An-Najah National University

Faculty of Graduate Studies

Topological concepts on certain fuzzy topological spaces including intuitionistic fuzzy topological spaces

By

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Supervised

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الإهداء

أما ثمرة بحثي هذا فأهديها

إلى الرجل الحصين ذو القامة العالية الذي بذل حباب عرقه وكد واجتهد ليمدني بالعون والرشاد، الذي علمني أن الأعمال الكبيرة لا تتم إلا بالصبر والعزيمة والإصرار، اسأل الله أن يمد في عمره ويجزيه عني خير الجزاء. والدي العزيز د. بسام مناصرة

إلى من نذرت عمرها في أداء رسالة صنعتها من أوراق الصبر وطرزتها في ظلام الدهر بلا فتور أو كلل، رسالة تعلم العطاء كيف يكون العطاء وتعلم الوفاء كيف يكون الوفاء إليك أمي الحبيبة اهدي هذه الرسالة وشتان بين رسالة ورسالة

إلى زهرة الحياة ونورها، البلسم الشافي خيمة الحنان وغيمة المكان تحملني دائماً بين يديها دعاء متصل للسماء، خطيبتي وزوجتي المستقبلية بإذن الله

إلى أخ لم تلده أمي بل ولدته لي الأيام، يا من كنت خير عونٍ لي في غربتي وألهمتني الثبات، صديقي ورفيق دربي الوفي المخلص أ. هشام محمد ظريفة

إلى النجوم الخفية، سندي وعوني ومصدر سعادتي أخوتي وأخواتي

إلى روح جدتي الحاجة عائشة صوالحة " أم بسام " اسأل الله أن يجعل قبرها روضة من رياض الجنة ويجمعها مع النبيين والصديقين والشهداء

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الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

Topological concepts on certain fuzzy topological spaces including intuitionistic fuzzy topological spaces

أقر بأن ما اشتملت عليه هذه الرسالة إنما هو من نتاج جهدي الخاص بإستثناء ما تمت الإشارة اليه حيثما ورد، وإن هذه الرسالة ككل، أو أي جزء منها لم يقدم لنيل أي درجة أو لقب علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى .

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's Name: **إسم الطالب:**

Signature: **التوقيع:**

Date: **التاريخ:**

Contents

Topological concepts on certain fuzzy topological spaces including intuitionistic fuzzy topological spaces

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Abstract

In this thesis the topological concepts of fuzzy topological spaces and intuitionistic fuzzy topological spaces were investigated and have been associated with their duals in classical topological spaces.

Fuzzy sets, fuzzy points, fuzzy functions and fuzzy relations were presented along with their properties. Many topological concepts and properties were proved to be true in non fuzzy setting.

Intuitionistic Fuzzy sets, Intuitionistic fuzzy points, Intuitionistic fuzzy functions and Intuitionistic fuzzy relations were presented along with their properties. Many topological concepts and properties were proved to be true in fuzzy setting.

Also different approaches of separation axioms were investigated using Q-neighborhood and fuzzy points and Intuitionistic fuzzy points, and we studied another types of separation axioms on fuzzy setting and Intuitionistic fuzzy setting.

Finally, fuzzy compactness and Intuitionistic fuzzy compactness were introduced with a theorem proved the way they are related.

Chapter One

Introduction To Fuzzy Sets

Chapter One
Introduction To Fuzzy Sets

Introduction

Fuzzy sets, in mathematics, are sets having elements with a degree of membership. This concept was first generalized by Zadeh in 1965 in his famous paper [26], where the concept of fuzzy sets was introduced.

In classical set theory, an element either belongs or doesn't belong to the set, but in fuzzy set it is different, here, the element has a degree of membership between zero and one, which leads to a new definition of characteristic function.

1.1 Fuzzy Sets

In set theory a subset A of a set **X** can be identified with the characteristic function χ_A that maps **X** to {0,1} by taken all elements in A to 1, while taken an elements in **X** – A to 0.

$$i.e \quad \chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in X - A \end{cases}$$

zadeh in [26] extended the definition of the characteristic function by replacing the set {0,1} by the closed interval [0,1] which is the basis to the new definition of fuzzy sets.

Definition 1.1.1: [26]

Let **X** be nonempty (crisp) regular set, a fuzzy subset μ_A of X is characterized by a membership function $\mu_A(x)$ that maps X to the closed interval [0,1]; $i.e$ $\mu_A : X \to [0,1]$ and the value of $\mu_A(x)$ at x representing the grade of membership of x in μ_A.

In the case of characteristic function $\chi_A : X \to \{0,1\}$, if $\chi_A(x) = 0$, then the grade of membership is 0, and this means that x doesn't belong to A, while if $\chi_A(x) = 1$, then the grade of membership is 1, and this means that x belongs to A. But in the case of fuzzy sets; $\mu_A(x)$ could be any other number from 0 to 1, the value 0 is used to represent complete nonmembership, the value 1 is used to represent complete membership, and values in between are used to represent intermediate degrees of membership.

Example 1.1.2:

Let X be nonempty regular set, μ_A is fuzzy subset of X. $\mu_A(x) = 0.95$ may mean that x is more likely to be in μ_A, $\mu_A(x) = 0.5$ then x may be half way between belonging to μ_A and not belonging to μ_A.

It's clear that fuzzy subsets of X are generalize of regular subsets of X, on the other word, regular subset of X are a special case of fuzzy sets called crisp fuzzy sets where $\mu_A(x) \in \{0,1\} \subset [0,1]$.

We can represent a fuzzy subset of X by using different ways, in the following example we describe some of these ways:

Example 1.1.3:

Consider the regular set X where $X = \{a, b, c, d\}$ and let μ_A be a fuzzy subset of X that maps X to [0,1] by mapping:

$$a \to 0.3, \quad b \to 0.8, \quad c \to 0, \quad d \to 0.5$$

We may represent μ_A as the set of order pairs :

$$\mu_A = \{(a, 0.3), (b, 0.8), (c, 0), (d, 0.5)\}$$

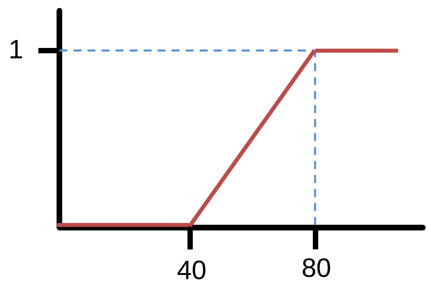
Or we may write it as: $\mu_A = \{a_{0.3}, b_{0.8}, c_0, d_{0.5}\}$.

Example 1.1.4:

Take X to be a set of people of age 25, a fuzzy subset "FAT" may be defined to be the answer of the question "to what degree a person x is fat"? the answer could come on a membership function based on a person's fat:

$$\mathrm{FAT}(x) = \begin{cases} 0 & \text{if } \ x < 40 \\ \frac{weight\ of\ x - 40}{40} & \text{if } \ 40 \le x < 80 \\ 1 & \text{if } \ x \ge 80 \end{cases}$$

Graphically:

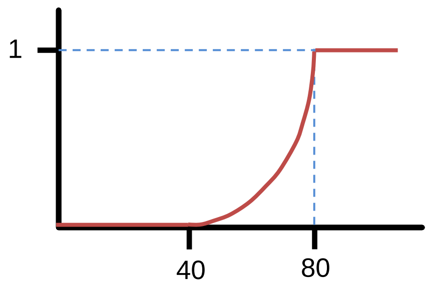


We may say that the percentage of belonging for any person with weight > 80 to being FAT is 100%, while a person with weight 70 kg fat has a percentage 75% and we write:

$$\mathrm{FAT}(70) = 0.75 \text{ or } 75\%$$

This grade of membership function is linear, but we may have the nonlinear function that reflects the importance of the fat needed. For example:

$$\mathrm{FAT}(x) = \begin{cases} 0 & \text{if } \ x < 40 \\ \frac{(weight\ of\ x-40)^2}{1600} & \text{if } \ 40 \leq x < 80 \\ 1 & \text{if } \ x \geq 80 \end{cases}$$



There are other types of fuzzy subsets, one of them is the fuzzy constant subset of X, which is the function that takes all elements of X to a constant c, where $c \in [0,1]$, and it is denoted by $\overline{c}$.

Special fuzzy constant subsets are $\overline{1}$ and $\overline{0}$.

1.2 Operations on Fuzzy sets

After these new concepts of fuzzy sets were defined, we extend the usual operations on classical sets; including the union, intersection and complementation, to fuzzy sets as follows:

Definition 1.2.1: [26]

Let μ_A and μ_B be two fuzzy subsets of X, we say:

$$\mu_A \subseteq \mu_B \quad \text{iff} \quad \mu_A(x) \leq \mu_B(x) \quad \text{for all } x \text{ in X.}$$

Definition 1.2.2: [26]

Let μ_A and μ_B be two fuzzy subsets of X, the intersection, union and complement of fuzzy subsets, denotes respectively as $\mu_A \wedge \mu_B$, $\mu_A \vee \mu_B$, ${\mu_A}^c$ are also a fuzzy subsets of X and defined as follows:

Intersection: $(\mu_A \wedge \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}$

Union: $(\mu_A \vee \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\}$

Complement: ${\mu_A}^c(x) = 1 - \mu_A(x)$

These definition are generalized to any number of fuzzy subsets of X; so, for any family $\{ \mu_{A_\alpha} : \alpha \in \Gamma \}$ of fuzzy subsets of X, where Γ is an index set, then we define:

$\left(\vee_\alpha \mu_{A_\alpha}\right)(x) = \sup\{ \mu_{A_\alpha} : \alpha \in \Gamma \}$

$\left(\wedge_\alpha \mu_{A_\alpha}\right)(x) = \inf \{ \mu_{A_\alpha} : \alpha \in \Gamma \}$

We illustrate the previous definitions by the following examples:

Example 1.2.3:

(1) Consider the regular set X where X$=\{\, a, b, c, d, e\,\}$, take the fuzzy subsets:

$$\mu_A = \{a_{0.5}, b_{0.3}, c_{0.9}, d_0, e_{0.7}\}$$

$$\mu_B = \{a_{0.1}, b_{0.6}, c_{0.9}, d_{0.4}, e_{0.5}\}$$

Then $$\mu_A \vee \mu_B = \{a_{0.5}, b_{0.6}, c_{0.9}, d_{0.4}, e_{0.7}\}$$

$$\mu_A \wedge \mu_B = \{a_{0.1}, b_{0.3}, c_{0.9}, d_0, e_{0.5}\}$$

And $${\mu_A}^c = \{a_{0.5}, b_{0.7}, c_{0.1}, d_1, e_{0.3}\}$$

(2) consider the regular set X where X$=\{\, a, b, c\,\}$, take an infinite number of fuzzy subsets:

$$\mu_{A_1} = \{\, a_{0.69}, b_{0.11}, c_{0.5}\,\}$$

$$\mu_{A_2} = \{\, a_{0.699}, b_{0.101}, c_{0.5}\,\}$$

$$\mu_{A_3} = \{\, a_{0.6999}, b_{0.1001}, c_{0.5}\,\}$$

$$\vdots$$

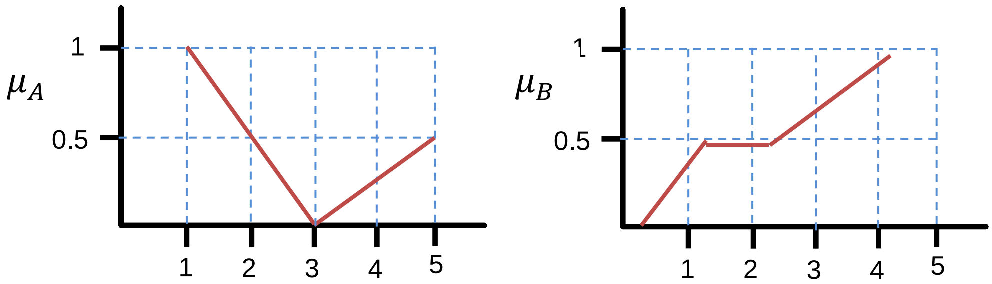
Then $\bigvee_{i=1}^{\infty} \mu_{A_i} = \{\, a_{0.7}, b_{0.11}, c_{0.5}\,\}$

And $\bigwedge_{i=1}^{\infty} \mu_{A_i} = \{\, a_{0.69}, b_{0.1}, c_{0.5}\,\}$

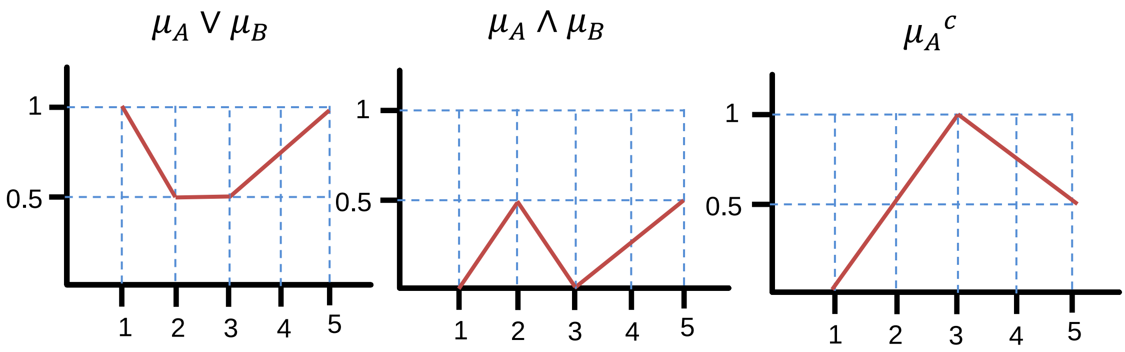
In case of continuous graph:

Example 1.2.4:

Take X = [0, 5], μ_A and μ_B are as follow :



Then



To show that this definition extends the union, intersection and complementation applied on regular subset of X, we have:

$$(\mu_{A\cap B})(x) = \min\{\, \mu_A(x), \mu_B(x)\,\}$$

If $x \in A$ and $x \in B$ then $\mu_A(x) = 1$ and $\mu_B(x) = 1$, which implies that $\min\{\, \mu_A(x), \mu_B(x)\,\} = 1$; so $(\mu_{A\cap B})(x) = 1$; i.e $x \in A \cap B$.

