An- Najah National University

Faculty of Graduate Studies

Analytical and Numerical Aspects of Wavelets

By

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Dedication

I dedicate this research to:

The most wonderful person in presence, and the most precious in my life, and the most caring, my father.

The person who has strengthened me with her prayers, blessed me with her love, encouraged me with her hope, and the cause of happiness in my life, my loving mother.

My beloved brothers ” Diyaa, Noor, Bader, Jinan and Rayyan ”.

My lovely sister ” Anwaar ”.

All my family.

My friends who encourage and support me.

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Also, I thank Dr. Ahmed Hasasne and Dr. Sameer Matter for supporting and helping me.

Last but not least, I would like to thank my family for supporting me.

إقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان :

Analytical and Numerical Aspects of Wavelets

أقر بأن ما اشتملت عليه هذه الرسالة إنما هي نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد، وأن هذه الرسالة ككل، أو أي جزء منها لم يقدم من قبل لنيل أي درجة علمية أو بحث علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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Table of contents

Analytical and Numerical Aspects of Wavelets

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Abstract

Almost every physical phenomenon can be described via a waveform –a function of time, space or some other variables, in particular, sound waves. The Fourier transform gives us a unique and powerful way of viewing these waveforms.

Nowadays, wavelet transformation is one of the most popular candidates of the time-frequency-transformations. There are three types of wavelet transforms, namely: continuous, discrete and fast wavelet transforms.

In this work we will study Fourier transform together with its properties and present the connections between Fourier transform and wavelet transform. Moreover, we will show how the Wavelet-Galerkin method can be used to solve ordinary differential equations and partial differential equations. For the applications of wavelet transform we will consider two applications; first signal decomposition and reconstruction: in this section we use two filters to decompose a signal using the wavelet decomposition algorithm and then we use similar process to rebuild the original signal using the wavelet reconstruction algorithm. A second application is the audio fingerprint. Assume we have an audio. We read this audio and then convert it into signals. These signals are then divided into a number of frames. Next, we decompose each frame of this audio signal into five layer wavelets. Finally we use the wavelet coefficients to compute the variance, zero crossing, energy and centroid.

Introduction

Wavelets have been initially introduced in the beginning of 1980's. They were developed in their initial stage in France by the so called " French School " by J. Morlet [18], A. Grossmann [33] and Y. Meyer [17].

In 1807, the French mathematician, Joseph Fourier, discovered that all periodic functions could be expressed as a weighted sum of basic trigonometric functions. The first known connection to modern wavelets dates back to Joseph Fourier.

The concepts of a wavelet, which was not introduced until the beginning of the 1980's, was first studied by Alfred Haar [1] in 1909, afterwards called the Haar wavelet.

Wavelets, or '' Ondelettes '' as they are called in French, are used as a tool for signal analysis for seismic data [7, 18]. They were introduced in seismology to provide a time dimension to seismic analysis, where Fourier analysis fails [18].

The name wavelet comes from the requirement that should integrate to zero, waving above and below x-axis [23]. Wavelets are mathematical tools that cut up data or functions into different frequency components, and then study each component with a resolution matching to its scale [1, 18].

In 1981, Morlet teamed up with Alex Grossmann developed the continuous wavelet transform in 1984 [21].

In 1985, Yves Meyer discovered the first smooth orthogonal wavelet basis functions with better time and frequency localization [23].

In 1986, Stephane Mallat, a former student of Yves Meyer, collaborated with Yves Meyer to develop multiresolution analysis theory (MRA), discrete wavelet transform and wavelet construction techniques [1, 12].

Ingrid Daubechies became involved in 1986. She introduced the interaction between signal analysis and the mathematical aspects of dilations and translations [11].

A major breakthrough was provided in 1988 when Daubechies managed to construct a family of orthonormal wavelets with compact support. This result was inspired by the work of Meyer and Mallat in the field of multiresolution analysis [7, 21]. Since then, mathematicians, physicists and applied scientists became more and more excited about the ideas.

Wavelets are currently being used in fields such as signal and image processing, human and computer vision, data compression, and many others.

This thesis is organized as follows:

In chapter one, we study the Fourier transform and wavelet transform. Types of wavelet transform, namely: continuous, discrete and fast wavelet transform will be considered in chapter two. Chapter three includes multiresolution analysis and solving ordinary differential equations and partial differential equations using Wavelet-Galerkin Methods. In chapter four, we present some applications of wavelets. These include: decomposition and reconstruction of signals and the audio fingerprint.

Chapter One

Fourier Transform and Wavelet Transform

Chapter One

Fourier Transform and Wavelet Transform

1.1 Introduction

Frequency measures how often a thing repeats over time [3]. A frequency domain is a plane on which signal strength can be represented graphically as a function of frequency, instead of a function of time. All signals have a frequency domain representation. In 1822, Baron Jean Baptiste Fourier detailed the theory that any real world waveform can be generated by the addition of sinusoidal waves. This was arguably proposed first by Gauss in 1805. Signals can be transformed between the time and the frequency domain through various transforms.

1.2 Fourier Transform

A wave is usually defined as an oscillation function of time or space, such as a sinusoid. The Fourier transform is a tool that breaks a waveform (a function or signal) into an alternate representation, characterized by sines and cosines. Any waveform can be re-written as the sum of sinusoidal functions as the Fourier transform shows.

The Fourier Transform of a function $h(t)$ is defined by

$$F\big(h(t)\big) = H(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi i f t}dt \qquad (1.1)$$

$H(f)$ gives how much power $h(t)$ contains at the frequency f, and is often called the spectrum of h. The result of Eq.(1.1) is a frequency or function of f. We can define the inverse of Fourier transform as:

$$F^{-1}\big(H(f)\big) = h(t) = \int_{-\infty}^{\infty} H(f)e^{2\pi i f t}df \qquad (1.2)$$

Eq.(1.2) states that we can obtain the original function $h(t)$ from the function $H(f)$. As a result, $h(t)$ and $H(f)$ form a Fourier pair, that is, they are distinct representations of the same underlying identity [27].

we can write this equivalence via the following symbol : $h \overset{F}{\Leftrightarrow} H$

Definition (1.1)

The amplitude of a signal is its maximum value.

Example (1)

The signal $f(t) = 5\cos\left(\frac{\pi}{2}t\right)$ has an amplitude 5 as shown in Figure 1.1

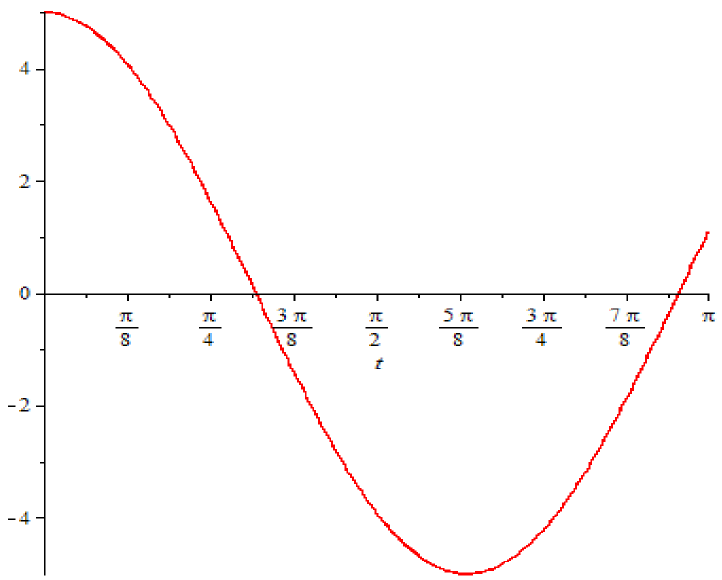


Figure 1.1 $f(t) = 5\cos\left(\frac{\pi}{2}t\right)$

The Fourier transform can be illustrated by the so called a box function (square pulse or square wave) [8].

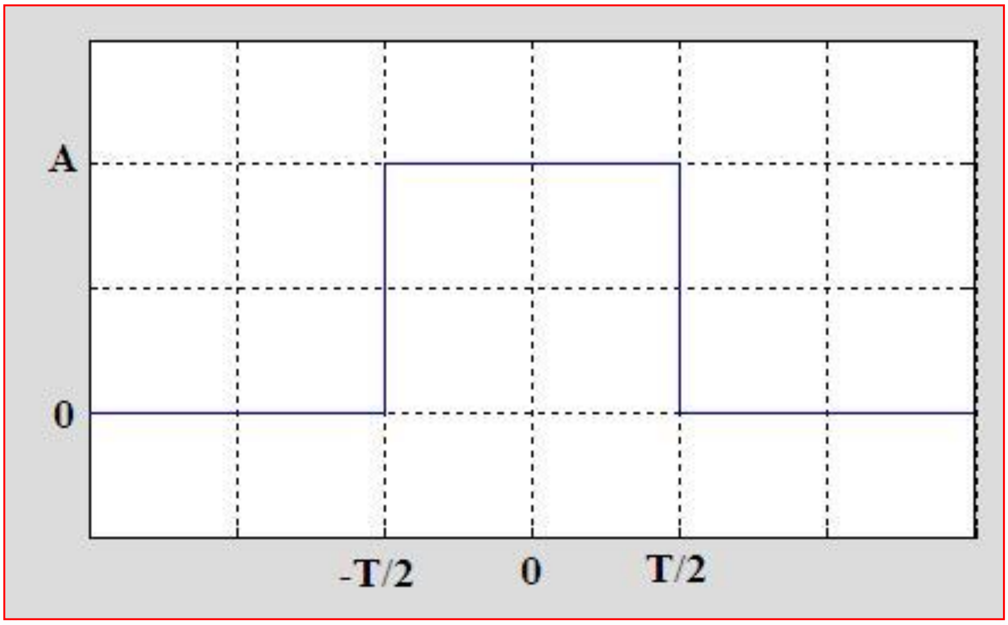


Figure 1.2 The box function

In Fig.(1.2) , the function $h(t)$ has amplitude of A, and extends from

$$t = -\frac{T}{2}\ to\ \ t = \frac{T}{2}\ \text{. For } |t|\ > \frac{T}{2}\ , h(t) =\ 0$$

Using the definition of the Fourier transform (eq. (1.1)) for calculating $H(f)$, the integral is:

$$F\big(h(t)\big) = H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-2\pi i f t} dt$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} A e^{-2\pi i f t} dt$$

$$= \frac{A}{-2\pi i f}\ \left[e^{-2\pi i f t}\right]_{\frac{-T}{2}}^{\frac{T}{2}}$$

$$= \frac{A}{-2\pi i f}\left[e^{-\pi i f T} - e^{\pi i f T}\right]$$

$$= \frac{AT}{\pi f T}\left[\frac{e^{\pi i f T} - e^{-\pi i f T}}{2i}\right]$$

$$= \frac{AT}{\pi f T}\sin(\pi f T)$$

$$= AT[sinc(fT)]\,.$$

The solution, $H(f)$ is often written as a sinc function, defined as : $sinc(t) = \frac{\sin(\pi t)}{\pi t}$.

Fig. 1.3 shows the Fourier transform of the box function such that the Fourier transform of $h(t)$ is $H(f)$

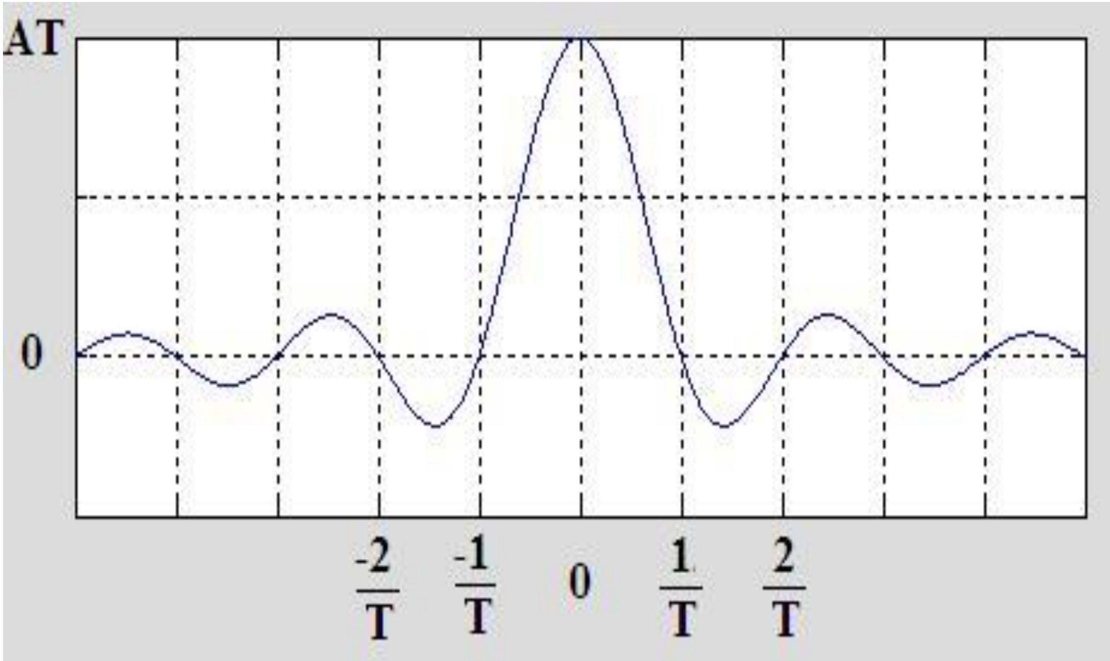


Figure 1.3 The sinc function is the Fourier Transform of the box function

We can illustrate the Fourier transform by considering the square pulses defined for T=10, and T=1. The box functions with their Fourier transforms are shown in Figures 1.4 and 1.5 for the amplitude A=1.

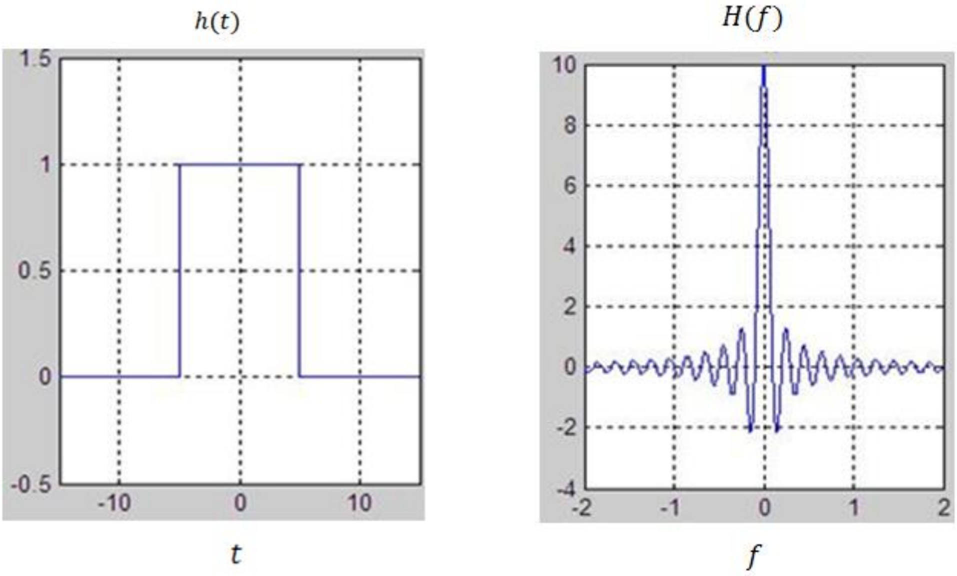


Figure 1.4 The box function with T=10, and its Fourier transform.

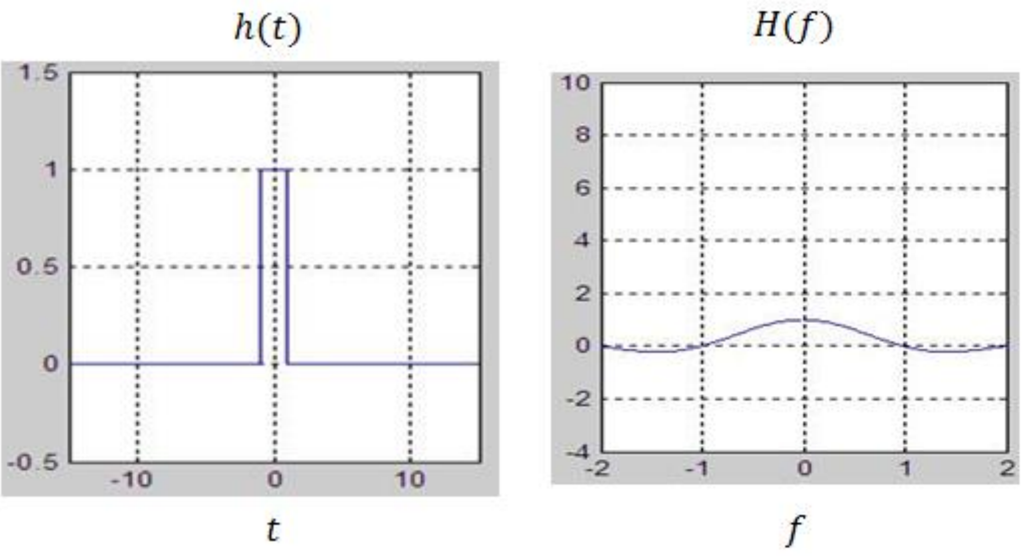


Figure 1.5 The box function with T=1, and its Fourier transform.

