An-Najah National University

Faculty of Graduate Studies

The Zero Divisor Graphs of Specific Commutative Rings

By

Laila Sufian Abdullah Mosleh

Supervisor

Dr. Khalid Adarbeh

This Thesis is Submitted in Partial Fulfillment of the Requirements for the degree of Master of Mathematics, Faculty of Graduate Studies, An-Najah National University, Nablus, Palestine.

2021

The Zero Divisor Graphs of Specific Commutative Rings

By

Laila Sufian Abdullah Mosleh

This Thesis was Defends Successfully on 26/12/2021 and approved by

Defense Committee Members	**Signature**
- **Dr. Khalid Adarbeh /Supervisor**	**………………**
- **Prof. Dr. Mohammad Saleh /External Examiner**	**………………**
- **Dr. Mohammad Borini Yasin /Internal Examiner**	**………………**

Dedication

This thesis is a dedication to my parents and my husband for their support, and to my son who enlightened my life, as well as to my whole family and friends. With respect and love.

Acknowledgement

First, I want to thank Allah for giving me the strength and courage to complete this work. I am grateful to my supervisor Dr. Khalid Adarbeh for his continuous help and guidance. Finally, I would like to thank all the stuff member of Mathematics Department at An-najah National University for their contribution during my studies.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل عنوان

The Zero Divisor Graphs of Specific Commutative Rings

أقر بأن ما اشتملت عليه هذه الرسالة إنما هي نتاج جهدي الخاص، باستثناء ما تم الإشارة إليه حيثما ورد، وأن هذه الرسالة ككل، أو أي جزء منها لم يقدم لنيل أي درجة أو لقب علمي أو بحثي لدى أي مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degrees or qualifications.

Student's Name: ليان سفيان مصلح **:اسم الطالب**

Signature ليان مصلح **:التوقيع**

Date ٢٠٢١/١٢/٢٦ **:التاريخ**

List of Content

Dedication III
Acknowledgement.. IV
Declaration V
List of Content.. VI
Abstract VII
Chapter One: Introduction of the zero divisor graph 1
Introduction:.. 1
Chapter Two: Some Properties of The Zero Divisor Graph of A Commutative Ring 11
Preview 11
2.1 Examples.. 11
2.2 Finite zero divisor graph:.. 19
2.3 Γ(A) has a spanning tree which is a star graph: 21
2.4 Complete zero divisor graph:.. 23
2.5 Cycle and line graphs zero divisor: 27
Chapter Three: The Central Sets and The Radius of The Zero Divisor Graph of Commutative Ring 33
Preview 33
3.1 The ring of Gaussian integers modulo n We start this section by the definition of the Gaussian integers. 33
3.2 THE CENTER MEDIAN AND THE RADIUS OF Γ(A) 35
3.3 MULTIPLE DOMINATION AND 2-PACKING OF Γ(A) 46
3.4 PERFECT ZERO DIVISOR GRAPH.. 51
Chapter Four: The Zero divisor Graph of Some Special Rings.................. 59
Preview 59
4.1 The zero Divisor Graph of Boolean and k-Boolean rings............ 59
4.2 The zero Divisor Graph of nilpotent ring 63
References 67
الملخص .... ب

The Connected Zero Divisor Graphs of Specific Commutative Rings

By
Laila Sufian Abdullah Mosleh
Supervisor
Dr. Khalid Adarbeh

Abstract

Let A be a commutative ring with 1. In 1998, David F. Anderson and Philip S. Livingston associated to A a graph $\Gamma(A)$ and they called it the zero divisor graph of A. The vertices of $\Gamma(A)$ is the set $Z(A)^* = Z(A) - \{0\}$, where $Z(A)$ denotes the set of all zero divisors of A, and for $x \neq y$ in $Z(A)^*$, the vertices x and y are adjacent if and only if $xy = 0$ [3]. In this thesis, we provide a study of the effect of some basic ring theoretic properties of a ring A on it's zero divisor graph ($\Gamma(A)$) by reproducing and illustrating using new examples, the main work done in [3, 12]. Moreover, in the last chapter, we investigate for the first time, the interplay between the ring-theoretic properties of some special rings; such as Boolean, K −Boolean, and nilpotent rings; and the graph theoretic properties of their zero divisor graphs.

Chapter One

Introduction of the zero divisor graph

Introduction:

In this chapter, we recall some basic information from graph theory and also from ring theory that will be used frequently in this master thesis. We start by the definition of a ring.

Definition 1.1*:* A ring A is an algebraic structure which consists of a set A with two binary operations addition $(+)$ and multiplication $(.)$ such that:

1) $(A, +)$ is a belian group.
   - $+$ is associative.
   - A has an additive identity called 0 ($0 + x = x$ for all $x \in A$).
   - Each element x of A has an additive inverse called $-x$ ($x + -x = 0$).
   - The addition is commutative ($x + y = y + x$ for every $x, y \in A$).
2) Multiplication is associative ($x(yz) = (xy)z,\ \forall\, x, y, z \in A$).
3) Multiplication distributes over addition$\begin{cases} x(y+z) = xy + xz \\ (x+y)z = xz + yz \end{cases}, \forall x, y, z \in A$.
4) If A contains a multiplicative identity, then it is called the unity and is denoted by 1 (i.e. $x.1 = 1.x = x,\ \forall x \in A$). In this case the ring is called a ring with unity.
5) If the multiplication is commutative ($xy = yx,\ \forall x, y \in A$), then A is called a commutative ring.

Examples of rings are:

1. The set of integers ($\mathbb{Z}$), real numbers ($\mathbb{R}$), and rational numbers ($\mathbb{Q}$) is under the usual addition and multiplication of reals.
2. The set $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$ under the addition and multiplication modulo n.
3. If A is any ring, then the polynomial ring $A[X]$ which consists of all polynomials with coefficients from A under the usual addition and multiplication of polynomials.
4. The set of all $n \times n$ matrices with real entries under the usual addition and multiplication of matrices is a non-commutative ring.
5. Cartesian product of any two rings under the component wise addition and multiplication is a ring.

Throughout this thesis, our rings are commutative with 1.

Now, we recall the definition of the graph.

Definition 1.2: A graph G consists of vertices which are connected by edges. The vertices are denoted by $V(G)$ and the edges are denoted by $E(G)$. We denote a graph G by the pair $G = (V, E)$, where the elements of V are the vertices of G; $V(G)$ and those of E are the edges of G; $E(G)$. [8]

In this thesis, we are interested in studying a special kind of graphs. Those graphs are issued from commutative rings, and to introduce the definition of these graphs, we need the following ring theory definition:

Definition 1.3: Let A be a commutative ring with 1. An element a of A is called a zero divisor if there is a non zero element b of A such that $ab = 0$. The set of all zero divisors of A is denoted by $\mathrm{Z(A)}$.

In 1998, David F. Anderson and Philip S. Livingston associated to a ring A a graph $\Gamma(\mathrm{A})$ called the zero divisor graph of A, which mainly depends on the set $\mathrm{Z(A)}$. Next is the definition:

Definition 1.4: Let A be a commutative ring with 1. The non-zero, zero divisor graph of A; denoted by $\Gamma(\mathrm{A})$; is the graph with vertices $\mathrm{Z(A)}^* = \mathrm{Z(A)} - \{0\}$, and for $x \neq y$ in $\mathrm{Z(A)}^*$, the vertices x and y are adjacent if and only if $xy = 0$. The edge set of $\Gamma(A)$ is $E\big(\Gamma(A)\big) = \{xy\colon x, y \text{ in } V(\Gamma(\mathrm{A})) \text{ and } xy = 0\}$. [3]

For more details about the zero divisor graph of rings, we refer the reader to [5,8,16,21].

To make things moreclear, we provide the following example which displays the zero divisor graph of $\mathbb{Z}_6$.

Example 1.5: The ring $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ under addition and multiplication module 6 has the three distinct vertices: $2, 3$ and 4. Where 2 is adjacent to 3, and 3 is adjacent to 4, while 2 is not adjacent to 4. Below is a sketch of $\Gamma(\mathbb{Z}_6)$.

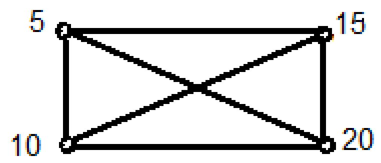
It deserves to mention that the zero divisor graph of a commutative ring was first introduced by Beck [4] who was interesting in coloring a graph with vertex set is the whole of the ring A, which makes sense, as he defined two vertices to be adjacent if and only if their product is zero and according to this definition, every element of A is a member (vertex) of the graph and is adjacent to zero.

Next, we recall the definitions of connected and complete graphs.

Definition 1.6: Let Γ be a graph.

1. Γ is called connected if there is a path between any two vertices.
2. Γ is called complete if any two distinct vertices are adjacent. We usually denote the complete graph by K_n. Where n is the number of the graphs vertices.

It is very clear that $\Gamma(\mathbb{Z}_6)$ is connected and not complete (2 is not adjacent to 4). But $\Gamma(\mathbb{Z}_{25})$ is a connected and complete graph. Below is the zero divisor graph of $\mathbb{Z}_{25}$ (see example 1.5)



$\Gamma(\mathbb{Z}_{25})$

Below are some basic definitions related to a connected graph G.

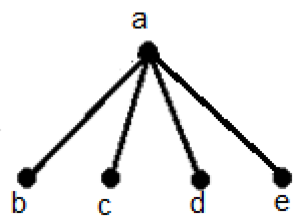
Definition 1.7 [13] **:** Let G be a connected graph.

1. The length from point x to point y; $L(x, y)$, is the number of edges when we move from x to y. (The number of edges of a path, and the path of length n is denoted by P^n).[19]
2. The distance $d(x, y)$ between two vertices x and y is the minimum of the lengths of all $x - y$ paths of G. That is $(d(x, y) = min\{L(x, y): x,y \in V(G)\})$. By [3], $d(x, y) = \infty$ if there is no path between them.

