**An-Najah National University**
**Faculty of Graduate Studies**

# A Dynamic Programming Approach to Control Heat Equation with Random Walk Process Using HJB Equation

**By**

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# Dedication

I dedicate this thesis to my parents, especially my mother who has been a great source of inspiration and support. This thesis is also dedicated to my dear brothers Ali, Ala', Yaqoot and Ahmad. To my friends especially Batool and I thank my doctors and everyone who contributed to stand by my side for the completion of this work.

# Acknowledgments

Foremost, I thank almighty Allah for ability to finish this thesis. After that, I would like to express my sincere gratitude to my supervisors Dr.Mohammad Najeeb Ass'ad and Prof.Dimitrios Tsagkarogiannis for their patience, motivation, immense knowledge and continuous support.

# الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

**A Dynamic Programming Approach to Control Heat Equation with Random Walk Process Using HJB Equation**

اقر بأن ما اشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد، وأن هذه الرسالة ككل، أو أي جزء منها لم يقدم من قبل لنيل أية درجة علمية أو بحث علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى.

# Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

**Student's Name:** **اسم الطالب:**

**Signature:** **التوقيع:**

**Date:** **التاريخ:**

# Contents

# A Dynamic Programming Approach to Control Heat Equation with Random Walk Process Using HJB Equation

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## Abstract

The heat equation is considered with Random Walk and Brownian motion under the assumption of Bernoulli's, Binomial, Geometric and Poisson distributions for Markov chain.

Some numerical methods are also used to find a numerical solution of heat equation under certain conditions as finite difference method (explicit and implicit), Crank Nicolson method and method of lines.

Separation of variables method also used to determine an analytic solution of heat equation.

In addition, we have used the Hamilton Jacobi Bellman equation (HJB) and algebraic Riccati equation that arises in the linear quadratic regulator (LQR) to obtain the optimal control function for heat equation.

Finally, a comparison between exact and approximate solution for state space equation using Euler's method.

# Preface

- **Preface**

The heat equation is an important partial differential equation that describes the distribution of heat or change in temperature in a particular region over time [24].

The heat equation has the general form:

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} \tag{1}$$

Where

$U = U(x, t)$ is the temperature and is a function of space and time.

$D$ is a constant.

The heat equation predicts that if a hot object is placed in a box of cold water, the body temperature will decrease, and eventually after an unlimited time, without any external sources of heat, the temperature becomes homogeneous.

The heat equation is of fundamental importance in different scientific fields. In mathematics, it is the prototypical parabolic partial differential equation. In statistics, the heat equation is connected with the study of Brownian motion. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes [16]. Heat and mass transfer is used to understand how drug delivery devices work, how kidney dialysis works, and how to control heat for temperature-sensitive things [24].

Many scientists have worked on the heat equation, for example Jean Biot (1774-1862) has studied the heat conduction equation but he was unsuccessful to deal with the problem of incorporating external convection effects in heat conduction analysis. Joseph Fourier (1768-1830) determined how to solve the problem and gave the well-known Fourier's law. Ernst Schmidt (1892-1975) was a German scientist and pioneer in the field of Engineering thermodynamic, especially in heat and mass transfer.

Chamkha and Khaled (2000) have studied the effect of magnetic field on the coupled heat and mass transfer by mixed convection in a linearly stratified stagnation flow in the presence of an internal heat generation or absorption. EL-Hakiem (2000) studied thermal radiation effects on hydromagnetic free convection and flow through a highly porous medium bounded by a vertical plane surface. Chamkha (2000) has analyzed hydromagnetic mixed convection from a permeable semi-infinite vertical plate embedded in porous medium in heat dimension[24]. In recent years, many researchers have worked on the heat conduction equation (see for example [1, 15, 20, 22]).

In this thesis we have studied how to control the heat equation. In the first chapter of this thesis we derived the heat equation using stochastic process specifically random walk with Bernoulli's, Binomial, Geometric and Poisson distribution. Additionally, we used Brownian motion to derive the heat equation.

In the second chapter we solved the heat equation in one dimension using the separation of variables method subject to some specific boundary and initial conditions (Dirichlet and Neumann conditions). In the third chapter we solved the heat equation by well-known numerical methods. Namely the Finite Difference Method, Theta Method, Crank-Nicolson Method and Method of Lines.

In the last chapter we have studied the optimal control of heat equation. Optimal control is the process of determining optimal control function and state trajectories for a dynamic system over a period of time to minimize a performance index. Optimal control is closely related to the theory of calculus of variations. Some important scientists who contributed to the theory of optimal control Isaac Newton (1642-1727),Johann Bernoulli (1667-1748), Leonhard Euler (1707-1793), Ludovico Lagrange (1736-1813), Andrien Legendre (1752-1833), Carl Jacobi (1804-1851), William Hamilton (1805-1865), Karl Weierstrass (1815-1897), Adolph Mayer (1839-1907), and Oskar Bolza (1857-1942). Some important milestones in the development of optimal control in the $20^{th}$ century include the formulation dynamic programming by Richard Bellman (1920-1984) in the 1950s, the development of the minimum principle by Lev Pontryagin (1908-1988), and the formulation of the linear quadratic regulator by Rudolf Kalman in the 1960s.

Optimal control has found applications in many different fields, including robotics, process control, aerospace, engineering, finance, eco-

nomics and management science (see for example [6, 11, 13, 21]).

There are two types of control systems, the first is closed loop system and the other is open loop system [13].

Closed loop control systems are also called feedback control systems are very common in process control and electronic control systems.

In closed-loop system, a controller is used to compare the output of a system with the required condition and convert the error into a control action designed to reduce the error and bring the output of the system back to the desired response. Closed-loop control systems use feedback to determine the actual input to the system and can have more than one feedback loop.

Open loop systems are defined by the fact that the output signal or condition is neither measured nor feedback for comparison with the input signal or system set point. Therefore open loop systems are commonly referred to as non feedback systems. Such that the output has no influence or effect on the control action of the input signal.

Also, as an open-loop system does not use feedback to determine if its required output was achieved, it assumes that the desired goal of the input was successful because it cannot correct any errors it could make, and so cannot compensate for any external disturbances to the system. In this thesis we take the closed loop case.

# Chapter One

# Random Walk and the Heat Equation

## 1.1 Markov Chain Definition and Examples

A discrete time stochastic process is a sequence of random variables, $\{x_n\}_{n=0}^{\infty} = \{x_0, x_1, x_2, ...\}$, where $x_n \in S$ and $S$ is the discrete state space.

In general, $x_n$ are dependent. Therefore in order to describe the stochastic process, we need to know all the joint probability density functions: $P(x_0 = i_0, ..., x_n = i_n), \forall n = 0, 1, 2, ...$ and $\forall i_0, i_1, ..., i_n \in S$.

Consider a random walk starts at $x_0 = 0$, the conditional probability density functions can be specified as:

$P(x_1 = i_1)$

$P(x_2 = i_2 | x_1 = i_1)$

$P(x_3 = i_3 | x_1 = i_1, x_2 = i_2)$

$\vdots$

$P(x_n = i_n | x_1 = i_1, x_2 = i_2, ..., x_{n-1} = i_{n-1})$

recall that $P(x_2 = i_2 | x_1 = i_1) = \frac{P(x_2 = i_2, x_1 = i_1)}{P(x_1 = i_1)}$

**Definition 1.1.** *A stochastic process $\{x_n, n = 0, 1, 2, ...\}$ with discrete state space $S$ is called a discrete Markov chain if*

$$P(x_n = i_n | x_{n-1} = i_{n-1}, ..., x_0 = i_0) = P(x_n = i_n | x_{n-1} = i_{n-1}) \forall i_0, ..., i_n \in S$$

*which is called memoryless property ([14, 19])*

The conditional probability $P(x_n = i_n | x_{n-1} = i_{n-1})$ is called Transi-

tion Probability and it is denoted by $P^{n-1,n}_{i_{n-1},i_n}$

The transition matrix P for the Markov chain is the $N \times N$ matrix for finite Markov chain, whose $(i, j)$ entry is $P_{ij}$. The matrix P is a stochastic matrix, that is,

$$0 \leq P_{ij} \leq 1, \quad 0 \leq i, j \leq N \tag{1.1}$$

$$\sum_{j=0}^{N} P_{ij} = 1, \quad 0 \leq i \leq N \tag{1.2}$$

Any matrix satisfying (1.1) and (1.2) is called the transition matrix for a Markov chain.

where:

$P_{ij}$ is the probability that the system is at state $j$ out of state $i$ in one step.

**Definition 1.2.** *A discrete time Markov chain is said to be homogeneous in time if the probability of going from one state to another is independent of the time at which the step is being made. That is, for all states $i, j$*

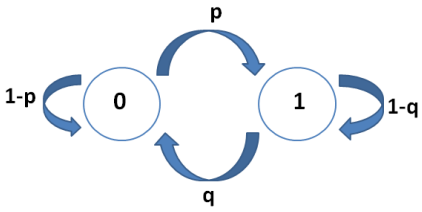
$$P(x_n = j | x_{n-1} = i) = P(x_{n+k} = j | x_{n+k-1} = i)$$

*$\forall k = -(n-1), -(n-2), ..., -1, 0, 1, ...$ otherwise the Markov chain is said to be non-homogeneous.*

**Example 1.3.** *A model for the state of the phone such that $x_n = 1$ or 0, when $x_n = 0$ that means at time $n$ the phone is free and $x_n = 1$ that means at time $n$ the phone is busy. Also at each time interval, we*

*assume that the probability of call comes in is p and no more than one call comes in the particular time interval that means if the phone is busy the incoming call doesn't occur at that time interval, also if the phone is busy at time interval the probability to be free is q at that time [19]. So this example gives a Markov chain with state space* $S = \{0, 1\}$ *and matrix:*

$$\mathrm{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}$$




**Example 1.4.** *Consider a random walker moving along the locations* $\{0, ..., N\}$ *such that at each step the walker moves one step to the right with probability p and one step to the left with probability* $q = 1 - p$*, but at the boundary points the walker moves inward with probability* 1*, [19]*

*so the transition matrix is given by*

$P(j, j+1) = p,\ P(j, j-1) = 1 - p,\ 0 < j < N$

$P(0, 1) = 1,\ P(N, N-1) = 1$

$P(i, j) = 0$*, for other sites.*

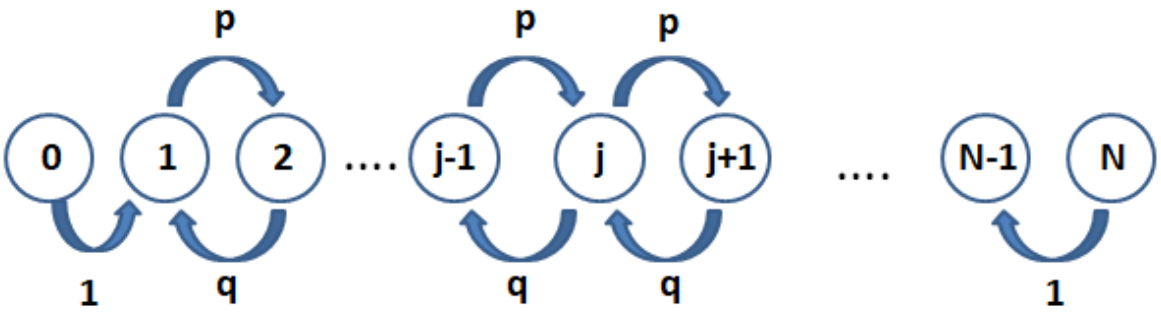



**Figure 1.1: Random walk with reflecting boundary**

## 1.2 Random Walk Definition and Examples

The concept of random walk was introduced for the first time by Karl Pearson 1905. However, in 1880, Lord Rayleigh applied this process without naming it to analyze a particular random vibration problem. Random walk is the best way to express the path that consist of a series of random steps. It may also indicate the path followed by a Brownian particle as it moves through a liquid.

Some fields of random walk implementations are finance, mathematics, economics, biology and physics.

**Definition 1.5.** *A random walk is a stochastic sequence* $\{S_n\}$ *that is defined by:*

$$S_n = x_1 + x_2 .... + x_n = \sum_{k=1}^{n} x_k$$

$$S_0 = 0$$

*Where* $\{x_k\}$ *are independent and identical distributed random variables.*

Consider a particle moves along the real line by steps, each step has

two choices either $x = 1$ or $x = -1$ , for example, one step to the right is with probability equal $p$ and to the left equal $1 - p = q$ that is $P(x = 1) = p$ and $p(x = -1) = 1 - p = q$, in this case it is called simple random walk, see figure (**??**), it is called symmetric random walk if $p = q = \frac{1}{2}$.

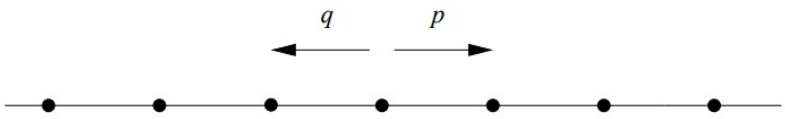



**Figure 1.2: Simple random walk**

The Random walk can happen in many situations, and is useful for analyzing different scenarios. One of the basic models of random walk, simple random walk on the integer lattice $\mathbb{Z}^d$ (d=1, one dimension). At each time step, a walker flips a fair coin and moves one step to the left or one step to the right depending on whether the coin comes up heads or tails.

Let $S_n$ denote the position of the walker at time $n$ [20].

If the walker starts at the origin ($x = 0$), so

$$S_n = x_1 + x_2 + ... + x_n$$

where $x_i$ equals $\pm 1$ represents the change in position and

$P(x_i = 1) = P(x_i = -1) = \frac{1}{2}$, see figure (1.3).

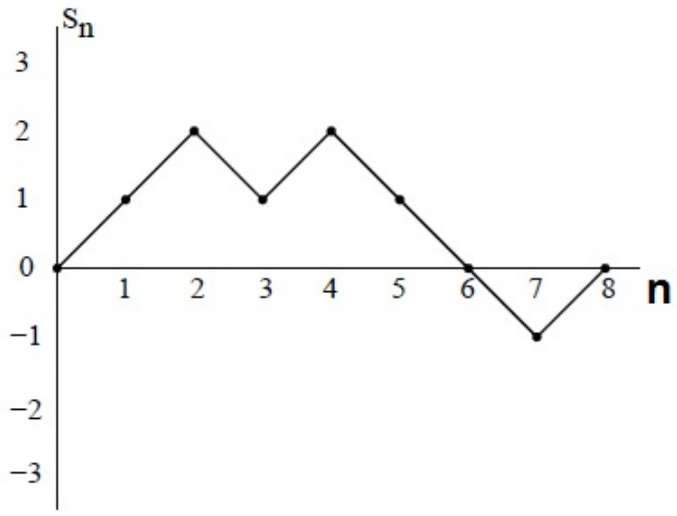



**Figure 1.3: Random walk with $S_0 = 0$**

## 1.3 Classifications of States

- Accessibility

  **Definition 1.6.** *A state $k$ is accessible from state $j$ if*

  $$P^n(j,k) = P(x_n = k | x_0 = j) > 0 \; for \; some \; n \geq 0$$

  *and it is denoted by $j \to k$ .*

  where

  $P^n(j,k)$ is the probability that at stage $n$, the system is in state $k$ given that it is initially in state $j$

- Communicability

  **Definition 1.7.** *A state $j$ communicates with state $k$, if state $j$ is accessible from state $k$, and state $k$ is accessible from state $j$, and it*

*is denoted by* $j \longleftrightarrow k$ *. In other words, two states communicate if and only if each state has a positive probability of eventually being reached by a chain starting in the other state.*

Properties of the symbol $\longleftrightarrow$ :

1. $i \longleftrightarrow i$ (reflexive).

2. if $i \longleftrightarrow j$, then $j \longleftrightarrow i$ (symmetric).

3. if $i \longleftrightarrow j$ and $j \longleftrightarrow k$, then $i \longleftrightarrow k$. (transitive).

So the relation $\longleftrightarrow$ is an equivalence relation on the state space. Also it partitions the state space into disjoint sets called *communication classes.*

**Definition 1.8.** *Chapman Kolmogorov equation is an identity relating the joint probability distributions of different sets of coordinates on a stochastic process, that is:*

$P^{n+m}(i,j) = \sum_{k \in S} P^n(i,k) P^m(k,j)$

*Proof.* 1. reflexive, since $P^0(i,i) = 1 > 0$.

2. symmetric, this is immediate from definition.

3. transitive, if $i \longleftrightarrow j$ then $P^n(i,j) > 0$, similarly, since

$j \longleftrightarrow k$, then $P^m(j,k) > 0$, therefore by the Chapman-Kolmogorov equation, we get:

$P^{n+m}(i,k) = \sum_{j \in S} P^n(i,j) P^m(j,k) \geq P^n(i,j) P^m(j,k) > 0.$ □

**Example 1.9.** *Given the following Markov chain*

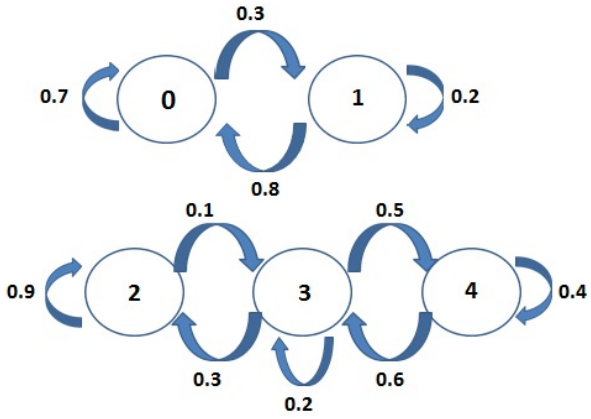



**Figure 1.4: Transition diagram**

$$\mathrm{p} = \begin{bmatrix} 0.7 & 0.3 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0.3 & 0.2 & 0.5 \\ 0 & 0 & 0 & 0.6 & 0.4 \end{bmatrix}$$

*For example, states 0 and 1 are accessible from state 0,*

*states 2, 3, and 4 are accessible from state 3*

$0 \longleftrightarrow 1,\ 2 \longleftrightarrow 3 \longleftrightarrow 4$

*there have two communication classes* $\{0, 1\}$, $\{2, 3, 4\}$

- Irreducibility

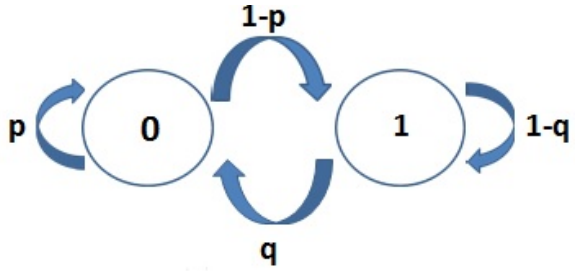
**Definition 1.10.** *A Markov chain is called irreducible if there is only one class of communication (all states belong to one class), otherwise it is called reducible.*

**Remark 1.11.** *If the Markov chain is irreducible then for any pair of $i$ and $j$, $\exists n \in N$ such that $P^n(i,j) > 0$*

Consider example (1.9), the classes $\{0,1\}$, $\{2,3,4\}$ are communicate but disjoint, therefore the chain is reducible.

**Example 1.12.** *Consider a Markov chain*

$$\mathrm{p} = \begin{bmatrix} p & 1\text{-}p \\ q & 1\text{-}q \end{bmatrix}$$




*this Markov chain is irreducible.*

- Recurrence and transience

  If $f_{ii} = P(x_n = i \ for some \ n \geq 1 | x_0 = i)$, is the probability that the Markov chain will eventually be found in state $i$ if it starts from state $i$.

