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Numerical Methods for Solving Nonlinear Fredholm Integral Equations

By

Hiba Jalal Mahmoud Odeh

Supervisor

Prof. Naji Qatanani

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This Thesis was defended successfully on 14/11/2019 and approved by:

Defense Committee Members	**Signature**
Prof. Naji Qatanani / Supervisor	**……**
Prof. Maher Qarawani / External examiner	**……**
Dr. Mohammad Yassin / Internal examiner	**……**

Dedication

To my parents Jalal and Nabeela Odeh, to my lovely husband Ghassan Estaitih, who have always responded to my side and supported me a lot, who gave me enough support to complete this work.

To my honey daughter Hala Estaitih, to all my brothers and sisters, to my parents in law, to all my brothers and sisters in law, to my friends, who stood by me, encouraged me and believed in me .

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الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

Numerical Methods for Solving Nonlinear Fredholm Integral Equations

أقر بأن ما اشتملت عليه هذه الرسالة إنما هي نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد، وأن هذا الرسالة ككل أو أي جزء منها لم يقدم من قبل لنيل أي درجة علمية أو بحث علمي لدى أي مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any degree or qualification.

Student's Name: **اسم الطالب:**

Signature: **التوقيع:**

Date: **التاريخ:**

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List of symbols

$A \times B$	The set $\{(a, b): a \in A, b \in B\}$, where A and B are nonempty sets
R	The set of all real numbers
R_+	The set of all nonnegative real numbers, $R_+ = [0, \infty)$
R^n	The space of n dimensional vectors whose components are real numbers, $R^n = R \times \ldots \times R, n - times$
N	The set of nonnegative integers
$f: A \to B$	A mapping from A to B, where A and B are arbitrary sets
$C(A, B)$	The space of continuous functions from A to B
$\|f\|$	$max\{\|f(x)\|: x \in [a, b]\} = max_{x \in [a,b]}\{\|f(x)\|\}$, where $f \in C([a, b], R)$

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Numerical Methods for Solving Nonlinear Fredholm Integral Equations

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Hiba Jalal Mahmoud Odeh
Supervisor
Prof. Dr. Naji Qatanani

Abstract

In this thesis we focus on the numerical treatment of nonlinear Fredholm integral equation of the second kind due to their enormous importance in many applications in various fields.

After addressing the basic concepts of nonlinear Fredholm integral equation of the second kind, we focus on the numerical treatment of this equation. This will be accomplished by implementing two numerical methods, namely, Haar Wavelet method and Homotopy Analysis method (HAM). The mathematical framework of these numerical methods will be presented.

These numerical methods will be illustrated by solving some numerical examples with known exact solutions.

Numerical results show clearly that the Homotopy analysis method is more effective in solving nonlinear Fredholm integral equations in comparison with its counter parts.

Introduction

The subject of integral equations is one of the most important mathematical tools in both pure and applied mathematics. Integral equations play a very important role in modern science such as numerous problems in engineering and mechanics. They have attracted the attention of many scientists and researchers in recent years. In fact, many physical problems are modeled in the form of linear and nonlinear integral equations specially Fredholm integral equation. Fredholm integral equation has received considerable attention in recent years, as in a potential theory and Dirichlet problem [3] and [36], electrostatics [34], mathematical problems of radiative equilibrium [19], the particle transport problems of astrophysics and reactor theory [22], and radiative heat transfer problems [40].

Various numerical methods for solving Fredholm integral equations have been developed by many researchers. In [6] Babolian and Shahsavaran proposed a method for solving nonlinear Fredholm integral equations of the second kind based on the Haar wavelets. Ahmad Shahsavaran and Akbar Shahsavaran [44] proposed a method based on Lagrange interpolations for solving nonlinear Fredholm integral equations of the power function type. In [13] Jafari Emamzadeh approximated the solution of the nonlinear Fredholm integral equation using Quadrature methods. Borzabadi and Fard [9] introduced an approach via optimization methods to find approximate solutions for nonlinear Fredholm integral

equations of the first kind. In [42], some numerical methods were used for solving nonlinear Fredholm integral equation of the second kind as: The Direct Computation method, The Series solution method, The Adomian Decomposition method, and The Successive Approximations method.

In this work, some numerical methods for solving the nonlinear Fredholm integral equation of the second kind will be investigated. These methods are: Haar wavelet method and Homotopy analysis method (HAM).

The Haar wavelet method is one of the most important numerical method which was used in recent years. Alfred Haar was the first pioneer who proposed the Haar wavelet method. Many types of wavelets were used for solving integral equations like Daubechies [46], Hermit-type trigonometric [10], Walsh functions [43], Cohen [30], and Albert [21] wavelets. Ulo Lepik and Enn Tamme in [26] used Haar wavelet method to solve the integral equations. Then in [41], Reihani and Abadi proposed another method based on Haar wavelet for solving linear Fredholm and Volterra integral equations of the second kind. In [7], E. Babolian and A. Shahsavaran presented a numerical method for solving nonlinear Fredholm and Volterra integral equations of the second kind which is based on the use of Haar wavelets and collocation method. In [51], Mingxu Yi and Yiming Chen proposed a Haar wavelet operational matrix for solving Fractional Partial differential equations. In [17], G. Hariharan applied the Haar wavelet method for solving linear and nonlinear Klein-Gordon equations. Imran Aziz and Siraj-ul-Islam in [4] proposed a two new

algorithms based on Haar wavelets for solving nonlinear Fredholm and Volterra integral equations.

Homotopy analysis method is also an important numerical method which was used in recent years. In fact, Shijun Liao was the first to introduce the Homotopy analysis method for nonlinear problems in general in [31]. Hossein Zadeh, Jafari and Karimi in [20] used Homotopy analysis method for solving integral and integro-differential equations. In [32], Shijun Liao used Homotopy analysis method in nonlinear differential equations. Then in [2], Allahviranloo and Ghanbari introduced the discrete Homotopy analysis method for solving nonlinear Fredholm integral equations. In [18], Edyta Hetmaniok, Damian Słota, Tomasz Trawi´nski and Roman Wituła presented an application of the homotopy analysis method for solving the nonlinear and linear integral equations of the second kind.

This thesis is organized as follows: In chapter one, we introduce some basic concepts of integral equations and investigate the existence and uniqueness of the solution of the nonlinear Fredholm integral equation of the second kind. In chapter two, we present some numerical methods for solving the nonlinear Fredholm integral equation of the second kind. These include: Haar Wavelet method and Homotopy Analysis method (HAM). Numerical examples implementing the aforementioned numerical methods together with a comparison between the analytical and numerical results are presented in chapter three. Conclusions have been drawn.

Chapter One

Theoretical Background

Chapter One

Theoretical Background

Basic Definitions and Mathematical Introductions

1.1 Preliminaries

Definition (1.1) [49]: Integral Equation:

An integral equation is an equation in which the unknown function appears under an integral sign. The general form of an integral equation can be written as:

$$h(x)u(x) = g(x)f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt \qquad (1.1)$$

where $u(x)$ is unknown function called the solution of the integral equation, $\alpha(x)$ and $\beta(x)$ are limits of integration, λ is a nonzero constant parameter, $g(x)$ is a function determines the homogeneity of the equation, $h(x), f(x)$ and $k(x,t)$ are known functions and $k(x,t)$ is called the kernel or the nucleus of the integral equation.

1.1.1 Classification of integral equations

There are several types of integral equations. These are:

1. **Volterra Integral equation**

This equation has the general form:

$$h(x)u(x) = g(x)f(x) + \lambda \int_{a}^{\beta(x)} k(x,t)u(t)dt \qquad (1.2)$$

where the upper limit of integration is variable.

- If $h(x) = 0$, then equation (1.2) is called Volterra integral equation of the first kind.

- If $h(x) = 1$, then equation (1.2) is called Volterra integral equation of the second kind.

2. Fredholm integral equation

The general form of this type is:

$$h(x)u(x) = g(x)f(x) + \lambda \int_D k(x,t)u(t)dt \qquad (1.3)$$

where the limit of integration D is a closed bounded set in R.

- If $h(x) = 0$, then equation (1.3) is called Fredholm integral equation of the first kind.

- If $h(x) = 1$, then equation (1.3) is called Fredholm integral equation of the second kind.

3. Volterra-Fredholm integral equation

The standard form of this type is:

$$u(x) = f(x) + \int_0^x k_1(x,t)u(t)dt + \int_a^b k_2(x,t)u(t)dt \qquad (1.4)$$

where $k_1(x,t)$ and $k_2(x,t)$ are two given functions called kernels of the equation (1.4).

4. **Singular integral equation**

It is an integral equation in which the kernel becomes infinity in the domain of integration or when one or both limits of integration are infinite.

5. **Integro-differential equation**

The general form of this type is:

$$h(x)u^{(m)}(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt \qquad (1.5)$$

where $m = 1,2,\dots,n$ represents the m^{th} derivative of $u(x)$.

Equation (1.5) can be classified into either Fredholm integro-differential equation, Volterra integro-differential equation or Volterra-Fredholm integro-differential equation [3], [48].

1.1.2 Linearity concept of integral equations

Definition (1.2) [49]: If the exponent of the unknown function $u(x)$ inside the integral sign is one then the equation is called linear. On the other hand if the unknown function $u(x)$ has exponent other than one, or the equation contains nonlinear functions of $u(x)$ then the equation is called nonlinear.

1.1.3 Nonlinearity of Fredholm integral equation of the second kind

Nonlinear Fredholm integral equation of the second kind has many general forms, which depends on the presence of the unknown function $u(x)$ under the integral sign.

There are some well-known examples:

1. *Urysohn form*: the unknown function $u(x)$ is part of the kernel k, has the general form:

$$u(x) = f(x) + \lambda \int_a^b k\big(x, t, u(t)\big)dt \qquad a \le x \le b \tag{1.6}$$

where $f: [a, b] \to R$.

2. *Hammerstein form*: the unknown function $u(x)$ is separate from the kernel k, has the general form:

$$u(x) = f(x) + \lambda \int_a^b k(x, t)\emptyset(u(t))dt \qquad a \le x \le b \tag{1.7}$$

where $f: [a, b] \to R$, $\emptyset$ is a function depends on u. The following equations are examples of Hammerstein form:

i. Equation with power nonlinearity, has the form:

$$u(x) = f(x) + \lambda \int_a^b k(x, t)u^n(t)dt$$

ii. Equation with exponential nonlinearity, has the form:

$$u(x) = f(x) + \lambda \int_a^b k(x, t)\exp(\beta u(t))dt$$

iii. Equation with trigonometric nonlinearity, has the form:

$$u(x) = f(x) + \lambda \int_a^b k(x, t)\sin(\beta u(t))dt$$

$$u(x) = f(x) + \lambda \int_a^b k(x, t)\cos(\beta u(t))dt$$

$$u(x) = f(x) + \lambda \int_a^b k(x, t)\tan(\beta u(t))dt$$

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\cot(\beta u(t))dt$$

iv. Equation with logarithmic nonlinearity, has the form:

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\ln(\beta u(t))dt$$

v. Equation with hyperbolic nonlinearity, has the form:

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\sinh(\beta u(t))dt$$

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\cosh(\beta u(t))dt$$

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\tanh(\beta u(t))dt$$

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\coth(\beta u(t))dt$$

3. Other equations of general forms:

i. $u(x) = f(x) + \lambda \int_a^b k(|x,t|)\varphi(u(t))dt$

ii. $u(x) = f(x) + \lambda \int_a^b k(x,t)\varphi(t,u(t))dt$

iii. $u(x) = f(x) + \lambda \int_a^b u(xt)\varphi(t,u(t))dt$

iv. $u(x) = f(x) + \lambda \int_a^b u(x+\beta t)\varphi\big(t,u(t)\big)dt$

1.1.4 Homogeneity concept of integral equations

Definition (1.3) [49]: An integral equation is called homogeneous if $f(x)$ is identically zero. On the other hand if $f(x)$ is not identically zero then the equation is called nonhomogeneous.

1.1.5 n-dimensional integral equation

For the n-independent variables $X = (x_1, x_2, \dots, x_n)$, the n-dimensional integral equation has the general form:

$$h(X)u(X) = f(X) + \int_G k(X,S)u(S)\,dS \tag{1.8}$$

where $X, S \in R, G \subseteq R^n$.

1.2 Types of Kernels

Integral equations involve different types of kernels. These are the following:

1. **Degenerate Kernel (Separable kernel)**

Definition (1.4): Separable kernel [37]: The kernel $k(x,y)$ is called degenerate kernel if it has the form:

$$k(x,y) = \sum_{i=0}^{n} X_i(x)Y_i(y) \tag{1.9}$$

2. **Symmetric (Hermitian) kernel**

Definition (1.5): Symmetric kernel [37]: The real kernel is symmetric if:

$$k(x,y) = k(y,x) \tag{1.10}$$

3. **Cauchy kernel**

Definition (1.6): Cauchy kernel: The form of this kernel is:

$$k(x,y) = \frac{H(x,y)}{x-y} \tag{1.11}$$

where $H(x,y)$ is a differentiable function of (x,y) with

$H(x,y) \neq 0$.

4. **Hilbert kernel**

Definition (1.7): Hilbert kernel: The form of this kernel is:

$$k(x,y) = \cot\frac{x-y}{2}, \qquad x \geq 0, y \leq 2\pi \tag{1.12}$$

Hilbert kernel has a relation with Cauchy kernel with the following simple relation in the case of the unit circle:

$$\frac{dt}{t-\tau} = \frac{1}{2}(\cot\frac{x-y}{2} + i)dx$$

where $t = e^{ix}, \tau = e^{iy}$.

5. **Skew-symmetric kernel**

Definition (1.8): Skew-symmetric kernel: It is of the form:

$$k(x,y) = -k(y,x) \tag{1.13}$$

1.3 Overview of spaces

Definition (1.9) [47]: Lipschitz condition: Let f be a function such that $f: R \to R$ then f is called a Lipschitz-continuous function on R if there exists a non-negative constant M such that:

$$|f(x) - f(y)| \leq M|x - y|, \qquad x, y \in R \tag{1.14}$$

M is constant called a Lipschitz constant for the function f on R.

Definition (1.10) [11]: Fixed point: A point $\xi \in X$ is a fixed point of $F: X \to X$ *if and only if* $F(\xi) = \xi$.

- **Metric space:**

Definition (1.11): Let X be a nonempty set, and a metric d such that $d: X \times X \to R_+$, then (X, d) is called a metric space if d satisfies the following conditions:

$d(x, y) = 0 \Leftrightarrow x = y,$ (Positive definiteness)

$d(x, y) = d(y, x),$ (Symmetry)

$d(x, y) \leq d(x, z) + d(z, y),$ (Triangle inequality)

where $x, y, z \in X$.

Definition (1.12): Let X be a metric space, then a mapping $F: X \to X$ is called contraction *if and only if* there exist a constant $k \in (0,1)$ such that for all $x_1, x_2 \in X$, F satisfies:

$$d(F(x_1), F(x_2)) \leq kd(x_1, x_2) \tag{1.15}$$

- **Theorem (1.1): Contraction Mapping Theorem:**

Suppose that:

i. X is a nonempty metric space

ii. $F: X \to X$ is a contraction

iii. The sequence $\{x_n\}_{n=0}^{\infty}$ is defined by:

$$x_{n+1} = F(x_n)$$

where $x_0 \in X$ is arbitrary, then F has a unique fixed point $\xi \in X$ and $\lim_{n\to\infty} d(x_n, \xi) = 0$.

- **Vector space (linear space):**

Definition (1.13):Vector space: Let X be a nonempty set of vectors element, and F is a field, then a vector space over F is a nonempty set X that satisfy two algebraic operations: vector addition and multiplication of vectors such that:

1. *Vector addition:*

i. If $x, y \in X \; then \; x + y \in X$

ii. $x + y = y + x$, addition is commutative.

iii. $x + (y + z) = (x + y) + z$, addition is associative.

iv. $0 + x = x + 0 = x$, exist of identity element of addition.

v. $x + (-x) = (-x) + x = 0$, exist of addition inverse.