3. The eccentricity of a vertex x in G is the maximum distance from x to any vertex in G denoted by $e(x)$. $e(x) = \max\{d(x,y) : y \in V(G)\}$.
4. The radius of G; $rad(G)$, is the minimum eccentricity among the vertices of G. ($rad(G) = min\,\{e(x) : x \in V(G)\}$).
5. The diameter of G; $diam\ G$, is the maximum of the possible distances between all the vertices, ($diam(G) = max\{d(x,y) : x,y \in V(G)\}$).
6. The center of G is the set of vertices that have minimal eccentricity.
7. The open neighborhood of a vertex x in G is the set $N(x) = \{y: xy \in E(G)\}$ while the closed neighborhood of a vertex x in G is the set $N[x] = N(x) \cup \{x\}$.

The following example illustrates the above mentioned definitions.

Example 1.8: Take this graph (G):



There is only one edge between a and b and hence, $d(a,b) = 1$. While $d(b,e) = 2$, which is the maximum distance between any two distinct vertices thus $diam(G) = 2$.

It is very clear that the maximum distance from a to all other vertices is 1, and hence $e(a) = 1$. Similarly, we deduce that $e(b) = e(c) = e(d) = e(e) = 2$. So, the radius of the graph is $r(G) = 1$. Lastly, since a is the only vertex with eccentricity equals the radius, $\{a\}$ is the center of the graph.

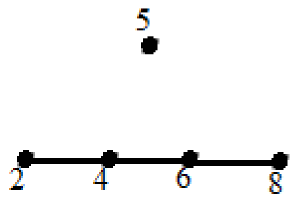
It is clear that the open neighborhood of the vertex c; $N(c) = \{a\}$, and hence the closed neighborhood of c; $N[c] = \{c, a\}$. We finish the example by notifying that the graph in this example can be realized as a zero divisor graph.

To provide an example of a disconnected graph. We appeal to the following definition.

Definition 1.9: Let A be a commutative ring. The complement graph $\overline{\Gamma(A)}$ is defined on the same vertex set but two distinct vertices x and y are adjacent if and only if $xy \neq 0$.[7]

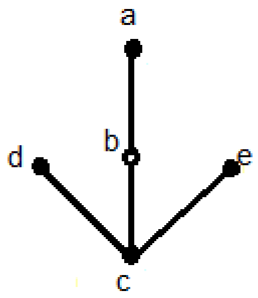
The following is an example of disconnected zero divisor graph.

Example 1.10: Take the ring $\mathbb{Z}_{10}$. Below is a sketch of $\overline{\Gamma(\mathbb{Z}_{10})}$. It is clear that there is no path between 5 and 2, hence $d(5,2) = \infty$. Which implies that the graph is disconnected.



Definition 1.11: A dominating set for a graph G is a subset D of vertices such that every vertex not in D is adjacent to at least one member of D.

Example 1.12: Take this graph.



In this graph the dominating set $D = \{a, c\}$, note that any vertex not in D is adjacent at least one vertex in D. Also, $\{d, b, e\}$ is another dominating.

Next, we introduce the definition of perfect graphs. For this purpose we need the following definition.

Definition 1.13: Let G be a contented graph:

1. The chromatic number of a graph Gdenoted by $\chi(G)$, is the minimum number of colors required to color the vertices of G such that any two adjacent vertices have different colors.
2. The clique number of graph G denoted by $\omega(G)$, is the size of the largest complete subgraph of G.

Definition 1.14: A subgraph of a graph is any subset of vertices together with any subset of edges containing those vertices.

Definition 1.15: Let $G = (V, E)$ be any graph, and let $S \subset V$ be any subset of vertices of G. Then the induced subgraph is the graph whose vertex set is S and whose edge set consists of all the edge in E that connecting pairs of vertices is S.(An induced subgraph is a subgraph maximal with respect to the number of edges).

Definition 1.16: A perfect graph is a graph G for which every induced subgraph H has chromatic number equal to its clique number.

Example 1.17: Take this graph:

In this graph, chromatic number equals 2 , and the clique number equals 2, hence this graph is a perfect graph.

Chapter two of this thesis is a reproducing of the work done by Anderson and Livingston in [3]. It consists of five sections: In the first section we provide several examples of zero divisor graphs for different rings and through these examples, we illustrate the effect of some basic properties of rings, such as finite rings and integral domains on the zero divisor graph of these rings. The second section contains the conditions under which the graph is finite.

The third section provides a graph that contain vertex adjacent to every other vertices. In the fourth section, we focused on the complete and the connected graphs. The fourth section also contains some properties of complete and complete bipartite graph. The fifth section provides a cycle zero divisor graph and discusses some properties cycle graph such as a girth.

Chapter three is devoted to study more properties of the zero divisor graph of commutative rings. The first section, is just a recalling of the definition of the ring of Gaussian integers modulo n, $Z_n[i]$, in addition to the fact that A

Gaussian prime integer is a unit multiple of one of the following: $(1 + i)$ or $(1 - i)$, A prime integer q in $\mathbb{Z}$ which $q \cong 3\ (mod\ 4)$. And $a + ib$, $a - ib$, where $p = a^2 + b^2$ and p is a prime integer in $\mathbb{Z}$ which $p \cong 1\ (mod\ 4)$. In the second section, the concepts of the center, median and the radius of the graphs are provided along with an illustrative example. Section 3.2 also contains the effect of the Noetherianity of ring on its zero divisor graph radius. The third section is about the domination and the 2-packing number of zero divisor graph and the relation with the radius. The last section is mainly a bout the perfect zero divisor graph .

The literature is very rich with the ring theoretic notions that are defined in terms of or depends on its zero divisors.

Definition 1.18: Let A be a commutative ring.

1. A is called a Boolean ring if $x^2 = x$ for all $x \in A$. (Notice that if $x^2 = x$, then $x(x-1) = 0$. So, if $x \neq 1$, it will be a zero divisor).[20]
2. A is called a $k-$Boolean ring if $x^{2k} = x$ for all $x \in A$, Where k is a positive integer. [25]
3. A is called a nilpotent ring if every element of A is nilpotent, where a nilpotent element is an element x such that there is a positive integer m such that $x^m = 0$.

In the last chapter, we investigate the interaction between the ring theoretic properties of the last mentioned rings and their zero divisor graphs. For example, in the first section, we will see that Boolean and $K-$Boolean rings share the property that their zero divisor graph contains a vertex which is adjacent to all other vertices if and only if $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Also, their zero divisor graphs are not complete. In the second section, we focus on the zero divisor graph of a nilpotent ring where we will see that those zero divisor graphs has diameter at most 2 and has a vertex adjacent to every other vertex. We will now show some basic definitions related to algebra.

Definition 1.19: Let A be a commutative ring. Then

1) $I \subset A$ is called an ideal if it is closed under subtraction and $ra \in I$ whenever $r \in A$ and $a \in I$.
2) A proper ideal P of A is called prime if whenever $xy \in P$, then $x \in P$ or $y \in P$.

Definition 1.20: Let A be a commutative ring with unity. And let $I \subset A$ be an ideal. Then I is annihilator ideal if $\forall x \in I: ax = 0$ where $a \in A$.

Definition 1.21: The rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively, and F_r will be the finite field with r elements.

Chapter Two

Some Properties of The Zero Divisor Graph of A Commutative Ring

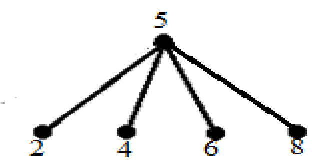
Preview

This chapter displays the interaction between the ring theoretic properties of a ring A and the graph theoretic properties of $\Gamma(A)$. Most of the results are inhanceing and reproducing results in [3] and the examples.

2.1 Examples.

In this section we provide some examples of graphs of different rings as well as the conditions under which the graph of a ring will be finite. Now we need the following example:

Example 2.1.1. Consider the ring $\mathbb{Z}_{10} = \{0,1,\dots,9\}$. $2,5$ are adjacent, since $2 \times 5 = 0$ and $5,4$ are adjacent, since $5 \times 4 = 0$. 2 and 3 are not adjacent, since $2 \times 3 = 6 \neq 0$. Below is a sketch of $\Gamma(\mathbb{Z}_{10})$.



The following example determines a necessary and sufficient conditions for a ring to have empty zero divisor graph.

Definition 2.1.2: An Integral domain is a nonzero commutative ring in which the product of any two nonzero elements is nonzero.

A is integral domain (ID) if $Z(A) = \{0\}$.

Example 2.1.3. If A is an integral domain, then $\mathrm{Z(A)} = \{0\}$, and hence $\Gamma(A)$ is empty. Actually, the converse of the last fact is also true. i.e., A is an integral domain if and only if $\Gamma(A)$ is the empty graph.

The following is an example of a zero divisor graph of one vertex.

Example 2.1.4: Take the ring $\mathbb{Z}_4$. The non-zero zero divisor graph of $\mathbb{Z}_4$ is Z*($\mathbb{Z}_4$)=$\{2\}$. Below is a sketch of $\Gamma(\mathbb{Z}_4)$.

Next, we recall the definition of an isomorphism of rings:

Definition 2.1.5. If A_1 and A_2 are rings then a ring homomorphism is a map $f: A_1 \longrightarrow A_2$ such that f is:

1)$f(a + b) = f(a) + f(b)$ for all a and b in A_1.

2)$f(ab) = f(a)f(b)$ for all a and b in A_1.

3)$f\left(1_{A_1}\right) = 1_{A_2}$.

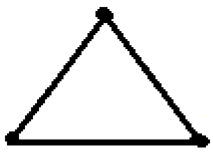
If the ring homomorphism is bijection (one-one and onto), then it is called a ring isomorphism. It is obvious that isomorphic rings have the same graph. This fact follows directly from the fact that the zero divisor property is preserved under the isomorphism (indeed, if $\varphi: A \to S$ and $x \in Z(A)$, then $\exists\ y$ such that $xy = 0$, it is clear that $\varphi(xy) = \varphi(x)\varphi(y) = 0$, thus $\varphi(x) \in Z(S)$. But this does not mean that non isomorphic rings cannot have the same graph. The following is a counter example to the last statement.