  **Definition 1.13.** *A state $i$ is recurrent if $f_{ii} = 1$ ,i.e., the system is certain to return to state $i$ if it starts at state $i$*

  **Definition 1.14.** *A state $i$ is transient if $f_{ii} < 1$,i.e., there is positive probability that the system starts at state $i$, fails to return to it.*

**Proposition 1.15.** *The state $j$ is recurrent if and only if*

$\sum_{n=1}^{\infty} P^n(j,j) = \infty$. *(in this case the chain returns to $j$ infinitely often with probability* 1 *).*

**Proposition 1.16.** *The state $j$ is transient if and only if*

$\sum_{n=1}^{\infty} P^n(j,j) < \infty$. *(in this case the chain returns to $j$ finitely many times with probability* 1 *).*

where $\sum_{n=1}^{\infty} P^n(j,j) = \mathbb{E}[number\, of\, visits\, to\, j | x_0 = j]$

Define $T_i = min\{n > 0 : x_n = i\}$ which is called first-passage time, that means the first time that the Markov chain is in state $i$, given that the process starts in state $i$.

If the chain does not return to state $i$, then $T_i = \infty$, therefore $T_i < \infty$ with probability one.

**Theorem 1.17.** *The following dichotomy holds:*

(i) *if $P(T_i < \infty) = 1$, then $i$ is recurrent.*

(ii) *if $P(T_i < \infty) < 1$, then $i$ is transient.*

In particular, every state is either transient or recurrent. (For more details see [3, 19, 26])

- Positive Recurrence and Null Recurrence

**Theorem 1.18.** *Let* P *be the transition matrix of a Markov chain, and let* $p^0$ *be the probability vector which represents the initial distribution. Then the probability that the chain is in state* $i$ *after* $n$ *steps* $p_i^n$ *is the* $i$*th entry in the vector*

$$p^n = p^0 \mathrm{P}^n$$

*where*

$p^n$ *is the distribution at time* $n$ *of the Markov chain.*

**Definition 1.19.** *A finite transition matrix* P *is regular if some power of* P *has only positive entries (i.e. strictly greater than zero). A Markov chain is a regular Markov chain if its transition matrix is regular.*

**Example 1.20.** *Consider the transition matrix*

$$\mathrm{P} = \begin{bmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{bmatrix}$$

*all entries of* P *are positive, therefore* P *is regular.*

**Theorem 1.21.** *For a regular transition matrix* P*, there exists some unique probability vector* $\bar{\pi}$ *such that:([3, 19])*

(i) *As $n$ becomes large, all of the rows of* $\mathrm{P}^n$ *approach* $\bar{\pi}$

$$\bar{\pi} = \lim_{n\to\infty} \mathrm{P}^n$$

(ii) $\bar{\pi}\mathrm{P} = \bar{\pi}$

(iii) $\bar{\pi}(j) > 0$

The probability vector $\bar{\pi}$ is called an invariant probability for transition matrix P, stationary, or steady state probability distribution.

**Example 1.22.** *Consider the two state Markov chain with transition matrix* $\mathrm{P} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}, 0 < p < 1, 0 < q < 1$

*The matrix* P *has eigenvalues* 1 *and* $1-p-q$, P *we can diagonalized as, [19]*

$$\mathrm{P} = QDQ^{-1}$$

*where*

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1-p-q \end{bmatrix}, Q = \begin{bmatrix} 1 & -p \\ 1 & q \end{bmatrix}, Q^{-1} = \begin{bmatrix} \frac{q}{q+p} & \frac{p}{q+p} \\ \frac{-1}{q+p} & \frac{1}{q+p} \end{bmatrix}$$

*when* P *is diagonalized it is easy to raise* P *to powers,*

$$\mathrm{P}^n = QD^nQ^{-1}$$

$$= Q \begin{bmatrix} 1 & 0 \\ 0 & (1-p-q)^n \end{bmatrix} Q^{-1}$$

*Since* $(1-p-q) < 1$, *then*

$$\lim_{n\to\infty} \mathrm{P}^n = \begin{bmatrix} \frac{q}{q+p} & \frac{p}{q+p} \\ \frac{q}{q+p} & \frac{p}{q+p} \end{bmatrix} = \begin{bmatrix} \bar{\pi} \\ \bar{\pi} \end{bmatrix}, \bar{\pi} = \begin{bmatrix} \frac{q}{q+p} & \frac{p}{q+p} \end{bmatrix}$$

**Definition 1.23.** *An infinite Markov chain is called null recurrent if it is recurrent but*

$$\lim_{n\to\infty} p^n(i,j) = 0 \ \forall i,j$$

*Otherwise, a recurrent chain is called positive recurrent.*

For positive recurrent Markov chain, $\forall i,j$ the limit

$$\lim_{n\to\infty} p^n(i,j) = \bar{\pi}$$

exists and is independent of the initial state $i$.

One way to determine whether or not a chain is positive recurrent is to try to find an invariant probability distribution. If a chain is positive recurrent, then there exists a unique steady state probability distribution, moreover, if a chain is not positive recurrent, there is no steady state probability distribution.

Recall the first passage time $T_i = min\{n > 0 : x_n = i\}$, if

(i) $\mathbb{E}(T_i) < \infty$, then the chain is positive recurrent.

(ii) $\mathbb{E}(T_i) = \infty$ and $P(T_i < \infty) = 1$, then the chain is null recurrent.

(iii) $P(T_i < \infty) < 1$, then the chain is transient.

| item | $P(x_n = i \mid x_0 = i)$ | $Z_i$ |
|---|---|---|
| Null recurrent state | 1 | $\infty$ |
| Positive recurrent state | 1 | $< \infty$ |
| Transient | $< 1$ | $< \infty$ |

where $Z_i$ is Expected number of visits to $i$ given $x_0 = i$.

- Absorbing states

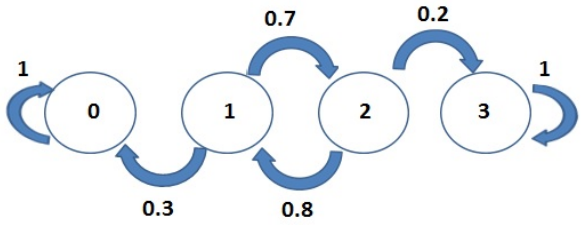
  **Definition 1.24.** *A state $j$ is called absorbing state when entering this state it is impossible to leave it. $P(x_n = j \mid x_0 = j) = 1$.*

  *If every state can reach an absorbing state, then the Markov chain is an absorbing Markov chain.*

- Ergodic Markov chain

  **Definition 1.25.** *A state $i$ is said to be ergodic if it is a periodic and positive recurrent. In other words, a state $i$ is ergodic if it is recurrent, has a period of 1, and has finite mean recurrence time. If all states in an irreducible Markov chain are ergodic, then the chain is said to be ergodic.*

**Example 1.26.** *Consider a Markov chain*




*the states {1,2} are transient states.*

*the states {0}, {3} are recurrent states*

*the states {0}, {3} are absorbing states*

*Non ergodic Markov chain.*

From example (1.12), the states {0,1} are recurrent states, ergodic Markov chain.

## 1.4 Random Walk and Diffusion

In this section the heat equation is derived in a different way from the usual physical way, since the basic model of heat diffusion uses the idea that heat spreads randomly in all directions at a given rate. Therefore, the heat equation is a partial differential equation derived from this intuition by calculating the average of a very large number of particles, so the heat equation will be derived by using the probabilities and stochastic processes.

In this section, a random walk with Bernoulli, Binomial, Geometric and Poisson distribution can be used to derive heat equation.

- Bernoulli distribution

  Consider a walker moves along the x-axis by steps. Each step has the length $h$ and time duration $\delta t$, in each step the walker can move only one step to the left (L) or one step to the right (R), such that the probability of moving to the left and moving to the right equally likely that is 1/2 (symmetric random walk).

  If there have $n$ events, then the number of events to the right (R) is $K$, so the number of events to the left (L) is $n-K$, therefor the probability density function for Bernoulli distribution is given by [31]:

  $P_n(K) = \frac{n!}{K!(n-K)!}\frac{1}{2^n}, k = 0, 1, 2, ...n, \qquad \text{with} \qquad \sum_{K=0}^{n} P_n(K) = 1$

  Let $U(x,t)$ be the probability that after time $t$ the particle will be at the state $x$. If the goal is to reach the state $x$ at time $t+\delta t$, according to the rule of motion we have two choices, the first one is the walker was at the state $x+h$ at time $t$ and moved to the left or the walker was at the state $x-h$ at time $t$ and moved to the right, as the position of the walker at time $t$ and the direction of its next movement are independent, so the first event happens with probability $\frac{1}{2}U(x+h,t)$, while the second with $\frac{1}{2}U(x-h,t)$.

  from transition diagram (figure 1.5) we obtain the following equa-

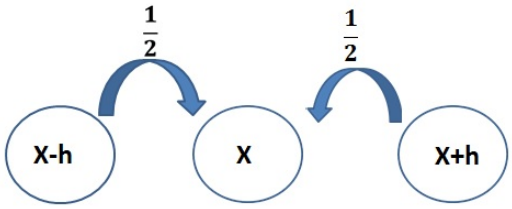



**Figure 1.5: Transition diagram for Bernoulli distribution**

tion:

$$U(x, t+\delta t) = \frac{1}{2}U(x+h, t) + \frac{1}{2}U(x-h, t) \tag{1.3}$$

we use the notation $U_t(x)$ for $U(x, t)$.

equation (1.3) can be written as:

$$U_{t+\delta t}(x) - U_t(x) = \frac{1}{2}\{U_t(x+h) - 2U_t(x) + U_t(x-h)\} \tag{1.4}$$

dividing equation (1.4) by $\delta t$ and multiplying the second part with $\frac{h^2}{h^2}$, equation (1.4) becomes:

$$\frac{U_{t+\delta t}(x) - U_t(x)}{\delta t} = \frac{h^2}{2\delta t}\frac{U_t(x+h) - 2U_t(x) + U_t(x-h)}{h^2} \tag{1.5}$$

let $D = \frac{h^2}{2\delta t}$, equation (1.5) becomes:

$$\frac{U_{t+\delta t}(x) - U_t(x)}{\delta t} = D\,\frac{U_t(x+h) - 2U_t(x) + U_t(x-h)}{h^2} \tag{1.6}$$

using Taylor series, $U(x, t+\delta t)$ and $U(x \pm h, t)$ can be written in the form:

$$U(x, t + \delta t) = U(x, t) + U_t(x, t)\delta t + o(\delta t)$$

$$U(x \pm h, t) = U(x, t) \pm U_x(x, t)h + o(h)$$

and taking the limits as $h \to 0$ and $\delta t \to 0$ in such away that the coefficient $D$ remains constant, equation (1.6) takes the form of diffusion (heat) equation:

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} \tag{1.7}$$

where $D$ is the diffusion coefficient.

- Binomial distribution

  At each step the probability of moving to the right is $p$ and moving to the left is $q = 1 - p$ (non-symmetric random walk).

  The number of events to the right (R) is $K$, so the number of events to the left (L) is $n - K$, therefor the probability density function for Binomial distribution is given by [31]:

  $P_n(K) = \frac{n!}{K!(n-K)!}p^K(1-p)^{n-K}$, with $\sum_{K=0}^{n} P_n(K) = 1$

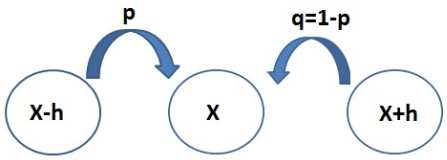



**Figure 1.6: Transition diagram for Binomial distribution**

from transition diagram (figure 1.6), we obtain the following equa-

tion:

$$U_{t+\delta t}(x) = (1-p)U_t(x+h) + pU_t(x-h) \tag{1.8}$$

Let $p = \frac{1}{2} + \epsilon$ and $1 - p = \frac{1}{2} - \epsilon$, where $\epsilon$ is a small positive infinitesimal quantity. Then equation (1.8) can be written as:

$$U_{t+\delta t}(x) = (\frac{1}{2} - \epsilon)U_t(x+h) + (\frac{1}{2} + \epsilon)U_t(x-h) \tag{1.9}$$

In similar way as equation (1.8), two other iterative relations can be written:

$$U_t(x-h) = (\frac{1}{2} + \epsilon)U_{t-\delta t}(x-2h) + (\frac{1}{2} - \epsilon)U_{t-\delta t}(x) \tag{1.10}$$

$$U_t(x+h) = (\frac{1}{2} + \epsilon)U_{t-\delta t}(x) + (\frac{1}{2} - \epsilon)U_{t-\delta t}(x+2h) \tag{1.11}$$

substituting equation (1.10) and equation (1.11) into equation (1.9), the result is:

$$\begin{aligned} U_{t+\delta t}(x) =& \frac{1}{4}\{U_{t-\delta t}(x-2h) + 2U_{t-\delta t}(x) + U_{t-\delta t}(x+2h)\} \\ &+ \epsilon\{U_{t-\delta t}(x-2h) - U_{t-\delta t}(x+2h)\} \\ &+ \epsilon^2\{U_{t-\delta t}(x-2h) - 2U_{t-\delta t}(x) + U_{t-\delta t}(x+2h)\} \end{aligned} \tag{1.12}$$

subtracting $U_{t-\delta t}(x)$ from both sides of (1.12), we get:

$$U_{t+\delta t}(x) - U_{t-\delta t}(x) =$$

$$\begin{aligned}&(\frac{1}{4}+\epsilon^2)\{U_{t-\delta t}(x-2h) - 2U_{t-\delta t}(x) + U_{t-\delta t}(x+2h)\}\\&+\epsilon\{U_{t-\delta t}(x-2h) - U_{t-\delta t}(x+2h)\}\end{aligned} \tag{1.13}$$

transforming equation (1.13) as follows

$$\frac{U_{t+\delta t}(x)-U_{t-\delta t}(x)}{2\delta t} =$$

$$\begin{aligned}&(\frac{1}{4}+\epsilon^2)\frac{(2h)^2}{2\delta t}\frac{U_{t-\delta t}(x-2h) - 2U_{t-\delta t}(x) + U_{t-\delta t}(x+2h)}{(2h)^2}\\&+\epsilon\frac{4h}{2\delta t}\frac{U_{t-\delta t}(x-2h) - U_{t-\delta t}(x+2h)}{4h}\end{aligned} \tag{1.14}$$

Let

$$D = (1+4\epsilon^2)\frac{h^2}{2\delta t} \qquad and \qquad \frac{D}{T}\frac{\partial V}{\partial x} = \frac{2h}{\delta t}\epsilon \tag{1.15}$$

substituting equation (1.15) into equation (1.14), we have:

$$\begin{aligned}\frac{U_{t+\delta t}(x) - U_{t-\delta t}(x)}{2\delta t} =& D\frac{U_{t-\delta t}(x-2h) - 2U_{t-\delta t}(x) + U_{t-\delta t}(x+2h)}{(2h)^2}\\&+\frac{D}{T}\frac{\partial V}{\partial x}\frac{U_{t-\delta t}(x-2h) - U_{t-\delta t}(x+2h)}{4h}\end{aligned} \tag{1.16}$$

This can be considered as difference equation of diffusion with drift, the exterior force ($\frac{\partial V}{\partial x}$) is the cause of the drift. The quantity $T$ is the temperature in units of the energy. According to equation (1.15),

$$D \propto \frac{h^2}{\delta t} \qquad while \qquad \frac{D}{T}\frac{\partial V}{\partial x} \propto \frac{h}{\delta t}. \tag{1.17}$$

in this case it is impossible to keep the diffusion coefficient constant [31]

$$D \propto \frac{h^2}{\delta t} = constant$$

However, can be taken the limit as $h \to 0$ and $\delta t \to 0$, such that

$$\frac{h}{\delta t} \propto v = constant$$

then

$$\frac{h^2}{\delta t} \propto D \to 0$$

and equation (1.16) becomes:

$$\frac{U_{t+\delta t}(x) - U_{t-\delta t}(x)}{2\delta t} = v\frac{U_{t-\delta t}(x-2h) - U_{t-\delta t}(x+2h)}{4h} \tag{1.18}$$

taking the limit as $h \to 0$ and $\delta t \to 0$, equation (1.18) takes the form:

$$\frac{\partial U}{\partial t} = -v\frac{\partial U}{\partial x}. \tag{1.19}$$

where

$$v = \epsilon\frac{4h}{2\delta t}$$

Equation (1.19) is called the advection equation.

- Geometric distribution

  Geometric distribution represents the number of failures before we get a success in a sequence of Bernoulli attempts; $k$, then the probability density function for $k$ is:

$$P(k) = pq^{k-1}, k = 1, 2, ...$$

  where

  $p$ : probability of success in any single experiment.

  $q = (1 - p)$ : probability of failure in any single experiment.

  The transition from the state $x-h$ to the state $x$ is with probability $p$ and from the state $x + h$ to the state $x$ is with probability $q$.

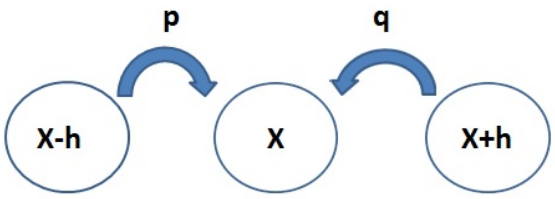



**Figure 1.7: Transition diagram for Geometric distribution**

  from the transition diagram (figure 1.7) we obtain the following equation:

$$U_{t+\delta t}(x) = pU_t(x - h) + qU_t(x + h) \tag{1.20}$$

subtracting $U_{t-\delta t}(x)$ from both sides of equation (1.20), the result is:

$$U_{t+\delta t}(x) - U_{t-\delta t}(x) = pU_t(x-h) + qU_t(x+h) - U_{t-\delta t}(x) \quad (1.21)$$

substituting $p = 1 - q$ in equation (1.21), we get:

$$U_{t+\delta t}(x) - U_{t-\delta t}(x) = U_t(x-h) - qU_t(x-h) + qU_t(x+h) - U_{t-\delta t}(x) \quad (1.22)$$

using Taylor expansion of $U_t(x-h)$ about $(x, t-\delta t)$:

$$U_t(x-h) = U_{t-\delta t}(x) - h\frac{\partial U}{\partial x} + \delta t\frac{\partial U}{\partial t} \quad (1.23)$$

substituting equation (1.23) into equation (1.22), we obtain:

$$U_{t+\delta t}(x) - U_{t-\delta t}(x) = q\{U_t(x+h) - U_t(x-h)\} - h\frac{\partial U}{\partial x} + \delta t\frac{\partial U}{\partial t} \quad (1.24)$$

dividing equation (1.24) by $2\delta t$ and multiplying the first part with $\frac{2h}{2h}$, we get:

$$\frac{U_{t+\delta t}(x) - U_{t-\delta t}(x)}{2\delta t} = \frac{q}{2\delta t}\frac{2h}{2h}\{U_t(x+h) - U_t(x-h)\} - \frac{h}{2\delta t}\frac{\partial U}{\partial x} + \frac{1}{2}\frac{\partial U}{\partial t} \quad (1.25)$$

taking the limits as $h \to 0$ and $\delta t \to 0$ and substituting $q = \frac{1}{2} - \epsilon$

into equation (1.25), equation (1.25) takes the form:

$$\frac{\partial U}{\partial t} = -\epsilon \frac{2h}{\delta t} \frac{\partial U}{\partial x} \tag{1.26}$$

equation (1.26) is an advection equation, that is defined by:

$$\frac{\partial U}{\partial t} = -v \frac{\partial U}{\partial x} \tag{1.27}$$

where

$$v = \epsilon \frac{2h}{\delta t}$$

- Poisson distribution

  ◦ Poisson process

  Consider $x(t)$ the number of customers arriving at a store by time $t$. Time is now continuous, so $t$ takes values in the non-negative real numbers. The rate at which customers arrive is subject to the following assumptions: ([3, 19])

  (i) The number of customers arriving during time interval does not affect the number arriving during a different time interval.

  (ii) The arrival rate at which customers arrive is constant.

  (iii) One arrived at a time.

  The first assumption can be expressed mathematically as: $s_1 \leq t_1 \leq s_2 \leq t_2 \leq ... \leq s_{n-1} \leq t_{n-1} \leq s_n \leq t_n$, that is, in the interval

$[s_i, t_i]$ the random variables $x(t_1) - x(s_1), x(t_2) - x(s_2), ..., x(t_{n-1}) - x(s_{n-1}), x(t_n) - x(s_n)$ are independent. Where $x(t_i) - x(s_i)$ represent the number of customers arrive in the time interval $[s_i, t_i]$.