The wider square pulse produces a narrower and more constrained spectrum (the Fourier Transform) as shown in Figure 1.4. Figure 1.5, shows that a thinner square pulse produces a wider spectrum than that of Figure 1.4. In general: rapidly changing functions require more high frequency content (as in Figure 1.5). Functions that are moving more slowly in time will have less high frequency energy (as in Figure 1.4).

Moreover, when the box function is shorter in time (as Figure 1.5), so that it has less energy, there appears to be less energy in its Fourier transform [8].

1.2.1 Properties of Fourier Transform [3, 4, 14]

1) Linearity of Fourier Transform

Let $g(t)$ and $h(t)$ be two functions where Fourier transforms are given by $G(f)$ and $H(f)$, respectively. Then the Fourier transform of any linear combination of g and h is given as:

$$F\{\, b_1 g(t) + b_2 h(t) \,\} = \; b_1 G(f) + b_2 H(f) \tag{1.3}$$

b_1 *and* b_2 are any constants (real or complex numbers). Eq.(1.3) can easily be shown by using the definition of the Fourier transform :

$$\begin{aligned} F\{\, b_1 g(t) + b_2 h(t) \,\} &= \int_{-\infty}^{\infty} [b_1 g(t) + b_2 h(t)] e^{-2i\pi f t}\, dt \\ &= \int_{-\infty}^{\infty} b_1 g(t) e^{-2i\pi f t}\, dt + \int_{-\infty}^{\infty} b_2 h(t) e^{-2i\pi f t}\, dt \\ &= b_1 \int_{-\infty}^{\infty} g(t) e^{-2i\pi f t}\, dt + \; b_2 \int_{-\infty}^{\infty} h(t) e^{-2i\pi f t}\, dt \\ &= b_1 G(f) + b_2 H(f). \end{aligned}$$

2) Shift Property of the Fourier Transform

The time shift is defined as:

$$F\{\,h(t-c)\} = \int_{-\infty}^{\infty} h(t-c)e^{-2i\pi ft}\,dt \qquad (1.4)$$

$$= \int_{-\infty}^{\infty} h(u)e^{-2i\pi f(u+c)}\,du$$

$$= e^{-2i\pi fc}\int_{-\infty}^{\infty} h(u)e^{-2i\pi fu}\,du$$

$$= e^{-2i\pi fc}H(f)\,.$$

if the original function $h(t)$ is shifted in time by a constant amount, then it should have the same magnitude of the spectrum, $H(f)$ (see Eq.(1.4)).

3) Scaling Property of the Fourier Transform

Let $h(t)$ have Fourier transform $H(f)$ scaled in time by a non-zero constant a, written as $h(at)$. The Fourier transform will be given by:

$$F\{\,h(at)\,\} = \frac{H\left(\frac{f}{a}\right)}{|a|} \qquad (1.5)$$

we can prove Eq.(1.5) by using the definition :

$$F\{\,h(at)\} = \int_{-\infty}^{\infty} h(at)e^{-2i\pi ft}\,dt$$

Substitute: $u = at\ \ , du = adt$

$$F\{\,h(at)\} = \int_{-\infty}^{\infty} \frac{h(u)}{a}e^{-2i\pi f\frac{u}{a}}du$$

Now, $if\ a\ is\ positive$, and $f\ > 0$ then

$$F\{\,h(at)\} = \int_{-\infty}^{\infty} \frac{h(u)}{a} e^{-2i\pi f \frac{u}{a}}\, du = \frac{H\left(\frac{f}{a}\right)}{a}$$

$if\ \ a\ is\ negative$,

$$F\{\,h(at)\,\} = \int_{\infty}^{-\infty} \frac{h(u)}{a} e^{-2i\pi f \frac{u}{a}}\, du = -\int_{-\infty}^{\infty} \frac{h(u)}{a} e^{-2i\pi f \frac{u}{a}}\, du = \frac{H\left(\frac{f}{a}\right)}{-a}$$

$$\rightarrow F\{\,h(at)\,\} = \frac{H\left(\frac{f}{a}\right)}{|a|}.$$

4) Derivative Property of the Fourier Transform

The Fourier transform of the derivative of $h(t)$ is given by:

$$F\left\{\frac{dh(t)}{dt}\right\} = 2i\pi f \times H(f) \qquad (1.6)$$

5) Convolution Property of the Fourier Transform

The convolution of two piecewise continuous functions $g(t)$ and $h(t)$ on $(-\infty, \infty)$ is a function in time defined by:

$$g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau \qquad (1.7)$$

The Fourier transform of the convolution of $g(t)\ and\ h(t)$ [with corresponding Fourier transform $G(f)\ and\ H(f)$] is given by :

$$F\{\,g(t) * h(t)\} = G(f)H(f) \qquad (1.8)$$

6) Modulation Property of the Fourier Transform

A function is " modulated " by another function if they are multiplied in time. The Fourier transform of the product is the convolution of the two functions in the frequency domain :

$$F\{\, g(t)h(t) \,\} = G(f) * H(f) \qquad (1.9)$$

7) Parseval's Theorem

We've seen how the Fourier transform gives a unique representation of the original underlying signal, $h(t)$. That is, $H(f)$ contains all the information about $h(t)$. To further cement the equivalence, we present Parseval's Identity for Fourier Transforms.

Let $h(t)$ have Fourier transform $H(f)$, then the following equation holds:

$$\int_{-\infty}^{\infty} |h(t)|^2 \, dt = \int_{-\infty}^{\infty} |H(f)|^2 \, df \qquad (1.10)$$

The integral of the squared magnitude of a function is known as the energy of the function. The Parseval’s identity states that the energy of $h(t)$ is the same as the energy contained in $H(f)$, as shown in eq.(1.10)

8) Duality

Suppose $h(t)$ has Fourier transform $H(f)$. Then the Fourier transform of the function $H(t)$ is calculated by :

$$F\{\, H(t) \,\} = h(-f) \qquad (1.11)$$

This is known as the duality property of the Fourier transform.

1.3 Wavelet Transform

Definition (1.2)

Let $p \geq 1$ be a real number. Then the $L^p - space$ is the set of all real-valued functions f on a domain I such that

$$\int |f(x)|^p \, dx < \infty \quad , over\ I$$

If $f \in L^p(I)$, then its $L^p - norm$ is defined as:

$$\|f\|_p = \left(\int |f(x)|^p \, dx\right)^{\frac{1}{p}}$$

Example (2)

If $I = \mathcal{R}$, then the space $L^2(\mathcal{R})$ is the set of all square integrable functions f on $\mathcal{R}$ with L^2 −norm defined by

$$\|f\|_2 = \left(\int |f(x)|^2 \, dx\right)^{\frac{1}{2}} < \infty$$

and the function is said to have finite energy.

A wavelet is a function that is localized in time and frequency with zero mean.

An oscillatory function $\psi(t) \in L^2(\mathcal{R})$, with zero mean is a wavelet if it has the following desirable properties :

1. Smoothness [6]: $\psi(t)$ is n times differentiable and the derivatives are continuous. This smoothness of the wavelet increases with the number of vanishing moment.

2. The important property which gave wavelets their name $i.e.$ the admissibility condition. It can be shown that $\psi(t)$ satisfies the admissibility condition if

$$\int \frac{|\psi(\omega)|^2}{|\omega|}\, d\omega \; < \; +\infty$$

$\psi(\omega)$ is the Fourier transform of $\psi(t)$. Now, by using the admissibility condition we can write $|\psi(\omega)^2|\big|_{\omega=0} = 0$, this means that the Fourier transform of $\psi(t)$ vanishes at the zero frequency.

A zero at the zero frequency means that the average value of the wavelet in the time domain must be zero

$$\int \psi(t)\, dt = 0$$

and therefore it must be oscillatory. In other words, $\psi(t)$ must be a wave [18, 25, 34].

3. A wavelet must have finite energy [22]

$$E = \int_{-\infty}^{\infty} |\psi(t)|^2 \; dt < \; \infty$$

Definition (1.3) [33]

A function $\psi \in L^2(\mathcal{R})$ which satisfies admissibility condition is called an (admissible) wavelet.

Definition (1.4) [11]

Given a function $g(t)$, we define the following:

1. Translation : $T_a g(t) = g(t-a) \quad for\ all\ a \in \mathcal{R}$.
2. Modulation : $M_a g(t) = e^{2\pi iat} g(t) \quad for\ all\ a \in \mathcal{R}$.

3. Dilation : $D_a g(t) = |a|^{-\frac{1}{2}} g\left(\frac{t}{a}\right)$ $\ for\ all\ a \in \mathcal{R}\ /\{0\}$.

Applying the translation and dilation operations on a wave can take place as follows:

$$T_b D_a g(t) = |a|^{-\frac{1}{2}} g\left(\frac{t-b}{a}\right) \quad and \quad D_a T_b g(t) = |a|^{-\frac{1}{2}} g\left(\frac{t}{a} - b\right).$$

Example (3)

The signal $f(t) = 4\cos(\frac{\pi}{3}t)$ has amplitude 4 as in figure 1.6, and has two translation and dilation as in figure 1.7

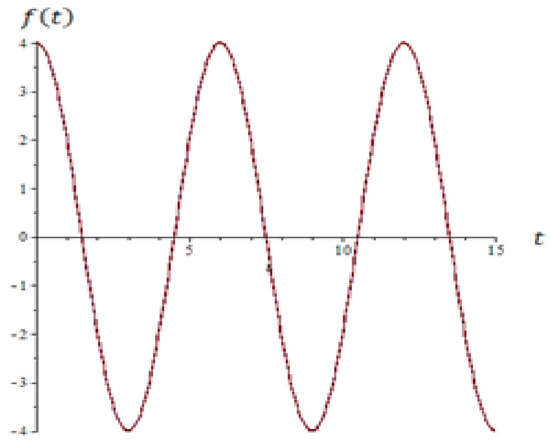
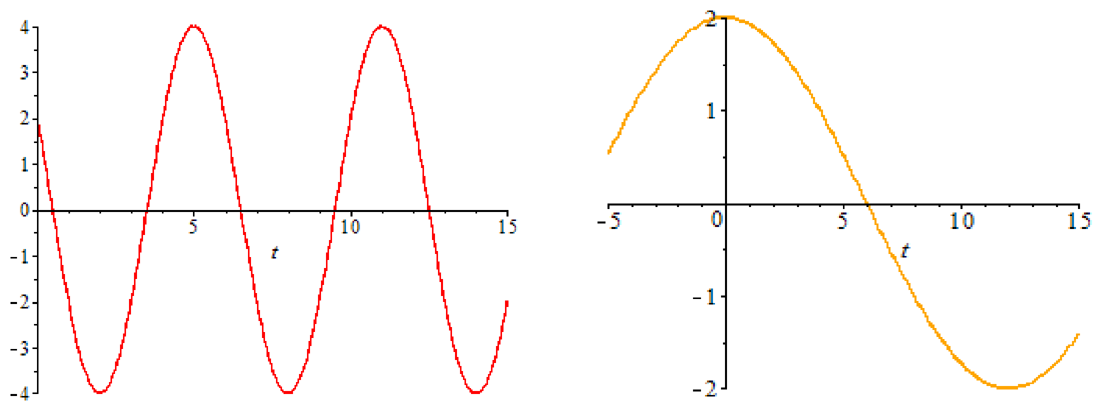


Figure 1.6 $f(t) = 4\cos(\frac{\pi}{3}t)$



$a)\ \ f(t) = 4\cos(\frac{\pi}{3}(t-5))$ $b)\ \ f(t) = 2\cos(\frac{\pi}{12}t)$

Figure 1.7

The wavelet basis is a family of functions based on a well-localized oscillating function $\psi(t)$ of the real variable t.

A wavelet is a function with zero average [19]:

$$\int_{-\infty}^{\infty} \psi(t)\, dt = 0 \tag{1.12}$$

Eq.(1.13) shows the family of functions generated from ψ by translation and dilation.

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\, \psi\left(\frac{t-b}{a}\right) \quad , a > 0\,, b \in \mathbb{R} \tag{1.13}$$

where b is the translation variable and a is the dilation variable ,

$\psi(t) \in L^2(\mathcal{R})$.

The function $\psi_{a,b}$ is called "wavelets" and sometimes called "mother wavelet" because all other wavelet functions within the family are generated from translations and dilations of $\psi(t)$. Actually, if the function has a dilation 1 and a zero translation then it is called the mother wavelet. Note that ψ is assumed to be real.

The input can be a real or complex function and the output also may be real or complex. In Fourier transform the input can be a real or complex function but its output is always complex [5].

A restriction on $\psi(t)$ is that it has a zero integral. Actually, a further restriction on $\psi(t)$ requires that the first $k+1$ moment vanish [17]. This gives a series of integral moments equal to zero, that is:

$$\int_{-\infty}^{\infty} \psi(t)\, dt = \;\ldots\ldots\ldots = \int_{-\infty}^{\infty} t^k\, \psi(t) dt = 0$$

A classical example of a wavelet is the Mexican hat function

$$\psi(t) = (1 - 2t^2)e^{-t^2}$$

Being the second derivative of a Gaussian, it has two vanishing moments, and $\psi(x)$ satisfies (1.12) [5].

Definition (1.5) [15]

Let U be an open set in $\mathcal{R}^n$, and let $f : U \to \mathcal{R}$ be a continuous function,

the support of f is $supp\, f = \left\{ \overline{x \in U : f(x) \neq 0} \right\}$.

The function f is compactly supported if $supp\, f$ is bounded**.**

Now, we say that ψ has compact support on I if it's vanish outside this interval.

1.3.1 Mother wavelet

For practical applications, and for efficiency reasons, one prefers continuously differentiable functions with compact support as mother wavelet. However, to satisfy analytical requirements and in general for theoretical reasons, one chooses the wavelet functions from a subspace of the space $L^1(\mathcal{R}) \cap L^2(\mathcal{R})$. This is the space of measurable functions that are absolutely and square integrable with

$$\int_{-\infty}^{\infty} |\psi(t)|\, dt < \infty \;\; and \;\; \int_{-\infty}^{\infty} |\psi(t)|^2\, dt < \infty$$

This space ensures that one can formulate the conditions of square norm one and zero mean:

$\int_{-\infty}^{\infty} \psi(t)dt = 0$ (condition for zero mean) , and

$\int_{-\infty}^{\infty} |\psi(t)|^2\, dt = 1$ (condition for square norm one [21]).

For ψ to be a wavelet for the continuous wavelet transform, the mother wavelet must satisfy the admissibility criterion in order to get a stable invertible transform.

In most situations, it is useful to restrict ψ to a continuous function with a higher number M of vanishing moments [17], i.e. for all integers $m < M$

$$\int_{-\infty}^{\infty} t^m\, \psi(t)dt = 0$$

The mother wavelet is dilated (or scaled) by a factor of a and translated (or shifted) by a factor of b to give:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$

1.3.2 Wavelet Transform

The wavelet transform is a tool that cuts up data or functions or operators into different frequency components . The wavelet transform of a signal evolving in time depends on two variables: scale (or frequency) and time, it provides a similar time-frequency description of Fourier transform with a few important differences, and is defined as [7, 18]:

$$W(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(t)\,\psi\left(\frac{t-b}{a}\right) dt \qquad (1.14)$$

The wavelet coefficients $C_{j,k}$ are given by

$$C_{j,k} = W\left(2^{-j}, k2^{-j}\right) \qquad (1.15)$$

here $a = 2^{-j}$ is called the dyadic dilation or binary dilation and $b = k2^{-j}$ is the dyadic or binary position.

and we define:

$$W_{r,s} = a_0^{-\frac{r}{2}} \int_{-\infty}^{\infty} f(t)\,\psi(a_0^{-r}\,t - sb_0)dt \qquad (1.16)$$

In Eq. (1.14) and (1.16) it is assumed that ψ satisfies Eqn. (1.12).

Formula (1.16) is obtained from (1.14) by restricting $a\ and\ b$ to only discrete values: $= a_0^r$, $b = sb_0a_0^r$, with r, s ranging over $\mathbb{Z}\ and\ a_0 > 1$, $b_0 > 0$ fixed.