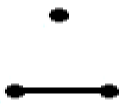
Example 2.1.6: The rings $\mathbb{Z}_9$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ have the same graph. Indeed, $Z^*(\mathbb{Z}_9) = \{3,6\}$, where 3 and 6 are adjacent, and $Z^*(\mathbb{Z}_2 \times \mathbb{Z}_2)=\{(1,0),(0,1)\}$, where (1,0) and (0,1) are adjacent. Below are the graphs:

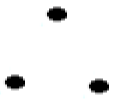
But it is obvious that $\mathbb{Z}_9$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not isomorphic since they have different cardinalities.

The possible graphs with three vertices are:

1) 

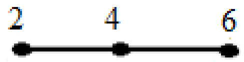
2) 

3) 

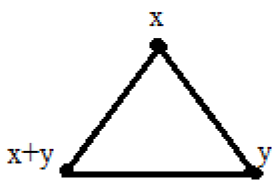
4) 

So, the possible connected graphs with three vertex are the first and the second graph. The following example ensure that these two graphs can be realized as a graph.

Example 2.1.7: **case1.** The zero divisor set of $\mathbb{Z}_8$ is $\{2,4,6\}$. It is clear that $2.4 = 0$, $4.6 = 0$ in $\mathbb{Z}_8$. Below are the graphs.



Case 2. For this case, we consider the polynomial ring$\frac{\mathbb{Z}_2[X,Y]}{(X^2,XY,Y^2)}$. The zero divisor set of this ring is$\{\bar{x},\bar{y},\overline{x+y}\}$ and it is clear that $\{\overline{xy}=\bar{0},\bar{x}.\bar{x}=\overline{x^2}=\bar{0},\ \overline{y^2}=\bar{0},\ \overline{x^2}+\bar{y}=\bar{0},\ \overline{y^2}+\bar{x}=\bar{0},\ \overline{x^2}+\overline{2xy}+\ \overline{y^2}=\bar{0}\}$. Hence the zero divisor graph of$\frac{\mathbb{Z}_2[X,Y]}{(X^2,XY,Y^2)}$is

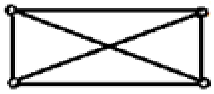


The following graphs are the possible disconnected graphs of 3 vertices.

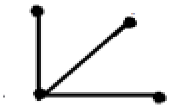
It can be easily notified from the examples 2.1.4, 2.1.6 and 2.1.7 that every connected graph with less than four vertices can be realized as $\Gamma(A)$ for some ring A. This observation fails in the situation of four vertices graphs.

Next we are interested in dealing with the situation of four vertices graphs. It is well known that there are twelve graphs with four vertices, 1)

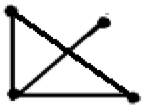
2)



3)

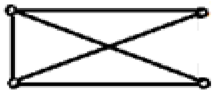


4)

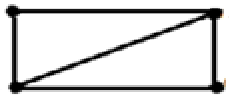


5)

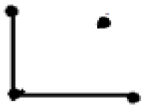
6)



7)

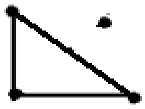


8)



9)

10)



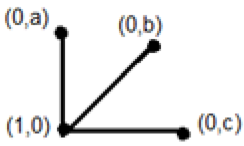
11)

12)

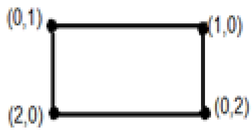
Notice that the graphs from $1 - 7$ are connected, and from the connected ones, only three can be realized as $\Gamma(A)$, for some commutative ring A. Below are the details:

The following example proves that the graphs from 1-3 can be realized as zero divisor graphs.

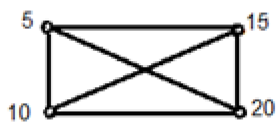
Example 2.1.8: The following graphs are connected with 4 vertices and can be realized as zero divisor graphs as shown:



$Z_2 \times F_4$

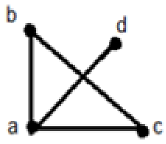


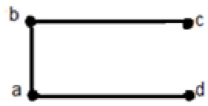
$Z_3 \times Z_3$

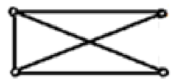


Z_{25}

The possible graphs of with 4 vertices cannot be realized as a zero divisor graph is:

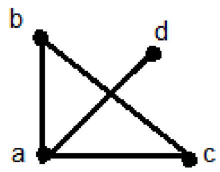






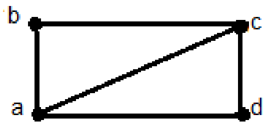
We provide the proof of the first two cases:

Case 1: The graph Γ, with vertices $\{a, b, c, d\}$ and edges: $a - b$, $a - c$, $a - d$, $b - c$, can not be realized as $\Gamma(A)$.



Suppose that Γ is $\Gamma(A)$ for some commutative ring A. It is clear that $(a + c)b = ab + cb = 0$. Hence $(a + c)$ is adjacent to b or b or zero, and so that $a + c \in \{0, a, c, b)$. If $a + c = a$ or c, then $c = 0$ or $a = 0$. which is a contradiction. Similarly, the contradiction holds if $(a + c) = 0$. So, the only possible value for $a + c$ is b. Now $(\boldsymbol{a} + \boldsymbol{c}) = \boldsymbol{b}$. Similarly,$(b + d)a = ba + da = 0$, implies that $(b + d) \in \{0, a, b, c, d\}$. If $b + d = b \text{ } or \text{ } d$, then $d = 0$ or $b = 0$, that is a contradiction. If $(b + d) = 0$, then $c(b) = c(-d)$ which implies that $cd = 0$, which is a contradiction with c not adjacent to d. If $(b + d) = a$, then $c(b + d) = ca = 0$ Hence $cd = 0$, contrudiction. So, we end with only one possibility $b + d = c$. Lastly, $b = a + c = a + b + d$ implies that $b = a + b + d$, and hence $a + d = 0$ or $d = -a$. Thus $bd = (-a)b = 0$ Which is a contradiction with b not adjacent to d. Therefore Γ can not be realized as $\Gamma(A)$.∎

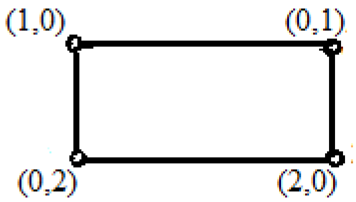
Case 2: The following graph also cannot be realized as a zero divisor graph:

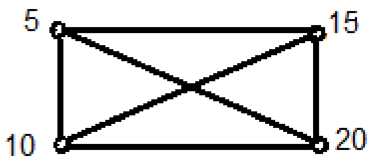


If the edges are: $a-b$, $a-c$, $a-d$, $c-d$, $\mathrm{b}-\mathrm{c}$. It is clear that $(a+b)c = ac + bc = 0$, so $(a+b) \in \{0, a, b, c, d\}$. If $a+b=a$ then $b=0$, which is acontradiction. Similarly, if $a+b=b$ then $a=0$, which is a contradiction. If $a+b=c$, then $d(a+b)=dc=0$ implies that $db=0$. Which is a contrudiction. Similarly, if $a+b=0$, contrudiction. Now take $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{d}.$ It is clear that $(b+d)a=ba+da=0$, then $(b+d)$ either $\{0, a, b, c, d\}$. If $b+d=b$ or d or 0, which is a contradiction. Then $b+d=c$ or $b+d=a$. If $b+d=c$ then $c=b+a+b = a+2b$, hence $c-a=2b$ then $2bd=(c-a)d$ which implies that $2bd=cd-ad$ and $2bd=0$. Hence $bd=0$, and hence there is an edge between b and d , which is a contradiction. Lastly, if $b+d=a$ then $b+a+b=a$ then $b=0$, contradiction.∎

The following example describes two complete zero divisor graphs of four vertices.

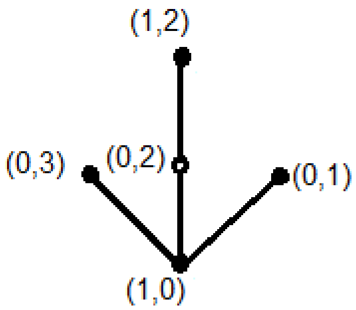
Example 2.1.9: The non-zero zero divisors of $\mathbb{Z}_3 \times \mathbb{Z}_3$ is $Z^*(\mathbb{Z}_3 \times \mathbb{Z}_3)=\{(1,0),(0,1),(2,0),(0,2)\}$, and the non-zero zero divisors of $\mathbb{Z}_{25}$ is $Z^*(\mathbb{Z}_{25})=\{5,10,15,20\}$. Below is the sketch of the zero divisor graphs of the mentioned rings. It is clear from the sketch that $\Gamma(\mathbb{Z}_{25})$ is complete but $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is not complete.





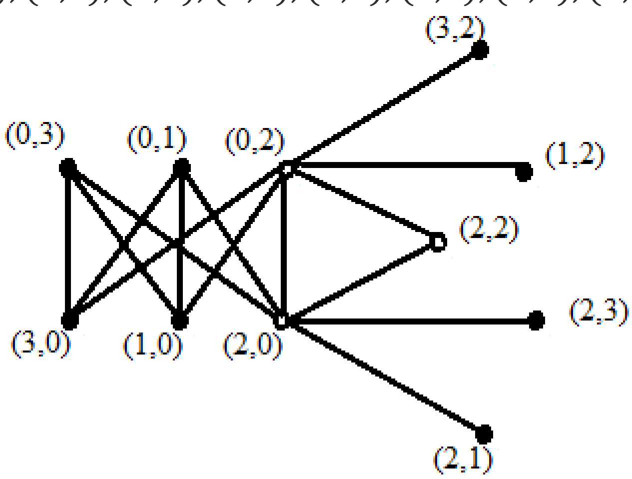
The following is an example of a zero divisor graph of five vertices.

Example 2.1.10: Take the ring of $\mathbb{Z}_2 \times \mathbb{Z}_4$. The non-zero, zero divisors $Z^*(\mathbb{Z}_2 \times \mathbb{Z}_4) = \{(1,2), (0,2), (1,0), (0,1), (0,3)\}$. Hence the graph takes this form.



