Degenerate Miller-Paris transformations

In the previous section, we define Miller Paris transformations for the generalized hypergeometric functions ${}_{r+1}F_{r+1}(z)$ and ${}_{r+2}F_{r+1}(z)$ as given in formulas (2.3), (2.7), and (2.9). However, it is important to note that these three transformations fail when $(c-b-m)_m = 0$, which occurs when $c-b \in \{1, \ldots, m\}$.

To address this limitation, Karp and Prilepkina in [1] provide a solution by introducing a separate theorem for each transformation. Before presenting the three theorems, we need to mention two lemmas that elucidate the behavior of the characteristic polynomials Q_m^ϵ and $\hat{Q}_m^\epsilon$ as $\epsilon \to 0$.

These lemmas are crucial in understanding the behavior of the transformations and pave the way for introducing the theorems that resolve the issues when $(c-b-m)_m = 0$.

Lemma 1. *Write Q_m^ϵ for the polynomial of* (2.4) *with $c-b-m = -q+\epsilon$ where $q \in \{0, 1, \ldots, m-1\}$, $\epsilon > 0$, and let $\zeta_1(\epsilon), \ldots, \zeta_m(\epsilon)$ denote its zeros. Suppose $f_j - b \notin \{m-q-m_j, m-q-m_j+1, \ldots, 0\}$ for $0 \leq j \leq r$ such that $m-q-m_j \leq 0$. Then the following are true as $\epsilon \to 0$*

1. *the first (after possible renumbering) $q+1$ zeros tend to consecutive non-positive in-*

tegers: $\zeta_1 \to 0,\ \zeta_2 \to -1, \dots, \zeta_{q+1} \to -q$.

2. *The remaining* $m-q-1$ *zeros (no such zeros remain if* $q = m-1$*) tend to the zero of the reduced polynomial*

$$R_{m-q-1}(t) = \sum_{k=0}^{m} (-1)^k (b)_k C_{k,r} (1-t-k)_{m-q-1} \tag{2.11}$$

of degree $m-q-1$.

3. *The limit relation*

$$\lim_{\epsilon \to 0} \frac{\epsilon}{\zeta_1(\epsilon)} = \frac{R_{m-q-1}(0)}{(m-q-1)!} \tag{2.12}$$

holds.

4. *The values* $Q_m^\epsilon(-l)$ *at the first* $q+1$ *non positive integers converges (as* $\epsilon \to 0$*) to the following expression :*

$$\lim_{\epsilon \to 0} {Q^\epsilon}_m(-l) = {Q^0}_m(-l) = \frac{1}{(-q)_l} \sum_{k=0}^{l} (b)_k C_{k,r} (-l)_k (m-q)_{l-k} \tag{2.13}$$

where $l = 0, 1, \dots, q$. *In particular* $Q_m^0(0) = 1$.

Lemma 2. *Write* $\hat{Q}_m^\epsilon$ *for the polynomial of* (2.10) *with* $c - a - m = -q + \epsilon$ *where* $q \in \{0, 1, \dots, m-1\}$, $\epsilon > 0$, *and let* $\eta_1(\epsilon), \dots, \eta_m(\epsilon)$ *denote its zeros. Suppose* $f_j - a \notin \{m-q-m_j, m-q-m_j+1, \dots, 0\}$ *for* $0 \le j \le r$ *such that* $m-q-m_j \le 0$. *Then the following are true as* $\epsilon \to 0$

1. *the first (after possible renumbering)* $q+1$ *zeros tend to consecutive non-positive integers:* $\eta_1 \to 0,\ \eta_2 \to -1, \dots, \eta_{q+1} \to -q$.

2. *The remaining* $m-q-1$ *zeros (no such zeros remain if* $q = m-1$*) tend to the zero of the reduced polynomial*

$$\hat{R}_{m-q-1}(t) = \frac{(b)_{q+1}}{(b-a)_{q+1}} \sum_{k=0}^{q} \frac{(-m+k)_{q-k+1} (a)_k C_{k,r}}{(q-k+1)!} \times {}_3F_2 \left[\begin{matrix} 1-m+q,\ t+q+1,\ 1-b-k \\ a-b+1,\ 2-k+q \end{matrix} ; 1 \right]$$

$$+\sum_{k=q+1}^{m}\frac{(-1)^k(a)_k(b)_kC_{k,r}(t+q+1)_{k-q-1}}{(a-b-q)_k(k-q-1)!}{}_3F_2\left[\begin{matrix}-m+k,\ t+k,\ -b-q\\ a-b-q+k,\ k-q\end{matrix};1\right] \tag{2.14}$$

of degree $m-q-1$.

3. *The limit relation*

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\eta_1(\epsilon)}=(-1)^{q+1}\hat{R}_{m-q-1}(0) \tag{2.15}$$

holds.

4. *The values* $\hat{Q}^{\epsilon}_m(-l)$ *at the first* $q+1$ *non positive integers converges as* $\epsilon\to 0$ *to the following expression :*

$$\lim_{\epsilon\to 0}\hat{Q}^{\epsilon}_m(-l)=\hat{Q}^{0}_m(-l)=\sum_{k=0}^{l}\frac{(-1)_k(a)_k(b)_kC_{k,r}(-l)_k}{(-q)_k(a-b-q)_k}{}_3F_2\left[\begin{matrix}k-m,\ k-l,\ -b-q\\ k-q,\ -b-q+a+k\end{matrix};1\right] \tag{2.16}$$

where $l=0,1,\dots,q$. *In particular* $\hat{Q}^{0}_m(0)=1$.

Now we can introduce our three theorems. For the detailed proofs of the preceding two lemmas and the following three theorems refer to [1].

2.2.1 First Karp and Prilepkina theorem to solve the degeneracy in the first Miller-Paris transformations

As we are aware, equation (2.3) becomes invalid when $(c-b-m)_m=0$. In Theorem (1), we shall present a solution that addresses this issue, ensuring that equation (2.3) remains valid even when $(c-b-m)_m=0$.

Theorem 1. *Suppose* $q\in\{0,1,\dots,m-1\}$, $b+m-q\notin-\mathbb{N}_0$ *and* $f_j-b\notin\{m-q-m_j,m-q-m_j+1,\dots,0\}$, *for* $0\le j\le r$ *such that* $m-q-m_j\le 0$. *Then for all* x *such that* $x\in\mathbb{C}-[1,\infty)$, *the following holds:*

$$(1-x)^a{}_{r+2}F_{r+1}\left[\begin{matrix}a,\ b,\ \boldsymbol{f}\textbf{+}\boldsymbol{m}\\ b+m-q,\ \boldsymbol{f}\end{matrix};x\right]=\sum_{j=0}^{q}\frac{(a)_j(-q)_jQ^0_m(-j)}{(b+m-q)_jj!}\left(\frac{x}{x-1}\right)^j$$

$$+\frac{x^{q+1}(a)_{q+1}R_{m-q-1}(-q-1)}{(x-1)^{q+1}(b+m-q)_{q+1}(m-q-1)!}{}_{m-q+1}F_{m-q}\left[\begin{matrix}1,\ a+q+1,\ \boldsymbol{\lambda}+q+2\\ b+m+1,\ \boldsymbol{\lambda}+q+1\end{matrix};\frac{x}{x-1}\right] \tag{2.17}$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{m-q-1})$ *are the zeros of* $R_{m-q-1}(t)$ *defined in* (2.11). *The value of* $R_{m-q-1}(-q-1)$ *is calculated by*

$$R_{m-q-1}(-q-1) = (-1)^{m-q-1} \sum_{k=0}^{q+1} (-1)^k (b)_k C_{k,r} (k-m)_{m-q-1} \tag{2.18}$$

where $C_{k,r}$ *defined in* (2.5).

2.2.2 Second Karp and Prilepkina theorem to solve the degeneracy in the second Miller-Paris transformations

Now as we do above, we will introduce a solution, where equation (2.7) becomes valid when $(c-b-m)_m = 0$.

Theorem 2. *Suppose* $q \in \{0, 1, \ldots, m-1\}$, $b+m-q \notin -\mathbb{N}_0$ *and* $f_j - b \notin \{m-q-m_j, m-q-m_j+1, \ldots, 0\}$, *for* $0 \le j \le r$ *such that* $m-q-m_j \le 0$. *Then for all* $x \in \mathbb{C}$ *,the following holds:*

$$e^{-x} {}_{r+1}F_{r+1}\left[\begin{matrix} b,\ \boldsymbol{f+m} \\ b+m-q,\ \boldsymbol{f} \end{matrix}; x\right] = \sum_{j=0}^{q} \frac{(-q)_j Q_m^0(-j)}{(b+m-q)_j j!} (-x)^j$$

$$+ \frac{(-x)^{q+1} R_{m-q-1}(-q-1)}{(b+m-q)_{q+1}(m-q-1)!} {}_{m-q}F_{m-q}\left[\begin{matrix} 1,\ \boldsymbol{\lambda}+q+2 \\ b+m+1,\ \boldsymbol{\lambda}+q+1 \end{matrix}; -x\right] \tag{2.19}$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{m-q-1})$ *are the zeros of* $R_{m-q-1}(t)$ *defined in* (2.11). *The value of* $R_{m-q-1}(-q-1)$ *is calculated from* (2.18).

2.2.3 Third Karp and Prilepkina theorem to solve the degeneracy in the third Miller-Paris transformations

Here we will introduce a new theorem, which solves the invalidity of equation (2.9) when $(c-a-m)_m = 0$.

Theorem 3. *Suppose* $q \in \{0, 1, \ldots, m-1\}$, $a+m-q \notin -\mathbb{N}_0$, $a-b \notin \{q+1-m, \ldots, q\}$ *and* $f_j - a \notin \{m-q-m_j, m-q-m_j+1, \ldots, 0\}$, *for* $0 \le j \le r$ *such that* $m-q-m_j \le 0$. *Then for all* x *such that* $x \in \mathbb{C} - [1, \infty)$, *the following holds:*

$$(1-x)^{b+q} {}_{r+2}F_{r+1}\left[\begin{matrix} a,\ b,\ \boldsymbol{f+m} \\ a+m-q,\ \boldsymbol{f} \end{matrix}; x\right] = \sum_{j=0}^{q} \frac{(-q)_j (a-b-q)_j \hat{Q}_m^0(-j)}{(a+m-q)_j j!} x^j$$

$$+\frac{x^{q+1}\hat{R}_{m-q-1}(-q-1)(b-a)_{q+1}}{(a+m-q)_{q+1}}\,{}_{m-q+1}F_{m-q}\left[\begin{matrix}1,\ a-b+1,\ \boldsymbol{\gamma}+q+2\\ a+m+1,\ \boldsymbol{\gamma}+q+1\end{matrix};x\right] \tag{2.20}$$

where $\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_{m-q-1})$ *are the zeros of* $\hat{R}_{m-q-1}(t)$ *defined in* (2.14). *The value of* $\hat{R}_{m-q-1}(-q-1)$ *is calculated by*

$$\hat{R}_{m-q-1}(-q-1)=\frac{(b)_{q+1}}{(b-a)_{q+1}}\sum_{k=0}^{q+1}\frac{(a)_k C_{k,r}(k-m)_{q-k+1}}{(q-k+1)!} \tag{2.21}$$

where $C_{k,r}$ *defined in* (2.5).

Chapter Three

New Transformations for Kampé de Fériet using Miller-Paris Transformation Formulas

3.1 Transformations for Kampé de Fériet using the generalized hypergeometric functions ${}_{r+2}F_{r+1}$, ${}_{r+1}F_{r+1}$

In this section, we will consider the Kampé de Fériet function [1] ([2], p.27)

$$F^{p:q:k}_{l:m:n}\left[\begin{matrix}(a_p):(b_q):(c_k)\\(\alpha_l):(\beta_m):(\gamma_n)\end{matrix}\middle| x,y\right]=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!} \tag{3.1}$$

where for convergence

- $p+q<l+m+1, p+k<l+n+1, |x|<\infty, |y|<\infty$

  or

- $p+q=l+m+1, p+k=l+n+1$ and

  (a) $|x|^{1/(p-l)}+|y|^{1/(p-l)}<1$, if $p>l$,

  (b) $\max\{|x|,|y|\}<1$, if $p\leq l$.

and use the Miller-Paris transformation formulas (2.3), (2.7), and (2.9) in order to derive new transformation formulas for the Kampé de Fériet function. To do that we will choose a suitable Kampé de Fériet function and write its double series expansion, after that, we will write it as a single sum that involves one of the generalized hypergeometric functions ${}_{r+2}F_{r+1}$, ${}_{r+1}F_{r+1}$. Then, as we obtain one of the generalized hypergeometric functions we can apply to the generalized hypergeometric function one of Miller-Paris transformation formulas (2.3), (2.7), and (2.9). By doing so, we will obtain a single sum that involves one of the generalized hypergeometric functions ${}_{m+2}F_{m+1}$, ${}_{m+1}F_{m+1}$. Finally, by rewriting this single sum that involves one of the generalized hypergeometric functions ${}_{m+2}F_{m+1}$, ${}_{m+1}F_{m+1}$ as a double sum with two variables, we can rewrite this double sum in terms of the Kampé de Fériet function. These steps will help us in deriving new transformation

[1]The two variables x and y are real numbers

formulas for the Kampé de Fériet function. To calculate the radius of convergence of transformation formulas, we will take the intersection between the radius of convergence for Kampé de Fériet function we chose at the beginning, the radius of convergence for the Miller-Paris transformation and the radius of convergence for Kampé de Fériet function we obtain at the end.

1. **First Transformation**

   Let us take

$$\begin{cases} \mathbf{m} = (b_1 - d_1, \dots, b_p - d_p), \quad vector\ of\ positive\ integers \\ m = \sum_{i=1}^{p} m_i = \sum_{i=1}^{p} b_i - d_i \\ a_2 + l \neq d_i, \quad \forall\, l = 0, 1, 2, \dots \\ (c_1 - a_2 - m)_m \neq 0 \end{cases} \tag{3.2}$$

   $(\boldsymbol{\zeta}) = (\zeta_1, \dots, \zeta_m)$ to be the zeros of the characteristic polynomial

$$Q_m(t) = \frac{1}{(c_1 - a_2 - m)_m} \sum_{k=0}^{m} (a_2 + l)_k C_{k,p}(t)_k (c_1 - a_2 - m - t)_{m-k} \tag{3.3}$$

   with $C_{0,p} = 1, C_{m,p} = [(d_1)_{m_1} \dots (d_p)_{m_p}]^{-1}$,

$$C_{k,p} = \frac{(-1)^k}{k!} {}_{p+1}F_p \left[\begin{matrix} -k, \ b_1, \ \dots, \ b_p \\ d_1, \ \dots, \ d_p \end{matrix} ; 1 \right] \tag{3.4}$$

   Now, if we consider the Kampé de Fériet function $F^{2:p:q}_{1:p:q}$. By expressing it in terms of a single sum involving ${}_{p+2}F_{p+1}(x)$, and applying the transformation provided in equation (2.3), we can rewrite the resulting equation using a double series representation. So the transformation will be as follows:

$$F^{2:p:q}_{1:p:q} \left[\begin{matrix} a_1, a_2 : b_1, \dots, b_p : r_1, \dots, r_q \\ c_1 \quad : d_1, \dots, d_p : k_1, \dots, k_q \end{matrix} \middle| x, y \right]$$

$$= \sum_{l=0}^{\infty} \frac{(a_1)_l (a_2)_l (r_1)_l \dots (r_q)_l}{(c_1)_l (k_1)_l \dots (k_q)_l} \frac{y^l}{l!} \, {}_{p+2}F_{p+1} \left[\begin{matrix} a_1 + l, \ a_2 + l, \ b_1 \dots, \ b_p \\ c_1 + l, \ d_1, \ \dots, \ d_p \end{matrix} ; x \right]$$

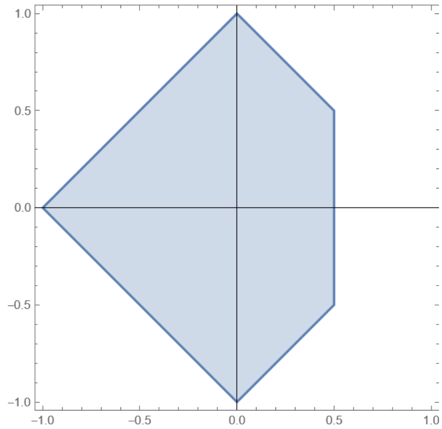
$$= \sum_{l=0}^{\infty} \frac{(a_1)_l (a_2)_l (r_1)_l \dots (r_q)_l}{(c_1)_l (k_1)_l \dots (k_q)_l} \frac{y^l}{l!} (1-x)^{-a_1-k}$$
$$\times {}_{m+2}F_{m+1}\left[\begin{matrix} a_1+l, \ c_1-a_2-m, \ (\boldsymbol{\zeta})+1 \\ c_1+l, \ (\boldsymbol{\zeta}) \end{matrix}; \frac{x}{x-1}\right]$$

$$= (1-x)^{-a_1} F^{1:m+1:q+1}_{1:m:q}\left[\begin{matrix} a_1 : c_1-a_2-m, (\boldsymbol{\zeta})+1 : a_2, r_1, \dots, r_q \\ c_1 : (\boldsymbol{\zeta}) : k_1, \dots, k_q \end{matrix}\middle| \frac{x}{x-1}, \frac{y}{1-x}\right] \tag{3.5}$$

where $|x| < 1, x < \frac{1}{2}$, $|x| + |y| < 1$ and $max \left\{ \left|\frac{x}{x-1}\right|, \left|\frac{y}{1-x}\right| \right\} < 1$.

Figure 2

Region of convergence for the first Kampé de Fériet transformation



Remark: In the previous transformation, we express the Kampé de Fériet function $F^{2:p:q}_{1:p:q}$ as a single sum involving ${}_{p+2}F_{p+1}(x)$, if we consider the same Kampé de Fériet function $F^{2:p:q}_{1:p:q}$ and express it as single sum involving ${}_{p+2}F_{p+1}(y)$, its important to notice that we must define

$$\begin{cases} \mathbf{m} = (r_1-k_1, \dots, r_p-k_p), \ \ vector\ of\ positive\ integers \\ m = \sum_{i=1}^{p} m_i = \sum_{i=1}^{q} r_i - k_i \\ a_2 + l \neq k_i \quad \forall\, l = 0, 1, 2, \dots \\ (c_1-a_2-m)_m \neq 0 \end{cases} \tag{3.6}$$

and take

$$C_{k,q} = \frac{(-1)^k}{k!} {}_{q+1}F_q\left[\begin{matrix} -k, \ r_1, \ \dots, \ r_q \\ k_1, \ \dots, \ k_q \end{matrix}; 1\right] \tag{3.7}$$

with $C_{0,q} = 1, C_{m,q} = [(k_1)_{m_1}, \dots (k_q)_{m_q}]^{-1}$.

2. **Second Transformation**

Let us take

$$\begin{cases} \mathbf{m} = (b_1 - d_1, \ldots, b_p - d_p), \ \ vector\ of\ positive\ integers \\ m = \sum_{i=1}^{p} m_i = \sum_{i=1}^{p} b_i - d_i \\ (c_1 - a_1 - m)_m \neq 0 \\ (c_1 - a_2 - m)_m \neq 0 \\ (1 + a_1 + a_2 - c_1 + l)_m \neq 0, \quad \forall\, l = 0, 1, 2, \ldots \end{cases} \tag{3.8}$$

$(\boldsymbol{\eta}) = (\eta_1, \ldots, \eta_m)$ to be the zeros of the characteristic polynomial

$$\hat{Q}_m(t) = \sum_{k=0}^{m} \frac{(-1)^k C_{k,p} (a_1 + l)_k (a_2 + l)_k (t)_k}{(c_1 - a_1 - m)_k (c_1 - a_2 - m)_k} \times {}_3F_2\left[\begin{matrix} -m + k, \ t + k, \ c_1 - a_1 - a_2 - l - m \\ c_1 - a_1 - m + k, \ c_1 - a_2 - m + k \end{matrix}; 1\right] \tag{3.9}$$

with $C_{0,p} = 1, C_{m,p} = [(d_1)_{m_1} \ldots (d_p)_{m_p}]^{-1}$,

$$C_{k,p} = \frac{(-1)^k}{k!} {}_{p+1}F_p\left[\begin{matrix} -k, \ b_1, \ \ldots, \ b_p \\ d_1, \ \ldots, \ d_p \end{matrix}; 1\right] \tag{3.10}$$

and consider the same Kampé de Fériet function $F^{2:p:q}_{1:p:q}$. By expressing it in terms of a single sum involving the generalised hypergeometric function ${}_{p+2}F_{p+1}(x)$, we can apply the transformation provided in equation (2.9). By doing so, we can rewrite the resulting equation using a double-series representation. So the transformation will be as follows:

$$F^{2:p:q}_{1:p:q}\left[\begin{matrix} a_1, a_2 : b_1, \ldots, b_p : r_1, \ldots, r_q \\ c_1 \ \ : d_1, \ldots, d_p : k_1, \ldots, k_q \end{matrix}\middle| x, y\right]$$

$$= \sum_{l=0}^{\infty} \frac{(a_1)_l (a_2)_l (r_1)_l \ldots (r_q)_l}{(c_1)_l (k_1)_l \ldots (k_q)_l} \frac{y^l}{l!} \, {}_{p+2}F_{p+1}\left[\begin{matrix} a_1 + l, \ a_2 + l, \ b_1 \ldots, \ b_p \\ c_1 + l, \ d_1, \ \ldots, \ d_p \end{matrix}; x\right]$$

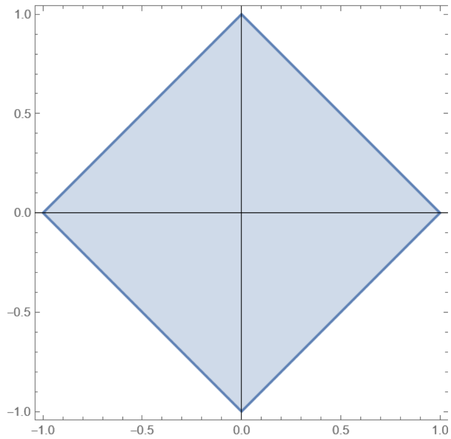
$$= \sum_{l=0}^{\infty} \frac{(a_1)_l (a_2)_l (r_1)_l \ldots (r_q)_l}{(c_1)_l (k_1)_l \ldots (k_q)_l} \frac{y^l}{l!} (1 - x)^{c_1 - a_1 - a_2 - l - m} \times {}_{m+2}F_{m+1}\left[\begin{matrix} c_1 - a_1 - m, \ c_1 - a_2 - m, \ (\boldsymbol{\eta}) + 1 \\ c_1 + l, \ (\boldsymbol{\eta}) \end{matrix}; x\right]$$

$$= F^{0:m+2:q+2}_{1:m:q}\left[\begin{matrix} \text{——} : c_1 - a_1 - m, c_1 - a_2 - m, (\boldsymbol{\eta}) + 1 : a_1, a_2, r_1, \ldots, r_q \\ c_1 : (\boldsymbol{\eta}) : k_1, \ldots, k_q \end{matrix}\middle| x, \frac{y}{1-x}\right] \times (1-x)^{c_1 - a_1 - a_2 - m} \tag{3.11}$$

where $|x| < 1$, $|x| + |y| < 1$ and $max\left\{|x|, \left|\frac{y}{1-x}\right|\right\} < 1$.

Figure 3

Region of convergence for the second Kampé de Fériet transformation



3. **Third Transformation**

Let us take

$$\begin{cases} \mathbf{m} = (b_1 - d_1, \ldots, b_p - d_p), \ \ \textit{vector of positive integers} \\ m = \sum_{i=1}^{p} m_i = \sum_{i=1}^{p} b_i - d_i \\ a_1 + l \neq d_i, \quad \forall\, l = 0, 1, 2, \ldots \\ (c_1 - a_1 - m)_m \neq 0 \end{cases} \tag{3.12}$$

$(\boldsymbol{\zeta}) = (\zeta_1, \ldots, \zeta_m)$ to be the zeros of the characteristic polynomial

$$Q_m(t) = \frac{1}{(c_1 - a_1 - m)_m} \sum_{k=0}^{m} (a_1 + l)_k C_{k,p}(t)_k (c_1 - a_1 - m - t)_{m-k} \tag{3.13}$$

with $C_{0,p} = 1, C_{m,p} = [(d_1)_{m_1} \ldots (d_p)_{m_p}]^{-1}$ and

$$C_{p,r} = \frac{(-1)^k}{k!}\, {}_{p+1}F_p\left[\begin{matrix} -k, & b_1, & \ldots, & b_p \\ & d_1, & \ldots, & d_p \end{matrix}; 1\right] \tag{3.14}$$

and consider the Kampé de Fériet function $F^{1:p:q}_{1:p:q}$. By expressing it in terms a single sum involving the generalized hypergeometric function ${}_{p+1}F_{p+1}(x)$, we can apply the transformation provided in equation (2.7), and rewrite the resulting equation using a double series representation. so the transformation will be as follows:

$$F^{1:p:q}_{1:p:q}\left[\begin{matrix} a_1, : b_1, \ldots, b_p : r_1, \ldots, r_q \\ c_1 \quad : d_1, \ldots, d_p : k_1, \ldots, k_q \end{matrix}\middle| x, y\right]$$

$$= \sum_{l=0}^{\infty} \frac{(a_1)_l (r_1)_l \ldots (r_q)_l}{(c_1)_l (k_1)_l \ldots (k_q)_l} \frac{y^l}{l!} \, {}_{p+1}F_{p+1}\left[\begin{matrix} a_1 + l, \; b_1 \ldots, \; b_p \\ c_1 + l, \; d_1, \; \ldots, \; d_p \end{matrix}; x\right]$$

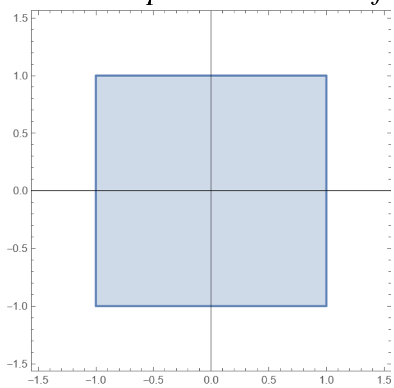
$$= \sum_{l=0}^{\infty} \frac{(a_1)_l (r_1)_l \ldots (r_q)_l}{(c_1)_l (k_1)_l \ldots (k_q)_l} \frac{y^l}{l!} e^x \, {}_{m+1}F_{m+1}\left[\begin{matrix} c_1 - a_1 - m, \; (\boldsymbol{\zeta}) + 1 \\ c_1 + l, \; (\boldsymbol{\zeta}) \end{matrix}; -x\right]$$

$$= e^x F^{0:m+1:q+1}_{1:m:q}\left[\begin{matrix} \text{——} : c_1 - a_1 - m, (\boldsymbol{\zeta}) + 1 : a_1, r_1, \ldots, r_q \\ c_1 : (\boldsymbol{\zeta}) : k_1 \ldots, k_q \end{matrix}\middle| -x, y\right] \tag{3.15}$$

where $max\{|x|, |y|\} < 1$.