For the second assumption, if $\lambda$ is the rate at which customers arrive, i.e., in a small time interval $[t, t+\delta t]$, a new customer arrives with probability $\lambda \delta t$.

The third assumption states that the probability that more than one customer arrive during a small time interval is too small. Therefore, as $\delta t \to 0$,

$$P(x(t+\delta t) = x + m | x(t) = x) = \begin{cases} \lambda \delta t + o(\delta t) & if m = 1 \\ o(\delta t) & if m > 1 \\ 1 - \lambda \delta t + o(\delta t) & if m = 0 \end{cases}$$

where $o(\delta t)$ represents some function that is much smaller than $\delta t$.

**Definition 1.27.** *A stochastic process $x(t)$ with $x(0) = 0$ satisfying the previous assumptions* (i − iii) *is called a Poisson process with rate parameter $\lambda$.*

**Theorem 1.28.** *For every $t \geq 0$, the random variable $x(t)$ has a Poisson distribution with rate $\lambda t$ that is*

$$P(x(t) = k) = \begin{cases} \frac{e^{-\lambda t}(\lambda t)^k}{k!} & k = 0, 1, 2, ... \\ 0 & otherwise \end{cases}$$

So, a Poisson process is a collection of random variables each of one has a Poisson distribution.

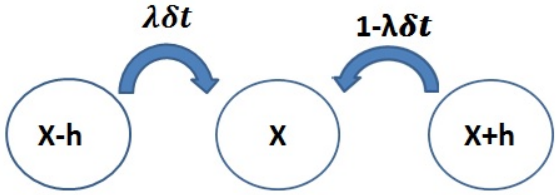



**Figure 1.8: Transition diagram for Poisson distribution**

from transition diagram (1.8) we obtain the following equation:

$$U_{t+\delta t}(x) = (1 - \lambda\delta t)U_t(x + h) + \lambda\delta t U_t(x - h) \tag{1.28}$$

$$U_{t+\delta t}(x) - U_t(x + h) = -\lambda\delta t(U_t(x + h) - U_t(x - h)) \tag{1.29}$$

using Taylor expansion of $U_t(x + h)$ about $(x, t + \delta t)$, we get:

$$U_t(x + h) = U_{t+\delta t}(x) + h\frac{\partial U}{\partial x} - \delta t\frac{\partial U}{\partial t} \tag{1.30}$$

substituting equation (1.30) into equation (1.29), we obtain:

$$\delta t\frac{\partial U}{\partial t} - h\frac{\partial U}{\partial x} = -\lambda\delta t(U_t(x+h) - U_t(x-h)) \tag{1.31}$$

$$\frac{\partial U}{\partial t} = \frac{h}{\delta t}(1 - 2\lambda\delta t)\frac{\partial U}{\partial x} \tag{1.32}$$

using $(1 - \lambda\delta t) = \frac{1}{2} - \epsilon \Longrightarrow -\lambda\delta t = -\frac{1}{2} - \epsilon$, the result is:

$$\frac{\partial U}{\partial t} = -\epsilon\frac{2h}{\delta t}\frac{\partial U}{\partial x} \tag{1.33}$$

Let $v = \epsilon\frac{2h}{\delta t}$, equation (1.33) can be written as:

$$\frac{\partial U}{\partial t} = -v\frac{\partial U}{\partial x} \tag{1.34}$$

equation (1.34) is called advection equation

## 1.5 Brownian Motion

Brownian motion is a stochastic process that describes the continuous random motion. The first model was proposed by Einstein in 1905 after the British scientist R. Brown in 1827 observed that the random movement of pollen particles in water. It is also called Wiener process relative to the N. Wiener who invented the mathematical construction of Brownian motion.

Let $X_t$ represent the position of a particle at time t. In this case $t$ takes on values in the non-negative real numbers. The Brownian will be an example of a stochastic process with both continuous state space and continuous time.

**Definition 1.29.** *A Brownian motion (Wiener process) with variance parameter $\sigma^2$ is a stochastic process $X_t$ with values in $\mathbb{R}$ satisfying: ([19, 26])*

(i) $X_0 = 0$

(ii) *$X_t$ has independent increments.*

*That means, for any time intervals $[s_1, t_1), [t_1, s_2), [s_2, t_2), ...$ the random variables $X_{t_1} - X_{s_1}, X_{s_2} - X_{t_1}, X_{t_2} - X_{s_2}, ...$ are independent.*

(iii) *For any $s < t$, the random variable $X_t - X_s$ has a normal distribution with mean $0$ and variance $(t-s)\sigma^2$*

(iv) *$X_t$ is almost surely continuous.*

The process with $\sigma^2 = 1$ is called standard Brownian motion.

Suppose the process $X_t$ satisfies these conditions $(\mathrm{i} - \mathrm{iv})$, our goal is to find a distribution of $X_t$, we take the case $t = 1$, so $X_1$ can be written as:

$$X_1 = X_{\frac{1}{n}} - X_0 + X_{\frac{2}{n}} - X_{\frac{1}{n}} + ... + X_1 - X_{1-\frac{1}{n}} \tag{1.35}$$

In other words, $X_1$ can be written as the sum of $n$ independent, identically distributed random variables.

In addition, if $n$ is large, each of the random variables are small.

Let

$$Y_n = max\{|X_{\frac{1}{n}} - X_0|, ..., |X_1 - X_{1-\frac{1}{n}}|\}$$

then as $n \to \infty$, $Y_n \to 0$, this is because of the last condition: $X_t$ is a continuous function of $t$, (if $Y_n$ did not go to 0 then there would be a jump in the path of $X_t$). Then, by applying the CLT (Central Limit Theorem), $X_t$ is shown to have a normal distribution (For more details see, [19, 26]).

**Definition 1.30.** *Central Limit Theorem (CLT) establishes that, when independent random variables are added, the distribution of their sum tends toward a normal distribution even if the original variables themselves are not normally distributed.*

If a Brownian motion starting at $x$, then the Brownian motion is a process satisfying the previous conditions (i$-$iv in definition 1.28) except the first condition, whereas the initial condition $X_0 = x$.

If $X_t$ is a Brownian motion starting at $X_0 = 0$, then $M_t = X_t + x$ is a Brownian motion starting at $x$.

Let $P_t(x, y)$ denote the transition density for any $x, y \in \mathbb{R}$ and $t > 0$, i.e., the density of $X_t$ for Brownian motion starting at $x$.

Since $X_t - X_0$ is normal with mean 0 and variance $t$, then $P_t(x, y)$ is given by:

$$P_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \tag{1.36}$$

Properties:

1. The transition density $P_t(x, y)$ satisfies the diffusion equation:

$$\frac{\partial P_t}{\partial t} = \frac{1}{2}\frac{\partial^2 P_t}{\partial y^2} \tag{1.37}$$

for any $x, y \in \mathbb{R}$ and $t > 0$.

differentiating equation (1.36), the result is:

$$\frac{\partial P_t(x,y)}{\partial t} = \frac{y^2 - 2yx + x^2 - t}{2t^2} P_t(x,y)$$

$$\frac{\partial P_t(x,y)}{\partial y} = \frac{x - y}{t} P_t(x,y)$$

$$\frac{\partial^2 P_t(x,y)}{\partial y^2} = \frac{y^2 - 2yx + x^2 - t}{t^2} P_t(x,y)$$

2. The probability density of $X_t$ is given by

$$f_{X_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

such that the mean of $X_t$ is zero and the variance is $t$ (standard Brownian motion).

3. $\mathbb{E}[X_s X_t] = min\{s, t\}$

4. For any $0 \leq s < t$ the increment $X_t - X_s$ is independent of the $\sigma$-field $\mathcal{F}_s = \sigma\{X_r : 0 \leq r \leq s\}$.

- Stopping time

  **Definition 1.31.** *A random variable $\tau \in [0, \infty)$ is a stopping time for the Brownian motion $X_t$ if for each $t$, the event $\{\tau \leq t\}$ is measurable with respect to $\mathcal{F}_t$.*

  It means, to know that the process has stopped before time $t$ or not, we only need to look at the Brownian motion up to time $t$.

  The most important example will be stopping time of the form

  $$\tau_x = inf\{t : X_t = x\}$$

- Strong Markov property

  Let $Y_t = X_{\tau+t} - X_\tau$, for $\tau$ being a stopping time.

  The strong Markov property states that $Y_t$ is a Brownian motion independent of $\mathcal{F}_\tau$, where $\mathcal{F}_\tau$ is the information contained in the Brownian motion up the stopping time $\tau$.

  To see how to use this property, we take the following example:

  **Example 1.32.** *Find the probability that $\exists\ t$, $0 \leq t \leq 1$ such that $X_t \geq 1$. [19]*

  *solution:*

  *Let $\tau = inf\{t, X_t = 1\}$*

*our goal is to find the probability of the set* $\{\tau \leq 1\}$. *We related to the set* $\{X_1 \geq 1\}$, *we have:*

$$P(X_1 \geq 1) = P(X_1 \geq 1 | \tau \leq 1) P(\tau \leq 1)$$

*by the strong Markov property after the random time* $\tau$, *the random variable* $X_1 - X_\tau = X_1 - 1$ *has the normal distribution with mean* 0, *hence by symmetry:*

$$P(X_1 - 1 \geq 0 | \tau \leq 1) = \frac{1}{2}$$

*we get:*

$$P(\tau \leq 1) = 2P(X_1 \geq 1) = 2 \int_1^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

This result is a particular case of the reflection principle.

**Definition 1.33.** *(Reflection Principle)*

*suppose* $X_t$ *is a Brownian motion with variance parameter* $\sigma^2$ *starting at* $z$ *and* $z < c$, *where* $z$ *and* $c$ *are constant, then for any* $t > 0$,

$$P(X_s \geq c \ for \ some \ 0 \leq s \leq t) = 2P(X_t \geq c | x_0 = z)$$

$$= 2 \int_c^\infty \frac{1}{\sqrt{2\pi t \sigma^2}} e^{-\frac{(x-z)^2}{2\sigma^2 t}} dx$$

## 1.6 Brownian Motion and the Heat Equation

Imagine the temperature is determined by a very large number of heat particles that move a Brownian motion, and the initial temperature is given by $g(x)$. Let $U(x,t)$ be the temperature at position $x$ at time $t$.

If there are $g(y)$ particles starting from the site $y$ and moving to site $x$ at time $t$. The probability of moving from $y$ to $x$ in time $t$ is the same as the probability of a particles move from $x$ to $y$ at the same time $t$. By calculating the average of a very large number of particles over all possible of $y$, $U(x,t)$ takes the form [20]:

$$U(x,t) = \mathbb{E}[g(X_t)|X_0 = 0] \tag{1.38}$$

If $X_t$ is a Brownian motion starting at x, then for a fixed $t$, $X_t$ is a random variable with probability density function

$$P_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$$

If $x$, $t$ are fixed and $p_t(x,y)$ is considered as a function of $y$. Symmetry is seen by noting that $p_t(x,y) = p_t(y,x)$.

equation (1.38) can be written as:

$$U(x,t) = \int_{\mathbb{R}} g(y) P_t(x,y) dy = \int_{\mathbb{R}} g(y) P_t(y,x) dy \tag{1.39}$$

define $U(x,0) = g(x)$ and $U(x,t)$ for $t > 0$ define by equation (1.38). If $t = 0$, $x = 0$ , $g(0) = 0$. Then we get the equation:

$$\lim_{s\to 0+} \frac{U(0,s) - g(0)}{s} = \lim_{s\to 0+} \frac{\mathbb{E}[g(X_s)|X_0 = 0]}{s} \tag{1.40}$$

Suppose that $g\, is\, C^2$ and approximating $g$ by the second order Taylor polynomial,

$$g(x) = g'(0)x + \frac{1}{2}g''(0)x^2 + o(x^2),\ x \to 0. \tag{1.41}$$

then,

$$\mathbb{E}[g(X_s)] = g'(0)\mathbb{E}[X_s] + \frac{1}{2}g''(0)\mathbb{E}[{X_s}^2] + o({X_s}^2) \tag{1.42}$$

but $\mathbb{E}[X_s] = 0$, $\mathbb{E}[{X_s}^2] = s$ and $o({X_s}^2) = o(s)$, dividing equation (1.42) by $s$ and taking the limit as $s \to 0$, the limit becomes $\frac{1}{2}g''(0)$. Therefore in general the partial differential equation that $U(x,t)$ satisfies is:

$$\frac{\partial U(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 U(x,t)}{\partial x^2} \tag{1.43}$$

equation (1.43) is the heat equation with $D = \frac{1}{2}$ and initial condition $U(x,0) = g(x)$.

# Chapter Two

# Analytical Methods for Solving the Heat Equation

- Separation of variables method

  The method of separation of variables (sometimes called Fourier method) is an appropriate method for solving the heat equation (1), it is basically based on the assumption that the solution $U(x,t)$ can be separated as a product of two functions $G(x)H(t)$ one depends on $x$ and the other on $t$.

$$U(x,t) = G(x)H(t)$$

  This separates out the partial differential equation into two or three ordinary differential equations, which are related to a common constant ([16, 25, 29]).

## 2.1 The Dirichlet Condition

Consider the following heat equation with the Dirichlet boundary conditions on the finite interval $0 < x < L$ [29]:

$$\begin{aligned} &U_t - DU_{xx} = 0,\ \ 0 < x < L, t > 0 \\ &U(x,0) = \psi(x) \\ &U(0,t) = U(L,t) = 0\,(\textit{homogeneous Dirichlet Boundary conditions}) \end{aligned} \tag{2.1}$$

By separation of variables, let $U(x,t) = G(x)H(t)$, substituting it in equation (2.1), the result is:

$$G(x)H'(t) - DG''(x)H(t) = 0$$

Dividing by $DG(x)H(t)$, we obtain:

$$\frac{H'(t)}{DH(t)} = \frac{G''(x)}{G(x)} \tag{2.2}$$

The left hand side is a function of $t$, the right hand side is a function of $x$, both sides of equation (2.2) are equal to some constant value $-\lambda$, therefore equation (2.2) becomes:

$$\frac{H'(t)}{DH(t)} = \frac{G''(x)}{G(x)} = -\lambda \tag{2.3}$$

Equation (2.3) is a pair of separate ordinary differential equations

$$G''(x) + \lambda G(x) = 0$$

and

$$H'(t) + D\lambda H(t) = 0$$

from the Dirichlet boundary conditions $U(0,t) = U(L,t) = 0$, we get:

$U(0,t) = G(0)H(t) = 0 \Longrightarrow G(0) = 0$

$U(L,t) = G(L)H(t) = 0 \Longrightarrow G(L) = 0$

therefore, the two ordinary differential equations become:

$$G''(x) + \lambda G(x) = 0, G(0) = G(L) = 0 \tag{2.4}$$

$$H'(t) + D\lambda H(t) = 0 \tag{2.5}$$

Initial condition can not be used, since $U(x,0) = G(x)H(0) = \psi(x)$ does not imply that $H(0)$

The characteristic equation for equation (2.4) is:

$$m^2 + \lambda = 0 \rightarrow m = \pm\sqrt{-\lambda}$$

If $\lambda = 0$, then $m = 0$ (repeated root with multiplicity = 2) $\Longrightarrow G(x) = c_1 x + c_2$

$G(0) = 0 \Longrightarrow c_2 = 0$

$G(L) = 0 \Longrightarrow c_1 = 0$

Therefore the only solution is $G(x) \equiv 0$ (trivial solution), as the result $U(x,t) \equiv 0$ and this does not satisfy the initial condition.

If $\lambda = -\sigma^2 < 0$, then $m = \pm\sigma$ (real distinct roots), so the solution of equation (2.4) is given by:

$$G(x) = c_1 e^{\sigma x} + c_2 e^{-\sigma x}$$

or

$$G(x) = c_1 cosh(\sigma x) + c_2 sinh(\sigma x)$$

since $G(0) = 0, c_1 cosh(0) + c_2 sinh(0) = 0$ but $sinh(0) = 0$ and

$cosh(0) = 1 \Longrightarrow c_1 = 0$

since $G(L) = 0, c_1 cosh(\sigma L) + c_2 sinh(\sigma L) = 0$ but $c_1 = 0$

$\Longrightarrow c_2 sinh(\sigma L) = 0 \Longrightarrow c_2 = 0$

since $\sigma L \neq 0 \Longrightarrow sinh(\sigma L) \neq 0$, so the solution is $G(x) \equiv 0$

If $\lambda = \sigma^2 > 0$, then $m = \pm\sigma i$ (complex roots), therefore the solution of equation (2.4) is given by:

$$G(x) = c_1 cos(\sigma x) + c_2 sin(\sigma x)$$

since $G(0) = 0 \Longrightarrow c_1 cos(0) + c_2 sin(0) = 0 \Longrightarrow c_1 = 0$

since $G(L) = 0 \Longrightarrow c_1 cos(\sigma L) + c_2 sin(\sigma L) = 0$, since $c_1 = 0 \Longrightarrow$

$c_2 sin(\sigma L) = 0$

for a nontrivial solution, $sin(\sigma L)$ must equal to 0, therefore

$sin(\sigma L) = 0 \Longrightarrow \sigma L = n\pi$, where n is a positive integer $\Longrightarrow \sigma_n = \frac{n\pi}{L}$

that is $\lambda_n = (\frac{n\pi}{L})^2$

If $G_n(x)$ is a nontrivial solution corresponding to $\lambda_n$, then $cG_n(x), c \neq 0$ is also a nontrivial solution corresponding to $\lambda_n$.

therefore,

$$G_n(x) = c_2 sin(\frac{n\pi}{L}x), n = 1, 2, 3, ...$$

For the second ordinary differential equation (2.5)

$$H_n'(t) + D\lambda_n H_n(t) = 0, \ (first\ order\ separable)$$

This ODE is easy to solve, so

$$H_n(t) = A_n e^{-D\lambda_n t}, A_n = e^{c_n}$$

$$H_n(t) = A_n e^{-D(\frac{n\pi}{L})^2 t}, n = 1, 2, 3, ...$$

Each value of $\sigma_n$ yields an independent solution satisfying the heat equation, we have an infinite number of independent solutions $U_n(x, t)$.