The following is an example of a zero divisor graph of eleven vertices.

Example 2.1.11: Consider the ring $\mathbb{Z}_4 \times \mathbb{Z}_4$. The non-zero zero divisors is $\{(0,1), (0,2), (0,3), (1,0), (2,0), (3,0), (1,2), (2,1), (2,2), (3,2), (2,3)\}$.



The following is an example of infinite zero divisor graph.

Example 2.1.12: Consider the ring $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times ...$, an infinite number of times. It is clear that the vertex $a = (1,0,0 \dots.)$ is adjacent to every other vertex with the first coordinate is zero. Hence, a adjacent to an infinite number of vertices. Thus, the zero divisor graph is infinite.

2.2 Finite zero divisor graph:

This section provides the necessary and sufficient conditions under which a ring A will have a finite zero divisor graph. We start by the following main theorem.

Theorem 2.2.1. Let A be a commutative ring. Then $\Gamma(A)$ is finite if and only if either A is finite or an integral domain**.** Especially, if $1 \leq |\Gamma(A)| < \infty$, then A is finite and not a field.[3]

Proof**:** If A is an integral domain, then as we shown in example 2.1.3, $\Gamma(A) = \emptyset$ which is finite. If A is finite then trivially $Z(A)$ is finite and the graph is finite.

Suppose that $\Gamma(A) = Z^*(A)$ is finite and nonempty, and suppose that A is infinite. There are nonzero x, y in A such that $xy = 0$. Let $I = ann(x)$. Then I is contained in $Z(A)$, and hence I is finite (since $Z(A)$ is finite). But $xy = 0$ implies that $y \in \text{ann}(x)$, and consequently, $ry \in I$ for all $r \in A$. Now A is infinite and I is finite force the existence of $i \in I$ such that the set J={$r \in A\backslash$ ry=i} is infinite. But for any $r, s \in J$, $(r - s)y = ry - sy = i - i = 0$ which implies that $(r - s) \in ann(y)$. Lastly, J is infinite set implies that $ann(y)$ is also infinite. We finish by observing that the infinite set $ann(y)$

is contained in $Z(A)$ which is a contradiction with $Z(A)$ is finite. Thus A must be finite. $\blacksquare$

The special case of finite graphs that form a closed geometric figure with n-sides are called n-gon. So the triangle is a 3-gon, the square is a 4-gon,…etc. We can use example 2.1.7 and 1.17 to observe that the zero divisor graph of a ring can be a triangle (3-gon) or a square (4-gon). Now, the question if the zero divisor graph of a ring can be n-gon where $n \geq 5$ is answered negatively below.

Proposition 2.2.2: There is no ring A for which $\Gamma(A)$ is an n-gon for any $n \geq 5$.[17]

Proof: Indeed by examples 2.1.7 and 1.17 the graph can be a triangle or a square.

First take the case $n = 5$. Suppose that $Z(A) = \{0, a, b, c, d, e\}$. The edges $a - b$, $b - c$, $c - d$, $d - e$, $e - a,$ and no other zero divisor relation, $ab = 0$ implies that $(-a)b = 0$, and consequently, $a = -a$. Similar arguments lead to $e = -e$, …. Thus, $-x = x$ for all x in $Z(A)$. Now $(b + e)a = ba + ea = 0$, hence $(b + e)$ belongs to $\{0, a, b, e\}$. If $b + e = b$ or e, then $b = 0$ or $e = 0$, that is a contradiction. If $b + e = 0$, then $b = -e = e$, contradiction. Lastly, if $b + e = a$, then in view of $(b + e)a = 0$, we have $a^2 = 0$. Similarly, $x^2 = 0$ for all x in $Z(A)$. Thus, $Z(A) = nil(A) = \{0, a, b, c, d, e\}$. Now, A being finite implies that $Z(A)$ is the unique prime ideal of A, hence $Z(A) = ann(x)$ for some x in $Z(A)$. Thus, $nil(A) = ann(x)$ for some non zero x in $Z(A)$. But $|ann(x)| = 4$, at case $x = a$, the

$ann(a) = b, e, a, 0$. Similar for every non zero x in $Z(A)$, which is a contradiction with the fact that $nil(A) = ann(x)$ and cardinality of $|nil(A)| = 6$. The case for $n > 5$ is similar.$\blacksquare$

2.3 $\boldsymbol{\Gamma(A)}$ has a spanning tree which is a star graph:

This section provides the ring theoretic conditions that must be satisfied by a ring A to have a zero divisor graph $\Gamma(A)$ which has a vertex adjacent to every other vertex. We start by the following preliminary lemma:

Lemma 2.3.1: Let A be a ring in which there is a element x with $x^2 = x$. Then $A = Ax \oplus A(1 - x)$.

Proof: It is clear that $A = Ax \oplus A(1 - x)$ as if $r \in A$, then $r = rx + r(1 - x)$. Remains to show that $Ax \cap A(1 - x) = \{0\}$. For that let $y = ax = b(1 - x) \in Ax \cap A(1 - x)$. Then $xy = xax = xb(1 - x)$. Hence $xy = ax^2 = b(x - x^2)$. But $x = x^2$, implies that $ax^2 = ax$ and $b(x - x^2) = 0$, Which leads to $xy = y = 0$.$\blacksquare$

The following theorem provides necessary and sufficient conditions for a ring A to have a zero divisor graph in which there is a vertex adjacent to all other vertex.

Theorem 2.3.2) Let A be a commutative ring. Then there is a vertex of A which is adjacent to every other vertex if and only if either $A \cong \mathbb{Z}_2 \times F$ where F is an integral domain, or $Z(A)$ is prime ideal.[3]

Proof: Suppose that $Z(A)$ is not an annihilator ideal. Let $0 \neq a \in Z(A)$ be an element which is adjacent to every other element. Notice that $a^2 \neq 0$, (if $a \in ann(a)$, then $Z(A) = I$ would be an annihilator ideal. Thus I is maximal among annihilator ideal and hence is prime). If $a^2 \neq a$, then a^2 is a zero divisor in $ann(a)$, thus $a^3 = a^2.a = 0$ since $ann(a)$ is prime. This implies that $a \in ann(a)$, which is a contradiction.

Thus $a^2 = a$, and consequently, $A = Aa \oplus A(1-a)$ (by lemma 2.3.1). So, we may assume that $A = A_1 \times A_2$, and $(1,0)$ is adjacent to all nonzero–zero divisor. For any $1 \neq c \in A_1$, $(c,0)$ is a zero divisor, since $(c,0)(0,b) = 0$ for all $b \in A_2$. But this implies that $(c,0) = (c,0)(1,0) = 0$, contradiction. Unless $c = 0$, hence $A_1 \cong \mathbb{Z}_2$.

If A_2 is not an integral domain, then there is a non zero $b \in Z(A_2)$. Then $(1,b)$ must be a zero divisor. But $(1,b)$ can not adjacent to $(1,0)$, a contradiction. Thus A_2 must be an integral domain. (Note that if $Z(A)$ is an annihilator ideal, then it is certainly maximal among annihilator ideals and hence is prime).

If $A \cong \mathbb{Z}_2 \times F$ for F an integral domain. Then the element $(1,0)$ is adjacent to every other vertex, since each has the form $(0,a)$ where a is non-zero. If $Z(A) = ann(x)$ for some non zero $x \in A$, then x is adjacent to every other vertex. ∎

In proving the previous theorem, if a vertex x of $\Gamma(A)$ is adjacent to every other vertex then either x is idempotent or $Z(A) = ann(x)$.

Definition 2.3.3: Let A be a commutative ring. A is Noetherian if it satisfies the following three equivalent conditions:[2]

1) Every ideal in A is finitely generated.
2) Every non empty set of ideals in A has maximal element.
3) Every ascending chain of ideals in A is stationary.

Example 2.3.4: The real numbers, and the complex numbers, are a Noetherian ring.

The following is a corollary of ***Theorem 2.3.2*** which concerns with the Noetherian case.

Corollary 2.3.5) Let A be a commutative Noetherian ring. Then $\Gamma(A)$ has a vertex a adjacent to all other vertex if and only if either $A \cong \mathbb{Z}_2 \times F$, where F is an (Noetherian) integral domain or $Z(A)$ is an (prime) ideal of A.

Proof: This is a direct consequence of the fact that in the Noetherian context, $Z(A)$ is an annihilator ideal if and only if it is an prime ideal.[10]

2.4 Complete zero divisor graph:

This section is about the complete zero divisor graphs. We will see different ring theoretic properties of the rings that have complete zero divisor graph.

For x, y in $Z(A)$, define $x \sim y$ if $xy = 0$ or $x = y$. The relation $\sim$ is always reflexive and symmetric, but not transitive in general. This relation can be used to characterize the complete zero divisor graphs through the following proposition.[3]

Proposition 2.4.1: The $\sim$ is transitive if and only if $\Gamma(A)$ is complete.

Proof: Suppose that $\sim$ is transitive and $x \sim y$,$y \sim z$. Then $x \sim z$. Note that $xy = yz = xz = 0$. It is clear that any two vertices are adjacent. Thus, the graph is complete.

Conversely, assume that $\Gamma(A)$ is complete. If $\{x, y, z\}$ any vertices in the graph, then $xy = yz = xz = 0$. Which implies that $x \sim y$, $y \sim z$ and $x \sim z$ .Thus the relation $\sim$ is transitive.$\blacksquare$

The following is an illustrative example to ***Proposition 2.4.1***

Example 2.4.2: It is very clear that the relation $\sim$ in proposition 2.4.1 is not transitive over $Z(\mathbb{Z}_8)$. Indeed, $2 \times 4 = 0$, $4 \times 6 = 0$, but $2 \times 6 = 4 \neq 0$ and $2 \neq 6$. Thus, by Proposition 2.4.1, $\Gamma(\mathbb{Z}_8)$ is not complete.

The following theorem provides another characterization of a complete zero divisor graph.

Theorem 2.4.3) Let A be a commutative ring. Then $\Gamma(A)$ is complete if and only if either $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(A)$.[17]

proof:$\Leftarrow$ By definition.