Figure 4

Region of convergence for the third Kampé de Fériet transformation



4. **Fourth Transformation**

let us take

$$\begin{cases} \boldsymbol{m_1} = (b_1 - d_1, \ldots, b_r - d_r), \;\; vector\ of\ positive\ integers \\ m_1 = \sum_{i=1}^{r} m_i = \sum_{i=1}^{r} b_i - d_i \\ a_2 \neq d_i \quad \forall\, i = 0, 1, \ldots, r \\ (e_1)_{m_1} = (c_1 - a_2 - m_1)_{m_1} \neq 0 \end{cases}$$

$$\begin{cases} \boldsymbol{m_2} = (b_1{}' - d_1{}', \ldots, b_s{}' - d_s{}'), \;\; vector\ of\ positive\ integers \\ m_2 = \sum_{j=1}^{s} b_i{}' - d_i{}' \\ a_2{}' \neq d_i{}' \quad \forall\, i = 0, 1, \ldots, s \\ (e_2)_{m_2} = (c_1{}' - a_2{}' - m_2)_{m_2} \neq 0 \end{cases} \tag{3.16}$$

$(\boldsymbol{\zeta}) = (\zeta_1, \ldots, \zeta_{m_1})$ to be the zeros of the characteristic polynomial

$$Q_{m_1}(t) = \frac{1}{(c_1 - a_2 - m_1)_{m_1}} \sum_{k=0}^{m_1} (a_2)_k C_{k,r}(t)_k (c_1 - a_2 - m_1 - t)_{m_1 - k} \tag{3.17}$$

with $C_{0,r} = 1, C_{m_1,r} = [(d_1)_{m_1} \ldots (d_r)_{m_r}]^{-1}$

$$C_{k,r} = \frac{(-1)^k}{k!} {}_{r+1}F_r \left[\begin{matrix} -k, \; b_1, \; \ldots, \; b_r \\ d_1, \; \ldots, \; d_r \end{matrix} ; 1 \right] \tag{3.18}$$

$(\boldsymbol{\zeta'}) = (\zeta_1{}', \ldots, \zeta_{m_2}{}')$ to be the zeros of the characteristic polynomial

$$Q_{m_2}(t) = \frac{1}{(c_1{}' - a_2{}' - m_2)_{m_2}} \sum_{k=0}^{m_2} (a_2{}')_k C'{}_{k,s}(t)_k (c_1{}' - a_2{}' - m_2 - t)_{m_2 - k} \tag{3.19}$$

with $C'{}_{0,s} = 1, C'{}_{m_2,s} = [(d_1)_{m_1} \ldots (d_s)_{m_s}]^{-1}$

$$C'{}_{k,s} = \frac{(-1)^k}{k!} {}_{s+1}F_s \left[\begin{matrix} -k, \; b_1{}', \; \ldots, \; b_s{}' \\ d_1{}', \; \ldots, \; d_s{}' \end{matrix} ; 1 \right] \tag{3.20}$$

and consider another formula for the Kampé de Fériet function $F^{0:r+2:s+2}_{0:r+1:s+1}$. Then by expressing it in terms of two sums, one involves the generalised hypergeometric function ${}_{p+2}F_{p+1}(x)$, while the other sum involves another generalised hypergeometric function ${}_{p+2}F_{p+1}(y)$ and applying the transformation provided in (2.3) to the ${}_{p+2}F_{p+1}(x)$ and ${}_{p+2}F_{p+1}(y)$, we can rewrite the resulting equation using a double series representation. The transformation will be as follows:

$$F^{0:r+2:s+2}_{0:r+1:s+1} \left[\begin{matrix} \text{——} : a_1, a_2, b_1, \ldots, b_r : a_1{}', a_2{}', b_1{}', \ldots, b_s{}' \\ \text{——} : c_1, d_1, \ldots, d_r : c_1{}', d_1{}', \ldots, d_s{}' \end{matrix} \middle| x, y \right]$$

$$= {}_{r+2}F_{r+1} \left[\begin{matrix} a_1, \; a_2, \; b_1, \; \ldots, \; b_r \\ c_1, \; d_1, \; \ldots, \; d_r \end{matrix} ; x \right] {}_{s+2}F_{s+1} \left[\begin{matrix} a_1{}', \; a_2{}', \; b_1{}', \; \ldots, \; b_s{}' \\ c_1{}', \; d_1{}', \; \ldots, \; d_s{}' \end{matrix} ; y \right]$$

$$= (1-x)^{-a_1} {}_{m_1+2}F_{m_1+1}\left[\begin{matrix} a_1,\ c_1-a_2-m_1,\ (\boldsymbol{\zeta})+1 \\ c_1,\ (\boldsymbol{\zeta}) \end{matrix}; \frac{x}{x-1}\right]$$

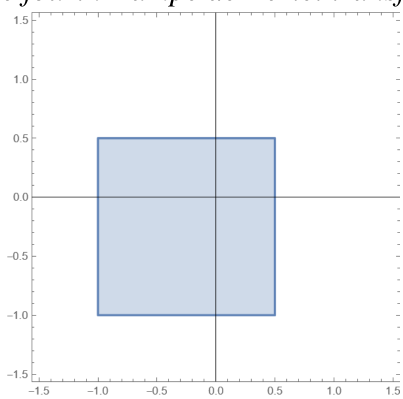
$$\times(1-y)^{-a_1'} {}_{m_2+2}F_{m_2+1}\left[\begin{matrix} a_1',\ c_1'-a_2'-m_2,\ (\boldsymbol{\zeta}')+1 \\ c_1',\ (\boldsymbol{\zeta}') \end{matrix}; \frac{y}{y-1}\right]$$

$$= F^{0:m_1+2:m_2+2}_{0:m_1+1:m_2+1}\left[\begin{matrix} \text{——} : a_1, e_1, (\boldsymbol{\zeta})+1 : a_1', e_2, (\boldsymbol{\zeta'})+1 \\ \text{——} : c_1, (\boldsymbol{\zeta}) : c_1', (\boldsymbol{\zeta'}) \end{matrix}\middle| \frac{x}{x-1}, \frac{y}{y-1}\right]$$
$$\times(1-x)^{-a_1}(1-y)^{-a_1'} \qquad (3.21)$$

where $x < \frac{1}{2},\ y < \frac{1}{2}$, $max\,\{|x|,|y|\} < 1$ and $max\left\{\left|\frac{x}{x-1}\right|, \left|\frac{y}{y-1}\right|\right\} < 1.$

Figure 5

Region of convergence for the fourth Kampé de Fériet transformation



5. **Fifth Transformation**

Let us take

$$\begin{cases} \boldsymbol{m_1} = (b_1-d_1,\ldots,b_r-d_r),\ \ vector\ of\ positive\ integers \\ m_1 = \sum_{i=1}^r b_i - d_i \\ (e_1)_{m_1} = (c_1-a_1-m_1)_{m_1} \neq 0 \\ (e_2)_{m_1} = (c_1-a_2-m_1)_{m_1} \neq 0 \\ (1+a_1+a_2-c_1)_{m_1} \neq 0 \\ \boldsymbol{m_2} = (b_1'-d_1',\ldots,b_s'-d_s'),\ \ vector\ of\ positive\ integers \\ m_2 = \sum_{j=1}^s b_i' - d_i' \\ (e_3)_{m_2} = (c_1'-a_1'-m_2)_{m_2} \neq 0 \end{cases}$$

$$\begin{cases} (e_4)_{m_2} = (c_1' - a_2' - m_2)_{m_2} \neq 0 \\ (1 + a_1' + a_2' - c_1')_{m_2} \neq 0 \end{cases} \tag{3.22}$$

$(\boldsymbol{\eta}) = (\eta_1, \ldots, \eta_{m_1})$ to be the zeros of the characteristic polynomial

$$\hat{Q}_{m_1}(t) = \sum_{k=0}^{m_1} \frac{(-1)^k C_{k,r} (a_1)_k (a_2)_k (t)_k}{(c_1 - a_1 - m_1)_k (c_1 - a_2 - m_1)_k} \times {}_3F_2\left[\begin{matrix} -m_1 + k,\ t + k,\ c_1 - a_1 - a_2 - m_1 \\ c_1 - a_1 - m_1 + k,\ c_1 - a_2 - m_1 + k \end{matrix}; 1\right] \tag{3.23}$$

with $C_{0,r} = 1, C_{m_1,r} = [(d_1)_{m_1} \ldots (d_r)_{m_r}]^{-1}$,

$$C_{k,r} = \frac{(-1)^k}{k!} {}_{r+1}F_r\left[\begin{matrix} -k,\ b_1,\ \ldots,\ b_r \\ d_1,\ \ldots,\ d_r \end{matrix}; 1\right] \tag{3.24}$$

$(\boldsymbol{\eta}') = (\eta_1', \ldots, \eta_{m_2}')$ to be the zeros of the characteristic polynomial

$$\hat{Q}_{m_2}(t) = \sum_{k=0}^{m_2} \frac{(-1)^k C'_{k,s} (a_1')_k (a_2')_k (t)_k}{(c_1' - a_1' - m_2)_k (c_1' - a_2' - m_2)_k} \times {}_3F_2\left[\begin{matrix} -m_2 + k,\ t + k,\ c_1' - a_1' - a_2' - m_2 \\ c_1' - a_1' - m_2 + k,\ c_1' - a_2' - m_2 + k \end{matrix}; 1\right] \tag{3.25}$$

with $C'_{0,s} = 1, C'_{m_2,s} = [(d_1')_{m_1} \ldots (d_s)_{m_s}]^{-1}$,

$$C'_{k,s} = \frac{(-1)^k}{k!} {}_{s+1}F_s\left[\begin{matrix} -k,\ b_1',\ \ldots,\ b_s' \\ d_1',\ \ldots,\ d_s' \end{matrix}; 1\right] \tag{3.26}$$

and consider the Kampé de Fériet function $F^{0:r+2:s+2}_{0:r+1:s+1}$. Then by expressing it in terms of two sums, one involves the generalised hypergeometric function ${}_{p+2}F_{p+1}(x)$, while the other sum involves another generalised hypergeometric function, ${}_{p+2}F_{p+1}(y)$ and applying the transformation provided in (2.9) to the ${}_{p+2}F_{p+1}(x)$ and ${}_{p+2}F_{p+1}(y)$, we can rewrite the resulting equation using a double series representation. The transformation will be as follows:

$$F^{0:r+2:s+2}_{0:r+1:s+1}\left[\begin{matrix} \text{——} : a_1, a_2, b_1, \ldots, b_r : a_1', a_2', b_1', \ldots, b_s' \\ \text{——} : c_1, d_1, \ldots, d_r : c_1', d_1', \ldots, d_s' \end{matrix}\middle| x, y\right]$$

$$= {}_{r+2}F_{r+1}\left[\begin{matrix} a_1,\ a_2,\ b_1,\ \ldots,\ b_r \\ c_1,\ d_1,\ \ldots,\ d_r \end{matrix}; x\right] {}_{s+2}F_{s+1}\left[\begin{matrix} a_1',\ a_2',\ b_1',\ \ldots,\ b_s' \\ c_1',\ d_1',\ \ldots,\ d_s' \end{matrix}; y\right]$$

$$= (1-x)^{c_1-a_1-a_2-m_1} {}_{m_1+2}F_{m_1+1}\left[\begin{matrix} c_1-a_1-m_1,\ c_1-a_2-m_1,\ (\boldsymbol{\eta})+1 \\ c_1,\ (\boldsymbol{\eta}) \end{matrix} ;x\right]$$

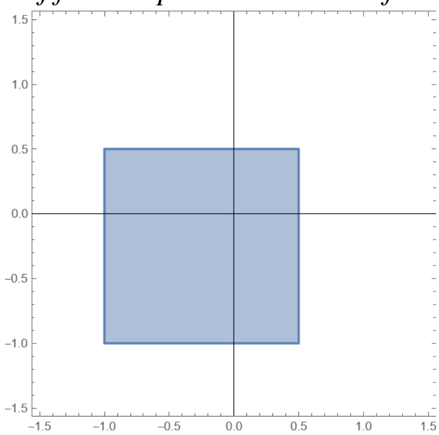
$$\times(1-y)^{c_1'-a_1'-a_2'-m_2} {}_{m_2+2}F_{m_2+1}\left[\begin{matrix} c_1'-a_1'-m_2,\ c_1'-a_2'-m_2,\ (\boldsymbol{\eta}')+1 \\ c_1',\ (\boldsymbol{\eta}') \end{matrix} ;y\right]$$

$$= F^{0:m_1+2:m_2+2}_{0:m_1+1:m_2+1}\left[\begin{matrix} \text{——} : e_1, e_2, (\boldsymbol{\eta})+1 : e_3, e_4, (\boldsymbol{\eta'})+1 \\ \text{——} : c_1, (\boldsymbol{\eta}) : c_1', (\boldsymbol{\eta'}) \end{matrix}\middle| x, y\right]$$
$$\times(1-x)^{c_1-a_1-a_2-m_1}(1-y)^{c_1'-a_1'-a_2'-m_2} \tag{3.27}$$

where $x < \frac{1}{2}$, $y < \frac{1}{2}$ and $max\,\{|x|, |y|\} < 1$.

Figure 6

Region of convergence for the fifth Kampé de Fériet transformation



6. **Sixth Transformation**

As observed earlier, we derived the transformations (3.21) and (3.27) by applying the first transformation formula for Miller Paris (2.3) to the expression (3.21) for both variables, x and y. Similarly, we apply the third transformation formula for Miller Paris (2.9) on the expression (3.27) for both variables, x and y. Now, our next step is to apply the first transformation formula for Miller Paris (2.3) to the generalized hypergeometric function ${}_{p+2}F_{p+1}(x)$ and the third transformation formula for Miller Paris (2.9) to the generalized hypergeometric function ${}_{p+2}F_{p+1}(y)$, considering the same Kampé de Fériet function. Before proceeding, we will take

$$\begin{cases} \boldsymbol{m_1} = (b_1-d_1, \ldots, b_r-d_r), \ \ vector\ of\ positive\ integers \\ m_1 = \sum_{i=1}^{r} b_i - d_i \end{cases}$$

$$
\begin{cases}
a_2 \neq d_i \\
(e_1)_{m_1} = (c_1 - a_2 - m_1)_{m_1} \neq 0 \\
\boldsymbol{m_2} = (b_1{}' - d_1{}', \dots, b_s{}' - d_s{}'), \;\; vector\ of\ positive\ integers \\
m_2 = \sum_{i=1}^{s} b_i{}' - d_i{}' \\
(e_2)_{m_2} = (c_1{}' - a_1{}' - m_2)_{m_2} \neq 0 \\
(e_3)_{m_2} = (c_1{}' - a_2{}' - m_2)_{m_2} \neq 0 \\
(1 + a_1{}' + a_2{}' - c_1{}')_{m_2} \neq 0
\end{cases}
\tag{3.28}
$$

$(\boldsymbol{\zeta}) = (\zeta_1, \dots, \zeta_{m_1})$ to be the zeros of the characteristic polynomial

$$
Q_{m_1}(t) = \frac{1}{(c_1 - a_2 - m_1)_{m_1}} \sum_{k=0}^{m_1} (a_2)_k C_{k,r} (t)_k (c_1 - a_2 - m_1 - t)_{m_1 - k} \tag{3.29}
$$

with $C_{0,r} = 1, C_{m_1,r} = [(d_1)_{m_1} \dots (d_r)_{m_r}]^{-1}$

$$
C_{k,r} = \frac{(-1)^k}{k!} {}_{r+1}F_r \left[\begin{matrix} -k, \; b_1, \; \dots, \; b_r \\ d_1, \; \dots, \; d_r \end{matrix} ; 1 \right] \tag{3.30}
$$

$(\boldsymbol{\eta}') = (\eta_1{}', \dots, \eta_{m_2}{}')$ to be the zeros of the characteristic polynomial

$$
\begin{aligned}
\hat{Q}_{m_2}(t) &= \sum_{k=0}^{m_2} \frac{(-1)^k C'_{k,s} (a_1{}')_k (a_2{}')_k (t)_k}{(c_1{}' - a_1{}' - m_2)_k (c_1{}' - a_2{}' - m_2)_k} \\
&\times {}_3F_2 \left[\begin{matrix} -m_2 + k, \; t + k, \; c_1{}' - a_1{}' - a_2{}' - m_2 \\ c_1{}' - a_1{}' - m_2 + k, \; c_1{}' - a_2{}' - m_2 + k \end{matrix} ; 1 \right]
\end{aligned} \tag{3.31}
$$

with $C'_{0,s} = 1, C'_{m_2,s} = [(d_1{}')_{m_1} \dots (d_s{}')_{m_s}]^{-1}$,

$$
C'_{k,s} = \frac{(-1)^k}{k!} {}_{s+1}F_s \left[\begin{matrix} -k, \; b_1{}', \; \dots, \; b_s{}' \\ d_1{}', \; \dots, \; d_s{}' \end{matrix} ; 1 \right] \tag{3.32}
$$

By following the above steps, we can rewrite the resulting equation using a double-series representation. The transformation will be as follows:

$$
F^{0:r+2:s+2}_{0:r+1:s+1} \left[\begin{matrix} \text{——} : a_1, a_2, b_1, \dots, b_r : a_1{}', a_2{}', b_1{}', \dots, b_s{}' \\ \text{——} : c_1, d_1, \dots, d_r : c_1{}', d_1{}', \dots, d_s{}' \end{matrix} \middle| x, y \right]
$$

$$= {}_{r+2}F_{r+1}\left[\begin{matrix} a_1,\ a_2,\ b_1,\ \ldots,\ b_r \\ c_1,\ d_1,\ \ldots,\ d_r \end{matrix}; x\right] {}_{s+2}F_{s+1}\left[\begin{matrix} a_1',\ a_2',\ b_1',\ \ldots,\ b_s' \\ c_1',\ d_1',\ \ldots,\ d_s' \end{matrix}; y\right]$$

$$= (1-x)^{-a_1} {}_{m_1+2}F_{m_1+1}\left[\begin{matrix} a_1,\ c_1-a_2-m_1,\ (\boldsymbol{\zeta})+1 \\ c_1,\ (\boldsymbol{\zeta}) \end{matrix}; \frac{x}{x-1}\right]$$

$$\times (1-y)^{c_1'-a_1'-a_2'-m_2} {}_{m_2+2}F_{m_2+1}\left[\begin{matrix} c_1'-a_1'-m_2,\ c_1'-a_2'-m_2,\ (\boldsymbol{\eta}')+1 \\ c_1',\ (\boldsymbol{\eta}') \end{matrix}; y\right]$$

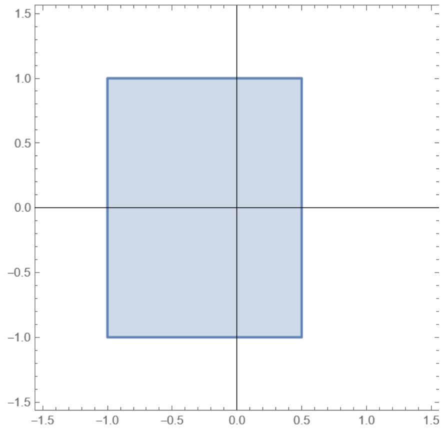
$$= F^{0:m_1+2:m_2+2}_{0:m_1+1:m_2+1}\left[\begin{matrix} \text{——} : a_1, e_1, (\boldsymbol{\zeta})+1 : e_2, e_3, (\boldsymbol{\eta}')+1 \\ \text{——} : c_1, (\boldsymbol{\zeta}) : c_1', (\boldsymbol{\eta}') \end{matrix} \middle| \frac{x}{x-1}, y\right]$$

$$\times\ (1-x)^{-a_1}(1-y)^{c_1'-a_1'-a_2'-m_2} \tag{3.33}$$

where $x < \frac{1}{2}$, $max\left\{|x|, |y|\right\} < 1$ and $max\left\{\left|\frac{x}{x-1}\right|, |y|\right\} < 1.$

Figure 7

Region of convergence for the sixth Kampé de Fériet transformation



7. **Seventh Transformation**

Let us take

$$\begin{cases} \boldsymbol{m_1} = (b_1-d_1, \ldots, b_r-d_r),\ \ \textit{vector of positive integers} \\ m_1 = \sum_{i=1}^{r} b_i - d_i \\ a_1 \neq d_i \\ (e_1)_{m_1} = (c_1-a_1-m_1)_{m_1} \neq 0 \\ \boldsymbol{m_2} = (b_1'-d_1', \ldots, b_s'-d_s'),\ \ \textit{vector of positive integers} \\ m_1 = \sum_{j=1}^{s} b_i' - d_i' \end{cases}$$

$$\begin{cases} a_1' \neq d_i' \\ (e_2)_{m_2} = (c_1' - a_1' - m_2)_{m_2} \neq 0 \end{cases} \tag{3.34}$$

$(\boldsymbol{\zeta}) = (\zeta_1, \ldots, \zeta_{m_1})$ to be the zeros of the characteristic polynomial

$$Q_{m_1}(t) = \frac{1}{(c_1 - a_1 - m_1)_{m_1}} \sum_{k=0}^{m_1} (a_1)_k C_{k,r}(t)_k (c_1 - a_1 - m_1 - t)_{m_1 - k} \tag{3.35}$$

with $C_{0,r} = 1, C_{m_1,r} = [(d_1)_{m_1} \ldots (d_r)_{m_r}]^{-1}$ and

$$C_{k,r} = \frac{(-1)^k}{k!} {}_{r+1}F_r \left[\begin{matrix} -k, \ b_1, \ \ldots, \ b_r \\ d_1, \ \ldots, \ d_r \end{matrix} ; 1 \right] \tag{3.36}$$

$(\boldsymbol{\zeta}') = (\zeta_1', \ldots, \zeta_{m_2}')$ to be the zeros of the characteristic polynomial

$$Q_{m_2}(t) = \frac{1}{(c_1' - a_1' - m_2)_{m_2}} \sum_{k=0}^{m_2} (a_1')_k C'_{k,s}(t)_k (c_1' - a_1' - m_2 - t)_{m_2 - k} \tag{3.37}$$

with $C'_{0,s} = 1, C'_{m_2,s} = [(d_1')_{m_1} \ldots (d_s')_{m_s}]^{-1}$ and

$$C'_{k,s} = \frac{(-1)^k}{k!} {}_{s+1}F_s \left[\begin{matrix} -k, \ b_1', \ \ldots, \ b_s' \\ d_1', \ \ldots, \ d_s' \end{matrix} ; 1 \right] \tag{3.38}$$

and consider the Kampé de Fériet function $F^{0:r+1:s+1}_{0:r+1:s+1}$. Then by expressing it in terms of two sums, one involves the generalised hypergeometric function ${}_{p+1}F_{p+1}(x)$, while the other sum involves another generalised hypergeometric function ${}_{p+1}F_{p+1}(y)$ and applying the transformation provided in (2.7) to the ${}_{p+1}F_{p+1}(x)$ and ${}_{p+1}F_{p+1}(y)$, we can rewrite the resulting equation using a double series representation. The transformation will be as follows:

$$F^{0:r+1:s+1}_{0:r+1:s+1} \left[\begin{matrix} \text{——} : a_1, b_1, \ldots, b_r : a_1', b_1', \ldots, b_s' \\ \text{——} : c_1, d_1, \ldots, d_r : c_1', d_1', \ldots, d_s' \end{matrix} \middle| x, y \right]$$

$$= {}_{r+1}F_{r+1} \left[\begin{matrix} a_1, \ b_1 \ldots, \ b_r \\ c_1, \ d_1, \ \ldots, \ d_r \end{matrix} ; x \right] {}_{s+1}F_{s+1} \left[\begin{matrix} a_1', \ b_1' \ldots, \ b_s' \\ c_1', \ d_1', \ \ldots, \ d_s' \end{matrix} ; y \right]$$

$$= e^{x+y} \ {}_{m_1+1}F_{m_1+1} \left[\begin{matrix} e_1, \ (\boldsymbol{\zeta}) + 1 \\ c_1, \ (\boldsymbol{\zeta}) \end{matrix} ; -x \right] {}_{m_2+1}F_{m_2+1} \left[\begin{matrix} e_2, \ (\boldsymbol{\zeta}') + 1 \\ c_1', \ (\boldsymbol{\zeta}') \end{matrix} ; -y \right]$$

$$= e^{x+y} F^{0:m_1+1:m_2+1}_{0:m_1+1:m_2+1}\left[\begin{array}{c} \text{——}: e_1, (\boldsymbol{\zeta})+1 : e_2, (\boldsymbol{\zeta'})+1 \\ \text{——}: c_1, (\boldsymbol{\zeta}) : c_1', (\boldsymbol{\zeta'}) \end{array}\middle| -x, -y\right] \tag{3.39}$$

for any values of $x,\ y$.

Remark: I have thoroughly tested all of the above Kampé de Fériet transformations using various parameter values and variables within the radius of convergence on Mathematica https://www.wolframcloud.com/obj/s12255413/Published/testing

Chapter Four
Radius of Convergence for the Double and Triple Hypergeometric Series

In this chapter, we will shift our focus to the double hypergeometric series, specifically the Appell hypergeometric series denoted as F_1, F_2, F_3, F_4, as well as the Horn series encompassing G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_5, H_6, H_7. Additionally, some of the triple hypergeometric series which introduced in ([2], p.59- p.65 and p70- p.107). Our aim is to try to find the radius of convergence for these series representations and compare our results with the findings presented in ([2], p.59- p.65 and p70- p.107), ensuring the reliability and accuracy of our analysis.

4.1 Radius of convergence for the double hypergeometric series

At the beginning, we will consider the Appell hypergeometric series F_1, F_2, F_3, F_4 and the Horn series G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_5, H_6, H_7. It's worth mentioning that to calculate the radius of convergence for these series one can use Horn's theorem ([2], p.56-p.57) which states that

Theorem 4. *If we have*

$$F(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C(m,n)x^m y^n \tag{4.1}$$

and define

$$f(m,n) = \frac{C(m+1,n)}{C(m,n)} \tag{4.2}$$

$$g(m,n) = \frac{C(m,n+1)}{C(m,n)} \tag{4.3}$$

which are rational functions. Let us define further

$$\rho(m,n) = |\lim_{u\to\infty} f(mu,nu)|^{-1} \tag{4.4}$$

$$\sigma(m,n) = |\lim_{u\to\infty} g(mu,nu)|^{-1} \tag{4.5}$$

which are rational functions too, and construct the following two subsets of $\mathbb{R}_+^2$

$$C = \{(r,s)|0 < r < \rho(1,0) \wedge 0 < s < \sigma(0,1)\} = K[\rho(1,0), \sigma(0,1)] \tag{4.6}$$

$$Z = \{(r,s)|\forall(m,n) \in \mathbb{R}_+^2 : r < \rho(m,n) \vee 0 < s < \sigma(m,n)\} \tag{4.7}$$

then the union of $C \cap Z$ *and its projections upon the coordinate axes is the representation in the absolute quadrant* $\mathbb{R}_+^2$ *of the region of convergence in* $\mathbb{C}^2$ *for the series F.*

However, the problem arises when constructing Z as it becomes challenging to find a direct relation (relations) between r and s without m, n. In ([2], p.59- p.65 and p70- p.107) the author (Srivastava) provides a method for finding such a relation for some of these cases, but not for others, leaving only the r, s relationship.

In this section, we aim to determine the radius of convergence for these functions using alternative approaches. Specifically, we plan to express each of these functions in terms of the Srivastava-Daoust series and leverage its radius of convergence to determine the radius of convergence for the given double series.

The Srivastava-Daoust series [9, 10], mentioned above, is expressed in the following form

$$F^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}} \left[\begin{matrix} [(a_A):\theta^{(1)},\ldots,\theta^{(n)}] : [(b^{(1)}_{B^{(1)}}):\phi^{(1)}];\ldots;[(b^{(n)}_{B^{(n)}}):\phi^{(n)}]; \\ [(c_C):\psi^{(1)},\ldots,\psi^{(n)}] : [(d^{(1)}_{D^{(1)}}):\delta^{(1)}];\ldots;[(d^{(n)}_{D^{(n)}}):\delta^{(n)}]; \end{matrix} x_1 \ldots x_n \right]$$

$$= \sum_{m_1,m_2,\ldots,m_n=0}^{\infty} \Omega(m_1,m_2,\ldots,m_n) \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} \cdots \frac{x_n^{m_n}}{m_n!} \tag{4.8}$$

where

$$\Omega(m_1,m_2,\ldots,m_n) = \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\cdots+m_n\theta_j^{(n)}} \prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}} \cdots \prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\cdots+m_n\psi_j^{(n)}} \prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}} \cdots \prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}} \tag{4.9}$$

and the coefficients $\theta_j^i, \phi_j^i, \psi_j^i, \delta_j^i,\ i \in \{1,\ldots,n\}$ are real and positive.

In order to find the radius of convergence of (4.8), one can use Horn's theorem for the double hypergeometric series and then we can extend to obtain various convergence conditions for the multiple hypergeometric series, refer to [10]. Consequently, to establish the set Z in Horn's theorem, we will define

$$E_i = \left(\mu_i^{1+\sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)}}\right) \frac{\prod_{j=1}^{C} \left(\sum_{l=1}^{n} \mu_l \, \psi_j^{(l)}\right)^{\psi_j^{(i)}} \prod_{j=1}^{D^{(i)}} (\delta_j^{(i)})^{\delta_j^{(i)}}}{\prod_{j=1}^{A} \left(\sum_{l=1}^{n} \mu_l \, \theta_j^{(l)}\right)^{\theta_j^{(i)}} \prod_{j=1}^{B^{(i)}} (\phi_j^{(i)})^{\phi_j^{(i)}}} \tag{4.10}$$

and

$$\Delta_i = 1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} \quad i \in \{1, \ldots, n\} \tag{4.11}$$

Now, the radius of convergence of (4.8) can be defined depending on the following three cases:

1. When $\Delta_i \; > 0 \;\; , \forall i \in \{1, \ldots, n\}$
2. When $\Delta_i \; = 0 \;\; , \forall i \in \{1, \ldots, n\}$
3. When $\Delta_i \; < 0 \;\; , \forall i \in \{1, \ldots, n\}$

We are interested in the case in which $\Delta_1 = \Delta_2 = \cdots = \Delta_n$ = 0. Here in this case, the set Z will determined by:

$$|x_1| < \varrho_1, \; |x_2| < \varrho_2, \; \ldots, \; |x_n| < \varrho_n \tag{4.12}$$

where

$$\varrho_i = \min_{\mu_1, \ldots, \mu_n > 0} \{E_i\} \tag{4.13}$$

For further information about the other cases and the special cases, you can refer to [9, 10].

Now, the first group of double series we will begin with, the Appell hypergeometric series which includes F_1, F_2, F_3, F_4. In fact, obtaining the radius of convergence for these series is relatively straightforward using Horn's theorem, and determining the relation between r and s is also easy. However, starting with them will allow us to understand how we can choose the coefficients $\theta_j^i, \phi_j^i, \psi_j^i, \delta_j^i$ for $i \in \{1, \ldots, n\}$.

4.1.1 Radius of convergence for Appell hypergeometric series F_1

The first Appell series we will start with is F_1, which takes the following form:

$$F_1(\alpha, \beta_1, \beta_2; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n}} \frac{y^m}{m!} \frac{y^n}{n!} \tag{4.14}$$

and its radius of convergence is given by, max $\{|x|, |y|\} < 1$ ([2], p.59).