2. *multiplication of vectors:*

i. $a(x + y) = ax + ay$

ii. $1 * x = x, \forall x \in X$

iii. $(ab)x = a(bx)$

iv. $(a + b)x = ax + bx$

where $x, y, z \in X$ and $a, b \in F$.

- **Linear operator:**

Definition (1.14): Linear operator: Let X, Y be two vector spaces, then the operator $T: X \to Y$ is linear if it is satisfy the following:

i. $T(x_1 + x_2) = T(x_1) + T(x_2)$ for all $x_1, x_2 \in X$.

ii. $T(ax) = aT(x)$ for all $a \in F$ and $x \in X$.

- **Inner product space:**

Definition (1.15): Inner product space: Let X be a vector space, and F be a field ($F = R \; or \; C$), then X is called inner vector product over F with the map $\langle .,. \rangle: X \times X \to F$ if $\langle .,. \rangle$ satisfy the following:

i. $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

ii. $\langle x, x \rangle \geq 0$

iii. $\langle ax, y \rangle = a\langle x, y \rangle$

iv. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

v. $\langle x, y \rangle = \overline{\langle y, x \rangle}$, where $\overline{\langle y, x \rangle}$ is the conjugate of $\langle x, y \rangle$.

where $x, y, z \in X, a \in F$.

- **Orthogonality:**

Definition (1.16): Orthogonality: Let x, y be two vectors in inner product space X, then x is orthogonal to y if $\langle x, y \rangle = 0$, we denoted it by $x \perp y$.

- **Orthonormal sets:**

Definition (1.17): Orthonormal sets: Let $X = \{x_1, x_2, \dots, x_i\} \in R^n$ is a set of vectors, then X is orthonormal if:

$$\langle x_i, x_j \rangle = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$$

- **Normed space:**

Definition (1.18): Normed space: Let X be a vector space and let F be a field ($F = R\ or\ C$), X is a vector space over F with a function $\|.\|: X \to F$ then X is a normed space if:

i. $\|x\| = 0 \iff x = 0$

ii. $\|ax\| = |a|\|x\|$

iii. $\|x + y\| \leq \|x\| + \|y\|$

where $x, y \in X, a \in F$, and a function $\|.\|$ is called the norm.

- **Cauchy sequence:**

Definition (1.19): Cauchy sequence: Let X be a space, then the sequence $\{x\}_{n=0}^{\infty} \subseteq X$ is called a Cauchy sequence if for every positive number $\epsilon > 0$ there exists a natural number N such that for all $n, m \geq N$, then:

$|x_n - x_m| < \epsilon.$

- **Convergent sequence:**

Definition (1.20): Convergent sequence: Let X be a space, then the sequence $\{x\}_{n=0}^{\infty} \subseteq X$ is convergent in X $if\ and\ only\ if$ there exist $x \in X$ such that for any $\epsilon > 0$ there exists a natural number N, such that for all $n \geq N$ then:

$|x_n - x| < \epsilon$ and,

$\lim_{n\to\infty} |x_n - x| = 0$

- **Convergence uniformly:**

Definition (1.21): Convergent uniformly: For the real valued function, let $\{f_n\}$ be a sequence of real valued functions defined on $D \subseteq R$, then $\{f_n\}$ is convergence uniformly to the function f $if\ and\ only\ if$ $\forall x \in D$ such that for any $\epsilon > 0$ there exists a natural number N, such that for all $n \geq N$ then:

$|f_n(x) - f(x)| < \epsilon$ and,

$\lim_{n\to\infty} |f_n(x) - f(x)| = 0$

- **Complete space:**

Definition (1.22): Complete space: Let X be a space, then X is called a complete space if every Cauchy sequence of points in X has a limit in X or every Cauchy sequence in X converges in X.

- **Banach space:**

Definition (1.23): Banach space: It is a complete normed vector space.

- **Hilbert space:**

Definition (1.24): Hilbert space: It is complete inner product space.

- **L^2 functions and L^2 spaces:**

Definition (1.25): L^2 functions: Let x be a real variable on the interval $[a, b]$, then L^2 functions is a complex valued function $f(x)$ such that:

$$\int_a^b |f(x)|^2 dx < \infty \tag{1.16}$$

Definition (1.26): Function space: L^2: The set of all functions in (1.16) is called the function space L^2on $[a, b]$, such that:

$$L^2[a, b] = \left\{ f : [a, b] \to C;\ \int_a^b |f(x)|^2 dx < \infty \right\} \tag{1.17}$$

where C is the complex numbers.

- L^2 is a Hilbert space with the inner product:

$$\langle f, g \rangle_2 = \int_a^b f(x)\overline{g(x)}dx\ ,\ f, g \in L^2$$

and the corresponding norm is:

$$\|f\|_2 = \int_a^b |f(x)|^2 dx$$

For more details, see [38], and [24].

1.4 Existence and uniqueness of the solution of nonlinear Fredholm integral equation of the second kind

The previous definitions and the following theorems are necessary for the existence and the uniqueness of the nonlinear Fredholm integral equation:

Theorem (1.2) [5]: Existence and Uniqueness of the solution:

If equation (1.3) satisfies the following conditions:

i. The function $f(x)$ is continuous and bounded, $|f(x)| < R$,

in $a \leq x \leq b$.

ii. There exist a constant K such that the function $k\big(x,t,u(t)\big)$ is integrable and bounded where $\big|k\big(x,t,u(t)\big)\big| < K$, $a \leq x,t \leq b$.

iii. The function $k\big(x,t,u(t)\big)$ satisfies the Lipschitz condition (1.14) $|k(x,t,z) - k(x,t,z')| < M|z - z'|$.

If $\lambda < \frac{1}{l(b-a)}$, where l is the largest number of $K\left(1 + \frac{R}{|\lambda|K(b-a)}\right)$ and M, then there exists a unique solution $u \in C([a,b],R)$. Moreover, for any starting function $u_0 \in C([0,1],R)$, the sequence $\{u_n\}_{n=0}^{\infty}$ such that:

$$u_{n+1}(x) = f(x) + \lambda \int_0^1 g\big(x,t,u_n(t)\big)dt \qquad (1.18)$$

satisfies:

$$\lim_{n\to\infty} max_{x\in[0,1]}|u_n(x) - u(x)| = 0 \qquad (1.19)$$

Proof:

Define $X = C([0,1], R)$ and $d: X \times X \to R_+$ such that $d(x_1, x_2) = \|x_1 - x_2\|$ for all $x_1, x_2 \in X$ where $\|.\|$ is a norm defined for $x \in C([0,1], R)$ by:

$$\|x\| = \max\{|x(s)|: s \in [0,1]\}$$

Then X is a complete metric space and also, the first condition of the contraction mapping theorem (1.1) is satisfied.

Now, to verify whether the second condition of contraction mapping theorem is satisfied we define $F: X \to X$ by:

$$(Fu)(x) = f(x) + \lambda \int_0^1 g\big(x, t, u(t)\big)dt \qquad (1.20)$$

where $u \in X$ and $x \in [0,1]$, then:

$$d\big((Fu_1), (Fu_2)\big) = \|(Fu_1) - (Fu_2)\| = max_{x\in[0,1]}|(Fu_1) - (Fu_2)(x)|$$

$$= max_{x\in[0,1]}|(Fu_1)(x) - (Fu_2)(x)|$$

Now, from (1.18) we have:

$$d\big((Fu_1), (Fu_2)\big) = max_{x\in[0,1]} \Big|f(x) + \lambda \int_0^1 g\big(x, t, u_1(t)\big)dt - f(x) - \lambda \int_0^1 g\big(x, t, u_2(t)\big)dt\Big|$$

$$= max_{x\in[0,1]} \Big|\lambda \int_0^1 g\big(x, t, u_1(t)\big)dt - \lambda \int_0^1 g\big(x, t, u_2(t)\big)dt\Big|$$

$$= |\lambda| max_{x\in[0,1]} \Big|\int_0^1 g\big(x, t, u_1(t)\big)dt - \int_0^1 g\big(x, t, u_2(t)\big)dt\Big|$$

$$\leq |\lambda| max_{x\in[0,1]} \int_0^1 \big|g\big(x, t, u_1(t)\big) - g\big(x, t, u_2(t)\big)\big|dt$$

Then by Lipchitz Condition (1.14) we have:

$$d\big((Fu_1),(Fu_2)\big) \leq |\lambda| max \int_0^1 M|u_1(t) - u_2(t)|dt$$

$$= |\lambda| \int_0^1 M|u_1(t) - u_2(t)|dt \leq |\lambda|M \int_0^1 max_{x\in[0,1]}|u_1(x) - u_2(x)|dt$$

$$= |\lambda|M\|u_1 - u_2\| \int_0^1 dt = |\lambda|Md(u_1, u_2)$$

It is clear that if $|\lambda|M < 1$ then F satisfies (1.14) with the constant $|\lambda|M$, then the second condition of the contraction mapping theorem satisfied.

By theorem (1.3), the sequence $\{u_n\}_{n=0}^{\infty}$ of functions $u_n \in C([0,1], R)$ defined by (1.18) $u_{n+1} = F(u_n)$ with any starting function $u_0 \in X$ converges to some $u \in C([0,1], R)$. So,

$$\lim_{n\to\infty} d(u_n, u) = \lim_{n\to\infty} \|u_n - u\|$$

$$= \lim_{n\to\infty} max_{x\in[0,1]}|u_n(x) - u(x)| = 0.$$

u is a unique fixed point of F satisfy definition (1.10) ($F(u) = u$ and $F(u(x)) = u(x)$) for all $x \in [0,1]$, therefore u is a unique solution of the nonlinear equation (1.3). This complete the proof of theorem (1.3).

Chapter two

Numerical methods for solving nonlinear Fredholm integral equations of the second kind

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Numerical methods for solving nonlinear Fredholm integral equations of the second kind

There are several numerical methods for solving nonlinear integral equations. In this chapter we propose two methods, namely; Haar wavelet method and the Homotopy analysis method.

2.1 Haar Wavelet Method

Definition (2.1): Wavelet [45]: The wavelet defined as a small wave, which oscillates rapidly, and must satisfy the following conditions:

i. $\int_{-\infty}^{\infty}|\psi(t)|dt < \infty$ (2.1)

ii. $\int_{-\infty}^{\infty}|\psi(t)|dt = 0$ (2.2)

iii. $\int_{-\infty}^{\infty}\frac{|\widehat{\psi}(t)|^2}{|\omega|}d\omega < \infty$ (2.3)

where $\widehat{\psi}$ is the Fourier transform of ψ.

Definition (2.2) [45]: The Mother wavelet function ψ is given by:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right) \qquad a, b \in \mathbb{R}, a \neq 0 \tag{2.4}$$

where a is the scaling parameter and b is location parameter.

2.1.1 Wavelet Transform

Wavelet transforms are generally divided into two types: The Continuous Wavelet Transform (CWT), and the Discrete Wavelet Transform (DWT).

1. Continuous wavelet transform:

Definition (2.3): Continuous wavelet transform [1]: Let $f(x)$ be any square integrable function, the continuous wavelet transform $W_\psi(f)$ of the function $f \in L_2(R)$ with respect to ψ is defined as:

$$W_\psi f(a,b) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{|a|}} \overline{\psi(\frac{t-b}{a})} dt \tag{2.5}$$

where $\bar{\psi}$ is the complex conjugate of the function ψ.

Definition (2.4) [1]: If $W_\psi f(a,b)$ is a continuous wavelet transform of a function $f(x)$ with respect to a mother wavelet ψ, then the inverse of continuous wavelet transform can be reconstructed by:

$$f(x) = \frac{2}{C_\psi} \int_0^{\infty} \left[\int_{-\infty}^{\infty} \frac{1}{|a|^2} W_\psi f(a,b) \psi_{a,b}(x) da db \right] \tag{2.6}$$

where C_ψ is a constant called the admissibility constant, given by:

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega)|}{|\omega|} d\omega < \infty \tag{2.7}$$

The inverse of continuous wavelet transform exists if C_ψ is positive and finite.

2. Discrete wavelet transform

Instead of use a family of wavelets in (2.4), for the discrete wavelet transform we use the family of wavelets:

$$\psi_{j,k}(x) = a_0^{-j/2}\psi\left(a_0^{-j}x - kb_0\right) \tag{2.8}$$

where $a_0 > 1,\ b_0 > 0,\ fixed\ f\ \ or\ \ j,k \in Z$

Definition (2.5) [23] [25]: For $f \in L^2$, the discrete wavelet transform of the function f is define by:

$$W_\psi f\left(2^{-j}, k\,2^{-j}\right) = 2^{-\frac{j}{2}}\int_{-\infty}^{\infty} f(t)\psi(2^j t - k)dt \tag{2.9}$$

In equation (2.5) we replace the scaling parameter a by 2^{-j} and the translation parameter b by $k\ 2^{-j}$ to get equation (2.9).

2.1.2 Wavelet Transform and Fourier Transform

The main differences between Wavelet transform and Fourier transform are that in the Fourier transform we have no time localization parameter (i.e. when we use Fourier transform to convert signals from the time domain to the frequency domain the time information will be lost, either in the case of wavelet conversion the time information will not be lost which makes it very important in some fields [28]), and that in Fourier transform we have cosine and sine functions instead of a wavelet function[1].

2.1.3 Wavelet Series and Wavelet Coefficients

Definition (2.6) [45]: For $f \in L^2$, the wavelet series of f is given by:

$$\sum_{j\in Z}\sum_{k\in Z}\langle f,\psi_{j,k}\rangle\psi_{j,k}(t) \tag{2.10}$$

where $\langle f,\psi_{j,k}\rangle$ is the wavelet coefficients of the function f.

Definition (2.7) [45]: The wavelet coefficients of f is given by:

$$\langle f,\psi_{j,k}\rangle = d_{j,k} = \int_{-\infty}^{\infty} f(t)\,\psi_{j,k}(t)dt \tag{2.11}$$

2.1.4Haar Wavelet

Haar scaling function, Mother Haar wavelet, and Haar wavelet family

Definition (2.8) [8]: Haar scaling function $\varphi(t)$ can be described as:

$$\varphi(t) = h_0(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & otherwise. \end{cases} \tag{2.12}$$

and the graph of the scaling function $\varphi(t)$ is given as:

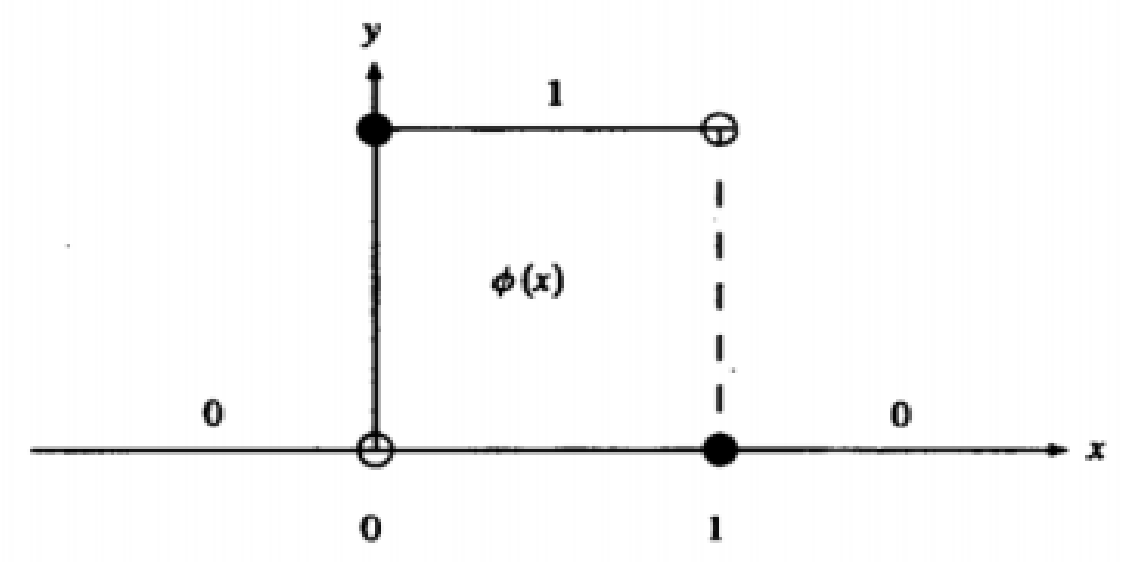

Fig.2.1: The scaling function $\varphi(t)$

Definition (2.9): The scaling function can be written as:

$$\varphi(x) = \varphi(2x) + \varphi(2x-1) \tag{2.13}$$

Definition (2.10) [8]: Let j be a nonnegative integer. The space of step function at level j is the set V_j , which is the space of piecewise constant functions of finite support with discontinuities contained in the set: $\left\{\frac{k}{2^j}\right\}_{k\in Z}$. $V_j$is defined to be the space spanned by the set $\{\varphi(2^j x-k)\}_{k\in Z}$.