But; if $x \notin A$ or $x \notin B$, then $\mu_A(x) = 0$ or $\mu_B(x) = 0$, which implies that $\min\{\, \mu_A(x), \mu_B(x)\,\} = 0$; so $(\mu_{A\cap B})(x) = 0$; i.e $x \notin A \cap B$.

Which complies with the classical definition of " Intersection".

In similar manner, we may show the same for union and complementation.

The next theorem shows that we can extend Demorgan's laws from regular(crisp) sets to fuzzy sets:

Theorem 1.2.5: [26]

Let $\mu_A(x)$ and $\mu_B(x)$ be two fuzzy subsets of X, we have:

(1) $(\mu_A \vee \mu_B)^c(x) = ({\mu_A}^c \wedge {\mu_B}^c)(x)$

(2) $(\mu_A \wedge \mu_B)^c(x) = ({\mu_A}^c \vee {\mu_B}^c)(x)$

Proof: (1) $(\mu_A \vee \mu_B)^c(x) = 1 - (\mu_A \vee \mu_B)(x)$

$$= 1 - \max\{\mu_A(x), \mu_B(x)\}$$

$$= \begin{cases} 1 - \mu_A(x), & \text{if } \ \mu_A(x) \geq \mu_B(x) \\ 1 - \mu_B(x), & \text{if } \ \mu_B(x) \geq \mu_A(x) \end{cases}$$

$$= \begin{cases} 1 - \mu_A(x), & \text{if } \ 1 - \mu_A(x) \leq 1 - \mu_B(x) \\ 1 - \mu_B(x), & \text{if } \ 1 - \mu_B(x) \leq 1 - \mu_A(x) \end{cases}$$

$$= \min\{1 - \mu_A(x),\ 1 - \mu_B(x)\}$$

$$= \min\{{\mu_A}^c(x),\ {\mu_B}^c(x)\}$$

$$= ({\mu_A}^c \wedge {\mu_B}^c)(x)$$

(2) $(\mu_A \wedge \mu_B)^c(x) = 1 - (\mu_A \wedge \mu_B)(x)$

$$= 1 - \min\{\mu_A(x), \mu_B(x)\}$$

$$= \begin{cases} 1 - \mu_A(x), & \text{if } \ \mu_A(x) \leq \mu_B(x) \\ 1 - \mu_B(x), & \text{if } \ \mu_B(x) \leq \mu_A(x) \end{cases}$$

$$= \begin{cases} 1-\mu_A(x), & \text{if } 1-\mu_A(x) \ge 1-\mu_B(x) \\ 1-\mu_B(x), & \text{if } 1-\mu_B(x) \ge 1-\mu_A(x) \end{cases}$$

$$= \max\{1-\mu_A(x),\ 1-\mu_B(x)\}$$

$$= \max\{\ {\mu_A}^c(x),\ \ {\mu_B}^c(x)\}$$

$$= ({\mu_A}^c \vee {\mu_B}^c)(x)$$

This theorem can be generalized to any family of fuzzy subsets of X, specifically:

$$\left(\vee_\alpha \mu_{A_\alpha}\right)^c = (\wedge_\alpha {\mu_{A_\alpha}}^c) \text{ and } \left(\wedge_\alpha \mu_{A_\alpha}\right)^c = (\vee_\alpha {\mu_{A_\alpha}}^c).$$

Notion of α – level is one of the basic notions of fuzzy sets, defining in the following definition:

Definition 1.2.6: [24]

The α – level set of a fuzzy subset μ_A denoted by ${\mu_A}^\alpha$ is a non-fuzzy subset of X, such that the grade of membership of its elements ≥ α, where α > 0, that is:

$${\mu_A}^\alpha = \{x \in X :\ \mu_A(x) \ge \alpha\} \quad \text{where} \quad \alpha > 0.$$

Also, we define 0 – level in case of X is the real line by:

$${\mu_A}^0 = \text{the closure of}\,(\,\{x \in X :\ \mu_A(x) > 0\}\,) \text{ in } \mathbb{R}^1.$$

The support of μ_A, denoted by $supp(\mu_A)$, is a crisp subset of X whose elements all have nonzero membership grades in μ_A.

$$i.e \ \ supp(\mu_A) = \{x \in X : \ \mu_A(x) > 0\}$$

It is obvious that $supp(\mu_A) = \emptyset$ iff $\mu_A = 0$ $i.e$ $\mu_A(x) = 0$ for all $x \in X$.

We say that a fuzzy subset μ_A of X, where X is infinite, is countable whenever $supp(\mu_A)$ is countable.

How we found the α – level ? the answer of this question in the next example:

Example 1.2.7:

(1) In discrete case:

Consider a regular set X, where X={ a, b, c, d }, let $\mu_A = \{a_{0.3}, b_{0.7}, c_{0.8}, d_{0.4}\}$ be a fuzzy subset of X, then

The 0.4 – level =$\mu_A{}^{0.4} = \{ \ b, c, d \ \}$

The 0.8 – level =$\mu_A{}^{0.2} = \{ \ c \ \}$

The 0.9 – level =$\mu_A{}^{0.9} = \emptyset$

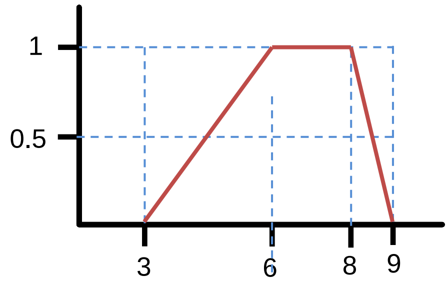
And the $supp(\mu_A) = \{ \ a, b, c, d \ \}$ = X.

(2) In continuous case:

Consider X = $\mathbb{R}^1 = (-\infty, \infty)$, and the fuzzy subset of X given as

$$\mu_A(x) = \begin{cases} \dfrac{x-3}{3} & \text{if} \quad x \in [3,6] \\ 1 & \text{if} \quad x \in [6,8] \\ 9-x & \text{if} \quad x \in [8,9] \\ 0 & \text{elsewhere} \end{cases}$$

Graphically:



The 0.3 – level at this fuzzy subset is $\mu_A{}^{0.3} = \{x \in X : \mu_A(x) \geq 0.3\}$

$$\frac{x-3}{3} \geq 0.3 \quad \Longrightarrow x \geq 3.9$$

$$9 - x \geq 0.3 \quad \Longrightarrow x \leq 8.7$$

Hence; $\mu_A{}^{0.3} = [\, 3.9,\ 8.7 \,]$.

In general, (with respect to the previous example); the α – level can be found as follows: $\mu_A{}^{\alpha} = [\, x_1^{\alpha}, x_2^{\alpha} \,]$

$$\alpha = \frac{x_1^{\alpha}-3}{3} \Longrightarrow x_1^{\alpha} = 3\alpha + 3$$

and $\quad \alpha = 9 - x_2^{\alpha} \Longrightarrow x_2^{\alpha} = 9 - \alpha$

So, $\mu_A{}^{\alpha} = [\, 3\alpha + 3 \,, 9 - \alpha \,]$.

1.3 Convex Fuzzy Sets

The convex of fuzzy sets was introduced by Zadeh in his famous paper, we assume for concreteness that X is a real Euclidean space $\mathbb{R}^n$.

Definition 1.3.1: [12]

Let $\mu_A : \mathbb{R}^n \rightarrow [\, 0, 1]$

(1) A fuzzy set μ_A is convex if and only if

$$\mu_A(tx_1 + (1-t)x_2) \geq \min\{\, \mu_A(x_1),\ \mu_A(x_2) \,\}$$

For $x_1, x_2 \in \mathbb{R}^n$, $t \in [\,0,1]$

(2) A fuzzy set μ_A is strongly convex if and only if

$$\mu_A(tx_1 + (1-t)x_2) > \min\{\, \mu_A(x_1),\ \mu_A(x_2) \,\}$$

For $x_1, x_2 \in \mathbb{R}^n$, $t \in (\,0,1)$.

Note that any strongly convex fuzzy set is convex, but the converse is not.

A basic property of convex fuzzy sets is expressed by the following theorem:

Theorem 1.3.2: [12]

If μ_A and μ_B are two convex fuzzy sets, then their intersection also a convex fuzzy set.

Proof: Let $\mu_C = \mu_A \wedge \mu_B$

$$\mu_C(tx_1 + (1-t)x_2) = \min\{\, \mu_A(tx_1 + (1-t)x_2),$$

$$\mu_B(tx_1 + (1-t)x_2) \,\}$$

Now; since μ_A and μ_B are convex

$$\mu_A(tx_1 + (1-t)x_2) \geq \min\{\, \mu_A(x_1),\ \mu_A(x_2) \,\}$$

$$\mu_B(tx_1 + (1-t)x_2) \geq \min\{\, \mu_B(x_1),\ \mu_B(x_2) \,\}$$

And hence

$\mu_C(tx_1 + (1-t)x_2) \geq$ min { min { $\mu_A(x_1),\ \mu_A(x_2)$ },

min { $\mu_B(x_1),\ \mu_B(x_2)$ }}

= min {min { $\mu_A(x_1),\ \mu_B(x_1)$ },min{ $\mu_A(x_2),\ \mu_B(x_2)$}}

= min { $\mu_C(x_1),\ \mu_C(x_2)$ }

Thus $\mu_C(tx_1 + (1-t)x_2) \geq$ min { $\mu_C(x_1),\ \mu_C(x_2)$ }

Therefor; μ_C is convex.

Theorem 1.3.3: [12]

(a) If μ_A is convex fuzzy set, then support of μ_A ($supp(\mu_A)$) is a convex set.

(b) If μ_A is strongly convex fuzzy set, then $supp(\mu_A) = \mathbb{R}^n$.

Proof: (a) It is implied directly by definition of convex fuzzy set and definition of support of μ_A.

(b) If μ_A is strongly convex fuzzy set, then from definition of strongly fuzzy set, for any $x_1,\ x_2 \in \mathbb{R}^n,\ x_2 \neq 0$ and $t = \frac{1}{2}$, we obtain:

$$\mu_A(x_1) = \mu_A\left(\frac{1}{2}(x_1 + x_2) + \frac{1}{2}(x_1 - x_2)\right)$$

$$> \min\{\mu_A(x_1 + x_2),\ \mu_A(x_1 - x_2)\} \geq 0$$

$i.e\ \ x \in supp(\mu_A)$

Therefor; $supp(\mu_A) = \mathbb{R}^n$. Q.E.D

Remark: From now we will replace a notation of fuzzy subset μ_A by $\mathcal{A}$ for more simply.

1.4 Fuzzy Points

There are many types of fuzzy subsets, one of them is a fuzzy point, it's defined by Wong [24], and later on, other definitions were presented by Srivatava [22], Ming and Liu [19].

Definition 1.4.1: [24]

Let X be a regular set, let x be a fixed element of X, a fuzzy point $\wp$ is a fuzzy subset of X with the membership function :

$$\wp(x) = \begin{cases} \lambda & \text{if} \quad x = a \quad \text{where } \lambda \in (0,1] \\ 0 & \text{elsewhere} \end{cases}$$

On the other word; a fuzzy point $\wp$ is a fuzzy subset of X that take an element α to a number $\lambda \in (0,1]$ and the remaining elements to 0, and it will be denoted by $\wp = a_\lambda$.

It is clear that $supp(\wp) = a$, $\wp(a) = \lambda$ and $\wp(X - \{a\}) = 0$.

For example: If X $= \{a, b, c\}$, then the point $b_{0.7}$ is a fuzzy subset $\{a_0, b_{0.7}, c_0\}$.

Definition 1.4.2: [24]

Let $\mathcal{p} = a_\lambda$ be a fuzzy point and $\mathcal{A}$ a fuzzy subset of X, then we may say $\mathcal{p}$ in $\mathcal{A}$ or $\mathcal{A}$ contains $\mathcal{p}$ denoted $\mathcal{p} \in \mathcal{A}$ if and only if $\lambda \leq \mathcal{A}(a)$.

$i.e$ $a_\lambda \in \mathcal{A} \iff \lambda \leq \mathcal{A}(a)$.

For example: if X = $\{ a, b, c \}$ and $\mathcal{A} = \{a_{0.5}, b_{0.3}, c_{0.9}\}$ then $a_{0.3} \in \mathcal{A}$ but $b_{0.6} \notin \mathcal{A}$.

Definition 1.4.3: [24]

(1) A fuzzy point $\mathcal{p} = a_\lambda$ in X is quasi-coincident with the fuzzy subset $\mathcal{A}$ of X, denoted by $\mathcal{p}\, \boldsymbol{Q}\, \mathcal{A}$ if and only if $\lambda + \mathcal{A}(a) > 1$ and it's clear that $a_\lambda\, \boldsymbol{Q}\, \mathcal{A} \iff a_\lambda \notin \mathcal{A}^c$.

For example: Let X = $\{ a, b, c, d \}$ and $\mathcal{A} = \{a_{0.1}, b_{0.3}, c_{0.6}, d_{0.9}\}$, then $c_{0.6}\, \boldsymbol{Q}\, \mathcal{A}$ since $0.5 \nleq 0.4$

(2) A fuzzy subset $\mathcal{A}$ of X is quasi-coincident with the fuzzy subset $\mathcal{B}$ of X if and only if $\exists x \in X$ such that $\mathcal{A}(x) + \mathcal{B}(x) > 1$, and we write $\mathcal{A}\, \boldsymbol{Q}\, \mathcal{B}$.

Note that:

(i) $\mathcal{A}\, \boldsymbol{Q}\, \mathcal{B} \iff \mathcal{B}\, \boldsymbol{Q}\, \mathcal{A}$.

(ii) $\mathcal{A}\, \boldsymbol{Q}\, \mathcal{B} \implies \mathcal{A} \cap \mathcal{B} \neq \emptyset$.

(iii) $\mathcal{p}\, \boldsymbol{Q}\, \mathcal{A}$ and $\mathcal{A} \subseteq \mathcal{B}$ ($i.e$ $\mathcal{A}(x) \leq \mathcal{B}(x)$, for each x in X, then $\mathcal{p}\, \boldsymbol{Q}\, \mathcal{B}$.

(iv) $\mathcal{A} \subseteq \mathcal{B} \iff \mathcal{A}$ and $\mathcal{B}^c$ are not quasi-coincident.

Proposition 1.4.4

We can write any fuzzy set A of X as the union of all fuzzy points in A, i.e $A = \bigcup_{p \in A} p$.