As a result, $U_n(x, t) = G_n(x)H_n(t)$ is a solution of the heat equation for $n = 1, 2, 3, ...$

The general solution that satisfies the Dirichlet boundary conditions is:

$$U(x, t) = \sum_{n=1}^{\infty} U_n(x, t) = \sum_{n=1}^{\infty} G_n(x)H_n(t)$$

$$U(x, t) = \sum_{n=1}^{\infty} A_n e^{-D(\frac{n\pi}{L})^2 t} sin(\frac{n\pi}{L}x)$$

The initial condition is used to determine the coefficient:

$$U(x, 0) = \psi(x)$$

$$\psi(x) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi}{L}x) \tag{2.6}$$

equation (2.6) is a Fourier sine series expansion of $\psi(x)$ on $[0, L]$ so,

$$A_n = \frac{2}{L}\int_0^L \psi(x) sin(\frac{n\pi}{L}x)dx, n = 1, 2, 3, ...$$

## 2.2 The Neumann Condition

For problems in one dimension, the homogenous Neumann conditions are given by [29]:

$$U_x(0,t) = 0 \;\; and \;\; U_x(L,t) = 0$$

Consider the following heat equation with the Neumann conditions on the finite interval $0 < x < L$:

$$\begin{aligned} &U_t - DU_{xx} = 0, 0 < x < L, t > 0 \\ &U(x,0) = \psi(x) \\ &U_x(0,t) = U_x(L,t) = 0 \, (homogeneous \, Neumann \, Boundary \, conditions) \end{aligned} \tag{2.7}$$

If we let $U(x,t) = G(x)H(t)$, and using the Neumann boundary conditions, then we have the following pair of ordinary differential equations:

$$G''(x) + \lambda G(x) = 0 \, with \, G'(0) = G'(L) = 0 \tag{2.8}$$

$$H'(t) + D\lambda H(t) = 0 \tag{2.9}$$

the characteristic equation for equation (2.8) is:

$$m^2 + \lambda = 0 \rightarrow m = \pm\sqrt{-\lambda}$$

If $\lambda = 0$, then $m = 0$, (a repeated root with multiplicity = 2)

$\Longrightarrow G(x) = c_1 x + c_2$

since $G'(0) = 0$ and $G(x) = c_1 x + c_2 \Longrightarrow G'(x) = c_1 \Longrightarrow c_1 = 0$

since $G'(L) = 0$ and $c_1 = 0 \Longrightarrow c_2\, arbitrary$

therefore, when $\lambda = 0 \Longrightarrow G(x) = constant$

If $\lambda = -\sigma^2 < 0$, then $m = \pm\sigma$ (real distinct roots), so the solution of equation (2.8) is given by:

$$G(x) = c_1 cosh(\sigma x) + c_2 sinh(\sigma x)$$

$$G'(x) = c_1 \sigma sinh(\sigma x) + c_2 \sigma cosh(\sigma x)$$

since $G'(0) = 0 \Longrightarrow c_1 \sigma sinh(0) + c_2 \sigma cosh(0) = 0$ since $sinh(0) = 0$ and $cosh(0) = 1 \Longrightarrow c_2 \sigma = 0 \Longrightarrow c_2 = 0$

since $G'(L) = 0 \Longrightarrow c_1 \sigma sinh(\sigma L) + c_2 \sigma cosh(\sigma L) = 0$ since $c_2 = 0 \Longrightarrow c_1 \sigma sinh(\sigma L) = 0$, since $\sigma L \neq 0 \Longrightarrow sinh(\sigma L) \neq 0 \Longrightarrow c_1 = 0$, therefore the solution is $G(x) \equiv 0$, as the result $U(x,t) \equiv 0$ and this does not satisfy the initial condition.

If $\lambda = \sigma^2 > 0$, then $m = \pm\sigma i$ (complex roots), therefore the solution of equation (2.8) is given by:

$$G(x) = c_1 cos(\sigma x) + c_2 sin(\sigma x)$$

$$G'(x) = -c_1 \sigma sin(\sigma x) + c_2 \sigma cos(\sigma x)$$

since $G'(0) = 0 \Longrightarrow -c_1 \sigma sin(0) + c_2 \sigma cos(0) = 0 \Longrightarrow c_2 = 0$

since $G'(L) = 0 \Longrightarrow -c_1\sigma sin(\sigma L) + c_2\sigma cos(\sigma L) = 0$ since $c_2 = 0 \Longrightarrow -c_1\sigma sin(\sigma L) = 0$

For a nontrivial solution, $sin(\sigma L) = 0 \rightarrow \sigma L = n\pi \Longrightarrow \sigma_n = \frac{n\pi}{L} \Longrightarrow \lambda_n = (\frac{n\pi}{L})^2$, therefore

$$G_n(x) = cos(\frac{n\pi}{L}x), n = 1, 2, 3, ...$$

In fact, we include $n = 0, (n = 0, 1, 2, ...)$, since $\lambda = 0$ with $G_0 = cos(0) = 1$

for the second ODE eqaution (2.9):

$$H'(t) + D\lambda_n H(t) = 0,\ n = 0, 1, 2, ...\ (first\ order\ separable\ )$$

this ODE is easy to solve, therefore

$$H_n(t) = A_n e^{-D\lambda_n t}, A_n = e^{c_n}$$

$$H_n(t) = A_n e^{-D(\frac{n\pi}{L})^2 t},\ n = 0, 1, 2, 3, ...$$

Note that $U_0(x, t) = G_0(x)H_0(t) = 1.A_0 = A_0$, but for ease, $U_0(x, t)$ can be written as $\frac{1}{2}.A_0 = \frac{A_0}{2}$

Therefore, the general solution that satisfies the Neumann boundary conditions is:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) = \sum_{n=0}^{\infty} G_n(x)H_n(x)$$

$$U(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-D(\frac{n\pi}{L})^2 t} cos(\frac{n\pi}{L}x)$$

The initial condition $U(x,0) = \psi(x)$, therfore

$$\psi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n cos(\frac{n\pi}{L}x) \tag{2.10}$$

equation (2.10) is a Fourier cosine series of $\psi(x)$ on $[0, L]$ so,

$$A_n = \frac{2}{L}\int_0^L \psi(x) cos(\frac{n\pi}{L}x)dx,\ n = 0, 1, 2, ...$$

**Example 2.1.** *Solve the following heat equation (Neumann condition):*

$$U_t - \frac{1}{4}U_{xx} = 0, 0 < x < 1, t > 0$$

$$U_x(0,t) = U_x(1,t) = 0$$

$$U(x,0) = 10x$$

*Solution:*

$L = 1$ , $D = \frac{1}{4}$, $\psi(x) = 10x$,

$$A_0 = 2\int_0^1 10x\, cos(0)dx = 10$$

$$A_n = 2\int_0^1 10x cos(n\pi x)dx = \frac{20((-1)^n - 1)}{n^2\pi^2}, n \geq 1$$

$$A_n = \begin{cases} \frac{-40}{n^2\pi^2} & n\, is\, odd \\ 0 & n\, is\, even \end{cases}$$

*therefore,*

$$U(x,t) = \begin{cases} 5 - \frac{40}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-\frac{(n^2\pi^2)t}{4}}cos(n\pi x) & n\, is\, odd \\ 5 & n\, is\, even \end{cases}$$

**Example 2.2.** *Solve the following heat equation (Dirichlet condition):*

$$U_t = U_{xx}, 0 < x < 1, t > 0$$

$$U(0,t) = U(1,t) = 0$$

$$U(x,0) = sin(2\pi x)$$

*Solution:*

$L = 1, D = 1, \psi(x) = sin(2\pi x)$

$$A_n = 2\int_0^1 sin(2\pi x)sin(n\pi x)dx = \begin{cases} 1 & n = 2 \\ 0 & otherwise \end{cases}$$

*therefore,*

$$U(x,t) = \begin{cases} e^{-4\pi^2 t}sin(2\pi x) & n = 2 \\ 0 & otherwise \end{cases}$$

# Chapter Three

# Numerical Methods for Solving the Heat Equation

## 3.1 Finite Difference Method

The finite difference approximations are one of the simplest and oldest methods for solving partial differential equations. It was known by L. Euler in 1768, in one dimension of space and was developed into two dimensions by C. Runge in 1908. The finite difference techniques began in numerical applications in the early 1950s.

It is based on replacement the partial derivatives by finite difference approximations, whether forward, centred or backward difference approximation.

### 3.1.1 Explicit Method

The idea of explicit forward difference method is to use the first order forward difference approximation for the time derivative and second order centred difference approximation for the space derivative ([16, 22, 28]).

Consider the heat equation:

$$\begin{aligned} &U_{xx} = U_t + g(x,t), 0 < x < L, t > 0 \\ &U(x,0) = \psi(x)\ (\textit{initial condition}) \\ &U(0,t) = U(L,t) = 0\ (\textit{boundary conditions}) \end{aligned} \tag{3.1}$$

$g(x,t), \psi(x)$, and $L$ are given.

Note that the computer can not run forever, so we must decide how much time it takes, therefore assume that the interval is $0 < t < T$.

Determining the points in the domain at which the solution is to be approximated. These points are called grid points.

Let $x_i = idx, i = 0, 1, 2, 3, ..., N + 1$

and $t_j = jdt, j = 0, 1, 2, 3, ..., M$

where $dx = \frac{L}{N+1}$ and $dt = \frac{T}{M}$ are the step sizes.

The result is a grid shown in figure (3.1):

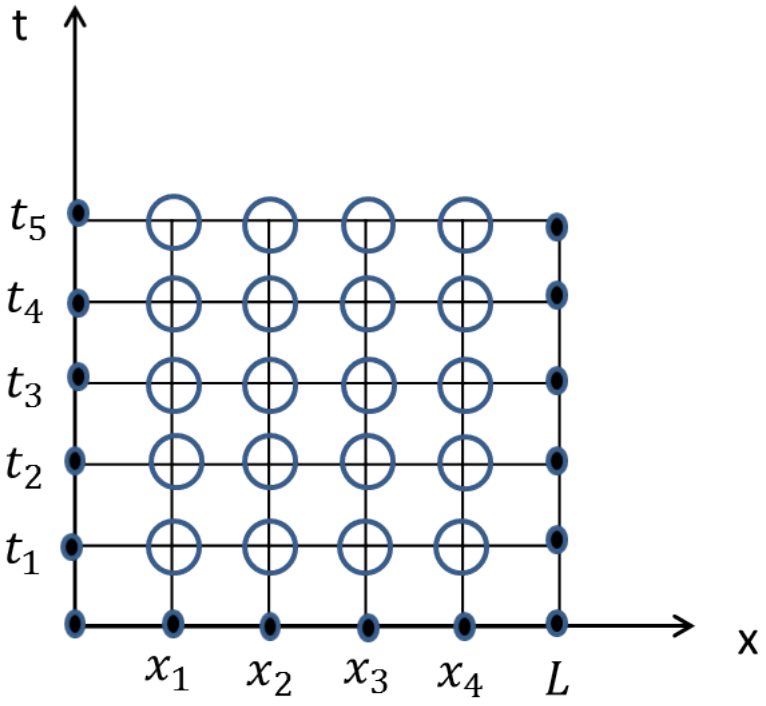



**Figure 3.1: Example of the grid system , N=4 and M=5**

Note that the solid grid points are either initial or boundary points where the solution is given. Initial it is given by $\psi(x_i)$, on the boundaries $(x = 0 \, or \, x = L)$ the solution is zero unless the boundary conditions are not homogeneous. The other grid points are the points at which the solution is to be approximated.

Evaluating the equation at each interior point $(x_i, t_j), i = 1, 2, 3, ..., N$ and $j = 1, 2, 3, ..., M$ by using the first order forward difference approximation for the time derivative $U_t(x_i, t_j)$ and second order centered difference approximation for the space derivative $U_{xx}(x_i, t_j)$, we get:

$$U_{xx}(x_i, t_j) = U_t(x_i, t_j) + g(x_i, t_j) \tag{3.2}$$

$$U_t(x_i, t_j) = \frac{U(x_i, t_{j+1}) - U(x_i, t_j)}{dt} - \frac{dt}{2} U_{tt}(x_i, \xi_j), \xi_j \in [t_j, t_{j+1}] \tag{3.3}$$

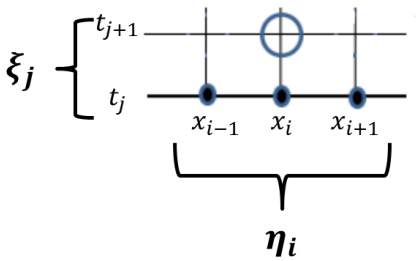
$$U_{xx}(x_i, t_j) = \frac{U(x_{i+1}, t_j) - 2U(x_i, t_j) + U(x_{i-1}, t_j)}{(dx)^2} - \frac{(dx)^2}{12} U_{xxxx}(\eta_i, t_j), \tag{3.4}$$

$\eta_i \in [x_{i-1}, x_{i+1}]$

substituting both equations (3.3) and (3.4) into equation (3.2), we have:

$$\frac{U(x_{i+1}, t_j) - 2U(x_i, t_j) + U(x_{i-1}, t_j)}{(dx)^2} - \frac{(dx)^2}{12} U_{xxxx}(\eta_i, t_j)$$

$$= \frac{U(x_i, t_{j+1}) - U(x_i, t_j)}{dt} - \frac{dt}{2} U_{tt}(x_i, \xi_j) + g(x_i, t_j)$$



$$\frac{U(x_{i+1},t_j)-2U(x_i,t_j)+U(x_{i-1},t_j)}{(dx)^2}+\zeta_{i,j}=\frac{U(x_i,t_{j+1})-U(x_i,t_j)}{dt}+g(x_i,t_j) \qquad (3.5)$$

where $\zeta_{i,j}$ is the truncation error, given by:

$$\zeta_{i,j}=\frac{dt}{2}U_{tt}(x_i,\xi_j)-\frac{(dx)^2}{12}U_{xxxx}(\eta_i,t_j)=O(dt)+O(dx)^2$$

multiplying equation (3.5) by $dt$ and we let $\lambda=\frac{dt}{(dx)^2}$, the result is: $\lambda[U(x_{i+1},t_j)-2U(x_i,t_j)+U(x_{i-1},t_j)]+dt\zeta_{i,j}=U(x_i,t_{j+1})-U(x_i,t_j)+dtg(x_i,t_j)$

$$U(x_i,t_{j+1})=\lambda U(x_{i+1},t_j)+(1-2\lambda)U(x_i,t_j)+\lambda U(x_{i-1},t_j)-dtg(x_i,t_j)+dt\zeta_{i,j} \qquad (3.6)$$

dropping the truncation error in equation (3.6) and let $U(x_i,t_j)\cong U_{i,j}$ be the value that satisfies the new difference equation (3.7), we get:

$$U_{i,j+1}=\lambda U_{i+1,j}+(1-2\lambda)U_{i,j}+\lambda U_{i-1,j}-dtg_{i,j} \qquad (3.7)$$

$i=1,2,3,...,N$ and $j=0,1,2,3,...,M-1$

from the initial condition $U(x,0)=\psi(x)$ and the boundary conditions $U(0,t)=U(L,t)=0$, we get:

$U_{i,0}=\psi(x_i)$ (initial condition) and $U_{0,j}=U_{N+1,j}=0$ (boundary conditions)

Equation (3.7) is an iterative formula for an explicit forward difference method for the heat equation. It is explicit since $U_{i,j+1}$ at the time level $j+1$ is given explicitly by values at the time level $j$.

The stencil is given in the following diagram:

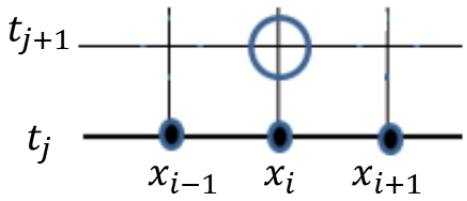



**Figure 3.2: The stencil for explicit forward difference method**

The time level $j = 0$ is used to approximate the solution at time level $j = 1$. Then time level $j = 1$ is used to approximate the solution at time level $j = 2$ and so on until the solution is approximated for the all time levels.

**Example 3.1.** *Consider the heat equation*

$$U_{xx} = U_t, 0 < x < 1, 0 < t < T$$

$$U(0,t) = U(1,t) = 0$$

$$U(x,0) = sin(2\pi x) \tag{3.8}$$

*the exact solution using separation of variables from example (2.2) is*

*given by:*

$$U(x,t) = e^{-4\pi^2 t} sin(2\pi x)$$

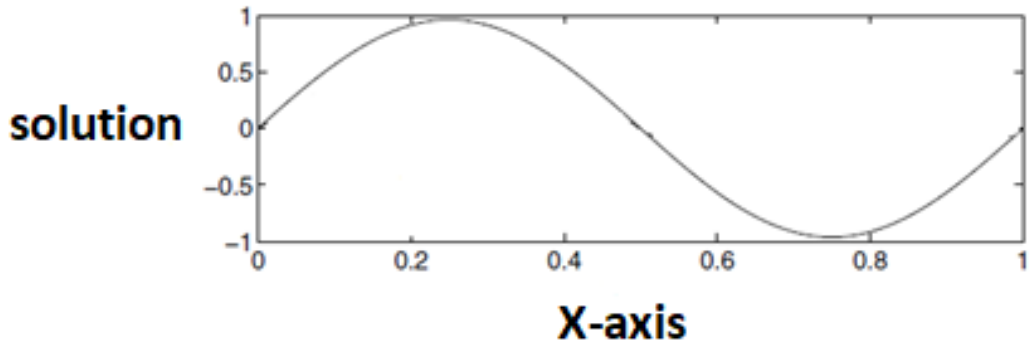



**Figure 3.3: The exact solution at t=0.001**

*using the explicit forward difference method with* $N = 20$, $M = 5, 10, 20$ *and* $T = 0.1$ *for each* $t = 0.02, 0.04, 0.1$, *the numerical solution of the heat equation (3.8) is given in Figure (3.4). [22]*

Figure (3.4), shows that as $M$ increases from 5 to 20 the approximated solution dashed line for $M = 20$ becomes closer to the exact solution, the solid line.

However, for the same $M = 20$ we have smaller time step $dt$ and it should give more accurate results, but at $t = 0.1$ the solution becomes unstable and far from the exact one $(-60 \leq U(x,t) \leq 60)$.

The effect of instability did not appear at smaller time $t = 0.04$ and $t = 0.02$.

The explicit forward difference is conditionally stable with stability

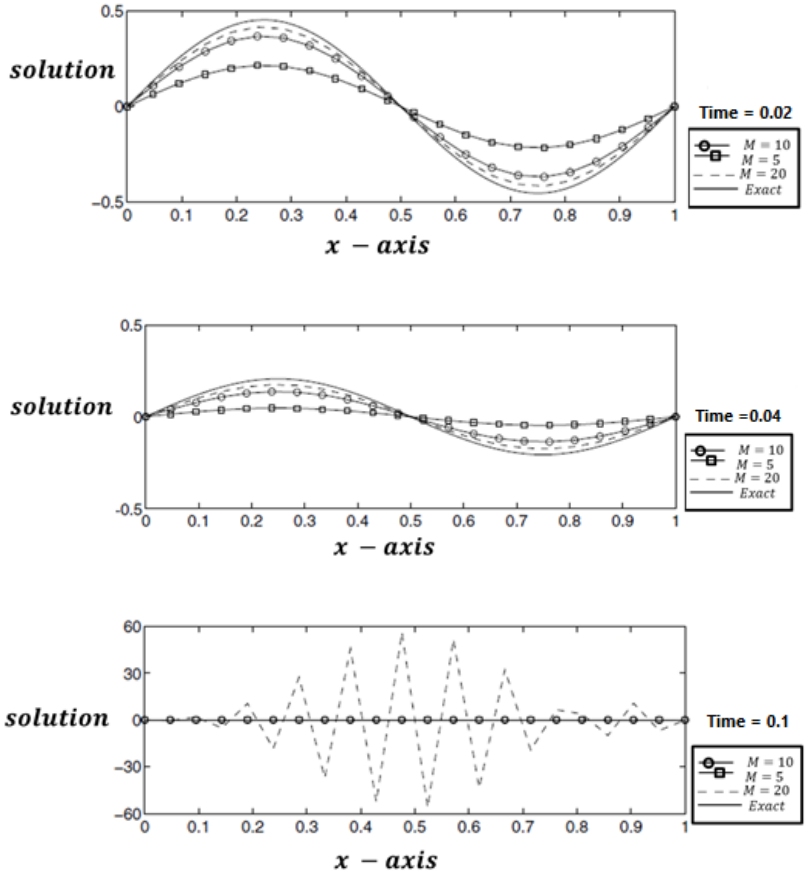



**Figure 3.4: Solution of the heat equation by explicit method**

condition: (for more details see [22])

$$\lambda = \frac{dt}{(dx)^2} \leq \frac{1}{2}$$

Therefore, for stability of this method the time and space step sizes must satisfy the relation

$$2dt \leq (dx)^2$$

**Example 3.2.** *Recall our example (3.1). Instability occurred for* $t = 0.1$ *even when* $M = 20$ *is used.*

$N = 20 \rightarrow dx = \frac{1}{21}$

$M = 20 \rightarrow dt = \frac{0.1}{20} = 0.005$

*therefore* $2dt = 0.01$ *and* $(dx)^2 = (\frac{1}{21})^2 \cong 0.00676 \rightarrow 2dt > (dx)^2$

*the stability condition is not satisfied.*

*To overcome the instability, we choose* $M$ *such that*

$$2(\frac{0.1}{M}) \leq (\frac{1}{21})^2$$

*which implies that*

$$M \geq 2(0.1)(21)^2 = 88.2$$

*for* $N = 20$, $M$ *must be at least 89 (*$M$ *integer), as a result, resolving example (3.1) using* $M = 90$ *and* $T = 0.1$ *for* $t = 0.02, 0.04, 0.1$, *the result is given in figure (3.5), for this calculation* $\lambda = 0.49$

*Figure (3.5) shows that the stability condition is satisfied when*

$M = 90$

The explicit forward difference Method:

Advantages: explicit and simple.