We can observe that the Srivastava-Daoust series involves n variables. Therefore, to rewrite F_1 as the Srivastava-Daoust series, we will set $n = 2$ and consider the following:

$$\begin{cases} A = C \\ 1 + D^{(1)} = B^{(1)} \\ 1 + D^{(2)} = B^{(2)} \end{cases} \tag{4.15}$$

and the coefficients $\theta_j^i, \phi_j^i, \psi_j^i, \delta_j^i$ as follows

$$\begin{cases} \theta_1^{(1)} = 1,\ \theta_1^{(2)} = 1 \\ \psi_1^{(1)} = 1,\ \psi_1^{(2)} = 1 \\ \phi_1^{(1)} = 1,\ \phi_1^{(2)} = 1 \\ \theta_{j+1}^{(1)} = \psi_{j+1}^{(1)} \quad j = 1, \ldots, C-1 \\ \theta_{j+1}^{(2)} = \psi_{j+1}^{(2)} \quad j = 1, \ldots, C-1 \\ \phi_{j+1}^{(1)} = \delta_j^{(1)} \quad j = 1, \ldots, D^{(1)} \\ \phi_{j+1}^{(2)} = \delta_j^{(2)} \quad j = 1, \ldots, D^{(2)} \end{cases} \tag{4.16}$$

Also the values of the vectors $b_j^{(1)}, b_j^{(2)}, d_j^{(1)}, d_j^{(2)}$

$$\begin{cases} a_{j+1} = c_{j+1} \quad j = 1, \ldots, C-1 \\ b_{j+1}^{(1)} = d_j^{(1)} \quad j = 1, \ldots, D^{(1)} \\ b_{j+1}^{(2)} = d_j^{(2)} \quad j = 1, \ldots, D^{(2)} \end{cases} \tag{4.17}$$

Depending on (4.15) and (4.17), we can write for $i = 1, 2$

$$c = (c_1, a_2, a_3, \ldots, a_C), \quad a = (a_1, a_2, a_3, \ldots, a_C)$$

$$d^{(i)} = (b_2^{(i)}, b_3^{(i)}, \ldots, b_{1+D^{(i)}}^{(i)}), \quad b^{(i)} = (b_1^{(i)}, b_2^{(i)}, \ldots, b_{D^{(2)}}^{(i)}, b_{1+D^{(i)}}^{(i)})$$

Finally, if we substitute (4.15)-(4.17) in (4.8), we will obtain F_1

$$F_{C:D^{(1)};D^{(2)}}^{C:1+D^{(1)};1+D^{(2)}} \left[\begin{matrix} [(a_C) : \theta^{(1)}, \theta^{(2)}] : [(b_{1+D^{(1)}}^{(1)}) : \phi^{(1)}]; [(b_{1+D^{(2)}}^{(2)}) : \phi^{(2)}]; \\ [(c_C) : \psi^{(1)}, \psi^{(2)}] : [(d_{D^{(1)}}^{(1)}) : \delta^{(1)}]; [(d_{D^{(2)}}^{(2)}) : \delta^{(2)}]; \end{matrix} \; x_1, x_2 \right]$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(a_1)_{m_1+m_2} (a_2)_{m_1+m_2} \ldots (a_C)_{m_1+m_2} \;\; (b_1^{(1)})_{m_1} (b_2^{(1)})_{m_1} \ldots (b_{1+D^{(1)}}^{(1)})_{m_1}}{(c_1)_{m_1+m_2} (a_2)_{m_1+m_2} \ldots (a_C)_{m_1+m_2} \;\; (b_2^{(1)})_{m_1} \ldots (b_{1+D^{(1)}}^{(1)})_{m_1}} \times \frac{(b_1^{(2)})_{m_2} (b_2^{(2)})_{m_2} \ldots (b_{1+D^{(2)}}^{(2)})_{m_2}}{(b_2^{(2)})_{m_2} \ldots (b_{1+D^{(2)}}^{(2)})_{m_2}} \frac{x^{m_1}}{m_1!} \frac{y^{m_2}}{m_2!}$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(a_1)_{m_1+m_2} \;\; (b_1^{(1)})_{m_1} \;\; (b_1^{(2)})_{m_2}}{(c_1)_{m_1+m_2}} \frac{x^{m_1}}{m_1!} \frac{y^{m_2}}{m_2!} \tag{4.18}$$

$$= F_1 \left[\begin{matrix} a_1; \; b_1^{(1)}; \; b_1^{(2)} \\ c_1; \text{——}; \text{——} \end{matrix} \middle| x, y \right]$$

Now, we need to calculate Δ_1 and Δ_2 to determine the radius of convergence of F_1. Depending on (4.15) and (4.16), we will obtain

$$\Delta_1 = 1 + \sum_{j=1}^{C} \psi_j^{(1)} + \sum_{j=1}^{D^{(1)}} \delta_j^{(1)} - \sum_{j=1}^{A} \theta_j^{(1)} - \sum_{j=1}^{B^{(1)}} \phi_j^{(1)}$$
$$= 1 + 1 + \sum_{j=2}^{C} \psi_j^{(1)} + \sum_{j=1}^{D^{(1)}} \delta_j^{(1)} - 1 - \sum_{j=2}^{C} \theta_j^{(1)} - 1 - \sum_{j=2}^{1+D^{(1)}} \phi_j^{(1)} = 0$$

$$\Delta_2 = 1 + \sum_{j=1}^{C} \psi_j^{(2)} + \sum_{j=1}^{D^{(2)}} \delta_j^{(2)} - \sum_{j=1}^{A} \theta_j^{(2)} - \sum_{j=1}^{B^{(2)}} \phi_j^{(2)}$$
$$= 1 + 1 + \sum_{j=2}^{C} \psi_j^{(2)} + \sum_{j=1}^{D^{(2)}} \delta_j^{(2)} - 1 - \sum_{j=2}^{C} \theta_j^{(2)} - 1 - \sum_{j=2}^{1+D^{(2)}} \phi_j^{(2)} = 0$$

we are in the case that we mentioned above. To contract the set Z, we will use the equations (4.10)-(4.12) and (4.15)-(4.17). By doing so, we will obtain $|x| < 1$ and $|y| < 1$,

$$\varrho_1 = \min_{\mu_1,\mu_2>0} \left\{ \left(\mu_1^{1+\sum_{j=1}^{D^{(1)}} \delta_j^{(1)} - 1 - \sum_{j=2}^{1+D^{(1)}} \phi_j^{(1)}} \right) \frac{\prod_{j=1}^{C} \left(\sum_{l=1}^{n} \mu_l \, \psi_j^{(l)} \right)^{\psi_j^{(1)}} \prod_{j=1}^{D^{(1)}} (\delta_j^{(1)})^{\delta_j^{(1)}}}{\prod_{j=1}^{C} \left(\sum_{l=1}^{n} \mu_l \, \theta_j^{(l)} \right)^{\theta_j^{(1)}} \prod_{j=1}^{1+D^{(1)}} (\phi_j^{(1)})^{\phi_j^{(1)}}} \right\}$$

$$= 1$$

$$\varrho_2 = \min_{\mu_1,\mu_2>0} \left\{ \left(\mu_2^{1+\sum_{j=1}^{D^{(2)}} \delta_j^{(2)} - 1 - \sum_{j=2}^{1+D^{(2)}} \phi_j^{(2)}} \right) \frac{\prod_{j=1}^{C} \left(\sum_{l=1}^{n} \mu_l \, \psi_j^{(l)} \right)^{\psi_j^{(2)}} \prod_{j=1}^{D^{(2)}} (\delta_j^{(2)})^{\delta_j^{(2)}}}{\prod_{j=1}^{C} \left(\sum_{l=1}^{n} \mu_l \, \theta_j^{(l)} \right)^{\theta_j^{(2)}} \prod_{j=1}^{1+D^{(2)}} (\phi_j^{(2)})^{\phi_j^{(2)}}} \right\}$$

$$= 1$$

which means that the Z can be determined by $|x| < 1$ and $|y| < 1$, which is equivalent to $\max\{|x|, |y|\} < 1$. To construct the C we will use Horn's theorem as follows:

$$\begin{aligned} f(m,n) = \frac{C(m+1,n)}{C(m,n)} &= \frac{(\alpha)_{m+n+1}(\beta_1)_{m+1}(\beta_2)_n}{(\gamma)_{m+n+1}(m+1)!n!} \frac{(\gamma)_{m+n} m! n!}{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n} \\ &= \frac{(\alpha+m+n)(\beta_1+m)}{(\gamma+m+n)(m+1)} \end{aligned}$$

$$\begin{aligned} g(m,n) = \frac{C(m,n+1)}{C(m,n)} &= \frac{(\alpha)_{m+n+1}(\beta_1)_m(\beta_2)_{n+1}}{(\gamma)_{m+n+1} m!(n+1)!} \frac{(\gamma)_{m+n} m! n!}{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n} \\ &= \frac{(\alpha+m+n)(\beta_2+n)}{(\gamma+m+n)(n+1)} \end{aligned}$$

Therefor, $\rho(m,n)$ and $\sigma(m,n)$ will be:

$$\begin{aligned} \rho(m,n) = |\lim_{u\to\infty} f(mu,nu)|^{-1} &= \left| \lim_{u\to\infty} \frac{(\alpha+mu+nu)(\beta_1+mu)}{(\gamma+mu+nu)(mu+1)} \right|^{-1} \\ &= \left| \lim_{u\to\infty} \frac{m(m+n)}{m(m+n)} \right|^{-1} = 1 \end{aligned}$$

$$\begin{aligned} \sigma(m,n) = |\lim_{u\to\infty} g(mu,nu)|^{-1} &= \left| \lim_{u\to\infty} \frac{(\alpha+mu+nu)(\beta_2+nu)}{(\gamma+mu+nu)(nu+1)} \right|^{-1} \\ &= \left| \lim_{u\to\infty} \frac{n(m+n)}{n(m+n)} \right|^{-1} = 1 \end{aligned}$$

As a result, the set C will be determined by:

$$\begin{aligned} C &= \{(r,s)|0 < r < \rho(1,0) \wedge 0 < s < \sigma(0,1)\} \\ &= \{(r,s)|0 < r < 1 \wedge 0 < s < 1\} = K[\rho(1,0),\sigma(0,1)] = K[1,1] \end{aligned}$$

since the set $Z \supset C$, the radius of convergence for the double hypergeometric function F_1 will be $\max\{|x|,|y|\} < 1$.

4.1.2 Radius of convergence for Appell hypergeometric series F_2

The second Appell series we will consider is F_2

$$F_2(\alpha,\beta_1,\beta_2;\gamma_1\gamma_2;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{4.19}$$

and its radius of convergence is given by, $|x|+|y| < 1$ ([2], p.59).

Similarly, we will set $n = 2$ and consider the following:

$$\begin{cases} A = C+1 \\ D^{(1)} = B^{(1)} \\ D^{(2)} = B^{(2)} \end{cases} \tag{4.20}$$

and the coefficients $\theta_j^i, \phi_j^i, \psi_j^i, \delta_j^i$ as follows

$$\begin{cases} \theta_1^{(1)} = 1, \ \theta_1^{(2)} = 1 \\ \delta_1^{(1)} = 1, \ \delta_1^{(2)} = 1 \\ \phi_1^{(1)} = 1, \ \phi_1^{(2)} = 1 \\ \theta_{j+1}^{(1)} = \psi_j^{(1)} \quad j = 1,\ldots,C \\ \theta_{j+1}^{(2)} = \psi_j^{(2)} \quad j = 1,\ldots,C \\ \phi_{j+1}^{(1)} = \delta_{j+1}^{(1)} \quad j = 1,\ldots,D^{(1)}-1 \\ \phi_{j+1}^{(2)} = \delta_{j+1}^{(2)} \quad j = 1,\ldots,D^{(2)}-1 \end{cases} \tag{4.21}$$

Also the values of the vectors $a_j, c_j, b_j^{(1)}, b_j^{(2)}, d_j^{(1)}, d_j^{(2)}$

$$\begin{cases} a_{j+1} = c_j \quad j = 1, \ldots, C \\ b_{j+1}^{(1)} = d_{j+1}^{(1)} \quad j = 1, \ldots, D^{(1)} - 1 \\ b_{j+1}^{(2)} = d_{j+1}^{(2)} \quad j = 1, \ldots, D^{(2)} - 1 \end{cases} \tag{4.22}$$

Depending on (4.20) and (4.22), we can write for $i = 1, 2$

$$c = (a_2, a_3, \ldots, a_{1+C}), \quad a = (a_1, a_2, a_3, \ldots, a_{1+C})$$

$$d^{(i)} = (d_1^{(i)}, b_2^{(i)}, \ldots, b_{D^{(i)}}^{(i)}), \quad b^{(i)} = (b_1^{(i)}, b_2^{(i)}, \ldots, b_{D^{(i)}}^{(i)})$$

Similarly, if we substitute equations (4.20)-(4.22) into (4.8), we will obtain F_2

$$F^{C+1:D^{(1)};D^{(2)}}_{C:D^{(1)};D^{(2)}} \left[\begin{matrix} [(a_{C+1}) : \theta^{(1)}, \theta^{(2)}] : [(b_{D^{(1)}}^{(1)}) : \phi^{(1)}]; [(b_{D^{(2)}}^{(2)}) : \phi^{(2)}]; \\ [(c_C) : \psi^{(1)}, \psi^{(2)}] : [(d_{D^{(1)}}^{(1)}) : \delta^{(1)}]; [(d_{D^{(2)}}^{(2)}) : \delta^{(2)}]; \end{matrix} \; x, y \right]$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(a_1)_{m_1+m_2} \ (b_1^{(1)})_{m_1} \ (b_1^{(2)})_{m_2}}{(d_1^{(1)})_{m_1} \ (d_1^{(2)})_{m_2}} \frac{x^{m_1}}{m_1!} \frac{y^{m_2}}{m_2!} \tag{4.23}$$

$$= F_2 \left[\begin{matrix} a_1; \ b_1^{(1)}; \ b_1^{(2)} \\ \text{——}; \ d_1^{(1)} \ ; \ d_1^{(2)} \end{matrix} \middle| x, y \right]$$

To construct the set Z, it is clear that we are in the same case since $\Delta_1 = \Delta_2 = 0$. Therefore, we can use equations (4.10)-(4.13), to obtain $|x| + |y| < 1$. Calculating the set $C = K[1, 1]$ can be easily achieved using Horn's theorem. Consequently, the radius of convergence will be $|x| + |y| < 1$, because $Z \subset C$.

4.1.3 Radius of convergence for Appell hypergeometric series F_3

The third Appell hypergeometric series we will take is F_3

$$F_3(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_n (\beta_1)_m (\beta_2)_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{4.24}$$

and its radius of convergence is given by, $\max\{|x|, |y|\} < 1$ ([2], p.59).

Similarly, we will set $n=2$ and consider the following:

$$\begin{cases} 1+A=C \\ 2+D^{(1)}=B^{(1)} \\ 2+D^{(2)}=B^{(2)} \end{cases} \tag{4.25}$$

and the coefficients $\theta^i_j, \phi^i_j, \psi^i_j, \delta^i_j$ as follows

$$\begin{cases} \psi^{(1)}_1=1,\ \psi^{(2)}_1=1 \\ \phi^{(1)}_1=1,\ \phi^{(1)}_2=1 \\ \phi^{(2)}_1=1,\ \phi^{(2)}_2=1 \\ \theta^{(1)}_j=\psi^{(1)}_{j+1} \quad j=1,\dots,A \\ \theta^{(2)}_j=\psi^{(2)}_{j+1} \quad j=1,\dots,A \\ \phi^{(1)}_{j+2}=\delta^{(1)}_j \quad j=1,\dots,D^{(1)} \\ \phi^{(2)}_{j+2}=\delta^{(2)}_j \quad j=1,\dots,D^{(2)} \end{cases} \tag{4.26}$$

Also the values of the vectors $a_j, c_j, b^{(1)}_j, b^{(2)}_j, d^{(1)}_j, d^{(2)}_j$

$$\begin{cases} c_{j+1}=a_j \quad j=1,\dots,A \\ b^{(1)}_{j+2}=d^{(1)}_j \quad j=1,\dots,D^{(1)} \\ b^{(2)}_{j+2}=d^{(2)}_j \quad j=1,\dots,D^{(2)} \end{cases} \tag{4.27}$$

Depending on (4.25) and (4.27), we can write for $i=1,2$

$$c=(c_1,c_2,\dots,c_{1+A}), \quad a=(c_2,c_3,\dots,c_{1+A})$$

$$b^{(i)}=(b^{(i)}_1,b^{(i)}_2,b^{(i)}_3,\dots,b^{(i)}_{D^{(i)}+2}), \quad d^{(i)}=(b^{(i)}_3,b^{(i)}_4,\dots,b^{(i)}_{D^{(i)}+2})$$

If we substitute equations (4.25)-(4.27) into (4.8), we will obtain F_3

$$F^{A:2+D^{(1)};2+D^{(2)}}_{1+A:D^{(1)};D^{(2)}}\left[\begin{matrix} [(a_A):\theta^{(1)},\theta^{(2)}]:[(b^{(1)}_{2+D^{(1)}}):\phi^{(1)}];[(b^{(2)}_{2+D^{(2)}}):\phi^{(2)}]; \\ [(c_{1+A}):\psi^{(1)},\psi^{(2)}]:[(d^{(1)}_{D^{(1)}}):\delta^{(1)}];[(d^{(2)}_{D^{(2)}}):\delta^{(2)}]; \end{matrix} x,y\right]$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(b_1^{(1)})_{m_1} (b_2^{(1)})_{m_1} (b_1^{(2)})_{m_2} (b_2^{(2)})_{m_2}}{(c_1)_{m_1+m_2}} \frac{x^{m_1}}{m_1!} \frac{y^{m_2}}{m_2!} \tag{4.28}$$

$$= F_3 \left[\begin{matrix} \text{——};\ b_1^{(1)},\ b_2^{(1)};\ b_1^{(2)},\ b_2^{(2)} \\ c_1;\ \text{——};\ \text{——} \end{matrix} \middle| x, y \right]$$

Since $\Delta_1 = \Delta_2 = 0$, we can use equations (4.10)-(4.13), to obtain $\frac{1}{|x|} + \frac{1}{|y|} > 1$. The set $C = K[1, 1]$ can be easily calculated using Horn's theorem. Consequently, the radius of convergence will be $\frac{1}{|x|} + \frac{1}{|y|} > 1$, because $C \subset Z$.

4.1.4 Radius of convergence for Appell hypergeometric series F_4

The last Appell hypergeometric series we will take is F_4

$$F_4(\alpha, \beta; \gamma_1, \gamma_2; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{(\gamma_1)_m (\gamma_2)_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{4.29}$$

and its radius of convergence is given by, $\sqrt{|x|} + \sqrt{|y|} < 1$ ([2], p.59).

If we set $n = 2$ and consider the following:

$$\begin{cases} 2 + C = A \\ 1 + B^{(1)} = D^{(1)} \\ 1 + B^{(2)} = D^{(2)} \end{cases} \tag{4.30}$$

and the coefficients $\theta_j^i, \phi_j^i, \psi_j^i, \delta_j^i$ as follows

$$\begin{cases} \theta_1^{(1)} = 1,\ \theta_1^{(2)} = 1 \\ \theta_2^{(1)} = 1,\ \theta_2^{(2)} = 1 \\ \delta_1^{(1)} = 1,\ \delta_1^{(2)} = 1 \\ \theta_{j+2}^{(1)} = \psi_j^{(1)} \quad j = 1, \dots, C \\ \theta_{j+2}^{(2)} = \psi_j^{(2)} \quad j = 1, \dots, C \\ \phi_j^{(1)} = \delta_{j+1}^{(1)} \quad j = 1, \dots, B^{(1)} \\ \phi_j^{(2)} = \delta_{j+1}^{(2)} \quad j = 1, \dots, B^{(2)} \end{cases} \tag{4.31}$$

Also the values of the vectors $a_j, c_j, b_j^{(1)}, b_j^{(2)}, d_j^{(1)}, d_j^{(2)}$

$$\begin{cases} c_j = a_{j+2} \quad j = 1, \ldots, C \\ b_j^{(1)} = d_{j+1}^{(1)} \quad j = 1, \ldots, B^{(1)} \\ b_j^{(2)} = d_{j+1}^{(2)} \quad j = 1, \ldots, B^{(2)} \end{cases} \tag{4.32}$$

Depending to (4.30) and (4.32), we can write for $i = 1, 2$

$$c = (a_3, a_4, \ldots, a_{2+C}), \quad a = (a_1, a_2, \ldots, a_{2+C})$$

$$b^{(i)} = (d_2^{(i)}, d_3^{(i)}, \ldots, d_{B^{(i)}+1}^{(i)}), \quad d^{(i)} = (d_1^{(i)}, d_2^{(i)}, \ldots, d_{B^{(i)}+1}^{(i)})$$

If we substitute equations (4.30)-(4.32) into (4.8), we will obtain F_4

$$F_{C:1+B^{(1)};1+B^{(2)}}^{2+C:B^{(1)};B^{(2)}} \left[\begin{matrix} [(a_{2+C}) : \theta^{(1)}, \theta^{(2)}] : [(b_{B^{(1)}}^{(1)}) : \phi^{(1)}]; [(b_{B^{(2)}}^{(2)}) : \phi^{(2)}]; \\ [(c_C) : \psi^{(1)}, \psi^{(2)}] : [(d_{1+B^{(1)}}^{(1)}) : \delta^{(1)}]; [(d_{1+B^{(2)}}^{(2)}) : \delta^{(2)}]; \end{matrix} x, y \right]$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(a_1)_{m_1+m_2} \, (a_2)_{m_1+m_2}}{(d_1^{(1)})_{m_1} \, (d_1^{(2)})_{m_2}} \frac{x^{m_1}}{m_1!} \frac{y^{m_2}}{m_2!} \tag{4.33}$$

$$= F_4 \left[\begin{matrix} a_1, \ a_2; \text{——}; \text{——} \\ \text{——}; \ d_1^{(1)}; \ d_1^{(2)} \end{matrix} \middle| x, y \right]$$

As well, since $\Delta_1 = \Delta_2 = 0$, we can use equations (4.10)-(4.13), to obtain $\sqrt{|x|} + \sqrt{|y|} < 1$. The set $C = K[1, 1]$ can be calculated using Horn's theorem. Consequently, the radius of convergence will be $\sqrt{|x|} + \sqrt{|y|} < 1$, because $Z \subset C$.

After we study the group of Appell hypergeometric series F_1, F_2, F_3 and F_4, we will study another group of double hypergeometric series known as Horn series, which includes G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_5, H_6, and H_7. Our objective remains the same: determining the radius of convergence for these series by expressing each of them in terms of the Srivastava-Daoust series and leveraging its radius of convergence to determine the radius of convergence for the given Horn series.

As we proceed to the next sections, we will observe that some of these Horn series, such as H_3 and H_4, are easily expressed as Srivastava-Daoust series. However, for the other functions, this process is not as straightforward. Therefore, we will perform a decomposition for the double sum, as we will see.

4.1.5 Radius of convergence for Horn series H_3

At the beginning, let us start with H_3:

$$H_3(\alpha,\beta;\gamma;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{4.34}$$

and its region of convergence is given by:

$$|x|<\frac{1}{4} \quad \text{and} \quad |y|<\frac{1}{2}+\frac{1}{2}\sqrt{1-4|x|}$$

Actually, we can obtain the above radius of convergence using Mathematica, as shown in [11]. However, in this thesis, we will focus on obtaining the radius of convergence by rewriting the Horn series as Srivastava-Daoust series.
To do this, we set $n=2$ and consider the following:

$$\begin{cases} A=C \\ D^{(1)}=B^{(1)} \\ 1+D^{(2)}=B^{(2)} \end{cases} \tag{4.35}$$

and the coefficients $\theta_j^i,\phi_j^i,\psi_j^i,\delta_j^i$ as follows

$$\begin{cases} \theta_1^{(1)}=2,\ \theta_1^{(2)}=1 \\ \psi_1^{(1)}=1,\ \psi_1^{(2)}=1 \\ \phi_{1+D^{(2)}}=1 \\ \theta_{j+1}^{(1)}=\psi_{j+1}^{(1)} \quad j=1,\dots,C-1 \\ \theta_{j+1}^{(2)}=\psi_{j+1}^{(2)} \quad j=1,\dots,C-1 \\ \phi_j^{(1)}=\delta_j^{(1)} \quad j=1,\dots,B^{(1)} \end{cases}$$

$$\begin{cases} \phi_j^{(2)} = \delta_j^{(2)} \quad j = 1, \ldots, D^{(2)} \end{cases} \tag{4.36}$$

Also the values of the vectors $b_j^{(1)}, b_j^{(2)}, d_j^{(1)}, d_j^{(2)}$

$$\begin{cases} a_{j+1} = c_{j+1} \quad j = 1, \ldots, C-1 \\ b_j^{(1)} = d_j^{(1)} \quad j = 1, \ldots, B^{(1)} \\ b_j^{(2)} = d_j^{(2)} \quad j = 1, \ldots, D^{(2)} \end{cases} \tag{4.37}$$

After we define all values we want for equation (4.8), we can substitute them and then do some calculations to obtain H_3

$$F^{A:D^{(1)};1+D^{(2)}}_{A:D^{(1)};D^{(2)}} \left[\begin{matrix} [(a_A) : \theta^{(1)}, \theta^{(2)}] : [(b^{(1)}_{D^{(1)}}) : \phi^{(1)}]; [(b^{(2)}_{1+D^{(2)}}) : \phi^{(2)}]; \\ [(c_A) : \psi^{(1)}, \psi^{(2)}] : [(d^{(1)}_{D^{(1)}}) : \delta^{(1)}]; [(d^{(2)}_{D^{(2)}}) : \delta^{(2)}]; \end{matrix} x, y \right]$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(a_1)_{2m_1+m_2} \; (b^{(2)}_{1+D^{(2)}})_{m_2}}{(c_1)_{m_1+m_2}} \frac{x^{m_1}}{m_1!} \frac{y^{m_2}}{m_2!} \tag{4.38}$$

$$= H_3 \left[\begin{matrix} a_1; \text{——}; b^{(2)}_{1+D^{(2)}} \\ c_1; \text{——}; \text{——} \end{matrix} \middle| x, y \right]$$

To calculate the radius of convergence of H_3, we need to evaluate Δ_1 and Δ_2. After performing the calculations, we find that $\Delta_1 = \Delta_2 = 0$. As a result, the set Z will determined by:

$$|x| < \frac{m(m+n)}{(2m+n)^2}, \quad |y| < \frac{m+n}{2m+n} \tag{4.39}$$

where $m > 0$, $n > 0$. Now we can write (4.39) as the follows

$$1 - 4|x| > \frac{n^2}{(2m+n)^2}, \quad |y| - \frac{1}{2} < \frac{n}{2(2m+n)} \tag{4.40}$$

which implies

$$|x| < \frac{1}{4} \;\; and \;\; |y| < \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4|x|} \tag{4.41}$$

Let us plot the boundaries that determined the radius of convergence for H_3 based on (4.39) and observe the results. We can use Mathematica to perform this task by generating two values for m and n (both greater than 0) and then locating the corresponding point (x, y) in the $xy-$plane. We will repeat this process until we have a sufficient number of generated values, for example, 10000. It is worth mentioning that when plotting the boundaries, we must consider the boundaries for the set $C = K[\frac{1}{4}, 1]$, as indicated in figure (11b) in Appendix (4.2.3).

```
numPoints = 10000;
(*Generate random values for m and n*)
mValues = RandomReal[{0.000000000001,1},numPoints];
nValues = RandomReal[{0.000000000001,1},numPoints];
(*Initialize lists to store the x and y coordinates*)
xValues1 = ConstantArray[0,numPoints]; yValues1 = ConstantArray[0,
    numPoints];
xValues2 = ConstantArray[0,numPoints]; yValues2 = ConstantArray[0,
    numPoints];
(*Calculate x and y coordinates for each point*)
Do[
 m = mValues[[i]];
 n = nValues[[i]];
 x1 = (m)*(m+n)/(2*m+n)^2;
 y1 = (m+n)/(2*m+n);
 x2 = 1/4;
 y2 = m;
(*Store the coordinates*)
 xValues1[[i]] = x1; yValues1[[i]] = y1;
 xValues2[[i]] = x2; yValues2[[i]] = y2;
 ,{i,numPoints}];
(*Plot the points on the XY plane with zoomed-in region*)
ListPlot[{Transpose[{xValues1,yValues1}],Transpose[{xValues2,yValues2}]},
    PlotStyle->{PointSize[Small],PointSize[Small]},AxesLabel->{"X","Y"},
    AspectRatio->1,PlotLabel->"Points in XY Plane",PlotRange
    ->{{0,1},{0,1}}]
```

4.1.6 Radius of convergence for Horn series G_1

Let us consider the double series G_1

$$G_1(\alpha, \beta, \beta'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha)_{m+n} (\beta)_{n-m} (\beta')_{m-n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{4.42}$$

It is evident that we cannot directly rewrite G_1 as a Srivastava-Daoust series due to the presence of terms like $(\beta)_{n-m}$ and $(\beta')_{m-n}$, where the coefficients θ_j^i, ϕ_j^i, ψ_j^i, and δ_j^i cannot be negative. However, in ([12], p.43-p.44) another approach was used to eliminate the negative sign from the double series. By applying the same technique to G_1 (See Appendix 4.2.3), we obtain equation (4.43). I have verified this formula using Mathematica, and I found that the left-hand side and the right-hand side are equal for various values of α, β, β', x, and y, where x and y fall within the radius of convergence for both sides. https://www.wolframcloud.com/obj/s12255413/Published/Testing

$$\begin{aligned} G_1(\alpha, \beta, \beta'; x, y) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_{m+2n} (\beta')_m (1)_m}{(1)_{m+n} (1-\beta)_m} \frac{(-x)^m}{m!} \frac{(xy)^n}{n!} \\ &+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_{2m+n} (\beta)_n (1)_n}{(1)_{m+n} (1-\beta')_n} \frac{(xy)^m}{m!} \frac{(-y)^n}{n!} - \sum_{n=0}^{\infty} \frac{(\alpha)_{2n}}{(1)_n} \frac{(xy)^n}{n!} \end{aligned} \tag{4.43}$$

As the negative sign in (4.42) does not appear anywhere in (4.43), we can rewrite the first two double sums and the third sum using the Srivastava-Daoust series as follows:

- The first double sum

$$S_1 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_{m+2n} (\beta')_m (1)_m}{(1)_{m+n} (1-\beta)_m} \frac{(-x)^m}{m!} \frac{(xy)^n}{n!} \tag{4.44}$$

  Set $n = 2$ in Srivastava-Daoust series and consider the following:

$$\begin{cases} A = C \\ 1 + D^{(1)} = B^{(1)} \\ D^{(2)} = B^{(2)} \end{cases} \tag{4.45}$$

and the coefficients $\theta_j^i, \phi_j^i, \psi_j^i, \delta_j^i$ as follows

$$\begin{cases} \theta_1^{(1)} = 1,\ \theta_1^{(2)} = 2 \\ \psi_1^{(1)} = 1,\ \psi_1^{(2)} = 1 \\ \phi_1^{(1)} = 1,\ \phi_2^{(1)} = 1 \\ \delta_1^{(1)} = 1 \\ \theta_{j+1}^{(1)} = \psi_{j+1}^{(1)} \quad j = 1, \dots, A-1 \\ \theta_{j+1}^{(2)} = \psi_{j+1}^{(2)} \quad j = 1, \dots, A-1 \\ \phi_{j+2}^{(1)} = \delta_{j+1}^{(1)} \quad j = 1, \dots, D^{(1)}-1 \\ \phi_j^{(2)} = \delta_j^{(2)} \quad j = 1, \dots, D^{(2)} \end{cases} \tag{4.46}$$

Also the values of the vectors $a_j, c_j, b_j^{(1)}, b_j^{(2)}, d_j^{(1)}, d_j^{(2)}$

$$\begin{cases} a_1 = \alpha,\ c_1 = 1 \\ b_1^{(1)} = \beta',\ b_2^{(1)} = 1,\ d_1^{(1)} = 1 - \beta \\ c_{j+1} = a_{j+1} \quad j = 1, \dots, A-1 \\ b_{j+2}^{(1)} = d_{j+1}^{(1)} \quad j = 1, \dots, D^{(1)}-1 \\ b_j^{(2)} = d_j^{(2)} \quad j = 1, \dots, D^{(2)} \end{cases} \tag{4.47}$$

Now we can calculate the radius of convergence for the first double sum as we do before, we will obtain that

$$|x| < \frac{(m+n)}{(m+2n)}, \quad |xy| < \frac{n(m+n)}{(m+2n)^2} \tag{4.48}$$

where $m > 0$, $n > 0$. It is clear that we can apply the same approach used for H_3 to obtain the following relation between x and y

$$|xy| < \frac{1}{4} \ \wedge \ |x| < \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4|xy|} \tag{4.49}$$

- The Second double sum

$$S_2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n(1)_n}{(1)_{m+n}(1-\beta')_n} \frac{(xy)^m}{m!} \frac{(-y)^n}{n!} \tag{4.50}$$

Set $n = 2$ in Srivastava-Daoust series and consider the following:

$$\begin{cases} A = C \\ D^{(1)} = B^{(1)} \\ 1 + D^{(2)} = B^{(2)} \end{cases} \tag{4.51}$$

and the coefficients $\theta_j^i, \phi_j^i, \psi_j^i, \delta_j^i$ as follows

$$\begin{cases} \theta_1^{(1)} = 2,\ \theta_1^{(2)} = 1 \\ \psi_1^{(1)} = 1,\ \psi_1^{(2)} = 1 \\ \phi_1^{(2)} = 1,\ \phi_2^{(2)} = 1 \\ \delta_1^{(2)} = 1 \\ \theta_{j+1}^{(1)} = \psi_{j+1}^{(1)} \quad j = 1, \ldots, A-1 \\ \theta_{j+1}^{(2)} = \psi_{j+1}^{(2)} \quad j = 1, \ldots, A-1 \\ \phi_j^{(1)} = \delta_j^{(1)} \quad j = 1, \ldots, B^{(1)} \\ \phi_{j+2}^{(2)} = \delta_{j+1}^{(2)} \quad j = 1, \ldots, D^{(2)} - 1 \end{cases} \tag{4.52}$$

Also the values of the vectors $a_j, c_j, b_j^{(1)}, b_j^{(2)}, d_j^{(1)}, d_j^{(2)}$

$$\begin{cases} a_1 = \alpha,\ c_1 = 1 \\ b_1^{(2)} = \beta,\ b_2^{(2)} = 1,\ d_1^{(2)} = 1 - \beta' \\ c_{j+1} = a_{j+1} \quad j = 1, \ldots, A-1 \\ b_j^{(1)} = d_j^{(1)} \quad j = 1, \ldots, B^{(1)} \\ b_{j+2}^{(2)} = d_{j+1}^{(2)} \quad j = 1, \ldots, D^{(2)} - 1 \end{cases} \tag{4.53}$$

Similarly, if we can calculate the radius of convergence for the Second double sum, we will obtain that

$$|y| < \frac{(m+n)}{(2m+n)}, \quad |xy| < \frac{m(m+n)}{(2m+n)^2} \tag{4.54}$$

where $m > 0$, $n > 0$.