1.5 Fuzzy Membership:

In 1974, C.K Wong [24] define the concept of belonging of fuzzy point to a fuzzy set. Later on, different definitions of the same concept were added by M.Sarker [21] and Wong [24]. These definition were given independently. At the first look the definitions seem to be the same but after investigation they are found to be different in many aspects.

Proposition 1.5.1:

Let $\wp = x_\lambda$ be a fuzzy point in X and let A be any fuzzy set in X, then we can write A as a union of its fuzzy points.

i.e; $A = \bigvee\{(x, \lambda) : 0 < \lambda \leq A(x)$ where $x \in$ supp$(A)\}$

or we may write $A = \bigvee \wp : \wp \leq A$.

In the following definition we classify the different definitions of the relation “belonging” or $\in$.

Definition 1.5.2:

Let A be a fuzzy subset of a nonempty set X and let $\wp = x_\lambda$ be a fuzzy point in X, we define the relation $\in$ as follow:

1) $x_\lambda \in_1 A$ if and only if $\lambda < A(x)$

2) $x_\lambda \in_2 A$ if and only if $\lambda \leq A(x)$

3) $x_\lambda \in_3 A$ if and only if $\lambda = A(x)$

Remarks 1.5.3:

1) $x_\lambda \in A \vee B$ if and only if $x_\lambda \in A$ or $x_\lambda \in B$, which is true for all definitions of " $\in$ ".

For $\in_1$: $x_\lambda \in_1 A \vee B$ means $\lambda <$ max$\{ A(x), B(x) \}$ so $\lambda < A(x)$ or $\lambda < B(x) \Leftrightarrow x_\lambda \in_1 A$ or $x_\lambda \in_1 B$.

The same will be true if we replace $\in_1$ by $\in_2$ and $\in_3$.

2) $x_\lambda \in A \wedge B$ if and only if $x_\lambda \in A$ and $x_\lambda \in B$, which is true for all definitions of " $\in$ ".

For $\in_1$: $x_\lambda \in_1 A \wedge B$ means $\lambda <$ min$\{ A(x), B(x) \}$ so $\lambda < A(x)$ and $\lambda < B(x) \Leftrightarrow x_\lambda \in_1 A$ and $x_\lambda \in_1 B$.

The same will be true if we replace $\in_1$ by $\in_2$ and $\in_3$.

We can extend the previous remark to any finite number of fuzzy sets $A_1, A_2, \dots, A_n$.

In the case of arbitrary families of fuzzy sets $\{ A_\alpha : \alpha \in \Gamma \}$, there is a different between definitions of " $\in$ ", we will explain this different in the following:

Lemma 1.5.4:

Let $\{ A_\alpha : \alpha \in \Gamma \}$ be a family of fuzzy subset of X, then $x_\lambda \in A_\alpha$ for some α has three cases about union:

1) $x_\lambda \in_1 A_\alpha$ for some $\alpha \iff x_\lambda \in_1 \bigvee A_\alpha$.

2) $x_\lambda \in_2 A_\alpha$ for some $\alpha \implies x_\lambda \in_1 \bigvee A_\alpha$.

3) The statement $x_\lambda \in_3 A_\alpha$ for some $\alpha \Leftrightarrow x_\lambda \in_3 \bigvee A_\alpha$ is false.

Proof: (1) let $x_\lambda \in_1 A_\alpha$ for some α then $\lambda < A_\alpha(x)$ for some α, so $\lambda < \sup\{A_\alpha(x): \alpha \in \Gamma\}$, then $\lambda < (\bigvee A_\alpha)(x)$.

Therefor $x_\lambda \in_1 \bigvee A_\alpha$.

Conversely, let $x_\lambda \in_1 \bigvee A_\alpha$ then $\lambda < \sup\{A_\alpha(x): \alpha \in \Gamma\}$, that is there exist at least one $A_\alpha(x)$ say $A_{\alpha_0}(x)$ s.t $\lambda < A_{\alpha_0}(x) < \sup\{A_\alpha(x): \alpha \in \Gamma\}$

Therefor $x_\lambda \in_1 A_\alpha$ for some α.

(2) straightforward.

The converse of (2) may not be true, the following examples show that:

Example [1] : let $X = \{a, b\}$

and $A_1 = \{a_{0.5}, b_{0.69}\}$

$$A_2 = \{a_{0.5}, b_{0.699}\}$$

$$\vdots$$

Then $\bigvee_i A_i = \{a_{0.5}, b_{0.7}\}$

Now; $b_{0.7} \in_2 \bigvee_i A_i$ but $b_{0.7} \notin_2 A_i$ for each $i.$

Example [2] : let $X = \{\, m \,\}$ and $A_i = \left\{m_{\frac{i}{i+1}}\right\} : i = 1,2, \ldots$

Then $\bigvee_i A_i = \{\, m_1 \,\}$ and $m_1 \in_2 \bigvee_i A_i$ but $m_1 \notin_2 A_i$ for each $i.$

(3) To show the statement is not true, we give a counter example:

Example: let $X = \{a, b\}$

and $A_1 = \{a_{0.5}, b_{0.69}\}$

$$A_2 = \{a_{0.5}, b_{0.699}\}$$

$$\vdots$$

Then $\bigvee_i A_i = \{a_{0.5}, b_{0.7}\}$

We observe that $b_{0.7} \in_3 \bigvee_i A_i$ but $b_{0.7} \notin_3 A_i$ for each i and $b_{0.69} \in_3 A_1$ but $b_{0.69} \notin_3 \bigvee_i A_i$.

Lemma 1.5.5: [25]

Let $\{\, A_\alpha : \alpha \in \Gamma \,\}$ be a family of fuzzy subset of X, then $x_\lambda \in \bigwedge A_\alpha$ has three cases:

1) $x_\lambda \in_1 \bigwedge A_\alpha \implies x_\lambda \in_1 A_\alpha$ for all α.

2) $x_\lambda \in_2 \wedge A_\alpha \iff x_\lambda \in_2 A_\alpha$ for all α.

3) The statement $x_\lambda \in_3 \wedge A_\alpha \Longrightarrow x_\lambda \in_3 A_\alpha$ for all α is not true.

1.6 Functions on Fuzzy Sets

The concept of fuzzy function was defined between two families of fuzzy subsets corresponding to a function between two regular sets.

Remark: we will use F(X) to be the family of all fuzzy subsets of X.

Definition 1.6.1: [25]

Let X and Y be two regular sets, and let f be any function from X into Y ($i, e\ f: X \longrightarrow Y$), for any fuzzy subset $\mathcal{A}$ of X and for any fuzzy subset $\mathcal{B}$ of Y, we define f^* to be a fuzzy function between F(X) and F(Y) that takes a fuzzy subset of X to a fuzzy subset of Y, by :

$f^*: F(X) \longrightarrow F(Y)$

$$f^*(\mathcal{A})(y) = \begin{cases} \sup\{\mathcal{A}(x) : x \in f^{-1}(y)\} & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{if} \quad f^{-1}(y) = \emptyset \end{cases}$$

So; if $\mathcal{A}$ is a fuzzy subset of X, then $f^*(\mathcal{A})$ is a fuzzy subset of Y.

Also we define the fuzzy function $(f^*)^{-1}$ as $(f^*)^{-1}(\mathcal{B})$ by :

$$(f^*)^{-1}(\mathcal{B})(x) = \mathcal{B}(\,f(x))$$

So; if $\mathcal{B}$ is a fuzzy subset of Y, then $(f^*)^{-1}(\mathcal{B})$ is a fuzzy subset of X.

Example 1.6.2:

Consider X = $\{ a, b, c, d, e \}$ and Y = $\{ t, u, v, w \}$ and $f: X \longrightarrow Y$ by

$a \to t, \quad b, c \to u, \quad d, e \to v$;

Let $\mathcal{A}$ be the fuzzy subset of X such that $\mathcal{A} = \{a_{0.7}, b_{0.4}, c_{0.5}, d_0, e_{0.8}\}$

Then $f^*(\mathcal{A})$ is the fuzzy subset of Y defined as :

$$f^*(\mathcal{A}) : Y \longrightarrow [0,1]$$

$t \to 0.7, \quad u \to \max\{\mathcal{A}(b), \mathcal{A}(c)\} = \max\{0.4, 0.5\} = 0.5, \quad v \to 0.8$

and $w \to 0$, since $f^{-1}(w) = \emptyset$.

Example 1.6.3:

Consider X = $\{ a, b, c, d, e \}$ and Y = $\{ t, u, v, w \}$ and $f: X \longrightarrow Y$ by

$a \to t, \quad b, c \to u, \quad d, e \to v$;

Let $\mathcal{B}$ be the fuzzy subset of Y such that $\mathcal{B} = \{t_{0.5}, u_{0.4}, v_0, w_{0.8}\}$

Then $(f^*)^{-1}(\mathcal{B})$ is the fuzzy subset of X defined as :

$$(f^*)^{-1}(\mathcal{B}) : X \to [0,1]$$

$a \to 0.5, \quad b \to 0.4, \quad c \to 0.4, \quad d \to 0, \quad e \to 0.$

We know that the image of fuzzy subset of X is a fuzzy subset of Y, and the invers image of a fuzzy subset of Y is also a fuzzy subset in X;(by definition).

In special case of fuzzy subset, fuzzy point, but the question is " Is this notion true for fuzzy point" ? that is, if $\wp$ is a fuzzy pint in X, is the image of $\wp$ a fuzzy point in Y ? and if q is a fuzzy point in Y, is the invers image of q a fuzzy point in X?

The answer of these questions are given in the following theorem:

Theorem 1.6.4: [25]

(1) If $\wp = a_\lambda$ is a fuzzy point in X, then $f^*(\wp)$ is a fuzzy point in Y, call it q, where $f^*(\wp) = f(a)_\lambda = q$. Such that $f(a)$ is the $supp(q)$ and λ is the value of q.

Proof: If $f^{-1}(y) = \phi, then\ q(y) = 0$.

If $f^{-1}(y) \neq \phi$, then $q(y) = \sup\{\wp(x) : x \in f^{-1}(y)\}$ and we have two cases:

Case (1) : If $a \in f^{-1}(y)$, then

$$q(y) = \sup\{\wp(x) : x \in f^{-1}(f(a))\} = \{\lambda, 0, 0, ...\} = \lambda$$

Case (1) : If $a \notin f^{-1}(y)$, then

$$q(y) = \sup\{0, 0, \dots\}.$$

Therefore; $f^*(\wp) = f(a)_\lambda = q$ is a fuzzy point in Y.

(2) If $q = b_r$ is a fuzzy point in Y, then $(f^*)^{-1}(q)$ may not be a fuzzy point in X.

The following examples explain this result:

Example (1): If $f^{-1}(b) = \emptyset$, then $(f^*)^{-1}(q) = \emptyset$ which is not a fuzzy point.

Example (2): If $f^{-1}(b)$ has at least two elements (not singleton), say $f^{-1}(b) = \{ l, m\}$, then $(f^*)^{-1}(q) = \{l_r, m_r, 0, 0, \ldots\}$ which is not a fuzzy point.

According to the previous two examples, the only case that imply "if $q = b_r$ is a fuzzy point in Y, then $(f^*)^{-1}(q)$ is a fuzzy point in X" is if $f^{-1}(b)$ is singleton.

$i.e$ if $f^{-1}(b)$ is singleton, then $(f^*)^{-1}(q)$ is a fuzzy point in X.

The following theorem shows what the fuzzy functions do on the quasi-coincident relation between fuzzy sets:

Theorem 1.6.5: [25]

Let $f: X \longrightarrow Y$ be a function, and $f^*: F(X) \longrightarrow F(Y)$ be a fuzzy function, then for any fuzzy point $p = a_\lambda$ in X and for any fuzzy subset $\mathcal{A}$ of X, we have : If $p\, \boldsymbol{Q}\, \mathcal{A}$ then $f^*(p)\ \boldsymbol{Q}\ f^*(\mathcal{A})$.

Proof: we have $p = a_\lambda$ this implies that $f^*(p) = f(a)_\lambda$

Now, we want to show that $f^*(p)\, \boldsymbol{Q}\ f^*(\mathcal{A})$, $i.e\ \lambda + f^*(\mathcal{A})(f(a)) > 1$

$$\lambda + f^*(\mathcal{A})\big(f(a)\big) = \lambda + \sup\{\mathcal{A}(x) : x \in f^{-1}\big(f(a)\big)$$

$$\geq \lambda + \mathcal{A}(a)$$

$$> 1 \qquad \text{since} \quad p\, \boldsymbol{Q}\, \mathcal{A}$$

The next theorem generalize the previous theorem for any fuzzy subsets:

Theorem 1.6.6: [25]

Let $f: X \longrightarrow Y$ be a function, and $f^*: F(X) \longrightarrow F(Y)$ be a fuzzy function, then for any fuzzy subsets $\mathcal{A}$ and $\mathcal{B}$ of X, we have:

If $\mathcal{A}\ \boldsymbol{Q}\ \mathcal{B}$ then $f^*(\mathcal{A})\ \boldsymbol{Q}\ f^*(\mathcal{B})$

Proof: Since $\mathcal{A}\,\boldsymbol{Q}\,\mathcal{B}$; let $m \in X$ be a fixed element such that $\mathcal{A}(m) + \mathcal{B}(m) > 1$

Consider $f^*(\mathcal{A})\big(f(m)\big) + f^*(\mathcal{B})\big(f(m)\big) = \sup\{\mathcal{A}(x) : x \in f^{-1}\big(f(m)\big)\}$ $+ \sup\{\mathcal{B}(x) : x \in f^{-1}\big(f(m)\big)\} \geq \mathcal{A}(m) + \mathcal{B}(m) > 1$.

Theorem 1.6.7: [25]

Under the previous assumption

Let $q = b_r$ be a fuzzy point in Y such that $f^{-1}(b) = \{a\}$ and $\mathcal{B}$ a fuzzy subset of Y, then we have: if $q\ \boldsymbol{Q}\ \mathcal{B}$ then $(f^*)^{-1}(q)\ \boldsymbol{Q}\ (f^*)^{-1}(\mathcal{B})$

Proof: $r + (f^*)^{-1}(\mathcal{B})(a) = r + \mathcal{B}\big(f(a)\big) = r + \mathcal{B}(b)$

But $q\ \boldsymbol{Q}\ \mathcal{B}$ $i.e\ r + \mathcal{B}(b) > 1$

That is, $(f^*)^{-1}(q)\ \boldsymbol{Q}\ (f^*)^{-1}(\mathcal{B})$.