Disadvantage: conditionally stable.

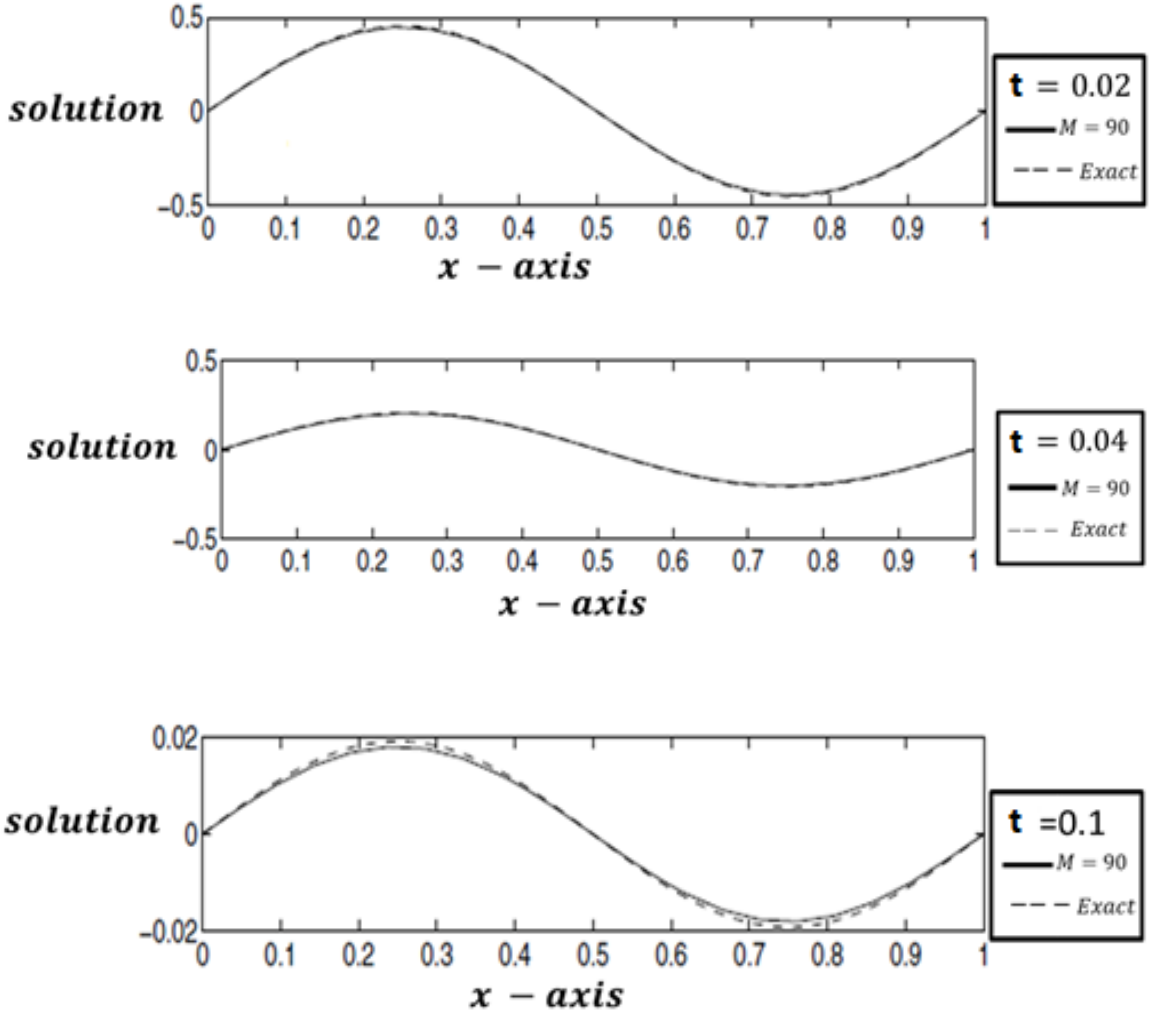



**Figure 3.5: Solution of heat equation (stability condition satisfied)**

### 3.1.2 Implicit Method

The idea of implicit backward difference method is to use backward difference approximation for time derivative and centred difference approximation for space derivative [22].

$$U_t(x_i, t_j) = \frac{U(x_i, t_j) - U(x_i, t_{j-1})}{dt} + O(dt)$$

$$U_{xx}(x_i, t_j) = \frac{U(x_{i+1}, t_j) - 2U(x_i, t_j) + U(x_{i-1}, t_j)}{(dx)^2} + O(dx)^2$$

dropping the error terms and let $U(x_i, t_j) \cong U_{i,j}$, then the heat equation (3.1) becomes:

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(dx)^2} = \frac{U_{i,j} - U_{i,j-1}}{dt} + g_{i,j}$$

$$\frac{dt}{(dx)^2}(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}) = U_{i,j} - U_{i,j-1} + dtg_{i,j} \tag{3.9}$$

Let $\lambda = \frac{dt}{(dx)^2}$, equation (3.9) can be rewritten as:

$$\lambda U_{i-1,j} - (1 + 2\lambda)U_{i,j} + \lambda U_{i+1,j} = -U_{i,j-1} + dtg_{i,j}$$

$i = 1, 2, 3, ..., N$ and $j = 1, 2, 3, ..., M$

For each interior point, we get a linear equation with 3 unknowns: $U_{i-1,j}, U_{i,j}$ and $U_{i+1,j}$ except near the boundaries and initial points which $U_{i,j-1}$ is known.

**Example 3.3.** *Recall example (3.1)*

*Using* $N = 20$, $M = 5, 10, 20$ *and* $T = 0.1$ *for each* $t = 0.02, 0.04, 0.1$, *the numerical solution of the heat equation (3.8) is given in figure (3.6):*

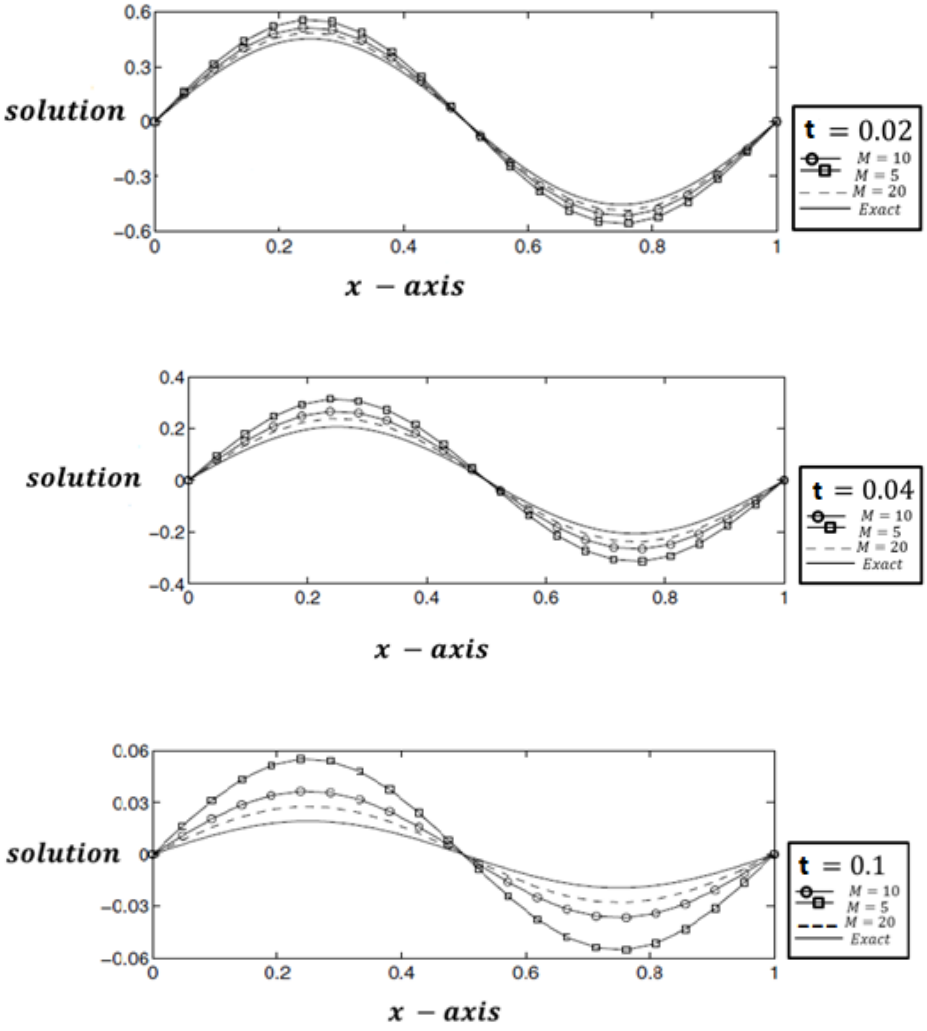



**Figure 3.6: Solution of the heat equation by implicit method**

The implicit method is stable , so no condition on the time and space step sizes.

## 3.2 Theta Method

The numerical results obtained from the explicit and implicit methods (figures (3.4) and (3.6)) shows that the exact solution lies between the results from both methods.

The idea of the theta method is to use both the explicit and the implicit to produce a more accurate method by adding a fraction $\theta$ of the explicit to $(1-\theta)$ of the implicit, $0 \leq \theta \leq 1$

The two methods (explicit and implicit) can be rewritten as:

Explicit: $U_{i,j+1} - U_{i,j} = V_{i,j}$

Implicit: $U_{i,j+1} - U_{i,j} = V_{i,j+1}$

where

$V_{i,j} = \lambda[U_{i+1,j} - 2U_{i,j} + U_{i-1,j}] - dtg_{i,j}$

Adding the explicit and the implicit terms together by adding a fraction $\theta$ of the explicit to $(1-\theta)$ of the implicit

$$\theta(explicit) + (1-\theta)(implicit)$$

$$U_{i,j+1} - U_{i,j} = \theta(U_{i,j+1} - U_{i,j}) + (1-\theta)(U_{i,j+1} - U_{i,j})$$

$$= \theta V_{i,j} + (1-\theta)V_{i,j+1}$$

$U_{i,j+1} - U_{i,j} = \theta[\lambda U_{i+1,j} - 2\lambda U_{i,j} + \lambda U_{i-1,j} - dtg_{i,j}] + (1-\theta)[\lambda U_{i+1,j+1} - 2\lambda U_{i,j+1} + \lambda U_{i-1,j+1} - dtg_{i,j+1}]$

$i = 1, 2, 3, ...N$ and $j = 0, 1, 2, ..., M-1$

This is called the theta method, $0 \leq \theta \leq 1$

when $\theta = 1 \rightarrow$ the explicit method.

when $\theta = 0 \rightarrow$ the implicit method.

when $\theta = \frac{1}{2} \rightarrow$ Crank Nicolson method.

The stability condition of the theta method is $\lambda(2\theta - 1) \leq \frac{1}{2}$

$0 \leq \theta \leq \frac{1}{2} \rightarrow$ the theta method is stable.

$\frac{1}{2} < \theta \leq 1 \rightarrow$ the theta method is conditionally stable [22].

## 3.3 Crank Nicolson Method

Crank Nicolson method was discovered by John Crank and Phyllis Nicolson in the mid-$20^{th}$ century. They apply it to the heat equation and they approximate the solution of the heat equation by approximating the time derivative and space derivative by finite differences.

The idea of Crank Nicolson method is to use centred difference approximation for space derivative and integrate the time derivative with respect to t ([9, 22]).

Consider the heat equation (3.1), it can be written as:

$$U_t(x,t) = G(x,t) \tag{3.10}$$

where

$$G(x,t) = U_{xx}(x,t) - g(x,t)$$

integrating equation (3.10) from $t_j$ to $t_{j+1}$, $j = 0, 1, 2, 3, ..., M-1$

$$\int_{t_j}^{t_{j+1}} U_t(x,t)dt = \int_{t_j}^{t_{j+1}} G(x,t)dt$$

using trapezoid quadrature rule:

$$\int_a^b g(x)dx = \frac{h}{2}[g(a) + g(b)] - \frac{h^3}{12}g''(c),\ c \in (a,b), h = b - a$$

$$U(x, t_{j+1}) - U(x, t_j) = \frac{dt}{2}[G(x, t_j) + G(x, t_{j+1})] - O(dt)^3$$

$U(x, t_{j+1}) - U(x, t_j) = \frac{dt}{2}[U_{xx}(x, t_j) - g(x, t_j) + U_{xx}(x, t_{j+1}) - g(x, t_{j+1})] - O(dt)^3$

The error $O(dt)^3$ is the local one, that is the error generated at each time step. However, the composite error for $j = 0, 1, ..., M-1$ is of order $O(dt)^2$

dropping the error term and using $2^{nd}$ order centered difference for the space derivative, we obtain:

$$U_{i,j+1} - U_{i,j} = \frac{dt}{2}[\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(dx)^2} + \frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{(dx)^2}]$$

$$-\frac{dt}{2}[g_{i,j} + g_{i,j+1}]$$

setting $\lambda = \frac{dt}{(dx)^2}$ and rearranging the terms, the result is:

$\lambda U_{i+1,j+1} - 2(1+\lambda)U_{i,j+1} + \lambda U_{i-1,j+1} = -\lambda U_{i+1,j} - 2(1-\lambda)U_{i,j} - \lambda U_{i-1,j} + dt(g_{i,j} + g_{i,j+1})$

$i = 1, 2, 3, ..., N,\ j = 0, 1, 2, 3, ..., M-1$

The error is $O(dx)^2 + O(dt)^2$

**Example 3.4.** *Recall the example (3.1)*

*Using* $N = 20$ , $M = 5, 20$ *and* $T = 0.1$ *for different time* $t = 0.02, 0.04, 0.1$, *the solution of the heat equation (3.8) by Crank Nicolson method is given in figure (3.7) [22].*

From figure (3.7) it is clear that Crank Nicolson has outperformed the explicit and the implicit methods. However, the method is not perfect, because it did not better at the jump discontinuities.

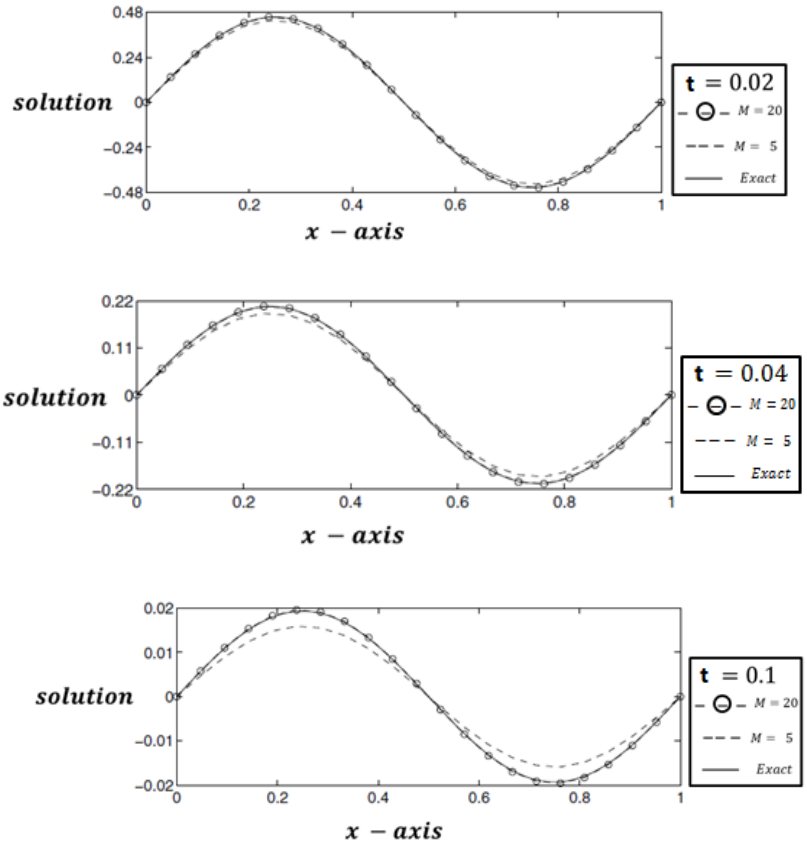



**Figure 3.7: Solution of the heat equation using the Crank Nicolson method**

**Example 3.5.** *Consider the heat equation with discontinuities in the initial condition:*

$$U_{xx} = U_t, 0 < x < 1, 0 < t < T$$

$$U(0,t) = U(1,t) = 0$$

$$U(x,0) = \begin{cases} 1 & \frac{1}{4} < x < \frac{3}{4} \\ 0 & otherwise \end{cases}, initial\ condition \tag{3.11}$$

*The initial condition has two discontinuities.*

*With* $N = 30$, $M = 5, 10, 20$ *and* $T = 0.1$ *for each time* $t = 0.02, 0.04, 0.1$, *the solution of the heat equation (3.11) using the Crank Nicolson method and the implicit method is given in figures (3.8) and (3.9) respectively.*

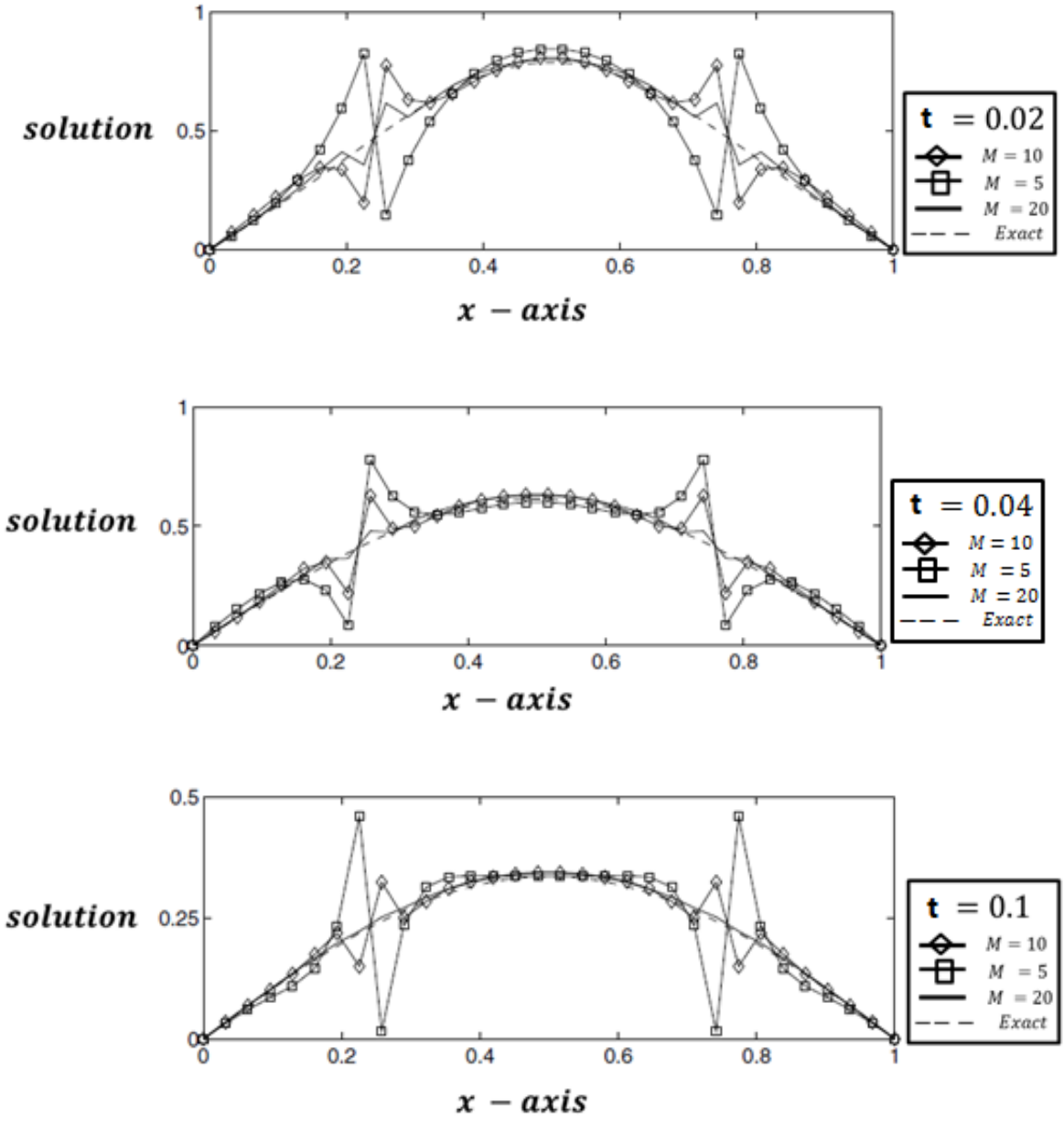



**Figure 3.8: Solution of the heat equation (example 3.5) using the Crank Nicolson method**

Both implicit and Crank Nicolson methods are stable, but the implicit method did better at the jump discontinuities. The explanation is in the type of stability.