$\Rightarrow$Suppose that $\Gamma(A)$ is complete. Then $xy = 0$ for all x, y distinct elements in $Z(A)$. We have to show that either $(Z(A))^2 = 0$ ($xy = 0$ for all $x, y \in Z(A)$) or $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. If $(Z(A))^2 \neq 0$, then since $\Gamma(A)$ is complete we must have an element $x \in Z(A)$ such that $x^2 \neq 0$, we claim that $x^2 = x$. On the contrary assume that $x^2 \neq x$. Then again since $\Gamma(A)$ is complete, $x^3 = x^2 . x = 0$. So that $x^3 + x^4 = 0$, and hence $x^2(x + x^2) = 0$. But $x^2 \neq 0$

implies that $x + x^2 \in Z(A)$. Now, if $x + x^2 = x$, then $x^2 = 0$ which contradicts the assumption. So $x + x^2 \neq x$ and since $\Gamma(A)$ is complete, we have $0 = x(x + x^2) = x^2 + x^3 = x^2$, a contradiction again. Thus $x = x^2$ and this implies that $A = Ax \oplus A(1 - x)$. (by lemma 2.3.1)

So we have $A \cong A_1 \times A_2$. We finish the proof by showing $A_1 \cong A_2 \cong \mathbb{Z}_2$. If $A_1 \neq \mathbb{Z}_2$, then there is $a \in A_1$ with $a \neq 1$. Now both of $(a, 0)$ and $(1,0)$ are in $Z(A_1 \times A_2)$ and $\Gamma(A)$ being complete forces $(a, 0)(1,0) = (0,0)$ and hence $a = 0$. So A_1 can have only two elements 0 and 1 and thus $A_1 \cong \mathbb{Z}_2$. In a similar way we show that $A_2 \cong \mathbb{Z}_2$.▌

Remark 2.4.4: If we exclude the case $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then in view of theorem 2.4.3, $\Gamma(A)$ is complete if and only if all the vertices are adjacent, even if they are not distinct, equivalently $\Gamma(A)$ is complete if and only if $xy = 0$ for all (not necessarly distinct) $x, y \in Z(A)$; Or $\Gamma(A)$ is complete if and only if $(Z(A))^2 = 0$.

The following corollary is an easy consonance of proposition 2.4.1. It gives an equivalent definition for the complete graphs in the language of relations.

Corollary 2.4.5) Let A be a commutative ring. For $x, y \in Z(A)$ define $x{\sim}^*y$ if $xy = 0$.The relation $\sim^*$ is an equivalence relation if and only if $\Gamma(A)$ is complete and $A \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

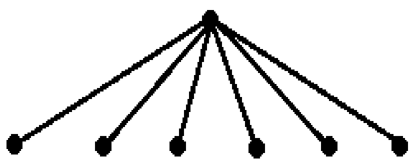
Proof: Suppose that the relation is equivalence, let $x, y \in Z(A)$. Then $xy = 0$. But the relation is equivalence implies that $yx = 0$. Now by definition of relation $x^2 = 0$. Hence $\Gamma(A)$ is complete.▌

Definition 2.4.6: A bipartite graph is a graph whose vertices V can be divided into two independent sets (there is no vertices in the same set are adjacent) X and Y and every edge connects one vertex in X to one vertex in Y.[23]

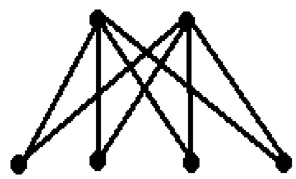
Definition 2.4.7: In bipartite graph if every vertex in X adjacent to every vertex in Y, then the graph is called a Complete Bipartite graph.[23] If X have n elements, Y have m elements then the complete bipartite graph denoted by $K^{n,m}$.[3]

If the complete bipartite graph takes the form $K^{1,n}$, then it is called a star graph.

Example 2.4.8: In the ring $A = \mathbb{Z}_3 \times \mathbb{Z}_5$. It is clear that the graph is complete bipartite graph. And the ring $A = \mathbb{Z}_2 \times \mathbb{Z}_7$ is a star graph. Below the sketch.



Z2xZ7



Z3xZ5

The following lemma proves that the graph of a product of two rings is a complete bipartite graph if and only if each ring is an integral domain.

Lemma 2.4.9: Let $A = A_1 \times A_2$. Then A is a complete bipartite graph if and only if A_1 and A_2 are integral domains.[6]

Proof: If $A = A_1 \times A_2$, where A_1 and A_2 are integral domains, then $Z(A_1 \times A_2) = C \cup B$ where $C = \{(x, 0)\backslash\, x \in A_1{}^*\}$ and $B = \{(0, y)\backslash y \in$

A_2*}. Hence the vertex set of $A_1 \times A_2$ is a union of two disjoint sets of vertices. Moreover, $(x, 0)(0, y) = (0,0)$ for all $x \in A_1$* and $y \in A_2$* implies that each vertex in C is adjacent to each vertex in B. Lastly, there is no other adjacency in $\Gamma(A_1 \times A_2)$ since both of A_1 and A_2 are domains $((x, 0)(y, 0) = (0,0)$ if and only if $x = 0$ or $y = 0)$.

Now if A_1 is not an integral domain, then there exist $x, y \in Z^*(A_1)$ such that $xy = 0$. Lastly, we have the 3-cycle $(x, 0)$ _$(y, 0)$_ $(0, 1)$ _ $(x, 0)$, so A is not a complete bipartite graph.

2.5 Cycle and line graphs zero divisor:

This section describes the graphs which is cycles or line graphs. And displays the relationship between the cycle graph and the girth values of the graph.

Recall that a subgraph of a graph is any subset of vertices together with any subset of edges containing those vertices. An induced subgraph is a subgraph maximal with respect to the number of edges. If an induced subgraph is itself complete, it is called a clique. The number of vertices in a maximal clique of a graph Γ is denoted by $cl(\Gamma)$.

Definition 2.5.1: A path of length n from a vertex a to a distinct vertex b is a sequence of $n + 1$ distinct vertices, $a = v_0, v_1, \dots, v_n = b$, such that v_i and v_{i+1} are adjacent for $0 \leq i \leq n - 1$.

If in a graph Γ there are vertices x and y such that $d(x, y) = \infty$ (no path between x and y). We say that the graph is disconnected.

Definition 2.5.2: If $x = y$ in a path $x = v_0, v_1, \dots, v_n = y$, then we call that the graph a cycle graph.

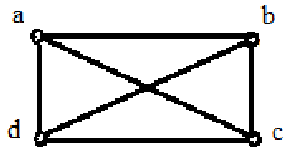


figure 1

In figure1, the sequence $a - b - c - d$ is an example of a path of length 3, the sequence $b - c - d - b$ defines a cycle of length 3.

This sequence $a - b - c - d - a$ in figure1, is a cycle of length 4.

A cycle of length three is called a triangle, and a cycle of length four is a square, and so on. The cycle $b - c - d - b$ in the figure1 is a triangle, and a cycle $a - b - c - d - a$ in figure1 is a square. It is clear that the graph has diameter one and girth three.

The line graph $L(G)$ of a graph G is defined to be the graph whose vertex set constitutes of the edges of G, Where two vertices are adjacent if the corresponding edges have a common vertex in G (A graph with points connected by lines).[15]

Remark 2.5.3: The line graph Γ_n can be realized as $\Gamma(A)$ if and only if $n \leq 3$.[3]

Proof: Suppose that $n = 4$, and $a - b - c - d$ is the only edge such that $ab = 0$, $bc = 0$, $cd = 0$. Now, $b(a + c) = ba + bc = 0$, hence $(a + c)$ is adjacent to b or b or zero, so that $(a + c) \in \{0, a, b, c\}$. If $(a + c) = 0$, then $a = -c$. Clearly that $da = -cd$ then $da = 0$, contradiction. If $(a + c) = a$, then $c = 0$.Which is a contradiction. Similarly, if $(a + c) = c$, then $a = 0$, contradiction. Now, $\boldsymbol{a + c = b}$**.** It is clear that $c(b + d) = cb + cd = 0$, hence $(b+d)$ is adjacent to c or c or zero, so $(b+d) \in \{0, c, b, d\}$. if $b+d=0$, then $b=-d$, clearly that $ab=-ad$ then $ad=0$, a contradiction. If $(b+d)=b$ then $d=0$, a contradiction. Similarly, if $(b+d)=d$. Now, if $(b+d)=c$ then $a+c+d=c$ implies that $a=-d$, hence $ca=-cd$ implies that $ca=0$. Contradiction with c not adjacent to a. Similarly, in $n \geq 5$, therefore we must have $n \leq 3$.$\blacksquare$

Proposition 2.5.4: Every graph G containing a cycle satisfies $g(G) \leq 2diam(G)+1$.[19]

Proof: Suppose that C is a shortest cycle in a graph G. Assume that $g(G) \geq 2diam(G)+2$, then C has two vertices x and y such that $d(x,y)$ in C at least $diam(G)+1$. In a graph G, x and y have a less distance, so any shortest path P between x and y is not a subgraph of a cycle C. Thus, the distance from x to y in G, $d(x,y) \leq diam(G)$. And the distance from x to y in C at least $diam(G)+1$. Together ($d(x,y)$ in G and in C) they form a cycle shortest than C. Which is a contradiction.$\blacksquare$

By examples we note that $\Gamma(A)$ is always connected with $diam(\Gamma(A)) \leq 3$, ($diam$ (Γ)= $sup\{d(x,y)$: x and y are distinct vertices of Γ). The following theorem prove it.