Chapter Two

Intuitionistic Fuzzy Sets

Chapter Two
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2.1 Intuitionistic Fuzzy Sets

Definition 2.1.1: [2]

Let X be a nonempty set, we define an intuitionistic fuzzy set (IFS for short) as an object having the form $\mathcal{A} = \langle A_1, A_2 \rangle$ where A_1 andA_2 are fuzzy subsets of X such that $0 \leq A_1(x) + A_2(x) \leq 1$ for all $x \in$ X.

$A_1(x)$ the degree of membership of x in $\mathcal{A}$.

$A_2(x)$ the degree of nonmembership of x in $\mathcal{A}$.

Note that, the ordinary fuzzy set A is special case of IFS, that can be written as $\langle A, A^c \rangle$, $i.e$ if A_2 is the complement of A_1 then the IFS become a fuzzy set.

Remark: The IFS $\tilde{0} = \langle \overline{0}, \overline{1} \rangle$ is the empty IFS, and $\tilde{1} = \langle \overline{1}, \overline{0} \rangle$ is the whole IFS.

Example 2.1.2:

Consider the regular set X, where X = $\{a, b, c, d\}$

And let $A_1 = \{a_{0.3}, b_{0.4}, c_{0.8}, d_{0.5}\}$ and $A_2 = \{a_{0.7}, b_{0.6}, c_{0.2}, d_{0.5}\}$

Then $\mathcal{A} = \langle A_1, A_2 \rangle$ is IFS and as well is a fuzzy subset of X, since $A_2 = {A_1}^c$.

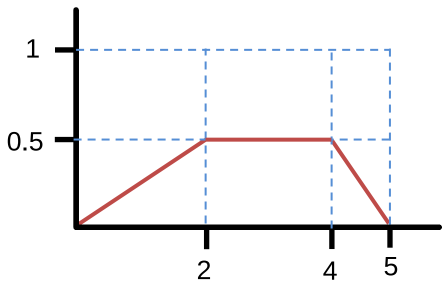
Example 2.1.3:

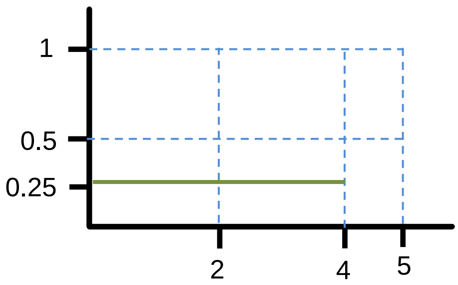
Let X = $\mathbb{R}^1 = (-\infty, \infty)$

Let $A_1(x) = \begin{cases} \frac{x}{4} & \text{if } \; 0 \le x \le 2 \\ \frac{1}{2} & \text{if } \; 2 \le x \le 4 \\ \frac{5-x}{2} & \text{if } \; 4 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$ be the degree of membership of x in $\mathcal{A}$.

And $A_2(x) = \begin{cases} \frac{1}{4} & \text{if } \; 0 \le x \le 4 \\ 0 & elsewhere \end{cases}$ be the degree of nonmembership of x in $\mathcal{A}$.

Then $\mathcal{A} = \langle A_1, A_2 \rangle$ is IFS of X and graphically as :





2.2 Some Basic Operations on IFS's

Definition 2.2.1: [4, 13]

Let $\mathcal{A} = \langle A_1, A_2 \rangle$ and $\mathcal{B} = \langle B_1, B_2 \rangle$ are two intuitionistic fuzzy subsets of X, we define inclusion, complement, union and intersection of them as:

(1) $\mathcal{A} \subseteq \mathcal{B} \iff A_1(x) \le B_1(x)$ and $A_2(x) \ge B_2(x)$; $\forall x \in$ X.

(2) $\mathcal{A}^c = \langle A_2, A_1 \rangle$

(3) $\mathcal{A} \cup \mathcal{B} = \langle\, A_1 \vee B_1,\ A_2 \wedge B_2 \,\rangle$

(4) $\mathcal{A} \cap \mathcal{B} = \langle\, A_1 \wedge B_1,\ A_2 \vee B_2 \,\rangle$

definitions (3) and (4) could be extended to any family of IFS's.

$i.e$ If $\mathcal{A}_i = \langle\, A_{i_1},\ A_{i_2} \rangle$ where $i \in \Delta$, then $\bigcup_i A_i = \langle\, \bigvee_i A_{i_1},\ \bigwedge_i A_{i_2} \rangle$ and $\bigcap_i A_i = \langle\, \bigwedge_i A_{i_1},\ \bigvee_i A_{i_2} \rangle$.

We illustrate the previous definition by the following example:

Example 2.2.2:

Consider the regular set X, where X = $\{\ a, b, c\ \}$, and let

$\mathcal{A} = \langle A_1, A_2 \rangle$ be IFS where $A_1 = \{a_{0.3}, b_{0.6}, c_{0.6}\ \}$, $A_2 = \{\ a_{0.5}, b_{0.3}, c_{0.4}\ \}$

And $\mathcal{B} = \langle B_1, B_2 \rangle$ where $B_1 = \{a_{0.6}, b_{0.7}, c_{0.8}\ \}$, $B_2 = \{a_{0.1}, b_{0.2}, c_{0.1}\ \}$

Since $A_1(x) \leq B_1(x)$ and $A_2(x) \geq B_2(x)$; $\forall x \in$ X then $\mathcal{A} \subseteq \mathcal{B}$.

$$\mathcal{A} \cup \mathcal{B} = \langle\, A_1 \vee B_1,\ A_2 \wedge B_2 \,\rangle = \langle\, B_1,\ B_2 \,\rangle$$

And $\mathcal{A} \cap \mathcal{B} = \langle\, A_1 \wedge B_1,\ A_2 \vee B_2 \,\rangle = \langle\, A_1,\ A_2 \,\rangle$

From this example we can see that since $\mathcal{A} \subseteq \mathcal{B}$; $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$, as in classical set theory.

The next theorem shows that we can extend Demorgan's laws from regular sets to IFSs:

Theorem 2.2.3: [3]

Let $\mathcal{A}$ and $\mathcal{B}$ be two IFSs of X, we have:

(1) $(\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c \cup \mathcal{B}^c$

(2) $(\mathcal{A} \cup \mathcal{B})^c = \mathcal{A}^c \cap \mathcal{B}^c$

Proof is clear.

Note: It can be generalize to any number of IFSs.

Here are the basic properties of inclusion and complementation:

Corollary 2.2.4: [3]

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{A}_i$ be IFSs in X, where $i \in \Gamma$, then:

1) $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{C} \implies \mathcal{A} \subseteq \mathcal{C}$

2) $\mathcal{A}_i \subseteq \mathcal{B}$ for each $i \in \Gamma \implies \bigcup_i \mathcal{A}_i \subseteq \mathcal{B}$

3) $\mathcal{B} \subseteq \mathcal{A}_i$ for each $i \in \Gamma \implies \mathcal{B} \subseteq \bigcap_i \mathcal{A}_i$

4) $\mathcal{A} \subseteq \mathcal{B} \iff \mathcal{B}^c \subseteq \mathcal{A}^c$

5) $(\tilde{0})^c = \tilde{1}$ and $(\tilde{1})^c = \tilde{0}$

2.3 Intuitionistic Fuzzy Points

This definition deals with a natural generalization of fuzzy point given by Pu-Liu [19]; that what we call "Intuitionistic Fuzzy Point".

Definition 2.3.1: [8]

Let X be a nonempty set and let $c \in$ X a fixed element in X. If $\alpha \in (0,1]$ and $\beta \in [0,1)$ such that $\alpha + \beta \leq 1$, then the intuitionistic fuzzy set $\tilde{\mathcal{p}} = c_{(\alpha,\beta)} = \langle c_\alpha, {c^c}_{1-\beta} \rangle$ is called an intuitionistic fuzzy point (IFP for short) in X, where α denotes the degree of membership of c in $\tilde{\mathcal{p}}$ and β the degree of nonmembership of c in $\tilde{\mathcal{p}}$.

Other definition of IFP:

Definition 2.3.2: [10]

Let X be a nonempty set, and let $\alpha, \beta \in [0,1]$ with $0 < \alpha + \beta \leq 1$. An intuitionistic fuzzy point $\tilde{\mathcal{p}}$ written as $c_{(\alpha,\beta)}$ is defined to be an intuitionistic fuzzy subset of X, given by:

$$\tilde{\mathcal{p}}(x) = c_{(\alpha,\beta)}(x) = \begin{cases} (\alpha,\beta) & \text{if} \quad x = c \\ (0,1) & \text{otherwise} \end{cases}$$

IFPs in X can sometimes be inconvenient when we express an IFS in X in terms of IFPs. This situation will occur if an IFS $\mathcal{A}$ contains some points $x \in$ X such that $A_1(x) = 0$ and $A_2(x) \in [0,1)$. Therefor we shall define "Vanishing IFPs" as follows:

Definition 2.3.3: [10]

Let X be a nonempty set and $c \in$ X a fixed element in X, if $\beta \in [0,1)$ then the IFS $\tilde{\tilde{p}} = c_{(\beta)} = \langle \overline{0}, c^c{}_{1-\beta} \rangle$ is called vanishing intuitionistic fuzzy point (VIFP for short) in X.

The following definition present some types of inclusion of an IFPs to an IFSs :

Definition 2.3.4: [8]

(1) let $\tilde{p} = c_{(\alpha,\beta)}$ be an IFP in X, and $\mathcal{A} = \langle A_1, A_2 \rangle$ be an IFS in X, we may say $\tilde{p}$ contained in $\mathcal{A}$ ($\tilde{p} \in \mathcal{A}$ for short) if and only if $\tilde{p} = c_{(\alpha,\beta)} \subseteq \mathcal{A}$.

On the other word, $\tilde{p} \in \mathcal{A}$ if and only if $\alpha \leq A_1(c)$ and $\beta \geq A_2(c)$.

(2) let $\tilde{\tilde{p}} = c_{(\beta)}$ be a VIFP in X, and $\mathcal{A} = \langle A_1, A_2 \rangle$ an IFS in X, $\tilde{\tilde{p}}$ is said to be in $\mathcal{A}$ ($\tilde{\tilde{p}} \in \mathcal{A}$ for short) if and only if $A_1(c) = 0$ and $\beta \geq A_2(c)$.

Result 2.3.5:

In intuitionistic fuzzy set theory, specifically in intuitionistic fuzzy points, we have in general an IFP $\tilde{p} = c_{(\alpha,\beta)}$ where $\alpha, \beta \in [0,1]$ such that $0 \leq \alpha + \beta \leq 1$, then we have the following cases:

1) If $\alpha \neq 0$ and $\beta \neq 1$, then $\tilde{p}$ is regular intuitionistic fuzzy point, and we call it intuitionistic fuzzy point (IFP).

2) If $\alpha = 0$ and $\beta \neq 1$, then $\tilde{\wp}$ become vanishing intuitionistic fuzzy point (VIFP) and we denote it by $\tilde{\tilde{\wp}}$.

For example: $\tilde{\tilde{\wp}} = \langle \overline{0}, \overline{0} \rangle$ it's VIFP.

3) If $\alpha = 1$, then its become a regular fuzzy point.

4) If $\beta = 1 - \alpha$, then its become a regular fuzzy point.

The following definition generalize the notion of quasi-coincident from fuzzy sets to IFS:

Definition 2.3.6: [10]

(1) An IFP $\tilde{\wp} = c_{(\alpha,\beta)}$ in X is said to be quasi-coincident with the IFS $\mathcal{A} = \langle A_1, A_2 \rangle$, denoted by $\tilde{\wp}\ \boldsymbol{\mathcal{Q}}\ \mathcal{A}$ if and only if $\alpha > A_2(c)$ or $\beta < A_1(c)$.

(2) Two IFSs $\mathcal{A} = \langle A_1, A_2 \rangle$ and $\mathcal{B} = \langle B_1, B_2 \rangle$ in X, are said to be quasi-coincident, denoted by $\mathcal{A}\ \boldsymbol{\mathcal{Q}}\ \mathcal{B}$ if and only if there exists an element $x \in$ X such that $A_1(x) > B_2(x)$ or $A_2(x) < B_1(x)$.

Note: we denote the negation of $\mathcal{A}\ \boldsymbol{\mathcal{Q}}\ \mathcal{B}$ by the symbol $\mathcal{A}\ \not{\boldsymbol{\mathcal{Q}}}\ \mathcal{B}$.

Example 2.3.7:

Let X = $\{u, v, w\}$ and consider the IFS

$\mathcal{A} = \langle\ (u_{0.4}, v_{0.5}, w_{0.7}), (u_{0.5}, v_{0.2}, w_{0.3})\ \rangle$ in X, take $c = u$ then we write:

$u_{(0.6,0.3)}\ \boldsymbol{\mathcal{Q}}\ \mathcal{A}$, $u_{(0.3,0.1)}\ \boldsymbol{\mathcal{Q}}\ \mathcal{A}$

but $u_{(0.2,0.5)} \not\subseteq \mathcal{A}$ since $0.2 < A_2(u)$ and $0.5 > A_1(u)$.

2.4 Intuitionistic Fuzzy Functions

We will define an IF functions between two families of IFSs by using a function between two fuzzy subsets corresponding to a function between two regular sets.

Remark: we will use IF(X) to be a family of all intuitionistic fuzzy subsets of X.

Definition 2.4.1: [14]

Let X and Y be two nonempty regular sets, and let $f: X \to Y$ and let $f^*: F(X) \to F(Y)$ be a fuzzy function, then we define $\overline{f}: IF(x) \to IF(Y)$ to be an IF function.