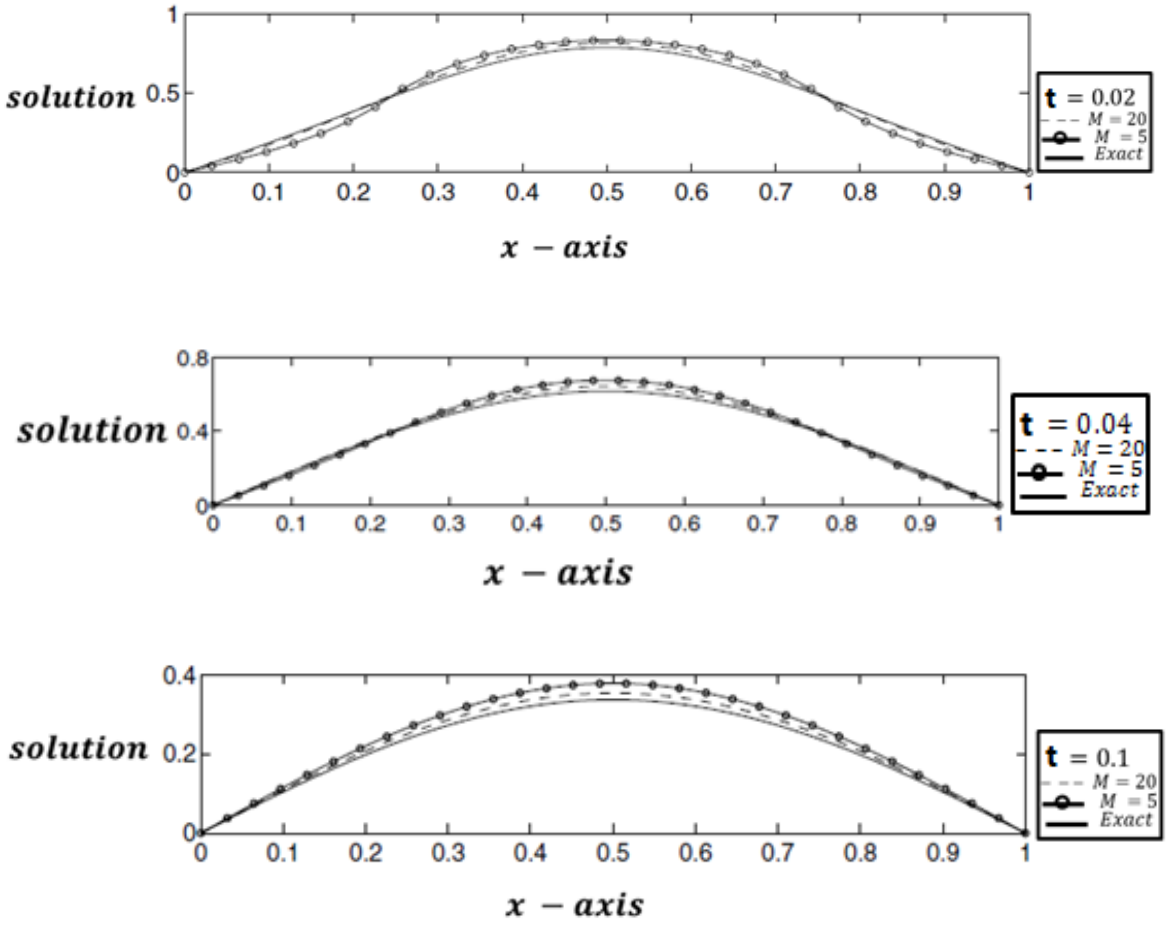



**Figure 3.9: Solution of the heat equation (example 3.5) using the implicit method**

There are two types of stability, stable and L-stable, the Crank Nicolson method is unconditionally stable but not L-stable. The implicit method is unconditionally stable and L-stable.

**Definition 3.6.** *L-stable methods: the method is stable and the amplification factor* $k \to 0$ *as the time step* $dt \to \infty$

## 3.4 Method of Lines

The method of lines was discovered in the early 1960s and it is known to experts in computational techniques in electromagnetism. It's applications has increased dramatically in the last few years, and we use this method to find a solution for the heat equation.

The idea of the method of lines is to reduce the problem to an initial value problem (IVP) by approximating the space derivative $U_{xx}$ using centered difference approximation ([22, 27]). Therefore equation (3.1) becomes:

$$U_t(x_i,t) = \frac{U(x_{i+1},t) - 2U(x_i,t) + U(x_{i-1},t)}{(dx)^2} - g(x_i,t) + O(dx)^2$$

dropping the error term, we obtain the IVP:

$$\frac{dU_i(t)}{dt} = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{(dx)^2} - g_i(t)$$

with $U_i(0) = \psi_i, i = 1, 2, 3, ..., N$

There are many numerical methods for IVPs. One of the most popular is the Runge Kutta method (RK4) which makes the method of lines with RK4 (LRK4).

This way we need to solve $N$ IVPs one for each $i$. This means for each $i = 1, 2, 3, ..., N$, approximate the solution on all time levels, then step to the next space level.

The method of lines is conditionally stable, but determining the stability condition is not so easy.

**Example 3.7.** *Recall the example (3.1), using the method of lines for solving the heat equation (3.8), using* $N = 18$, $M = 5, 10, 20$ *and* $T = 0.1$ *for each time* $t = 0.02, 0.04, 0.1$, *the result is given in figure (3.10).*

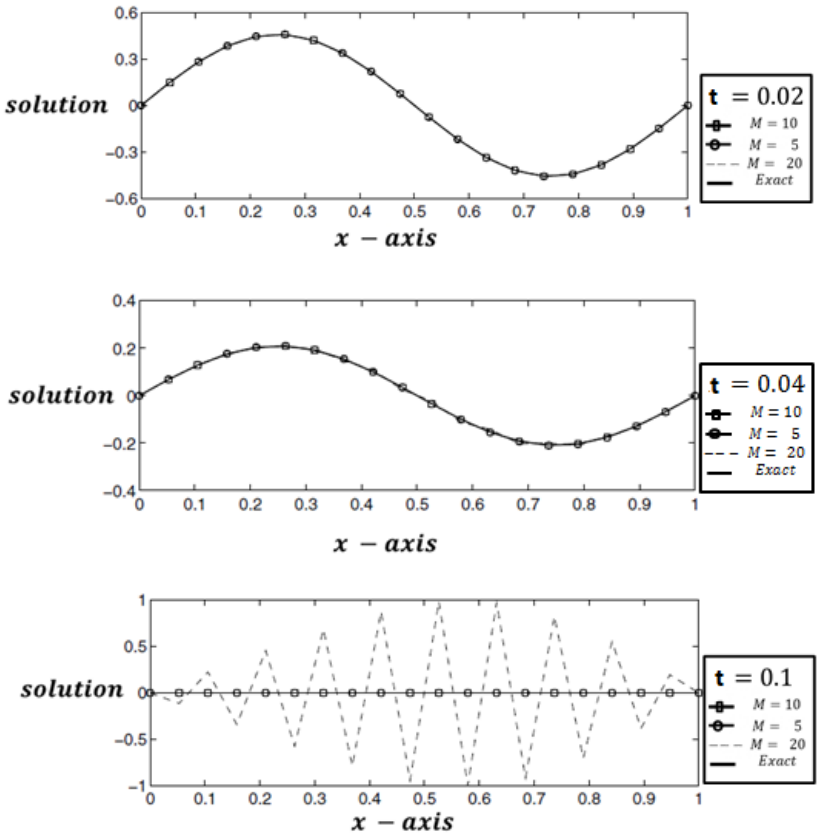



**Figure 3.10: Solution of the heat equation using the method of lines**

**Example 3.8.** *Consider the heat equation:*

$$U_{xx} = U_t, 0 < x < 1, 0 < t < T$$

$$U(0,t) = U(1,t) = 0$$

$$U(x,0) = sin(\pi x) \tag{3.12}$$

*the exact solution using separation of variables method is:*

$$A_n = 2\int_0^1 sin(\pi x)sin(n\pi x)dx = \begin{cases} 1 & n = 1 \\ 0 & otherwise \end{cases}$$

$$U(x,t) = \begin{cases} e^{-\pi^2 t}sin(\pi x) & n = 1 \\ 0 & otherwise \end{cases}$$

*Using Matlab the numerical solution of the heat equation (3.12) using* $N = 20$, $M = 1000$ *and* $T = 0.1$ *at time* $t = 0.0002$ *using explicit forward difference, implicit backward difference, Crank Nicolson and Method of Lines respectively is:*

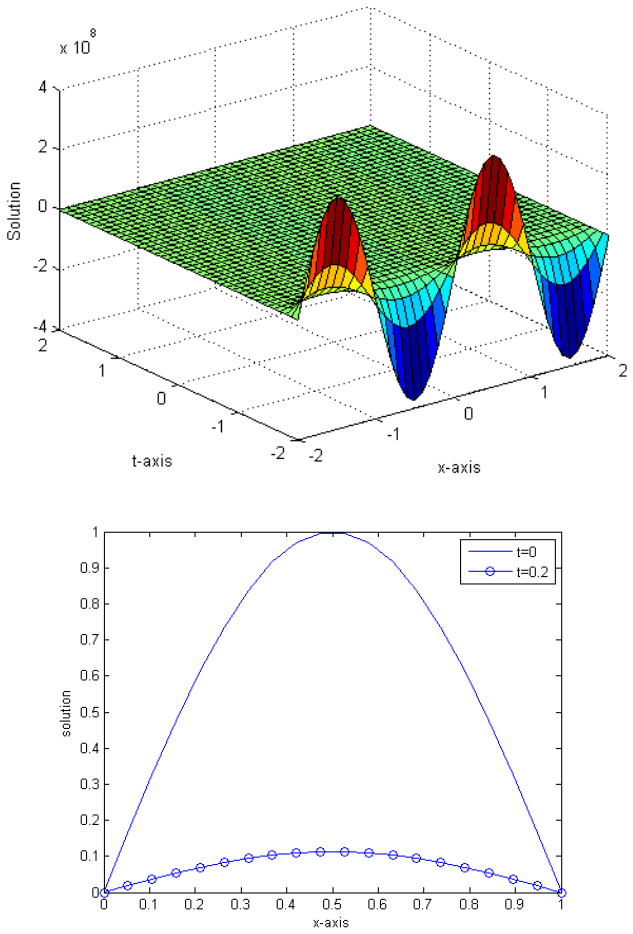



**Figure 3.11: The exact solution of the heat equation (3.12)**

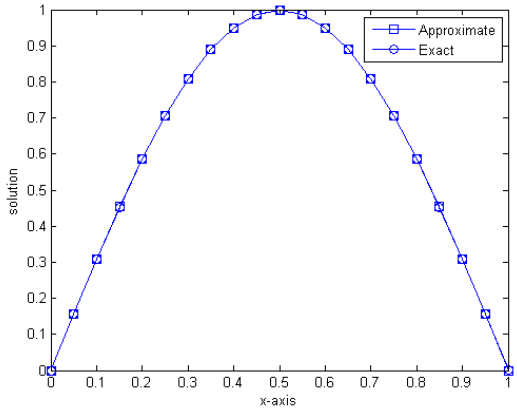



**Figure 3.12: The solution of the heat equation (3.12) using Explicit method**

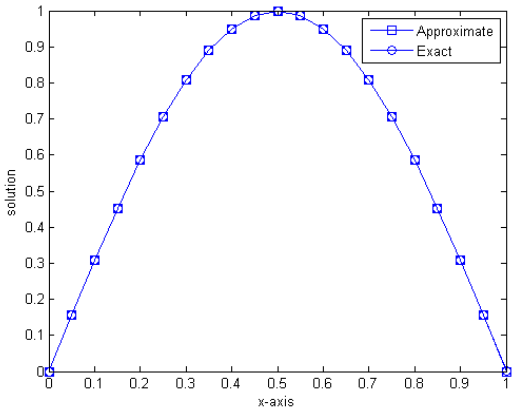



**Figure 3.13: The solution of the heat equation (3.12) using Implicit method**

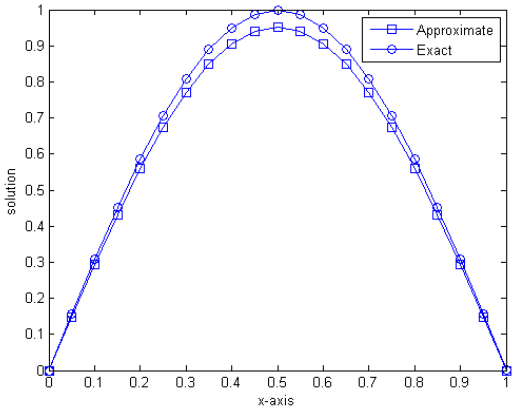



**Figure 3.14: The solution of the heat equation (3.12) using Crank Nicolson method**

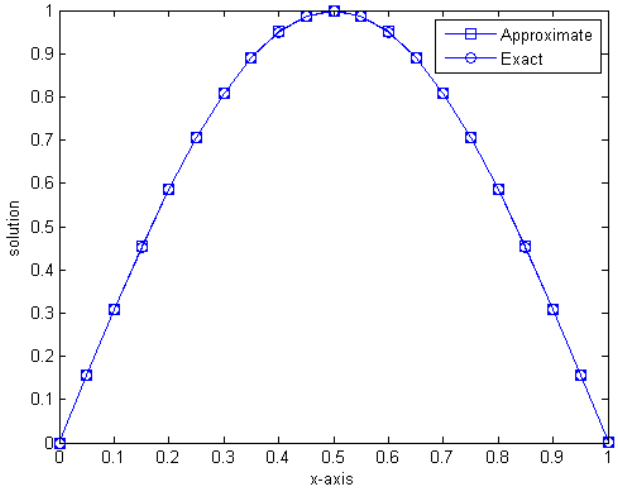



**Figure 3.15: The solution of the heat equation (3.12) using Method of Lines**

| Space step size | Explicit method | Implicit method | Crank Nicolson method | Method of Lines |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0.050000000000000 | 0.000154407266354 | 0.000000784508708 | 0.007208768409222 | 0.000269970079502 |
| 0.100000000000000 | 0.000305012513362 | 0.000001549700207 | 0.014240033015677 | 0.000533292599668 |
| 0.150000000000000 | 0.000448107340013 | 0.000002276732945 | 0.020920660749328 | 0.000783483686133 |
| 0.200000000000000 | 0.000580168276769 | 0.000002947704961 | 0.027086152383638 | 0.001014382804013 |
| 0.250000000000000 | 0.000697943545084 | 0.000003546094698 | 0.032584693052478 | 0.001220304450714 |
| 0.300000000000000 | 0.000798533126976 | 0.000004057167815 | 0.037280890435968 | 0.001396178151880 |
| 0.350000000000000 | 0.000879460173103 | 0.000004468339995 | 0.041059108568743 | 0.001537673313297 |
| 0.400000000000000 | 0.000938731991007 | 0.000004769486815 | 0.043826315181187 | 0.001641305854496 |
| 0.450000000000000 | 0.000974889111820 | 0.000004953193041 | 0.045514372462669 | 0.001704523998375 |
| 0.500000000000000 | 0.000987041227228 | 0.000005014935215 | 0.046081714840579 | 0.001725771104423 |
| 0.550000000000000 | 0.000974889111820 | 0.000004953193041 | 0.045514372462669 | 0.001704523998375 |
| 0.600000000000000 | 0.000938731991007 | 0.000004769486815 | 0.043826315181187 | 0.001641305854496 |
| 0.650000000000000 | 0.000879460173103 | 0.000004468339995 | 0.041059108568743 | 0.001537673313297 |
| 0.700000000000000 | 0.000798533126976 | 0.000004057167815 | 0.037280890435967 | 0.001396178151880 |
| 0.750000000000000 | 0.000697943545084 | 0.000003546094698 | 0.032584693052477 | 0.001220304450714 |
| 0.800000000000000 | 0.000580168276769 | 0.000002947704961 | 0.027086152383639 | 0.001014382804013 |
| 0.850000000000000 | 0.000448107340013 | 0.000002276732945 | 0.020920660749329 | 0.000783483686133 |
| 0.900000000000000 | 0.000305012513362 | 0.000001549700207 | 0.014240033015679 | 0.000533292599668 |
| 0.950000000000000 | 0.000154407266354 | 0.000000784508708 | 0.007208768409222 | 0.000269970079502 |
| 1.000000000000000 | 0.000000000000000 | 0.000000000000000 | 0.000000000000000 | 0.000000000000000 |

**Figure 3.16: Absolute error for each method used to solve the heat equation (3.12)**

# Chapter Four

# Optimal Control of Heat Equation

Optimal control is the process of determining optimal control function and state trajectories for a dynamic system over a period of time to minimize a performance index.

For example, consider a dynamic system:

$$\dot{x} = g(x(t), u(t), t) \tag{4.1}$$

$$x(t_0) = x_0$$

$$J(x(t), u(t), t) = S(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t)dt \tag{4.2}$$

Equation (4.1) is called state space equation and it can be linear and take the form:

$$\dot{x} = Ax + Bu$$

where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$ are constant matrices.

Equation (4.2) is called Performance index or cost function. The first part $S(x(t_f), t_f)$ is called terminal part and it is a function of final state and final time, moreover $L(x(t), u(t), t)$ is called trajectory part and it is a function of state, control function and time.

The optimal control problem in continuous time can be solved by two ways using the Pontryagin Maximum Principle or the Hamilton Jacobi Bellman (HJB) equation, in addition we use a dynamic programming approach for solving the (HJB) equation.

## 4.1 The Hamilton Jacobi Bellman (HJB) Equation

The Hamilton Jacobi Bellman equation is a partial differential equation which is the result of the dynamic programming theory by Richard Bellman in the 1950s [17]. The HJB is central to optimal control theory and the solution of HJB is the control function which afford the minimum cost for a given dynamical system with an associated performance index (cost function).

Consider a dynamic system described by state space equation:

$$\dot{x}(t) = g(x(t), u(t), t) \tag{4.3}$$

where

$$\dot{x} = \frac{dx}{dt}$$

$$x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T \in \mathcal{R}^n$$

is the state vector of the dynamical system.

$$u(t) \in \mathcal{R}^m$$

is called the control function (value function) which is the input function

of the dynamical system. Subject to the initial condition:

$$x(t_0) = x_0$$

and the performance index:

$$J(x(t), u(t), t) = S(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t)dt$$

The dynamic programming principle is used to derive HJB equation which is solved by a control function that minimize the performance index (4.2).

The optimal control is denoted by $u^\star$ such that

$$J(u^\star) \leq J(u)$$

substituting $u^\star$ in the state space equation (4.3), we obtain:

$$\dot{x} = g(x(t), u^\star(t), t)$$

the optimal solution of this equation is denoted by $x^\star$.