Remarks:

1. Function of finite support means that the function vanishes outside a finite interval.

2. If $f \in V_j$, then f is a finite sum of the form:

$$f = \sum_k a_k \varphi\left(2^j x-k\right), \quad a_k \in R \tag{2.14}$$

3. $V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots,$

   here the containment is strict (i.e. $V_j \subset V_{j+1}$ but $V_{j+1} \not\subset V_j$).

Theorem (2.1) [8]:

- A function $f(x)$ belongs to V_0 $if\ and\ only\ if$ $\left(2^j x\right)$ belongs to V_j.
- A function $f(x)$ belongs to V_j $if\ and\ only\ if$ $\left(2^{-j} x\right)$ belongs to V_0.

Proof: see [8], theorem (4.5).

Theorem (2.2) [8]: The set of functions $\left\{2^{\frac{j}{2}}\varphi(2^j x-k)\right\}$ form a basis of V_j , $j \in Z$.

Definition (2.11) [8]: The Haar wavelet function can be written as:

$$\psi(x) = \varphi(2x) - \varphi(2x-1) \tag{2.15}$$

Note: Equation (2.15) is true when ψ satisfies the following conditions:

- $\psi = \sum_k a_k \varphi\left(2^j x - k\right)$, where the sum is finite, $a_k \in R$.
- ψ is orthogonal to any element on V_0.

Definition (2.12) [8]: Let j be a nonnegative integer. The space of the function at level j is the set W_j , $W_j$defined to be the space spanned by the set $\{\psi(2^j x - k)\}_{k\in Z}$, then $f \in W_j$ can be expressed as:

$$f = \sum_k a_k \psi\left(2^j x - k\right), \quad k \in Z, a_k \in R \tag{2.16}$$

Theorem (2.3) [8]: W_j is orthogonal complement of V_j in V_{j+1} and $V_{j+1} = V_j \oplus W_j$.

Theorem (2.4) [8]: The space $L^2(R)$ can be decomposed as an infinite orthogonal direct sum, then:

$L^2(R) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots$

$f(x) \in L^2(R)$can be written as:

$$f(x) = f_0 + \sum_{k=0}^{\infty} w_k \tag{2.17}$$

where $f_0 \in V_0$ and $w_k \in W_k$.

Definition (2.13) [42]: The Haar wavelet family for $t \in [0,1)$ is defined as follows:

$$\psi_i(t) = \begin{cases} 1 & for\ t \in [\mathcal{E}_1(i), \mathcal{E}_2(i)] \\ -1 & for\ t \in [\mathcal{E}_2(i), \mathcal{E}_3(i)] \\ 0 & otherwise \end{cases} \tag{2.18}$$

where the notations:

$\mathcal{E}_1(i) = \frac{k-1}{m}$, $\mathcal{E}_2(i) = \frac{k-0.5}{m}$, $\mathcal{E}_3(i) = \frac{k}{m}$ are introduced.

The integer $m = 2^j (j = 0,1, \dots, J)$ indicates the level of the wavelet. $1 \leq k < m + 1$ is the translation parameter. J is an integer determining the maximal level of the resolution. The index i is calculated by the formula $i = m + k - 1$, where the maximal value is $i = 2M$ and $M = 2^J$.

For $i = 0$, the function $\psi_0(t)$ is the scaling function $\varphi(t)$ given in equation (2.12).

For $i = 1$, the function $\psi_1(t)$ is the mother wavelet function $\psi(t)$ given in following definition.

The first eight Haar functions on the interval [0,1) are illustrated in figure (2.1).

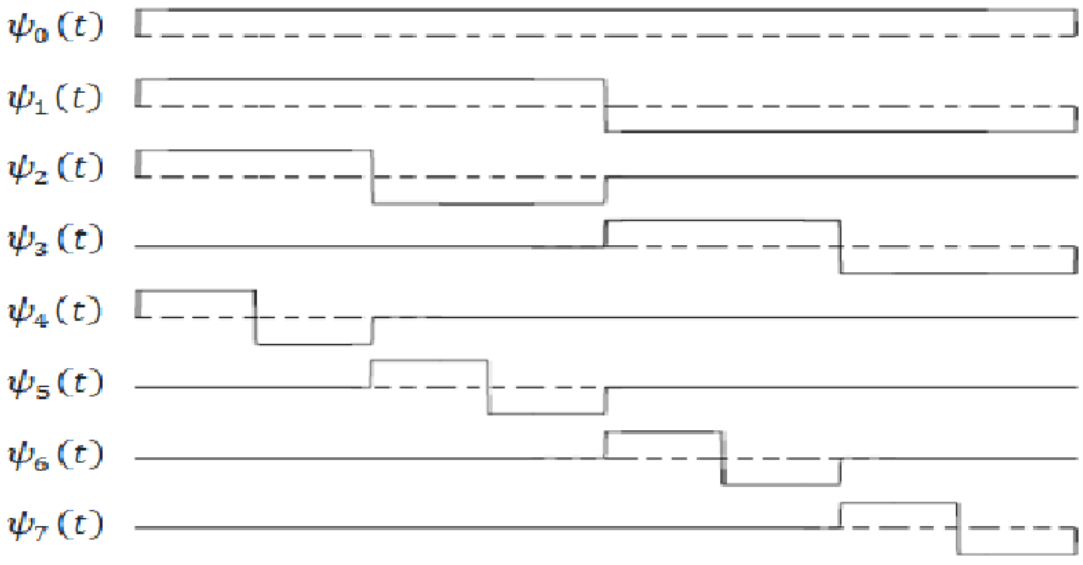

Fig.2.2: The first eight Haar function (Haar wavelet $\psi_i(t)$).

Note [50]: The sequence $\{\psi_i\}_{i=0}^{\infty}$ is a complete orthonormal system in $L^2[0,1]$, and for the series $\sum_i \langle f, \psi_n \rangle$, ψ_n is convergence uniformly to f, $f \in C[0,1]$, where $\langle f, \psi_n \rangle = \int_0^1 f\psi_n(t)dt$.

Definition (2.14) [8]: The mother wavelet function $\psi(t)$ on the interval $[0,1)$ can be described as:

$$\psi(t) = h_1(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \\ 0 & otherwise. \end{cases} \tag{2.19}$$

The graph of function $\psi(t)$ is shown in figure (2.3).

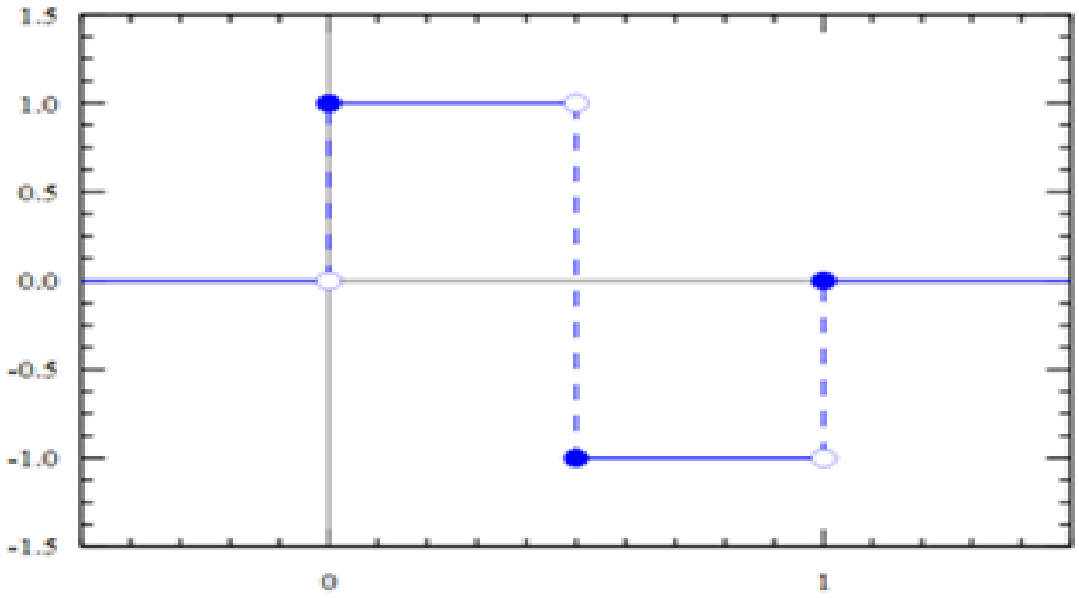

Fig.2.3: The Mother wavelet function $\psi(t)$

Theorem (2.5) [27] [29]: Haar wavelet functions are orthogonal to each other and also forms an orthogonal basis as:

$$\int_0^1 \psi_i(t)\psi_l(t)\,dt = \begin{cases} 2^{-j}, & i = l = 2^j + k + 1 \\ 0, & i \neq l \end{cases} \tag{2.20}$$

2.1.5 Integration of Haar function

Definition (2.15) [27]: The integrals of Haar functions on the interval $[A, B]$ are given by:

$$p_{v,i}(x) = \underbrace{\int_A^x \int_A^x \dots \int_A^x \psi_i(t)dt^v}_{v-times} = \frac{1}{(v-1)!}\int_A^x (x-t)^{v-1}\psi_i(t)dt \tag{2.21}$$

where $v = 1,2,\dots,n, \quad i = 1,2,\dots,2M.$

Equation (2.21) can be solved analytically, and then we obtain:

$$p_{\alpha,i}(x) = \begin{cases} 0 & x > \mathcal{E}_1(i) \\ \frac{1}{\alpha!}[x-\mathcal{E}_1(i)]^\alpha & x \in [\mathcal{E}_1(i), \mathcal{E}_2(i)] \\ \frac{1}{\alpha!}\{[x-\mathcal{E}_1(i)]^\alpha - 2[x-\mathcal{E}_2(i)]^\alpha\} & x \in [\mathcal{E}_2(i), \mathcal{E}_3(i)] \\ \frac{1}{\alpha!}\{[x-\mathcal{E}_1(i)]^\alpha - 2[x-\mathcal{E}_2(i)]^\alpha + [x-\mathcal{E}_3(i)]^\alpha\} & x > \mathcal{E}_3(i) \end{cases}$$

(2.22)

Formula (2.22) true when $i > 1$, for $i = 1$ we have $\mathcal{E}_1 = A, \mathcal{E}_2 = \mathcal{E}_3 = B$, then:

$$p_{\alpha,1}(x) = \frac{1}{\alpha!}[x-A]^\alpha \tag{2.23}$$

For the Haar wavelet we have the following integrals [28]:

$$p_{i,1}(t) = \int_0^t \psi_i(t)dt \tag{2.24}$$

$$p_{i,v}(t) = \int_0^t p_{i.v-1}(t)dt, \qquad v = 2,3,\dots \tag{2.25}$$

in virtue of equation (2.18) and by performing integration in (2.24) - (2.25) we get:

$$p_{i,1}(t) = \begin{cases} t - \mathcal{E}_1 & t \in [\mathcal{E}_1, \mathcal{E}_2] \\ \mathcal{E}_3 - t & t \in [\mathcal{E}_2, \mathcal{E}_3] \\ 0 & otherwise \end{cases} \tag{2.26}$$

$$p_{i,2}(t) = \begin{cases} 0 & t \in [0, \mathcal{E}_1] \\ \frac{1}{2}(t - \mathcal{E}_1)^2 & t \in [\mathcal{E}_1, \mathcal{E}_2] \\ \frac{1}{4m^2} - \frac{1}{2}(\mathcal{E}_3 - t)^2 & t \in [\mathcal{E}_2, \mathcal{E}_3] \\ \frac{1}{4m^2} & t \in [\mathcal{E}_3, 1] \end{cases} \tag{2.27}$$

$$p_{i,3}(t) = \begin{cases} 0 & t \in [0, \mathcal{E}_1] \\ \frac{1}{6}(t - \mathcal{E}_1)^3 & t \in [\mathcal{E}_1, \mathcal{E}_2] \\ \frac{1}{4m^2}(t - \mathcal{E}_2) - \frac{1}{6}(\mathcal{E}_3 - t)^3 & t \in [\mathcal{E}_2, \mathcal{E}_3] \\ \frac{1}{4m^2}(t - \mathcal{E}_2) & t \in [\mathcal{E}_3, 1] \end{cases} \tag{2.28}$$

$$p_{i,4}(t) = \begin{cases} 0 & t \in [0, \mathcal{E}_1] \\ \frac{1}{24}(t - \mathcal{E}_1)^4 & t \in [\mathcal{E}_1, \mathcal{E}_2] \\ \frac{1}{8m^2}(t - \mathcal{E}_2)^2 - \frac{1}{24}(\mathcal{E}_3 - t)^4 + \frac{1}{192m^4} & t \in [\mathcal{E}_2, \mathcal{E}_3] \\ \frac{1}{8m^2}(t - \mathcal{E}_2)^2 + \frac{1}{192m^4} & t \in [\mathcal{E}_3, 1] \end{cases} \tag{2.29}$$

2.1.6 Haar wavelet matrices

Definition (2.16) [16] [28]: Define the $2M \times 2M$ Haar wavelet coefficient matrix Has $H(i,j) = h_i(j)$.

$$H_1 = 1, \qquad H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Definition (2.17) [16] [28]: Define the $2M \times 2M$ Haar wavelet integral matrix P_v as $P_v(i,j) = p_{v,i}(j)$.

Note: Divide the interval [0,1) into $2M$ parts of equal length, then the grid points is given by:

$$t(j) = \left(j - \frac{1}{2}\right)\Delta t, \qquad j = 1,2,\dots,2M \tag{2.30}$$

where $\Delta t = \frac{1}{2M}$, and $M = 2^J$.

Definition (2.18) [16]: The row vector of Haar coefficient matrix is given by:

$$h_{(\mu)}(t) = \psi_{(\mu)}(t) = [\psi_1(t), \psi_2(t), \dots, \psi_{\mu-1}(t)]^T \tag{2.31}$$

where $\mu = 2M = 2^{J+1}$.

Definition (2.19) [41]: The Haar coefficient matrix H is given by:

$$H_{(\mu)} = [h_{(\mu)}\left(\frac{1}{2\mu}\right), h_{(\mu)}\left(\frac{3}{2\mu}\right), \dots, h_{(\mu)}\left(\frac{2\mu-1}{2\mu}\right)]^T \tag{2.32}$$

where $H_{(1)} = [1]$, $H_{(2)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Definition (2.20) [16]: The $2M$ square operational matrix of integration P, is defined by:

$$P_{\mu\times\mu}\psi_\mu(t) = \int_0^{t_j} \psi_j(t)dt \tag{2.33}$$

Definition (2.21) [28]: The operational matrix of integration P is defined as follow:

$$P_{\mu\times\mu} = \frac{1}{2\mu}\begin{bmatrix} 2\mu P_{\frac{\mu}{2}\times\frac{\mu}{2}} & -H_{\frac{\mu}{2}\times\frac{\mu}{2}} \\ H_{\frac{\mu}{2}\times\frac{\mu}{2}}{}^{-1} & O_{\frac{\mu}{2}\times\frac{\mu}{2}} \end{bmatrix} \tag{2.34}$$

where $O_{\frac{\mu}{2}\times\frac{\mu}{2}}$ is a square zero matrix of size $\frac{\mu}{2}$, and $P_1 = 0.5$.