We define the image and the preimage of IFSs by:

(i) If $\mathcal{A} = \langle A_1, A_2 \rangle$ is an IFS in X, then the image of $\mathcal{A}$ under $\overline{f}$ is an IFS in Y defined by :

$$\overline{f}(\mathcal{A}) = \langle\, f^*(A_1)\,,\ f^{*c}(1 - A_2)\,\rangle$$

where

$$f^*(A_1)(y) = \begin{cases} \sup\{\, A_1(x) :\ x \in f^{-1}(y)\,\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 \quad : & \text{other wise} \end{cases}$$

And

$$f^{*c}(1 - A_2)(y) = \begin{cases} \inf\{\, A_2(x) :\ x \in f^{-1}(y)\,\} & \text{if } f^{-1}(y) \neq \emptyset \\ 1 \quad : & \text{other wise} \end{cases}$$

That's to say

$$\overline{f}(\mathcal{A}) = \begin{cases} \langle \bigvee_{x \in f^{-1}(y)} A_1(x), \bigwedge_{A_2(x)} A_2(x) \rangle & \text{if } f^{-1}(y) \neq \emptyset \\ \tilde{0} & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

(ii) if $\mathcal{B} = \langle B_1, B_2 \rangle$ is an IFS in Y, then the preimage of $\mathcal{B}$ under $\overline{f}$ is an IFS in X defined by :

$$(\overline{f})^{-1}(\mathcal{B}) = \langle f^{*-1}(B_1), \quad f^{*-1}(B_2) \rangle$$

Where

$$f^{*-1}(B_1)(x) = B_1(f(x))$$

And

$$f^{*-1}(B_2)(x) = B_2(f(x))$$

Chapter Three

Fuzzy Topological Spaces

Chapter Three
Fuzzy Topological Spaces

3.1 Definition of fuzzy topology

Definition 3.1.1: [11]

A fuzzy topology on a nonempty set X is a family τ of fuzzy subset of X (i.e $\tau \leq F(X)$) satisfy the following conditions:

(i) $\overline{1}, \overline{0} \in \tau$.

(ii) If A, B $\in \tau$, then $A \wedge B \in \tau$.

(iii) If $\{A_\alpha : \alpha \in \Gamma\}$ is any family of fuzzy sets in τ, then $\vee_\alpha A_\alpha \in \tau$.

The pair (X, τ) is called a fuzzy topological space and the member of τ are called open fuzzy sets, and their complements are called closed fuzzy sets.

As (regular) topology; the indiscrete fuzzy topology contains only) $\overline{1}$ and $\overline{0}$, while the discrete fuzzy topology contains all fuzzy sets, and the set of all crisp fuzzy sets in X is also a fuzzy topology.

Theorem 3.1.2: [6]

1) If τ_1 and τ_2 are fuzzy topologies on a nonempty set X, then their intersection is a fuzzy topology on X.

2) Under the previous assumption, $\tau_1 \cup \tau_2$ may not be a fuzzy topology.

Proof: (1)(i) $\bar{1}$ and $\bar{0}$ belongs to τ_1 and τ_2, then $\bar{1} \in \tau_1 \cap \tau_2$ and $\bar{0} \in \tau_1 \cap \tau_2$.

(ii) Let $A, B \in \tau_1 \cap \tau_2$ then $A, B \in \tau_1$ and $A, B \in \tau_2$,Hence $A \wedge B \in \tau_1 \cap \tau_2$.

(iii) Let $\{A_\alpha : \alpha \in \Gamma\} \in \tau_1 \cap \tau_2$, then $A_\alpha \in \tau_1$ and $A_\alpha \in \tau_2$ for all $\alpha \in \Gamma$, Hence $\bigvee_{\alpha \in \Gamma} A_\alpha \in \tau_1 \cap \tau_2$.

(2) we consider the following example:

Let $X = \{a, b, c\}$, and

Let $\tau_1 = \left\{\bar{0}, \bar{1}, A_1 = \{a_{0.7}, b_0, c_1\}, A_2 = \{a_{0.7}, b_{0.5}, c_{0.3}\}, A_1 \cup A_2 = \{a_{0.7}, b_{0.5}, c_1\}, A_1 \cap A_2 = \{a_{0.7}, b_0, c_{0.3}\}\right\}$

$\tau_2 = \left\{\bar{0}, \bar{1}, B_1 = \{a_{0.5}, b_{0.5}, c_{0.5}\}\right\}$ are two fuzzy topologies on X.

$\tau_1 \cup \tau_2 = \{\bar{0}, \bar{1}, A_1, A_2, A_1 \cup A_2, A_1 \cap A_2, B_1\}$

Now $A_1 \cap B_1 \notin \tau_1 \cup \tau_2$

Then $\tau_1 \cup \tau_2$ is not a fuzzy topology.

The previous theorem can be generalize to any number of fuzzy topologies on X.

3.2 Neighborhood system:

Remark: we will use $\in$ for $\in_2$.

Definition 3.2.1: [19]

Let (X, τ) be a FTs, a fuzzy set G in (X, τ) is a neighborhood (nbd for short) of a fuzzy point $p = x_\lambda$ iff there exist a fuzzy open set U such that $x_\lambda \leq U \leq G$.

In general, we can say that G is a nbd of a fuzzy set A iff there exists a fuzzy open set U s.t. $A \leq U \leq G$.

Note : The nbd system is the family of all nbds of a fuzzy point x_λ.

Theorem 3.2.2: [19]

Let A be a fuzzy set in a fuzzy topological space (X, τ), then the following are equivalent:

1) A is fuzzy open.

2) For each $p = x_\lambda \in A$, A is a nbd of p.

Proof: (1) $\Longrightarrow$ (2) is a straightforward. It remains to show (2) $\Longrightarrow$ (1)

The assumption ensure that for each $p \in A$, there exists a fuzzy open set U_p s.t. $p \in U_p \leq A$, it follows that $\bigvee_{p \in A} p \leq \bigvee U_p \leq A$

by proposition 1.4.4 implies $A \leq \bigvee U_p \leq A$ and consequently $A = \bigvee U_p$ which is fuzzy open. Therefor A is fuzzy open.

Theorem 3.2.3: [21]

A fuzzy set A is open iff for each fuzzy set B contained in A, A is a nbd of B.

Proof: $\Longrightarrow$ clear.

$\Longleftarrow$ since $A < A$, there exist an open fuzzy set U s.t. $A < U < A$

Hence $A = U$ and A is open fuzzy set. Q.E.D

3.3 Interior and Closure of fuzzy sets

Definition 3.3.1: (Interior) [20]

Let A and B be fuzzy sets in FTs (X, τ) and let $B \subset A$, then B is called an interior fuzzy set of A iff A is a nbd of B.

The union of all interior fuzzy sets of A is called the interior of A and denoted by A°.

i.e $A^\circ = \bigvee\{U : U \in \tau,\ U \leq A\}$.

Theorem 3.3.2: [20]

Let A be a fuzzy set in a FTs (X, τ), then A° is fuzzy open and it's the largest open fuzzy set contained in A.

Proof: By definition 3.3.1, clearly, A° is itself an interior fuzzy set of A, hence there exists an open fuzzy set U s.t. $A^\circ \subset U \subset A$, but U is an interior fuzzy set of A (U interior because $U \subset U \subset A$), hence $U \subset A^\circ$, which implies that $A^{\circ} = U$. Thus A° is fuzzy open and it's the largest open fuzzy set contained in A.

Corollary 3.3.3: [20]

The fuzzy set A is open if and only if $A = A^{\circ}$.

Proof: $\Rightarrow$ A is open, $A \subseteq A^{\circ}$ and $A^{\circ} \subseteq A$, this implies $A = A^{\circ}$.

$\Leftarrow$ $A = A^{\circ}$ and A° is open, which implies A is open.

Definition 3.3.4: (Closure) [20]

Let (X, τ) be a FTs and let A be any fuzzy subset of X, then the closure of A denoted by cl(A) or $\bar{A}$ is defined by :

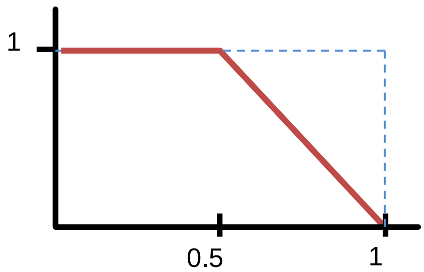
cl(A) = $\bigwedge\{F : F^c \in \tau, \ A \leq F\}$.

We will consider some examples to compute the closure and the interior of some fuzzy sets in a FTs:

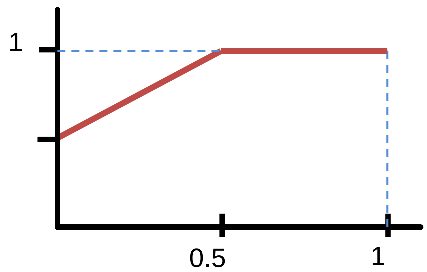
Example 3.3.5:

Given the following fuzzy sets A, B, C and D of $X = [\,0, 1\,]$

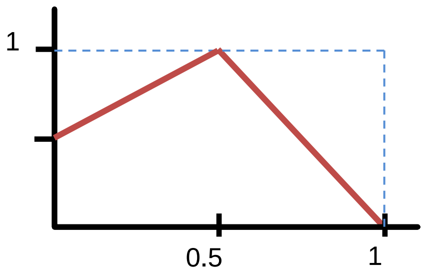
A



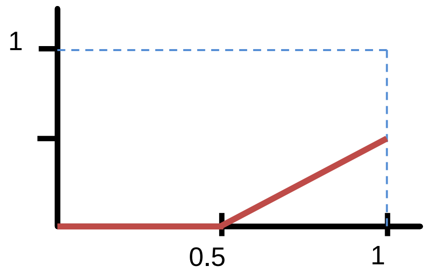
B



C



D

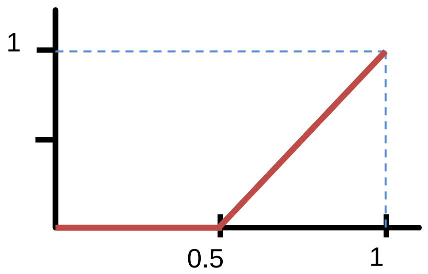


Where $\tau = \{\bar{0}, \bar{1}, A, B, C\}$

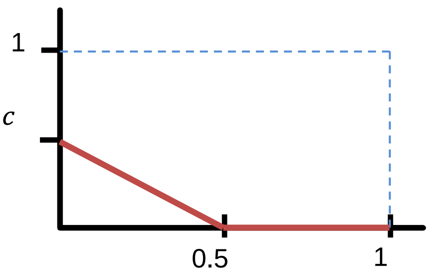
To find $D^{\circ}, \overline{D}$

Firstly we find the fuzzy closed sets which are the complements of the members of τ:

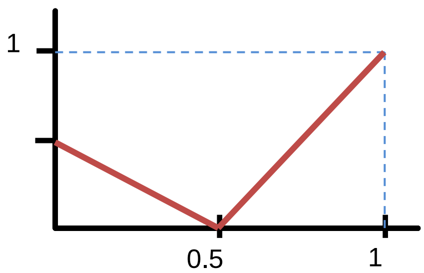
A^c



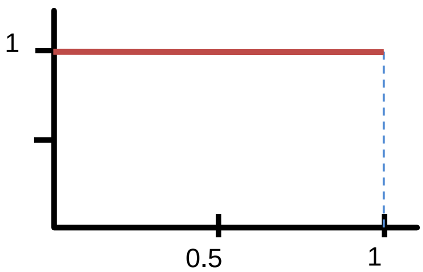
B^c



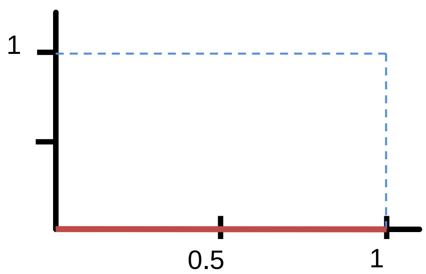
C^c



$\bar{1}$



$\bar{0}$



Now, the fuzzy closed set contaning in D are A^c, C^c and 1.

Therefor $\overline{D} = A^c \wedge C^c \wedge 1 = A^c$.

And the interior is $D^{\circ} = \overline{0}$.

Example 3.3.6:

Let τ be the fuzzy topology generated by the fuzzy sets A, B and C such that $A = \{a_{0.8}, b_{0.2}, c_1\}$

$B = \{a_{0.8}, b_{0.6}, c_{0.4}\}$

$C = \{a_{0.6}, b_{0.6}, c_{0.6}\}$

Then $\tau = \{\overline{0}, \overline{1}, A, B, C, \{a_{0.8}, b_{0.2}, c_{0.4}\}, \{a_{0.6}, b_{0.2}, c_{0.6}\}, \{a_{0.6}, b_{0.6}, c_{0.4}\},$

$\{a_{0.6}, b_{0.2}, c_{0.4}\}, \{a_{0.8}, b_{0.6}, c_1\}, \{a_{0.8}, b_{0.6}, c_{0.6}\}\}$

Now, its clear that $A^{\circ} = A$

To find $cl(B)$, we need the fuzzy closed sets which are :

$\overline{0}, \overline{1}, \{a_{0.2}, b_{0.8}, c_0\}, \{a_{0.2}, b_{0.4}, c_{0.6}\}, \{a_{0.4}, b_{0.4}, c_{0.4}\}, \{a_{0.2}, b_{0.8}, c_{0.6}\}$

$\{a_{0.4}, b_{0.8}, c_{0.4}\}, \{a_{0.4}, b_{0.4}, c_{0.6}\}, \{a_{0.4}, b_{0.8}, c_{0.6}\}, \{a_{0.2}, b_{0.4}, c_0\}$

$\{a_{0.2}, b_{0.4}, c_{0.4}\}$

Hence $cl(B) = \overline{1}$.

Lemma 3.3.7: [20]

Le A be a fuzzy set in a FTs (X, τ), then $\mathrm{cl}(A)$ is fuzzy closed set.

Proof: We have $\mathrm{cl}(A)$ = $\{\bigwedge_\alpha F_\alpha : F$ fuzzy closed and $A \leq F_\alpha\}$

$(\bigwedge_\alpha F_\alpha)^c = \bigvee_\alpha {F_\alpha}^c$ which is fuzzy open.

Theorem 3.3.8: [20]

Let (X, τ) be a fuzzy topological space, a fuzzy set A is fuzzy closed if and only if $A = \mathrm{cl}(A)$.

Proof: Assume that $A = \mathrm{cl}(A)$, but $\mathrm{cl}(A)$ is fuzzy closed, therefor A is fuzzy closed.

Conversely, assume A is fuzzy closed, then by definition 3.3.4(closure), $A \leq F_\alpha$ for each F_α. Hence $A \leq \bigwedge_\alpha F_\alpha$

Which implies $A \leq \mathrm{cl}(A)$

Now, $\mathrm{cl}(A) = \bigwedge\{F : F^c \in \tau,\ A \leq F\}$, that is $\mathrm{cl}(A)$ is the smallest closed fuzzy set containing A, but A is closed fuzzy set, this implies $\mathrm{cl}(A) \leq A$.