Consider the following equation:

$$J_t(x(t), t) = L(x(t), u(t), t) + J_x(x(t), t)g(x(t), u(t), t) \qquad (4.4)$$

Equation (4.4) is a partial differential equation which is called the Hamil-

ton Jacobi Bellman equation (HJB) ([8, 17]). Also it can be written as:

$$0 = J_t + \mathcal{H} \tag{4.5}$$

Where $\mathcal{H}$ is called the Hamiltonian function that is defined by:

$$\mathcal{H} = L + \lambda^T g$$

where $\lambda \in \mathcal{R}^n$ is called the costate variable.

we can find the optimal control that minimize the performance index (4.2) by the following theorem.

**Theorem 4.1.** *(Maximum Principle) if $x^\star$ and $u^\star$ is optimal solution, then there exist $\lambda^\star$ is also a solution such that: ([8, 17, 23])*

$$\frac{\partial \mathcal{H}}{\partial x} = -\dot{\lambda} \ (costate\, equation)$$
$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{x} \ (optimal\, state\, equation)$$
$$\frac{\partial \mathcal{H}}{\partial u} = 0 \ (optimal\, control\, equation)$$

Subject to the condition:

$$\mathcal{H}(x^\star, u^\star, \lambda^\star, t) \leq \mathcal{H}(x, u, \lambda, t)$$

**Example 4.2.** *Find the optimal control for the given system*

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

$$J = \int_0^2 \frac{1}{2}u^2 dt$$

$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, x(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

*solution:*

$$L = \frac{1}{2}u^2, g = \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

$$J_t + \mathcal{H} = 0$$

$$\mathcal{H} = L + \lambda^T g = \frac{1}{2}u^2 + [\lambda_1 \quad \lambda_2]g = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u$$

*by theorem (4.1)*

$$\frac{\partial \mathcal{H}}{\partial u} = u + \lambda_2 = 0 \rightarrow u = -\lambda_2$$

$$\mathcal{H} = -\frac{1}{2}{\lambda_2}^2 + \lambda_1 x_2$$

$$\frac{\partial \mathcal{H}}{\partial x_1} = 0 = -\dot{\lambda_1} \rightarrow \lambda_1 = c_3$$

$$\frac{\partial \mathcal{H}}{\partial x_2} = \lambda_1 = -\dot{\lambda_2} \rightarrow \dot{\lambda_2} = -c_3 \rightarrow \lambda_2 = -c_3 t + c_4$$

$$\frac{\partial \mathcal{H}}{\partial \lambda_1} = x_2 = \dot{x_1} \rightarrow x_1 = \frac{c_3}{6}t^3 - \frac{c_4}{2}t^2 + c_2 t + c_1$$

$$\frac{\partial \mathcal{H}}{\partial \lambda_2} = -\lambda_2 = \dot{x_2} = c_3 t - c_4 \rightarrow x_2 = \frac{c_3}{2} t^2 - c_4 t + c_2$$

*by the initial conditions:*

$$x_1(0) = 1 \rightarrow c_1 = 1$$

$$x_2(0) = 2 \rightarrow c_2 = 2$$

$$x_1(2) = 1 \rightarrow \frac{8}{6} c_3 - 2c_4 + 5 = 1$$

$$x_2(2) = 0 \rightarrow 2c_3 - 2c_4 + 2 = 0$$

$$\rightarrow c_3 = 3, c_4 = 4$$

*therefore*

$$\lambda_1{}^\star = 3$$

$$\lambda_2{}^\star = 4 - 3t$$

$$x_1{}^\star = \frac{1}{2} t^3 - 2t^2 + 2t + 1$$

$$x_2{}^\star = \frac{3}{2} t^2 - 4t + 2$$

$$u^\star = 3t - 4$$

## 4.2 Linear Quadratic Control Regulator (LQR)

In LQR the dynamic system is linear, also the performance index and the control function $u(t)$ is quadratic ([12, 30])

Consider the continuous linear dynamical system:

$$\dot{x}(t) = Ax + Bu \tag{4.6}$$

$$x(0) = x_0$$

$$J = \frac{1}{2}x^T(t)Rx(t) + \frac{1}{2}\int_0^{t_f}(x^THx + u^TQu)dt \tag{4.7}$$

$$J = \frac{1}{2}\int_0^{\infty}(x^THx + u^TQu)dt$$

without terminal part in the infinite time.

Where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$ are constant matrices and $A, B$ are controllable.

$Q > 0$ positive definite matrix.

$H, R \geq 0$ are positive semi definite matrices.

### 4.2.1 State Space Formulation

We discuss how to form a state space equation for any ordinary differential equation. Consider the general form of a system characterized by an $n^{th}$ order differential equation:

$$f^{(n)} + a_1 f^{(n-1)} + a_2 f^{(n-2)} + ... + a_{n-1}\dot{f} + a_n f = u \tag{4.8}$$

subject to

$$f(0), \dot{f}(0), ..., f^{(n-1)}(0)\, are\, known$$

such that

$$\dot{f} = \frac{df}{dt}, \ddot{f} = \frac{d^2f}{dt^2}, ..., f^{(n)} = \frac{d^nf}{dt^n}$$

Define $x_1 = f, x_2 = \dot{f}, ..., x_n = f^{(n-1)}$, equation (4.8) can be written as [5]:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = x_4$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - ... - a_1 x_n + u$$

We can write it as a vector matrix differential equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & ... & 0 & 1 \\ -a_n & -a_{n-1} & ... & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \qquad (4.9)$$

therefore,

$$\dot{x} = Ax + Bu$$

where $A, B$ and $x$ are defined in equation (4.9).

**Example 4.3.** *Form the state space equation for the following system*

$$\dddot{f} - 3\ddot{f} + 2\dot{f} - 2f = u$$

*Solution:*

$x_1 = f$, $x_2 = \dot{f}$ *and* $x_3 = \ddot{f}$, *then*

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = 2x_1 - 2x_2 + 3x_3 + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

In equation (4.6), the matrices $A$ and $B$ must be controllable. Therefore, we will discuss how we can know that $A$ and $B$ can be controlled or not.

Let $M = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$

**Definition 4.4.** *the system is controllable if the rank of matrix $M$ is $n$ (i.e. $M$ is with full rank), other wise the system is uncontrollable ([5, 18, 32]).*

**Definition 4.5.** *The matrix M is called the system controllability matrix.*

Consider the following example:

**Example 4.6.** *Determine whether the following system is controllable or not*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

*Solution:*

$$n = 2 \rightarrow M = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix}$$

*rank(M)=2 the system is controllable*

### 4.2.2 Solution of the State Space Equation by Laplace Transform

The state space equation can be written in a vector matrix form, therefore we determine how to find Laplace transform of a vector.

Let

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$\mathcal{L}\{x(t)\} = \begin{bmatrix} \mathcal{L}\{x_1(t)\} \\ \vdots \\ \mathcal{L}\{x_n(t)\} \end{bmatrix} = \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = X(s)$$

since

$$\mathcal{L}\{\dot{g}\} = \mathcal{L}\{\frac{dg}{dt}\} = sG(s) - g(0)$$

therefore

$$\mathcal{L}\{\dot{x}(t)\} = \begin{bmatrix} \mathcal{L}\{\dot{x}_1(t)\} \\ \vdots \\ \mathcal{L}\{\dot{x}_n(t)\} \end{bmatrix} = \begin{bmatrix} sX_1(s) - x_1(0) \\ \vdots \\ sX_n(s) - x_n(0) \end{bmatrix} = sX(s) - x(0)$$

to solve equation (4.6), we take Laplace transform for both sides of equation (4.6),to get [5]:

$$sX(s) - x(0) = AX(s) + BU(s) \rightarrow [sI - A]X(s) = x(0) + BU(s)$$

$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s) \tag{4.10}$$

$$x(t) = \mathcal{L}^{-1}[[sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s)] \tag{4.11}$$

**Example 4.7.** *solve the following system*

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -8 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

*where* $u(t) = 1$ *and* $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

*Solution:*

$$[sI - A] = \begin{bmatrix} s-1 & 1 \\ 8 & s+1 \end{bmatrix}$$

$$[sI - A]^{-1} = \begin{bmatrix} \frac{s+1}{s^2-9} & \frac{-1}{s^2-9} \\ \frac{-8}{s^2-9} & \frac{s-1}{s^2-9} \end{bmatrix}$$

$$U(s) = \mathcal{L}(u(t)) = \frac{1}{s}$$

$$X(s) = \begin{bmatrix} \frac{-1}{s(s^2-9)} \\ \frac{s-1}{s(s^2-9)} \end{bmatrix}$$

*taking the inverse Laplace transform, the result is:*

$$x(t) = \begin{bmatrix} \frac{1}{9} - \frac{cosh(3t)}{9} \\ \frac{1}{9} - \frac{cosh(3t)}{9} + \frac{sinh(3t)}{3} \end{bmatrix}$$

### 4.2.3 Determination the Optimal Control Using Maximum Principle to Derive Algebraic Riccati Equation

The optimal control for the system in equation (4.6) that minimize the cost function in equation (4.7) is determined as:

The Hamiltonian function for equatoin (4.7) is:

$$\mathcal{H} = \frac{1}{2}(x^T H x + u^T Q u) + \lambda^T (Ax + Bu)$$

the costate equation is given by:

$$\dot{\lambda}(t) = \frac{\partial \mathcal{H}}{\partial x} = -(Hx + A^T\lambda)$$

the optimal control equation is:

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \rightarrow u = -Q^{-1}B^T\lambda$$

the optimal state equation is:

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial \lambda} = Ax - BQ^{-1}B^T\lambda$$

These equations can be written in vector matrix form as:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BQ^{-1}B^T \\ -H & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{4.12}$$

But it is not easy to solve the system (4.12), therefore we guess the solution of (4.12) as:

$$\lambda(t) = P(t)x(t), P \in \mathcal{R}^{n\times n}$$

$$\dot{\lambda} = \dot{P}x + P\dot{x}$$

$$= \dot{P}x + P(Ax + Bu)$$

$$= \dot{P}x + P(Ax - BQ^{-1}B^T\lambda)$$

$$= \dot{P}x + P(Ax - BQ^{-1}B^TPx)$$

$$-(Hx + A^TPx) = (\dot{P} + PA - PBQ^{-1}B^TP)x$$

$$(\dot{P} + PA + A^TP - PBQ^{-1}B^TP + H)x = 0$$

From last equation, we get the equation:

$$\dot{P} + PA + A^TP - PBQ^{-1}B^TP + H = 0 \tag{4.13}$$

$$P(t_f) = R$$

for the infinite time horizon, there is no terminal part, therefore when the time approaches infinity, [23] we have:

$$lim_{t\to\infty}\dot{P} = 0 \tag{4.14}$$

from equations (4.14) and (4.13), we get:

$$PA + A^TP - PBQ^{-1}B^TP + H = 0 \tag{4.15}$$

Equation (4.15) is called Algebraic Riccati Equation(ARE) [12], where the positive definite matrix $P$ is the solution of the equation (4.15).

The control function $u$ that minimize the cost function $J$ is written

as:

$$u(t,x) = -(Q^{-1}B^T P(t))x(t) = -K(t)x(t)$$

substituting $u$ in the state space equation (4.6), the result is:

$$\dot{x} = (A - BQ^{-1}B^T P(t))x$$

**Example 4.8.** *Consider the following system*

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

*with*

$$H = \begin{bmatrix} a^2 & 0 \\ 0 & 0 \end{bmatrix}, Q = 1$$

*Let* $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ *be the solution of riccati equation, then the riccati equation becomes:*

$$\begin{bmatrix} -p_{12}^2 + a^2 & p_{11} - p_{12}p_{22} \\ p_{11} - p_{12}p_{22} & 2p_{12} - p_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

*the solution is:*

$$P = \begin{bmatrix} \sqrt{2a^3} & a \\ a & \sqrt{2a} \end{bmatrix}$$

*therefore*

$$K = Q^{-1}B^T P = \begin{bmatrix} a & \sqrt{2a} \end{bmatrix}$$

*the control function that minimizing the cost function is* $u = -Kx$

Consider the following procedure to solve ARE:

Take the following linear dynamical system:

$$\dot{x}(t) = Ax + Bu \tag{4.16}$$

$$x(0) = x_0$$

$$J = \frac{1}{2}x^T(t)Rx(t) + \frac{1}{2}\int_0^{t_f}(x^T Hx + u^T Qu)dt \tag{4.17}$$

equations (4.16) and (4.17) can be solved by [10]:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -BQ^{-1}B^T \\ -H & -A \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = Z \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$$

$x(0) = x_0$, $p(t_f) = Rx(t_f)$

The above system is the corresponding Hamiltonian system for equations (4.16) and (4.17).

The solution of equation $p(t_f) = Rx(t_f)$ in $t = t_f$ using state transient matrix ($\mathcal{L}^{-1}[SI - Z]^{-1}$) is:

$$\begin{bmatrix} x(t_f) \\ p(t_f) \end{bmatrix} = e^{z(t_f-t)} \begin{bmatrix} \Phi_{11}(t_f-t) & \Phi_{12}(t_f-t) \\ \Phi_{21}(t_f-t) & \Phi_{22}(t_f-t) \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{4.18}$$

$$P(t) = [\Phi_{22}(t_f-t) - R\Phi_{12}(t_f-t)]^{-1}[R\Phi_{11}(t_f-t) - \Phi_{21}(t_f-t)] \tag{4.19}$$

**Example 4.9.** *Consider the following system*

$$\dot{x}(t) = x(t) + u(t)$$

*find the control function $u$ that minimize the following cost function*

$$J = \frac{1}{2}8x^2(5) + \frac{1}{2}\int_0^5 u^2(t)dt$$

*we have:* $A = 1$, $B = 1$, $t_f = 5$, $R = 8$, $H = 0$ *and* $Q = 1$

*Hamiltonian matrix Z is:*

$$Z = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

*The transient matrix is:*

$$\begin{bmatrix} e^t & \frac{1}{2}e^{-t} - \frac{1}{2}e^t \\ 0 & e^{-t} \end{bmatrix}$$

$$K(t) = \frac{8e^{(5-t)}}{-3e^{-(5-t)} + 4e^{(5-t)}}$$

*the control function* $u(t,x) = -Kx$

## 4.3 Optimal Control of Heat Equation

In linear quadratic control, the state space equation is an ordinary differential equation. Therefore we must convert the heat equation to an ordinary differential equation, so by using the separation of variables method we can express the solution of the heat equation as a product of two functions each of them is a solution of ordinary differential equations, then use the riccati equation for solving each of them.

Recall the example (2.2):

$$U_t = U_{xx}, 0 < x < 1, 0 < t < 0.1$$

$$U(0,t) = U(1,t) = 0$$

$$U(x,0) = sin(2\pi x) \tag{4.20}$$

the exact solution is:

$$U(x,t) = e^{-4\pi^2 t} sin(2\pi x)$$

By separation of variables, we obtain:

$$U(x,t) = G(x)H(t)$$

where $G(x)$ is the solution of the following ordinary differential equation:

$$\frac{d^2G(x)}{dx^2} + 4\pi^2 G(x) = 0$$

and $H(t)$ is the solution of the following ordinary differential equation:

$$\frac{dH(t)}{dt} + 4\pi^2 H(t) = 0$$

To obtain the control function for each of ODE's, we take the case that each of them equal the control function $u_1, u_2$ respectively. For the first ordinary differential equation:

$$\frac{d^2G(x)}{dx^2} + 4\pi^2 G(x) = u_1, G(0) = G(1) = 0 \tag{4.21}$$

the state space equation for equation (4.21) is:

$$x_1 = G$$

$$x_2 = \dot{G}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -4\pi^2 x_1 + u_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4\pi^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1$$

$$\dot{x} = Ax + Bu_1$$

To verify that $A$ and $B$ are controllable or not:

$$M = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

rank(M)=2, so $A$ and $B$ are controllable.

The performance index for equation (4.21) is:

$$J = \frac{1}{2}\int_{0.0}^{0.1} (x^2 + u^2)dt$$

with $Q = 1,\ R = 0,\ H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$Z = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4\pi^2 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 4\pi^2 & 0 \end{bmatrix}$$

$$(sI - Z) = \begin{bmatrix} s & -1 & 0 & 0 \\ 4\pi^2 & s & 0 & 1 \\ 1 & 0 & s & 1 \\ 0 & 1 & -4\pi^2 & s \end{bmatrix}$$

$$(sI - Z)^{-1} = \begin{bmatrix} \frac{s}{(s^2+4\pi^2)} & \frac{1}{s^2+4\pi^2-1} & \frac{-4\pi^2}{(s^2+4\pi^2-1)(4\pi^2+s^2)} & \frac{-s}{(s^2+4\pi^2-1)(4\pi^2+s^2)} \\ \frac{-4\pi^2}{(s^2+4\pi^2)} & \frac{s}{s^2+4\pi^2-1} & \frac{-4\pi^2 s}{(s^2+4\pi^2-1)(4\pi^2+s^2)} & \frac{-s^2}{(s^2+4\pi^2-1)(4\pi^2+s^2)} \\ \frac{-1}{s^2+4\pi^2} & 0 & \frac{s}{s^2+4\pi^2} & \frac{-1}{s^2+4\pi^2} \\ 0 & \frac{-1}{s^2+4\pi^2-1} & \frac{4\pi^2}{4\pi^2+s^2-1} & \frac{s}{4\pi^2+s^2-1} \end{bmatrix}$$

$$P = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where

$$a_{11} = \frac{0.1591549431 \cos\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)\sqrt{4\pi^2-1}\,\sin(2\pi(0.1-t))}{\cos(2\pi(0.1-t))\cos\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)\sqrt{4\pi^2-1} - 6.283185309\sin(2\pi(0.1-t))\sin\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)}$$

$$a_{12} = \frac{0.1591549431 \sin(2\pi(0.1-t))\sin\left(\sqrt{4\pi^2-1}\,(0.1-t)\right)}{\cos(2\pi(0.1-t))\cos\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)\sqrt{4\pi^2-1} - 6.283185309\sin(2\pi(0.1-t))\sin\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)}$$

$$a_{21} = \frac{6.283185309 \sin\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)\sin(2\pi(0.1-t))}{\cos(2\pi(0.1-t))\cos\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)\sqrt{4\pi^2-1} - 6.283185309\sin(2\pi(0.1-t))\sin\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)}$$

$$a_{22} = \frac{\cos(2\pi(0.1-t))\sin\left(\sqrt{4\pi^2-1}\,(0.1-t)\right)}{\cos(2\pi(0.1-t))\cos\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)\sqrt{4\pi^2-1} - 6.283185309\sin(2\pi(0.1-t))\sin\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)}$$

$$u_1 = -\left[\begin{array}{c} \frac{6.283185309 \sin\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)\sin(2\pi(0.1-t))}{\cos(2\pi(0.1-t))\cos\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)\sqrt{4\pi^2-1} - 6.283185309\sin(2\pi(0.1-t))\sin\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)} \\ \frac{\cos(2\pi(0.1-t))\sin\left(\sqrt{4\pi^2-1}\,(0.1-t)\right)}{\cos(2\pi(0.1-t))\cos\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)\sqrt{4\pi^2-1} - 6.283185309\sin(2\pi(0.1-t))\sin\left(\sqrt{4\pi^2-1}\,(-0.1+t)\right)} \end{array}\right] X$$

For the second ordinary differential equation:

$$\frac{dH(t)}{dt} + 4\pi^2 H(t) = u_2 \tag{4.22}$$

the state space equation for (4.22) is:

$$x_1 = H$$

$$\dot{x}_1 = -4\pi^2 x_1 + u_2$$

$$A = -4\pi^2, B = 1$$

$$\dot{x} = Ax + Bu_2$$

$M = 1 \rightarrow rank(M) = 1 \rightarrow A$ and $B$ is controllable.