If we apply the same approach used for H_3, we will obtain the following relation between x and y

$$|xy| < \frac{1}{4} \ \wedge \ |y| < \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4|xy|} \tag{4.55}$$

- The third sum

$$\sum_{n=0}^{\infty} \frac{(\alpha)_{2n}}{(1)_n} \frac{(xy)^n}{n!} \tag{4.56}$$

We can employ the Srivastava-Daoust series by taking $n = 1$ in this case since we have one variable. Alternatively, we can use the ratio test, which is easier. By applying the ratio test, we find that $|xy| < \frac{1}{4}$.

Finally, by taking the intersection of the three radii of convergence,

$$\begin{gathered} |xy| < \frac{1}{4} \ \textit{and} \ |x| < \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4|xy|} \\ |xy| < \frac{1}{4} \ \textit{and} \ |y| < \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4|xy|} \\ |xy| < \frac{1}{4} \end{gathered} \tag{4.57}$$

and then considering the intersection of this result with the set $C = K[1, 1]$, we will obtain figure (12a) in Appendix (4.2.3), which represents the same radius of convergence for G_1 introduced in ([2], p.59). Similar to the previous approach, we can use Mathematica to plot the boundaries that determine the radius of convergence for G_1 based on (4.48), (4.54), and $xy = \frac{1}{4}$. The idea is as follows: for the first double sum, we can build a code to generate two values for m and n (both greater than 0) and then locate the corresponding point (x, y) in the xy-plane. We repeat this process to generate 10000 points for m and n. Similarly, we perform the same procedure for the second double sum and plot $xy = \frac{1}{4}$ for the third sum. Finally, we combine the three plots on one xy-plane. By doing this, we will obtain figure (12b) in Appendix (4.2.3). You can find below further information about the code we use,https://www.wolframcloud.com/obj/s12255413/Published/Radius

```
numPoints = 10000;
(*Generate random values for m and n*)
mValues = RandomReal[{0.000000000001,10000},numPoints];
```

```
nValues = RandomReal[{0.000000000001,10000},numPoints];
(*Initialize lists to store the x and y coordinates*)
xValues1 = ConstantArray[0,numPoints]; yValues1 = ConstantArray[0,numPoints];
xValues2 = ConstantArray[0,numPoints]; yValues2 = ConstantArray[0,numPoints];
xValues3 = ConstantArray[0,numPoints]; yValues3 = ConstantArray[0,numPoints];
(*Calculate x and y coordinates for each point*)
Do[
 m = mValues[[i]];
 n = nValues[[i]];
 x1 = (m+n)/(m+2*n);
 y1 = n/(m+2*n);
 x2 = m/(2*m+n);
 y2 = (m+n)/(2*m+n);
 x3 = 1/4;
 y3 = 1/(4*x3);
 (*Store the coordinates*)
 xValues1[[i]] = x1; yValues1[[i]] = y1;
 xValues2[[i]] = x2; yValues2[[i]] = y2;
 xValues3[[i]] = x3; yValues3[[i]] = y3;
 ,{i,numPoints}];
(*Plot the points on the XY plane with zoomed-in region*)
ListPlot[{Transpose[{xValues1,yValues1}],Transpose[{xValues2,yValues2}],Transpose[{xValues3,yValues3}]},PlotStyle->PointSize[Small],AxesLabel->{"X","Y"},AspectRatio->1,PlotLabel->"Points in XY Plane",PlotRange->{{0,1},{0,1}}]
```

4.1.7 Radius of convergence for Horn series G_2

After considering G_1, we will utilize the same approach for G_2

$$G_2(\alpha,\alpha'\beta,\beta';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_m(\alpha')_n(\beta)_{n-m}(\beta')_{m-n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{4.58}$$

Since we have terms such as $(\beta)_{n-m}$ and $(\beta')_{m-n}$ in G_2, we will rewrite G_2 using the technique presented in ([12], p.43-p.44) (See Appendix 4.2.3) and then proceed to rewrite it as the Srivastava-Daoust series. Upon verifying (4.59) with Mathematica, I found that the left-hand side and the right-hand side are equal for various values of $\alpha,\alpha',\beta,\beta'$, and x,y within the radius of convergence for both expressions https://www.wolframcloud.com/obj/s12255413/Pul

$$G_2(\alpha,\alpha'\beta,\beta';x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_{m+n}(\beta')_m(1)_m(\alpha')_n}{(1)_{m+n}(1-\beta)_m}\frac{(-x)^m}{m!}\frac{(xy)^n}{n!}$$

$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha')_{m+n}(\alpha)_m(\beta)_n(1)_n}{(1)_{m+n}(1-\beta')_n}\frac{(xy)^m}{m!}\frac{(-y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha')_n}{(1)_n}\frac{(xy)^n}{n!} \tag{4.59}$$

Now, we can rewrite the first two double sums and the third sum using the Srivastava-Daoust equation.

- The first double sum

$$S_1=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_{m+n}(\beta')_m(1)_m(\alpha')_n}{(1)_{m+n}(1-\beta)_m}\frac{(-x)^m}{m!}\frac{(xy)^n}{n!} \tag{4.60}$$

Set $n=2$ in the Srivastava-Daoust series and consider the following:

$$\begin{cases} A=C \\ 1+D^{(1)}=B^{(1)} \\ 1+D^{(2)}=B^{(2)} \end{cases} \tag{4.61}$$

and the coefficients $\theta^i_j,\phi^i_j,\psi^i_j,\delta^i_j$ as follows

$$\begin{cases} \theta^{(1)}_1=1,\ \theta^{(2)}_1=1 \\ \psi^{(1)}_1=1,\ \psi^{(2)}_1=1 \end{cases}$$

$$\begin{cases} \phi_1^{(1)} = 1,\ \phi_2^{(1)} = 1 \\ \delta_1^{(1)} = 1, \phi_1^{(2)} = 1 \\ \theta_{j+1}^{(1)} = \psi_{j+1}^{(1)} \quad j = 1, \dots, A-1 \\ \theta_{j+1}^{(2)} = \psi_{j+1}^{(2)} \quad j = 1, \dots, A-1 \\ \phi_{j+2}^{(1)} = \delta_{j+1}^{(1)} \quad j = 1, \dots, D^{(1)}-1 \\ \phi_{j+1}^{(2)} = \delta_j^{(2)} \quad j = 1, \dots, D^{(2)} \end{cases} \tag{4.62}$$

Also the values of the vectors $a_j, c_j, b_j^{(1)}, b_j^{(2)}, d_j^{(1)}, d_j^{(2)}$

$$\begin{cases} a_1 = \alpha,\ c_1 = 1 \\ b_1^{(1)} = \beta',\ b_2^{(1)} = 1,\ d_1^{(1)} = 1-\beta,\ b_1^{(2)} = \alpha' \\ c_{j+1} = a_{j+1} \quad j = 1, \dots, A-1 \\ b_{j+2}^{(1)} = d_{j+1}^{(1)} \quad j = 1, \dots, D^{(1)}-1 \\ b_{j+1}^{(2)} = d_j^{(2)} \quad j = 1, \dots, D^{(2)} \end{cases} \tag{4.63}$$

Now we can calculate the radius of convergence for this double series which is

$$|x| < 1,\ \ |xy| < 1 \tag{4.64}$$

- The second sum

$$S_2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_{m+n}(\alpha)_m(\beta)_n(1)_n}{(1)_{m+n}(1-\beta')_n} \frac{(xy)^m}{m!} \frac{(-y)^n}{n!} \tag{4.65}$$

It is clear that it is almost the same as the first sum. Therefore we can rewrite it as a Srivastava-Daoust series and find the radius of convergence which leads

$$|xy| < 1,\ \ |y| < 1 \tag{4.66}$$

- The third sum

$$S_3 = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha')_n}{(1)_n} \frac{(xy)^n}{n!} \tag{4.67}$$

Using the ratio test the radius of convergence for this series is $|xy| < 1$.

Finally, by taking the intersection of the three radii of convergence, and then considering the intersection of this result with the set $C = K[1, 1]$, we will obtain figure (13) in Appendix (4.2.3), which is the same radius of convergence for G_2 introduced in ([2], p.59).

4.1.8 Radius of convergence for Horn series H_1

Now let us consider the hypergeometric series H_1 which has the following form

$$H_1(\alpha, \beta, \gamma; \delta; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m} \frac{x^m}{m!} \frac{y^n}{n!} \tag{4.68}$$

Since we have the term $(\alpha)_{m-n}$ in H_1, we will rewrite it using the technique presented in ([12], p.43-p.44) (See Appendix 4.2.3) and then proceed to rewrite it as the Srivastava-Daoust series. Upon verifying the formula (4.69) with Mathematica, I found that the left-hand side and the right-hand side are equal for various values of $\alpha, \beta, \gamma, \delta$, and x, y within the radius of convergence for both expressions. https://www.wolframcloud.com/obj/s12255413/Publishe

$$H_1(\alpha, \beta, \gamma; \delta; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_{m+2n}(\alpha)_m(1)_m(\gamma)_n}{(\delta)_{m+n}(1)_{m+n}} \frac{x^m}{m!} \frac{(xy)^n}{n!}$$

$$+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_{2m+n}(\gamma)_{m+n}(1)_n}{(1)_{m+n}(\delta)_m(1-\alpha)_n} \frac{(xy)^m}{m!} \frac{(-y)^n}{n!} - \sum_{n=0}^{\infty} \frac{(\beta)_{2n}(\gamma)_n}{(1)_n(\delta)_n} \frac{(xy)^n}{n!} \tag{4.69}$$

As we did previously, we will rewrite each of these three summations in terms of the Srivastava-Daoust series. By doing so, we will obtain the following three radii of convergence:

- For the first double sum

$$S_1 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_{m+2n}(\alpha)_m(1)_m(\gamma)_n}{(\delta)_{m+n}(1)_{m+n}} \frac{x^m}{m!} \frac{(xy)^n}{n!}$$

  the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|x| < \frac{(m+n)^2}{m(m+2n)} \tag{4.70}$$

$$|xy| < \frac{(m+n)^2}{(m+2n)^2} \tag{4.71}$$

which implies

$$(|xy|^{-1/2} - 1)^2 > 1 - |x|^{-1} \tag{4.72}$$

- For the second double sum

$$S_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta)_{2m+n}(\gamma)_{m+n}(1)_n}{(1)_{m+n}(\delta)_m(1-\alpha)_n} \frac{(xy)^m}{m!} \frac{(-y)^n}{n!}$$

the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|xy| < \frac{m^2}{(2m+n)^2} \tag{4.73}$$

$$|y| < \frac{n}{2m+n} \tag{4.74}$$

which implies

$$2|xy|^{1/2} + |y| < 1 \tag{4.75}$$

- For the third sum

$$S_3 = \sum_{n=0}^{\infty} \frac{(\beta)_{2n}(\gamma)_n}{(1)_n(\delta)_n} \frac{(xy)^n}{n!}$$

its radius of convergence will be $|xy| < \frac{1}{4}$

Finally, by taking the intersection of the three radii of convergence, and then considering the intersection of this result with the set $C = K[1,1]$, we will obtain figure (14a) in Appendix (4.2.3), which represents the same radius of convergence for H_1 as introduced in ([2], p.59). Additionally, when we plot the boundaries that determine the radius of convergence for H_1, we will obtain figure (14b) Appendix (4.2.3).

```
numPoints = 10000;
(* Generate random values for m and n *)
mValues = RandomReal[{0.000000000001, 10000}, numPoints];
nValues = RandomReal[{0.000000000001, 10000}, numPoints];
(* Initialize lists to store the x and y coordinates *)
```

```
xValues1 = ConstantArray[0, numPoints]; yValues1 = ConstantArray[0, numPoints];
xValues2 = ConstantArray[0, numPoints]; yValues2 = ConstantArray[0, numPoints];
xValues3 = ConstantArray[0, numPoints]; yValues3 = ConstantArray[0, numPoints];
(* Calculate x and y coordinates for each point *)
Do[
  m = mValues[[i]];
  n = nValues[[i]];
  x1 = (m + n)^2/(m*(m + 2*n));
  y1 = m/(m + 2*n);
  x2 = (m^2)/(n*(2*m + n));
  y2 = n/(2*m + n);
  x3 = 1/4;
  y3 = 1/(4*x3);
  (* Store the coordinates *)
  xValues1[[i]] = x1;
  yValues1[[i]] = y1;
  xValues2[[i]] = x2;
  yValues2[[i]] = y2;
  xValues3[[i]] = x3;
  yValues3[[i]] = y3;
  , {i, numPoints}];
(* Plot the points on the XY plane with zoomed-in region *)
ListPlot[
  {Transpose[{xValues1, yValues1}], Transpose[{xValues2, yValues2}], Transpose[{xValues3,yValues3}]},
  PlotStyle -> PointSize[Small], AxesLabel -> {"X", "Y"}, AspectRatio -> 1,PlotLabel -> "Points in XY Plane", PlotRange -> {{0, 1}, {0, 1}}]
```

4.1.9 Radius of convergence for Horn series H_2

Let us consider the hypergeometric series H_2

$$H_2(\alpha,\beta,\gamma,\delta;\epsilon,x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{(\epsilon)_m}\frac{x^m}{m!}\frac{y^n}{n!} \tag{4.76}$$

As we have $(\alpha)_{m-n}$ in H_2, we will rewrite it using the technique presented in ([12],p.43-p.44) (See Appendix 4.2.3) and then proceed to rewrite it as the Srivastava-Daoust series. Upon verifying the formula (4.77) with Mathematica, I found that the left-hand side and the right-hand side are equal for various values of $\alpha,\beta,\gamma,\delta,\epsilon$, and x,y within the radius of convergence for both expressions. https://www.wolframcloud.com/obj/s12255413/Published/Testing

$$H_2(\alpha,\beta,\gamma,\delta;\epsilon,x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{m+n}(\alpha)_m(1)_m(\gamma)_n(\delta)_n}{(\epsilon)_{m+n}(1)_{m+n}}\frac{x^m}{m!}\frac{(xy)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}(\gamma)_{m+n}(\beta)_m(1)_n}{(1)_{m+n}(\epsilon)_m(1-\alpha)_n}\frac{(xy)^m}{m!}\frac{(-y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\beta)_n(\gamma)_n(\delta)_n}{(1)_n(\epsilon)_n}\frac{(xy)^n}{n!} \tag{4.77}$$

Similarly, we will rewrite each of these three summations in terms of the Srivastava-Daoust series. By doing so, we will obtain the following three radii of convergence:

- For the first double sum

$$S_1=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{m+n}(\alpha)_m(1)_m(\gamma)_n(\delta)_n}{(\epsilon)_{m+n}(1)_{m+n}}\frac{x^m}{m!}\frac{(xy)^n}{n!}$$

  the radius of convergence will be as follows for $m>0$ and $n>0$:

$$|x|<\frac{m+n}{m} \tag{4.78}$$

$$|xy|<\frac{m+n}{n} \tag{4.79}$$

  which implies

$$|xy|^{-1}+|x|^{-1}>1 \tag{4.80}$$

- For the second double sum

$$S_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\delta)_{m+n}(\gamma)_{m+n}(\beta)_m(1)_n}{(1)_{m+n}(\epsilon)_m(1-\alpha)_n} \frac{(xy)^m}{m!} \frac{(-y)^n}{n!}$$

the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|xy| < \frac{m}{m+n} \tag{4.81}$$

$$|y| < \frac{n}{m+n} \tag{4.82}$$

which implies

$$|xy| + |y| < 1 \tag{4.83}$$

- For the third sum

$$S_3 = \sum_{n=0}^{\infty} \frac{(\beta)_n(\gamma)_n(\delta)_n}{(1)_n(\epsilon)_n} \frac{(xy)^n}{n!}$$

its radius of convergence will be $|xy| < 1$

Finally, by taking the intersection of the three radii of convergence and then considering the intersection of this result with the set $C = K[1, 1]$, we will obtain figure (15a) in Appendix (4.2.3), which represents the same radius of convergence for H_1 as introduced in ([2], p.59). Also, when we plot the boundaries that determine the radius of convergence for H_2, we will obtain figure (15b) in Appendix (4.2.3).

```
numPoints = 10000;
(* Generate random values for m and n *)
mValues = RandomReal[{0.000000000001, 10000}, numPoints];
nValues = RandomReal[{0.000000000001, 10000}, numPoints];
(* Initialize lists to store the x and y coordinates *)
xValues1 = ConstantArray[0, numPoints]; yValues1 = ConstantArray[0,
    numPoints];
xValues2 = ConstantArray[0, numPoints]; yValues2 = ConstantArray[0,
    numPoints];
```

```
xValues3 = ConstantArray[0, numPoints]; yValues3 = ConstantArray[0,
    numPoints];
xValues4 = ConstantArray[1, numPoints]; yValues4 = mValues/nValues;
(* Calculate x and y coordinates for each point *)
Do[
  m = mValues[[i]];
  n = nValues[[i]];
  x1 = (m + n)/m; y1 = m/n;
  x2 = m/n; y2 = n/(m + n);
  x3 = 1; y3 = 1/x3;
  (* Store the coordinates *)
  xValues1[[i]] = x1; yValues1[[i]] = y1;
  xValues2[[i]] = x2; yValues2[[i]] = y2;
  xValues3[[i]] = x3; yValues3[[i]] = y3;
  ,{i, numPoints}];
(* Plot the points on the XY plane with zoomed-in region *)
ListPlot[{
 Transpose[{xValues1, yValues1}],Transpose[{xValues2, yValues2}],Transpose
     [{xValues3, yValues3}],Transpose[{xValues4, yValues4}]},PlotStyle ->
     {PointSize[Small], PointSize[Small], PointSize[Small], PointSize[
     Small]},AxesLabel -> {"X", "Y"},AspectRatio -> 1,PlotLabel -> "Points
      in XY Plane",PlotRange -> {{0, 1}, {0, 1}}]
```

4.1.10 Radius of convergence for Horn series H_4

Now let us see what happens if we consider H_4

$$H_4(\alpha, \beta; \gamma, \delta, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_m(\delta)_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{4.84}$$

As we can see, we can directly rewrite H_4 in terms of the Srivastava-Daoust series. If we do that, the radius of convergence for this series will be

$$|x| < \frac{m^2}{(2m+n)^2} \tag{4.85}$$

$$|y| < \frac{n}{2m+n} \tag{4.86}$$

for any $m > 0,\ n > 0$. Which implies

$$1 - 2|x|^{1/2} > \frac{n}{2m+n}, \quad |y| < \frac{n}{2m+n} \tag{4.87}$$

It is clear that the two inequalities (4.87) can be written as

$$2|x|^{1/2} + |y| < 1 \tag{4.88}$$

Finally, if we take the intersection of (4.88) and the set $C = K[\frac{1}{4}, 1]$, we will obtain figure (16a) in Appendix (4.2.3), which is the same radius of convergence for H_4 introduced in ([2], p.59). Also if we try to plot the boundaries that determine the radius of convergence for H_4, we will obtain figure (16b) in Appendix (4.2.3).

```
numPoints = 10000;
(* Generate random values for m and n *)
mValues = RandomReal[{0.000000000001, 10000}, numPoints];
nValues = RandomReal[{0.000000000001, 10000}, numPoints];
(* Initialize lists to store the x and y coordinates *)
xValues1 = ConstantArray[0, numPoints]; yValues1 = ConstantArray[0,
    numPoints];
(* Calculate x and y coordinates for each point *)
Do[
  m = mValues[[i]];
  n = nValues[[i]];
  (* Calculate x and y for equation 1 *)
  x1 = (m^2)/(2*m + n)^2;
  y1 = n/(2*m + n);
  (* Store the coordinates *)
  xValues1[[i]] = x1;
  yValues1[[i]] = y1;
  ,{i, numPoints}];
```

```
(* Plot the points on the XY plane with zoomed-in region *)
ListPlot[{Transpose[{xValues1, yValues1}]},PlotStyle -> PointSize[Small],
    AxesLabel -> {"X", "Y"},AspectRatio -> 1,PlotLabel -> "Points in XY
    Plane",PlotRange -> {{0, 1}, {0, 1}}]
```

4.1.11 Radius of convergence for Horn series H_5

Now, let us take another hypergeometric series, which is H_5

$$H_5(\alpha, \beta; \gamma; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_{n-m}}{(\gamma)_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{4.89}$$

As we have $(\beta)_{n-m}$ in H_5, we will rewrite it using the technique presented in ([12], p.43-p.44) (See Appendix 4.2.3) and then proceed to rewrite it as the Srivastava-Daoust series. Upon verifying the formula (4.90) with Mathematica, I found that the left-hand side and the right-hand side are equal for various values of α, β, γ, and x, y within the radius of convergence for both expressions. https://www.wolframcloud.com/obj/s12255413/Published/Testing

$$H_5(\alpha, \beta; \gamma; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{2m+3n}(1)_m}{(1)_{m+n}(1-\beta)_m(\gamma)_n} \frac{(-x)^m}{m!} \frac{(xy)^n}{n!}$$

$$+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{3m+n}(\beta)_n(1)_n}{(\gamma)_{m+n}(1)_{m+n}} \frac{(xy)^m}{m!} \frac{(y)^n}{n!} - \sum_{n=0}^{\infty} \frac{(\alpha)_{3n}}{(\gamma)_n(1)_n} \frac{(xy)^n}{n!} \tag{4.90}$$

After rewriting each of these three summations in terms of the Srivastava-Daoust series we will obtain the following three radii of convergence:

- For the first double sum

$$S_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{2m+3n}(1)_m}{(1)_{m+n}(1-\beta)_m(\gamma)_n} \frac{(-x)^m}{m!} \frac{(xy)^n}{n!}$$

  the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|x| < \frac{m(m+n)}{(2m+3n)^2} \tag{4.91}$$

$$|xy| < \frac{n^2(m+n)}{(2m+3n)^3} \tag{4.92}$$

- For the second double sum

$$S_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{3m+n}(\beta)_n(1)_n}{(\gamma)_{m+n}(1)_{m+n}} \frac{(xy)^m}{m!} \frac{(y)^n}{n!}$$

  the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|xy| < \frac{m(m+n)^2}{(3m+n)^3} \tag{4.93}$$

$$|y| < \frac{(m+n)^2}{n(3m+n)} \tag{4.94}$$

- For the third sum

$$S_3 = \sum_{n=0}^{\infty} \frac{(\alpha)_{3n}}{(\gamma)_n(1)_n} \frac{(xy)^n}{n!}$$

  its radius of convergence will be $|xy| < \frac{1}{27}$

Question: The problem here is whether we can find a relation between (4.91) and (4.92), (4.93) and (4.94), in order to find the intersection of these three radii of convergence. Afterwards, we can take this result and find its intersection with the set $C = k[\frac{1}{4}, 1]$ to determine the same radii of convergence for H_5??

Despite this challenge, if we attempt to plot the boundaries that determine the radius of convergence for H_5, we will obtain figure (17) in Appendix (4.2.3) where the orange, blue, and green curves represent the radius of convergence for the S_1, S_2 and S_3 respectively.

```
numPoints = 100000;
(*Generate random values for m and n*)
mValues = RandomReal[{0.000000000001, 10000}, numPoints];
nValues = RandomReal[{0.000000000001, 10000}, numPoints];
(*Initialize lists to store the x and y coordinates*)
xValues1 = ConstantArray[0, numPoints]; yValues1 = ConstantArray[0,
    numPoints];
xValues2 = ConstantArray[0, numPoints]; yValues2 = ConstantArray[0,
```

```
    numPoints];
xValues3 = ConstantArray[0, numPoints]; yValues3 = ConstantArray[0,
    numPoints];
(*Calculate x and y coordinates for each point*)
Do[
  m = mValues[[i]];
  n = nValues[[i]];
  x1 = (m*n)/((3*m + n)^2);
  y1 = ((m + n)^2)/(n*(3*m + n));
  x2 = (m*(m + n))/((2*m + 3*n)^2);
  y2 = (n^2)/(m*(2*m + 3*n));
  x3 = 1/(27*y2);
  y3 = 1/(27*x2);
  If[0 < x1 < 0.25 && 0 < y1 < 1,
  xValues1[[i]] = x1;
  yValues1[[i]] = y1;];
  If[0 < x2 < 0.25 && 0 < y2 < 1,
  xValues2[[i]] = x2;
  yValues2[[i]] = y2;];
  If[0 < x3 < 0.25 && 0 < y3 < 1,
  xValues3[[i]] = x3;
  yValues3[[i]] = y3;];
  ,{i, numPoints}];
(*Remove zero entries from xValues and yValues*)
xyData1 = Transpose[{xValues1, yValues1}] /. {0, 0} -> Sequence[];
xyData2 = Transpose[{xValues2, yValues2}] /. {0, 0} -> Sequence[];
xyData3 = Transpose[{xValues3, yValues3}] /. {0, 0} -> Sequence[];
(*Plot the points on the XY plane*)
ListPlot[
  {xyData1, xyData2, xyData3}, PlotStyle -> PointSize[Small],AxesLabel ->
      {"X", "Y"},AspectRatio -> 1,PlotLabel -> "Points in XY Plane",
      PlotRange -> {{0, 0.25}, {0, 1}}]
```

Remark: It is clear that we can use Horn's theorem to determine $\rho(m, n)$ and $\sigma(m, n)$ for H_5. By doing so, we can easily find that

$$\rho(m, n) = \frac{m|m-n|}{(2m+n)^2}, \quad \sigma(m, n) = \frac{n^2}{(2m+n)|m-n|} \tag{4.95}$$

Using the same approach(generating points),

```
numPoints = 10000;
(*Generate random values for m and n*)
mValues = RandomReal[{0.000000000001, 10000}, numPoints];
nValues = RandomReal[{0.000000000001, 10000}, numPoints];
(*Initialize lists to store the x and y coordinates*)
xValues = ConstantArray[0, numPoints]; yValues = ConstantArray[0,
    numPoints];
(*Calculate x and y coordinates for each point*)
Do[
  m = mValues[[i]];
  n = nValues[[i]];
  x = (m*(Abs[m - n]))/((2*m + n)^2); y = (n^2)/((2*m + n)*Abs[m - n]);
  (*Store the coordinates*)
  xValues[[i]] = x;
  yValues[[i]] = y,
  {i, numPoints}];
(*Plot the points on the XY plane*)
ListPlot[Transpose[{xValues, yValues}], PlotStyle -> PointSize[Small],
    AxesLabel -> {"X", "Y"}, PlotRange -> {{0, 0.25}, {0, 1}}, AspectRatio
     -> 1, PlotLabel -> "Points in XY Plane"]
```

we will obtain figure (17c) in Appendix (4.2.3). By comparing the two results, one obtained by expressing H_5 as a Srivastava-Daoust series, as seen in figure (17) in Appendix (4.2.3), and the other obtained directly using Horn's theorem as seen in figure (17c) in Appendix (4.2.3), we observe that the two results are identical.