Theorem 2.5.5) Let A be a commutative ring, then $\Gamma(A)$ is connected and $diam(\Gamma(A)) \leq 3$. Furthermore, if $\Gamma(A)$ contains a cycle, then $g(\Gamma(A)) \leq 7$.[3Thm 2.3]

Proof: Let $u, v \in Z^*(A)$ be distinct. If $uv = 0$, then $d(u, v) = 1$. Hence $diam(\Gamma(A)) \leq 3$. Now suppose that $uv \neq 0$. If $u^2 = v^2 = 0$, then $u - uv - v$ is a path of length 2. Thus $d(u, v) = 2$ and $diam((\Gamma(A)) \leq 3$. If $u^2 = 0$ and ${}^2v \neq 0$, then $\exists\, b \in Z^*(A) - \{u, v\}$ such that $bv = 0$. If $bu = 0$, then $u - b - v$ is a path of length 2. And if $bu \neq 0$ then $u - bu - v$ is a path of length 2, in either case $d(u, v) = 2$ and $diam((\Gamma(A)) \leq 3$.
Similarly if $u^2 \neq 0, v^2 = 0$. Lastly, if $u^2 \neq 0, v^2 \neq 0, uv \neq 0$: then $\exists\, a, b \in Z^*(A) - \{u, v\}$ such that $au = bv = 0$. If $a = b$ then $u - a - v$ is a path of length 2. And if $a \neq b$, then $ab = 0$ or $ab \neq 0$. If $ab = 0$, then $u - a - b - v$ is a path of length 3. And if $ab \neq 0$ then $u - ab - v$ is a path of length 2. Then $d(u, v) \leq 3$ in all cases. Hence $diam((\Gamma(A)) \leq 3$. Furthermore, if a graph contains a cycle, then by proposition 2.5.4, $g(G) \leq 2diam(G) + 1$. Which implies that $g(G) \leq 2(3) + 1$. Hence $g(G) \leq 7$.∎
As a consequence of theorem 2.5.5 For $a, b \in Z^*(A)$, either $ab = 0$, or $ac = cb = 0$ for some $c \in Z^*(A) - \{a, b\}$, or $ac_1 = c_1c_2 = c_2b = 0$ for some distinct c_1,$c_2 \in Z^*(R) - \{a, b\}$.

***Example 2.5.6*:** In $A = \mathbb{Z}_{25}$ all paths show that $diam(\Gamma(A)) = 1$. And in $A = \mathbb{Z}_6$ the path $2 - 3 - 4$ shows that $diam(\Gamma(A)) = 2$. And in $A = \mathbb{Z}_2 \times \mathbb{Z}_4$ the path $(0,3) - (1,0) - (0,2) - (1,2)$ shows that $diam(\Gamma(A)) = 3$.
Now we call that a ring A is Artinian if A satisfies the descending chain condition of ideals, that is no infinite descending sequence of ideals.

The following theorem explain that If $\Gamma(A)$ contains a cycle when A be a commutative Artinian ring, then the girth of $\Gamma(A)$ can not be grater than or equal 5.

Theorem 2.5.7) Let A be a commutative Artinian ring. If $\Gamma(A)$ contains a cycle, then $gr(\Gamma(A)) \leq 4$.[3]

Proof: Suppose that $\Gamma(A)$ contains a cycle, A is a commutative Artinian ring. Then A is a finite direct product of Artinian local rings [14, thm 8.7]. Now we have three cases:

Case 1: Suppose that A is local with unique maximal ideal $M \neq 0$. Then $M = ann(x)$, for some $x \neq 0$ in M [10 thm82]. If there are $y, z \in M^{*}-\{x\}$ with $yz = 0$, then $y - x - z - y$ is a triangle (cycle) in this case $gr(\Gamma(A)) = 3$. Other wise, $\Gamma(A)$ contains no cycle, contradiction.

Case 2: Suppose that $A = A_1 \times A_2$. If $|A_1| \geq 3$ and $|A_2| \geq 3$, we may choose $a_i \in A_i - \{0,1\}$ then $(1,0) - (0,1) - (a_1, 0) - (0, a_2) - (1,0)$ is a square (cycle), in this case $gr((\Gamma(A)) \leq 4$.

Case 3: Suppose that $A = Z_2 \times A_2$. If $|Z(A_2)| \leq 2$, then $\Gamma(A)$ contains no cycle, contradiction. Hence, we must have $|Z(A_2)| \geq 3$.

Since $\Gamma(A)$ is connected, there are two distinct vertices $x, y \in Z(A_2) - \{0\}$ such that $xy = 0$. Thus $(0, x) - (1,0) - (0, y) - (0, x)$ is a triangle (cycle), in this case $gr(\Gamma(A)) = 3$. Thus, in all cases $gr(\Gamma(A)) \leq 4$.∎

Corollary 2.5.8: Let A be a finite commutative ring. Then A has $gr(\Gamma(A)) = 4$ if and only if: $A \cong F \times K$, where F, K finite fields and $|F|, |K| \geq 3$. For example: $\mathbb{Z}_3 \times \mathbb{Z}_5$. Or $A \cong F \times D$, where F is finite field with $|F| \geq 3$ and D is finite ring with $|Z(D)| = 2$. For example: $\mathbb{Z}_3 \times \mathbb{Z}_4$.

Corollary 2.5.9: Let A be a finite commutative ring. Then A has $gr(\Gamma(A)) = \infty$, if and only if: $|\Gamma(A)| \leq 2$, $|\ \Gamma(A)| = 3$ and $\Gamma(A)$ is not complete. Or $A \cong \mathbb{Z}_2 \times F$, where F is a finite field or finite ring with $|Z(F)| = 2$. For example: $\mathbb{Z}_2 \times \mathbb{Z}_7$.

Chapter Three

The Central Sets and The Radius of The Zero Divisor Graph of Commutative Ring

Preview

This chapter display some properties of the zero divisor graph for a commutative ring. We define the ring of Gaussian integers modulo n, $Z_n[i]$. The center, the median, and the radius are determined. And we compute the domination and k −domination number and the $2-packing$ number of $\Gamma(A)$, where A is an Artinian ring. Perfect zero divisor graphs $\Gamma(A)$ are investigated.

3.1 The ring of Gaussian integers modulo n We start this section by the definition of the Gaussian integers.

Definition 3.1.1: The set of Gaussian integers denoted by, $\mathbb{Z}[i]$, is defined by $\mathbb{Z}[i] = \{a + bi: a, b \in \mathbb{Z} \; and \; i = \sqrt{-1}\}$.

Remark 3.1.2: The set $\mathbb{Z}[i]$ is a subring of the ring of complex numbers under the usual addition and multiplication of complex numbers.

Definition 3.1.3: A prime element P of a commutative ring A is an element which neither zero nor unit (has a multiplicative invers) and if Pdivides xy, where $x, y \in A$ then P divides x or P divides y. The prime elements of a Gaussian ring are called the Gaussian prime.

If x is a prime integer, then $x = 2$ or $x \cong 1 \, mod \, 4$ or $x \cong 3 \, mod \, 4$. In this thesis, p denotes a prime integer which is $p \cong 1 \, mod \, 4$ and q denotes a prime integer which is $q \cong 3 \, mod \, 4$. [6]

The following fact describes the Gaussian prime integer.

Fact 3.1.4: A Gaussian prime integer is a unit multiple of one of the following:[12]

(1) $(1 + i)$ or $(1 - i)$.

(2) A prime integer q in $\mathbb{Z}$ which $q \cong 3 \, (mod \, 4)$.

(3) $a + ib$ and $a - ib$, where $p = a^2 + b^2$ and p is a prime integer in $\mathbb{Z}$ which $p \cong 1 \, (mod \, 4)$.[6]

Now, p and p_j denote prime integers which are congruent to 1 modulo 4, while q and q_j denote prime integers which are congruent to 3 modulo 4.

Definition 3.1.5: Let n be a natural number greater than 1 and let $< n >$ be the principal ideal generated by n in $Z[i]$, and let $Z_n = \{0,1,2, \dots, n-1\}$ be the ring of integers modulo n. Then the factor ring $Z[i]/< n >$ is isomorphic to $Z_n[i] = \{\overline{a} + i \, \overline{b} : \overline{a}, \overline{b} \in Z_n\}$. Clearly, $Z_n[i]$ is a ring under addition and multiplication modulo n. This ring is called the ring of Gaussian integers modulo n.

Recall the Chinese remainder theorem by definition:

Definition 3.1.6: Let $x_1 \dots \dots x_n$ be ideals of a commutative ring A, with $x_i + x_j = A$, for every $i \neq j$. Then for every $a_1 \dots \dots a_n \in A$ there exist $a \in A$ such that $a \equiv a_i \, mod \, x_i$ for $1 \leq i \leq n$.

Theorem 3.1.7: If $\mathrm{n} = \mathrm{p}$ such that $(p \cong 1\, mod\ 4)$ or $\mathrm{n} = \mathrm{q_1 q_2}$ such that $(q_j \cong 3\, mod\ 4)$, then $\Gamma(\mathbb{Z}_n[\mathrm{i}])$ is complete bipartite graph .[6, Lemma16]

Proof: Suppose that $n = p$. Now $p \cong 1\ (mod\ 4)$, implies that $p = a^2 + b^2$, then $\mathbb{Z}_p[i] \cong Z[i]/< p >= Z[i]/< a^2 + b^2 > = Z[i]/< a + bi >< a - bi > = Z[i]/< a + bi > \times Z[i]/< a - bi >$, hence $\mathbb{Z}_p[i]$ is a product of two integral domain. By lemma 2.4.9 $\Gamma(\mathbb{Z}_p[i])$ is a complete bipartite graph.

If q_1 and q_2 are two primes such that $q_j \cong 3\ (mod\ 4)$, for each j, then $\mathbb{Z}_{q_1 q_2}[i] \cong \mathbb{Z}_{q_1}[i] \times \mathbb{Z}_{q_2}[i]$ (by definition 3.1.6), is a direct product of two fields. Which implies that $\Gamma(\mathbb{Z}_{q_1 q_2}[i])$ is a complete bipartite graph. So If $\mathrm{n} = \mathrm{p}$ or $\mathrm{n} = \mathrm{q_1 q_2}$, then $\Gamma(\mathbb{Z}_n[\mathrm{i}])$ is complete bipartite graph.$\blacksquare$

In any complete bipartite graph it is clear that any vertex v, $ecc(v) = 2$. Hence by theorem 3.1.7 if $\mathrm{n} = \mathrm{p}$ or $\mathrm{n} = \mathrm{q_1 q_2}$, then for every vertex in $\Gamma(\mathbb{Z}_n[i])$, $ecc(v) = 2$. Hence, the center of $\Gamma(\mathbb{Z}_n[i])$ is $V(\Gamma(\mathbb{Z}_n[i])$.