2.1.7 Functions expansion of Haar wavelet series

Definition (2.22) [17] [41]: For any function $f \in L^2[0,1)$, then $f(x)$ can be expanded into Haar wavelet series by the form:

$$f(x) = \sum_0^\infty a_i h_i(x), \quad i \in \{0\} \cup N \tag{2.35}$$

where a_i denotes the Haar wavelet coefficients given by the form:

$a_0 = \int_0^1 f(x)h_0(x)dx$, $a_n = 2^j \int_0^1 f(x)h_i(x)dx$ such that:

$i = 2^j + k - 1,\ j \geq 0,\ 0 \leq k < 2^j,\ x \in [0,1)$

The discrete form of (2.35) is:

$$\hat{f}(x_i) = \sum_0^{m-1} a_i h_i(x_i) = A^T{}_{(m)} h_{(m)}(x) \quad , m = 2^j, i \in \{0\} \cup N \tag{2.36}$$

where $A^T{}_{(m)} = [a_0, a_1, \dots, a_{m-1}]$ is the coefficient vector, and $h_{(m)}(x) = [h_0(x), h_1(x), \dots, h_{m-1}(x)]^T$ is the Haar function vector.

Definition (2.23) [17] [41]: Assume $f(x)$ is differentiable function with $|f(x)| \leq K,\ \forall x \in (a,b)\ such\ that\ \left|f\acute{}(x)\right| \leq K$.

where K is a positive constant. The Haar wavelet approximation function for $f(x)$ is given by:

$$f_M(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{2.37}$$

Definition (2.24) [17] [41]: The error function E of the wavelet approximations is defined as:

$$E = \int_0^1 [f(x) - \hat{f}(x)]^2 dx \tag{2.38}$$

$\hat{f}(x)$ denotes the approximation of $f(x)$.

The discrete form of (2.37) is:

$$E_i = \Delta x \sum_{i=0}^{\infty} [f(x_i) - \hat{f}(x_i)]^2 \tag{2.39}$$

2.1.8 Convergence analysis of Haar wavelet

Definition (2.25) [15]: The square of the error norm of wavelet approximation is given by:

$$\|f(x) - f_M(x)\|^2 = \frac{K^3}{12M^2} \tag{2.40}$$

The maximum absolute error $= E_\infty = max. \left| {f^e}_i(x) - {f^a}_i(x) \right|$.

The maximum relative error $= E_{rel} = \frac{E_\infty}{|{f^e}_i(x)|}$.

where ${f^e}_i$ is the exact Haar solution, ${f^a}_i$ is the approximate Haar solution at the i^{th} colocation point x_i , $i = 1,2, \dots ,2m$.

2.1.9 Numerical simulation of the Haar wavelet method

For Haar wavelet approximation of the function $f(x)$ we use the following grid (collocation) points:

$$x_j = \frac{j-0.5}{2M} , \quad j = 1,2, \dots ,2M \tag{2.41}$$

Definition (2.26) [4]: Any square integrable function can be approximated using Haar wavelet by:

$$f(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{2.42}$$

Substitute the collocation point (2.41) in equation (2.42) to get:

$$f(x_j) = \sum_{i=1}^{2M} a_i h_i(x_j) \ , \quad j = 1,2,\dots,2M \tag{2.43}$$

Equation (2.43) is a $2M \times 2M$ linear system of equations. In matrix form equation (2.43) is:

$$F = AH \tag{2.44}$$

where:

$$F = [f_1 \quad f_2 \quad \cdots \quad f_{2M}], f_i = f(x_i) \ , \quad A = [a_1 \quad a_2 \quad \cdots \quad a_{2M}],$$

$$H = [h_{ij}], \ \ h_{ij} = h_i(x_j), \ \ i,j = 1,2,\dots,2M.$$

Theorem (2.6) [4]: The solution of the system (2.44) is given by:

$$a_1 = \frac{1}{2M} \sum_{j=1}^{2M} f(x_j) \tag{2.45}$$

In general:

$$a_i = \frac{1}{\rho} \left(\sum_{j=\alpha}^{\beta} f(x_j) - \sum_{j=\beta+1}^{\gamma} f(x_j) \right), \quad i = 2,3,\dots,2M \tag{2.46}$$

where:

$$\alpha = \rho(\sigma - 1) + 1,$$

$$\beta = \rho(\sigma - 1) + \frac{\rho}{2},$$

$$\gamma = \rho\sigma,$$

$\rho = \frac{2M}{\tau}$,

$\sigma = i - \tau$,

$\tau = 2^{\lfloor \log_2(i-1) \rfloor}$.

Proof: see [4].

Substituting equations (2.45) and (2.46) into equation (2.42) to get:

$$f(x) = \frac{1}{2M}\sum_{j=1}^{2M} f(x_j)h_1(x) + \sum_{i=2}^{2M}\frac{1}{\rho}\left(\sum_{j=\alpha}^{\beta} f(x_j) - \sum_{j=\beta+1}^{\gamma} f(x_j)\right)h_i(x) \quad (2.47)$$

Note: For two variable functions $g(x, y)$; equation (2.42) becomes:

$$g(x, y) = \sum_{i=1}^{2M} a_i(x)h_i(y) \quad (2.48)$$

then substitute the collocation points (2.41) into equation (2.48) to get:

$$g(x, y_j) = \sum_{i=1}^{2M} a_i(x)h_i(y_j) \quad (2.49)$$

a_i is the variable coefficient can be evaluated using the following corollary.

Corollary (2.1) [4]: The unknown coefficients a_i in equation (2.49) are given by:

$$a_1(x) = \frac{1}{2M}\sum_{j=1}^{2M} g(x, y_j) \quad (2.50)$$

and in general:

$$a_i(x) = \frac{1}{\rho}\left(\sum_{j=\alpha}^{\beta} g(x, y_j) - \sum_{j=\beta+1}^{\gamma} g(x, y_j)\right), \quad i = 2,3, \dots ,2M \quad 2.51)$$

where $\alpha, \beta, \gamma, \rho, \sigma, and\ \tau$ are defined in theorem (2.6).

2.1.10 Haar wavelet method for nonlinear Fredholm integral equation

Consider the nonlinear Fredholm integral equation of the second kind:

$$u(x) = f(x) + \lambda \int_0^1 k(x,t)\emptyset(t,u(t))dt \tag{2.52}$$

where $k \in L^2[0,1] \times L^2[0,1]$ and $f \in L^2[0,1]$ are known real valued functions, $\emptyset$ is a nonlinear function, and $u(x)$ is the unknown function that is to be determined.

Approximating the function $\lambda k(x,t)\emptyset(u(t))$ using definition (2.21) to get:

$$\lambda k(x,t)\emptyset\big(u(t)\big) \cong \sum_{i=1}^{2M} a_i(x)h_i(t) \tag{2.53}$$

Substituting equation (2.53) into equation (2.52) we obtain:

$$u(x) = f(x) + \sum_{i=1}^{2M} a_i(x) \int_0^1 h_i(t)dt \tag{2.54}$$

Then equation (2.54) can be written as:

$$u(x) = f(x) + a_1(x) \tag{2.55}$$

By corollary (2.1), equation (2.55) becomes:

$$u(x) = f(x) + \frac{\lambda}{2M}\sum_{j=1}^{2M} k\big(x,t_j\big)\emptyset\left(u\big(t_j\big)\right) \tag{2.56}$$

Appling the collocation point (2.41) into equation (2.56) yields:

$$u(x_i) = f(x_i) + \frac{\lambda}{2M}\sum_{j=1}^{2M} k(x_i, t_j)\emptyset\left(u(t_j)\right) \tag{2.57}$$

where $i, j = 1,2, \dots, 2M$.

Equation (2.57) is a system of nonlinear algebraic equations with the following matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{2M} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{2M} \end{bmatrix} + \frac{\lambda}{2M}\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1\,2M} \\ k_{21} & k_{22} & \cdots & k_{2\,2M} \\ \vdots & \vdots & \ddots & \vdots \\ k_{2M\,1} & k_{2M\,2} & \cdots & k_{2M\,2M} \end{bmatrix}\begin{bmatrix} \emptyset_1 \\ \emptyset_2 \\ \vdots \\ \emptyset_{2M} \end{bmatrix} \tag{2.58}$$

where $u_i = u(x_i)$, $f_i = f(x_i)$, $k_{ij} = k(x_i, t_j)$, $\emptyset_i = \emptyset(u(t_i))$,

$i, j = 1,2, \dots, 2M$. System (2.58) can be solved for $u(x_i)$ using Newton's Raphson method [26].

2.2 Homotopy Analysis Method

Definition (2.25) [32]: Let f and g be two continuous functions such that$f, g \in C[a, b]$, then f can be deformed continuously into g which can construct a Homotopy as follows:

$H: f(x) \sim g(x)$

$$H(x; q) = (1 - q)f(x) + qg(x) \tag{2.59}$$

where $x \in [a, b]$ is independent variable, $q \in [0,1]$ is called the embedding parameter, $H(x; q)$ is called Homotopy, and $f(x)$ and $g(x)$ are called Homotopic.

Notes:

1. $H(x;q)$ not only depend into x but also into q.
2. When $q=0$ we have $H(x;0)=f(x)$ and when $q=1$ we have $H(x;1)=g(x)$. So that as q increase from 0 to 1, the real function $H(x;q)$ varies continuously from $f(x)$ to $g(x)$, (this kind of continuous variation is called deformation in topology).
3. $g(x)$ must be a continuous function, (i.e. a continuous real function can't be deformed continuously into a discontinuous function).

2.2.2 Homotopy derivatives and series

Here, some definitions and theorems will be presented for the deduced high-order deformation equation.

Definition (2.26) [33]: Let $N[u(t)]$ be a nonlinear equation, and $\emptyset$ be a function of the Homotopy parameter q whose Maclaurin series is given by:

$$\emptyset = \sum_{i=0}^{\infty} u_i q^i \tag{2.60}$$

(2.60) is called a Homotopy series.

Definition (2.27) [33]: The m^{th} order Homotopy derivatives of the function $\emptyset$ is denoted by $D_m(\emptyset)$ and it is of the form:

$$D_m(\emptyset) = \frac{1}{m!}\frac{\partial^m \emptyset}{\partial q^m}\Big|_{q=0} \tag{2.61}$$

where m is a positive integer.

Theorem (2.7) [33]: Malabahrami and Khani's theorem: For Homotopy series (2.60), it holds that:

$$D_m(\emptyset^k) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \dots \sum_{r_{k-2}=0}^{r_{k-3}} u_{r_{k-3}-r_{k-2}} \sum_{r_{k-1}=0}^{r_{k-2}} u_{r_{k-2}-r_{k-1}} u_{r_{k-1}} \tag{2.62}$$

where $m \geq 0$ and $k \geq 1$.

Proof: see [33].

Theorem (2.8) [33]: Let f and g be two functions independent of the Homotopy parameter q, then for Homotopy series:

$$\emptyset = \sum_{i=0}^{\infty} u_i q^i, \quad \psi = \sum_{j=0}^{\infty} v_j q^j$$

it holds that:

$$D_m(f\emptyset + g\psi) = fD_m(\emptyset) + gD_m(\psi) \tag{2.63}$$

Proof: see [33].

Theorem (2.9) [33]: For Homotopy series:

$$\emptyset = \sum_{i=0}^{\infty} u_i q^i, \quad \psi = \sum_{j=0}^{\infty} v_j q^j$$

it holds that:

1. $D_m(\emptyset) = u_m$
2. $D_m(q^k\emptyset) = D_{m-k}(\emptyset)$

3. $D_m(\emptyset\psi) = \sum_{i=1}^{m} D_i(\emptyset) D_{m-i}(\psi) = \sum_{i=1}^{m} D_i(\psi) D_{m-i}(\emptyset)$

4. $D_m(\emptyset^n \psi^l) = \sum_{i=1}^{m} D_i(\emptyset^n) D_{m-i}(\psi^l) = \sum_{i=1}^{m} D_i(\psi^l) D_{m-i}(\emptyset^n)$

where $m \geq 0, n \geq 0, l \geq 0$, and $0 \leq k \leq m$ are integers.

Proof: see [33].

Theorem (2.10) [33]: Let $\mathcal{L}$ be a linear operator independent of the Homotopy parameter q, then for the Homotopy series (2.60) it holds that:

$$D_m(\mathcal{L}\emptyset) = \mathcal{L}[D_m(\emptyset)] \tag{2.64}$$

Proof: see [33].

Theorem (2.11) [33]: For the Homotopy series (2.60), it holds that:

$D_0\left(e^{\emptyset}\right) = e^{u_0}$

$D_m\left(e^{\emptyset}\right) = \sum_{k=0}^{m-1}(1 - \frac{k}{m}) D_k\left(e^{\emptyset}\right) D_{m-k}(\emptyset)$

$D_0(\sin(\emptyset)) = \sin(u_0), \quad D_0(\cos(\emptyset)) = \cos(u_0)$

$D_m(\sin(\emptyset)) = \sum_{k=0}^{m-1}(1 - \frac{k}{m}) D_k(\sin(\emptyset)) D_{m-k}(\emptyset)$

$D_m(\cos(\emptyset)) = \sum_{k=0}^{m-1}(1 - \frac{k}{m}) D_k(\cos(\emptyset)) D_{m-k}(\emptyset)$

where $m \geq 1$ is an integer.

Proof: see [33].

Theorem (2.12) [33]: If the two Homotopy series:

$$\emptyset = \sum_{i=0}^{\infty} u_i q^i , \quad \psi = \sum_{j=0}^{\infty} v_j q^j$$

satisfy $\emptyset = \psi$ in the domain $q \in [0, a)$, then $D_m(\emptyset) = D_m(\psi)$ and $u_m = v_m$ for any integer $m \geq 0$ and any real number $a > 0$.

Proof: see [33].

2.2.3 Deformation equations

There are two types of deformation equations in HAM; zero-order deformation equation and high-order deformation equation.

In this section we will use the definitions and theorems of the previous section to deduce deformation equations.

Construct a Homotopy:

$$(1-q)\mathcal{L}[\emptyset(x;q) - u_0(x)] - qhH(x)N[\emptyset(x;q)] = \widehat{H}[\emptyset(x;q), u_0(x), H(x), h, q] \quad (2.65)$$

where $H(x) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator with property $\mathcal{L}\big(u(x)\big) = 0$ when $u(x) = 0$, $u_0(x)$ is an initial guess of $u(x)$, $q \in [0,1]$ is an embedding parameter, $\emptyset(x;q)$ unknown function, N nonlinear operator, and $h \neq 0$ is an auxiliary parameter which is very important in convergence of the Homotopy series (2.55), it is also called the convergent control parameter.

1. *Zero order deformation equation:*

Definition (2.28) [32]: Given two equations; the original equation u_1 and the initial equation u_0, with the solutions x_1 and x_0. If one can construct a Homotopy of equation $\emptyset(q): u_0 \sim u_1$ that as the Homotopy parameter $q \in [0,1]$ increase from 0 to 1, $\emptyset(q)$ deforms continuously from the initial equation u_0 to the original equation u_1, and the solutions varies continuously from the known solution x_0 to the unknown solution x_1, then this kind of Homotopy equations is called zeroth-order deformation equation.

Definition (2.29) [31]: The zero order deformation equation is of the form:

$$(1-q)\mathcal{L}[\emptyset(x;q) - u_0(x)] = qhH(x)N[\emptyset(x;q)] \tag{2.66}$$

where $\mathcal{L}, q, \emptyset, u_0, h, H(x), N$ are defined in (2.65).

Note: We have freedom to choose $\mathcal{L}, h, H(x), u_0$ according with the property $\mathcal{L}\big(u(x)\big) = 0$ when $u(x) = 0$.

In fact, when (2.65) is equivalent to zero we have the zero order deformation equation.

Now in equation (2.66), when $q = 0$ we have:

$$\emptyset(x;0) = u_0(x) \tag{2.67}$$

and when $q = 1$ we have:

$$\emptyset(x;1) = u(x) \tag{2.68}$$

So that as q increases from 0 to 1, $\emptyset(x;q)$ varies from $u_0(x)$ to $u(x)$.

Now, to find m^{th} order deformation equations take m^{th} order Homotopy derivative in both sides of zero order deformation equation which mentioned in the following.