4.1.12 Radius of convergence for Horn series H_6

Let us take the double hypergeometric series H_6

$$H_6(\alpha,\beta;\gamma;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{2m-n}(\beta)_{n-m}(\gamma)_n\frac{x^m}{m!}\frac{y^n}{n!} \tag{4.96}$$

As we have the terms $(\alpha)_{2m-n}$ and $(\beta)_{n-m}$ in H_6, we will rewrite it using the technique presented in ([12], p.43-p.44) (See Appendix 4.2.3).

$$H_6(\alpha,\beta,\gamma;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+n}(1)_m(\gamma)_n}{(1)_{m+n}(1-\beta)_m}\frac{(-x)^m}{m!}\frac{(xy)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m-n}(\beta)_n(\gamma)_{m+n}\frac{(x)^m}{m!}\frac{(y)^{m+n}}{(m+n)!}-\sum_{n=0}^{\infty}\frac{(\alpha)_n(\gamma)_n}{(1)_n}\frac{(xy)^n}{n!} \tag{4.97}$$

Notice that the second double sum also contains $(\alpha)_{m-n}$. To address this, we can apply the same technique as presented in ([12], p.43-p.44) to the second double sum. By doing so, we obtain (4.98). To validate this result, I have checked (4.98) using Mathematica and found that the left-hand side and the right-hand side are equal for various values of α,β,γ, and x,y, where x,y belong to the radius of convergence for both expressions. https://www.wolframcloud.com/obj/s12255413/Published/Testing

$$H_6(\alpha,\beta,\gamma;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+n}(1)_m(\gamma)_n}{(1)_{m+n}(1-\beta)_m}\frac{(-x)^m}{m!}\frac{(xy)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma)_{m+2n}(1)_m(\alpha)_m(1)_n(\beta)_n}{(1)_{m+n}(1)_{m+2n}}\frac{(xy)^m}{m!}\frac{(xy^2)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\beta)_n(\gamma)_{2n}}{(1)_{2n}}\frac{(xy^2)^n}{n!}$$
$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{m+n}(\gamma)_{2m+n}(1)_n}{(1)_{2m+n}(1-\alpha)_n}\frac{(xy^2)^m}{m!}\frac{(-y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_n(\gamma)_n}{(1)_n}\frac{(xy)^n}{n!} \tag{4.98}$$

After rewriting each of these sums in terms of the Srivastava-Daoust series we will obtain the following five radii of convergence:

- For the first double sum

$$S_1=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+n}(1)_m(\gamma)_n}{(1)_{m+n}(1-\beta)_m}\frac{(-x)^m}{m!}\frac{(xy)^n}{n!}$$

the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|x| < \frac{m(m+n)}{(2m+n)^2} \tag{4.99}$$

$$|xy| < \frac{(m+n)}{2m+n} \tag{4.100}$$

which implies

$$|x| < \frac{1}{4}, \;\; |xy| < \frac{1}{2} + \frac{1}{2}\sqrt{1-4x} \tag{4.101}$$

- For the second double sum

$$S_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\gamma)_{m+2n}(1)_m(\alpha)_m(1)_n(\beta)_n}{(1)_{m+n}(1)_{m+2n}} \frac{(xy)^m}{m!} \frac{(xy^2)^n}{n!}$$

the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|xy| < \frac{m+n}{m} \tag{4.102}$$

$$|xy^2| < \frac{m+n}{n} \tag{4.103}$$

which implies

$$|xy|^{-1} + |xy^2|^{-1} > 1 \tag{4.104}$$

- For the third double sum

$$S_3 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta)_{m+n}(\gamma)_{2m+n}(1)_n}{(1)_{2m+n}(1-\alpha)_n} \frac{(xy^2)^m}{m!} \frac{(-y)^n}{n!}$$

the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|xy^2| < \frac{m}{m+n} \tag{4.105}$$

$$|y| < \frac{n}{m+n} \tag{4.106}$$

which implies

$$|xy^2| + |y| < 1 \tag{4.107}$$

- For the fourth sum
$$S_4 = \sum_{n=0}^{\infty} \frac{(\beta)_n(\gamma)_{2n}}{(1)_{2n}} \frac{(xy^2)^n}{n!}$$
its radius of convergence will be $|xy^2| < 1$

- The fifth sum
$$S_5 = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\gamma)_n}{(1)_n} \frac{(xy)^n}{n!}$$
its radius of convergence will be $|xy| < 1$

Finally, by taking the intersection of the five radii of convergence and then considering the intersection of this result with the set $C = K[\frac{1}{4}, 1]$, we will obtain figure (18a) in Appendix (4.2.3), which represents the same radius of convergence for H_6 as introduced in ([2], p.59). Also, when we plot the boundaries that determine the radius of convergence for H_6, we will obtain figure (18b) in Appendix (4.2.3).

```
numPoints = 10000;
(* Generate random values for m and n *)
mValues = RandomReal[{0.000000000001, 10000}, numPoints];
nValues = RandomReal[{0.000000000001, 10000}, numPoints];
(* Initialize lists to store the x and y coordinates *)
xValues1 = ConstantArray[0, numPoints]; yValues1 = ConstantArray[0,
    numPoints];
xValues2 = ConstantArray[0, numPoints]; yValues2 = ConstantArray[0,
    numPoints];
xValues3 = ConstantArray[0, numPoints]; yValues3 = ConstantArray[0,
    numPoints];
xValues4 = ConstantArray[0, numPoints]; yValues4 = ConstantArray[0,
    numPoints];
xValues5 = ConstantArray[0, numPoints]; yValues5 = ConstantArray[0,
    numPoints];
(* Calculate x and y coordinates for each point *)
Do[
  m = mValues[[i]];
```

```
  n = nValues[[i]];
  (* Calculate x and y for equation 1 *)
  x1 = m*(m + n)/(2*m + n)^2;
  y1 = (2*m + n)/m;
  x2 = n*(m + n)/(m^2);
  y2 = m/n;
  x3 = m*(m + n)/n^2;
  y3 = n/(m + n);
  x4 = 1; y4 = 1/x4;
  x5 = 0.25;
  y5 = mValues[[i]]/nValues[[i]];
  (* Store the coordinates *)
  xValues1[[i]] = x1;
  yValues1[[i]] = y1;
  xValues2[[i]] = x2;
  yValues2[[i]] = y2;
  xValues3[[i]] = x3;
  yValues3[[i]] = y3;
  xValues4[[i]] = x4;
  yValues4[[i]] = y4;
  xValues5[[i]] = x5;
  yValues5[[i]] = y5;
  , {i, numPoints}];
(* Plot the points on the XY plane with zoomed-in region *)
ListPlot[
 {Transpose[{xValues1, yValues1}],Transpose[{xValues2,yValues2}],
 Transpose[{xValues3,yValues3}],Transpose[{xValues4,yValues4}],
 Transpose[{xValues5,yValues5}]},
 PlotStyle -> {PointSize[Small],PointSize[Small],PointSize[Small],
    PointSize[Small], PointSize[Small]},AxesLabel -> {"X", "Y"},
    AspectRatio -> 1,PlotLabel -> "Points in XY Plane",PlotRange -> {{0,
    1}, {0, 1}}]
```

4.1.13 Radius of convergence for Horn series H_7

Let us take the double hypergeometric series H_7

$$H_7(\alpha, \beta, \gamma, \delta; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{2m-n}(\beta)_n(\gamma)_n}{(\delta)_m} \frac{x^m}{m!} \frac{y^n}{n!} \tag{4.108}$$

As we have $(\alpha)_{2m-n}$ in H_7, we will rewrite it using the technique presented in ([12], p.43-p.44) (See Appendix 4.2.3).

$$H_7(\alpha, \beta, \gamma; \delta; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n(\gamma)_n}{(\delta)_{m+n}} \frac{(x)^{m+n}}{(m+n)!} \frac{(y)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_{m+n}}{(\delta)_m} \frac{(x)^m}{m!} \frac{(y)^{m+n}}{(m+n)!} - \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(\gamma)_n}{(\delta)_n(1)_n} \frac{(xy)^n}{n!} \tag{4.109}$$

Note that the second double sum also contains $(\alpha)_{m-n}$. To address this, we can apply the same technique as presented in ([12], p.43-p.44) to the second double sum. By doing so, we will obtain (4.110) (See Appendix 4.2.3). To validate this result, I have checked (4.110) using Mathematica and found that the left-hand side and the right-hand side are equal for various values of $\alpha, \alpha', \beta, \beta'$ and x, y, where x, y belong to the radius of convergence for both expressions https://www.wolframcloud.com/obj/s12255413/Published/Testing

$$H_7(\alpha, \beta, \gamma; \delta; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n}(1)_m(\beta)_n(\gamma)_n}{(1)_{m+n}(\delta)_{m+n}} \frac{(x)^m}{m!} \frac{(xy)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta)_{m+2n}(\gamma)_{m+2n}(\alpha)_m(1)_m(1)_n}{(\delta)_{m+n}(1)_{m+n}(1)_{m+2n}} \frac{(xy)^m}{m!} \frac{(xy^2)^n}{n!} - \sum_{n=0}^{\infty} \frac{(\beta)_{2n}(\gamma)_{2n}}{(\delta)_n(1)_{2n}} \frac{(xy^2)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta)_{2m+n}(\gamma)_{2m+n}(1)_n}{(1)_{2m+n}(\delta)_m(1-\alpha)_n} \frac{(xy^2)^m}{m!} \frac{(-y)^n}{n!} - \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(\gamma)_n}{(\delta)_n(1)_n} \frac{(xy)^n}{n!} \tag{4.110}$$

Now, if we rewrite each of these sums in terms of the Srivastava-Daoust series we will obtain the following five radii of convergence:

- For the first double sum

$$S_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n}(1)_m(\beta)_n(\gamma)_n}{(1)_{m+n}(\delta)_{m+n}} \frac{(x)^m}{m!} \frac{(xy)^n}{n!}$$

the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|x| < \frac{(m+n)^2}{(2m+n)^2} \tag{4.111}$$

$$|xy| < \frac{(m+n)^2}{n(2m+n)} \tag{4.112}$$

which implies

$$(|x|^{-1/2} - 1)^2 > 1 - |xy|^{-1} \tag{4.113}$$

- For the second double sum

$$S_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta)_{m+2n}(\gamma)_{m+2n}(\alpha)_m(1)_m(1)_n}{(\delta)_{m+n}(1)_{m+n}(1)_{m+2n}} \frac{(xy)^m}{m!} \frac{(xy^2)^n}{n!}$$

the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|xy| < \frac{(m+n)^2}{m(m+2n)} \tag{4.114}$$

$$|xy^2| < \frac{(m+n)^2}{(m+2n)^2} \tag{4.115}$$

which implies

$$(|xy|^{-1/2} - 1)^2 > 1 - |x|^{-1} \tag{4.116}$$

- For the third double sum

$$S_3 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta)_{2m+n}(\gamma)_{2m+n}(1)_n}{(1)_{2m+n}(\delta)_m(1-\alpha)_n} \frac{(xy^2)^m}{m!} \frac{(-y)^n}{n!}$$

the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|xy^2| < \frac{m^2}{(2m+n)^2} \tag{4.117}$$

$$|y| < \frac{n}{2m+n} \tag{4.118}$$

which implies

$$2|xy^2|^{1/2} + |y| < 1 \tag{4.119}$$

- for the fourth sum
$$S_4 = \sum_{n=0}^{\infty} \frac{(\beta)_{2n}(\gamma)_{2n}}{(\delta)_n(1)_{2n}} \frac{(xy^2)^n}{n!}$$
its radius of convergence will be $|xy^2| < \frac{1}{4}$

- For the fifth sum
$$S_5 = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(\gamma)_n}{(\delta)_n(1)_n} \frac{(xy)^n}{n!}$$
its radius of convergence will be $|xy| < 1$

Finally, by taking the intersection of the above five radii of convergence and then considering the intersection of this result with the set $C = K[\frac{1}{4}, 1]$, we will obtain figure (19a) in Appendix (4.2.3), which represents the same radius of convergence for H_7 as introduced in ([2], p.59). Also, when we plot the boundaries that determine the radius of convergence for H_7, we will obtain figure (19b) in Appendix (4.2.3). You can find below further about the code we use, https://www.wolframcloud.com/obj/s12255413/Published/Radius

```
numPoints = 100000;
(* Generate random values for m and n *)
mValues = RandomReal[{0.000000000001, 100}, numPoints];
nValues = RandomReal[{0.000000000001, 100}, numPoints];
(* Initialize lists to store the x and y coordinates *)
xValues1 = ConstantArray[0, numPoints]; yValues1 = ConstantArray[0,
    numPoints];
xValues2 = ConstantArray[0, numPoints]; yValues2 = ConstantArray[0,
    numPoints];
xValues3 = ConstantArray[0, numPoints]; yValues3 = ConstantArray[0,
    numPoints];
xValues4 = ConstantArray[0, numPoints]; yValues4 = ConstantArray[0,
    numPoints];
xValues5 = ConstantArray[0, numPoints]; yValues5 = ConstantArray[0,
    numPoints];
xValues6 = ConstantArray[0.25, numPoints]; yValues6 = 1/Sqrt[mValues*
    nValues];
(* Calculate x and y coordinates for each point *)
```

```
Do[
  m = mValues[[i]];
  n = nValues[[i]];
  x1 = (m + n)^2/(2*m + n)^2;
  y1 = (2*m + n)/n;
  x2 = (m + n)^2/m^2;
  y2 = m/(m + 2*n);
  x3 = m^2/n^2;
  y3 = n/(2*m + n);
  x4 = 1; y4 = 1/x4;
  x5 = 1/yValues6[[i]]^2; y5 = yValues6[[i]];
  (* Store the coordinates *)
  xValues1[[i]] = x1;
  yValues1[[i]] = y1;
  xValues2[[i]] = x2;
  yValues2[[i]] = y2;
  xValues3[[i]] = x3;
  yValues3[[i]] = y3;
  xValues4[[i]] = x4;
  yValues4[[i]] = y4;
  xValues5[[i]] = x5;
  yValues5[[i]] = y5;
  ,{i, numPoints}];
(* Plot the points on the XY plane with zoomed-in region *)
ListPlot[{Transpose[{xValues1, yValues1}],Transpose[{xValues2, yValues2}],
    Transpose[{xValues3, yValues3}],Transpose[{xValues4, yValues4}],
    Transpose[{xValues5, yValues5}],Transpose[{xValues6, yValues6}]},
    PlotStyle -> {PointSize[Small],PointSize[Small],PointSize[Small],
    PointSize[Small], PointSize[Small],PointSize[Small]},AxesLabel -> {"X"
    , "Y"},AspectRatio -> 1,PlotLabel -> "Points in XY Plane",PlotRange ->
     {{0, 1}, {0, 1}}]
```

4.1.14 Radius of convergence for Horn series G_3

The last Horn series we will consider is G_3

$$G_3(\alpha,\alpha';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{2n-m}(\alpha')_{2m-n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{4.120}$$

As we have $(\alpha)_{2n-m}$ and $(\alpha')_{2m-n}$ in G_3, we will rewrite it using the technique presented in ([12], p.43-p.44).

$$G_3(\alpha,\alpha';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{n-m}(\alpha')_{2m+n}\frac{x^{m+n}}{(m+n)!}\frac{y^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m+2n}(\alpha')_{m-n}\frac{x^m}{m!}\frac{y^{m+n}}{(m+n)!}-\sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha')_n}{(1)_n}\frac{(xy)^n}{n!} \tag{4.121}$$

Note that in the first double sum we have $(\alpha)_{n-m}$, and in the second double sum, we have $(\alpha')_{m-n}$. To handle this, we will apply the same technique presented in ([12], p.43-p.44) twice, once to the first double sum and once to the second double sum. By doing so, we obtain (4.122) (See Appendix 4.2.3). I have verified (4.122) using Mathematica and found that the left-hand side and the right-hand side are equal for various values of α,α', and x,y, where x,y belong to the radius of convergence for both expressions https://www.wolframcloud.com/obj/s12255413/Published/Testing

$$G_3(\alpha,\alpha';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{2n-m}(\alpha')_{2m-n}\frac{x^m}{m!}\frac{y^n}{n!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{3m+2n}(1)_n}{(1)_{2m+n}(1-\alpha')_n}\frac{(xy^2)^m}{m!}\frac{(-y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha')_{3n}}{(1)_{2n}}\frac{(x^2y)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+3n}(\alpha')_m(1)_m(1)_n}{(1)_{m+2n}(1)_{m+n}}\frac{(xy)^m}{m!}\frac{(xy^2)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_{3n}}{(1)_{2n}}\frac{(xy^2)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha')_{2m+3n}(1)_m}{(1)_{m+2n}(1-\alpha)_m}\frac{(-x)^m}{m!}\frac{(x^2y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha')_n}{(1)_n}\frac{(xy)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha')_{3m+n}(1)_m(1)_n(\alpha)_n}{(1)_{m+n}(1)_{2m+n}}\frac{(x^2y)^m}{m!}\frac{(xy)^n}{n!} \tag{4.122}$$

After rewriting each of these sums in terms of the Srivastava-Daoust series we will obtain the following seven radii of convergence:

- For the first double sum

$$S_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha')_{2m+3n}(1)_m}{(1)_{m+2n}(1-\alpha)_m} \frac{(-x)^m}{m!} \frac{(x^2y)^n}{n!}$$

  the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|x| < \frac{m(m+2n)}{(2m+3n)^2} \tag{4.123}$$

$$|x^2y| < \frac{n(m+2n)^2}{(2m+3n)^3} \tag{4.124}$$

- For the second double sum

$$S_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha')_{3m+n}(1)_m(1)_n(\alpha)_n}{(1)_{m+n}(1)_{2m+n}} \frac{(x^2y)^m}{m!} \frac{(xy)^n}{n!}$$

  the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|x^2y| < \frac{(2m+n)^2(m+n)}{(3m+n)^3} \tag{4.125}$$

$$|xy| < \frac{(2m+n)(m+n)}{n(3m+n)} \tag{4.126}$$

- For the third double sum

$$S_3 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+3n}(\alpha')_m(1)_m(1)_n}{(1)_{m+2n}(1)_{m+n}} \frac{(xy)^m}{m!} \frac{(xy^2)^n}{n!}$$

  the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|xy| < \frac{(m+n)(m+2n)}{m(m+3n)} \tag{4.127}$$

$$|xy^2| < \frac{(m+n)(m+2n)^2}{(m+3n)^3} \tag{4.128}$$

- For the fourth double sum

$$S_4 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{3m+2n}(1)_n}{(1)_{2m+n}(1-\alpha')_n} \frac{(xy^2)^m}{m!} \frac{(-y)^n}{n!}$$

the radius of convergence will be as follows for $m > 0$ and $n > 0$:

$$|xy^2| < \frac{m(2m+n)^2}{(3m+2n)^3} \tag{4.129}$$

$$|y| < \frac{n(2m+n)}{(3m+2n)^2} \tag{4.130}$$

- For the fifth sum

$$S_5 = \sum_{n=0}^{\infty} \frac{(\alpha')_{3n}}{(1)_{2n}} \frac{(x^2y)^n}{n!}$$

its radius of convergence will be $|x^2y| < \frac{4}{27}$

- For the sixth sum

$$S_6 = \sum_{n=0}^{\infty} \frac{(\alpha)_{3n}}{(1)_{2n}} \frac{(xy^2)^n}{n!}$$

its radius of convergence will be $|xy^2| < \frac{4}{27}$

- For the seventh sum

$$S_7 = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha')_n}{(1)_n} \frac{(xy)^n}{n!}$$

its radius of convergence will be $|xy| < 1$

Question: The problem here is whether we can find a relation between (4.123) and (4.124), (4.125) and (4.126),(4.127) and (4.128), (4.129) and (4.130), in order to find the intersection of these four radii of convergence. Afterwards, we can take this result and find its intersection with the set $C = k[\frac{1}{4}, \frac{1}{4}]$ to determine the same radii of convergence for G_3.

Despite this challenge, if we attempt to plot the boundaries that determine the radius of convergence for G_3, we will obtain the figures (20a) and (20b) in Appendix (4.2.3).

```
numPoints = 10000;
(*Generate random values for m and n*)
mValues1 = RandomReal[{0.000000000001, 10000}, numPoints];
nValues1 = RandomReal[{0.000000000001, 10000}, numPoints];
mValues2 = RandomReal[{0.000000000001, 10000}, numPoints];
nValues2 = RandomReal[{0.000000000001, 10000}, numPoints];
mValues3 = RandomReal[{0.000000000001, 10000}, numPoints];
nValues3 = RandomReal[{0.000000000001, 10000}, numPoints];
mValues4 = RandomReal[{0.000000000001, 10000}, numPoints];
nValues4 = RandomReal[{0.000000000001, 10000}, numPoints];
mValues5 = RandomReal[{0.000000000001, 10000}, numPoints];
nValues5 = RandomReal[{0.000000000001, 10000}, numPoints];
mValues6 = RandomReal[{0.000000000001, 10000}, numPoints];
nValues6 = RandomReal[{0.000000000001, 10000}, numPoints];
mValues7 = RandomReal[{0.000000000001, 10000}, numPoints];
nValues7 = RandomReal[{0.000000000001, 10000}, numPoints];
(*Initialize lists to store the x and y coordinates*)
xValues = ConstantArray[0, 7*numPoints];
yValues = ConstantArray[0, 7*numPoints];
(*Calculate x and y coordinates for each point*)
Do[
  m = mValues1[[i]]; n = nValues1[[i]];
  x = m*(m + 2*n)/(2*m + 3*n)^2;
  y = n*(2*m + 3*n)/(m^2);
  xValues[[i]] = x;
  yValues[[i]] = y;
  , {i, numPoints}];
Do[
  m = mValues2[[i]]; n = nValues2[[i]];
  x = (2*m + n)*n/(3*m + n)^2;
  y = (m + n)*(3*m + n)/n^2;
  xValues[[i + numPoints]] = x;
  yValues[[i + numPoints]] = y;
```

```
  , {i, numPoints}];
Do[
  m = mValues3[[i]];
  n = nValues3[[i]];
  x = (m + n)*(m + 3*n)/m^2;
  y = m*(m + 2*n)/(m + 3*n)^2;
  xValues[[i + 2*numPoints]] = x;
  yValues[[i + 2*numPoints]] = y;
  , {i, numPoints}];
Do[
  m = mValues4[[i]];
  n = nValues4[[i]];
  x = m*(3*m + 2*n)/(n^2);
  y = n*(2*m + n)/(3*m + 2*n)^2;
  xValues[[i + 3*numPoints]] = x;
  yValues[[i + 3*numPoints]] = y;
  , {i, numPoints}];
Do[
  x = Sqrt[4/27]; y = 1/x;
  xValues[[i + 4*numPoints]] = x;
  yValues[[i + 4*numPoints]] = y;
  , {i, numPoints}];
Do[
  y = Sqrt[4/27]/Sqrt[x];
  xValues[[i + 5*numPoints]] = x;
  yValues[[i + 5*numPoints]] = y;
  , {i, numPoints}];
Do[
  x = Sqrt[4/27]*y;
  xValues[[i + 6*numPoints]] = x;
  yValues[[i + 6*numPoints]] = y;
  , {i, numPoints}];
(*Plot the points on the XY plane*)
```

```
ListPlot[Transpose[{xValues, yValues}], PlotStyle -> PointSize[Small],
    AxesLabel -> {"X", "Y"},AspectRatio -> 1, PlotLabel -> "Points in XY
    Plane", PlotRange -> {{0, .25}, {0, .25}}]
```

Remark: It is clear that we can use Horn's theorem to determine $\rho(m,n)$ and $\sigma(m,n)$ for H_5. By doing so, we can easily find that

$$\rho(m,n) = \frac{m|m-2n|}{(2m-n)^2}, \quad \sigma(m,n) = \frac{n|2m-n|}{(2n-m)^2} \tag{4.131}$$

Using the same approach (generating points)

```
numPoints = 10000;
(*Generate random values for m and n*)
mValues = RandomReal[{0.000000000001, 100}, numPoints];
nValues = RandomReal[{0.000000000001, 100}, numPoints];
(*Initialize lists to store the x and y coordinates*)
xValues = ConstantArray[0, numPoints]; yValues = ConstantArray[0,
    numPoints];
(*Calculate x and y coordinates for each point*)
Do[
  m = mValues[[i]];
  n = nValues[[i]];
  x = (m*(Abs[m - 2*n]))/((2*m - n)^2);
  y = (n*(Abs[2*m - n]))/((2*n - m)^2);
  xValues[[i]] = x;
  yValues[[i]] = y,
  {i, numPoints}];
(*Plot the points on the XY plane*)
ListPlot[
  Transpose[{xValues, yValues}], PlotStyle -> PointSize[Small],AxesLabel
     -> {"X", "Y"}, PlotRange -> {{0, 0.25}, {0, .25}},AspectRatio -> 1,
     PlotLabel -> "Points in XY Plane"]
```

we will obtain also figure (20c) in Appendix (4.2.3). So the two results, one obtained by expressing G_3 as a Srivastava-Daoust series and the other obtained directly using Horn's theorem are identical.

4.2 Radius of convergence for the triple hypergeometric series

In this section, we will take specific triple hypergeometric series introduced in ([2], p.70- p.107). Our primary focus is to determine the radii of convergence for these series by reformulating them as Srivastava-Daoust series. It is noteworthy that, in the previous section we considered $n = 2$ in the Srivastava-Daoust series, here we extend our analysis by taking $n = 3$ to accommodate the triple hypergeometric series.

By employing this approach, we aim to ascertain the convergence properties of these triple hypergeometric series and establish their regions of validity. Furthermore, we seek to compare our results with the previous findings presented in ([2], p.70- p.107).

4.2.1 Radius of convergence for the triple hypergeometric series H_c

The triple hypergeometric series H_c is represented by ([2], p.78).

$$H_c(\alpha, \beta_1, \beta_2; \gamma, x, y, z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma)_{m+n+p}} \frac{x^m y^n z^p}{m!n!p!} \tag{4.132}$$

where the radius of convergence of the above triple series is given by ([2], p.93):

$$x < 1 \wedge y < 1 \wedge z < 1 \wedge x + y + z - 2\sqrt{(1-x)(1-y)(1-z)} < 2 \tag{4.133}$$

Our main goal is to rewrite equation (4.132) as a Srivastava-Daoust series. This endeavor will enable us to determine its radius of convergence and make a direct comparison with the above inequalities (4.133).