As a changes, the $\psi_{a,0}(t) = |a|^{-\frac{1}{2}}\ \psi\left(\frac{t}{a}\right)$ covers different frequency ranges (large values of the scaling parameter $|a|$ correspond to small frequencies or large scale $\psi_{a,0}$; small values of $|a|$ correspond to high frequencies or very fine scale $\psi_{a,0}$) [7].

Chapter Two

Types of Wavelet Transform

Chapter Two

Types of Wavelet Transform

2.1 Introduction

The wavelet transform was introduced by Morlet, who used it to evaluate seismic data. Since then, various types of wavelet transforms have been developed. There exists many different types of wavelet transform, all starting from the basic formulas (1.14) and (1.16):

1. The Continuous Wavelet Transform (CWT), also called the Integral Wavelet Transform: it performs a multiresolution analysis by contraction and dilatation of the wavelet functions.
2. The Discrete Wavelet Transform (DWT): uses filter banks for the construction of the multiresolution time-frequency plane.
3. The Fast Wavelet Transform (FWT)

2.2 Continuous Wavelet Transform (CWT)

Definition (2.1) (Parseval's Identity) [13]

Parseval's identity represents an important result on Fourier series. Namely;

$$\sum_{n=-\infty}^{\infty} |C_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

where the Fourier coefficients $C_n \; of \; f$ are given by

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-\,inx} dx$$

where f in $L^2[-\pi, \pi]$.

Definition (2.2) [17, 5]

The inner product of the vector space $L^2(\mathcal{R})$ of square integrable is defined by

$$< f(t), g(t) > = \int_{-\infty}^{\infty} f(t)g(t)dt$$

we defined:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right) \quad with\, a,b \in \mathcal{R} \;, a \neq 0 \qquad (2.1)$$

The continuous wavelet transform [33, 19] of a function $f(t) \in L^2(\mathcal{R})$ is defined as its inner product with a family of admissible wavelets $\psi_{a,b}(t)$, $i.e$ $\mathcal{W}(a,b) = < f \,, \psi_{a,b} >$ (2.2)

$$\mathcal{W}(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} \psi\left(\frac{t-b}{a}\right) f(t)\, dt \qquad (2.3)$$

where $a\ and\ b$ are the time and scale variables (respectively).

we can write (by using Parseval's identity [31, 33])

$$2\pi\, \mathcal{W}(a,b) = < \hat{f} \,, \hat{\psi}_{a,b} > \qquad (2.4)$$

where

$$\hat{\psi}_{a,b}(\omega) = \frac{a}{\sqrt{|a|}} e^{-i\omega b}\, \hat{\psi}(a\omega)$$

Suppose that the wavelet ψ satisfies the admissibility condition [34]

$$C_\psi = \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{\omega}\, d\omega \ < \ \infty$$

The inverse transform is given by

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{W}(a,b)\, \psi_{a,b}(x)\, da \frac{db}{a^2} \qquad (2.5)$$

Note that if $\psi \neq 0$, then all $\psi \in L^2(\mathcal{R})$ are admissible wavelets, $\hat{\psi}$ differentiable at 0 and $\hat{\psi}(0) = 0$. This has given ψ the name wavelet or "small wave" [7, 33].

We note that the reconstruction formula (2.5) and the expression for C_ψ in Eq.(2.5) are such that they are consistent with the definition of the wavelet family in Eq.(2.1).

In fact, the Fourier transform can be viewed as a special case of the continuous wavelet transform with the choice of the mother wavelet

$$\psi(t) = e^{-2\pi i t f} \ .$$

Definition (2.3) [33]

A function g is Hölder continuous of order β $(0 < \beta \leq 1)$ at a point t if $|g(t) - g(t+h)| = \mathcal{O}(h^\beta)$.

Now, if the continuous wavelet transform has an asymptotic behavior like

$$\mathcal{W}(c,d) = \mathcal{O}\left(c^{\frac{\beta+1}{2}}\right) \qquad for\ c \to 0$$

Then the function g is Hölder continuous of order β, $(0 < \beta \leq 1)$. The converse is true as well.

If higher order of Hölder continuous functions $(\beta \geq 1)$ exists [33]; then the number of vanishing moments of the wavelet has to be bigger than , i.e

$$\int_{-\infty}^{\infty} \psi(x)\, x^q dx = 0 \qquad for \quad 0 \leq q \leq \beta \quad and\; q \in \mathrm{Z}$$

2.2.1 Properties of Continuous Wavelet Transform

If ψ and ϕ are wavelets, and let $, g \in L^2(\mathcal{R})$, then the following properties hold:

1. Linearity : $W(\alpha f + \beta g) = \alpha W(f) + \beta W(g)\;\; , \alpha, \beta \in \mathcal{R}$.
2. Translation : $W(T_c f) = W(f)(a, b - c)$.
3. Dilation : $W(D_c f) = \frac{1}{\sqrt{c}}(Wf)\left(\frac{a}{c}, \frac{b}{c}\right)\; , \; c > 0$.
4. Symmetry : $W(f) = \overline{W_f \psi(\frac{1}{a}, \frac{-b}{a})}\quad , \; a \neq 0$.

Note that we denote W_ψ by W .

2.2.2 Popular Functions for CWT Analysis

There are two functions that are popular for CWT analysis [7, 33]:

1. The Mexican hat function

$$\psi(x) = (1 - x^2)e^{-\frac{x^2}{2}} = -\frac{d^2}{dx^2}\, e^{-\frac{x^2}{2}}$$

This is the second derivative of a Gaussian $e^{-\frac{x^2}{2}}$. This wavelet is smooth and has two vanishing moments.

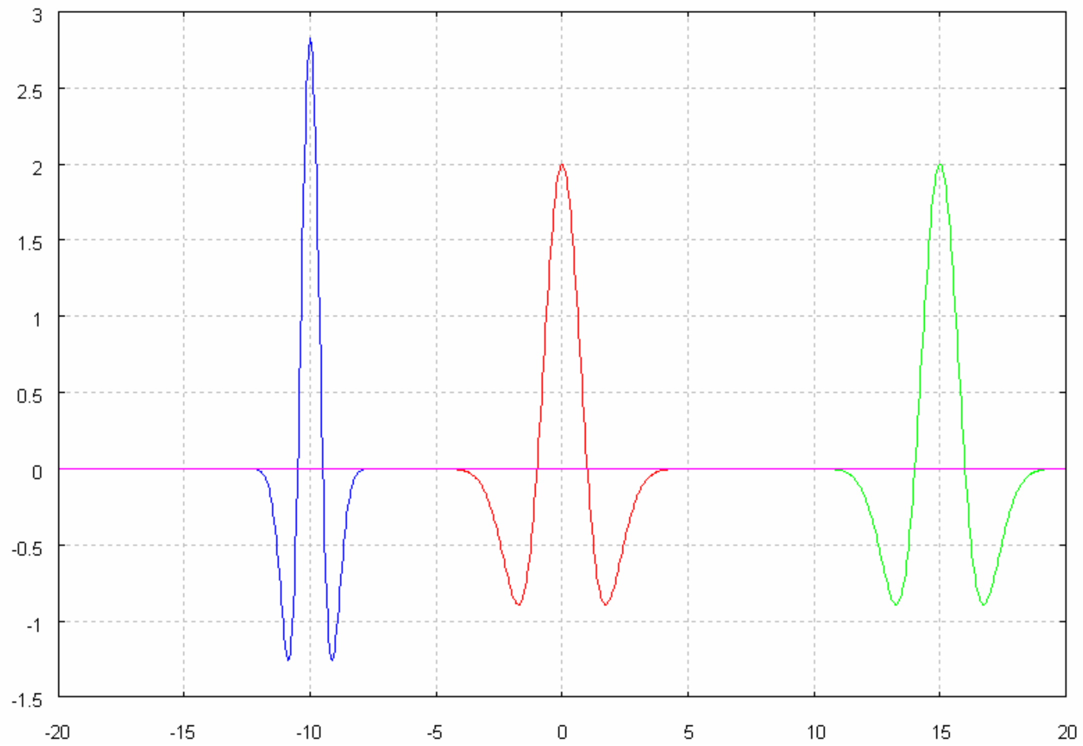


Figure 2.1 $\psi(x) = (1 - x^2)e^{-\frac{x^2}{2}}$ and its translation and dilation

$\psi(x)$ has two translated dilations.

Now, consider the Mexican hat wavelet:

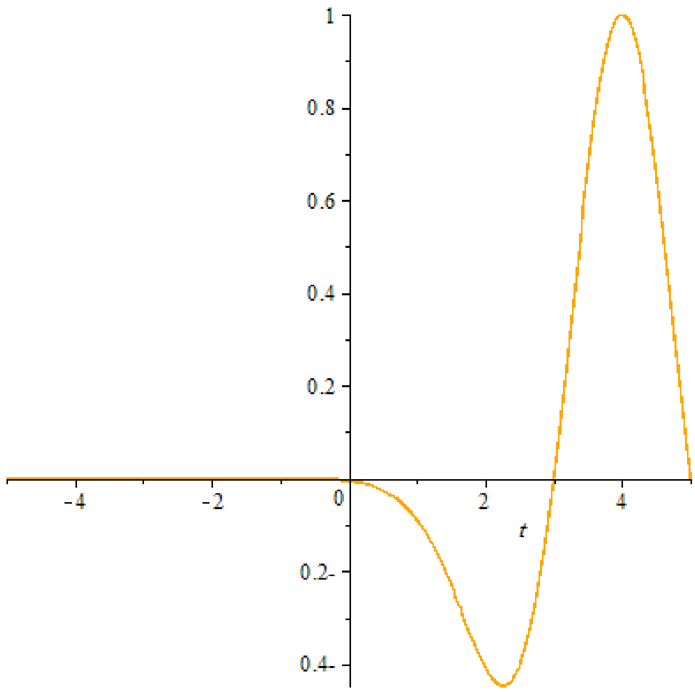
$$\psi(t) = (1 - t^2)e^{-\frac{t^2}{2}}$$

$$\psi_{a,b} = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right) = \frac{1}{\sqrt{|a|}}\left(1 - \left(\frac{t-b}{a}\right)^2\right)e^{-\frac{\left(\frac{t-b}{a}\right)^2}{2}}$$

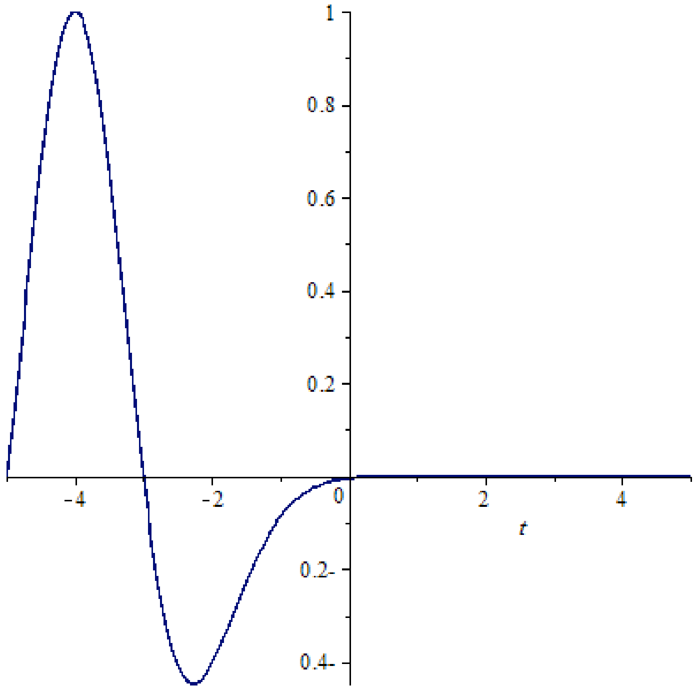
Fix $a = 1$ and the translating factor b could be any integer.

$$\psi_{1,b} = \psi(t-b) = (1 - (t-b)^2)e^{-\frac{(t-b)^2}{2}}$$

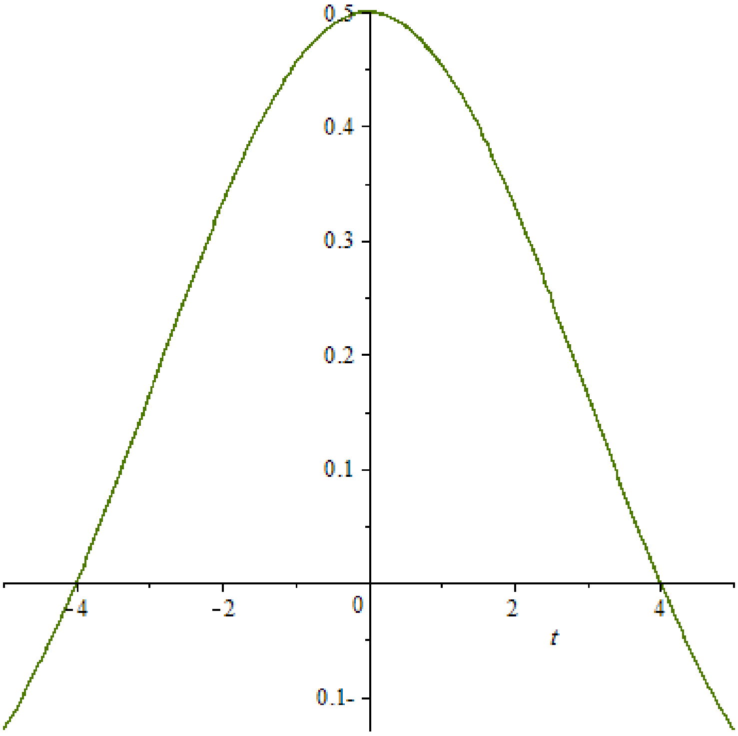
$$\psi_{1,4} = \psi(t-4) = (1 - (t-4)^2)e^{-\frac{(t-4)^2}{2}}$$



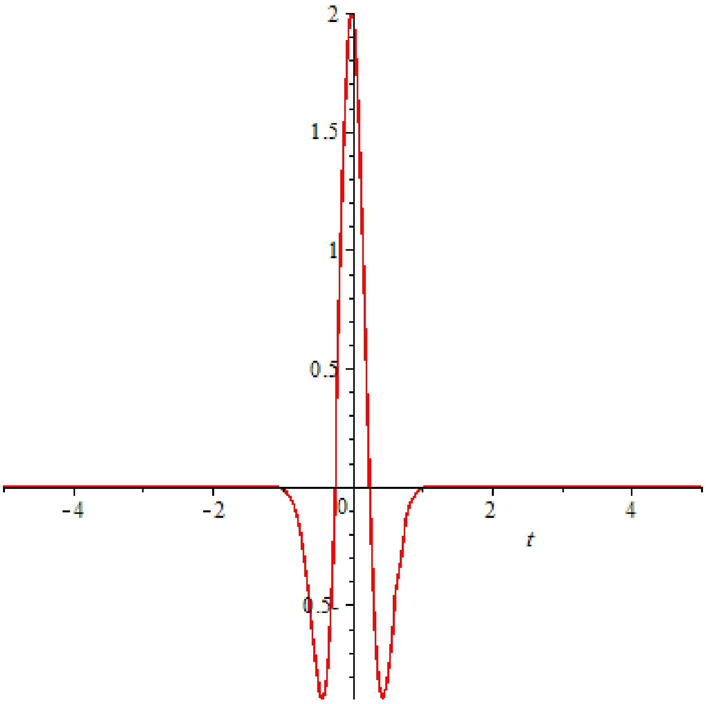
$(a)\ \psi_{1,4} = \left(1 - (t-4)^2\right)e^{-\frac{(t-4)^2}{2}}$



$(b)\ \psi_{1,-4} = \left(1 - (t+4)^2\right)e^{-\frac{(t+4)^2}{2}}$



$$(c)\ \psi_{4,0} = \frac{1}{2}\left(1 - \frac{t^2}{16}\right)e^{-\frac{t^2}{32}}$$



$$(d)\ \psi_{\frac{1}{4},0} = 2(1 - 16t^2)e^{-8*t^2}$$

Figure 2.2

Figure 2.2 (a) is the graph of the Mexican hat wavelet where $a = 1$ and $=$ 4 , Figure 2.2 (b) the same when $a = 1$ and $b = -4$.

Also, fixing $b = 0$ and the dilation parameter $\in \mathcal{R}$, and $a \neq 0$,

$$\psi_{a,0} = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t}{a}\right) = \frac{1}{\sqrt{|a|}}\left(1 - \frac{t^2}{a^2}\right) * e^{-\frac{t^2}{2*a^2}}$$

Notice, $a > 1$ extends the wave and $0 < a < 1$ shrinks the wave as we can see in Figure 2.2 (c) and (d) respectively.