The performance index for (4.22) is:

$$J = \frac{1}{2}\int_{0.0}^{0.1} (x^2 + u^2)dt$$

with $Q = 1, R = 0, H = 1$

$$Z = \begin{bmatrix} -4\pi^2 & -1 \\ -1 & 4\pi^2 \end{bmatrix}$$

$$[sI - Z] = \begin{bmatrix} s + 4\pi^2 & 1 \\ 1 & s - 4\pi^2 \end{bmatrix}$$

$$[sI - Z]^{-1} = \begin{bmatrix} \frac{4\pi^2 - s}{16\pi^4 - s^2 + 1} & \frac{1}{16\pi^4 - s^2 + 1} \\ \frac{1}{16\pi^4 - s^2 + 1} & \frac{-4\pi^2 - s}{16\pi^4 - s^2 + 1} \end{bmatrix}$$

$$Let\, a = cosh(\sqrt{16\pi^4 + 1}t) - \frac{4\pi^2 sinh(\sqrt{16\pi^4 + 1}t)}{\sqrt{16\pi^4 + 1}}$$

$$, b = \frac{-sinh(\sqrt{16\pi^4 + 1}t)}{\sqrt{16\pi^4 + 1}}$$

$$and\, c = cosh(\sqrt{16\pi^4 + 1}t) + \frac{4\pi^2 sinh(\sqrt{16\pi^4 + 1}t)}{\sqrt{16\pi^4 + 1}}$$

$$\mathcal{L}^{-1}[sI - Z]^{-1} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$P = \frac{sinh(\sqrt{16\pi^4 + 1}(0.1 - t))}{\sqrt{16\pi^4 + 1}cosh(\sqrt{16\pi^4 + 1}(0.1 - t)) + 4\pi^2 sinh(\sqrt{16\pi^4 + 1}(0.1 - t))}$$

$$u_2 = \left[ \frac{-sinh(\sqrt{16\pi^4+1}(0.1-t))}{\sqrt{16\pi^4+1}cosh(\sqrt{16\pi^4+1}(0.1-t)) + 4\pi^2 sinh(\sqrt{16\pi^4+1}(0.1-t))} \right] x$$

$u_1$ and $u_2$ are factors of control function of heat equation (4.20).

## 4.4 Illustrated Examples and Sensitivity Analysis

In this section, we will implement the Euler method to solve the state space equation.

- Euler method

  Euler method was created by Leonhard Euler in 1770 ([2, 4]), it is a numerical method to approximating the solution of differential equation:

$$y' = F(t, y), a \leq t \leq b, y(a) = y_0 \tag{4.23}$$

  The domain [a,b] is divided into a number of sub-intervals $M$, all sub-intervals have the same length $h = \frac{b-a}{M}$.

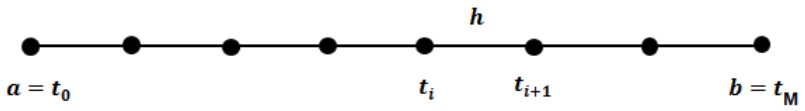



**Figure 4.1: Sub intervals of Euler method**

  from figure (4.1), note that $t_i = a + ih, i = 0, 1, ..., M$, to approximate the solution of the differential equation, we need to approximate the solution at each point $t_i, i = 0, 1, ..., M$, the solution at $t_0 = a$ is given by the initial condition $y(a) = y_0$.

  We use the notations:

  $y(t_i) = y_i$ for the exact solution at $t_i$.

  $R(t_i) = R_i$ for the approximate solution at $t_i$.

  approximating $y'$ at $t_i$ using first-order forward difference formula:

$$y_i' = \frac{y_{i+1} - y_i}{h} - \frac{h}{2} y''(\zeta_i), \zeta_i \in [t_i, t_{i+1}] \tag{4.24}$$

substituting (4.24) into equation (4.23), the result is:

$$\frac{y_{i+1}-y_i}{h}-\frac{h}{2}y''(\zeta_i)=F(t_i,y_i), i=0,1,...,M-1$$

$$y_{i+1}=y_i+hF(t_i,y_i)+\frac{h^2}{2}y''(\zeta_i) \tag{4.25}$$

dropping the remainder term, the new equation is not satisfied by $y_i$ but by $R_i$ where $R_i \cong y_i$, the equation (4.25) becomes:

$$R_{i+1}=R_i+hF(t_i,R_i), i=0,1,...,M-1 \tag{4.26}$$

$$R(a)=R_0$$

equation (4.26) is the iterative formula for Euler method for equation (4.23).

Starting with initial condition $R_0$ we compute $R_1$, then use $R_1$ to compute $R_2$ and so on until $R_M$.

**Example 4.10.** *Consider the differential equation,*

$$y'=t+2y, y(0)=0 \tag{4.27}$$

*using* $h=0.25, M=4$ *find the solution for equation (4.27) in [0.1] by Euler method.*

*Solution:*

$F(t, y) = t + 2y$ *and* $i = 0, 1, 2, 3$

$$R_{i+1} = R_i + 0.25(t_i + 2R_i), i = 0, .., 3$$

*The exact solution is:*

$$y = 0.25e^{2t} - 0.5t - 0.25$$

| Step size | Numerical solution | Exact solution | Error |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.250000000000000 | 0 | 0.037180317675032 | 0.037180317675032 |
| 0.500000000000000 | 0.062500000000000 | 0.179570457114761 | 0.117070457114761 |
| 0.750000000000000 | 0.218750000000000 | 0.495422267584516 | 0.276672267584516 |
| 1.000000000000000 | 0.515625000000000 | 1.097264024732663 | 0.581639024732663 |

**Figure 4.2: Numerical results for h=0.25**

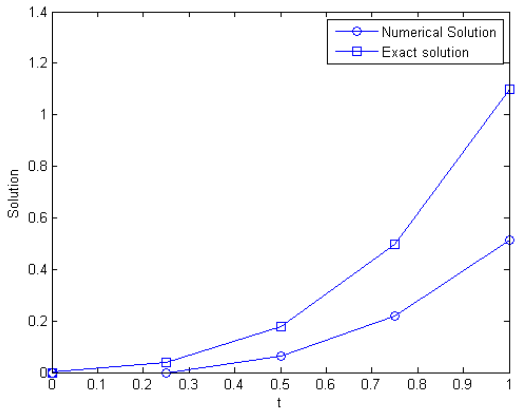



**Figure 4.3: Numerical Solution vs. Exact Solution, h=0.25**

*In figure (4.3) the numerical solution is so inaccurate, to reduce the error, we may reduce the step size h since the remainder term we dropped in the derivation depends on h.*

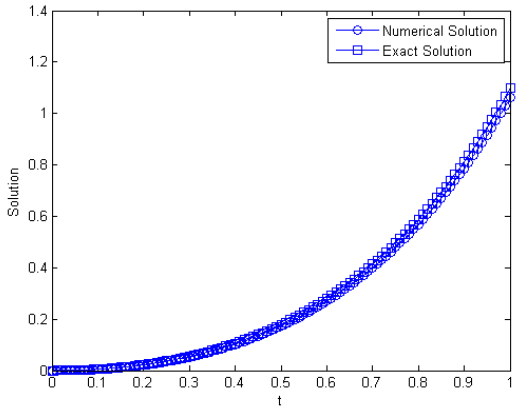



**Figure 4.4: Numerical Solution vs. Exact Solution, h=0.01**

*As we see in figure (4.4), the accuracy of this numerical solution is much higher than before.*

For solving the state space equation by Euler method, consider the example:

**Example 4.11.** *(Closed loop system)*

$$\dot{x} = 2x + u$$

$$x(0) = 1$$

$$J = \frac{1}{2}\int_0^1 u^2 dt$$

*Solution:*

$A = 2$,$B = 1$ *and* $Q = 1$

*The control function* $u$ *that minimize the cost function* $J$ *is taking the formula:*

$$u = -(Q^{-1}B^T P)x$$

*therefore, the state space equation becomes:*

$$\dot{x} = (A - BQ^{-1}B^T P)x$$

*Riccati equation for this example is:*

$$4P - P^2 = 0 \rightarrow P = 4 \, and \, P = 0 \, (rejected)$$

*The state space equation becomes:*

$$\dot{x} = -2x$$

*the exact solution is:*

$$x(t) = e^{-2t}$$

*by Euler method, using* $h = 0.01$ *and* $M = 100$*, the numerical solution and exact solution is given in figure (4.5).*

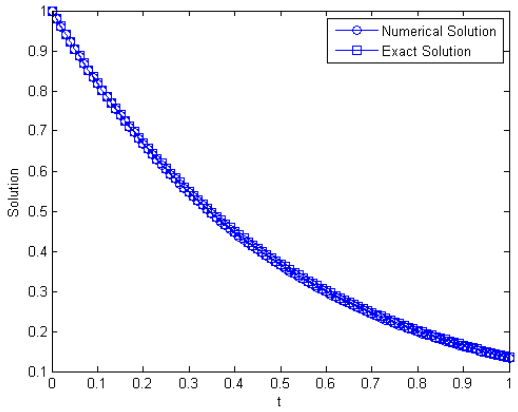



**Figure 4.5: Numerical Solution vs. Exact Solution, h=0.01**

**Example 4.12.**

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$J = \frac{1}{2} \int_0^1 ({x_1}^2 + u^2) dt$$

*Solution:*

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \; and \; Q = 1$$

*The solution of riccati equation would be:*

$$P = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$$

*The state space equation becomes:*

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x \tag{4.28}$$

*The exact solution is:*

$$x(t) = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$$

*By Euler method, using* $h = 0.01$ *and* $M = 100$*, we get:*

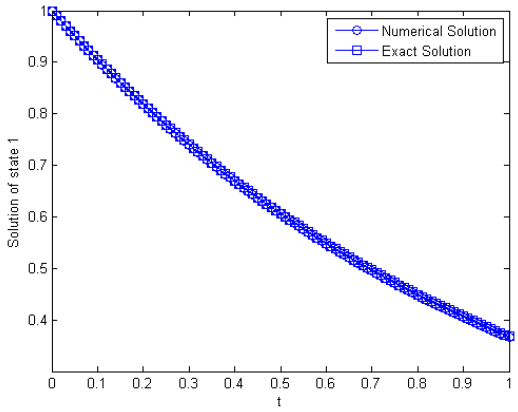



**Figure 4.6: Numerical Solution vs. Exact Solution for the first state, h=0.01**

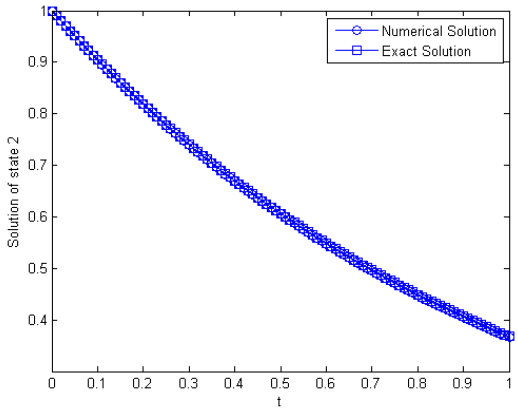



**Figure 4.7: Numerical Solution vs. Exact Solution for the second state, h=0.01**

we introduce the open control system through the following example:

**Example 4.13.** *( open loop system), Recall the example (4.7),*

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -8 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{4.29}$$

*where* $u(t) = 1$ *and* $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

*with output equation:*

$$y(t) = Cx(t) \tag{4.30}$$

*where* $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

*the exact solution by Laplace transform is:*

$$x(t) = \begin{bmatrix} \frac{1}{9} - \frac{cosh(3t)}{9} \\ \frac{1}{9} - \frac{cosh(3t)}{9} + \frac{sinh(3t)}{3} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ -8x_1 - x_2 + 1 \end{bmatrix} = F(t, x)$$

*using Euler method solve the system (4.29) using* $h = 0.01$, $t \in [0, 0.5]$, *the result is given in figures (4.8) and (4.9).*

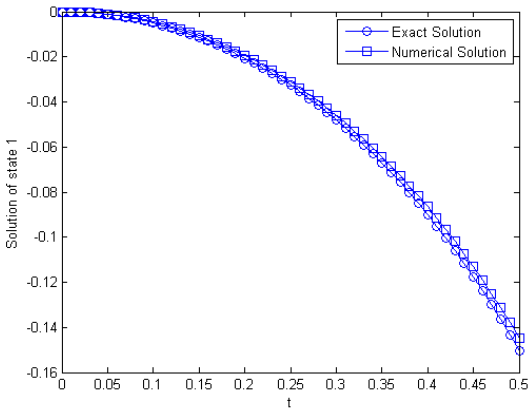



**Figure 4.8: Numerical Solution vs. Exact Solution for the first state, h=0.01**

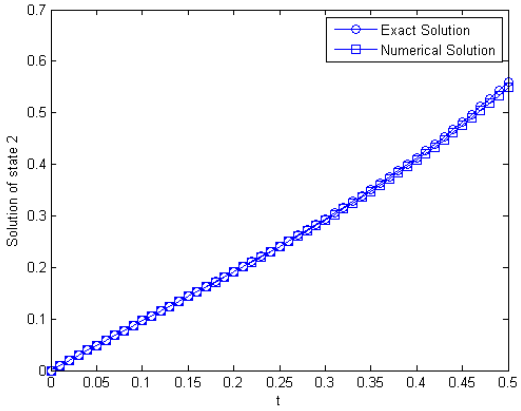



**Figure 4.9: Numerical Solution vs. Exact Solution for the Second state, h=0.01**

the solution of output equation (4.30) is given in figure (4.10)

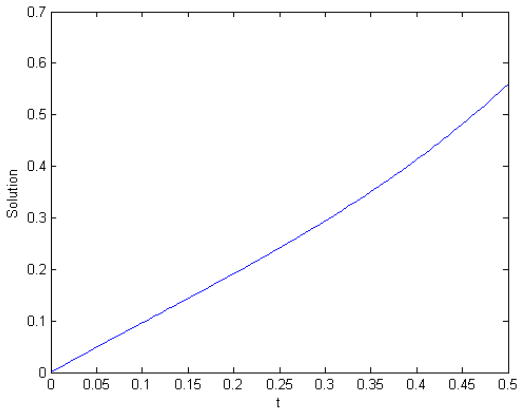



**Figure 4.10: Solution of output equation**

**Example 4.14.** *(CD-player)*

*Consider the example of the CD-player that could be found in [7], where the dimension of a matrix A is* $120 \times 120$

*The numerical solution using Euler method for the states 10, 60 and 120 respectively are given in figures (4.11).*

**Example 4.15.** *Consider the state space equation (4.28) where the matrix A is a* $200 \times 200$ *diagonal matrix in which the main diagonal entries are all -1.*

*Using Euler method, the solution of state 200 for example, is given by figure (4.12).*

*The absolute error for the results is between 0 and* $1.8 \times 10^{-3}$

$(0 \leq error \leq 1.8 \times 10^{-3})$.

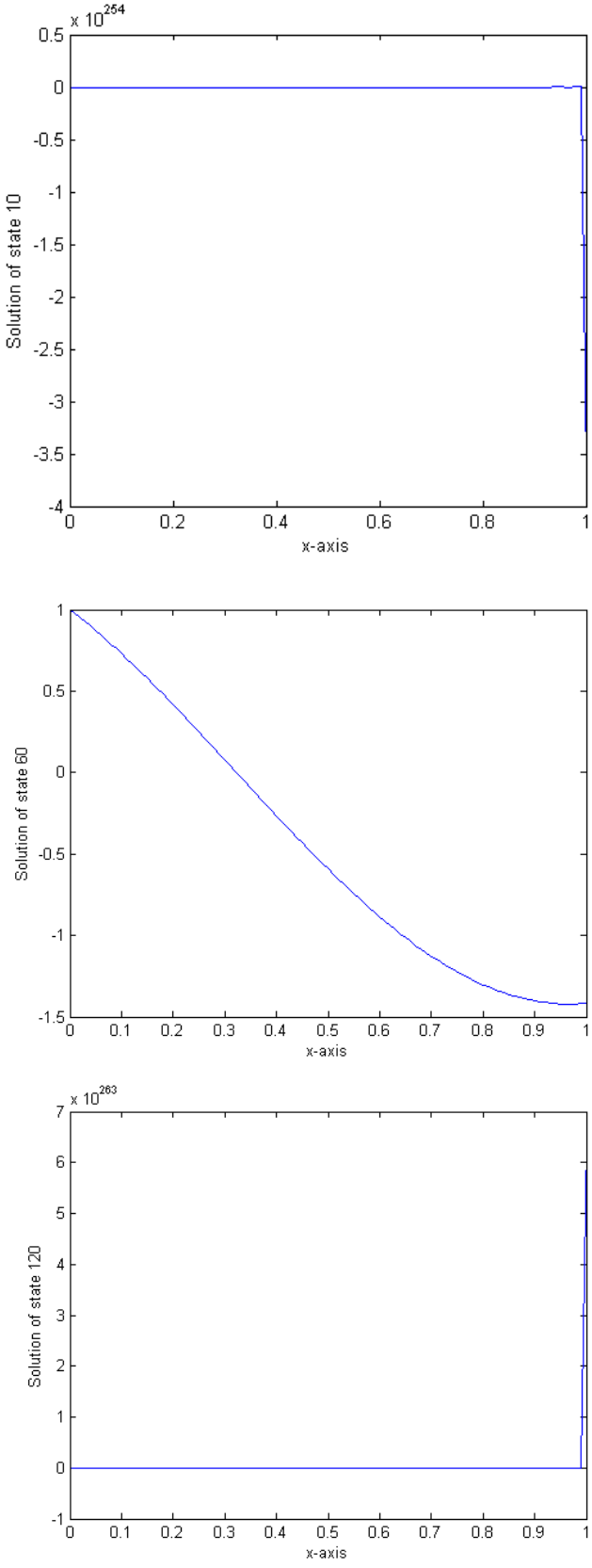



**Figure 4.11: Numerical results for CD-player (example(4.14))**

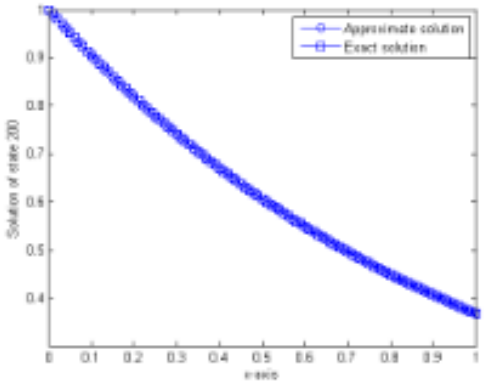



Figure 4.12: Numerical solution vs. Exact solution for state 200 (example(4.15))

# Conclusion

We study heat equation under Dirichlet, Neumann, Euler and HJB equation. We conclude that goodness of numerical solution and analytic solution where the tolerance is $10^{-3}$. The control function derived in the case of using algebraic Riccati equation found to be nonlinear function; trigonometric and hyperbolic functions depends on the initial conditions.

We recommended farther research to consider heat equation as laplacian equation to determine the control function using HJB equation. Also the heat equation could be discussed using other distribution functions for Markov chain.

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جامعة النجاح الوطنية

كلية الدراسات العليا

# استخدام البرمجة الديناميكية في السيطرة على المعادلة الحرارية في حالة عملية المشي العشوائي وذلك عن طريق معادلة هاملتون جاكوب بيلمان

إعداد

سالي محمد علي عنابسة

إشراف

د.محمد نجيب أسعد

أ.د.ديمتريوس تساجكاروجيينس

قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات المحوسبة بكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس–فلسطين

2019

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## الملخص

تم دراسة المعادلة الحرارية تحت شروط المشي العشوائي والحركة البراونية وتوزيعات برنولي, ذات الحدين, الهندسي وبواسون لسلسلة ماركوف.

لقد تم استخدام بعض الطرق العددية مثل طريقة الفرق المحدد, كرانك نيكلسون, وطرق الخطوط في إيجاد حل عددي للمعادلة الحرارية.

كما تم استخدام طريقة فصل المتغيرات لإيجاد حل تحليلي للمعادلة الحرارية. ولقد استخدمنا معادلة هاملتون جاكوب بيلمان ومعادلة ريكاتي لإيجاد اقتران السيطرة للمعادلة الحرارية.

وأخيرا تم مقارنة الحلول المختلفة مع بعضها وتحليلها.