After rewriting equation (4.132) as a Srivastava-Daoust series, we will find that the set Z determined by:

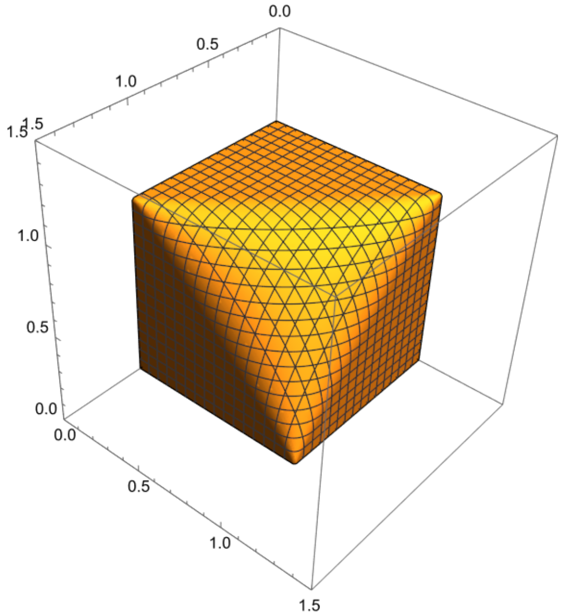
$$|x| < \frac{m_1(m_1+m_2+m_3)}{(m_1+m_3)(m_1+m_2)}$$

$$|y| < \frac{m_2(m_1+m_2+m_3)}{(m_1+m_2)(m_3+m_2)}$$

$$|z| < \frac{m_3(m_1+m_2+m_3)}{(m_1+m_3)(m_3+m_2)} \tag{4.134}$$

Figure 8

Radius of convergence for triple hypergeometric series H_c



for any values of $m_1, m_2, m_3 > 0$. If we try to plot the boundaries that determine the radius of convergence for H_c as we do in two variable case using the inequalities above (4.134) taking into consideration the set[1] $C = K[1,1,1]$, we will obtain figures (21a) and (21b) in (4.2.3), which represents the same boundaries that determine the radius of convergence for H_c as we see in figures (21c) in Appendix (4.2.3).

```
numPoints = 1000;
(*Generate random values for m1,m2,and m3*)
m1Values = RandomReal[{0.000000000001, 100}, numPoints];
m2Values = RandomReal[{0.000000000001, 100}, numPoints];
m3Values = RandomReal[{0.000000000001, 100}, numPoints];
(*Initialize lists to store the x,y,and z coordinates*)
xValues = ConstantArray[0, numPoints]; yValues = ConstantArray[0,
    numPoints]; zValues = ConstantArray[0, numPoints];
(*Calculate x,y,and z coordinates for each point*)
Do[
  m1 = m1Values[[i]]; m2 = m2Values[[i]]; m3 = m3Values[[i]];
```

[1]The set C can be easily determined by applying Horn's theorem to the triple hypergeometric functions. For more details about this theorem, you can refer to Srivastava's book.

```
  x= (m1*(m1 + m2 + m3))/((m1 + m3)*(m1 + m2));
  y= (m2*(m1 + m2 + m3))/((m1 + m2)*(m3 + m2));
  z= (m3*(m1 + m2 + m3))/((m1 + m3)*(m3 + m2));
  xValues[[i]] = x; yValues[[i]] = y; zValues[[i]] = z,
  {i, numPoints}];
(*Filter points that satisfy the initial conditions*)
initialIntersectionPoints = Select[Select[Transpose[{xValues, yValues,
    zValues}], #[[1]] < 1 && #[[2]] < 1 && #[[3]] < 1 &], # != {0,0, 0}
    &];
(*Plot the final intersection points in the XYZ space*)
ListPointPlot3D[initialIntersectionPoints,PlotStyle -> PointSize[Small],
    AxesLabel -> {"X", "Y", "Z"},PlotLabel -> "Final Intersection Points",
     ImageSize -> Medium,AspectRatio -> Full]
```

4.2.2 Radius of convergence for the triple hypergeometric series (11.c) in Srivastava's book

Now, let us delve deeper and consider the triple hypergeometric series (11.c) in ([2], p.77) that takes the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_p(b)_{m-p}}{(c)_{m+n}}\frac{x^m y^n z^p}{m!n!p!} \tag{4.135}$$

where the radius of convergence of the above triple series is given by ([2], p.92):

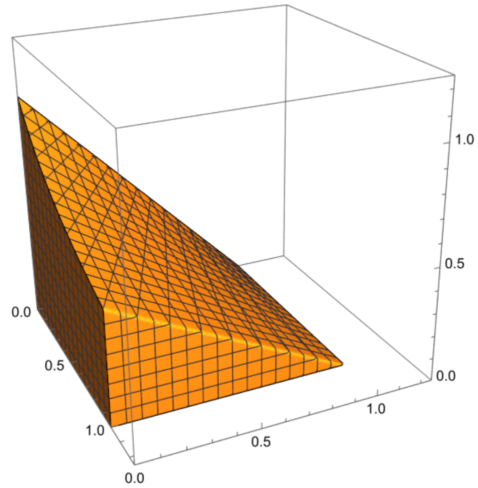
$$x < 1 \wedge y < 1 \wedge z < \frac{1-y}{1+x} \tag{4.136}$$

It is evident that we cannot directly rewrite (4.135) as a Srivastava-Daoust series due to the presence of $(b)_{m-p}$. Therefore, we will apply the technique presented on ([12], p.43-p.44) in order to be able to write it as a Srivastava-Daoust series (See Appendix 4.2.3). By doing so, we obtain

$$\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_p(b)_{m-p}}{(c)_{m+n}}\frac{x^m y^n z^p}{m!n!p!}$$

Figure 9

Radius of convergence for triple hypergeometric series (11.c) in Srivastava's book



$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_{n+p}(1)_m(b)_m(a_3)_p}{(c)_{m+n+p}(1)_{m+p}} \frac{x^m y^n (xz)^p}{m!n!p!}$$

$$- \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n}(a_3)_m}{(c)_{m+n}(1)_m} \frac{(xz)^m y^n}{m!n!}$$

$$+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n+p}(a_3)_{m+p}(1)_p}{(c)_{m+n}(1)_{m+p}(1-b)_p} \frac{(xz)^m y^n (-z)^p}{m!n!p!} \tag{4.137}$$

I have thoroughly tested the above formula for various parameter values and variables x, y and z using Mathematica. I found that the left-hand side and the right-hand side of the equation are equal, where x, y and z belong to the radius of convergence for both sideshttps://www.wolframcloud.com/obj/s12255413/Published/TestingWe can now rewrite each one of these summations as Srivastava-Daoust series. Upon doing so, we obtain the following results:

- The first triple sum given by

$$S_1 = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_{n+p}(1)_m(b)_m(a_3)_p}{(c)_{m+n+p}(1)_{m+p}} \frac{x^m y^n (xz)^p}{m!n!p!}$$

has a region of convergence determined by the values of x, y and z that satisfy the conditions:

$$
\begin{aligned}
|x| &< \frac{m_1 + m_3}{m_1} \\
|y| &< \frac{m_2}{m_2 + m_3} \\
|xz| &< \frac{m_1 + m_3}{m_2 + m_3}
\end{aligned}
\tag{4.138}
$$

where $m_1, m_2, m_3 > 0$. If we try to plot the boundaries that determine the radius of convergence for S_1 taking into consideration the set $C = K[1, 1, 1]$, we will obtain an empty box which means that the boundaries that determine the radius of convergence for S_1 lie outside the box whose dimensions are determined by the set $C = K[1, 1, 1]$. To know further about the code we use https://www.wolframcloud.com/obj/s12255413/Published/Plots

- The second triple sum given by

$$
S_2 = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n+p}(a_3)_{m+p}(1)_p}{(c)_{m+n}(1)_{m+p}(1-b)_p} \frac{(xz)^m y^n (-z)^p}{m!n!p!}
$$

has a region of convergence determined by the values of x, y and z that satisfy the conditions:

$$
\begin{aligned}
|xz| &< \frac{m_1}{m_1 + m_2 + m_3} \\
|y| &< \frac{m_2}{m_1 + m_2 + m_3} \\
|z| &< \frac{m_3}{m_1 + m_2 + m_3}
\end{aligned}
\tag{4.139}
$$

where $m_1, m_2, m_3 > 0$. Which implies

$$
|xz| + |y| + |z| < 1 \tag{4.140}
$$

Now, we can either directly plot the radius of convergence for S_2 using inequality (4.140), or plot the boundaries that determine the radius of convergence for S_2 described in (4.139). Choosing the first option will result in the figure (22a) in Appendix (4.2.3), while the second option will lead to the figure (22b) in Appendix (4.2.3). We can see that the two figures determine the same radius of convergence.

```
numPoints = 200000;
(*Generate random values for m1,m2,and m3*)
m1Values = RandomReal[{0.00000001, 1000}, numPoints];
m2Values = RandomReal[{0.00000001, 1000}, numPoints];
m3Values = RandomReal[{0.00000001, 1000}, numPoints];
(*Initialize lists to store the x,y,and z coordinates*)
xValues = ConstantArray[0, numPoints];
yValues = ConstantArray[0, numPoints];
zValues = ConstantArray[0, numPoints];
(*Calculate x,y,and z coordinates for each point*)
Do[
  m1 = m1Values[[i]];
  m2 = m2Values[[i]];
  m3 = m3Values[[i]];
  x = (m1)/(m3);
  y = (m2)/(m1 + m2 + m3);
  z = (m3)/(m1 + m2 + m3);
  xValues[[i]] = x;
  yValues[[i]] = y;
  zValues[[i]] = z,
  {i, numPoints}];
(*Filter points that satisfy the conditions x<1,y<1,and z<1/4*)
intersectionPoints = Select[
    Select[Transpose[
        {xValues, yValues,zValues}], #[[1]] < 1 && #[[2]] < 1 && #[[3]] <
             1 &], # != {0, 0, 0} &];
(*Plot the intersection points in the XYZ space*)
ListPointPlot3D[
  intersectionPoints, PlotStyle -> PointSize[Small], AxesLabel -> {"X",
     "Y", "Z"}, PlotLabel -> "Intersection Points", ImageSize -> Medium,
      AspectRatio -> Full]
```

- The first double sum given by

$$S_3 = \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n}(a_3)_m}{(c)_{m+n}(1)_m} \frac{(xz)^m y^n}{m!n!}$$

  has a region of convergence determined by the values of x, y and z that satisfies the conditions:

$$|xz| < \frac{m_1}{m_1 + m_2}$$
$$|y| < \frac{m_2}{m_1 + m_2} \tag{4.141}$$

  where $m_1, m_2 > 0$. Which implies

$$|xz| + |y| < 1 \tag{4.142}$$

Also, we can either directly plot the radius of convergence for S_3 using inequality (4.142), or plot the boundaries that determine the radius of convergence for S_3 described in (4.141). Choosing the first option will result in the figure (23a) in Appendix (4.2.3), while the second option will lead to the figure (23b) in Appendix (4.2.3).

```
numPoints = 100000;
(*Generate random values for m1,m2,and m3*)
m1Values = RandomReal[{0.00000001, 1}, numPoints];
m2Values = RandomReal[{0.00000001, 1}, numPoints];
m3Values = RandomReal[{0.00000001, 1}, numPoints];
(*Initialize lists to store the x,y,and z coordinates*)
xValues = ConstantArray[0, numPoints];
yValues = ConstantArray[0, numPoints];
zValues = ConstantArray[0, numPoints];
(*Calculate x,y,and z coordinates for each point*)
Do[
  m1 = m1Values[[i]];
  m2 = m2Values[[i]];
  m3 = m3Values[[i]];
```

```
  x = (m1)/(m3*(m1 + m2));
  y = (m2)/(m1 + m2);
  z = (m3);
  xValues[[i]] = x;
  yValues[[i]] = y;
  zValues[[i]] = z,
  {i, numPoints}];
(*Filter points that satisfy the conditions x<1,y<1,and z<1/4*)
intersectionPoints = Select[
    Select[Transpose[
    {xValues, yValues,zValues}], #[[1]] < 1 && #[[2]] < 1 && #[[3]] < 1 &], # != {0, 0, 0} &];
(*Plot the intersection points in the XYZ space*)
ListPointPlot3D[
   intersectionPoints, PlotStyle -> PointSize[Small], AxesLabel -> {"X", "Y", "Z"}, PlotLabel -> "Intersection Points", ImageSize -> Medium, AspectRatio -> Full]
```

Finally, we can see the intersection of the radii of convergence for S_1, S_2 and S_3 taking into consideration the set $C = K[1, 1, 1]$ is the same radius of convergence obtained from equation (4.136).

Also, to validate our calculations and ensure the accuracy of our conclusions, we plot the boundaries obtained from the S_2, S_3 and figure (9) in Appendix (4.2.3) on the same box as we see in figure (24) in Appendix (4.2.3).

```
numPoints = 10000;
(*Generate random values for m1,m2,and m3*)
m1Values = RandomReal[{0.00000000000001, 0.01}, numPoints];
m2Values = RandomReal[{0.00000000000001, 0.01}, numPoints];
m3Values = RandomReal[{0.00000000000001, 0.01}, numPoints];
(*Initialize lists to store the x,y,and z coordinates*)
xValues = ConstantArray[0, numPoints];
```

```
yValues = ConstantArray[0, numPoints];
zValues = ConstantArray[0, numPoints];
(*Calculate x,y,and z coordinates for each point*)
Do[
  m1 = m1Values[[i]];
  m2 = m2Values[[i]];
  m3 = m3Values[[i]];
  x = (m1)/(m3);
  y = (m2)/(m1 + m2 + m3);
  z = (m3)/(m1 + m2 + m3);
  xValues[[i]] = x;
  yValues[[i]] = y;
  zValues[[i]] = z,
  {i, numPoints}];
(*Filter points that satisfy the conditions x<1,y<1,and z<1/4*)
intersectionPoints = Select[Select[
    Transpose[{xValues, yValues, zValues}], #[[1]] < 1 && #[[2]] < 1 &&
        #[[3]]<1 &], # != {0, 0, 0} &];
(*Plot the intersection points in the XYZ space*)
Show[
  ListPointPlot3D[
    intersectionPoints, PlotStyle -> PointSize[Small], AxesLabel -> {"X",
        "Y", "Z"}, PlotLabel -> "Intersection Points", ImageSize -> Medium,
         AspectRatio -> Full],
 RegionPlot3D[
   x < 1 && y < 1 && z < ((1 - y)/(1 + x)), {x, 0, 1}, {y, 0, 1}, {z, 0,
      1}, AxesLabel -> {"x", "y", "z"}, PlotPoints > 100, Mesh -> None,
      PlotStyle -> Directive[Opacity[0.5], Red]]]
```

Question: Can we establish a relationship between x, y and z in (4.138) and then plot the intersection of this relation with inequalities (4.140), (4.142) and $C = K[1, 1, 1]$ in order to compare our finding with the result obtained in ([2], p.92)??

4.2.3 Radius of convergence for the triple hypergeometric series (36.h) in Srivastava's book

Now, let us turn our attention to the triple hypergeometric series (36.h) in ([2], p.83) which is given by the following formula

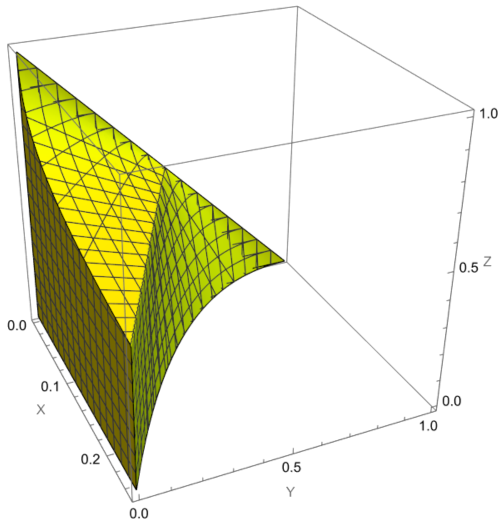
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{n+p}(b_1)_{2m+n-p}(b_2)_{p-n}}{(c)_m}\frac{x^m y^n z^p}{m!n!p!} \tag{4.143}$$

where the radius of convergence of the above triple series is given by ([2], p.99)

$$y+z<1\wedge\sqrt{x}<\frac{\sqrt{1-4yz}-|1-2z|}{4z} \tag{4.144}$$

Figure 10

Radius of convergence for triple hypergeometric series (36.h) in Srivastava's book



It is evident that we cannot directly rewrite (4.143) as a Srivastava-Daoust series due to the presence of $(b_1)_{2m+n-p}$ and $(b_2)_{p-n}$. Therefore, we will apply the technique presented on ([12], p.43-p.44) in order to be able to write it as a Srivastava-Daoust series (See Appendix 4.2.3). By doing so, we will obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{n+2p}(b_1)_{2m+n}(1)_n}{(1)_{n+p}(c)_m(1-b_2)_n}\frac{x^m(-y)^n(yz)^p}{m!n!p!}$$

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+2n}(b_1)_m(b_2)_m}{(1)_{m+n}(c)_m}\frac{(xz)^m(yz)^n}{m!n!}$$

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{2m+2n}(b_2)_{2m}}{(1)_{2m+n}(c)_m}\frac{(xz^2)^m(yz)^n}{m!n!}-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(b_1)_{2m}(a)_{2n}}{(c)_m(1)_n}\frac{(x)^m(yz)^n}{m!n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2n+p}(b_2)_{2m+p}(1)_p}{(1)_{2m+n+p}(c)_m(1-b_1)_p}\frac{(xz^2)^m(yz)^n(-z)^p}{m!n!p!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2n+p}(b_1)_{2m+p}(1)_m(b_2)_p(1)_p}{(c)_{m+p}(1)_{m+p}(1)_{n+p}}\frac{x^m(yz)^n(xz)^p}{m!n!p!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{m+2n+2p}(b_2)_{m+2p}(b_1)_m(1)_m(1)_p}{(c)_{m+p}(1)_{m+p}(1)_{m+n+2p}}\frac{(xz)^m(yz)^n(xz^2)^p}{m!n!p!} \tag{4.145}$$

I have also tested the above formula for various parameter values and variables x, y and z using Mathematica. I found that the left-hand side and the right-hand side of the equation are equal, where x, y and z belong to the radius of convergence for both sideshttps://www.wolframcloud.com/obj/s12255413/Published/TestingWe can now rewrite each one of these summations as Srivastava-Daoust series. Accordingly, we will obtain the following results:

- The first triple sum given by

$$S_1=\sum_{m,n,p=0}^{\infty}\frac{(a)_{n+2p}(b_1)_{2m+n}(1)_n}{(1)_{n+p}(c)_m(1-b_2)_n}\frac{x^m(-y)^n(yz)^p}{m!n!p!}$$

  has a region of convergence determined by the values of x, y and z that satisfy the conditions:

$$|x|<\frac{m_1^2}{(2m_1+m_2)^2}$$
$$|y|<\frac{m_2(m_2+m_3)}{(m_2+2m_3)(2m_1+m_2)} \tag{4.146}$$
$$|yz|<\frac{m_3(m_2+m_3)}{(m_2+2m_3)^2}$$

  where $m_1, m_2, m_3 > 0$. If we try to plot the boundaries that determine the radius of convergence for S_1 using (4.146) taking into consideration the set $C = K[\frac{1}{4}, 1, 1]$, we will obtain Figure (25) in Appendix (4.2.3).

```
numPoints = 100000;
(*Generate random values for m1,m2,and m3*)
m1Values = RandomReal[{0.000000000001, 10000}, numPoints];
m2Values = RandomReal[{0.000000000001, 10000}, numPoints];
m3Values = RandomReal[{0.000000000001, 10000}, numPoints];
(*Initialize lists to store the x,y,and z coordinates*)
xValues = ConstantArray[0, numPoints];
yValues = ConstantArray[0, numPoints];
zValues = ConstantArray[0, numPoints];
(*Calculate x,y,and z coordinates for each point*)
Do[
  m1 = m1Values[[i]];
  m2 = m2Values[[i]];
  m3 = m3Values[[i]];
  x = (m1^2)/((2 m1 + m2)^2);
  y = ((m2)*(m2 + m3))/((m2 + 2*m3)*(2 m1 + m2));
  z = (m3*(2 m1 + m2))/((m2)*(m2 + 2 m3));
  xValues[[i]] = x;
  yValues[[i]] = y;
  zValues[[i]] = z,
  {i, numPoints}];
(*Filter points that satisfy the conditions x<1,y<1,and z<1/4*)
intersectionPoints = Select[
        Select[Transpose[
          {xValues, yValues,zValues}], #[[1]] < 1/4 && #[[2]] < 1 &&
              #[[3]] < 1 &], # != {0, 0, 0} &];
(*Plot the intersection points in the XYZ space*)
ListPointPlot3D[
  intersectionPoints, PlotStyle -> PointSize[Small], AxesLabel -> {"X",
     "Y", "Z"}, PlotLabel -> "Intersection Points", ImageSize -> Medium,
      AspectRatio -> Full]
```

- The second triple sum given by

$$S_2 = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2n+p}(b_1)_{2m+p}(1)_m(b_2)_p(1)_p}{(c)_{m+p}(1)_{m+p}(1)_{n+p}} \frac{x^m(yz)^n(xz)^p}{m!n!p!}$$

has a region of convergence determined by the values of x, y and z that satisfy the conditions:

$$\begin{aligned} |x| &< \frac{(m_1+m_3)^2}{(2m_1+m_3)^2} \\ |yz| &< \frac{m_2(m_2+m_3)}{(2m_2+m_3)^2} \\ |xz| &< \frac{(m_1+m_3)^2(m_2+m_3)}{m_3(2m_2+m_3)(2m_1+m_3)} \end{aligned} \tag{4.147}$$

where $m_1, m_2, m_3 > 0$. If we try to plot the boundaries that determine the radius of convergence for S_2 taking into consideration the set $C = K[\frac{1}{4}, 1, 1]$, we will obtain an empty box which means that the boundaries that determine the radius of convergence for S_2 lie outside the box whose dimensions are determined by the set $C = K[\frac{1}{4}, 1, 1]$. To know further about the code we use https://www.wolframcloud.com/obj/s12255413/Published/Plots

- The third triple sum given by

$$S_3 = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+2n+2p}(b_2)_{m+2p}(b_1)_m(1)_m(1)_p}{(c)_{m+p}(1)_{m+p}(1)_{m+n+2p}} \frac{(xz)^m(yz)^n(xz^2)^p}{m!n!p!}$$

has a region of convergence determined by the values of x, y and z that satisfy the conditions:

$$\begin{aligned} |xz| &< \frac{(m_1+m_3)^2(m_1+m_2+2m_3)}{m_1(m_1+2m_3)(m_1+2m_2+2m_3)} \\ |yz| &< \frac{m_2(m_1+m_2+2m_3)}{(m_1+2m_2+2m_3)^2} \\ |xz^2| &< \frac{(m_1+m_3)^2(m_1+m_2+2m_3)^2}{(m_1+2m_2+2m_3)^2(m_1+2m_3)^2} \end{aligned} \tag{4.148}$$

where $m_1, m_2, m_3 > 0$. If we try to plot the boundaries that determine the radius of convergence for S_3 taking into consideration the set$C = K[\frac{1}{4}, 1, 1]$, we will obtain an empty box which means that the boundaries that determine the radius of convergence for S_3 lie outside the box whose dimensions are determined by the set $C = K[\frac{1}{4}, 1, 1]$.

- The fourth triple sum given by

$$S_4 = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+2n+p}(b_2)_{2m+p}(1)_p}{(1)_{2m+n+p}(c)_m(1-b_1)_p} \frac{(xz^2)^m(yz)^n(-z)^p}{m!n!p!}$$

has a region of convergence determined by the values of x, y and z that satisfy the conditions:

$$\begin{aligned} |xz^2| &< \frac{m_1^2(2m_1+m_2+m_3)^2}{(2m_1+m_3)^2(2m_1+2m_2+m_3)^2} \\ |yz| &< \frac{m_2(2m_1+m_2+m_3)}{(2m_1+2m_2+m_3)^2} \\ |z| &< \frac{m_3(2m_1+m_2+m_3)}{(2m_1+2m_2+m_3)(2m_1+m_3)} \end{aligned} \tag{4.149}$$

where $m_1, m_2, m_3 > 0$. If we try to plot the boundaries that determine the radius of convergence for S_4 using (4.149) taking into consideration the set $C = K[\frac{1}{4}, 1, 1]$, we will obtain Figure (26) in Appendix (4.2.3).

```
numPoints = 100000;
(*Generate random values for m1,m2,and m3*)
m1Values = RandomReal[{0.000000000001, 10000}, numPoints];
m2Values = RandomReal[{0.000000000001, 10000}, numPoints];
m3Values = RandomReal[{0.000000000001, 10000}, numPoints];
(*Initialize lists to store the x,y,and z coordinates*)
xValues = ConstantArray[0, numPoints]; yValues = ConstantArray[0,
    numPoints]; nzValues = ConstantArray[0, numPoints];
(*Calculate x,y,and z coordinates for each point*)
Do[
  m1 = m1Values[[i]];
  m2 = m2Values[[i]];
  m3 = m3Values[[i]];
  x = (m1^2)/(m3^2);
  y = (m2*(2 m1 + m3))/((2 m1 + 2 m2 + m3)*(m3));
  z = (m3*(2 m1 + m2 + m3))/((2 m1 + 2 m2 + m3)*(2 m1 + m3));
  xValues[[i]] = x;
```

```
  yValues[[i]] = y;
  zValues[[i]] = z,
  {i, numPoints}];
(*Plot the points in the XYZ space*)
ListPointPlot3D[Select[Transpose[{xValues, yValues, zValues}], # != {0,
   0, 0} &], PlotStyle -> PointSize[Small], AxesLabel -> {"X", "Y", "Z"
   },PlotLabel -> "Points in XYZ Space", PlotRange -> {{0, 1/4}, {0, 1},
    {0, 1}}, ImageSize -> Medium, AspectRatio -> Full]
```

- The first double sum given by

$$S_5 = \sum_{m,n=0}^{\infty} \frac{(a)_{m+2n}(b_1)_m(b_2)_m}{(1)_{m+n}(c)_m} \frac{(xz)^m(yz)^n}{m!n!}$$

has a region of convergence determined by the values of x, y and z that satisfy the conditions:

$$|xz| < \frac{m_1 + m_2}{m_1 + 2m_2}$$
$$|yz| < \frac{m_2(m_1 + m_2)}{(m_1 + 2m_2)^2} \tag{4.150}$$

where $m_1, m_2 > 0$. If we try to plot the boundaries that determine the radius of convergence for S_5 taking into consideration the set $C = K[\frac{1}{4}, 1, 1]$, we will obtain an empty box which means that the boundaries that determine the radius of convergence for S_5 lie outside the box whose dimensions are determined by the set $C = K[\frac{1}{4}, 1, 1]$.

- The second double sum given by

$$S_6 = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+2n}(b_2)_{2m}}{(1)_{2m+n}(c)_m} \frac{(xz^2)^m(yz)^n}{m!n!}$$

has a region of convergence determined by the values of x, y and z that satisfy the conditions:

$$|xz^2| < \frac{(2m_1 + m_2)^2}{(2m_1 + 2m_2)^2}$$
$$|yz| < \frac{m_2(2m_1 + m_2)}{(2m_1 + 2m_2)^2} \tag{4.151}$$

where $m_1, m_2 > 0$. If we try to plot the boundaries that determine the radius of convergence for S_6 taking into consideration the set $C = K[\frac{1}{4}, 1, 1]$, we will obtain an empty box which means that the boundaries that determine the radius of convergence for S_6 lie outside the box whose dimensions are determined by the set $C = K[\frac{1}{4}, 1, 1]$. To know further about the code we use https://www.wolframcloud.com/obj/s12255413/Published/Plots

- The third double sum given by

$$S_7 = \sum_{m,n=0}^{\infty} \frac{(b_1)_{2m}(a)_{2n}}{(c)_m(1)_n} \frac{(x)^m(yz)^n}{m!n!}$$

has a region of convergence determined by the values of x, y and z that satisfies $|x| < \frac{1}{4}$, $|yz| < \frac{1}{4}$.

Finally, to validate our calculations and ensure the accuracy of our conclusions, we plot the boundaries obtained from the S_1, S_4, S_7 and figure (9) on the same box as we see in figures (27), (28) and (29) in Appendix (4.2.3).

```
numPoints = 1000;
(*Generate random values for m1,m2,and m3 for the first shape*)
m1Values = RandomReal[{0.000000000001, 10}, numPoints];
m2Values = RandomReal[{0.000000000001, 10}, numPoints];
m3Values = RandomReal[{0.000000000001, 10}, numPoints];
(*Initialize lists to store the x,y,and z coordinates for the first \
    shape*)
xValues1 = ConstantArray[0, numPoints];
yValues1 = ConstantArray[0, numPoints];
zValues1 = ConstantArray[0, numPoints];
(*Calculate x,y,and z coordinates for the first shape*)
Do[
  m1 = m1Values[[i]];
  m2 = m2Values[[i]];
  m3 = m3Values[[i]];
  x = (m1^2)/(m3^2);
```

```
  y = (m2*(2 m1 + m3))/((2 m1 + 2 m2 + m3)*(m3));
  z = (m3*(2 m1 + m2 + m3))/((2 m1 + 2 m2 + m3)*(2 m1 + m3));
  xValues1[[i]] = x;
  yValues1[[i]] = y;
  zValues1[[i]] = z,
  {i, numPoints}];
(*Filter points that satisfy the conditions x<1,y<1,and z<1/4 for the \
first shape*)
intersectionPoints1 =
  Select[Select[
    Transpose[{xValues1, yValues1,
      zValues1}], #[[1]] < 1/4 && #[[2]] < 1 && #[[3]] <
       1 &], # != {0, 0, 0} &];
(*Generate random values for m1,m2,and m3 for the second shape*)
m1Values = RandomReal[{0.000000000001, 1}, numPoints];
m2Values = RandomReal[{0.000000000001, 1}, numPoints];
m3Values = RandomReal[{0.000000000001, 1}, numPoints];
(*Initialize lists to store the x,y,and z coordinates for the second \
shape*)
xValues2 = ConstantArray[0, numPoints];
yValues2 = ConstantArray[0, numPoints];
zValues2 = ConstantArray[0, numPoints];
(*Calculate x,y,and z coordinates for the second shape*)
Do[
  m1 = m1Values[[i]];
  m2 = m2Values[[i]];
  m3 = m3Values[[i]];
  x = (m1^2)/((2 m1 + m2)^2);
  y = ((m2)*(m2 + m3))/((m2 + 2*m3)*(2 m1 + m2));
  z = (m3*(2 m1 + m2))/((m2)*(m2 + 2 m3));
  xValues2[[i]] = x;
  yValues2[[i]] = y;
  zValues2[[i]] = z,
```

```
 {i, numPoints}];
(*Plotting*)
Show[ListPointPlot3D[intersectionPoints1, PlotStyle -> Red, PlotLabel -> "Shape 1"],
 ListPointPlot3D[
  Select[Transpose[{xValues2, yValues2, zValues2}], # != {0, 0, 0} &], PlotStyle -> Blue, PlotLabel -> "Shape 2"],
 RegionPlot3D[ y*z < 1/4 && y + z < 1 && Sqrt[x] < (Sqrt[1 - 4*y*z] - Abs[1 - 2*z])/(4*z), {x, 0.0000001, 1/4}, {y, 0.0000001, 1}, {z, 0.0000001, 1}, AxesLabel -> {"X", "Y", "Z"}, PlotPoints -> 100, PlotStyle -> Directive[Opacity[1], Yellow], Mesh -> None], AxesLabel -> {"X", "Y", "Z"}, ImageSize -> Medium, AspectRatio -> Full, PlotRange -> {{0, 1/4}, {0, 1}, {0, 1}}]
```

Question: Can we set up a relationship between x, y and z in (4.146), (4.147), (4.148), (4.149), (4.150) and (4.151) and then plot the intersection of these inequalities with $|x| < \frac{1}{4}$, $|yz| < \frac{1}{4}$ and $C = K[\frac{1}{4}, 1, 1]$ in order to compare our result with the one obtained in ([2], p.99) ??

Conclusion

In this thesis, we accomplished our two objectives. Firstly, we derived seven new transformations for the Kampé de Fériet function utilizing Miller-Paris transformations for generalized hypergeometric functions. Subsequently, we determined the radius of convergence for each transformation. To ensure precision, we conducted testing using the Mathematica across various parameter and variable values, confirming the accuracy of these transformations.

Furthermore, we endeavored to ascertain the radius of convergence for well-known double and triple hypergeometric functions by expressing them through the Srivastava-Daoust equation. We then compared our findings with those presented in Srivastava's book ([2], p.59-p.65 and p.70-p.107) to validate our results.