2. The Morlet wavelet

$$\psi(x) = e^{i\omega_0 x}\, e^{-\frac{x^2}{2}\omega_0^2}\,.$$

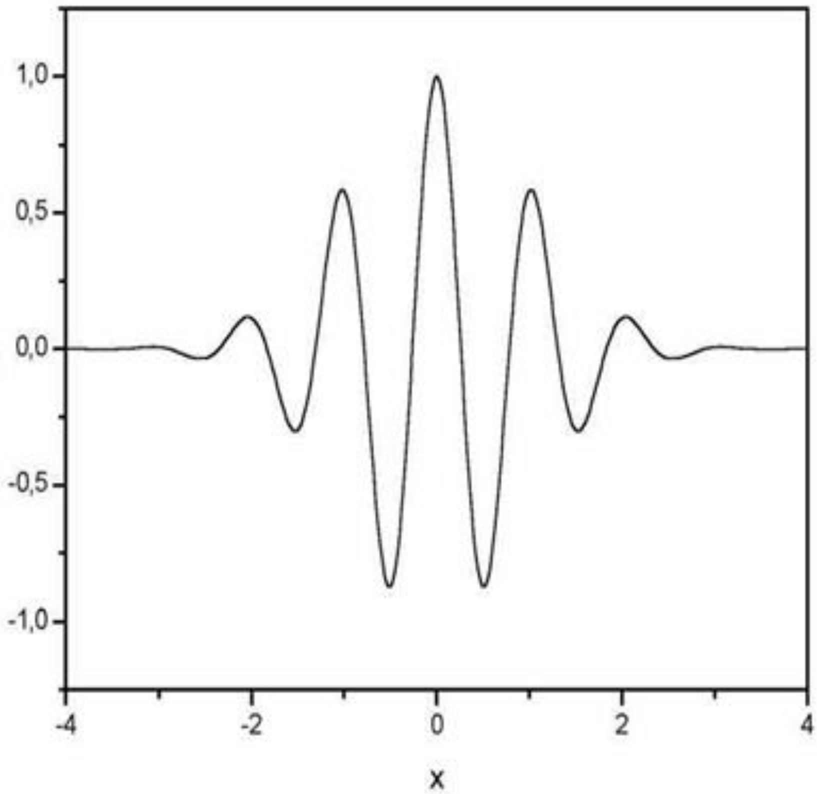


Figure 2.3 $\psi(x) = e^{i\omega_0 x}\, e^{-\frac{x^2}{2}\omega_0^2}$

(see [7, 19, 33] for more details).

2.3 Discrete Wavelet Transform (DWT)

2.3.1 Discretization of the Continuous Wavelet Transform

The discretization will allow for numerical solutions based on a summation rather than a continuous integral.

To discretize continuous wavelet transform [18, 5]

$$W(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(t)\,\psi\left(\frac{t-b}{a}\right) dt$$

we use binary discretization set $a = 2^{-j}$ and $b = 2^{-j}k$, $where\ j,k \in$ Z. The discretization of $W(a,b)$ becomes $W\left(2^{-j}, 2^{-j}k\right)$ such that $j,k \in$ Z, where the corresponding discrete wavelet functions are defined by

$$\psi_{jk}(x) = 2^{\frac{j}{2}}\,\psi\left(2^{j}x - k\right) \quad , j,k \in \ Z \tag{2.6}$$

2.3.2 Discrete Wavelet Transform

In the continuous wavelet transform, we consider the family

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right) \qquad a,b \in \mathcal{R}\ , a\ \neq 0$$

and ψ is admissible. In the discretization we restrict a, b to discrete values only, we choose: dilation parameter $a = a_0^u$, where $u \in$ Z, dilation step $a_0 \neq 1$ fixed and assume $a_0 > 1$.

Now if $u = 0$, we can discretize b by choosing integer (positive and negative) multiples of one fixed b_0 such that $b_0 > 0$, so that the $\psi(x - vb_0)$ "cover" the whole line.

$width\,(f)$ can be measured by using the formula $width\,(f) = [\int x^2\,|f(x)|^2\,dx]^{\frac{1}{2}}$ where it is assumed that $\int x\,|f(x)|^2\,dx = 0$. Now, for different values of , the width of $a_0^{-u/2} * \psi(a_0^{-u}x)$ is $a_0^u * width\,(\psi(x))$.

Choosing $b = vb_0a_0^u$ will ensure that the discretized wavelets at level u "cover" the line in the same way that the $\psi(x - vb_0)$ does.

Thus we choose $a = a_0^u\,, b = vb_0a_0^u$, where u, v range over Z, and $a_0 > 1$, $b_0 > 0$ are fixed; the appropriate choices for a_0, b_0 depend, of course, on the wavelet ψ [7]. This corresponds to

$$\psi_{u,v}(x) = a_0^{-\frac{u}{2}}\,\psi\left(\frac{x - vb_0a_0^u}{a_0^u}\right)$$

$$= a_0^{-\frac{u}{2}}\psi(a_0^{-u}\,x - vb_0)\,.$$

Discrete wavelets are not continuously scalable and translatable but can only be scaled and translated in discrete steps. This is achieved by modifying the wavelet representation

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\,\psi(\frac{t - b}{a})$$

to create

$$\psi_{j,k}(t) = \frac{1}{\sqrt{s_0^j}}\,\psi\left(\frac{t - k\tau_0 s_0^j}{s_0^j}\right) \tag{2.7}$$

where j and k are integers and $s_0 > 1$ is a fixed dilation step. τ_0 (the translation factor) depends on the dilation step. $\psi_{j,k}(t)$ is called a discrete wavelet, which is normally a (piecewise) continuous function. The effect of discretizing the wavelet is that the time-scale space is now sampled at discrete intervals [34].

Example (1)

We introduce a function that satisfies the admissibility condition, namely the Haar wavelet. The Haar wavelet is defined by:

$$\psi(x) = \begin{cases} 1\ , & x \in \left[0, \frac{1}{2}\right) \\ -1\ , & x \in \left[\frac{1}{2}, 1\right) \\ 0\ , & otherwise \end{cases}$$

Note that the Haar wavelet is discontinuous, has compact support, and has a zero mean.

We show $\psi(x)$ is an admissible wavelet by computing

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}}\left(\int_0^{\frac{1}{2}} e^{-ix\omega}\ dx - \int_{\frac{1}{2}}^{1} e^{-ix\omega}\ dx\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{1 + e^{-i\omega} - 2e^{-\frac{i\omega}{2}}}{i\omega}\right)$$

and so

$$\frac{\left|\hat{\psi}(b\omega)\right|^2}{b} = \frac{\left|1 + e^{-ib\omega} - 2e^{-\frac{ib\omega}{2}}\right|^2}{b^3\ \omega^2} = \frac{\left|e^{-\frac{ib\omega}{2}}\right|^2 \times \left|e^{\frac{ib\omega}{4}} - e^{-\frac{ib\omega}{4}}\right|^4}{b^3\ \omega^2}$$

$$= \frac{16sin^4(\frac{b\omega}{4})}{b^3\ \omega^2}$$

Integrating by parts yields [37]:

$$C_\psi = \lim_{m\to\infty} \int_0^{\frac{m\omega}{4}} \frac{sin^4(x)}{x^3}\ dx$$

$$= \lim_{m\to\infty} \left.\frac{-sin^4(x)}{2x^2}\right]_{x=0}^{\frac{m\omega}{4}} + \lim_{m\to\infty} \frac{1}{4} \int_0^{\frac{m\omega}{4}} \frac{2sin(2x) - \sin(4x)}{x^2}\ dx$$

$$= \lim_{m\to\infty} \left.\frac{\sin(4x) - 2\sin(2x)}{4x}\right|_{x=0}^{\frac{m\omega}{4}} + \lim_{m\to\infty} \int_0^{\frac{m\omega}{4}} \frac{\cos(2x) - \cos(4x)}{x}\ dx$$

$$= \lim_{m\to\infty} \left(\int_0^{\frac{m\omega}{4}} \frac{\cos(2x) - 1}{x}\ dx - \int_0^{\frac{m\omega}{2}} \frac{\cos(4x) - 1}{x}\ dx \right)$$

$$= ln2$$

The Discrete Wavelet Transform (DWT) of $g(t)$ with respect to a wavelet $\psi(t)$ is defined as [2]

$$d(u,v) = \frac{1}{{a_0}^{\frac{u}{2}}} \int g(t)\ \psi({a_0}^{-u}\ t - vb_0)dt \qquad (2.8)$$

where u : dilation parameter, v : translation parameter, a_0, b_0 depend on the wavelet used.

As a special case, if $a = 2^u$ and $= v2^u$, the wavelet expressed as

$$\psi_{a,b}(t) = \psi\left(\frac{t-b}{a}\right) = \psi\left(\frac{t-v2^u}{2^u}\right) = \psi(2^{-u}t - v) = \psi_{u,v}(t)$$

which is the dilated and translated version of the mother wavelet.

The DWT is:

$$d(u,v) = \frac{1}{2^{u/2}} \int_{2^u v}^{2^u(v+1)} g(t)\ \psi(2^{-u}\, t - v)\, dt$$

Here, $d(u,v)$ is equivalent to CWT $W(a,b)$ when $a = 2^u$ and $= v2^u$. Inverse operation (IDWT) is expressed as:

$$g(t) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} d(u,v) 2^{-\frac{u}{2}}\ \psi(2^{-u}\, t - v)$$

The DWT is computed by successive low pass and high pass filtering (also known as filter bank) of the discrete time-domain signal along with down sampling by two provides approximation and detail components.

Now, if the impulse response of high pass filter is $g(v)$ and low pass filter is $h(v)$ then we can express a two level decomposition tree of DWT by the Fig. 2.4. The reverse process of decomposition yields the reconstruction of original sequence $x(v)$. Fig. 2.5 shows two levels wavelet reconstruction tree. The approximation and detail coefficients are up sampled by two at every level then passed through the high pass and low pass synthesis filters of impulse response of $\acute{g}(v)$ and $\acute{h}(v)$ then added. This process is continued to obtain the original signal through the same number of levels as in the decomposition process [29, 31].

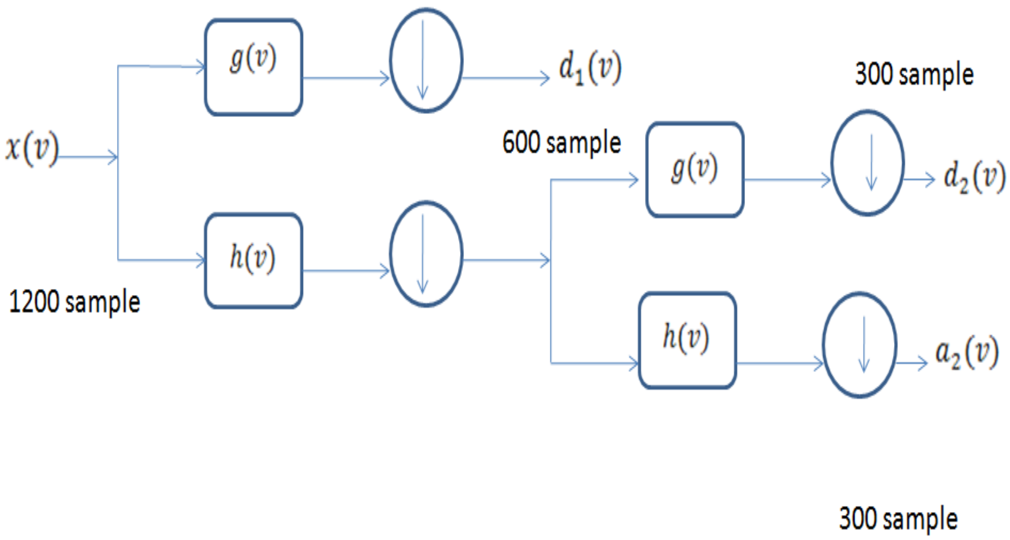


Fig. 2.4 Decomposition tree of DWT

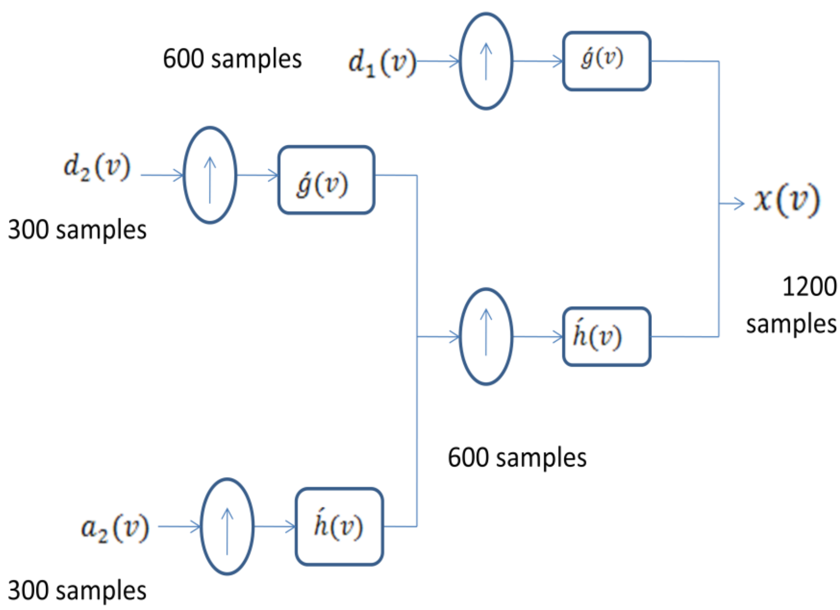


Fig. 2.5 Reconstruction tree of DWT

Note that, formulas (2.3) and (2.8) assume that ψ satisfies

$$\int \psi(t)dt = 0$$

We can consider the formal wavelet definitions, with the Fourier transform (1.1) and (1.2):

1. A wavelet is a function $\psi(t)$ in $L^2(\mathcal{R})$ whose Fourier transform $\Psi(f)$ satisfies the condition (almost everywhere)

$$\int_0^\infty \frac{|\Psi(tf)|^2}{t}\,dt = 1$$

2. A wavelet is a function $\psi(t)$ in $L^2(\mathcal{R})$ such that $2^{p/2}\psi(2^p x - q)$, $p, q \in \mathrm{Z}$ is an orthonormal basis for $L^2(\mathcal{R})$.

For more details see ([7], [5] and [34]).

2.4 Fast Wavelet Transform (FWT)

Any wavelet function can be expressed as a weighted sum of shifted, double-resolution scaling functions. That is, we can write

$$\phi(x) = \sqrt{2}\sum_m g_\phi(m)\ \phi(2x - m)$$

where $g_\phi(m)$ are called the wavelet function coefficients and g_ϕ is the wavelet vector.

If the function being expanded is a sequence of numbers, like samples of a continuous function $f(x)$, the resulting coefficients are called the discrete wavelet transform (DWT) of $f(x)$. The DWT coefficients of $f(x)$ are defined as [5, 19]:

$$D_\phi(j,k) = \frac{1}{\sqrt{M}} \sum_x f(x)\ \phi_{j,k}(x)$$

Now consider the multiresolution equation

$$\psi(x) = \sqrt{2} \sum_m g_\psi(m)\ \psi(2x - m)$$

By scaling x by 2^j, translation of x by k units, and letting $n = 2k + m$ we would get

$$\psi\left(2^j x - k\right) = \sqrt{2} \sum_m g_\psi(m)\ \psi\left(2\left(2^j x - k\right) - m\right)$$

$$= \sqrt{2} \sum_n g_\psi(n - 2k)\ \psi\left(2^{j+1} x - n\right)$$

Similarly,

$$\phi\left(2^j x - k\right) = \sqrt{2} \sum_n g_\phi(n - 2k)\ \phi\left(2^{j+1} x - n\right)$$

Now consider the DWT coefficient functions $D_\phi(j,k)$. By changing variable we can get

$$D_\phi(j,k) = \frac{1}{\sqrt{M}} \sum_x f(x)\ 2^{\frac{j}{2}}\, \phi\left(2^j x - k\right)$$

Now replacing $\phi\left(2^j x - k\right)$, it becomes

$$D_\phi(j,k) = \frac{1}{\sqrt{M}} \sum_x f(x)\, 2^{\frac{j}{2}} \left\{ \sum_n g_\phi(n - 2k) \sqrt{2}\, \phi\left(2^{j+1} x - n\right) \right\}$$

$$D_\phi(j,k) = \sum_n g_\phi(n - 2k) \left\{ \frac{1}{\sqrt{M}} \sum_x f(x)\, 2^{\frac{j+1}{2}}\ \phi\left(2^{j+1} x - n\right) \right\}$$

where the bracketed quantity is identical to $D_{\phi}(j_0, k)$ with $j_0 = j + 1$. We can thus write

$$D_{\phi}(j,k) = \sum_{n} g_{\phi}(n - 2k)\ D_{\phi}(j + 1, k)$$

Similarly [19], the approximation coefficients are written by

$$D_{\psi}(j,k) = \sum_{n} g_{\psi}(n - 2k)\ D_{\psi}(j + 1, k)\,.$$

Chapter Three

Solving ODEs and PDEs Using Wavelets

Chapter Three

Solving ODEs and PDEs Using Wavelets

3.1 Introduction

The concept of Multiresolution is intuitively related to the study of signals or images at different levels of resolution. The resolution of a signal is a qualitative description associated with its frequency content. An objective of a Multiresolution analysis is to construct a wavelet system, which is a complete orthonormal set in $L^2(\mathcal{R})$. Many applications of mathematics require the numerical approximation of solutions of differential equations. We discuss how to solve Ordinary Differential Equation and Partial Differential Equation by Wavelet-Galerkin Method.