2. *High order deformation equation*

Definition (2.30) [32]: Given a nonlinear equation $N[u(x)]$ which has at least one solution $u(x)$. Let $\emptyset(q)$ denoted the zeroth-order deformation equation which has a solution, then the Homotopy Maclaurin series (2.60) can be written as:

$$\emptyset(x;q) \sim u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m \tag{2.69}$$

and the Homotopy series $\emptyset(x;1)$ is:

$$\emptyset(x;1) \sim u_0(x) + \sum_{m=1}^{\infty} u_m(x) \tag{2.70}$$

The equations related to the unknown $u_m(x)$ are called the m^{th}-order deformation equations.

Definition (2.31) [31]: Let $\vec{u}_n = \{u_0(x), u_1(x), \dots, u_n(x)\}$ be a vector, the m^{th} order deformation equation is of the form:

$$\mathcal{L}[u_m(x) - \chi_m u_{m-1}(x)] = hH(x) R_m\left(\vec{u}_{m-1}(x)\right), \ m = 1,2, \dots \tag{2.71}$$

where R_m is called the m^{th} Homotopy derivative of (2.66), and:

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\emptyset(x;q)]}{\partial q^{m-1}}\Big|_{q=0} \quad (2.72)$$

$$\chi_m = \begin{cases} 0 & ,m \leq 1 \\ 1 & ,m > 1 \end{cases} \quad (2.73)$$

Definition (2.32) [39]: Define the linear right inverse operator $\mathcal{L}^*$ of $\mathcal{L}$ such that:

$$\mathcal{L}^*\mathcal{L}[u(x)] = u(x) + k(x), \;\; \mathcal{L}^*\mathcal{L} = I \quad (2.74)$$

where I is the identity linear operator. Then the m^{th} order deformation equation is given by:

$$u_m(x) = \chi_m u_{m-1}(x) + h\mathcal{L}^*\left[H(x)R_m\left(u_{m-1}(x)\right)\right] + k(x) \quad (2.75)$$

Definition (2.33) [32]: If the solution of the zeroth-order deformation equation exists, then we have the Homotopy series solution:

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) \quad (2.76)$$

The n^{th} -order approximate solution and the exact solution respectively is:

$$\hat{u}_n(x) = u_0(x) + \sum_{m=1}^{n} u_m(x) \quad (2.77)$$

$$u(x) = \lim_{n\to\infty} \hat{u}_n(x) \quad (2.78)$$

2.2.4 Convergence theorem

Theorem (2.13) [14]: If the following series:

$$u_0(x) + \sum_{m=1}^{\infty} u_m(x)$$

is convergent, then the limit of this series is an exact solution of the nonlinear equation $N[u(t)] = 0$.

Proof: see [14].

Theorem (2.14) [39]: If (2.76) is convergent then we have the sequence:

$X_n = \sum_{m=1}^{n} R_m(x)$ convergent to zero, where $R_m(x)$ is defined on (2.72).

Proof: see [39].

2.2.5 Homotopy analysis method for nonlinear Fredholm integral equation

Consider the nonlinear Fredholm integral equation of the second kind (2.47).

First choose an initial approximation $u_0(x) = f(x)$ and the auxiliary linear operator $L[\emptyset(x;q)] = \emptyset(x;q)$.Then according to equation (2.47) we define the nonlinear operator:

$$N[\emptyset(x;q)] = \emptyset(x;q) - f(x) - \lambda \int_a^b k(x,t)F[\emptyset(t;q)]dt \qquad (2.79)$$

with assumption $H(x) = 1$, construct the zero-order deformation equation as:

$$(1-q)[\emptyset(x;q) - f(t)] = hpN[\emptyset(x;q)] \qquad (2.80)$$

when $q = 0$, $\emptyset(x;0) = f(x)$ and when $q = 1$, $\emptyset(x;1) = u(x)$.

Now, to find the m^{th}-order deformation equation; by using the m^{th}-order Homotopy derivative (2.61):

$$u_m(x) = \frac{1}{m!}\frac{\partial^m \emptyset(x;q)}{\partial q^m}\Big|_{q=0}$$

and differentiate m −times with respect to q then divide by $m!$, we obtain:

$$u_m(x) = \chi_m u_{m-1}(x) + hR_m(\vec{u}_{m-1}), \quad m \geq 1 \tag{2.81}$$

where

$$R_m(\vec{u}_{m-1}) = \chi_m u_{m-1}(x) - \lambda \int_a^b k(x,t)[\frac{1}{(m-1)!}\frac{\partial^{m-1}F[\emptyset(x;q)]}{\partial q^{m-1}}]dt \tag{2.82}$$

and $\chi_m = \begin{cases} 0 & ,m = 1 \\ 1 & ,m > 1 \end{cases}$, then:

$$\frac{1}{(m-1)!}\frac{\partial^{m-1}F[\emptyset(x;q)]}{\partial q^{m-1}}\Big|_{q=0} = \frac{1}{(m-1)!}\frac{\partial^{m-1}F[\sum_{i=0}^{\infty}u_i(x)q^i]}{\partial q^{m-1}}\Big|_{q=0} = \tag{2.83}$$

$$A_{m-1}[u_0, u_1, \dots, u_{m-1}] = A_{m-1}(t)$$

where $A_{m-1}(t)$ is called the Adomian polynomials. For more details about Adomian polynomials see [49].

Then we can write equation (2.82) as:

$$R_m(\vec{u}_{m-1}) = \chi_m u_{m-1}(x) - \lambda \int_a^b k(x,t)A_{m-1}(t)dt\,, \quad m \geq 1 \tag{2.84}$$

where $A_i(t)$ are the Adomain polynomial for the nonlinear term $F[u(t)]$ in equation (2.47).

Now, using equation (2.81) and according to equation (2.84) we can compute u_m and the Homotopy series solution of equation (2.47) is:

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) \tag{2.85}$$

and the n^{th}-order approximation solution is:

$$\hat{u}_n(x) = u_0(x) + \sum_{m=1}^{n} u_m(x), \quad n \geq 1 \tag{2.86}$$

According to theorem (2.13) equation (2.86) is the exact solution of equation (2.47) [2].

2.2.6 h-Curve

As mentioned previously, the value of h is very important for the convergence of equation (2.87). So that, the series solution converges fast enough in a large enough region.

When we solve equation (2.47) we have a series of solution with two variables x and h. In order to determine the optimal value of h we use the so-called h-curve.

Because h is an independent variable, it is easy to plot a curve versus h. There will be an interval over that. By taking different values of h from this interval, they all go to the exact value of the solution. Therefore, if the solution is unique, all the values of h are converge to the same value with different speed of convergent. So in $h-$curve there is a horizontal line segment corresponds to an area called the area valid for h denote by R_h.

In conclusion, $h-$curve provides us with a suitable way to show the effect of h on the convergence area and the rate of solution of the series[31].

Chapter Three

Numerical Examples and Results

Chapter Three

Numerical Examples and Results

In this chapter we will present some illustrative examples and carry out comparison between the accuracy and convergence of the two methods; Haar Wavelet method and Homotopy Analysis method.

These examples are based on a methodology for finding a solution of nonlinear Fredholm integral equation of the second kind using Maple and Matlab software.

Example 3.1

Consider the nonlinear Fredholm integral equation of the second kind:

$$u(x) = \cos \pi x + \frac{14}{9}\frac{(x \cos \pi x)}{\pi^2} + \int_0^1 xt \cos \pi x \, u^3(t)dt \qquad (3.1)$$

with the exact solution $u(x) = \cos \pi x$.

The Haar wavelet method and Homotopy Analysis method were used to solve equation (3.1).

(a) Haar Wavelet method

Here we use $l = 2M$ for simplicity.

To solve equation (3.1) using Haar wavelet method, the following algorithm was implemented using the Matlab software:

Algorithm (3.1):

1. Input the fixed positive integer l, Maximum iteration, and Tolerance.
2. Calculate the collocation points (2.41) $x_r \; and \; t_p, \;\; r, p = 0,1,..,l$
3. Input $f(x), k(x,t)$
4. Calculate $f(x_r), k(x_r, t_p)$
5. Input $\emptyset(u(t))$
6. Calculate $\emptyset(u(t_p))$
7. Calculate the algebraic nonlinear system $u(x_r)$

$$u(x_r) = f(x_r) + \frac{\lambda}{l}\sum_{j=1}^{l} k(x_r, t_p)\emptyset\left(u(t_p)\right)$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{2M} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{2M} \end{bmatrix} + \frac{\lambda}{2M}\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1\,2M} \\ k_{21} & k_{22} & \cdots & k_{2\,2M} \\ \vdots & \vdots & \ddots & \vdots \\ k_{2M\,1} & k_{2M\,2} & \cdots & k_{2M\,2M} \end{bmatrix}\begin{bmatrix} \emptyset_1 \\ \emptyset_2 \\ \vdots \\ \emptyset_{2M} \end{bmatrix}$$

8. Solve the Algebraic nonlinear system $u(x_r)$using Newton Raphson method for nonlinear equations to get the solution of the system.

Therefore, the following results have been obtained:

Figure (3.1) shows a comparison plot between exact and approximate solutions at $l = 2$.

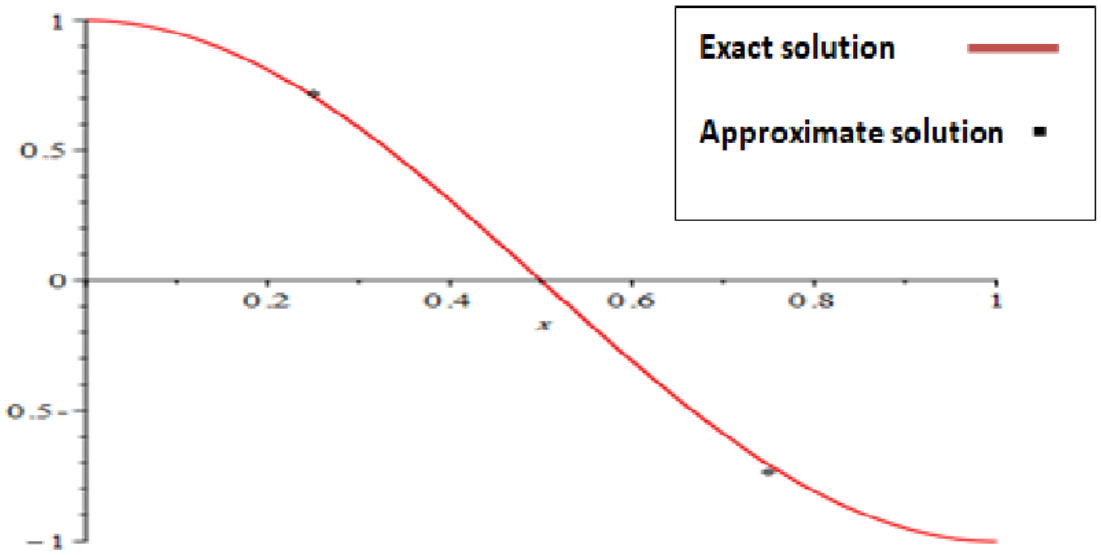

Fig. (3.1): Comparison between exact and approximate solutions at $l = 2$.

Table (3.1): Shows the exact and approximate solutions, errors and relative errors for equation (3.1) at $l = 8$:

x	Exact solution	Approximate solution	Error	Relative error
0.0625	0.9807852804	0.9808419174	5.7×10^{-5}	5.8×10^{-5}
0.1875	0.8314696123	0.8316136561	1.4×10^{-4}	1.7×10^{-4}
0.3125	0.5555702329	0.5557306446	1.6×10^{-4}	2.9×10^{-4}
0.4375	0.1950903220	0.1951691827	7.9×10^{-5}	4.04×10^{-4}
0.5625	-0.1950903220	-0.1951917143	1.01×10^{-4}	5.2×10^{-4}
0.6875	-0.5555702329	-0.5559231386	3.6×10^{-4}	6.4×10^{-4}
0.8125	-0.8314696123	-0.8320938020	6.2×10^{-4}	7.5×10^{-4}
0.9375	-0.9807852804	-0.9816348366	8.5×10^{-4}	8.7×10^{-4}

Figure (3.2) shows a comparison plot between exact and approximate solutions at $l = 8$.

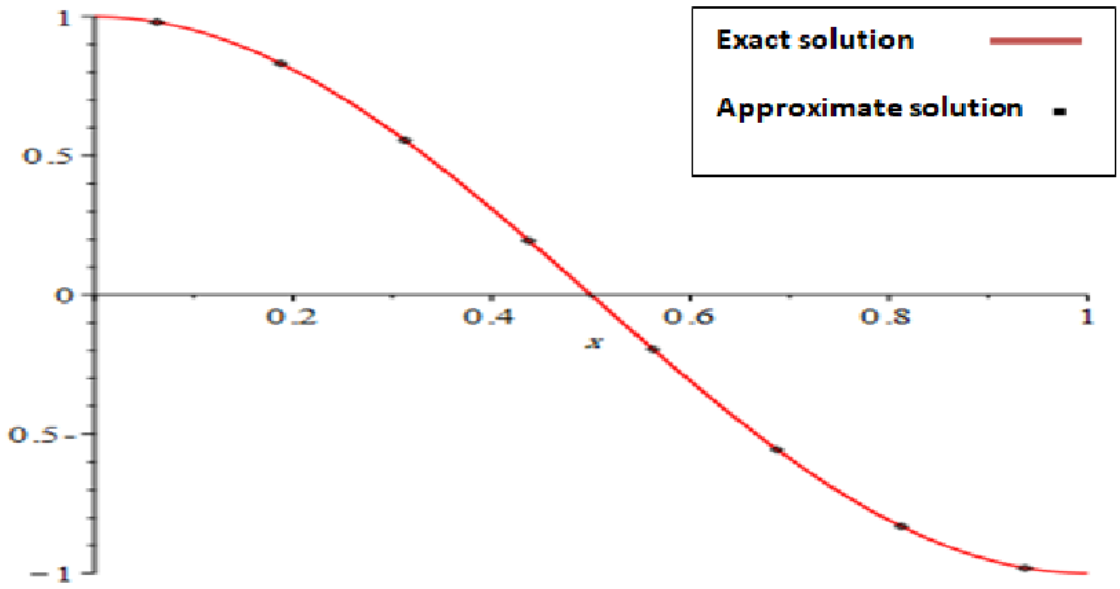

Fig. (3.2): Comparison plot of exact and approximate solution at $l = 8$.

(b) Using Homotopy Analysis method:

Here we take the initial approximation $u_0(x) = f(x)$

To solve equation (3.1) using Homotopy Analysis method the following Algorithm was implemented with help of Maple software.