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Appendices

Appendix A

Removing the negative sign from the Pochhammer symbols in the double hypergeometric series G_1, G_2, G_3, H_1, H_2, H_5, H_6, H_7

1. Remove the negative sign from the double hypergeometric series G_1

$$G_1(\alpha,\beta,\beta';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m+n}(\beta)_{n-m}(\beta')_{m-n}\frac{x^m}{m!}\frac{y^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\alpha)_{m+2n}(\beta)_{-m}(\beta')_m\frac{x^{m+n}}{(m+n)!}\frac{y^n}{n!}-\sum_{n=0}^{\infty}(\alpha)_{2n}\frac{x^n}{n!}\frac{y^n}{n!}$$

$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\alpha)_{2m+n}(\beta)_n(\beta')_{-n}\frac{x^m}{m!}\frac{y^{m+n}}{(m+n)!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_{m+2n}(\beta')_m(1)_m}{(1)_{m+n}(1-\beta)_m}\frac{(-x)^m}{m!}\frac{(xy)^n}{n!}$$

$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_n(1)_n}{(1)_{m+n}(1-\beta')_n}\frac{(xy)^m}{m!}\frac{(-y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_{2n}}{(1)_n}\frac{(xy)^n}{n!}$$

2. Remove the negative sign from the double hypergeometric series G_2

$$G_2(\alpha,\alpha'\beta,\beta';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_m(\alpha')_n(\beta)_{n-m}(\beta')_{m-n}\frac{x^m}{m!}\frac{y^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\alpha)_{m+n}(\alpha')_n(\beta)_{-m}(\beta')_m\frac{x^{m+n}}{(m+n)!}\frac{y^n}{n!}-\sum_{n=0}^{\infty}(\alpha)_n(\alpha')_n\frac{x^n}{n!}\frac{y^n}{n!}$$

$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\alpha)_m(\alpha')_{m+n}(\beta)_n(\beta')_{-n}\frac{x^m}{m!}\frac{y^{m+n}}{(m+n)!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_{m+n}(\beta')_m(1)_m(\alpha')_n}{(1)_{m+n}(1-\beta)_m}\frac{(-x)^m}{m!}\frac{(xy)^n}{n!}$$

$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha')_{m+n}(\alpha)_m(\beta)_n(1)_n}{(1)_{m+n}(1-\beta')_n}\frac{(xy)^m}{m!}\frac{(-y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha')_n}{(1)_n}\frac{(xy)^n}{n!}$$

3. Remove the negative sign from the double hypergeometric series H_1

$$H_1(\alpha,\beta,\gamma;\delta;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m}\frac{x^m}{m!}\frac{y^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_{m+2n}(\gamma)_n}{(\delta)_{m+n}}\frac{x^{m+n}}{(m+n)!}\frac{y^n}{n!}-\sum_{n=0}^{\infty}\frac{(\beta)_{2n}(\gamma)_n}{(\delta)_n}\frac{x^n}{n!}\frac{y^n}{n!}$$

$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_{-n}(\beta)_{2m+n}(\gamma)_{m+n}}{(\delta)_m}\frac{x^m}{m!}\frac{y^{m+n}}{(m+n)!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{m+2n}(\alpha)_m(1)_m(\gamma)_n}{(\delta)_{m+n}(1)_{m+n}}\frac{x^m}{m!}\frac{(xy)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{2m+n}(\gamma)_{m+n}(1)_n}{(1)_{m+n}(\delta)_m(1-\alpha)_n}\frac{(xy)^m}{m!}\frac{(-y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\beta)_{2n}(\gamma)_n}{(1)_n(\delta)_n}\frac{(xy)^n}{n!}$$

4. Remove the negative sign from the double hypergeometric series H_2

$$H_2(\alpha,\beta,\gamma,\delta;\epsilon,x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{(\epsilon)_m}\frac{x^m}{m!}\frac{y^n}{n!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\beta)_{m+n}(\gamma)_n(\delta)_n}{(\epsilon)_{m+n}}\frac{x^{m+n}}{(m+n)!}\frac{y^n}{n!}-\sum_{n=0}^{\infty}\frac{(\beta)_n(\gamma)_n(\delta)_n}{(\epsilon)_n}\frac{x^n}{n!}\frac{y^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{-n}(\beta)_m(\gamma)_{m+n}(\delta)_{m+n}}{(\epsilon)_m}\frac{x^m}{m!}\frac{y^{m+n}}{(m+n)!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{m+n}(\alpha)_m(1)_m(\gamma)_n(\delta)_n}{(\epsilon)_{m+n}(1)_{m+n}}\frac{x^m}{m!}\frac{(xy)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}(\gamma)_{m+n}(\beta)_m(1)_n}{(1)_{m+n}(\epsilon)_m(1-\alpha)_n}\frac{(xy)^m}{m!}\frac{(-y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\beta)_n(\gamma)_n(\delta)_n}{(1)_n(\epsilon)_n}\frac{(xy)^n}{n!}$$

5. Remove the negative sign from the double hypergeometric series H_5

$$H_5(\alpha,\beta;\gamma;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_{n-m}}{(\gamma)_n}\frac{x^m}{m!}\frac{y^n}{n!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+3n}(\beta)_{-m}}{(\gamma)_n}\frac{x^{m+n}}{(m+n)!}\frac{y^n}{n!}+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{3m+n}(\beta)_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^{m+n}}{(m+n)!}$$

$$-\sum_{n=0}^{\infty}\frac{(\alpha)_{3n}}{(\gamma)_n}\frac{x^n}{n!}\frac{y^n}{n!}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+3n}(1)_m}{(1)_{m+n}(1-\beta)_m(\gamma)_n}\frac{(-x)^m}{m!}\frac{(xy)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{3m+n}(\beta)_n(1)_n}{(\gamma)_{m+n}(1)_{m+n}}\frac{(xy)^m}{m!}\frac{(y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_{3n}}{(\gamma)_n(1)_n}\frac{(xy)^n}{n!}$$

6. Remove the negative sign from the double hypergeometric series H_6

$$H_6(\alpha,\beta;\gamma;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{2m-n}(\beta)_{n-m}(\gamma)_n\frac{x^m}{m!}\frac{y^n}{n!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{2m+n}(\beta)_{-m}(\gamma)_n\frac{x^{m+n}}{(m+n)!}\frac{y^n}{n!}-\sum_{n=0}^{\infty}(\alpha)_n(\gamma)_n\frac{x^n}{n!}\frac{y^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m-n}(\beta)_n(\gamma)_{m+n}\frac{x^m}{m!}\frac{y^{m+n}}{(m+n)!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{2m+n}(\beta)_{-m}(\gamma)_n\frac{x^{m+n}}{(m+n)!}\frac{y^n}{n!}-\sum_{n=0}^{\infty}(\beta)_n(\gamma)_{2n}\frac{x^n}{n!}\frac{y^{2n}}{(2n)!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{-n}(\beta)_{m+n}(\gamma)_{2m+n}\frac{x^m}{m!}\frac{y^{2m+n}}{(2m+n)!}-\sum_{n=0}^{\infty}(\alpha)_n(\gamma)_n\frac{x^n}{n!}\frac{y^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_m(\beta)_n(\gamma)_{m+2n}\frac{x^{m+n}}{(m+n)!}\frac{y^{m+2n}}{(m+2n)!}=\ -\sum_{n=0}^{\infty}\frac{(\alpha)_n(\gamma)_n}{(1)_n}\frac{(xy)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+n}(1)_m(\gamma)_n}{(1)_{m+n}(1-\beta)_m}\frac{(-x)^m}{m!}\frac{(xy)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma)_{m+2n}(1)_m(\alpha)_m(1)_n(\beta)_n}{(1)_{m+n}(1)_{m+2n}}\frac{(xy)^m}{m!}\frac{(xy^2)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{m+n}(\gamma)_{2m+n}(1)_n}{(1)_{2m+n}(1-\alpha)_n}\frac{(xy^2)^m}{m!}\frac{(-y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\beta)_n(\gamma)_{2n}}{(1)_{2n}}\frac{(xy^2)^n}{n!}$$

7. Remove the negative sign from the double hypergeometric series H_7

$$H_7(\alpha,\beta,\gamma,\delta;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m-n}(\beta)_n(\gamma)_n}{(\delta)_m}\frac{x^m}{m!}\frac{y^n}{n!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_n(\gamma)_n}{(\delta)_{m+n}}\frac{x^{m+n}}{(m+n)!}\frac{y^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(\gamma)_n}{(\delta)_n}\frac{x^n}{n!}\frac{y^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_{m+n}}{(\delta)_m}\frac{x^m}{m!}\frac{y^{m+n}}{(m+n)!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_n(\gamma)_n}{(\delta)_{m+n}}\frac{x^{m+n}}{(m+n)!}\frac{y^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(\gamma)_n}{(\delta)_n}\frac{x^n}{n!}\frac{y^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{-n}(\beta)_{2m+n}(\gamma)_{2m+n}}{(\delta)_m}\frac{x^m}{m!}\frac{y^{2m+n}}{(2m+n)!}-\sum_{n=0}^{\infty}\frac{(\beta)_{2n}(\gamma)_{2n}}{(\delta)_n}\frac{x^n}{n!}\frac{y^{2n}}{(2n)!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\beta)_{m+2n}(\gamma)_{m+2n}}{(\delta)_{m+n}}\frac{x^{m+n}}{(m+n)!}\frac{y^{m+2n}}{(m+2n)!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{2m+n}(1)_m(\beta)_n(\gamma)_n}{(1)_{m+n}(\delta)_{m+n}}\frac{(x)^m}{m!}\frac{(xy)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(\gamma)_n}{(\delta)_n(1)_n}\frac{(xy)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{2m+n}(\gamma)_{2m+n}(1)_n}{(1)_{2m+n}(\delta)_m(1-\alpha)_n}\frac{(xy^2)^m}{m!}\frac{(-y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\beta)_{2n}(\gamma)_{2n}}{(\delta)_n(1)_{2n}}\frac{(xy^2)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{m+2n}(\gamma)_{m+2n}(\alpha)_m(1)_m(1)_n}{(\delta)_{m+n}(1)_{m+n}(1)_{m+2n}}\frac{(xy)^m}{m!}\frac{(xy^2)^n}{n!}$$

8. Remove the negative sign from the double hypergeometric series G_3

$$G_3(\alpha,\alpha';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{2n-m}(\alpha')_{2m-n}\frac{x^m}{m!}\frac{y^n}{n!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{n-m}(\alpha')_{2m+n}\frac{x^{m+n}}{(m+n)!}\frac{y^n}{n!}+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m+2n}(\alpha')_{m-n}\frac{x^m}{m!}\frac{y^{m+n}}{(m+n)!}$$

$$-\sum_{n=0}^{\infty}(\alpha)_n(\alpha')_n\frac{x^n}{n!}\frac{y^n}{n!}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{-m}(\alpha')_{2m+3n}\frac{x^{m+2n}}{(m+2n)}\frac{y^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_n(\alpha')_{3m+n}\frac{x^{2m+n}}{(2m+n)!}\frac{y^{m+n}}{(m+n)!}-\sum_{n=0}^{\infty}(\alpha')_{3n}\frac{x^{2n}}{(2n)!}\frac{y^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m+3n}(\alpha')_m\frac{x^{m+n}}{(m+n)!}\frac{y^{m+2n}}{(m+2n)!}-\sum_{n=0}^{\infty}(\alpha)_{3n}\frac{x^n}{n!}\frac{y^{2n}}{(2n)!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{3m+2n}(\alpha')_{-n}\frac{x^m}{m!}\frac{y^{2m+n}}{(2m+n)!}-\sum_{n=0}^{\infty}(\alpha)_n(\alpha')_n\frac{x^n}{n!}\frac{y^n}{n!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{3m+2n}(1)_n}{(1)_{2m+n}(1-\alpha')_n}\frac{(xy^2)^m}{m!}\frac{(-y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha')_{3n}}{(1)_{2n}}\frac{(x^2y)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+3n}(\alpha')_m(1)_m(1)_n}{(1)_{m+2n}(1)_{m+n}}\frac{(xy)^m}{m!}\frac{(xy^2)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_{3n}}{(1)_{2n}}\frac{(xy^2)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha')_{2m+3n}(1)_m}{(1)_{m+2n}(1-\alpha)_m}\frac{(-x)^m}{m!}\frac{(x^2y)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha')_n}{(1)_n}\frac{(xy)^n}{n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha')_{3m+n}(1)_m(1)_n(\alpha)_n}{(1)_{m+n}(1)_{2m+n}}\frac{(x^2y)^m}{m!}\frac{(xy)^n}{n!}$$

Appendix B

Removing the negative sign from the Pochhammer symbols in the triple hypergeometric series (11.c) and (36.h) in Srivastava's book

1. Remove the negative sign from the triple hypergeometric series (11.c) in Srivastava's book

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_p(b)_{m-p}}{(c)_{m+n}}\frac{x^m y^n z^p}{m!n!p!}$$

$$=\sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_{n+p}(a_3)_p(b)_m}{(c)_{m+n+p}}\frac{x^{m+p}y^n z^p}{(m+p)!n!p!}\right.$$

$$+\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{m+n+p}(a_3)_{m+p}(b)_{-p}}{(c)_{m+n}}\frac{x^m y^n (z)^{m+p}}{m!n!(m+p)!}$$

$$\left.-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{m+n}(a_3)_m}{(c)_{m+n}}\frac{x^m y^n z^m}{m!n!m!}\right]$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_{n+p}(1)_m(b)_m(a_3)_p}{(c)_{m+n+p}(1)_{m+p}}\frac{x^m y^n (xz)^p}{m!n!p!}$$

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{m+n}(a_3)_m}{(c)_{m+n}(1)_m}\frac{(xz)^m y^n}{m!n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{m+n+p}(a_3)_{m+p}(1)_p}{(c)_{m+n}(1)_{m+p}(1-b)_p}\frac{(xz)^m y^n (-z)^p}{m!n!p!}$$

2. Remove the negative sign from the triple hypergeometric series (36.h) in Srivastava's book

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{n+p}(b_1)_{2m+n-p}(b_2)_{p-n}}{(c)_m}\frac{x^m y^n z^p}{m!n!p!}$$

$$=\sum_{m=0}^{\infty}\left[\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{n+2p}(b_1)_{2m+n}(b_2)_{-n}}{(c)_m}\frac{x^m}{m!}\frac{y^{n+p}}{(n+p)!}\frac{z^p}{p!}\right.$$

$$\left.+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2n+p}(b_1)_{2m-p}(b_2)_p}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^{p+n}}{(p+n)!}-\sum_{n=0}^{\infty}\frac{(a)_{2n}(b_1)_{2m}}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^n}{n!}\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{n+2p}(b_1)_{2m+n}(b_2)_{-n}}{(c)_m} \frac{x^m}{m!} \frac{y^{n+p}}{(n+p)!} \frac{z^p}{p!}$$

$$- \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2n}(b_1)_{2m}}{(c)_m} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^n}{n!}$$

$$+ \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{2n+p}(b_1)_{2m+p}(b_2)_p}{(c)_{m+p}} \frac{x^{m+p}}{(m+p)!} \frac{y^n}{n!} \frac{z^{n+p}}{(n+p)!} \right.$$

$$- \sum_{m=0}^{\infty} \frac{(a)_{m+2n}(b_1)_m(b_2)_m}{(c)_m} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^{m+n}}{(m+n)!}$$

$$\left. + \sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{m+2n+p}(b_1)_{m-p}(b_2)_{p+m}}{(c)_m} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^{m+n+p}}{(m+n+p)!} \right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{n+2p}(b_1)_{2m+n}(b_2)_{-n}}{(c)_m} \frac{x^m}{m!} \frac{y^{n+p}}{(n+p)!} \frac{z^p}{p!}$$

$$- \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+2n}(b_1)_m(b_2)_m}{(c)_m} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^{m+n}}{(m+n)!}$$

$$+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{2n+p}(b_1)_{2m+p}(b_2)_p}{(c)_{m+p}} \frac{x^{m+p}}{(m+p)!} \frac{y^n}{n!} \frac{z^{n+p}}{(n+p)!}$$

$$- \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2n}(b_1)_{2m}}{(c)_m} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^n}{n!}$$

$$+ \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{m+2n+2p}(b_1)_m(b_2)_{m+2p}}{(c)_{m+p}} \frac{z^{m+n+2p}}{(m+n+2p)!} \frac{x^{m+p}}{(m+p)!} \frac{y^n}{n!} \right.$$

$$+ \sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{2m+2n+p}(b_1)_{-p}(b_2)_{2m+p}}{(c)_m} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^{2m+n}}{(2m+n)!}$$

$$\left. - \sum_{m=0}^{\infty} \frac{(a)_{2m+2n}(b_2)_{2m}}{(c)_m} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^{2m+n}}{(2m+n)!} \right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{n+2p}(b_1)_{2m+n}(1)_n}{(1)_{n+p}(c)_m(1-b_2)_n} \frac{x^m(-y)^n(yz)^p}{m!n!p!}$$

$$- \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+2n}(b_1)_m(b_2)_m}{(1)_{m+n}(c)_m} \frac{(xz)^m(yz)^n}{m!n!}$$

$$- \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2m+2n}(b_2)_{2m}}{(1)_{2m+n}(c)_m} \frac{(xz^2)^m(yz)^n}{m!n!} - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(b_1)_{2m}(a)_{2n}}{(c)_m(1)_n} \frac{(x)^m(yz)^n}{m!n!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2n+p}(b_2)_{2m+p}(1)_p}{(1)_{2m+n+p}(c)_m(1-b_1)_p}\frac{(xz^2)^m(yz)^n(-z)^p}{m!n!p!}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2n+p}(b_1)_{2m+p}(1)_m(b_2)_p(1)_p}{(c)_{m+p}(1)_{m+p}(1)_{n+p}}\frac{x^m(yz)^n(xz)^p}{m!n!p!}$$

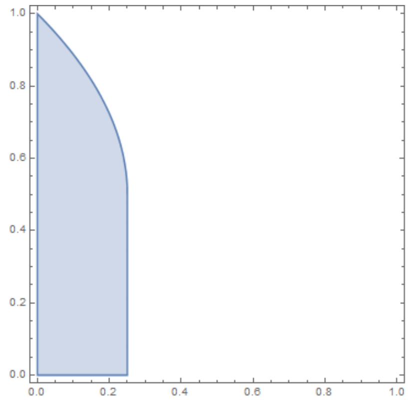
$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{m+2n+2p}(b_2)_{m+2p}(b_1)_m(1)_m(1)_p}{(c)_{m+p}(1)_{m+p}(1)_{m+n+2p}}\frac{(xz)^m(yz)^n(xz^2)^p}{m!n!p!}$$

Appendix C

Radius of convergence for Horn series G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_5, H_6 and H_7

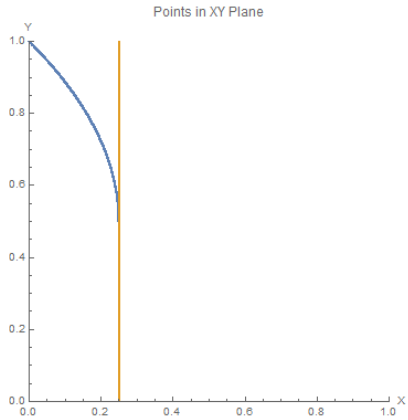
Figure 11

Radius of convergence for double hypergeometric series H_3



(a)

Radius of convergence for double hypergeometric series H_3 using (4.41)

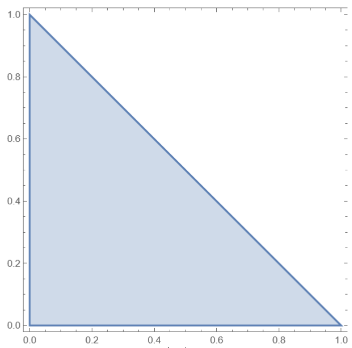


(b)

Boundaries that determined the radius of convergence for double hypergeometric series H_3 using (4.39)

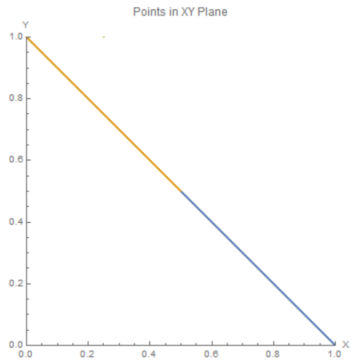
Figure 12

Radius of convergence for double hypergeometric series G_1



(a)

Radius of convergence for double hypergeometric series G_1 using (4.57) *and* $C = k[1, 1]$



(b)

Boundaries that determine the radius of convergence for double hypergeometric series G_1 using (4.48), (4.54), $xy = \frac{1}{4}$ *and* $C = k[1, 1]$

Figure 13

Radius of convergence for double hypergeometric series G_2 using (4.64) *and* (4.66)

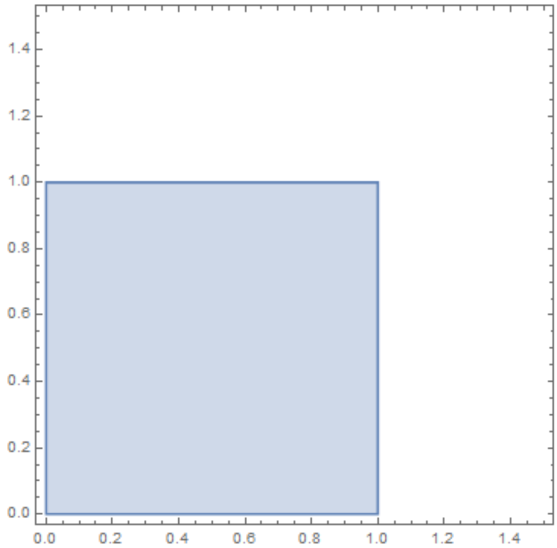
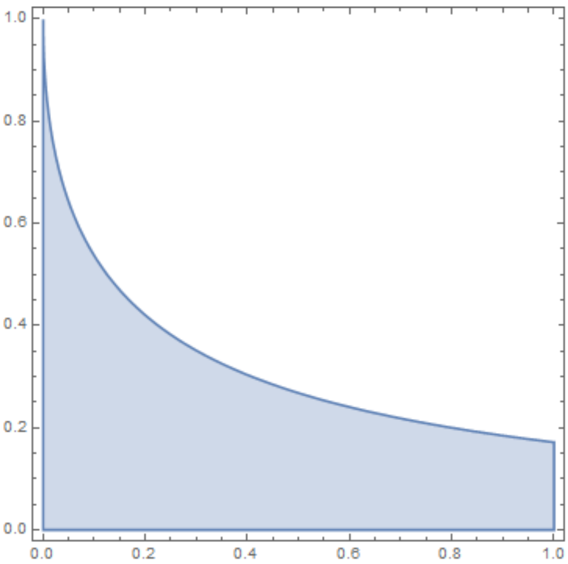
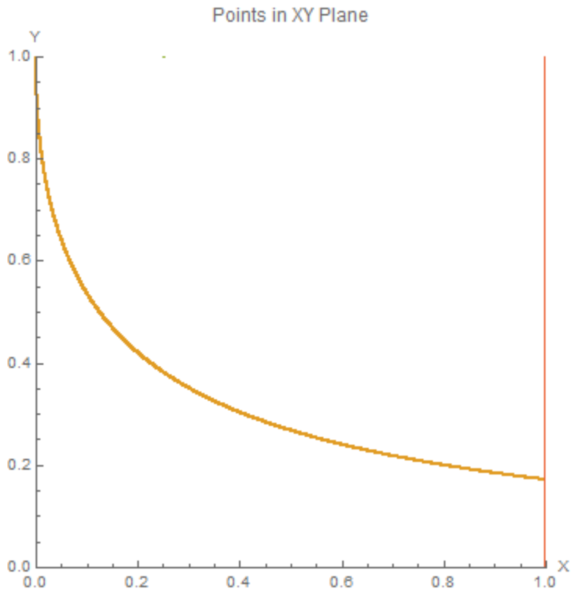


Figure 14

Radius of convergence for double hypergeometric series H_1





(b)

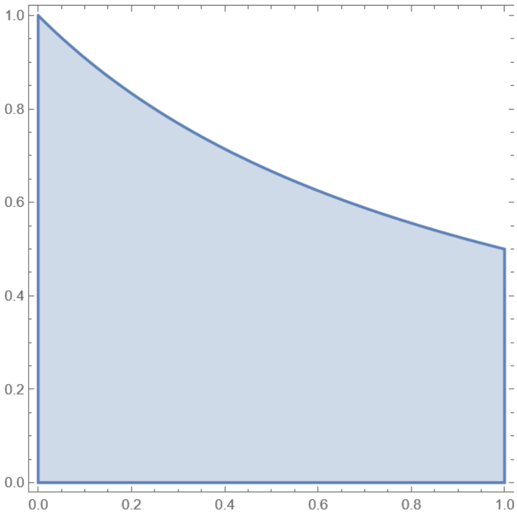
Boundaries that determine the radius of convergence for double hypergeometric series H_1 using (4.70), (4.71), (4.73), (4.74), $xy = \frac{1}{4}$ *and* $C = K[1, 1]$

(a)

Radius of convergence for double hypergeometric series H_1 using (4.72), (4.75), $xy < \frac{1}{4}$ *and* $C = K[1, 1]$

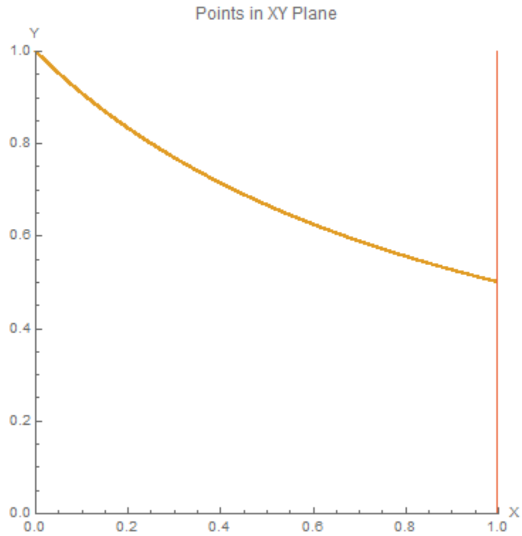
Figure 15

Radius of convergence for double hypergeometric series H_2



(a)

Radius of convergence for double hypergeometric series H_2 using (4.80), (4.83), $xy < 1$ *and* $C = K[1, 1]$

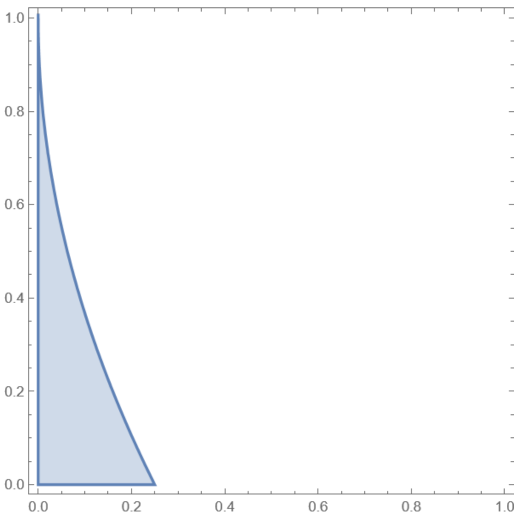


(b)

Boundaries that determine the radius of convergence for double hypergeometric series H_2 using (4.78), (4.79), (4.81), (4.82), $xy = 1$ *and* $C = K[1, 1]$

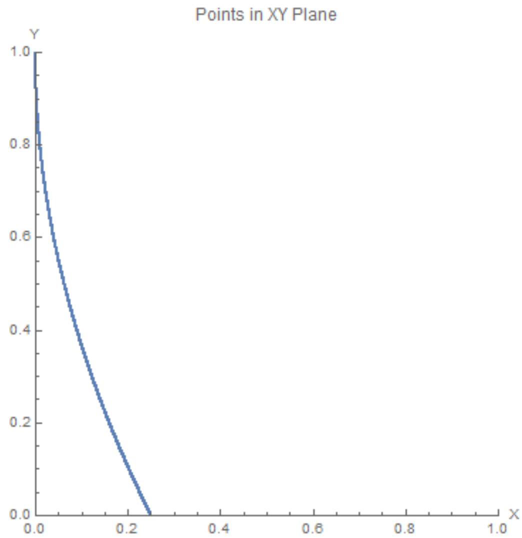
Figure 16

Radius of convergence for double hypergeometric series H_4



(a)

Radius of convergence for double hypergeometric series H_4 using (4.88) *and* $C = K[\frac{1}{4}, 1]$

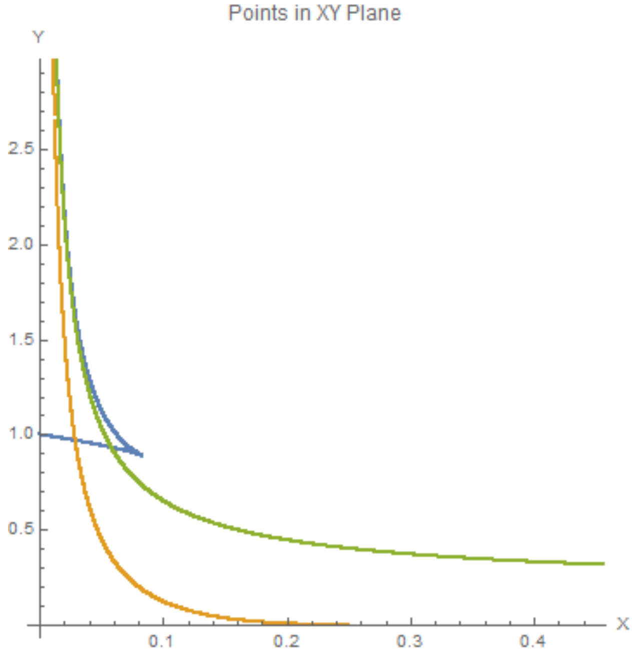


(b)

Boundaries that determine the radius of convergence for double hypergeometric series H_4 using (4.87) *and* $C = K[\frac{1}{4}, 1]$

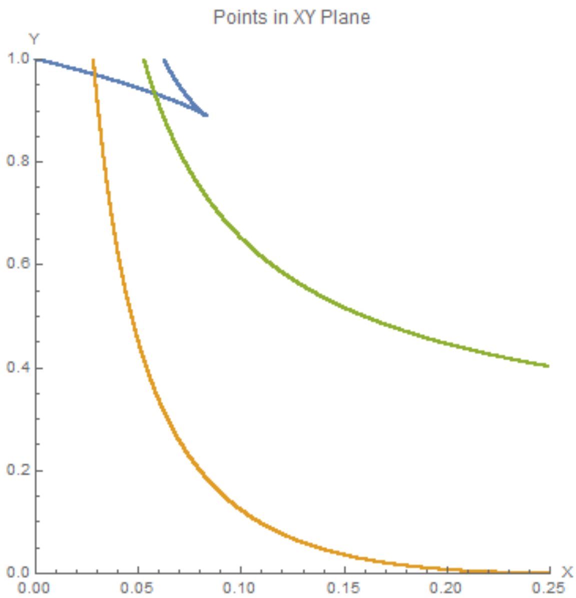
Figure 17

Radius of convergence for double hypergeometric series H_5



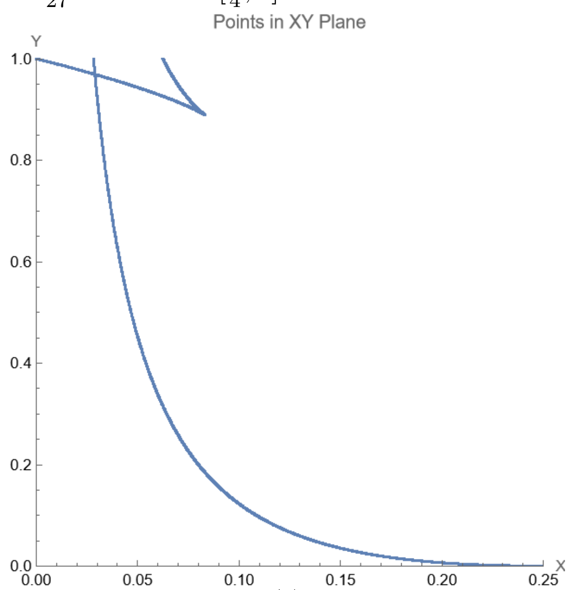
(a)