3.2 Multiresolution Analysis and Construction of Wavelets

The basic principle of the multiresolution analysis (MRA) deals with the decomposition of the whole function space into individual subspaces $W_m \subset W_{m+1}$ [5, 6, 7, 18, 19, 25].

Definition (3.1) [9]

The space spanned by basis function $\{\phi_i(t)\}$ is

$$span\,\{\phi_i(t)\} = \sum c_i\,\phi_i(t) \quad for\ any\ constant\ c_i\ .$$

Definition (3.2)

The wavelet set $\{\psi_{a,b}\}$ forms an orthogonal system if

$$< \psi_{a,b} , \psi_{a,c} > = \int \psi_{a,b}(t)\, \psi_{a,c}(t) dt = 0 \quad , \quad b \neq c$$

If in addition

$$< \psi_{a,b} , \psi_{a,c} > = \int \psi_{a,b}(t)\, \psi_{a,c}(t) dt = 1$$

then the system is called orthonormal.

Definition (3.3)

Given $\{\psi_{a,b}(t)\}$ is an orthogonal system, we call it complete if a function $f(t)$ satisfies $< f, \psi_{a,b} > = 0$ implies that $f \equiv 0$, or more precisely, that

$$\|f\|^2 = \int f^2(t)\, dt = 0 \,.$$

Definition (3.4) [6, 18]

A multiresolution analysis (MRA) in $L^2(\mathcal{R})$ is an increasing sequence of closed subspaces $W_j\,, j \in \mathbb{Z},\ in\ L^2(\mathcal{R})$

$$.. \subset W_0 \subset W_1 \subset W_2 \subset W_3 \subset ..$$

satisfying the properties:

1. $W_j \subset W_{j+1}$
2. Dilation property $f(t) \in W_j \leftrightarrow f(2t) \in W_{j+1}\ for\ all\ j \in \mathbb{Z}$
3. Intersection property:

$$\lim_{j \to +\infty} W_j = \bigcap_{j \in \mathbb{Z}} W_j = \{0\}.$$

4. Density Property :

$$\lim_{j\to -\infty} W_j = \overline{\bigcup_{j\in\mathbb{Z}} W_j} = L^2(\mathcal{R})$$

means $\bigcup_{j\in\mathbb{Z}} W_j$ is dense in $L^2(\mathcal{R})$.

5. Existence of a scaling function. There exists a function $\phi \in W_0$ such that $\{\ \phi(x-m) : m \in \mathbb{Z}\ \}$ is an orthonormal basis for W_0 .

$$W_0 = span\ \{\phi(x-m)\}$$

$$W_0 = \left\{ \sum_{k\in\mathbb{Z}} \beta_k\ \phi(x-m) :\ \{\beta_k\}_{k\in\mathbb{Z}}\ \in\ l^2(\mathbb{Z}) \right\}$$

the function ϕ is called the scaling function or father wavelet of the given MRA.

6. For every wavelet ψ, we can consider $W_j \in L^2(\mathcal{R})$, $\forall\ j \in$ Z,

$W_j = \cdots + V_{j-2} + V_{j-1}$ and $W_{j+1} = W_j + V_j$, $j \in$ Z .

Condition 4 means that for any $\in L^2(\mathcal{R})$, there exists a sequence $\{f_m\}_{m=1}^{\infty}$ such that each $f_m \in \bigcup_{j\in\mathbb{Z}} W_j$ and $\{f_m\}_{m=1}^{\infty}$ converges to f in $L^2(\mathcal{R})$, that is , $\|f_m - f\| \to 0\ \ as\ m \to \infty$.

In condition 5, we have a multiresolution analysis with a Riesz basis if we assume ($\{\ \phi(x-m) : m \in \mathbb{Z}\ \}$) a Riesz basis for W_0

Condition 2 implies that

$$f(x) \in W_j\ \leftrightarrow f(2^m x) \in W_{j+m}\ for\ all\ j,m\ \in \mathbb{Z}$$

in particular,

$$f(x) \in W_0 \leftrightarrow f(2^j x) \in W_j$$

The set of all possible approximations of functions at the resolution 2^{-j} represents the space W_j. MRA is then obtained by computing the approximation of signals at various resolutions with orthogonal projections onto different spaces $\{W_j\}_{j\in Z}$.

In order to calculate the approximation, the orthogonal basis of each space W_j is generated by dilating and translating a single function ϕ called scaling function, $i.e.$

$$\phi_{j,m} = 2^{-\frac{j}{2}} \phi(2^{-j}t - m) \quad , m \in \mathbb{Z} \tag{3.1}$$

Defining the orthonormal projection operator P_j from $L^2(\mathcal{R})$ onto W_j by

$$P_j(f)(x) = \sum_{k\in\mathbb{Z}} < f, \phi_{j,k} > \phi_{j,k}(x).$$

we have two cases: First, if the resolution 2^{-j} goes to 0, then the condition(3) implies that we lose all the details of f and

$$\lim_{j\to +\infty} \|P_j f\| = 0$$

Second, if the resolution 2^{-j} goes to $+\infty$, then the property (4) ensures that

$$\lim_{j\to -\infty} P_j f = f \ \ in\, L^2(\mathcal{R}), \qquad \lim_{j\to -\infty} \|f - P_j f\| = 0$$

If we replace 2^{-j} by 2^{j} in Eq.(3.1), then we have

$$\lim_{j\to+\infty} P_j(f) = f \quad and \quad \lim_{j\to-\infty} P_j\,(f) = 0$$

The projection $P_j(f)$ can be considered as an approximation of f at the scale 2^{-j} .

The real importance of a multiresolution analysis lies in the simple fact that it enables us to construct an orthonormal basis for $L^2(\mathcal{R})$ [1].

In order to prove this statement, first assume that $\{W_m\}$ is a multiresolution analysis. Since $W_0 \subset W_1$, define V_0 as the orthogonal complement of W_0 in W_1 , that is , $W_1 = W_0 \oplus V_0$.

Since, $W_m \subset W_{m+1}$, define V_m as the orthogonal complement of W_m in W_{m+1} for every $m \in \mathbb{Z}$ so that we have

$$W_{m+1} = W_m \oplus V_m \quad for\ each\ m \in \mathbb{Z}$$

Since $W_m \to \{0\}$ as $m \to -\infty$, we see that

$$W_{m+1} = W_m \oplus V_m \quad = \bigoplus_{l=-\infty}^{\infty} V_l \quad for\ all\ m \in \mathbb{Z}$$

Since $\bigcup_{j\in\mathbb{Z}} W_j$ is dense in $L^2(\mathcal{R})$, we may take the limit as $m \to \infty$ to obtain

$$L^2(\mathcal{R}) = \bigoplus_{l=-\infty}^{\infty} V_l$$

where $\oplus$ means direct sum.

To find an orthonormal wavelet, therefore, all we need to do is to find a function $\psi \in V_0$ such that $\{\, \psi(x-k) : \ k \in \mathbb{Z} \,\}$ is an orthonormal basis for V_0 . In fact, if this is the case, then

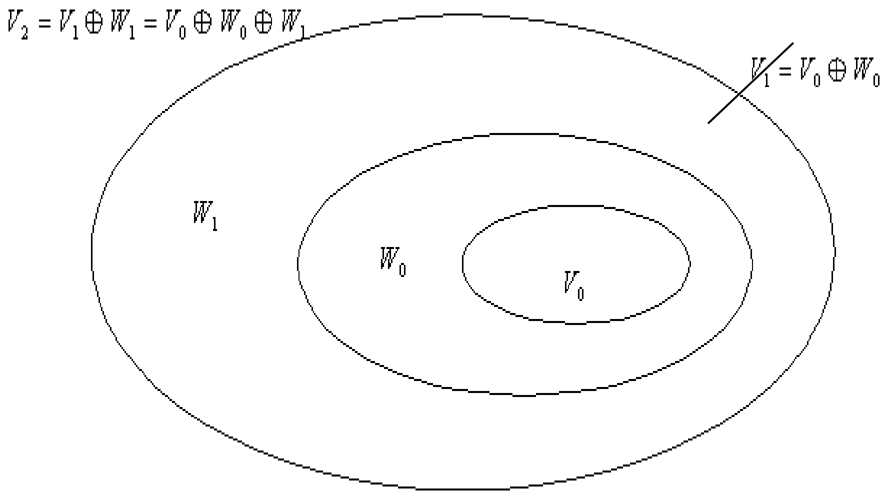
$$\left\{ \psi_{j,k}\,(x) = 2^{\frac{j}{2}}\,\psi\big(2^j x - k\big) : \ k \in \mathbb{Z} \ \right\}$$

is an orthonormal basis for V_j for all $j \in \mathbb{Z}$ due to the condition in the definition of multiresolution analysis and definition of V_j .

hence

$$\left\{ \psi_{j,k}\,(x) = 2^{\frac{j}{2}}\,\psi\big(2^j x - k\big) : \ k, j \in \mathbb{Z} \ \right\}$$

is an orthonormal basis for $L^2(\mathcal{R})$, which shows that ψ is an orthonormal wavelet on $\mathcal{R}$.



Daubechies has constructed, for an arbitrary integer N, an orthonormal basis for $L^2(\mathcal{R})$ of the form

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\,\psi\left(2^j x - k\right) : \; k, j \in \mathbb{Z}$$

that satisfies the following properties [1, 5]:

1. $supp\ (\psi_N) = [-N+1\ , N]$
2. ψ_N has λN continuous derivatives for large N , where

$$\lambda = 1 - \frac{ln3}{ln4} \cong .2075$$

3. ψ_N has N vanishing moments

$$\int_{-\infty}^{\infty} x^k\,\psi(x)dx = 0 \quad for\ k = 0,1, \dots \dots, N.$$

3.3 Wavelets and Differential Equations

We consider the class of ordinary differential equations of the form

$$L\,v(x) = f(x) \qquad for \quad x \in [0,1] \tag{3.2}$$

where L is called differential operator, represented as a finite linear combination and its derivatives containing higher degree, L is denoted by

$$L = \sum_{j=0}^{m} c_j(x)\,D^j$$

where $c_j(x)$ are the coefficients of the operator, and with appropriate boundary conditions on $v(x)$ for $x = 0{,}1$. [1, 18, 23].

Definition (3.5) [1, 23]

Let B be an $m \times m$ matrix. Define $\|B\|$ the norm of B by

$$\|B\| = Sup\,\frac{\|Bx\|}{\|x\|}$$

where the supremum is taken over all nonzero vectors in C^m. Equivalently,

$$\|B\| = Sup\,\{\,\|Bx\| : \|x\| = 1\,, x \in C^m\,\}$$

Definition (3.6) [32]

Let B be an $m \times m$ matrix. Define $C_\#(B)$, the condition number of the matrix B by $C_\#(B) = \|B\|\,\|B^{-1}\|$

If B is not invertible , the condition number of the matrix B equals ∞ .

Note that $C_\#(B)$ is scale invariant, $i.e$, for $c \neq 0$,

$$C_\#(cB) = C_\#(B)$$

Definition (3.7) [28]

The transpose of B is B^T where $B^T = \{b_{ji}\}$. Thus if B is $n \times p$ the transpose B^T is $p \times n$ with i, j element equal to the j, i element of B.

Definition (3.8) [28]

The conjugate transpose of B is B^* where $B^* = \{\,\bar{b}_{ji}\,\}$, and $\bar{b}$ is the complex conjugate of b . Thus if the order of $B\ is\ n \times p$, the conjugate transpose B^*, is $p \times n$ with i, j element equal to the complex conjugate of the j, i element of B.

If B is a matrix with real elements then $B^* = B^T$.

B is normal if $BB^* = B^*B$.

Lemma (3.1) [1]

Suppose that B is an $m \times m$ normal invertible matrix. Let

$$|\lambda|_{max} = \max \{ |\lambda| : \lambda \ is \ an \ eigenvalue \ of \ B \}$$

and

$$|\lambda|_{min} = \min \{ |\lambda| : \lambda \ is \ an \ eigenvalue \ of \ B \}$$

$$C_{\#}(B) = \frac{|\lambda|_{max}}{|\lambda|_{min}}$$

3.3.1 Wavelet–Galerkin Method for Differential Equations

The classical Galerkin methods have the disadvantage that the stiffness matrix becomes ill conditioned as the problem size grows. To overcome this disadvantage, we use wavelets as basis functions in a Galerkin method. Then, the result is a linear system that is sparse because of the compact support of the wavelets, and that, after preconditioning, it has a condition number independent of problem size because of the multiresolution structure [1, 18, 19, 23].

We consider the class of ordinary differential equations (known as Sturm–Liouville equations) of the form [17, 18]

$$Lv(t) = -c(t)\acute{\acute{v}}(t) - \acute{c}(t)\acute{v}(t) + d(t)v(t) \qquad (3.3)$$

$$Lv(t) = -\frac{d}{dt}\left(c(t)\frac{dv}{dt}\right) + d(t)v(t) = f(t)\,, \qquad 0 \le t \le 1$$

with Dirichlet boundary conditions

$$v(0) = v(1) = 0$$

Let $c(t), d(t)$ *and* $f(t)$ be real-valued functions, such that $f(t)$ *and* $d(t)$ are continuous functions and the function $c(t)$ has a continuous derivative on [0,1] .

Then, there exist finite constants A_1, A_2, A_3 such that [20]

$$0 \le A_1 \le c(t) \le A_2 \quad and \quad 0 \le d(t) \le A_3 \qquad \forall t \in [0,1]$$

the operator is called uniformly elliptic.

Note that $L^2[0,1]$ is a Hilbert space with inner product

$$< f, g > = \int_0^1 f(t)\ \overline{g(t)}\, dt$$

where $\overline{g(t)}$ is the complex conjugate of $g(t)$.

For the Galerkin method, we suppose that $\{u_j\}_j$ is a complete orthonormal system (orthonormal basis) for $L^2[0,1]$, and that every u_j is C^2 on [0,1] and it satisfies

$$u_j(0) = u_j(1) = 0 \qquad (3.4)$$

we select some finite set Γ of indices j and consider the subspace

$$S = Span\left\{ u_j : j \in \Gamma \right\}$$

$i.e$ the set of all finite linear combination of the elements $\{u_j\}, j \in \Gamma$.

looking for an approximation for v_s of the exact solution v of the equation (3.3) in the form [23]

$$v_s = \sum_{k \in \Gamma} x_k \, u_k \quad \in S \qquad (3.5)$$

where each x_k is a scalar. Our criterion for determining the coefficients x_k is that v_s should behave like the true solution v on the subspace S , that is,

$$< L\, v_s, u_j > \; = \; < f, u_j > \quad for\ all\ j \ \in \Gamma \qquad (3.6)$$

By linearity, it follow that

$$< L\, v_s, g > \; = \; < f, g \; > \quad for\ all\ g \ \in S$$

Note that the approximate solution v_s automatically satisfies the boundary conditions $v_s(0) = v_s(1) = 0$, because of equation (3.4). Using these results we get

$$< L \left(\sum_{k \in \Gamma} x_k \, u_k \right), u_j \; > = < f, u_j > \quad for\ all\ j \in \Gamma$$

and if we substitute equation (3.5) in (3.6) then we obtain

$$\sum_{k \in \Gamma} < L\, u_k \, , u_j > x_k = < f \, , u_j > \qquad for\ all\ j \in \Gamma \qquad (3.7)$$

Let x denote the vector $(x)_{k \in \Gamma}$, and y be the vector $(y_k)_{k \in \Gamma}$, where $y_k = < f \, , u_j >$, and let A be the matrix with rows and columns indexed by Γ, that is, $A = [\, a_{j,k} \,]_{j,k \in \Gamma}$, where

$$a_{j,k} = < L\, u_k \, , u_j > .$$

Thus , (3.7) is linear system of equations

$$\sum_{k \in \Gamma} a_{j,k} \;\, x_k = \; y_j \quad for\ all\ j \in \Gamma, or \quad Ax = y \qquad (3.8)$$

In the Galerkin method, for each subset Γ we obtain an approximation $v_s \in S$, by solving the linear system (3.8) for *x* and using these components to determine v_s by (3.5). We expect that as we increase our set Γ in some systematic way, our approximation v_s will converge to the exact solution v.