Therefor $A = \mathrm{cl}(A)$.

Lemma 3.3.9: [20]

Let (X, τ) be a fuzzy topological space, then for any A and B fuzzy subsets of X the following are true:

1) $\overline{A \vee B} = \bar{A} \vee \bar{B}$.

2) $(A \vee B)^{\circ} \geq A^{\circ} \vee B^{\circ}$.

3) $(A^{\circ})^{c} = \overline{A^{c}}$.

4) $(\bar{A})^{c} = (A^{c})^{\circ}$.

Proof: (1) $\overline{A \vee B} = \bigwedge_{\alpha} F_{\alpha} : F_{\alpha}$ closed and $A \vee B \leq F_{\alpha}$

But

$$\bar{A} \vee \bar{B} = \bigwedge_{\substack{F \text{ closed} \\ A \leq F}} F \quad \bigvee \quad \bigwedge_{\substack{K \text{ closed} \\ B \leq K}} K$$

$$= \bigwedge_{\substack{F_i, K_j \\ A \leq F_i \\ B \leq K_j}} (F_i \vee K_j)$$

$$= \bigwedge_{\substack{L_t \text{ closed} \\ A \vee B \leq L_t}} L_t$$

$$= \overline{A \vee B}$$

(2) $A \leq A \vee B \Longrightarrow A^{\circ} \leq (A \vee B)^{\circ}$

$B \leq A \vee B \Longrightarrow B^{\circ} \leq (A \vee B)^{\circ}$

Hence $A^{\circ} \vee B^{\circ} \leq (A \vee B)^{\circ}$.

(3) $1 - A^{\circ} = 1 - \bigvee \{ U : U \in \tau,\ U \leq A \}$

$= \bigwedge \{ 1 - U : U \in \tau,\ U \leq A \}$

$= \bigwedge \{ 1 - U : U \in \tau,\ 1 - U \geq 1 - A \}$

$= \bigwedge \{ F : F^{c} \in \tau,\ F \geq 1 - A \}$

$= \overline{A^{c}}$.

(4) $1 - \bar{A} = 1 - \bigwedge \{ F : 1 - F \in \tau,\ A \leq F \}$

$= \bigvee \{ 1 - F : 1 - F \in \tau,\ A \leq F \}$

$= \bigvee \{ U : U \in \tau,\ U \leq 1 - A \}$

$= (A^{c})^{\circ}$.

Corollary 3.3.10: [20]

Let (X, τ) be a fuzzy topological space and let A, B be two fuzzy sets, then the following are true :

1) $\overline{A \wedge B} = \bar{A} \wedge \bar{B}$.

2) $(A \wedge B)^{\circ} \leq A^{\circ} \wedge B^{\circ}$.

Proof : Straightforward.

Chapter four

Intuitionistic fuzzy topological spaces

Chapter four

Intuitionistic fuzzy topological spaces

4.1 Definition of intuitionistic fuzzy topology

Recall : (1) IFS(X) means all intuitionistic fuzzy sets of a set X.

(2) τ_F means fuzzy topology.

Definition 4.1.1: [9]

Let X be a nonempty set and let $\tau \subseteq$ IFS(X) then τ is called an intuitionistic fuzzy topology on X (IFT, for short) if its satisfies the following conditions:

1) $\tilde{0}, \tilde{1} \in \tau$

2) If $A, B \in \tau$, then $A \cap B \in \tau$.

3) If $\{ A_\alpha : \alpha \in \Gamma \} \in \tau$, then $\bigcup_\alpha A_\alpha \in \tau$.

The pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS) and the members of τ are called intuitionistic fuzzy open sets (IFO) and their complement are called intuitionistic fuzzy closed sets (IFC).

It's clear that if (X, τ_F) is a fuzzy topological space, then (X, τ) is an IFTS, where $\tau = \{ \langle A, A^c \rangle : A \in \tau_F \}$.

Example 4.1.2

(1) Let X be a nonempty set and let $\tau_{\text{ind}} = \{\tilde{0}, \tilde{1}\}$, then clearly τ_{ind} is an IFTS. In this case, τ_{ind} is called intuitionistic fuzzy indiscrete topology.

(2) Let X be a nonempty set and let $\tau_{\text{dis}} =$ IFS(X), then clearly τ_{dis} is an IFTS. In this case, τ_{dis} is called intuitionistic fuzzy discrete topology.

Example 4.1.3

Let $X = \{a, b, c\}$

$$A = \langle \{a_{0.5}, b_{0,5}, c_{0.4}\}, \{a_{0.2}, b_{0.4}, c_{0.4}\} \rangle$$

$$B = \langle \{a_{0.4}, b_{0.6}, c_{0.2}\}, \{a_{0.5}, b_{0.3}, c_{0.3}\} \rangle$$

$$C = \langle \{a_{0.5}, b_{0.6}, c_{0.4}\}, \{a_{0.2}, b_{0.3}, c_{0.3}\} \rangle$$

$$D = \langle \{a_{0.4}, b_{0.5}, c_{0.2}\}, \{a_{0.5}, b_{0.4}, c_{0.4}\} \rangle$$

Then the family $\tau = \{\tilde{0}, \tilde{1}, A, B, C, D\}$ is an IFT on X.

Lemma 4.1.4: [9]

If (X, τ_F) be a fuzzy topological space such that τ_F be not indiscrete, then we can construct two IFTSs on X as follow:

(1) $\tau^1 = \{\tilde{0}, \tilde{1}\} \cup \{ \langle A_\alpha, \bar{0} \rangle : A_\alpha \in \tau_F \}$.

(2) $\tau^2 = \{\tilde{0}, \tilde{1}\} \cup \{ \langle \bar{0}, {A_\alpha}^c \rangle : A_\alpha \in \tau_F \}$.

Where $\tau_F = \{\bar{0}, \bar{1}\} \cup \{ A_\alpha \}$ where $\alpha \in \Gamma$.

Proof : straightforward by definition of IFT.

Notation 4.1.5:

(a) IFO(X) denotes the set of all IFOs in X.

(b) IFC(X) denotes the set of all IFCs in X.

Theorem 4.1.6: [15]

Let (X, τ) be IFTS, then the following are true:

1) $\tilde{0}, \tilde{1} \in$ IFC(X)

2) If $A_1, A_2 \in$ IFC(X), then $A_1 \cup A_2 \in$ IFC(X).

3) If $\{ A_\alpha : \alpha \in \Gamma \} \in$ IFC(X), then $\bigcap_\alpha A_\alpha \in$ IFC(X).

Proof: (1) $\tilde{0} \in \tau \Longrightarrow \tilde{0}^c = \tilde{1} \in$ IFC(X).

$\tilde{1} \in \tau \Longrightarrow \tilde{1}^c = \tilde{0} \in$ IFC(X).

(2) suppose $A_1,\ A_2 \in$ IFC(X), then ${A_1}^c,\ {A_2}^c \in$ IFO(X)

$${A_1}^c \cap\ {A_2}^c = (A_1 \cup\ A_2)^c \in \text{IFO(X)}$$

So $A_1 \cup\ A_2 \in$ IFC(X).

(3) suppose $\{ A_\alpha : \alpha \in \Gamma \} \in$ IFC(X), then ${A_\alpha}^c \in$ IFO(X) for $\alpha \in \Gamma$.

So $\bigcup_\alpha {A_\alpha}^c \in$ IFO(X)

But $\bigcup_\alpha {A_\alpha}^c = (\ \bigcap_\alpha A_\alpha)^c$

Hence $\bigcap_{\alpha} A_{\alpha} \in$ IFC(X).

From the previous theorem we conclude the following result.

Result 4.1.7

If (X, τ) is finite IFTS, then the complement of elements in τ forms an IFTS, and its true for fuzzy topology and general topology.

Definition 4.1.8: [9, 16]

Let (X, τ_1), (X, τ_2) be two IFTSs, then τ_1 is said to be contained in τ_2 if and only if for each $A \in \tau_1$ implies $A \in \tau_2$.

Theorem 4.1.9: [9]

(1) Let $\{\tau_i : i \in \Delta\}$ be a family of IFTs on X. Then $\bigcap_{i\in\Delta} \tau_i$ is also an IFT on X.

(2) Let τ_1, τ_2 be two IFTs on X. Then $\tau_1 \cup \tau_2$ may not be an IFT on X.

Proof: (1) Let $\{\tau_i : i \in \Delta\}$ be a family of IFTs on X, we want to show that $\bigcap_{i\in\Delta} \tau_i$ is an IFT on X.

[i] $\tilde{0} \in \tau_i : \forall i \in \Delta \implies \tilde{0} \in \bigcap_{i\in\Delta} \tau_i$.

Similarly, $\tilde{1} \in \bigcap_{i\in\Delta} \tau_i$.

[ii] Let $A_1, A_2 \in \bigcap_{i\in\Delta} \tau_i$, then $A_1, A_2 \in \tau_i$ for each $i \in \Delta$ and

hence $A_1 \cap A_2 \in \tau_i$ for each $i \in \Delta$

Thus, $A_1 \cap A_2 \in \bigcap_{i\in\Delta} \tau_i$.

[iii] Let $\{ A_\alpha : \alpha \in \Gamma \} \subseteq \bigcap_{i\in\Delta} \tau_i$,then $\{ A_\alpha : \alpha \in \Gamma \} \subseteq \tau_i : \forall i \in \Delta$

and hence $\bigcup_{\alpha\in\Gamma} A_\alpha \in \tau_i$ for every $i \in \Delta$.

Thus $\bigcup_{\alpha\in\Gamma} A_\alpha \in \bigcap_{i\in\Delta} \tau_i$.

(2) By counter example:

Let X= $\{a, b\}$

Let $\tau_1 = \{\tilde{0}, \tilde{1}, \langle\{a_{0.5}, b_{0.5}\}, \{a_{0.2}, b_{0.4}\}\rangle \}$,

Let $\tau_2 = \{\tilde{0}, \tilde{1}, \langle\{a_{0.3}, b_{0.2}\}, \{a_{0.6}, b_{0.3}\}\rangle \}$,

Then $\tau_1 \cup \tau_2 = \{\tilde{0}, \tilde{1}, \langle\{a_{0.5}, b_{0.5}\}, \{a_{0.2}, b_{0.4}\}\rangle, \langle\{a_{0.3}, b_{0.2}\}, \{a_{0.6}, b_{0.3}\}\rangle \}$

$\langle\{a_{0.5}, b_{0.5}\}, \{a_{0.2}, b_{0.4}\}\rangle \cap \langle\{a_{0.3}, b_{0.2}\}, \{a_{0.6}, b_{0.3}\}\rangle \notin \tau_1 \cup \tau_2$

Then $\tau_1 \cup \tau_2$ is not an IFT.

Theorem 4.1.10

If $(X, \mathscr{f})$ be any fuzzy topological space, then $\tau = \{ \langle A, A^c \rangle : A \in \mathscr{f}\}$ is an IFTS.

Proof: (i) $\emptyset \in \mathscr{f} \Longrightarrow \langle \emptyset, \emptyset^c \rangle = \langle \emptyset, X \rangle = \langle \bar{0}, \bar{1} \rangle = \tilde{0} \in \tau$.

and $X \in \mathscr{f} \Longrightarrow \langle X, X^c \rangle = \langle X, \emptyset \rangle = \langle \bar{1}, \bar{0} \rangle = \tilde{1} \in \tau$.

(ii) let $A_1, A_2 \in \mathscr{f} \Longrightarrow \langle A_1, {A_1}^c \rangle \cap \langle A_2, {A_2}^c \rangle = \langle A_1 \wedge A_2,\ {A_1}^c \vee {A_2}^c \rangle$

But $A_1 \wedge A_2 \in \not{f}$ and $A_1{}^c \vee A_2{}^c \in \not{f}$

Thus $\langle A_1 \wedge A_2,\ A_1{}^c \vee A_2{}^c \rangle \in \tau$.

(iii) let $\{ A_\alpha : \alpha \in \Gamma \} \subseteq \not{f}$

Now, $\bigcup_\alpha \langle A_\alpha, A_\alpha{}^c \rangle = \langle \bigvee_\alpha A_\alpha, \bigwedge_\alpha A_\alpha{}^c \rangle = \langle \bigvee_\alpha A_\alpha, (\bigvee_\alpha A_\alpha)^c \rangle \in \tau$.

4.2 Basis and subbasis for IFTS

Proposition 4.2.1: [23]

We can write any IFS A in X as the union of all IFPs in A.

i.e $A = \bigcup_{\tilde{p} \in A} \tilde{p}$

Definition 4.2.2: [23]

Let (X, τ) be an IFTS, then the collection $\mathfrak{B} \subseteq \tau$ is called a base of IFT τ if for every $\tilde{p} \in$ G, where G is any IFOS $\Longrightarrow \exists\, \mathcal{B} \in \mathfrak{B}$ such that $\tilde{p} \in \mathcal{B} \subseteq$ G.

The following is another definition of basis of IFT:

Definition 4.2.3: [23]

A collection $\mathfrak{B}$ of IFSs on X is said to be basis (or base) for an IFT τ on X if :

(i) for every $\tilde{p}$ in X, $\exists\, \mathcal{B} \in \mathfrak{B}$ such that $\tilde{p} \in \mathcal{B}$.

(ii) if $\tilde{p} \in \mathcal{B}_1 \cap \mathcal{B}_2$ where $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, then $\exists\ \mathcal{B}_3 \in \mathfrak{B}$ such that $\tilde{p} \in \mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$.

Theorem 4.2.4

Let (X, τ) be an IFTS and let $\mathfrak{B}$ is a base for τ, then the IFS G $\in \tau \Leftrightarrow$ G = $\bigcup_{\alpha\in\Gamma} \mathcal{B}_\alpha$ where $\mathcal{B}_\alpha \in \mathfrak{B}, \forall \alpha \in \Gamma$.

Proof : $\Longrightarrow$ suppose G $\in \tau$, then for any $\tilde{p} \in G$, $\exists\ \mathcal{B} \in \mathfrak{B}$ such that $\tilde{p} \in \mathcal{B} \subseteq$ G (by definition 4.2.2)

Now, by proposition 4.2.1 we have $G = \bigcup_{\tilde{p}\in G} \mathcal{B}$.