Algorithm (3.2):

1. Input $\emptyset\big(u(t)\big), u_0(t), k(x,t), f(x)$, the positive fixed integer m and $Lampda$
2. Calculate Adomian polynomial $A_i(t)$.
3. Input χ_m.
4. Calculate $R_m,\ and\ u_m$, we get the following results:

$u_1(x) = 0.2433\ h\ x\ cos(\pi x)$

$u_2(x) = 0.3938\ h^2\ x\cos(\pi x) + 0.2433\ h\ x\cos(\pi x)$

$u_3(x) = 0.6664h^3\ x\cos(\pi x) + 0.7876\ h^2\ x\cos(\pi x)\ +$

$0.2433\ h\ x\cos(\pi x)$

$$u_4(x) = 1.1747\, h^4\, x\cos(\pi x) + 1.9994\, h^3\, x\cos(\pi x) + 1.1814\, h^2\, x\cos(\pi x) + 0.2433\, h\, x\cos(\pi x)$$

$$u_5(x) = 2.1462\, h^5 x\cos(\pi x) + 4.6991\, h^4\, x\cos(\pi x) + 3.9988 h^3\, x\cos(\pi x) + 1.5752\, h^2\, x\cos(\pi x) + 0.2433\, h\, x\cos(\pi x)$$

$$\vdots$$

figure (3.3) shows the $8^{th}, 9^{th} and\ 10^{th}$ order approximation solutions with respect to h at $x = 1$ (the h-curve):

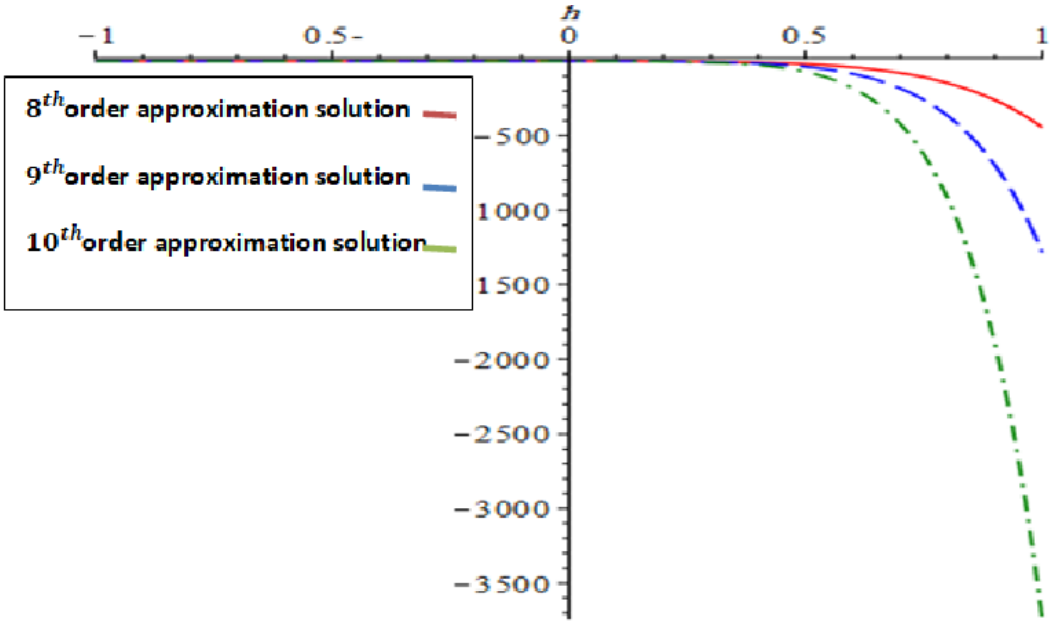

Fig.(3.3): the h-curve of $8^{th}, 9^{th}, and\ 10^{th}$ order approximation solutions with respect to h at $x = 1$.

Table (3.2): Shows the values of the percentage relative errors in reconstruction of the exact solution at several values of h when $x = 1$:

m	$h = -1$	$h = -0.7$	$h = -0.5$	$h = -0.3$	$h = -0.1$
2	6.5×10^{-2}	9.9×10^{-3}	1.3×10^{-2}	4.7×10^{-2}	0.1129
4	5.6×10^{-2}	1.7×10^{-3}	1.8×10^{-3}	1.6×10^{-2}	8.1×10^{-2}
10	0.1099	4.02×10^{-5}	1.1×10^{-5}	8.7×10^{-4}	3.2×10^{-2}
16	0.3960	1.7×10^{-6}	3.4×10^{-8}	6.2×10^{-5}	1.3×10^{-2}

From table (3.2) it is clear that at $h = -0.5$ gives the smallest relative error.

Table (3.3): The following table shows the exact and approximate solutions, errors and relative errors at different points of $x \in [0, 1]$ with $h = -0.5$ and $m = 8$:

x	Exact solution	Approximate solution	Error	Relative error
0	1	1	0	0
0.1	0.9510565163	0.9510619365	5.4×10^{-6}	5.7×10^{-6}
0.2	0.8090169943	0.8090262154	9.2×10^{-6}	1.1×10^{-5}
0.3	0.5877852522	0.5877953016	1×10^{-5}	1.7×10^{-5}
0.4	0.3090169938	0.3090240382	7.04×10^{-6}	2.3×10^{-5}
0.5	0	-2.05×10^{-10}	2.05×10^{-10}	–
0.6	−0.3090169942	−0.3090275606	1.05×10^{-5}	3.4×10^{-5}
0.7	−0.5877852527	−0.5878087012	2.3×10^{-5}	4×10^{-5}
0.8	−0.8090169945	−0.8090538791	3.7×10^{-5}	4.6×10^{-5}
0.9	−0.9510565165	−0.9511052969	4.9×10^{-5}	5.1×10^{-5}
1	−1	−1.000056990	5.7×10^{-5}	5.7×10^{-5}

Table (3.4): The following table shows the exact and approximate solutions, and errors at different point of $x \in [0, 1]$ with $h = -0.5$ and $m = 16$:

x	Exact solution	Approximate solution	Error	Relative error
0	1	1	0	0
0.1	0.9510565163	0.9510565132	3.1×10^{-9}	3.3×10^{-9}
0.2	0.8090169943	0.8090169886	5.7×10^{-9}	7.05×10^{-9}
0.3	0.5877852522	0.5877852460	6.2×10^{-9}	1.05×10^{-8}
0.4	0.3090169938	0.3090169896	4.2×10^{-9}	1.4×10^{-8}
0.5	0	-2.05×10^{-10}	2.05×10^{-10}	–
0.6	−0.3090169942	−0.3090169878	6.4×10^{-9}	2.1×10^{-8}
0.7	−0.5877852527	−0.5877852385	1.4×10^{-8}	2.4×10^{-8}
0.8	−0.8090169945	−0.8090169720	2.3×10^{-8}	2.8×10^{-8}
0.9	−0.9510565165	−0.9510564854	3.1×10^{-8}	3.3×10^{-8}
1	−1	−0.9999999660	3.4×10^{-8}	3.4×10^{-8}

In figure (3.4) we plot the exact and approximate solutions of example (3.1) at $m = 16$:

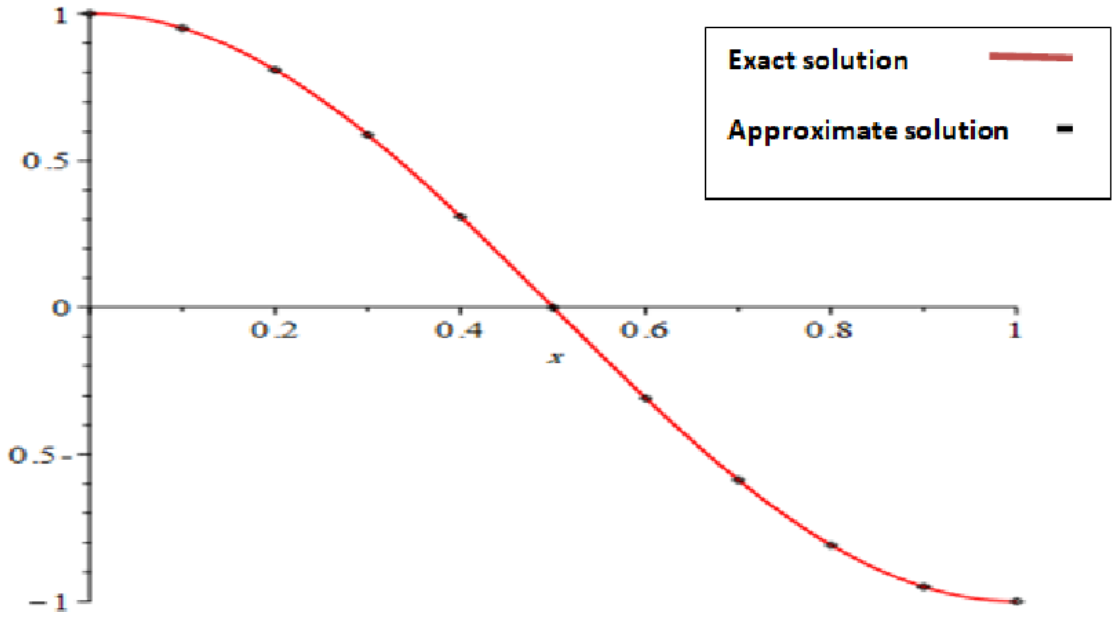

Fig.(3.4): Comparison between exact solution (Solid line) and approximate solution (Dot line) at $m = 16$ ($x \in [0, 1]$).

Results in tables (3.1) and (3.3) show clearly the maximum relative errors are 8.7×10^{-4} at $l = 8$ and 5.7×10^{-5} at $m = 8$ respectively.

As a result, it is clear that in example (3.1) Homotopy analysis method is more accurate than Haar wavelet method.

Example 3.2

Consider the nonlinear Fredholm integral equation of the second kind:

$$u(x) = -x^2 - \frac{x}{3}\left(2\sqrt{2} - 1\right) + 2 + \int_0^1 xt\sqrt{u(t)}dt \qquad (3.2)$$

with exact solution $u(x) = 2 - x^2$.

The Haar wavelet method and Homotopy Analysis method were used to solve equation (3.2).

(a) Haar wavelet method:

Algorithm (3.1) implements the Haar Wavelet method using Matlab software. Therefore, we obtain the following results:

Figure (3.5) shows a comparison plot between exact and approximate solutions at $l = 2$.

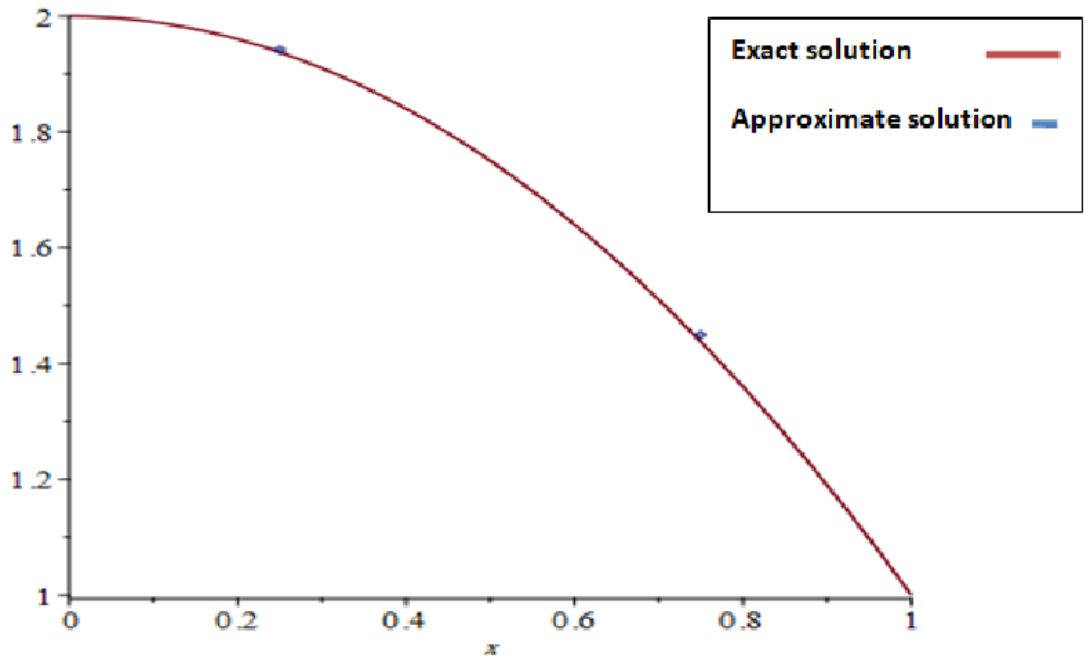

Fig. (3.5): Comparison between Exact and Approximate solutions at $l = 2$.

Table (3.5): Shows the exact and approximate solutions, errors and relative errors at $l = 8$:

x	Exact solution	Approximate solution	Error	Relative error
0.0625	1.996093750	1.9961605859	6.6×10^{-5}	3.3×10^{-5}
0.1875	1.964843750	1.9650442577	2×10^{-4}	1×10^{-4}
0.3125	1.902343750	1.9026779296	3.3×10^{-4}	1.7×10^{-4}
0.4375	1.808593750	1.8090616015	4.6×10^{-4}	2.5×10^{-4}
0.5625	1.683593750	1.6841952733	6×10^{-4}	3.5×10^{-4}
0.6875	1.527343750	1.5280789452	7.3×10^{-4}	4.8×10^{-4}
0.8125	1.339843750	1.3407126170	8.6×10^{-4}	6.4×10^{-4}
0.9375	1.121093750	1.1220962889	1×10^{-3}	8.9×10^{-4}

Figure (3.6) shows a comparison plot between exact and approximate solutions at $l = 8$.

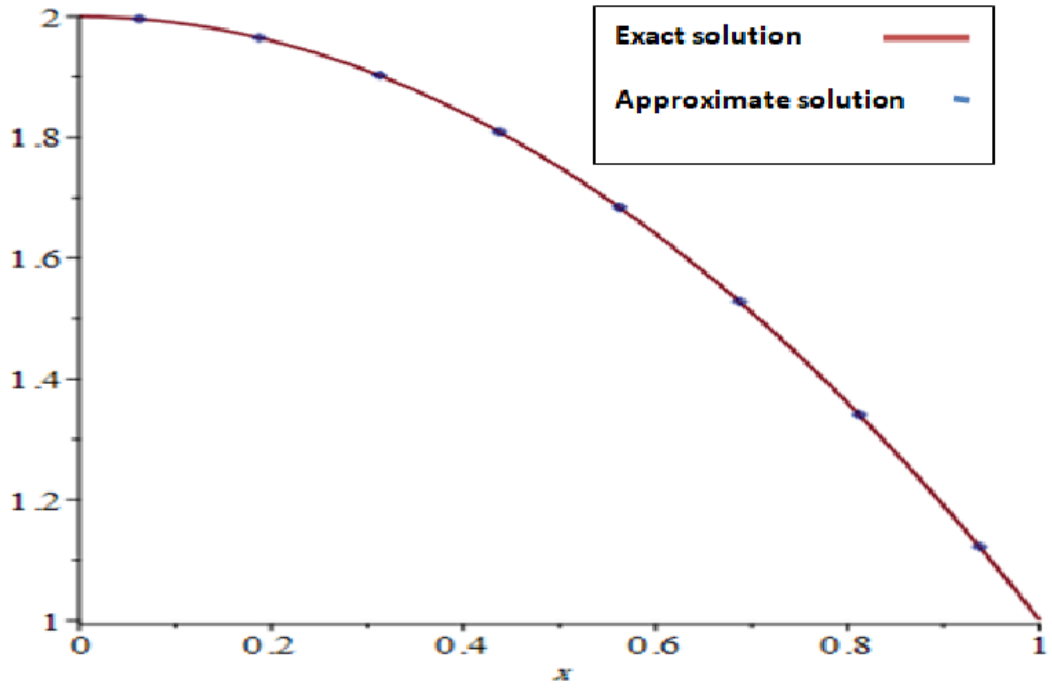

Fig. (3.6): Comparison between exact and approximate solutions at $l = 8$.

(b) Homotopy Analysis method:

Here we take the initial approximation $u_0(x) = f(x)$

Then, Algorithm (3.2) implements the Homotopy Analysis method using Maple software. Therefore, we obtain the following results:

$$u_1(x) = -0.512hx$$

$$u_2(x) = -0.418h^2x - 0.512hx$$

$$u_3(x) = -0.328h^3x - 0.836h^2x - 0.512hx$$

$$u_4(x) = -0.239h^4x - 0.984h^3x - 1.255h^2x - 0.512hx$$

$$u_5(x) = -0.152h^5x - 0.959h^4x - 1.968h^3x - 1.673h^2x - 0.512hx$$

$$\vdots$$

Figure (3.7) shows the $6^{th}, 7^{th}$ *and* 8^{th} order approximation solutions with respect to h at $x = 1$:

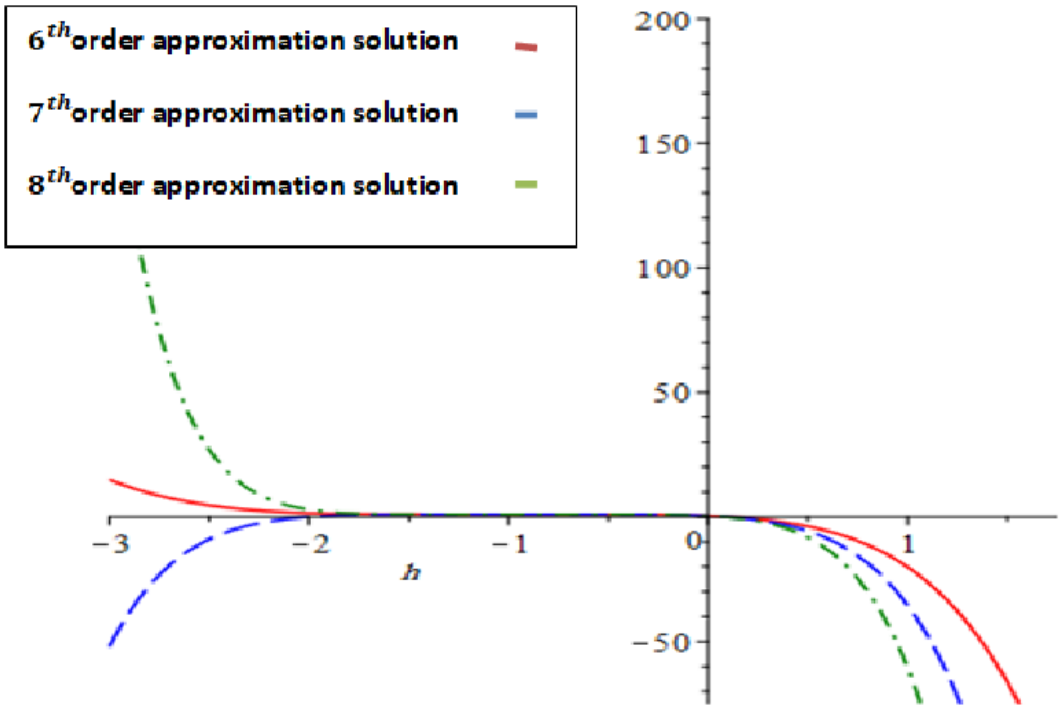

Fig.(3.7): The h-curve of $6^{th}, 7^{th}$ *and* 8^{th}order approximation solutions with respect to h at $x = 1$.