The nature of the linear system depends upon choosing a wavelet basis for the Galerkin method. There are two properties that the matrix A in the linear system (3.8) should have [20]:

First, A should have a small condition number to obtain stability of the solution under small perturbations in the data. Second, A should be sparse for quick calculations, which means that A should have a high proportion of entries that are 0 .

The condition number of A measures how unstable the linear system $Ax = y$ is under perturbation of the data y. In applications, a small condition number ($i.e.$ near 1) is desirable.

If the condition number of A is high, it would be convenient to replace the linear system $Ax = y$ by an equivalent system $Mz = v$ whose matrix M has a low condition number.

Now, let $\phi_{j,k}(t) = 2^{j/2}\,\phi(2^j\,t - k)$ be a wavelet basis for $L^2([0,1])$ with boundary condition

$$\phi_{j,k}(0) = \phi_{j,k}(1) = 0$$

for each $(j,k) \in \Gamma$, $\phi_{j,k}\ is\ C^2$.

The scale of ϕ approximates 2^{-j} and is centralized near point $2^{-j}k$ and equates to zero outside the interval centred at $2^{-j}k$ of length proportional to 2^{-j} .

In wavelet Galerkin method, equations (3.5) and (3.6) may thus be replaced by

$$v_s = \sum_{j,k\in\Gamma} x_{j,k}\,\phi_{j,k}$$

and

$$\sum_{j,k\in\Gamma} < L\,\phi_{j,k}\,,\phi_{l,m} > \; x_{j,k} = < f,\phi_{l,m} > \qquad \forall l,m \in \Gamma$$

So that $AX = Y$, where

$$A = \left[a_{(l,m);(j,k)}\right]_{(l,m),(j,k)\in\Gamma} \,, X = \left(x_{j,k}\right)_{(j,k)\in\Gamma} \quad , Y = \left(y_{l,m}\right)_{(l,m)\in\Gamma}$$

Now, in $a_{(l,m);(j,k)} = < L\phi_{j,k}\,,\phi_{l,m} >$, $y_{l,m} = < f\,,\phi_{l,m} >$, The pairs (l,m) and (j,k) represent respectively rows and columns of A.

Consider A to be sparse. Represent $AX = Y$ by equivalent $MZ = V$. In which case M has relatively low condition number, if A has not. This system is now well conditioned. Again M to be sparse is desirable.

The matrix M has condition number bounded independently of Γ. So we increase Γ to approximate solution with more accuracy, the condition number maintains its bounded ness [23].

3.4 Solution of Partial Differential Equations Using Wavelets

For using Wavelet-Galerkin method for PDEs, we consider the following time dependent problem [18]

$$u_t(x,t) + Lu(x,t) = f(x,t)\,, x \in \Gamma \;\,, t > 0 \qquad (3.9)$$

with boundary conditions on $u(x,t)$.

we approximate the solution of (3.9) in the following form –by using wavelets Galerkin method-

$$u(x,t) = \sum_{(j,k)\in\Lambda} a_{j,k}(t)\;\psi_{j,k}(x) \qquad (3.10)$$

substituting the solution (3.10) in eq. (3.9), we get

$$\sum_{(j,k)\in\Lambda} \frac{d}{dt}\, a_{j,k}(t)\psi_{j,k}(x) + L \sum_{(j,k)\in\Lambda} a_{j,k}(t)\;\psi_{j,k}(x) = f(x,t)$$

then, applying the orthogonality condition, we have

$$\sum_{(j,k)\in\Lambda} \frac{da_{j,k}(t)}{dt} < \psi_{j,k}, \psi_{l,m} > + \sum_{(j,k)\in\Lambda} a_{j,k}(t) < L\psi_{j,k}, \psi_{l,m} > = < f, \psi_{l,m} > \qquad (3.11)$$

$$for\ all\ (l,m) \in \Lambda$$

The system (3.11) is a system of linear or nonlinear ODEs, and we get two assumptions:

1. If we get a linear system then we can solve it by Runge- Kutta 4*th* order method.
2. If we get a non linear system, then we can't solve it by $RK4$, and be complicated due to the product of integrals called connection coefficients.

Chapter Four

Applications of Wavelets

4.1 Filter Bank

4.2 Signals Decomposition and Reconstruction

4.3 Audio Fingerprint

Chapter Four

Applications of Wavelets

In this chapter we will consider some applications of wavelets.

4.1 Filter Bank

A filter bank is a set of filters, which split up the signal's frequency components into different signals, each with a subset of frequencies. A simple filter bank consists of one high pass filter and one low pass filter, both having a cut off frequency at half the frequency bandwidth. Applying this filter bank to signal results into two new signals, one with the lower half frequencies and one with the upper half frequencies.

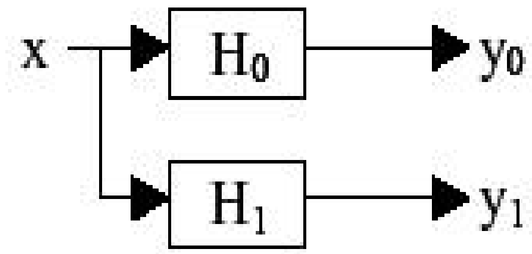


Fig. (4.1) Simple filter bank

where x is input signal, and y_0 , y_1 are output signals. H_1 and H_0 are high and low pass filters, respectively. To construct a filter bank with more than two frequency bands, the output signal y_0 could be filtered again by two filters, one low pass filter and a high pass filter which divide the bands up again into two bands. The number of samples (lengths of output signals) have doubled. The solution is to downsample (decimates) [2, 31]. The downsampling operation, which is done in the analysis bank, shall remove the odd-numbered components and save only the even-numbered components of the two outputs, as shown in Eq. (4.1)

$$(\downarrow 2)\begin{bmatrix} \cdot \\ v(-2) \\ v(-1) \\ v(0) \\ v(1) \\ v(2) \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ v(-4) \\ v(-2) \\ v(0) \\ v(2) \\ v(4) \\ \cdot \end{bmatrix} \tag{4.1}$$

The downsampling operator is usually denoted by $\downarrow 2$. Decimating results in a signal with half the number of samples that represent the same time interval as the original signal. Thus, the sample rate is halved, too.

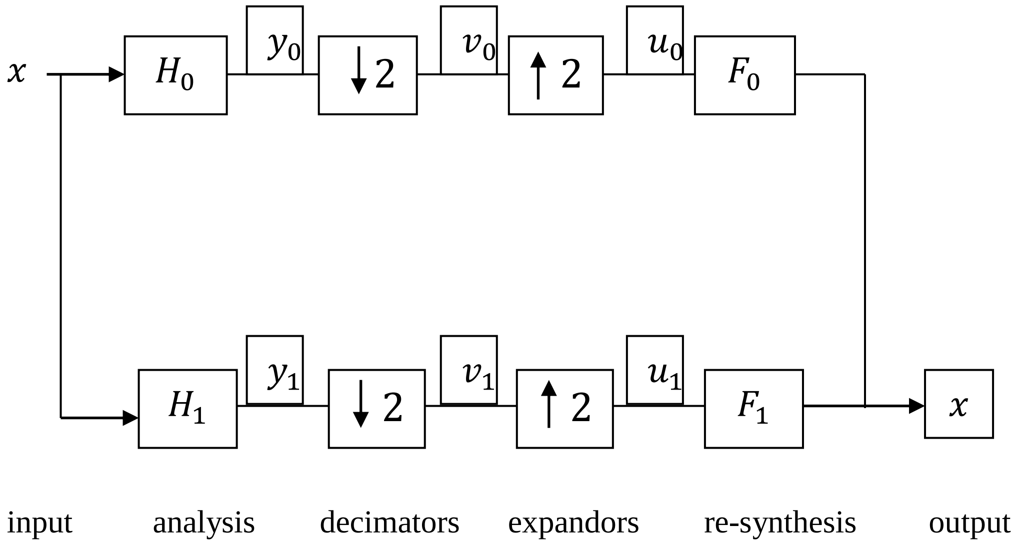


Fig. (4.2) Two-Channel Analysis/Re-Synthesis Filter Bank.

The decimated output can then be filtered again with the same filters to split it up into lower and higher frequency contents.

For reconstruction, up sampling (expanding) must be done in order to undo the decimation by inserting a zero after each sample. Introducing zeros between samples is to create a longer signal. Additionally, two re-synthesis filters F_0 and F_1 are needed to smooth out the zeros, reversing the analysis of low pass and high pass filters. The resulting samples are obtained by adding the outputs of the re-synthesis filters.

Applied to a half- length vector v, up sampling inserts zeros as in Eqn. (4.2), where $\uparrow 2$ indicates up sampling [31].

$$(\uparrow 2)\begin{bmatrix} \cdot \\ v(0) \\ v(1) \\ v(2) \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ v(0) \\ 0 \\ v(1) \\ 0 \\ v(2) \\ \cdot \end{bmatrix} \tag{4.2}$$

Figure (4.2) shows a two-channel filter bank analysis followed by re-synthesis.

As discrete filters do not have an ideal cut off, the low pass and high pass filters' frequency responses overlap: the low pass lets through frequency components of the high pass band, conversely, the high pass filter lets through low frequencies, see Figure (4.3). This aspect, causes aliasing when down sampled. The solution for perfect reconstruction is to design the reconstruction filters F_0 and F_1 in such a way that they cancel out the aliasing of the analysis filters.

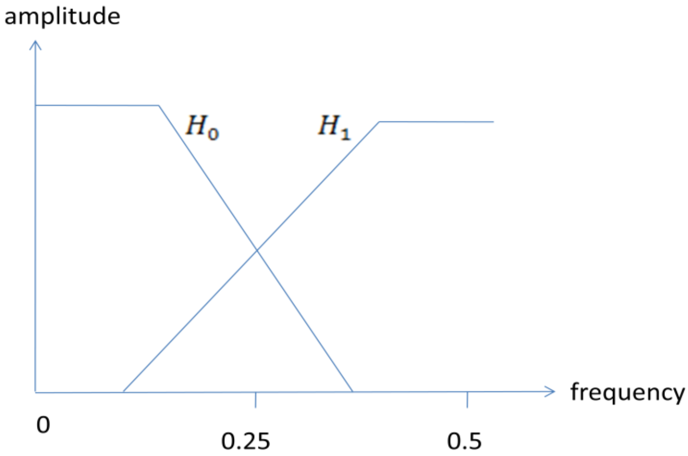
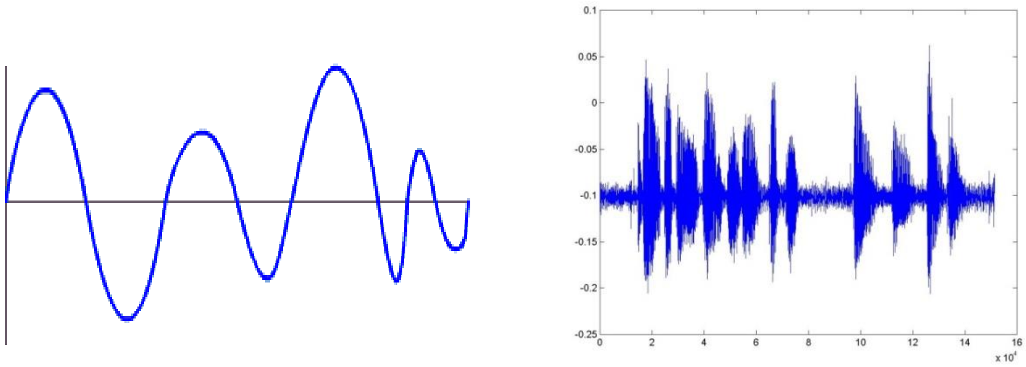


Fig (4.3) Overlapping low pass and high pass filter responses

4.2 Signals Decomposition and Reconstruction

4.2.1 Why Developing Wavelets ?

Approximating signal data with functions is not a new concept. Joseph Fourier developed a method using sines and cosines to represent other functions in the early 1800's. Fourier analysis is very good for analyzing signal data that does not change with time or involve jump discontinuities, because of the smooth and periodic behavior of sines and cosines. The graph on the right in Figure 4.4 displays a more complex signal that includes many jump discontinuities and appears to be dampening with time. The graph on the left in Figure 4.4 displays a well behaved signal, which might represent sound from a musical instrument. Fourier analysis would easily approximate the signal on the left, but not the one on the right [12].



Signal Complex signal

Figure 4.4

4.2.2 The Haar Wavelet

The Haar scaling function and wavelet are defined respectively as:

$$\phi(x) = \begin{cases} 1 & if \quad 0 \le x < 1 \\ 0 & otherwise \end{cases} \qquad and \qquad \psi(x) = \begin{cases} 1 & if \quad 0 \le x < \dfrac{1}{2} \\ -1 & if \quad \dfrac{1}{2} \le x < 1 \\ 0 & otherwise \end{cases}$$

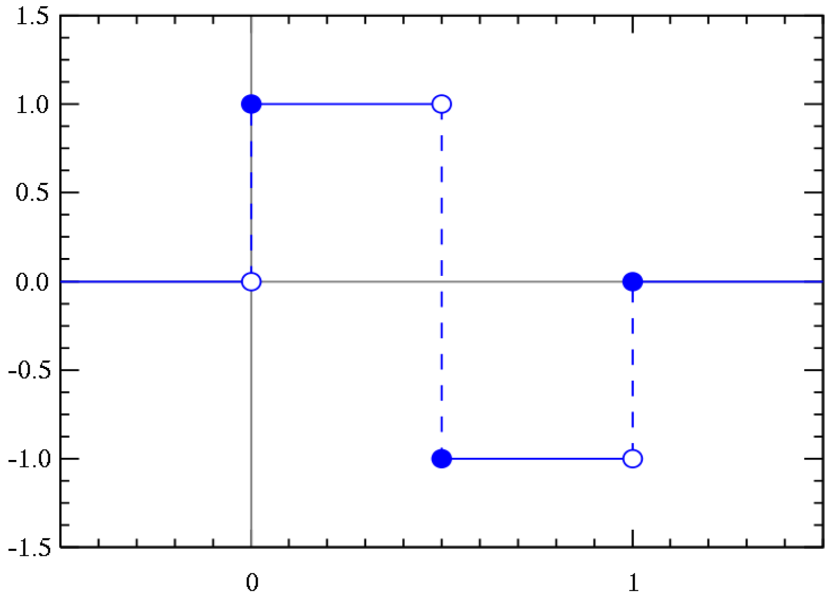


Figure 4.5 Haar Wavelet

Figure 4.5 shows the Haar wavelet. The Haar wavelet and Haar scaling function are used to generate sets of basis functions, which are used to break up or reconstruct a signal. The basis functions are similar to the original scaling function and wavelet, except that they are shifted and have different heights and widths. For example, $\phi(x-n)$ has the same graph as $\phi(x)$ but shifted to the right n units, $m\phi(x)$ has the same graph as $\phi(x)$ but with height m instead of 1 , and $\phi(bx)$ has the same graph as $\phi(x)$ but takes value 1 for the range $[0,\frac{1}{b})$ and 0 elsewhere. We can rewrite the Haar wavelet using the scaling function by combining the previous concepts as:

$$\psi(x) = \phi(2x) - \phi(2x-1)$$

where $\phi(x)\ and\ \phi(2x-1)$ are orthogonal.

Figure 4.6 explains how the Haar scaling function can be used to approximate a signal. The right side of Figure 4.6 displays one possible approximation of the signal on the left using building blocks based on the Haar scaling function. The left side of Figure 4.6 shows a signal that contains a small amount of noise. Although this is a simple example, it highlights basic concepts of approximating a signal using a multiresolution analysis.

The Haar scaling function and wavelet are discontinuous, and do not approximate smooth signals well. Daubechies developed wavelets that are localized in behavior, continuous, and yield better approximations to smooth signals [12].