$\Longleftarrow$ clearly $\mathfrak{B} \subseteq \tau$, since τ is an IFT on X, therefor any arbitrary union of members of $\mathfrak{B}$ belongs to τ. That is $\bigcup_{\alpha\in\Gamma} \mathcal{B}_\alpha \in \tau$ as $\mathcal{B}_\alpha \in \mathfrak{B}$.

Definition 4.2.5: [23]

Let (X, τ) be an IFTS, then a subfamily $\not{f}$ of τ is called a subbase for τ if the family of finite intersection of members of $\not{f}$ forms a base for τ.

Given any collection $\not{f}$ of IFSs in X containing $\tilde{0}$ and $\tilde{1}$, then the set τ consisting of arbitrary unions of finite intersection of members of $\not{f}$ forms an IFT on X. This is the smallest IFT on X containing $\not{f}$ and is called the IFT generated by $\not{f}$.

4.3 Intuitionistic fuzzy neighborhood

Definition 4.3.1: [16]

Let $\tilde{p}$ be an IFP of an IFTS (X, τ). An IFS $\mathcal{N}$ is called an Intuitionistic fuzzy neighborhood (IFN for short) of $\tilde{p}$ if there is an IFOS G in X such that $\tilde{p} \in G \subseteq \mathcal{N}$.

Theorem 4.3.2: [16]

Let (X, τ) be an IFTS, then an IFS A of X is an IFOS if and only if A is an IFN of $\tilde{p}$ for every $\tilde{p} \in A$.

Proof : let A be an IFOS, clearly A is an IFN of every $\tilde{p} \in A$.

Conversely, suppose that A is an IFN of every IFP belonging to A. Let $\tilde{p} \in A$, since A is an IFN of $\tilde{p}_\alpha$, there is an IFOS $G_{\tilde{p}_\alpha}$ in X such that $\tilde{p} \in G_{\tilde{p}_\alpha} \subseteq A$.

So we have $A = \bigcup_\alpha \{ \tilde{p}_\alpha \colon \tilde{p}_\alpha \in A\} \subseteq \bigcup_\alpha \{G_{\tilde{p}_\alpha} \colon \tilde{p}_\alpha \in A\} \subseteq A$ and hence $A = \bigcup_\alpha \{G_{\tilde{p}_\alpha} \colon \tilde{p}_\alpha \in A\}$. Since each $G_{\tilde{p}_\alpha}$ is an IFOS, then A is also an IFOS in X.

4.4 Interior and closure of IFS

Definition 4.4.1: [9]

Let (X, τ) be an IFTS and A be an IFS in X, then the Intuitionistic fuzzy interior and Intuitionistic fuzzy closure of A are defining by :

$\text{int}(A) = \bigcup \{G : G \text{ is an IFOS in X and } G \subseteq A\}$

$\text{cl}(A) = \bigcap \{F : F \text{ is an IFCS in X and } A \subseteq F\}$

We denote the interior of A by $\text{int}(A)$ or A°. Also we denote the closure of A by $\text{cl}(A)$ or $\overline{A}$.

Note that $\text{int}(A)$ is an IFOS and $\text{cl}(A)$ is an IFCS in X.

Theorem 4.4.2: [9]

Let (X, τ) be an IFTS and A be an IFS in X, then :

1) A is an IFOS in X if and only if $A^{\circ} = A$.

2) A is an IFCS in X if and only if $\text{cl}(A) = A$.

Proof is trivial.

Now we consider the following example to compute the interior and closure of IFS in IFTS X:

Example 4.4.3: [9]

In example 4.1.3, if $K = \langle \{a_{0.55}, b_{0,55}, c_{0.45}\}, \{a_{0.3}, b_{0.4}, c_{0.3}\} \rangle$ then :

$$\text{int}(K) = \bigcup \{G : G \in \tau \text{and } G \subseteq K\} = D$$

and $$\text{cl}(K) = \bigcap \{F : F \text{ is an IFCS in X and } K \subseteq F\} = \tilde{1}.$$

Theorem 4.4.4: [9]

For any IFS A in an IFTS (X, τ), we have :

1) $\text{cl}(A^c) = (A^\circ)^c$

2) $(A^c)^\circ = (\overline{A})^c$

Proof : (1) let $A = \langle A_1, A_2 \rangle$ and suppose that the family $\{G_\alpha = \langle G_{\alpha_1}, G_{\alpha_2} \rangle :$ $\alpha \in \Gamma\}$ be the IFOSs contained in A.

Then $A^\circ = \langle \vee_\alpha G_{\alpha_1} , \wedge_\alpha G_{\alpha_2} \rangle$

Hence $(A^\circ)^c = \langle \wedge_\alpha G_{\alpha_2} , \vee_\alpha G_{\alpha_1} \rangle$ ………………… (*)

Now $A^c = \langle A_2, A_1 \rangle$ and $G_{\alpha_1} \leq A_1$ and $G_{\alpha_2} \geq A_2$ for every $\alpha \in \Gamma$, we obtain that $\{{G_\alpha}^c = \langle G_{\alpha_2}, G_{\alpha_1} \rangle : \alpha \in \Gamma\}$ is the family of IFCSs containing A^c, which implies $\text{cl}(A^c) = \langle \wedge_\alpha G_{\alpha_2} , \vee_\alpha G_{\alpha_1} \rangle$ ……………… (**)

Hence from (*) and (**) we get $\text{cl}(A^c) = (A^\circ)^c$.

(2) let $A = \langle A_1, A_2 \rangle$ and suppose that the family $\{F_\alpha = \langle F_{\alpha_1}, F_{\alpha_2} \rangle \ \alpha \in \Gamma\}$ be the IFOSs containing A.

Then $\overline{A} = \langle \wedge_\alpha F_{\alpha_1} , \ \vee_\alpha F_{\alpha_2} \rangle$

Hence $(\overline{A})^c = \langle \vee_\alpha F_{\alpha_2} , \ \wedge_\alpha F_{\alpha_1} \rangle$ .... (*)

Now $A^c = \langle A_2, A_1 \rangle$ and $A_1 \leq F_{\alpha_1}$ and $A_2 \geq F_{\alpha_2}$ for every $\alpha \in \Gamma$, we obtain that $\{F_\alpha{}^c = \langle F_{\alpha_2}, F_{\alpha_1} \rangle : \ \alpha \in \Gamma\}$ is the family of IFOSs contained in A^c, that is $(A^c)^\circ = \langle \vee_\alpha F_{\alpha_2} , \ \wedge_\alpha F_{\alpha_1} \rangle$ .... (**)

Hence from (*) and (**) we get $(A^c)^\circ = (\overline{A})^c$.

Theorem 4.4.5: [9]

Let (X, τ) be an IFTS and A, B be an IFSs in X, then the following hold :

1) $\operatorname{int}(A) \subseteq A$.

2) $A \subseteq \operatorname{cl}(A)$.

3) $A \subseteq B \Longrightarrow \operatorname{int}(A) \subseteq \operatorname{int}(B)$.

4) $A \subseteq B \Longrightarrow \operatorname{cl}(A) \subseteq \operatorname{cl}(B)$.

5) $\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$.

6) $\operatorname{cl}(\operatorname{cl}(A)) = \operatorname{cl}(A)$.

7) $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$.

8) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

9) $\mathrm{int}(\tilde{1}) = \tilde{1}$ and $\mathrm{cl}(\tilde{1}) = \tilde{1}$.

10) $\mathrm{cl}(\tilde{0}) = \tilde{0}$ and $\mathrm{int}(\tilde{0}) = \tilde{0}$.

Chapter five

Compactness and separation axioms for fuzzy topology

Chapter five

Compactness and separation axioms for fuzzy topology

5.1 compact fuzzy topological spaces

Definition 5.1.1: [5]

A family C of fuzzy sets is a cover of fuzzy set B if and only if $B \subset \{c: c \in C\}$.

It is an open cover if and only if each members of C is an open fuzzy set.

A subcover of C is a subfamily of C which is also a cover.

Definition 5.1.2: [5]

A fuzzy topological space (X, τ) is compact if every cover of X by elements of τ contains a finite subcover.

i.e if $A_i \in \tau : i \in \Gamma$ and $\vee_{i\in\Gamma} A_i = 1$, then there are finitely many index $i_1, i_2, \dots, i_n \in \Gamma$ such that $\vee_{j=1}^{n} A_{i_j} = 1$.

On the other world (X, τ) is compact if and only if each open cover of X has a finite subcover.

Definition 5.1.3: [5]

A fuzzy topological space (X, τ) will be called regular if for each fuzzy point $\wp$ and each fuzzy closed set F such that $\wp \wedge F = \bar{0}$, there exists fuzzy open set U and V such that $\wp \in U$ and $F \subset V$.

Definition 5.1.4: [1]

Let (X,τ) be a fuzzy topology and Y fuzzy subset of X, then the pair (Y,τ_y) is called a fuzzy topological subspace if the family $\tau_y = \{G \cap Y: G \in \tau\}$ satisfies the following conditions:

1) $\forall\, H \in \tau_y, \exists\, F_H \in \tau^c$ s.t. $Y - H = F_H \cap Y.$

2) $\forall\, F \in \tau^c, \exists\, G_F \in \tau$ s.t. $Y - (F \cap Y) = G_F \cap Y.$

Theorem 5.1.5:

Every subspace of regular space is also regular.

Proof : Let X be a fuzzy regular space and A be a subspace of X. We have to prove that A is regular. Recall that τ_A = $\{G_A$: G ∈ τ}, where G = {(x, $\mu_G(x)$: x ∈ X} and G_A = {(x, $\mu_{G|A}(x)$) : x ∈ A}. Let $\wp$ be fuzzy point in A and F_A is closed set of A such that $\wp \notin F_A$. Since A is a subspace of X, therefore $\wp$ ∈ X and there is a closed set F in X, which generated the closed subset F_A of A. Since X is regular space and $\wp \wedge F = \bar{0}$ there exist open sets U and V such that $\wp \subseteq$ U = (x, μ_U) and F ⊆ V = (x, μ_V). Thus U_A = (x, $\mu_{U|A}$), V_A = (x, $\mu_{V|A}$)are open sets in A such that $\wp \subseteq U_A$ and $F_A \subseteq V_A$.

Hence A is a regular subspace of X.

Definition 5.1.6: [7]

A fuzzy topological space (X,τ) is normal if for each pair of closed sets F_1, F_2 such that $F_1 \wedge F_2 = \bar{0}$, there exist fuzzy open sets G_1 and G_2 such that $F_i \subseteq G_i : i = 1,2$ and $G_1 \wedge G_2 = \bar{0}$.

Theorem 5.1.7

A closed subset of normal space is normal.

Proof : let (X,τ) be a fuzzy normal space and let A be closed subset of X, then (X,τ_A) is a subspace.

Take F_1, F_2 any two fuzzy closed subsets of A with $F_1 \subseteq A - F_2$, since A is fuzzy closed subset of X $\Longrightarrow F_1 \subseteq X - F_2$ and since (X,τ) is normal then there exist G_1 , $G_2 \in \tau$ such that $F_i \subseteq G_i : i = 1,2$ and $G_1 \wedge G_2 = \bar{0}$.

Now, $A \wedge G_1$ and $A \wedge G_2$ are two fuzzy open subset of τ_A such that $F_1 \subset A \wedge G_1$, $F_2 \subset A \wedge G_2$ and $(A \wedge G_1) \wedge (A \wedge G_2) = \bar{0}$.

5.2 Separation axioms

Definition 5.2.1: [22]

A fuzzy topological space (X,τ) is said to be fuzzy T_0 if and only if $\forall x, y \in X,\ x \neq y, \exists\, U \in \tau$ such that either $U(x) = 1$ and $U(y) = 0$ or $U(x) = 0$ and $U(y) = 1$.

Another definition for fuzzy T_0:

Definition 5.2.2: [22]

A fuzzy topological space (X, τ) is said to be fuzzy T_0 if and only if for any x_λ, y_s two fuzzy singletons with $x \neq y$, there exist a fuzzy open set U such that $x_\lambda \leq U \leq {y_s}^c$ or $y_s \leq U \leq {x_\lambda}^c$.

Definition 5.2.3 [21]

A fuzzy topological space (X, τ) is said to be fuzzy T_1 if and only if $\forall\, x, y \in X, x \neq y,\ \exists\ U, V \in \tau$ such that $(x) = 1$, $U(y) = 0$ and $V(y) = 1$, $V(x) = 0$.

Another definition for fuzzy T_1:

Definition 5.2.4 [21]

A fuzzy topological space (X, τ) is said to be fuzzy T_1 if and only if for any x_λ, y_s two fuzzy singletons with $x \neq y$, there exists two fuzzy open set U, V such that $x_\lambda \leq U \leq {y_s}^c$ and $y_s \leq V \leq {x_\lambda}^c$.

It is obvious that (X, τ) is fuzzy $T_1 \Longrightarrow (X, \tau)$ is fuzzy T_0.

The following example shows a T_0 space may not be T_1:

Example 5.2.5

Let $X = \{a, b\}$, $\tau = \{\overline{0}, \overline{1}, \{a_{0.9}, b_{0.3}\}, \{a_{0.99}, b_{0.3}\}, \dots\}$

For any a_λ, b_r there exist U neighborhood of a_λ such that $a_\lambda \in U \leq {b_r}^c = \{a_1, b_{0.7}\}$.

Therefor τ is fuzzy T_0.

But it is not fuzzy T_1 by taking a_λ, $b_{0.4}$, there is no $V \in \tau$ such that $b_{0.4} \in V \leq {a_\lambda}^c = \{a_{1-\lambda}, b_1\}$.

Definition 5.2.6: [7]

A fuzzy topological space (X, τ) is said to be fuzzy strong - T_1 (in short T_s or F -T_1) if and only if every fuzzy singleton is closed fuzzy set.

Example 5.2.7

Let $X = \{a, b\}$, $\tau = \{\bar{0}, \bar{1}, \{a_\lambda, b_1\}, \{a_1, b_r\}, \{a_\lambda, b_r\} : \forall\lambda, r \in (0,1)\}$

Then τ is T_s space because every fuzzy singleton is closed.

Theorem 5.2.8 [21]

A fuzzy topological space (X, τ) is fuzzy T_1 if and only if every crisp singleton is closed.

It is clear that if (X, τ) is fuzzy T_s then (X, τ) is fuzzy T_1.