Table (3.6): Shows the values of the percentage relative errors in reconstruction of the exact solution at several values of h when $x = 2$:

m	$h = -1$	$h = -0.8$	$h = -0.75$	$h = -0.5$	$h = -0.2$
2	3.7×10^{-3}	5.7×10^{-2}	7.6×10^{-2}	0.2019	0.4213
4	3.6×10^{-4}	4.1×10^{-3}	7.8×10^{-3}	6.3×10^{-2}	0.2901
5	2.1×10^{-4}	1.04×10^{-3}	2.4×10^{-3}	3.5×10^{-2}	0.2403
8	7.9×10^{-5}	1.2×10^{-5}	5.9×10^{-5}	5.8×10^{-3}	0.1361
10	4.5×10^{-5}	2.2×10^{-6}	5.9×10^{-6}	1.7×10^{-3}	9.2×10^{-2}
16	5.2×10^{-5}	4.9×10^{-7}	4.7×10^{-6}	1.7×10^{-3}	9.2×10^{-2}

From table (3.6) it is clear that as m goes to infinity $h = -0.8$ gives the smallest relative error.

Table (3.7): the following table shows the exact solution, approximate solution, and errors at different point of $x \in [0, 1]$ with $h = -0.8$ and $m = 8$:

x	Exact solution	Approximate solution	Error	Relative Error
0	2	2	0	0
0.1	1.99	1.989998712	1.2×10^{-6}	6.4×10^{-7}
0.2	1.96	1.959997424	2.5×10^{-6}	1.3×10^{-6}
0.3	1.91	1.909996137	3.8×10^{-6}	2.02×10^{-6}
0.4	1.84	1.839994848	5.1×10^{-6}	2.8×10^{-6}
0.5	1.75	1.749993560	6.4×10^{-6}	3.6×10^{-6}
0.6	1.64	1.639992272	7.7×10^{-6}	4.7×10^{-6}
0.7	1.51	1.509990982	9.01×10^{-6}	5.9×10^{-6}
0.8	1.36	1.359989696	1.03×10^{-5}	7.5×10^{-6}
0.9	1.91	1.189988410	1.1×10^{-5}	9.7×10^{-6}
1	1	0.9999871194	1.2×10^{-5}	1.2×10^{-5}

In Figure (3.8) we plot the exact and approximate solutions of example (3.1) at $m = 8$:

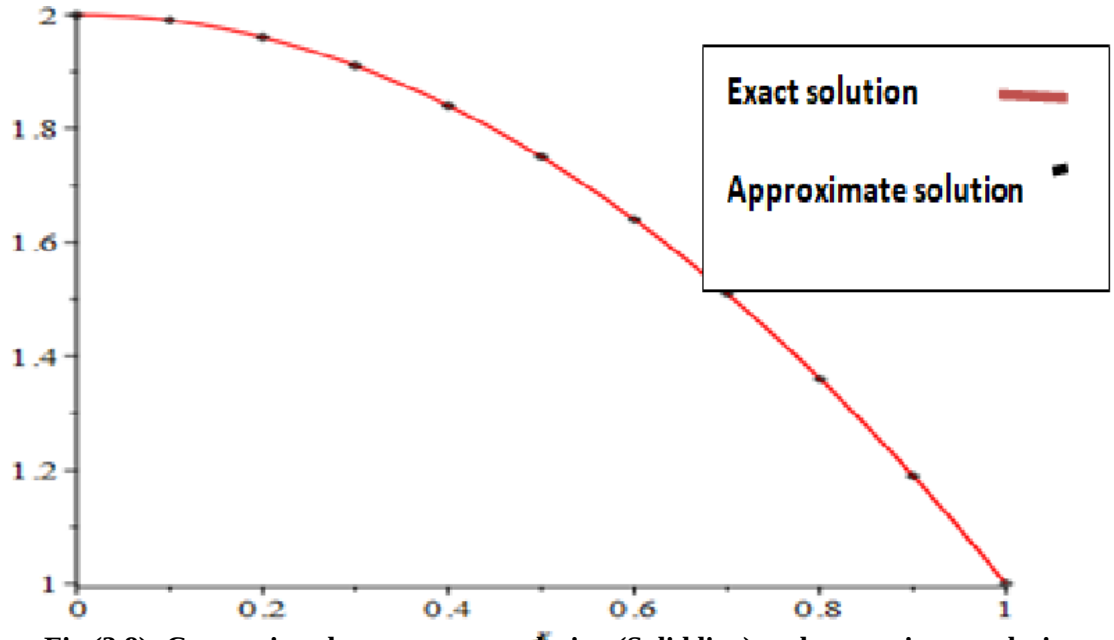

Fig.(3.8): Comparison between exact solution (Solid line) and approximate solution (Dot line) at $m = 8$ ($x \in [0, 1]$).

Results in tables (3.5) and (3.7) show clearly the maximum relative errors are 8.9×10^{-4} at $l = 8$ and 1.2×10^{-5} at $m = 8$ respectively.

As a result, it is clear that in example (3.2) Homotopy analysis method is more accurate than Haar wavelet method.

Example 3.3

Consider the nonlinear Fredholm integral equation of the second kind:

$$u(x) = x\ln(2) + \int_0^1 \frac{xt}{1+u(t)}\,dt \tag{3.3}$$

with exact solution $u(x) = x$.

The Haar wavelet method and Homotopy Analysis method were used to solve equation (3.3).

(a) Haar wavelet method:

Algorithm (3.1) implements the Haar wavelet method using Matlab software. Therefore, we obtain the following results:

Figure (3.9) displays both the exact and approximate solutions at $l = 2$.

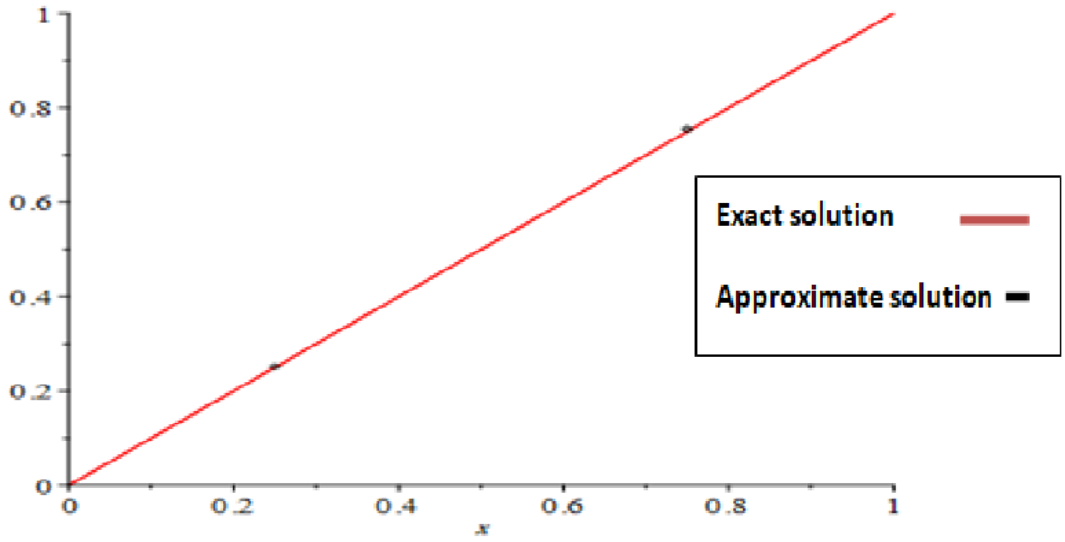

Fig. 3.9: Comparison between Exact and Approximate solutions at $l = 2$.

Table (3.8): Shows the exact and approximate solutions, errors and relative errors at $l = 32$:

x	Exact solution	Approximate solution	Error	Relative error
0.0156	0.015625	0.0156254280	4.3×10^{-7}	2.7×10^{-5}
0.1718	0.171875	0.1718797087	4.7×10^{-6}	2.7×10^{-5}
0.3281	0.328125	0.3281339893	8.9×10^{-6}	2.7×10^{-5}
0.4843	0.484375	0.4843882700	1.3×10^{-5}	2.7×10^{-5}
0.6406	0.640625	0.6406425507	1.6×10^{-5}	2.7×10^{-5}
0.7968	0.796875	0.7968968313	2.2×10^{-5}	2.7×10^{-5}
0.9531	0.953125	0.9531511120	2.6×10^{-5}	2.7×10^{-5}
0.9843	0.984375	0.9844019681	2.8×10^{-5}	2.7×10^{-5}

Figure (3.10) shows a comparison plot between exact and approximate solutions at $l = 32$.

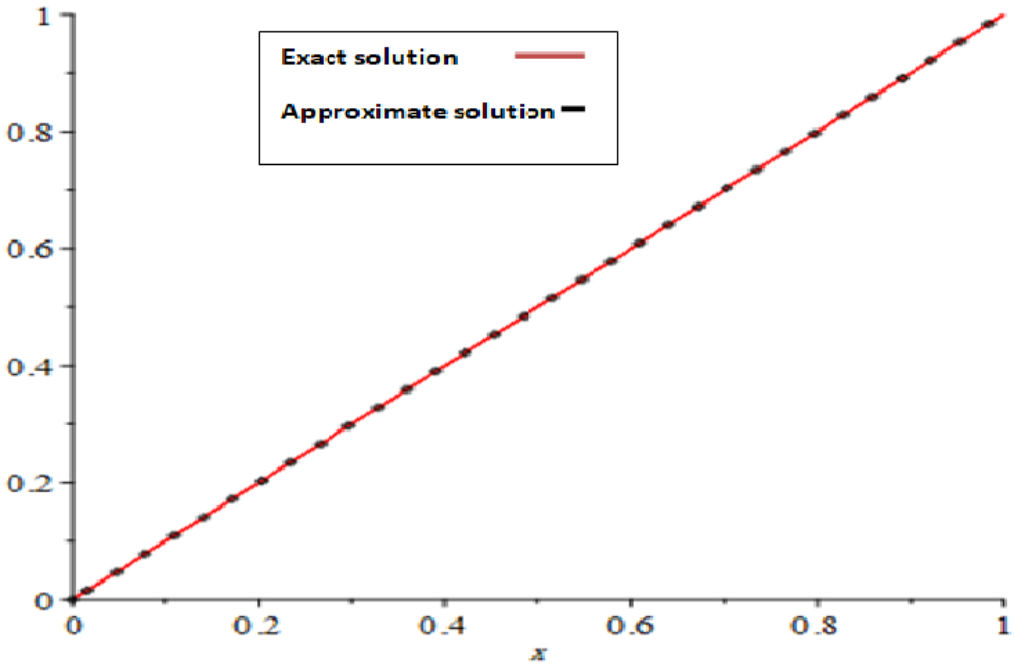

Fig. 3.10: Comparison between exact and approximate solutions at $l = 32$.

(b) Homotopy Analysis method:

Here we take the initial approximation $u_0(x) = f(x)$

Then, Algorithm (3.2) implements the Homotopy Analysis method using Maple software. Therefore, we obtain the following results:

$$u_1(x) = -0.3467hx$$

$$u_2(x) = -0.3981h^2x - 0.3467hx$$

$$u_3(x) = -0.6454h^3x - 0.7961h^2x - 0.3467hx$$

$$u_4(x) = -0.5548h^4x - 1.3961h^3x - 1.1942h^2x - 0.3467hx$$

$$u_5(x) = -0.6753h^5x - 2.2192h^4x - 2.7922h^3x - 1.5923h^2x - 0.3467hx$$

$$\vdots$$

Figure (3.11) show the $6^{th}, 7^{th}$ *and* 8^{th}order approximation solutions with respect to h at $x = 1$:

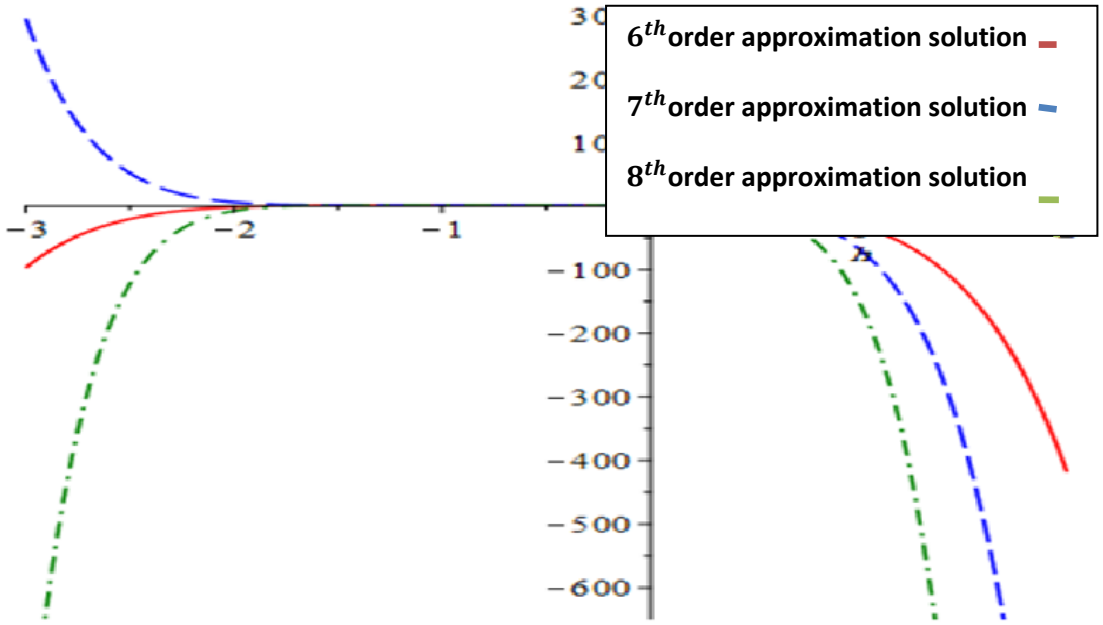

Fig.(3.11): The h-curve of $6^{th}, 7^{th}$ *and* 8^{th}order approximation solutions with respect to h at $x = 1$.