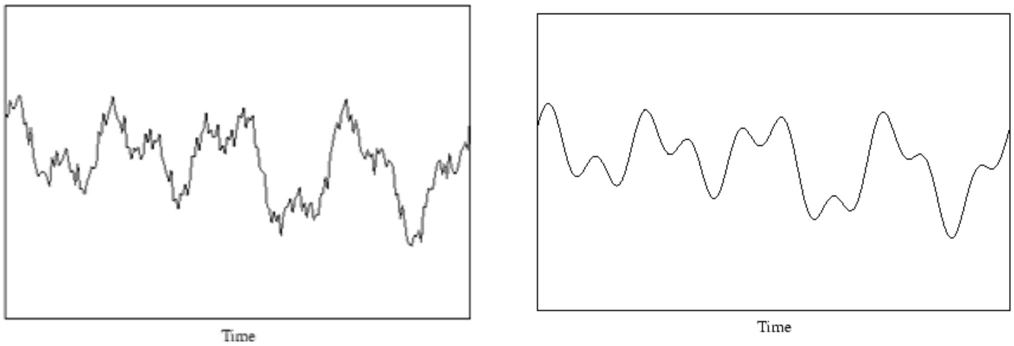


Figure 4.6

4.2.3 Multiresolution Analysis and the Discrete Wavelet Transform

How can we use the Haar wavelet and scaling functions to actually analyze a signal ? Let's use the points of data $a_3 = [5\ \ 6\ \ 25\ \ 30\ \ 20\ \ 7\ \ 25\ \ 5]$ to perform a discrete wavelet transform to illustrate the concepts behind a multiresolution analysis [36].

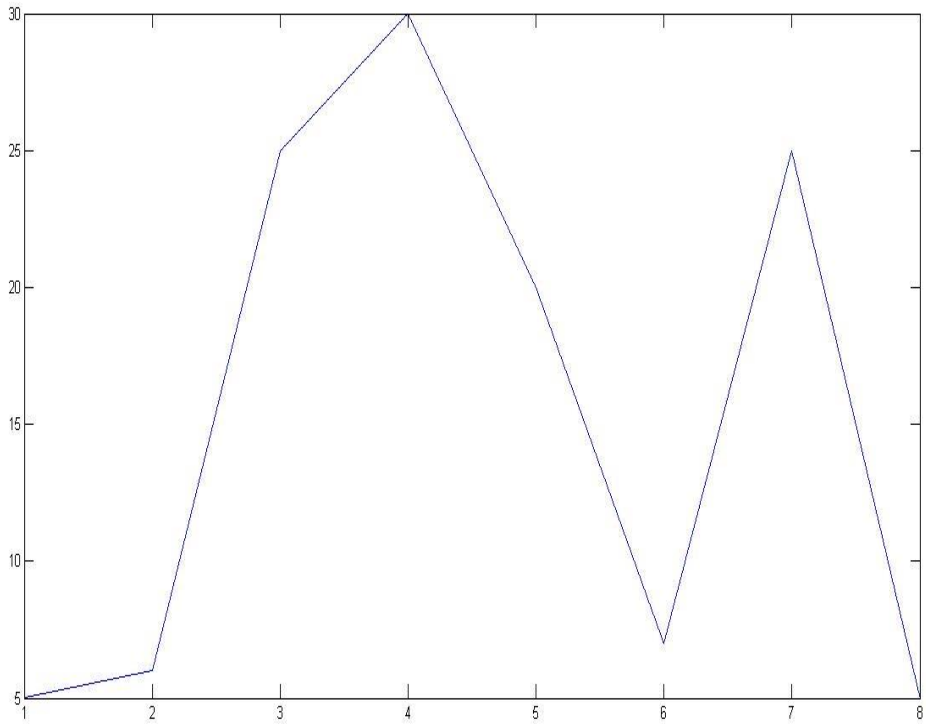


Figure 4.7: A plot of a_3

Wavelet Decomposition Algorithm

Two filters are necessary to decompose a signal using the wavelet decomposition algorithm [36]. The low pass filter, L, is for averaging, and the high pass filter, H, is for differencing. Deriving the low and high pass filters based on the Haar scaling function and wavelet yields $L = [0.5\ \ 0.5]$ and $H = [-0.5\ \ 0.5]$.

Step 1: Calculate the convolutions of L and H with the signal. By Matlab [24] we get :

$$conv(L, a_3) = [2.5\ \ 5.5\ \ 15.5\ \ 27.5\ \ 25\ \ 13.5\ \ 16\ \ 15\ \ 2.5]$$

$$conv(H, a_3) = [-2.5\ \ -.5\ \ -9.5\ \ -2.5\ \ \ 5\ \ \ 6.5\ \ -9\ \ 10\ \ 2.5]$$

Step 2: downsampling, which means discarding the odd numbered coefficients. The resulting vectors, a_2 and b_2 are respectively referred to as the scaling and wavelet coefficients.

$$a_2 = [5.5\ \ 27.5\ \ 13.5\ \ 15]$$

$$b_2 = [-.5\ \ -2.5\ \ 6.5\ \ 10]$$

The first coefficient in a_2 is 5.5 or $\frac{5+6}{2}$, and the second coefficient is 27.5 or $\frac{25+30}{2}$. The $conv.$ of a_3 and the low pass filter yields the averages of neighboring data points. The first coefficient of b_2 is $-.5$ which is difference of 5 *and* 6 with their average 5.5 . The last coefficient in b_2 is 10 which is the difference of 25 and 5, with their averages 15 . The

$convolution$ of a_3 and the high pass filter yielded the differences of neighboring data points and their average.

The process continues by decomposing the scaling coefficient vector using the same 2 steps, and finishes when 1 coefficient remains.

$$a_1 = [16.5 \quad 14.25]$$

$$b_1 = [-11 \quad -.75]$$

$$a_0 = [15.375]$$

$$b_0 = [1.125]$$

It is not a coincidence that the original signal data, a_3, has $2^3 = 8$ data points, or that the first level decomposition contains $2^2 = 4$ coefficients. All signals that are analyzed using a discrete wavelet transform must have length equal to some power of 2, which is referred to as dyadic length.

Why not stop with the first level decompositions, a_2 and b_2? The largest coefficients in b_2 are 6.5 and 10, which are associated with the changes from fifth data point, 20, to the sixth data point, 7, and from the seventh data point, 25, to the eighth data point, 5. If the goal is to detect large changes in the signal, then the first decomposition missed the change from the second data point, 6, to the third data point, 25. The change is detected in the largest coefficient of b_1, -11, which is associated with the coefficients 5.5 and 27.5 in a_2, or the shift from 6 to 25 in the original signal. Multiresolution analysis uses different scales of resolution to build a complete picture of the original signal.

Wavelet Reconstruction Algorithm

A similar process can be used to rebuild the original signal using the wavelet reconstruction algorithm [36]. A new low pass filter, LT, and a new high pass filter, HT, are needed. $LT = [1 \;\; 1]$ and $HT = [1 \; -1]$ based on the Haar scaling function and wavelet.

Step 1: Up sample the scaling and wavelet coefficient vectors by adding zeros.

$$Up(a_0) = [15.375 \quad 0]$$

$$Up(b_0) = [\,1.125 \quad 0]$$

Step 2: Calculate the convolutions of the scaling coefficients and LT, and the wavelet coefficients and HT. By Matlab we get :

$$conv.\big(LT, Up(a_0)\big) = [15.375 \quad 15.375]$$

$$conv.\big(HT, Up(b_0)\big) = [1.125 \quad -1.125]$$

Step 3: Add the new average and difference vectors to yield the reconstructed average vector, a_1

$$ra_1 = [15.375 \quad 15.375] + [1.125 \quad -1.125] = [16.5 \quad 14.25]$$

The reconstructed version of a_1 and the difference vector, b_1, can be used to reconstruct a_2.

$$Up(ra_1) = [16.5 \quad 0 \quad 14.25 \quad 0]$$

$$Up(b_1) = [-11 \quad 0 \quad -.75 \quad 0]$$

$$ra_2 = conv(LT, Up(ra_1)) + conv(HT, Up(b_1))$$

$$= [16.5 \quad 16.5 \quad 14.25 \quad 14.25] + [-11 \quad 11 \quad -.75 \quad .75]$$

$$= [5.5 \quad 27.5 \quad 13.5 \quad 15]$$

Finally, we can reconstruct the original signal.

$$ra_3 = conv(LT, Up(ra_2)) + conv(HT, Up(b_2))$$

$$= [5 \quad 6 \quad 25 \quad 30 \quad 20 \quad 7 \quad 25 \quad 5]$$

There were 4 coefficients in b_2, 2 coefficients in b_1, and 1 coefficient in a_0 and b_0. Eight coefficients from the wavelet decomposition were necessary to reconstruct the original signal, which contained 8 data points.

Example (1)

Wavelet Decomposition Algorithm

Let $b_3 = [2 \quad 5 \quad 25 \quad 20 \quad 4 \quad 3 \quad 15 \quad 2]$

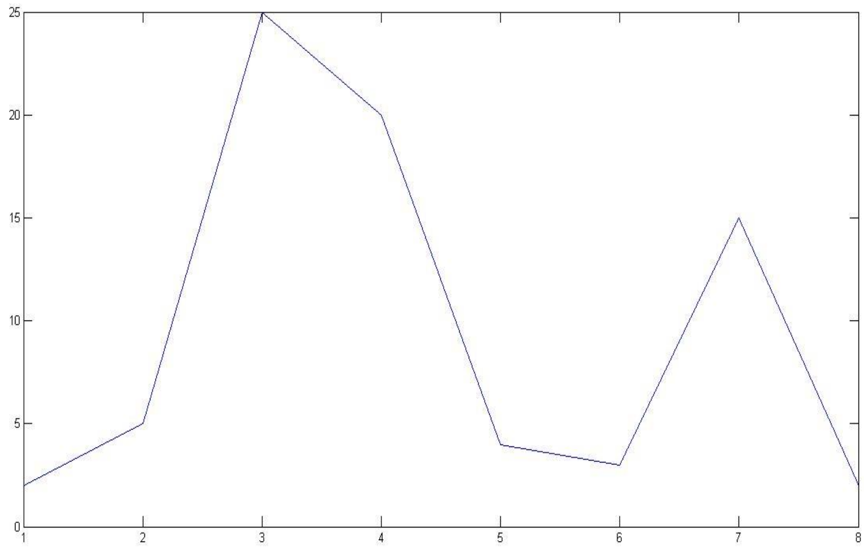


Figure 4.8: A plot of b_3

and L low pass filter, H high pass filter

$$L = [0.5 \quad 0.5]$$

$$H = [-0.5 \quad 0.5]$$

Calculate $conv$ by Matlab [24]

$$conv(L, b_3) = [1 \quad 3.5 \quad 15 \quad 22.5 \quad 12 \quad 3.5 \quad 9 \quad 8.5 \quad 1]$$

$$conv(H, b_3) = [-1 \quad -1.5 \quad -10 \quad 2.5 \quad 8 \quad 0.5 \quad -6 \quad 6.5 \quad 1]$$

again,

$$b_2 = [3.5 \quad 22.5 \quad 3.5 \quad 8.5]$$

$$c_2 = [-1.5 \quad 2.5 \quad 0.5 \quad 6.5]$$

$$b_1 = [13 \quad 6]$$

$$c_1 = [-9.5 \quad -2.5]$$

$$b_0 = [9.5]$$

$$c_0 = [3.5]$$

Wavelet Reconstruction Algorithm

$$LT = [1 \quad 1]$$

$$HT = [1 \quad -1]$$

$$Up(b_0) = [9.5 \quad 0]$$

$$Up(c_0) = [3.5 \quad 0]$$

$$conv\big(LT, Up(b_0)\big) = [9.5 \quad 9.5 \quad 0]$$

$$conv\big(HT, Up(c_0)\big) = [3.5 \quad -3.5 \quad 0]$$

$$rb_1 = [9.5 \quad 9.5 \quad 0] + [3.5 \quad -3.5 \quad 0] = [13 \quad 6 \quad 0]$$

$$Up(rb_1) = [13 \quad 0 \quad 6 \quad 0]$$

$$Up(c_1) = [-9.5 \quad 0 \quad -2.5 \quad 0]$$

$$conv\big(LT, Up(rb_1)\big) = [13 \quad 13 \quad 6 \quad 6 \quad 0]$$

$$conv\big(HT, Up(c_1)\big) = [-9.5 \quad 9.5 \quad -2.5 \quad 2.5 \quad 0]$$

$$rb_2 = conv\big(LT, Up(rb_1)\big) + conv\big(HT, Up(c_1)\big)$$

$$[13 \quad 13 \quad 6 \quad 6 \quad 0] + [-9.5 \quad 9.5 \quad -2.5 \quad 2.5 \quad 0]$$
$$= [3.5 \quad 22.5 \quad 3.5 \quad 8.5 \quad 0]$$

$$rb_3 = conv\big(LT, Up(rb_2)\big) + \ conv\big(HT, Up(c_2)\big)$$

$$[3.5 \quad 3.5 \quad 22.5 \quad 22.5 \quad 3.5 \quad 3.5 \quad 8.5 \quad 8.5 \quad 0] + [-1.5 \quad 1.5 \quad 2.5$$
$$-2.5 \quad 0.5 \quad -0.5 \quad 6.5 \quad -6.5 \quad 0]$$

$$= [2 \quad 5 \quad 25 \quad 20 \quad 4 \quad 3 \quad 15 \quad 2]$$

we get the original signal.

4.3 Audio Fingerprint

Audio Fingerprint or content-based audio identification (CBID) has been studied since the 1990s.

An audio fingerprint can be seen as a short summary of an audio object using a limited number of bits, $i.e$ a fingerprint function F should map an audio subject X consisting of a large number of bits into a fingerprint of only a limited number of bits. We can draw a map with hash functions H from object X (large) to a hash value (small).

Hash function allow comparison of two large subjects X and Y, by just comparing their respective hash value $H(X)$ and $H(Y)$ [10, 30].

Definition (4.1)

The formula of Hanning Window is

$$\omega(n) = \begin{cases} 0.5 * \left(1 - \cos\left(\frac{2\pi n}{N-1}\right)\right) & , 0 \leq n \leq N-1 \\ 0 & , \quad else \end{cases}$$

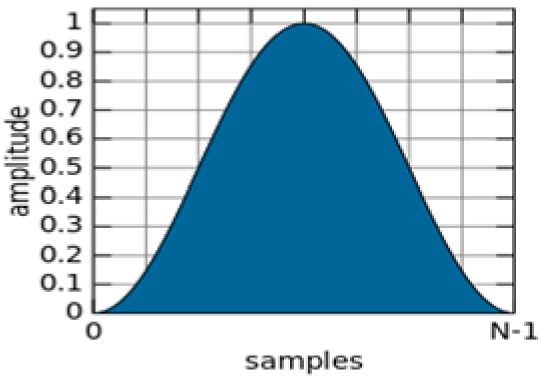


Figure 4.9 Hanning Window

There are various steps in audio fingerprint which are as follows [16, 26]:

1^{st} step : Pre-processing or Pretreatment: As the input to the algorithm, the audio file is given. The Pretreatment involves the conversion of audio signal into mono signal, filtering using a low-pass filter, and down sampled whose down sampling frequency is 5KHZ.

2^{nd} step : Framing, Windowing and Overlapping : The signal must be divided into a number of frames. The length of the frame is 0.375, using Hanning window, the overlap factor is p=28/32 .The number of frame depends on the audio. The rate at which frames are computed per second is called frame rate. A window function is applied to each block to minimize the discontinuities at the beginning and end.

3^{rd} step : Decomposition: if the number of vanishing moments is 4 then we denote it by $db4$, using the wavelet based on $db4$ to decompose each frame of audio signal in 5 layer wavelet. We get a six components, one is approximation component $cA5$ and five details components $cD1, \_\_\_, cD5$.

4^{th} step : we calculate the following [35]:

1. The variance of the wavelet coefficients :

The formula is :

$$\sigma(i,j) = \frac{1}{N} \sum_{j=1}^{N} \left(cD_j - \overline{cD}\right)^2$$

where

$$\overline{cD} = \sum_{j=1}^{N} \sum cD_j$$

2. The zero crossing rate of wavelet coefficients :

The formula is :

$$zcr_m = \frac{1}{2}\sum_{m} |sign[x(n)] - sign[x(n-1)]|\, \omega(n-m)$$

where $\omega(n)$ is the window function, and N is the length of window function, and $x(n)$ is the n^{th} value of the wavelet coefficients in the m^{th} frame, which separately correspond to cA_5 and cD_5, and

$$sign\,[x(n)] = \begin{cases} 1 & x(n) \geq 0 \\ 0 & otherwise \end{cases}$$

3. The centroid of the wavelet domain

The centroid of the wavelet domain is expressed as the center of energy distribution. In wavelet domain, the centroid of the audio signal changes with time, so it can be the characteristic of reflecting the non-stationarity of audio signal.

The computational formula of the centroid is:

$$centroid = \frac{\sum_{i=1}^{N} i|x(i)|^2}{\sum_{i=1}^{N} |x(i)|^2}$$

where $x(i)$ is the i^{th} wavelet coefficient.