Boundaries that determine the radius of convergence for double hypergeometric series H_5 using (4.91), (4.92), (4.93), (4.94), $xy < \frac{1}{27}$ *and* $C = K[\frac{1}{4}, 1]$



(b)

Zoom in on Figure (17a) *on the region* $0 < x < \frac{1}{4}$ *and* $0 < y < 1$

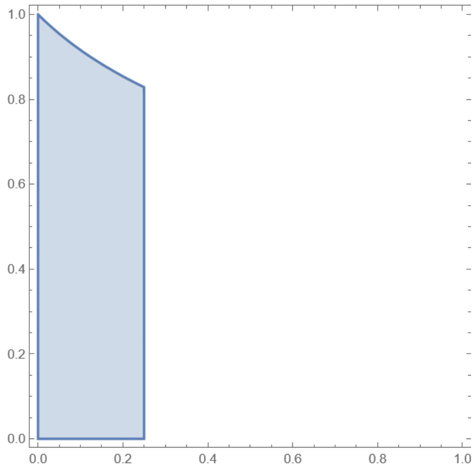


(c)

Radius of convergence for double hypergeometric series H_5 using Horn's theorem

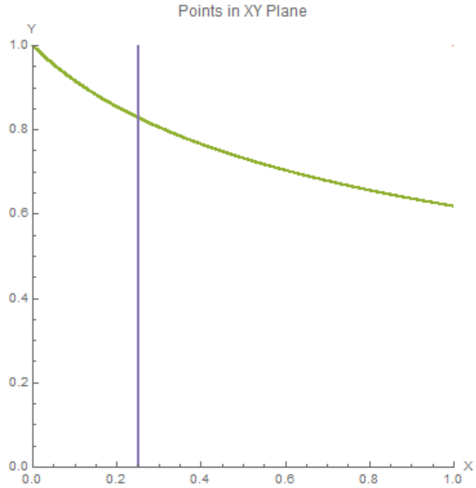
Figure 18

Radius of convergence for double hypergeometric series H_6



(a)

Radius of convergence for double hypergeometric series H_6 using (4.101), (4.104), (4.107), $xy < 1$, $xy^2 < 1$ *and* $C = K[\frac{1}{4}, 1]$

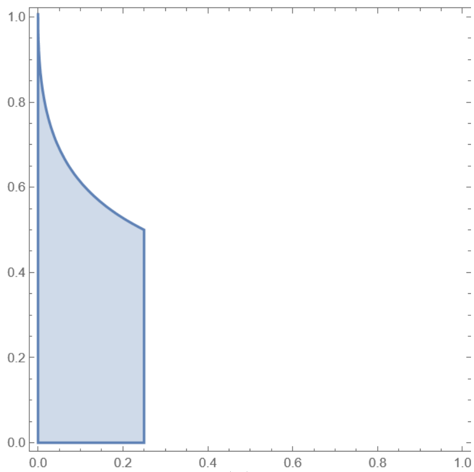


(b)

Boundaries that determine the radius of convergence for double hypergeometric series H_6 using (4.99), (4.100), (4.102), (4.103), (4.105), (4.106), $xy = 1$, $xy^2 = 1$ *and* $C = K[\frac{1}{4}, 1]$

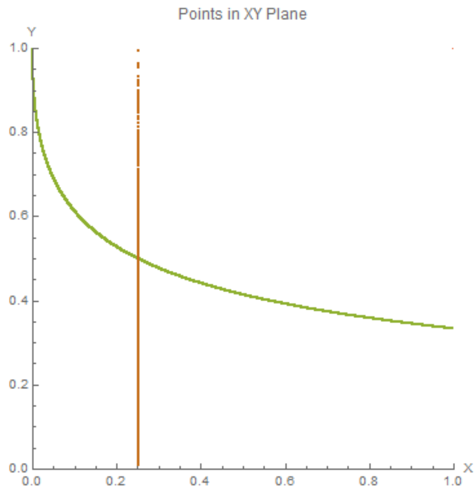
Figure 19

Radius of convergence for double hypergeometric series H_7



(a)

Radius of convergence for double hypergeometric series H_7 using (4.113), (4.116), (4.119), $xy < 1$, $xy^2 < \frac{1}{4}$ *and* $C = K[\frac{1}{4}, 1]$

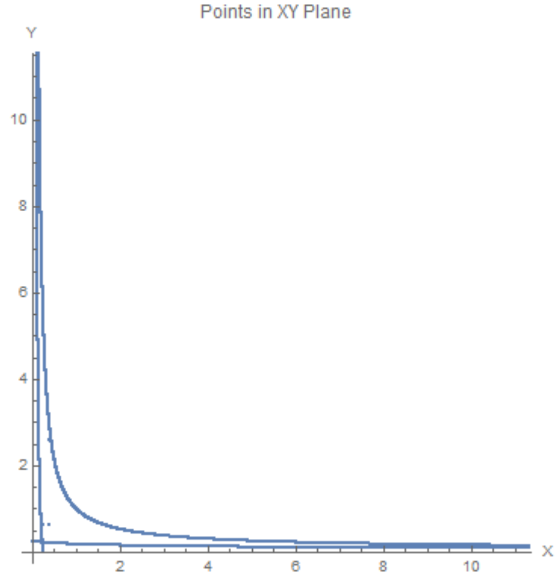


(b)

Boundaries that determine the radius of convergence for double hypergeometric series H_7 using (4.111), (4.112), (4.114), (4.115), (4.117), (4.118), $xy = 1$, $xy^2 = \frac{1}{4}$ *and* $C = K[\frac{1}{4}, 1]$

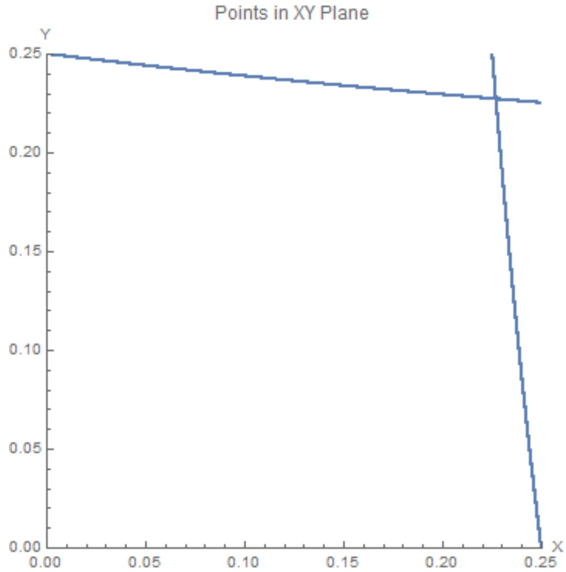
Figure 20

Radius of convergence for double hypergeometric series G_3



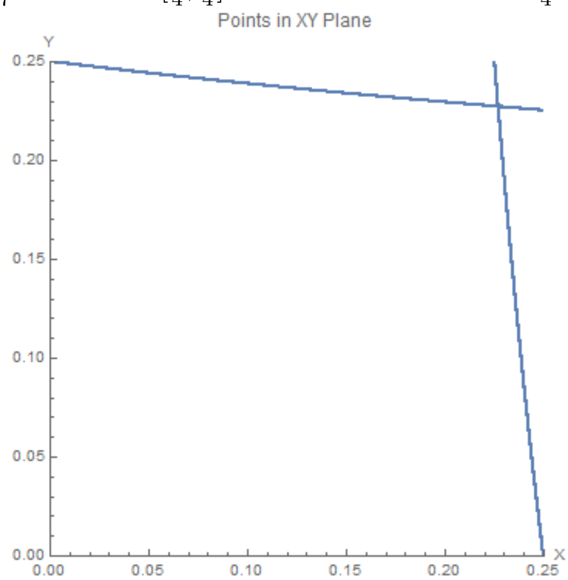
(a)

Boundaries that determine the radius of convergence for double hypergeometric series G_3 using (4.123), (4.124), (4.125), (4.126), (4.127), (4.128), (4.129), (4.130), $xy < 1$, $xy^2 < \frac{4}{27}$, $x^2y < \frac{4}{27}$ *and* $C = K[\frac{1}{4}, \frac{1}{4}]$



(b)

Zoom in on Figure (20b) *on the region* $0 < x < \frac{1}{4}$ *and* $0 < y < \frac{1}{4}$



(c)

Boundaries that determine the radius of convergence for double hypergeometric series G_3 using Horn's theorem

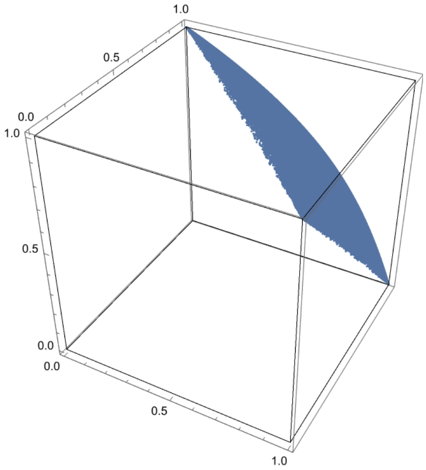
Appendix D

Comparison between our results and Srivastava's ones for triple hypergeometric functions H_c, (11.c) and (36.h)

1. Radius of convergence for triple hypergeometric functions H_c

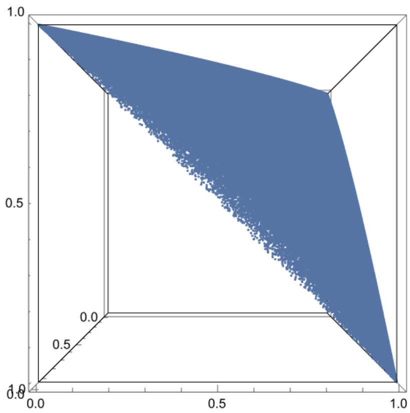
Figure 21

Boundaries that determine the radius of convergence for triple hypergeometric series H_c



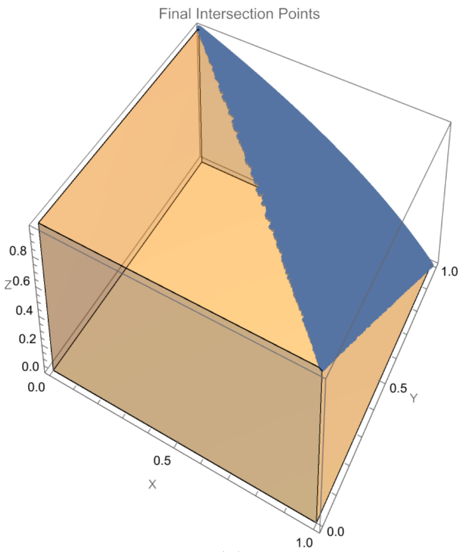
(a)

Front view of the boundaries that determine the radius of convergence for H_c



(b)

Top view of the boundaries that determine the radius of convergence for H_c



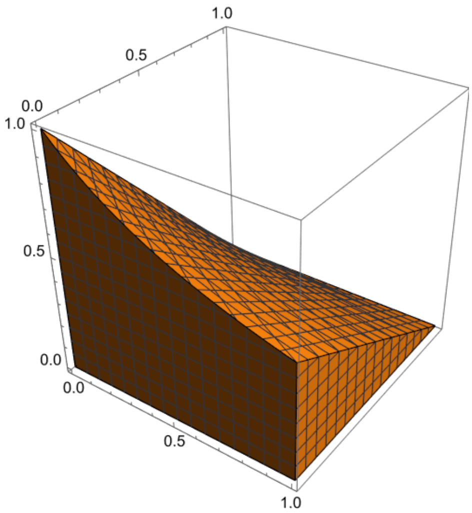
(c)

Boundaries that determine the radius of convergence for H_c and figure (4.134) *an the same box*

2. Radius of convergence for triple hypergeometric functions (11.c)

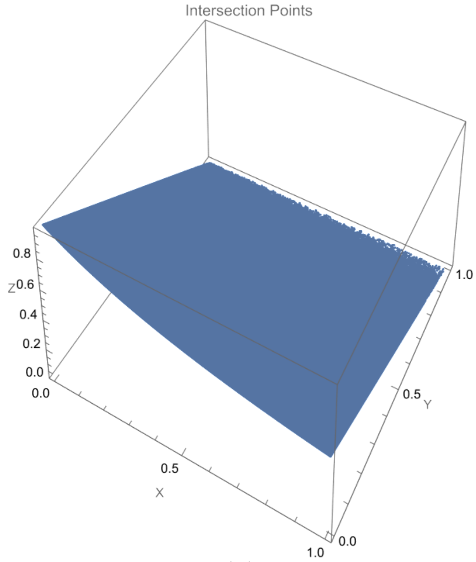
Figure 22

Radius of convergence for S_2 in triple hypergeometric series (11.c)



(a)

Radius of convergence for S_2 in triple hypergeometric series (11.c)

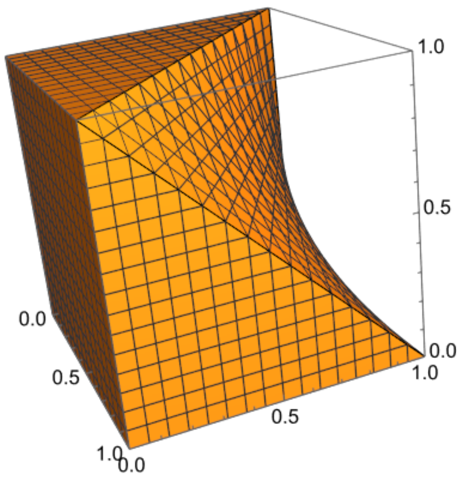


(b)

Boundaries that determine the radius of convergence for S_2 in triple hypergeometric series (11.c)

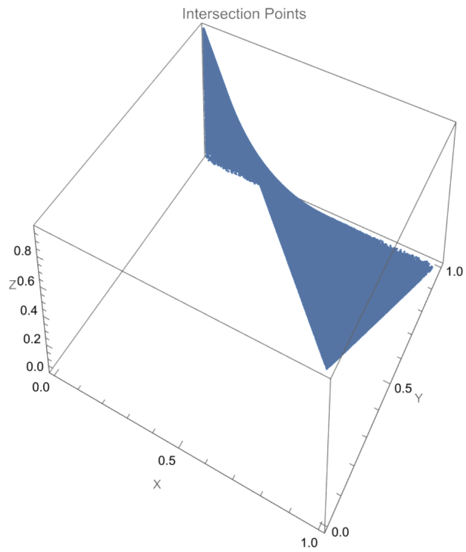
Figure 23

Radius of convergence for S_3 in triple hypergeometric series (11.c)



(a)

Radius of convergence for S_3 in triple hypergeometric series (11.c)

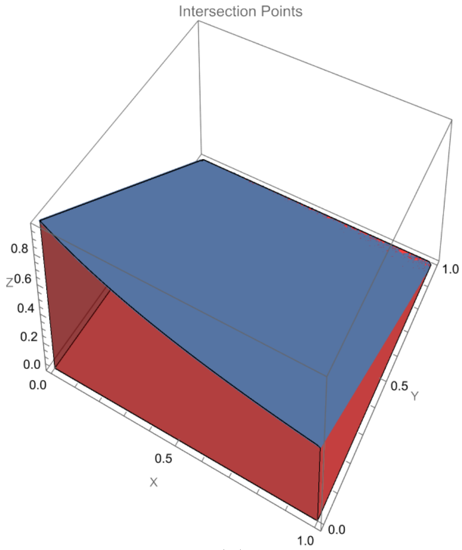


(b)

Boundaries that determine the radius of convergence for S_3 in triple hypergeometric series (11.c)

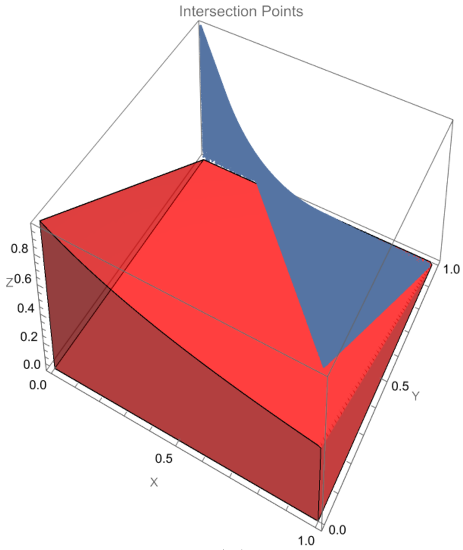
Figure 24

Radius of convergence for triple hypergeometric function (11.c)



(a)

Boundaries that determine the radius of convergence for S_2 and figure (9) on the same box.



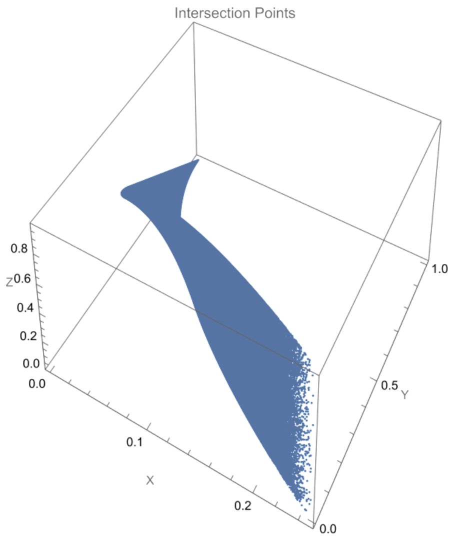
(b)

Boundaries that determine the radius of convergence for S_3 and figure (9) on the same box.

3. Radius of convergence for triple hypergeometric functions (36.h)

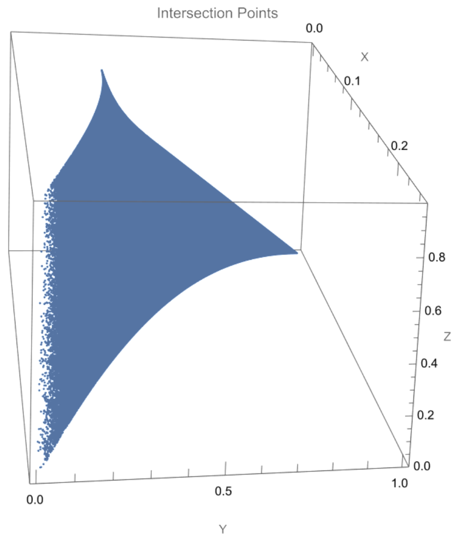
Figure 25

Boundaries that determine the radius of convergence for S_1 in triple hypergeometric series (36.h)



(a)

First view for the boundaries that determine the radius of convergence for S_1 in triple hypergeometric series (36.h)

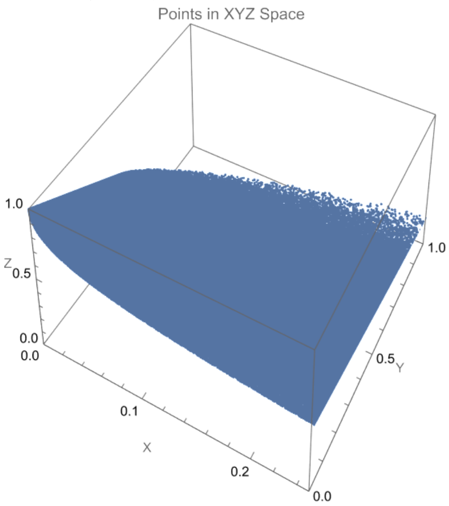


(b)

Second view for the boundaries that determine the radius of convergence for S_1 in triple hypergeometric series (36.h)

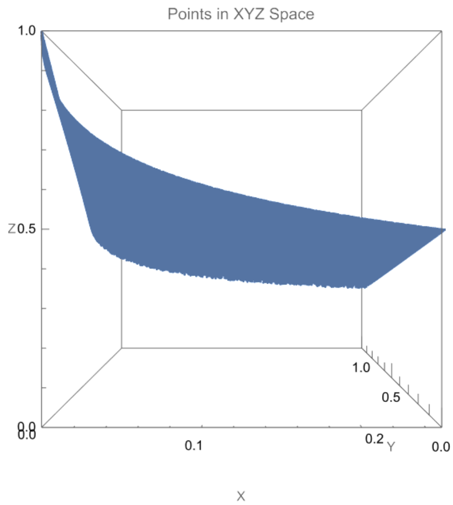
Figure 26

Boundaries that determine the radius of convergence for S_4 in triple hypergeometric series (36.h)



(a)

First view for the boundaries that determine the radius of convergence for S_4 in triple hypergeometric series (36.h)

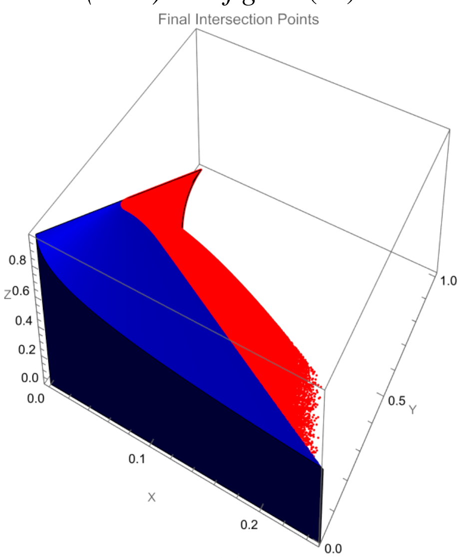


(b)

Second view for the boundaries that determine the radius of convergence for S_4 in triple hypergeometric series (36.h)

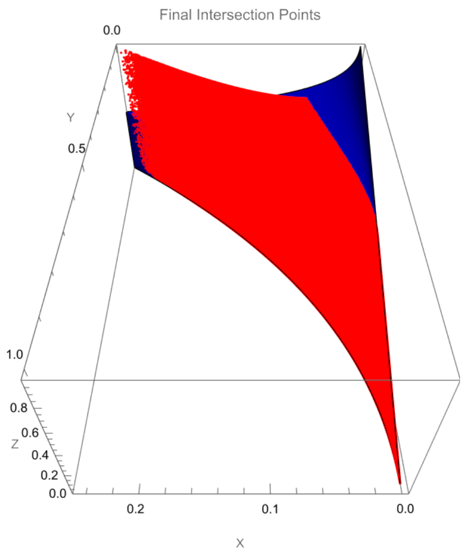
Figure 27

Boundaries that determine the radius of convergence for S_1, S_7 in triple hypergeometric series (36.h) and figure (10) *on the same box.*



(a)

First view for the boundaries that determine the radius of convergence for S_1, S_7 in triple hypergeometric series (36.h) and relation (4.144) *in the same box.*

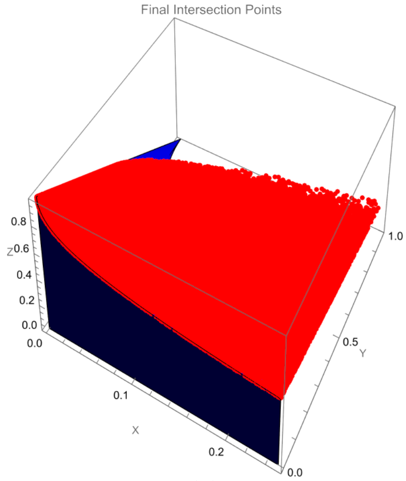


(b)

Second view for the boundaries that determine the radius of convergence for S_1, S_7 in triple hypergeometric series (36.h) and relation (4.144) *in the same box.*

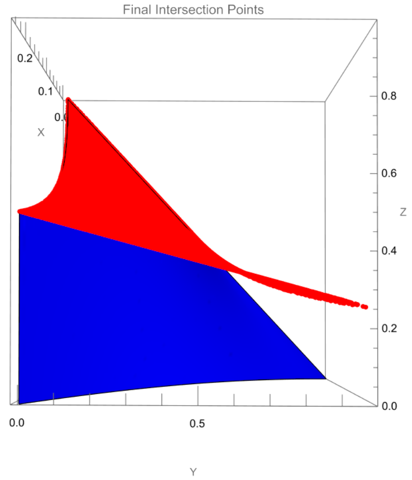
Figure 28

Boundaries that determine the radius of convergence for S_4, S_7 in triple hypergeometric series (36.h) and figure (10) *on the same box.*



(a)

First view for the boundaries that determine the radius of convergence for S_4, S_7 in triple hypergeometric series (36.h) and relation (4.144) *in the same box.*

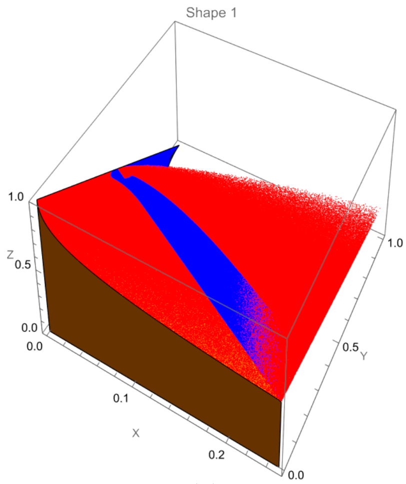


(b)

Second view for the boundaries that determine the radius of convergence for S_4, S_7 in triple hypergeometric series (36.h) and relation (4.144) *in the same box.*

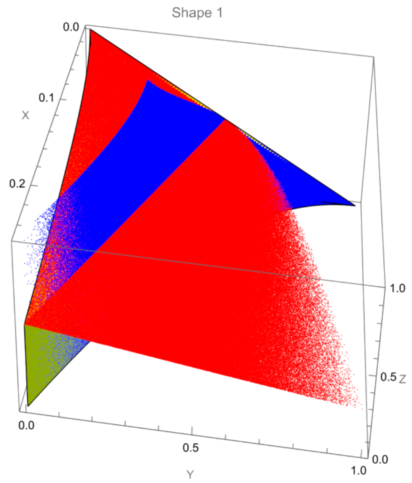
Figure 29

Boundaries that determine the radius of convergence for S_1, S_4, S_7 in triple hypergeometric series (36.h) and figure (10) *on the same box.*



(a)

First view for the boundaries that determine the radius of convergence for S_1, S_4, S_7 in triple hypergeometric series (36.h) and relation (4.144) *in the same box.*



(b)

Second view for the boundaries that determine the radius of convergence for S_1, S_4, S_7 in triple hypergeometric series (36.h) and relation (4.144) *in the same box.*

جـامعـــة النجاح الوطنية

كليـــة الدراســـات العليـا

نصف قطر التقارب لبعض الدوال الهندسية الفوقية (متعددة المتغيرات)

إعداد

ليث قشوع

إشراف

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قدمت هذه الرسالة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات،
من كلية الدراسات العليا، في جامعة النجاح الوطنية،
نابلس – فلسطين.

2024

نصف قطر التقارب لبعض الدوال الهندسية الفوقية
(متعددة المتغيرات)

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الملخص

الخلفية: الدوال الهندسية الفائقة هي فئة من الدوال الخاصة في الرياضيات التي تلعب دورًا حاسمًا في مختلف فروع العلوم والهندسة. وتكمن أهميتها في تنوعها وقدرتها على تمثيل مجموعة واسعة من الظواهر الرياضية والفيزيائية. تشمل الجوانب الرئيسية التي تؤكد أهميتها وتطبيقاتها حلول المعادلات التفاضلية، وحل معادلة شرودنغر لمختلف الأنظمة الفيزيائية، ودراسة التكاملات المعقدة والتكاملات الكنتورية، وحل المشكلات التي تنطوي على المجالات الكهرومغناطيسية وانتشار الموجات في الوسائط المختلفة ولها تطبيقات في الميكانيكا السماوية للتنبؤ بمواقع ومدارات الأجرام السماوية.

الأهداف: في هذه الأطروحة لدينا هدفان رئيسيان. الأول هو اشتقاق صيغ تحويل جديدة لدالة كامبي دي فيريوت مع الأخذ في الاعتبار نصف قطر التقارب لكل تحويل. بينما يقوم الهدف الآخر على تطوير طرق بديلة لتحديد نصف قطر التقارب للمتسلسلات الهندسية الفوقية متعددة المتغيرات (الثنائية والثلاثية).

الطرق: لتحقيق الأهداف التي نريدها، سوف نستخدم صيغ تحويل ميلر-باريس للدوال الهندسية الفوقية المعممة ونصف قطر تقارب كل منها لاشتقاق صيغ تحويل جديدة لدالة كامبي دي فيريوت. وأيضًا، سوف نستخدم برنامج ماثيماتيكا لتطوير طريقة تحدد نصف قطر التقارب لبعض المتسلسلات الهندسية الفائقة متعددة المتغيرات.

النتائج: أثناء اختبار تحويلات دالة كامبي دي فيريوت التي قمنا باشتقاقها، وجدنا أن الطرفان الأيسر الأيمن متساويان في جميع الحالات التي تم اختبارها. أيضًا، عندما حاولنا تحديد نصف قطر التقارب للمتسلسلات الهندسية الفوقية متعددة المتغيرات من خلال رسمها على برنامج ماثيماتيكا، تشير النتائج إلى أن النتائج التي توصلنا إليها متطابقة مع تلك المقدمة في كتاب سريفاستافا.

الكلمات المفتاحية: الدوال فوق الهندسية، دالة كامبي دي فيريوت، نصف قطر التقارب، تحويلات ميلر-باريس، متسلسلة سريفاستافا - داوست، متسلسلة هورن، الدوال الهندسية الفائقة لأبيل، الدوال الهندسية الفائقة الغاوسية، تكامل ميلين-بارنز.