Table (3.9): Shows the values of the percentage relative errors in reconstruction of the exact solution at several values of h when $x = 1$:

m	$h=-1$	$h=-0.9$	$h=-0.6$	$h=-0.5$	$h=-0.2$
2	1.2×10^{-2}	5.2×10^{-3}	3.4×10^{-2}	6×10^{-2}	0.1841
4	1.9×10^{-3}	4.1×10^{-4}	4.5×10^{-3}	1.3×10^{-2}	0.1110
5	8.7×10^{-4}	1.4×10^{-4}	1.7×10^{-3}	5.9×10^{-3}	8.6×10^{-2}
8	1.1×10^{-4}	7.5×10^{-6}	8.9×10^{-5}	6.2×10^{-4}	4.1×10^{-2}
10	3.2×10^{-5}	1.2×10^{-6}	1.3×10^{-5}	1.4×10^{-4}	2.5×10^{-2}
16	1.04×10^{-6}	6.5×10^{-9}	4.4×10^{-8}	1.9×10^{-6}	5.8×10^{-3}
20	1.2×10^{-7}	2.3×10^{-10}	1.03×10^{-9}	1.07×10^{-9}	2.2×10^{-3}

From table (3.9) it is clear that as m goes to infinity $h = -0.9$ gives the smallest relative error.

Table (3.10): The following table shows the exact and approximate solutions, and errors at $h = -0.9$ and $m = 20$:

x	Exact solution	Approximate solution	Error	Relative Error
0	0	0	0	–
0.1	0.1	0.0999999999	2.3×10^{-11}	2.3×10^{-10}
0.2	0.2	0.1999999999	4.5×10^{-11}	2.3×10^{-10}
0.3	0.3	0.2999999999	6.8×10^{-11}	2.3×10^{-10}
0.4	0.4	0.3999999999	9.07×10^{-11}	2.3×10^{-10}
0.5	0.5	0.4999999999	1.1×10^{-10}	2.3×10^{-10}
0.6	0.6	0.5999999999	1.4×10^{-10}	2.3×10^{-10}
0.7	0.7	0.6999999999	1.6×10^{-10}	2.3×10^{-10}
0.8	0.8	0.7999999999	1.8×10^{-10}	2.3×10^{-10}
0.9	0.9	0.8999999999	2.04×10^{-10}	2.3×10^{-10}
1	1	0.9999999999	2.3×10^{-10}	2.3×10^{-10}

In figure (3.12) we plot the exact and approximate solutions of example (3.1) at $m = 20$:

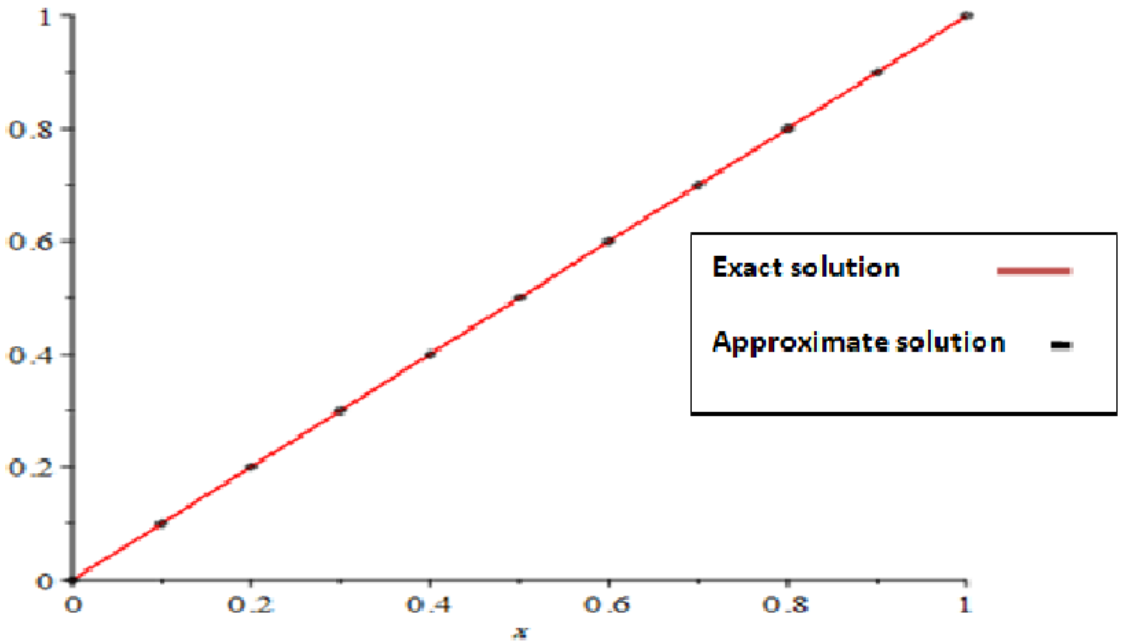

Fig.(3.12): Comparison between exact solution (Solid line) and approximate solution (Dot line) at $m = 20$ ($x \in [0, 1]$).

Results in tables (3.8) and (3.10) show clearly the maximum relative errors are 2.7×10^{-5} at $l = 32$ and 2.3×10^{-10} at $m = 20$ respectively.

As a result, it is clear that in example (3.3) Homotopy analysis method is more accurate than Haar wavelet method.

Conclusions

In this work, two numerical methods for solving nonlinear Fredholm integral equation of the second kind are presented. These are: Haar Wavelet method and Homotopy Analysis method (HAM).

In addition, these numerical methods we implemented in a form of algorithms to solve some numerical examples with known analytical solutions. Numerical results have shown to be in a close agreement with the analytical ones. Moreover, the Homotopy Analysis Method is one of the most powerful numerical technique for solving nonlinear Fredholm integral equation of the second kind in comparison with other numerical methods.

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Appendix

Matlab Code for example (3.1) (a):

```
clc;
clearall;
M =2;
z = 1;
MaxIter = 400;
tol = .00001;
fori= 1: 1: 2*M
x(z) = (i - 0.5)/(2*M);
   z = z+1;
end
Y =@(x) cos(pi*x)+(14/9)*((x*cos(pi*x))/(pi^2));
k=1;
fori=1:2*M
F(k) = Y(x(i));
   k=k+1;
end
F=F';
YY = @(x,y) x*y*cos(pi*x);
for j=1:2*M
fori=1:2*M
U1(j,i) = 1/(2*M)*YY(x(j),x(i));
end
end
U=[eye(2*M,2*M),-U1];
A = sym('A', [M 4]);
A=A';
A=A(:);
A=sort(A');
k =1;
fori = (2*M+1):4*M
A(i) = A(k)^3;
   k =k+1;
end
newsystem=A*U';
newsystem = [newsystem-F'];
system=[U,F];
formatlong
 Anew = [A(1:2*M)];

fori = 1: 2*M
p0(1,i) =.5;
end
J = jacobian(newsystem);
w= zeros(MaxIter,2*M);
w(1,:) = p0;
dsnorm = inf;
```

```
iter = 1;
whiledsnorm>tol&&iter<MaxIter
    q=subs(J,Anew,w(iter,:));
ds = -inv(q)*subs(newsystem,Anew,w(iter,:))';

w(iter+1,:) = w(iter,:) + ds';
dsnorm = norm(ds,inf);
iter = iter+1;

end
w = w(1:iter,:)
```

Maple Code for example (3.1) (b):

```
> restart
> with(Student) :
> Digits := 10 :
> Uex := x→cos(Pi·x) :
> F := u→u^3 :
> u[0] := cos(π·x) + 14/9 · x·cos(π·x)/π^2 :
> u1[0] := cos(π·x) + 14/9 · x·cos(π·x)/π^2 :
> k := (x, t)→x·t·cos(Pi·x) :
> m := 4 : a := 0 : b := 1 : Lampda := 1 : v := 1 :
> A[0] := F(u[0]) :
> for n from 0 to m do A[n] := 1/n! · d^n/dp^n ( F( sum(u[i]·p^i, i = 0 .. n) ) ); od:
> p := 0 :
> for i from 0 to m do A[i] := subs(x = t, A[i]); od:
> xm := Array(0 ..m) :
> xm[0] := 0 : xm[1] := 0 :
> for i from 2 to m do xm[i] := 1; od:
>
   for l from 1 to m do
   R[l] := evalf( xm[l]·u1[l − 1] − Lampda · int(k(x, t)·A[l − 1], t = a .. b) );
   u1[l] := evalf(xm[l]·u1[l − 1] + h·R[l]);
   u[l] := subs(x = t, u1[l]);
   A[l − 1] := A[l − 1];
   od:

> U22 := evalf(sum(u1[s], s = 0 ..m − 2)) :
> U33 := evalf(sum(u1[s], s = 0 ..m − 1)) :
> U44 := evalf(sum(u1[s], s = 0 ..m)) :
> U2 := unapply(U22, x, h) : U3 := unapply(U33, x, h) : U4 := unapply(U44, x, h) :
> plot([U2(v, h), U3(v, h), U4(v, h)], h = −1 ..1, color = ["Red", "Blue", "Green"])
```

Matlab Code for example (3.2) (a):

```
clc;
clearall;
M =2;
z = 1;
MaxIter = 400;
tol = .00001;
fori= 1: 1: 2*M
x(z) = (i - 0.5)/(2*M);
   z = z+1;
end
Y =@(x) -x^2-(x/3)*(2*sqrt(2)-1)+2;
k=1;
fori=1:2*M
F(k) = Y(x(i));
   k=k+1;
end
F=F';
YY = @(x,y) x*y;
for j=1:2*M
fori=1:2*M
U1(j,i) = 1/(2*M)*YY(x(j),x(i));
end
end
U=[eye(2*M,2*M),-U1];
 A = sym('A', [M 4]);
A=A';
A=A(:);
A=sort(A');
k =1;
fori = (2*M+1):4*M
A(i) = sqrt(A(k));
   k =k+1;
end
newsystem=A*U';
newsystem = [newsystem-F'];
system=[U,F];
formatlong
 Anew = [A(1:2*M)];
fori = 1: 2*M
p0(1,i) =.5;
end
J = jacobian(newsystem);
w= zeros(MaxIter,2*M);
w(1,:) = p0;
dsnorm = inf;
iter = 1;
whiledsnorm>tol&&iter<MaxIter
```

```
    q=subs(J,Anew,w(iter,:));
ds = -inv(q)*subs(newsystem,Anew,w(iter,:))';

w(iter+1,:) = w(iter,:) + ds';
dsnorm = norm(ds,inf);
iter = iter+1;
end
w = w(1:iter,:)
```

Maple Code for example (3.2) (b):

```
> restart
> with(Student) :
> Digits := 10 :
> Uex := x→2 − x^2 :
> F := u→√u :
> u[0] := -x^2 − x/3·(2·√2 − 1) + 2 :
> u1[0] := -x^2 − x/3·(2·√2 − 1) + 2 :
> k := (x, t)→x·t :
> m := 4 : a := 0 : b := 1 : Lampda := 1 : v := 1 :
> A[0] := F(u[0]) :
> for n from 0 to m do A[n] := 1/n!·d^n/dp^n (F(sum(u[i]·p^i, i = 0 .. n))); od:
> p := 0 :
> for i from 0 to m do A[i] := subs(x = t, A[i]); od:
> xm := Array(0 .. m) :
> xm[0] := 0 : xm[1] := 0 :
> for i from 2 to m do xm[i] := 1; od:
>
   for l from 1 to m do
   R[l] := evalf(xm[l]·u1[l − 1] − Lampda·int(k(x, t)·A[l − 1], t = a .. b));
   u1[l] := evalf(xm[l]·u1[l − 1] + h·R[l]);
   u[l] := subs(x = t, u1[l]);
   A[l − 1] := A[l − 1];
   od:

> U22 := evalf(sum(u1[s], s = 0 .. m − 2)) :
> U33 := evalf(sum(u1[s], s = 0 .. m − 1)) :
> U44 := evalf(sum(u1[s], s = 0 .. m)) :
> U2 := unapply(U22, x, h) : U3 := unapply(U33, x, h) : U4 := unapply(U44, x, h) :
> plot([U2(v, h), U3(v, h), U4(v, h)], h = -3 .. 2, color = ["Red", "Blue", "Green"])
```

Matlab Code for example (3.3) (a):

```
clc;
clearall;
```

```
M =2;
z = 1;
MaxIter = 400;
tol = .00001;
fori= 1: 1: 2*M
x(z) = (i - 0.5)/(2*M);
   z = z+1;
end
Y =@(x) log(2)*x;
k=1;
fori=1:2*M
F(k) = Y(x(i));
   k=k+1;
end
F=F';
YY = @(x,y) x*y;
for j=1:2*M
fori=1:2*M
U1(j,i) = 1/(2*M)*YY(x(j),x(i));
end
end
U=[eye(2*M,2*M),-U1];
 A = sym('A', [M 4]);
A=A';
A=A(:);
A=sort(A');
k =1;
fori = (2*M+1):4*M
A(i) = 1/(1+A(k));
   k =k+1;
end
newsystem=A*U';
newsystem = [newsystem-F'];
system=[U,F];
formatlong
 Anew = [A(1:2*M)];
fori = 1: 2*M
p0(1,i) =.5;
end
J = jacobian(newsystem);
w= zeros(MaxIter,2*M);
w(1,:) = p0;
dsnorm = inf;
iter = 1;
whiledsnorm>tol&&iter<MaxIter
     q=subs(J,Anew,w(iter,:));
ds = -inv(q)*subs(newsystem,Anew,w(iter,:))';

w(iter+1,:) = w(iter,:) + ds';
dsnorm = norm(ds,inf);
```

```
iter = iter+1;

end
w = w(1:iter,:)
```

Maple Code for example (3.3) (b):

```
> restart
> with(Student) :
> Digits := 100 :
> Uex := x→x :
> F := u→1/(1+u) :
> u[0] := ln(2)·x :
> u1[0] := ln(2)·x :
> k := (x,t)→x·t :
> m := 5 : a := 0 : b := 1 : Lampda := 1 : v := 1 :
> A[0] := F(u[0]) :
> for n from 0 to m do A[n] := 1/n! · d^n/dp^n ( F( sum(u[i]·p^i, i=0..n) ) ); od:
> p := 0 :
> for i from 0 to m do A[i] := subs(x=t, A[i]); od:
> xm := Array(0..m) :
> xm[0] := 0 : xm[1] := 0 :
> for i from 2 to m do xm[i] := 1; od:
>
  for l from 1 to m do
  R[l] := evalf( xm[l]·u1[l-1] - Lampda·int(k(x,t)·A[l-1], t=a..b) );
  u1[l] := evalf(xm[l]·u1[l-1] + h·R[l]);
  u[l] := subs(x=t, u1[l]);
  A[l-1] := A[l-1];
  od:

> U22 := evalf(sum(u1[s], s=0..m-2)) :
> U33 := evalf(sum(u1[s], s=0..m-1)) :
> U44 := evalf(sum(u1[s], s=0..m)) :
> U2 := unapply(U22, x, h) : U3 := unapply(U33, x, h) : U4 := unapply(U44, x, h) :
> plot([U2(v,h), U3(v,h), U4(v,h)], h=-3..2, color=["Red", "Blue", "Green"])
```

جامعة النجاح الوطنية
كلية الدراسات العليا

طرق عددية لحل معادلة فريدهولم التكاملية غير الخطية من النوع الثاني

إعداد

هبة جلال محمود عودة

إشراف

أ. د. ناجي قطناني

قدمت هذه الرسالة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات المحوسبة بكلية الدراسات العليا في جامعة النجاح الوطنية، نابلس–فلسطين.

2019

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أ. د. ناجي قطناني

الملخص

في هذه الرسالة، تم التركيز على المعالجة العددية لمعادلة فريدهولم التكاملية غير الخطية من النوع الثاني بسبب أهميتها الهائلة في العديد من التطبيقات في مختلف المجالات.

بعد معالجة المفاهيم الأساسية لمعادلة فريدهولم التكاملية غير الخطية من النوع الثاني، تم التركيز على المعالجة العددية لهذه المعادلة. حيث تم تحقيق ذلك من خلال تطبيق طريقتين عدديتين، هما طريقة مويجات هار وطريقة هوموتوبي التحليلية. حيث تم عرض الإطار الرياضي لهذه الطرق العددية.

تم توضيح هذه الطرق العددية عن طريق حل بعض الأمثلة العددية وتم إجراء مقارنة بينها. أظهرت النتائج العددية بوضوح أن طريقة هوموتوبي التحليلية كانت أكثر فاعلية في حل معادلات فريدهولم التكاملية غير الخطية من النوع الثاني مقارنة بطريقة مويجات هار.