Theorem 5.2.9

Every subspace of T_1- space is T_1.

Proof : let X be a T_1 fuzzy topological space and A be a subspace of X. so $\tau_A = \{G_A : G_A = (x, \mu_{G|A}), G \in \tau\}$.

Let $x, y \in A$ such that $x \neq y$, then $x, y \in X$ are two distinct points and as X is T_1, there exist $U, V \in \tau$ such that $U(x) = 1$, $U(y) = 0$ and $V(y) = 1$,

$V(x) = 0$. Then U_A and V_A are fuzzy open set of A such that $U_A(x) = 1$, $U_A(y) = 0$ and $V_A(y) = 1$, $V_A(x) = 0$.

This shows that A is T_1.

Definition 5.2.10 [21]

A fuzzy topological space (X, τ) is said to be fuzzy Hausdorff or fuzzy T_2 if and only if for any two distinct fuzzy points $\wp, \mathscr{g} \in X$ there exist disjoint $U, V \in \tau$ with $\wp \in U$ and $\mathscr{g} \in V$.

Definition 5.2.11 [21]

A fuzzy topological space (X, τ) is said to be fuzzy T_2 if and only if for any x_λ, y_s two fuzzy singletons with $x \neq y$, there exists two fuzzy open set U, V such that $x_\lambda \leq U \leq {y_s}^c$ and $y_s \leq V \leq {x_\lambda}^c$ and $U \leq V^c$.

Definition 5.2.12: [7]

A fuzzy topological space (X, τ) is said to be fuzzy Urysohn (fuzzy $T_{2\frac{1}{2}}$) if and only if for every x_λ, y_s two fuzzy singletons with $x \neq y$, there exists two fuzzy open set U, V such that $x_\lambda \leq U \leq {y_s}^c$ and $y_s \leq V \leq {x_\lambda}^c$ and $\mathrm{cl}(U) \leq (\mathrm{cl}(V))^c$.

It is easy to show that if (X, τ) is fuzzy $T_{2\frac{1}{2}}$ then (X, τ) is fuzzy T_2.

Definition 5.2.13: [7]

A fuzzy topological space (X, τ) is said to be fuzzy strong - T_3 (F - T_3) if and only if it is T_1 or F - T_1 and regular.

In classical topological spaces, if we have a regular T_0 space (X,τ) then (X,τ) is T_3 space, but in fuzzy topological spaces, if we have a regular T_0 space (X,τ) then (X,τ) is fuzzy Urysohn space ($T_{2\frac{1}{2}}$).

Theorem 5.2.14

Every subspace of T_3 space is also T_3.

Proof : we know that T_3 is regular T_1 space, and every subspace of T_1-space is T_1 (by theorem 5.2.9) and every subspace of regular space is regular (by theorem 5.1.4), this implies that every subspace of T_3 space is T_3.

Definition 5.2.15: [7]

A fuzzy topological space (X,τ) is said to be fuzzy strong - T_4 (F - T_4) if and only if it is T_1 or F - T_1 and normal.

Theorem 5.2.16 [7]

Every closed subspace of T_4 space is also T_4.

Proof : we know that T_4 is normal T_1 space, and every subspace of T_1-space is T_1 (by theorem 5.2.9) and every closed subspace of normal space is normal, therefor every closed subspace of T_4 space is T_4.

Chapter six

Compactness and separation axioms for IFTS

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Compactness and separation axioms for IFTS

6.1 compact Intuitionistic fuzzy topological spaces

Definition 6.1.1: [9]

Let (X,τ) be an IFTS, if a family $\{\langle A_{1_i}, A_{2_i}\rangle : i \in \Gamma\}$ of IFOSs in X satisfy the condition $\cup\{\langle A_{1_i}, A_{2_i}\rangle : i \in \Gamma\} = \tilde{1}$, then its called an IF open cover of X.

A finite sub family of IFO cover $\{\langle A_{1_i}, A_{2_i}\rangle : i \in \Gamma\}$ of X, which is also an IFO cover of X is called a finite subcover of $\{\langle A_{1_i}, A_{2_i}\rangle : i \in \Gamma\}$.

Definition 6.1.2: [9]

An IFTS (X,τ) will be called IF compact if and only if every IFO cover of X has a finite subcover.

Example 6.1.3

Consider the IFTS (X,τ) where X $= \{a, b\}$.

$A_n = \langle \left(a_{\frac{n}{n+1}}, b_{\frac{n+1}{n+2}}\right), \left(a_{\frac{1}{n+2}}, b_{\frac{1}{n+3}}\right) \rangle$ and $\tau = \{\tilde{0}, \tilde{1}\} \cup \{A_n : n \in \mathbb{N}\}$

Note that $\bigcup_{n\in\mathbb{N}} A_n$ is an IFO cover for X but this cover has no finite subcover.

Therefor the IFTS (X,τ) is not compact.

Definition 6.1.4: [9]

Let (X,τ) be an IFTS and A an IFS in X , if a family $\left\{\langle A_{1_i}, A_{2_i}\rangle : i \in \Gamma\right\}$ of IFOSs in X satisfy the condition $A \subseteq \left\{\langle A_{1_i}, A_{2_i}\rangle : i \in \Gamma\right\}$, then its called an IF open cover of A.

A finite sub family of IFO cover $\left\{\langle A_{1_i}, A_{2_i}\rangle : i \in \Gamma\right\}$ of A, which is also an IFO cover of A is called a finite subcover of $\left\{\langle A_{1_i}, A_{2_i}\rangle : i \in \Gamma\right\}$.

Definition 6.1.5: [9]

An IFS $A = \langle A_1, A_2\rangle$ in an IFTS (X,τ) is called IF compact if and only if every IFO cover of A has a finite subcover.

Definition 6.1.6: [17]

An IFTS (X,τ) is called IF regular space if for every IFP $\tilde{p}$ and every IFCS F such that $\tilde{p} \cap F = \tilde{0}$, there exists an IFOSs U, V such that $\tilde{p} \in U$ and $F \subseteq V$.

Theorem 6.1.7: [17]

If (X,τ) is regular IFTS then for any IFOS U and an IFP $\tilde{p}$ such that $\tilde{p} \cap U^c = \tilde{0}$, there exist an IFOS V such that $\tilde{p} \in V \subseteq \overline{V} \subseteq U$.

Proof: suppose that (X,τ) is regular IFTS such that $\tilde{p} \cap U^c = \tilde{0}$, $U = \langle A_1, A_2\rangle$ then $U^c = \langle A_2, A_1\rangle$ is an IFCS in X. since X is regular, $\exists$ two IFOSs V, G such that $\tilde{p} \in V$, $U^c \subseteq G$ and $V \cap G = \tilde{0}$.

Now, G^c is an IFCS in X such that $V \subseteq G^c \subseteq U$. Thus $\tilde{p} \in V \subseteq \overline{V}$ and $\overline{V} \subseteq G^c \subseteq U$ so $\overline{V} \subseteq U$.

Hence $\tilde{p} \in V \subseteq \overline{V} \subseteq U$.

Theorem 6.1.8

Every subspace of IF regular space is also IF regular.

Definition 6.1.9: [17]

An IFTS (X, τ) is called normal IFTS if for every pair of IFCSs F_1, F_2 such that $F_1 \cap F_2 = \tilde{0}$ then there exists IFOSs G_1, G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \tilde{0}$.

Theorem 6.1.10: [17]

If (X, τ) is normal IFTS then for every IFCS F of X and any IFOS U of X such that $F \cap U^c = \tilde{0}$, there exists an IFOS G such that $F \subseteq G \subseteq \overline{G} \subseteq U$.

Proof: suppose that (X, τ) is normal IFTS. Let F be an IFCS in X and U be an IFOS in X such that $F \cap U^c = \tilde{0}$, then $F \subseteq U$.

Since X is normal and U^c is an IFCS in X, then ∃ two IFOSs G and G_1 such that $G \cap G_1 = \tilde{0}$ and $F \subseteq U$, $U^c \subseteq G_1$.

This implies that ${G_1}^c \subseteq U$ and $G \subseteq {G_1}^c$ but ${G_1}^c$ is IFCS, therefor $\overline{G} \subseteq {G_1}^c$. Thus $F \subseteq G \subseteq \overline{G} \subseteq U$.

6.2 separation axioms in IFTS

Definition 6.2.1: [23]

An IFTS (X,τ) is called IF T_0 if and only if for any $x,y \in X, x \neq y$, $\exists\, U,V \in \tau$ such that $U(x) = (1,0)$, $U(y) = (0,1)$ or $V(x) = (0,1)$, $V(y) = (1,0)$.

Definition 6.2.2: [23]

An IFTS (X,τ) is called IF T_1 if and only if for any $x,y \in X, x \neq y$, $\exists\, U,V \in \tau$ such that $U(x) = (1,0)$, $U(y) = (0,1)$ and $V(x) = (0,1)$, $V(y) = (1,0)$.

Definition 6.2.3: [18]

An IFTS (X,τ) is called IF T_2(or, Hausdorff) if for any pair of disjoint IFPs or VIFPs $\tilde{p}, \tilde{q}$ in X, there exist $U,V \in \tau$ such that $\tilde{p} \in U$, $\tilde{q} \in V$ and $U \cap V = \tilde{0}$.

Example 6.2.4: [23]

Let $X = \{a,b\}$ and let $\tau = \left\{ \tilde{0}, \tilde{1}, \ \langle (a_1,b_0),(a_0,b_1) \rangle, \ \langle (a_0,b_1),(a_1,b_0) \rangle \right\}$ then (X,τ) is an IFTS and it is an IF T_0, T_1 and T_2 spaces.

It obvious that if (X,τ) is IF $T_2 \Longrightarrow (X,\tau)$ is IF $T_1 \Longrightarrow (X,\tau)$ is IF T_0 but none of the implication are reversible.

Definition 6.2.5: [23]

An IFTS (X,τ) is called IF $q-T_2$ if for every pair of distinct IFPs or VIFPs $\tilde{p},\tilde{q}$ in X, there exist $U,V\in\tau$ such that $\tilde{p}\in U$, $\tilde{q}\in V$ and $U\subseteq V^c$.

We have (X,τ) is an IF $T_2 \Longrightarrow (X,\tau)$ is an IF $q-T_2$ but none of the implication are reversible.

Theorem 6.2.6: [23]

Every subspace of T_0 space is T_0.

Proof : let X be a T_0 IFTS and A be a subspace of X.

So $\tau_A=\left\{G_A=\langle\mu_{G|A},\nu_{G|A}\rangle: x\in A,\ G\in\tau\right\}$ where $G=\langle\mu_G,\nu_G\rangle$.

Let $x,y\ \in X$ such that $x\neq\ y$. Since X is T_0, then $\exists\ U,V\in\tau$ such that $U(x)=(1{,}0),\ U(y)=(0{,}1)$ or $V(x)=(0{,}1),\ V(y)=(1{,}0)$.

Thus there exist $U_A,V_A\in\tau_A$ such that $U_A(x)=(1{,}0),\ U_A(y)=(0{,}1)$ or $V_A(x)=(0{,}1),\ V_A(y)=(1{,}0)$.

This prove that the subspace A is IF T_0.

Theorem 6.2.7: [23]

Every subspace of T_1 space is T_1.

Proof : same as the previous theorem.

Theorem 6.2.8: [23]

Every subspace of T_2 space is T_2.

Proof :

let (X, τ) be an IF T_2 space and let A be a subspace of X where $\tau_A = \{G_A = \langle \mu_{G|A}, \nu_{G|A} \rangle : x \in A,\ G \in \tau\}$ where $G = \langle \mu_G, \nu_G \rangle$.

Let $\tilde{p}$ and $\tilde{q}$ be two distinct IFPs in A (they have distinct supports), then $\tilde{p}$, $\tilde{q}$ are also distinct IFPs in X but X is IF T_2, then there exist $U, V \in \tau$ such that $\tilde{p} \in U$, $\tilde{q} \in V$ and $U \cap V = \tilde{0}$.

Thus there exists $U_A, V_A \in \tau_A$ such that $\tilde{p} \in U_A$, $\tilde{q} \in V_A$ and $U_A \cap V_A = \tilde{0}$.

This prove that the subspace A is also T_2 space.

Conclusion

Through this study it was found that many properties of topological spaces in a regular setting were extended to topological spaces in fuzzy setting including intuitionistic fuzzy setting. However, some other properties were not extended to fuzzy setting, while its extended to IF setting and some properties was extended to IF setting but not in fuzzy setting, which motivated the researchers to put down new definitions to conclude parallel theorems.

There have been different definitions for the same property, this causes researches and studies to be scattered, there have to be unification of definitions of different properties that will orient the research by all interested people to be in one direction and all efforts would be strengthened.

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جامعة النجاح الوطنية

كلية الدراسات العليا

المفاهيم التبولوجية على الفراغات التبولوجية الضبابية التي تتضمن الفراغات التبولوجية الضبابية الحدسية

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إشراف

د. محمد العملة

قدمت هذه الأطروحة استكمالاً لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدراسات العليا، في جامعة النجاح الوطنية، نابلس– فلسطين.

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الملخص

في هذه الرسالة قمنا بدراسة المفاهيم والخصائص التبولوجية للفراغات التبولوجية الضبابية والفراغات التبولوجية الضبابية الحدسية وربطها بتلك المفاهيم والخصائص للفراغات التبولوجية الكلاسيكية.

أيضاً، تم عرض المجموعات والنقط والاقترانات والعلاقات الضبابية مع خصائصها، لقد تم اثبات ان العديد من المفاهيم و الخصائص التبولوجية هي توسعة لتلك الخصائص في البيئة غير الضبابية.

أيضاً، تم عرض المجموعات والنقط والاقترانات والعلاقات الضبابية الحدسية مع خصائصها. لقد تم اثبات ان العديد من المفاهيم و الخصائص التبولوجية هي توسعة لتلك الخصائص في البيئة الضبابية.

كذلك تم التحري عن المسارات المختلفة لفرضيات الانفصال باستخدام الجوار وكذلك النقاط الضبابية والنقاط الضبابية الحدسية، ولقد تم دراسة انواع اخرى من مسلمات الفصل في الفضاءات التبولوجية الضبابية والفضاءات التبولوجية الضبابية الحدسية.

وأخيراً، تم تقديم مفاهيم التراص الضبابي والتراص الحدسي واثباتات للعلاقات بينها.