4. The energy of sub-band in wavelet domain

The formula of calculating the energy of sub band [30] is as follows:

$$energy = \frac{1}{N}\sum_{i=1}^{N} |x(i)|^2$$

The change in amplitude of the audio signal is an important dynamic characteristic of the audio signal, and it can reflect the change of energy.

We can use the wavelet coefficients to measure the energy characteristics of audio because of the fact that the average rate of the wavelet coefficients corresponds to the average rate in time domain.

The principle framework of the algorithm is shown in Fig. 4.10

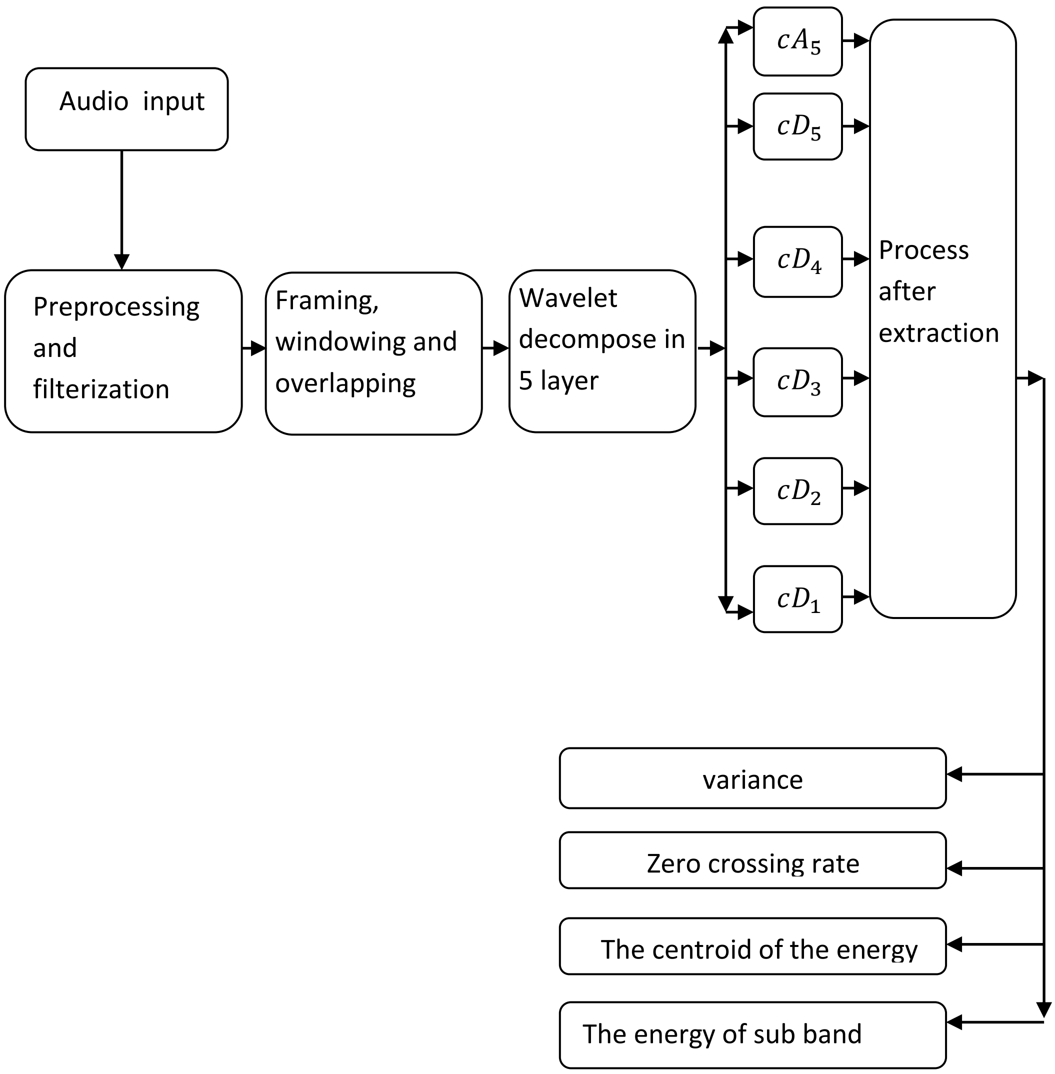


Fig. 4.10 : Principle framework of the algorithm

We can summarize the previous steps as follows:

The audio file is given as the input to the algorithm then we convert the audio into signals, then the signal is divided into a number of frames, next we decompose each frame of audio signal in 5 layer wavelet. Finally, we use the wavelet coefficient to compute the following parameters: variance, zero crossing rate, centroid and energy.

Now, we apply these steps in Matlab by:

```
function [ output_args ] = matlab_code( input_args )
% Student Name: Noora Hazem Janem
% Student ID: 11256149
% An najah University
% College of Science
% Department of Mathematics
% Supervisor: Prof. Naji Qatanani

% This Matlab code has been developed for my master thesis entitled
%" Analytical & Numerical Aspects of Wavelets "

% The following MATLAB code is to read an audio, filter, framing, and
apply
% Wavelet decomposition on it, then we use the wavelet coefficients to
% compute the following parameters: Variance, Zero Crossing, Centriod,
&
% Energy

% Step 1: Read an audio from a spcified directory
%[y, Fs] = wavread('C:\Users\My folder\Desktop\noora');

clc
fileName='C:\Users\My folder\Desktop\noora';
[wave, fs, nbits]=wavread(fileName);
fprintf('Information of the sound file "%s":\n', fileName);
fprintf('Duration = %g seconds\n', length(wave)/fs);
fprintf('Sampling rate = %g samples/second\n', fs);
fprintf('Bit resolution = %g bits/sample\n', nbits);


% Step 2: Pre-processing stage on the audio
%y = decimate(wave,2);


% Sampling frequency is 5 kHz
Fs = 50e3;
t = linspace(0,1,50e3);
% Lowpass filter everything below 5 kHz
% Specify the filter
```

```
hlpf = fdesign.lowpass('Fp,Fst,Ap,Ast',4e3,4.1e3,0.5,50,50e3);
% Design the filter
D = design(hlpf);
% apply the filter
y = filter(D,wave);
% figure;
% subplot(211)
% plot(psd(spectrum.periodogram,wave,'Fs',Fs,'NFFT',length(wave)));
% title('Original Signal PSD');
% subplot(212);
% plot(psd(spectrum.periodogram,y,'Fs',Fs,'NFFT',length(wave)));
% title('Filtered Signal PSD');


% Step 3: Break the signal into frames of frame length of 0.375 using
hanning
% window
frame_len = 0.375;
N = length(wave);
num_frames = floor(N*frame_len);

new_sig = zeros(N,1);
count = 0;
frame_len =3;
for k = 1: num_frames
    % Extract a frame of speech
    frame = wave((k-1)*frame_len+1 : frame_len*k)

    % Identify non silent frames by finding frames with max amplitude
more
    % than 0.03
   % max_val = max(frame);

   % if(max_val > 0.03)
        %this frame is not silent
        count = count +1;
        new_sig((count-1)*frame_len + 1 : frame_len*count) = frame;
  %  end
end


%subplot(211)
figure,plot(t(1:length(wave)),wave(1:length(wave))); title('Original
Waveform');
%subplot(212)
figure,plot(t(1:length(wave)),y(388:length(wave)+387));
title('Filtered Waveform');


% Reload the original One-dimensional signals and then compute the
number
% of signals
[h,w]=size(new_sig);


% Perform one-dimensional decomposition at 5 layer wavelet of the
signals
% using db4
```

```
for i=1:h
[c,l] = wavedec(new_sig(i),5,'db4')
coeff_num = size(c,2);
for j=1:coeff_num
wavelet_coefficients(i,j)=c(1,j);
end
end


% Compute the variance of the wavelet coefficients
Variance = var(var(wavelet_coefficients));


% Now, we need to calculate the zero crossing value of the wavelet
% coefficients
ZeroCrossingRate = mean(mean(abs(diff(sign(wavelet_coefficients)))));


% The centroid of the wavelet domain can be computed using the
following
% equation:
Centroid = mean(mean(wavelet_coefficients));


% Computing or finding the energy of sub-band in the wavelet domain
can be
% achieved using the following equation:
for i=1:h
[Ea(i,:),Ed(i,:)] = wenergy(wavelet_coefficients(i,:),l);
end


% To find the mean of the energy corresponding to the wavelet
coefficients
% details, we use:
energy = mean(mean(Ed));
end
```

We read audio named noora in the previous code

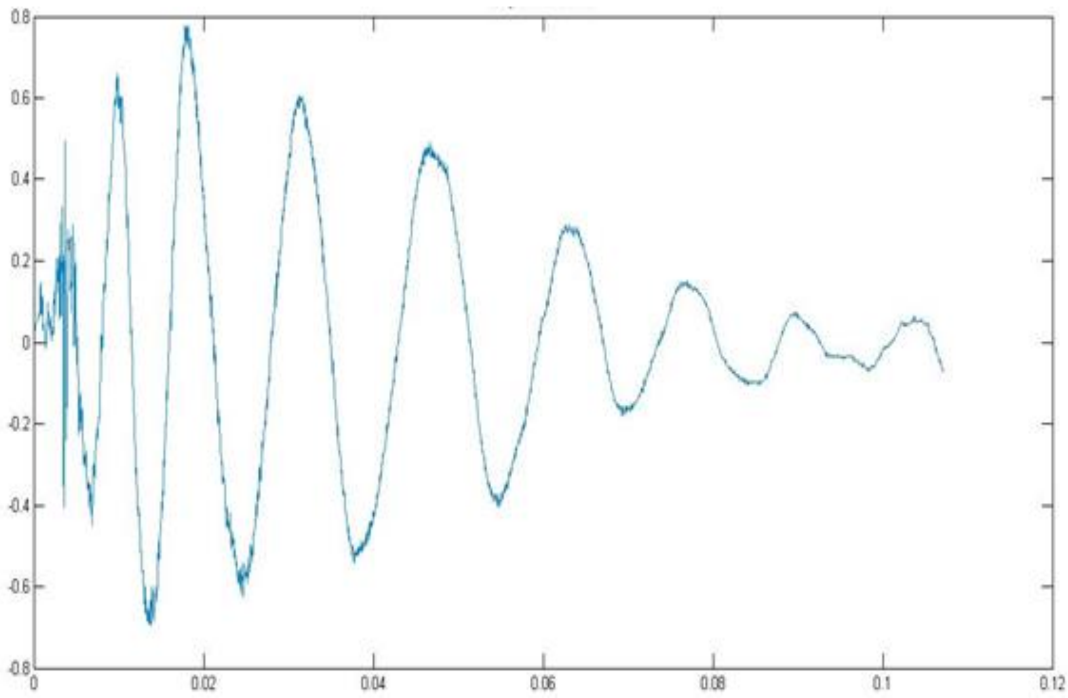
If we applied the 1^{st} step and use low pass filter with down sampling frequency 5KHZ, then we get

Duration = 0.121519 seconds

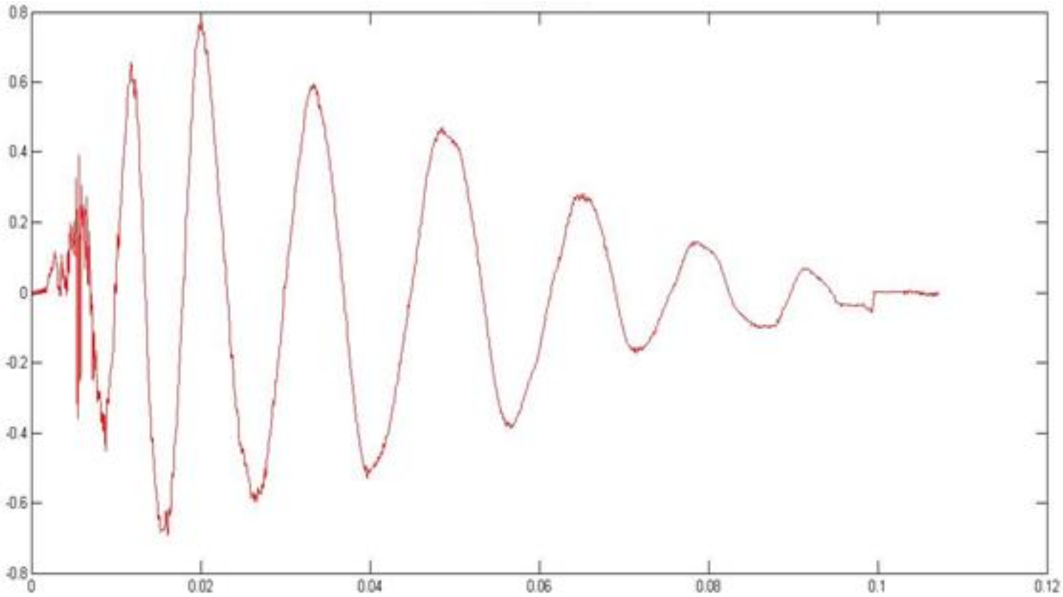
Sampling rate = 44100 samples / second

Bit resolution = 16 bits / sample

we get from 2^{nd} and 3^{rd} steps two graphs : the original waveform and the filtered waveform:



a) Original Waveform



b) Filtered Waveform

Figure 4.11 : The Original Waveform and Filtered Waveform

From Figure 4.11, we got filtered waveform from the original waveform, such that Figure 4.11 (b) becomes smoother than Figure 4.11 (a).

From 4^{th} step we get the following results:

$$variance = 1.0988$$

$$zero\ crossingrate = 0.7146$$

$$centroid = 0.0047$$

$$energy = 7.3362e - 032$$

Note that:

frame length $= \ 0.375$,

number of frame of this audio = 2009,

number of signals (h) $= 6027$,

coefficient number for each frame $= 33$.

Conclusion

The fundamental idea of wavelet transforms is the transformation that allows only changes in time extension, but not in shape. This is influenced by the choice of basis functions which satisfy that condition.

The wavelet transform is more accurate than the Fourier Transform. The Fourier Transform cannot provide any information about the changes of the spectrum with respect to time. Fourier transform assumes that the signal is stationary. Hence, we use the wavelet transform because it is more suitable for analyzing the non stationary signal, since it preserves the quality of the signal.

We have observed the importance of wavelet transform in various applications. These applications include the audio fingerprint. By filtering and down sampling we have obtained a filtered waveform from the original waveform. In other words, we de-noise the noisy signal to become smooth.

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نوره حازم عبد الحميد جانم

إشراف

أ. د. ناجي قطناني

قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية، نابلس، فلسطين

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الملخص

تحويل فوربيه (Fourier Transfrom) هو طريقة من أجل تمثيل الإشارات الدورية باستخدام سلسلة من اقترانات الجيب و جيب التمام تم تطويرها من أجل أي إشارة حتى لو لم تكن دورية بإنهاء دورها إلى اللانهاية حيث يقوم التحويل بنقل الإشارة من مجال الزمن إلى مجال التردد وبالعكس. ثم استخدم ما يعرف بالنافذة ثابتة العرض، من خلال تمثيل الإشارة زمنيا وتردديا على حساب دقتها الزمنية والترددية، بيد أنه عند استخدام نافذة صغيرة يتم الحصول على دقة عالية من أجل العناصر التي تتغير بسرعة، بينما لا تكون هذه الدقة عالية للعناصر المتغيرة ببطء، ولذلك تم تطوير ما يعرف بتحويل المويجات.

تحويل المويجات (Wavelet Transform) هو تطوير لتحويل فوربيه؛ إذ إنه يستخدم نافذة متغيرة العرض بدلا من استخدام نافذة ثابتة العرض، إذ يتم تغيير عرض النافذة للحصول على المعلومات مختلفة التردد على طول الموجة لإنتاج ما يعرف بالمويجات التي يختلف ترددها حسب عرض النافذة المستخدمة.

تقوم النافذة الصغيرة بإنتاج مويجة مضغوطة تتضمن العناصر ذات التردد المرتفع والتي تعرف بالعوامل التفصيلية. وتقوم النافذة الكبيرة بإنتاج مويجة ممددة تتضمن العناصر ذات التردد المنخفض والتي تعرف بالعوامل التقريبية.

يمكن تعريف المويجة على أنها إشارة محدودة الطول الزمني وتمتلك قيمة متوسطة تساوي الصفر. إن أحد التطبيقات الأساسية التي يتم إجراؤها على الإشارة بعد تحليلها عن طريق تحويل المويجات هو إزالة الضجيج الموجود في تلك الإشارة.

هناك نوعان رئيسيان من تحويل المويجات :تحويل المويجات المستمر و تحويل المويجات المتقطع. عن طريق استخدام المويجات تم حل المعادلات التفاضلية.