3.2 THE CENTER MEDIAN AND THE RADIUS OF $\boldsymbol{\Gamma}(\boldsymbol{A})$

This section describes some of the characteristics of each center, median and radius, and shows the relationship between center, median and radius. As well as the relationship between the diameter and the radius.

We start by recalling some basic definitions. For a connected graph G, let v be any vertex in a graph G. Then the degree of v, $deg(v)$ is the number of vertices adjacent to v. The minimum degree of G denoted by $\delta(G)$ is defined by $min\{deg(x) : x \in V\ (G)\}$. A graph in which all vertices have the same degree is called a regular graph. The distance $d(u, v)$ between two vertices

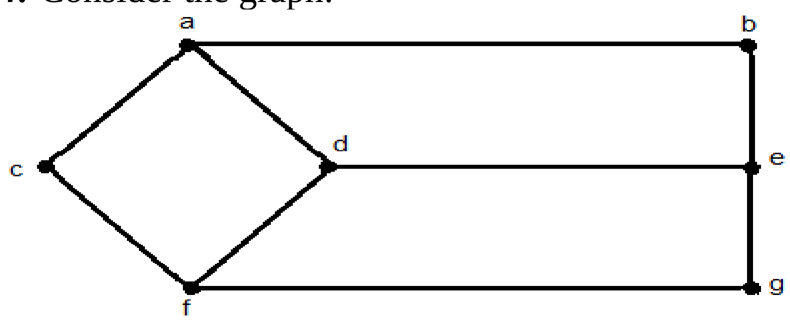
u and v is the minimum of the lengths of all $u - v$ paths of G. The eccentricity of a vertex v in G is the maximum distance from v to any vertex in G. The radius of G, $rad(G)$, is the minimum eccentricity among the vertices of G.The set of vertices with minimal eccentricity is called the center of the graph, and this minimum eccentricity value is the radius of G.

Definition 3.2.1: The minimum eccentricity from all the vertices is called the radius of the Graph G.

Definition 3.2.2: If the eccentricity of a vertex of a graph is equal to its radius ($e(V) = r(V)$), then this vertex is a central point of the graph.

Definition 3.2.3: The center of the graph is defined to be the set of all central points.

Example 3.2.4: Consider the graph:



In this graph, the distance between vertex e to vertex d is $1 (d(e, d) = 1)$ as we have one edge between them. There are many paths from d to e

- da, ab, be
- df, fg, ge
- de (this is the shortest path so that is considered for distance between the vertices)
- df, fc, ca, ab, be

- da, ac, cf, fg, ge

In the above graph, the eccentricity of a is 3. Since the distance from a to b is 1 (ab),

From a to c is 1 (ac),

From a to d is 1 (ad),

From a to e is 2($ab - be$) or ($ad - de$),

From a to f is 2 ($ac - cf$) or ($ad - df$),

From a to g is 3 ($ac - cf - fg$) or ($ad - df - fg$).

So the eccentricity is 3, which is a maximum distance from vertex a to any vertex (the distance between ag which is maximum).

Similarly,

$e(d) = 2$

$e(g) = 3$

$e(b) = 3$

$e(e) = 3$

$e(c) = 3$

$e(f) = 3$

In the above graph the radius $r(G) = 2$, which is the minimum eccentricity for d. And the diameter of a Graph $d(G) = 3$, which is the maximum eccentricity.

In the example, d is the central point of the graph. Since $e(d) = r(G) = 2$. We say that $\{d\}$ is the centre of the Graph.

Lastly, the Girth of the graph is 4 (number of edges in the shortest cycle of G), this is the shortest cycle in G: $a-c-f-d-a$ or $d-f-g-e-d$ or $a-b-e-d-a$.$\blacksquare$

If $Z(A)$ is an ideal, then the graph has a vertex which is adjacent to every other vertex (by theorem 2.3.2). In this case the radius equal zero if the graph has exactly one vertex (in this case the graph has no edge), for example $\mathbb{Z}_4$, and the radius equal one if $|Z*(A)| \geq 2$. Hence if $Z(A)$ is an ideal then the radius at most one.

The following theorem describes the radius of a Noetherian commutative ring.

Theorem 3.2.5**:** Let A be a commutative Noetherian ring with identity that is not integral domain. Then the radius of $\Gamma(A)$ is at most 2. [22]

Proof: Assume that A is not an integral domain and $Z(A)$ is not an ideal (otherwise, the graph has a vertex which is adjacent to every other vertex and hence the radius at most1). Now we have two cases:

Case1: Suppose A is reduced (it has no non-zero nilpotent elements). Now, $Z(A) = \cup_{i=1}^{n} p_i$, where each P_i is a minimal prime of A. Since $Z(A)$ is not an ideal, $n \geq 2$ (otherwise, $Z(A) = P_1$ which is an ideal). For $j = 1, \dots, n$, choose $0 \neq y_j \in \cap \{P_i \,/\, i = 1, \dots, j-1, j+1, \dots, n\}$, this y_j exist since P_j is a minimal prime ideal of A. Let $x \in Z(A)$. Then $x \in P_m$ for some $1 \leq m \leq n$. Clearly, $xy_m = 0$ since $xy_m \in \cap P_i = nil(R) = \{0\}$ [14prop1.8]. So that if $j \neq m$, then $y_m y_j = 0$. Thus, $d(y_j, x) = 1$ (if $j = m$) or $d(y_j, x) = 2$, (if $j \neq m$). Hence, the radius of $\Gamma(A)$ is at most 2.

Case2: Suppose A is not reduced ($nil(A) \neq 0$). $Z(A) = \bigcup_{i=1}^{n} p_i$, where each P_i is a minimal prime of A. For each $i = 1, \dots, N$, there is $0 \neq a_i \in A$ such that $P_i = ann(a_i)$ [10thm 86]. Choose $0 \neq v \in nil(A) \subseteq \cap P_i = \cap ann(a_i)$ where $i = 1, \dots, N$. Hence, $va_i = 0$ for each $i = 1, \dots, N$. Let $x \in Z(A)$. Then $x \in P_j$ for some j. Thus, either we have this path $x - v$ and in this case $xv = 0$. Or we have this path $x - a_j - v$ and in this case $xv \neq 0$.

Hence, the eccentricity of v is at most 2. Lastly x being arbitrary implies that the radius of $\Gamma(A)$ at most 2.$\blacksquare$

***Corollary 3.2.6*:** Let A be a commutative Noetherian ring with identity.

1) The radius of $\Gamma(A)$ is zero if and only if the graph has exactly one vertex.

2) The radius of $\Gamma(A)$ is one if and only if either $A \cong \mathbb{Z}_2 \times B$, where B is an integral domain, or $Z(A)$ is an ideal of A. [22]

Proof:

1) The radius is zero specially when the graph has exactly one vertex. Since the graph has no edges hence the diameter of the graph is zero.

2) Clearly that any graph G with radius 1 necessarily has at least one vertex adjacent to all other vertices of G. (by theorem 2.3.2) This case comes true if and only if either $A \cong \mathbb{Z}_2 \times B$, where B is an integral domain, or $Z(A)$ is an ideal of A.

Example 3.2.7: $rad(\mathbb{Z}_4) = 0$ and $rad(\mathbb{Z}_2 \times \mathbb{Z}_7) = 1$.

The following theorem describes the center of any ring of the form $A = A_1 \times \cdots \times A_n \times F_1 \times \cdots \times F_m$. Where A_i is a commutative Artinian local ring with identity that is not a field and each F_i is a field.

Theorem 3.2.8: Let n and m be positive integers. Let $A = A_1 \times \cdots \times A_n \times F_1 \times \cdots \times F_m$, where each A_i is a commutative Artinian local ring with identity that is not a field and each F_i is a field. For each $j = 1, \dots, m$. define the ideal $I_j = \{0\} \times \cdots \times \{0\} \times F_j \times \{0\} \times \cdots \times \{0\}$. Then the center of $\Gamma(A)$ is $J(R) \cup (\bigcup_{j=1}^{m} I_j) - \{(0,0,\dots,0)\}$. [22Thm 3.6]

Proof. Let $a = (a_1, \dots, a_n, b_1, \dots, b_m) \in V(\Gamma(A))$. By Corollary 3.2.6, it is enough to show $d(w,a) \leq 2$ for every element w in the above union. For each $i = 1, \dots, n$, let M_i be the maximal ideal of A_i. Then $J(A) = (M_1 \times \cdots \times M_n \times \{0\} \times \cdots \times \{0\})$. Let $x = (x_1, \dots, x_n, 0, \dots, 0) \in J(A)^*$. Without losing generality, say $x \neq a$. If each $x_i a_i = 0$ in A_i, then $d(x,a) = 1$. Suppose for some $1 \leq j \leq n$, $a_j \in M_j$ but $x_j a_j \neq 0$. Since A_j is local, $\Gamma(A_j)$ has radius 1 (A_j has a vertex which is adjacent to all other vertex). Thus, there is some nonzero $y_j \in A_j$ with eccentricity 1. Define $y = (0, \dots, 0, y_j, 0, \dots, 0) \in A$. (Clearly, $x \neq y$ and $a \neq y$). Then $y \in V(\Gamma(A))$, and $e(y_j) = 1$, hence y_j adjacent to every other vertex. So $ya = yx = 0$ then we have this path $x - y - a$ in $\Gamma(A)$. If each a_j is a unit in A_j, and $b_k = 0$ for some $1 \leq k \leq m$. Define $z = (0, \dots, 0, 1, 0, \dots, 0)$, where the nonzero input is the identity of F_k. Clearly $z \neq x$, $z \neq a$, then we have this path $x - z - a$ in $\Gamma(A)$. Hence, $d(x,a) \leq 2$.

Let $v \in I_j$ for some $j = 1, \dots, m$, say $v \neq a$. If $b_j = 0$, then $va = 0$ and $d(v, a) = 1$. If some other $b_k = 0$, define $y = (0, \dots, 0, 1, 0, \dots, 0)$, where the nonzero input is the identity of F_k. Then $y \neq v$, $y \neq a$ and $v - y - a$ is a path in $\Gamma(A)$. If every b_k is nonzero, some entry a_h must be a zero-divisor of A_h for some $1 \leq h \leq m$. Choose a nonzero $c_h \in A_h$ such that $a_h c_h = 0$. Define $c = (0, \dots, 0, c_h, \dots, 0, \dots, 0)$. Then $c \neq v$, $c \neq a$, and $v - c - a$ is a path in $\Gamma(A)$. Hence, in all cases, $d(v, a) \leq 2$.

Now, suppose $z = (d_1, \dots, d_n, f_1, \dots, f_m)$ not an element of the union above. In all possible cases, we have a vertex $w \in V(\Gamma(A))$ such that $d(z, w) > 2$. Note that this means $wz \neq 0$ and $ann(w) \cap ann(z) = \{0\}$, otherwise, $d(z, w)$ is one or two. Now we have three cases:

Case 1. There are index $1 \leq \mathrm{i} < j \leq m$ such that $\mathrm{f_i} \neq 0$ and $\mathrm{f_j} \neq 0$. Define $\mathrm{w} = (1, \dots, 1, 0, 1, \dots, 1)$, where zero is in index place $\mathrm{n} + \mathrm{i}$. Then $\mathrm{wz} \neq 0$ and $\mathrm{ann(w)} = \mathrm{I_i}$. Thus, $\mathrm{ann(z)} \cap \mathrm{ann(w)} = \{0\}$.

Case 2. For some index $1 \leq \mathrm{i} \leq \mathrm{n}$, $\mathrm{d_i}$ is a unit of $\mathrm{A_i}$. Choose r to be a nonzero zero-divisor of $\mathrm{A_i}$. Define $\mathrm{w} = (1, \dots, 1, \mathrm{r}, 1, \dots, 1)$. Then $\mathrm{wz} \neq 0$ and $\mathrm{ann(w)} \subseteq \{0\} \times \dots \{0\} \times A_i \times \{0\} \times \dots \times \{0\}$. Thus, $ann(w) \cap ann(z) = \{0\}$.

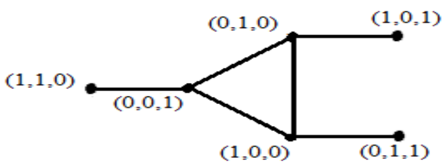
Case 3. Each $\mathrm{d_i} \in \mathrm{M_i}$ for $\mathrm{i} = 1, \dots, \mathrm{n}$, with some $d_i \neq 0$, and $f_j \neq 0$ for some index $1 \leq j \leq m$. Define $w = (1, \dots, 1, 0, 1, \dots, 1)$, where 0 is in index place $n + j$. Then $wz \neq 0$ and $ann(w) = I_j$. Thus, $ann(w) \cap ann(z) = \{0\}$. Since $d(z, w) > 2$ in any case, z cannot be in the center of $\Gamma(A)$.∎

Definition 3.2.9: The status of a vertex a, denoted $s(a)$, is the sum of the distances from a to the other vertices of G. $s(a) = \sum\{d(a,b): b \in V(G)\}$.

Definition 3.2.10: The median of a graph G is the set of vertices with minimal status. If the graph G has no edges, then the median of G is $V(G)$.

By the definition of the zero-divisor graph, $deg(a) = |ann(a)| - 2$ if $a^2 = 0$, otherwise $deg(a) = |ann(a)| - 1$.

Example 3.2.11: Consider the ring $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The zero divisor of this ring is $\{(1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1)\}$ and it is graph is shown below. It is very clear that the median of $\Gamma(A)$ is the set $\{(1,0,0), (0,1,0), (0,0,1)\}$.



The following theorem explains the relationship between the center, median and the radius of the zero divisor graph of a finite commutative ring.

***Theorem 3.2.12*:** Let A be a finite commutative ring which is not an integral domain. Then:[22]

1) If the radius of the $\Gamma(A)$ is at most 1, then the median and center of $\Gamma(A)$ are equal.

2) If the radius is two, then the median is a subset of the center.

***Proof*:** If the radius is zero, then the graph has exactly one vertex. Hence the result is clear (the median and the center is $V(\Gamma(A))$).

1**)** In any connected graph of radius1, we have a vertex x or some of vertices which is adjacent to every other vertex, those vertices are in the center. So, the distance between x (in the center) to any vertices is 1, hence the $(s(x) = |Z*(A)| - 1)$. Which implies that any vertex in the center has a minimal status and contains in the median. So the center and the median are equal.

2**)** Suppose that the radius of $\Gamma(A)$ is equal 2. Then (By corollary3.2.6) A is not isomorphic to $\mathbb{Z}_2 \times F$ for any finite field F and A is not ideal. Suppose that $A \approx A_1 \times \cdots \times A_n \times F_1 \times \cdots \times F_m$ be the Artinian decomposition of A. Let z be a vertex of $\Gamma(A)$ that is not in the center $z = (a_1, \ldots, a_n, b_1, \ldots, b_m)$. Suppose that x is in the center. We will prove that $s(x) < s(z)$. Note that if x is in the center of $\Gamma(A)$, then the $ecc(x)$ is 2 because the radius is 2. Hence,

$s(x) = deg(x) + 2(|Z*(A)| - 1 - deg(x)) = 2|Z*(A)| - deg(x) - 2.$ $(*)$

Clearly that the equation $(*)$ means that every vertex of the median must have the same degree. Since z is not in the center, there is some vertex u such that $d(z,u) = 3$. Hence,

$s(z) > deg(z) + 2(|Z*(A)| - 1 - deg(z) = 2|Z*A| - deg(z) - 2$ $(**)$ we have three cases:

If the $d(x,z) = 1$, then there is a vertex y such that $d(z,y) > 2$, since y not in the center. Suppose that $d(x,z) = 2$.

Case 1: $b_i \neq 0$ and $b_j \neq 0$ for some $1 \leq i < j \leq m$, Let $x = (0, \ldots, 0, 1, \ldots 0)$ where 1 is the identity of F_i. Then x is in the center of $\Gamma(A)$ and $ann(z) \subset ann(x)$. Since neither x nor z is nilpotent, this means

$deg(z) = |ann(z)| - 1 < |ann(x)| - 1 = deg(x)$, By $(*)$ and $(**)$, $s(z) > s(x)$.

Case 2: $b_j \neq 0$ for some $1 \leq j \leq m$. Suppose that M_i is the maximal ideal of A_i, each $a_i \in M_i$ with some $a_k \neq 0$ for some $1 \leq k \leq n$. Let $x = (0, \dots, 0, a_k, 0, \dots, 0)$. Then x is in the center of $\Gamma(A)$ and $ann(z) \subset ann(x)$. Therefore $deg(z) = |ann(z)| - 1 < |ann(x)| - 1 = deg(x)$. Hence, By $(*)$ and $(**)$, $s(z) > s(x)$.

Case 3: a_i is a unit in A_i for some $1 \leq i \leq n$. Let c be a nonzero element of the maximal ideal of A_i, and let $x = (0, \dots, 0, c, 0, .., 0)$. Then x is in the center of $\Gamma(A)$ and $ann(z) \subset ann(x)$. Therefore, $deg(z) = |ann(z)| - 1 < |ann(x)| - 1$. Hence $deg(z) \leq deg(x)$. By $(*)$ and $(**)$, $s(z) > s(x)$.

Hence, in each possible cases there is a vertex x of the center with s(x) $<$ $s(z)$. Hence, z cannot be in the median. Therefore, the median is subset of the center.∎

The following theory explains the relationship between the diameter and the radius of $\Gamma(A)$.

***Theorem 3.2.13*:** Let A be a commutative Artinian ring with identity that is not a domain.[22]

(1) The radius of $\Gamma(A)$ is zero if and only if the diameter of $\Gamma(A)$ is zero if and only if the graph has exactly one vertex.

(2) If the radius of $\Gamma(A)$ is 1, then the diameter of $\Gamma(A)$ is 1 if and only if $\Gamma(A)$ is complete. Otherwise, the diameter is 2.

(3) If the radius of $\Gamma(A)$ is 2, then the diameter of $\Gamma(A)$ is 2 if and only if $A \cong F_1 \times F_2$, where F_1 and F_2 are both fields and both not isomorphic to $\mathbb{Z}_2$. Otherwise, the diameter of $\Gamma(A)$ is 3.

Proof:

(1) This case is clear (in this case the graph has no edges and has only one vertex).

(2) If the radius of $\Gamma(A)$ is 1, then the diameter is at most 2 (since the graph has a vertex which is adjacent to all other vertex), and if the diameter equal 3 that is a contradiction, (since for any y in the center of $\Gamma(A)$ and for any vertices a and b, there is a path $(a - y - b)$). The diameter is 1 if and only if all the vertices of $\Gamma(A)$ are adjacent (the graph is complete). Suppose the radius of $\Gamma(A)$ is 2. Then the diameter is 2 or 3. If $A \cong F_1 \times F_2$ where F_1 and F_2 are fields and not isomorphic to $\mathbb{Z}_2$, then (by lemma 2.4.9) $\Gamma(A)$ is a complete bipartite graph that is not a star graph. It is customary to verify that such a graph has a diameter of 2.

Next, assume $A \ncong F_1 \times F_2$ where F_1 and F_2 are both fields and both not isomorphic to $\mathbb{Z}_2$. Consider the Artinian decomposition $A = A_1 \times \ldots \times A_n \times F_1 \times \ldots \times F_m$. In all possible cases, there is an element not in the center of $\Gamma(A)$. (Note that if $n = 0$ and $m = 2$ and $F_1 \cong \mathbb{Z}_2$ or $F_2 \cong \mathbb{Z}_2$, then $\Gamma(A)$ is a star graph and has radius 1.)

Case 1. $n \geq 1$ and $m \geq 1$. Let $x \neq 0 \in M_1$. Let $Y = (x, 0, \ldots, 0, 1, 0, \ldots 0)$, where the entry in position $n + 1$ is the identity of F_1. Then y is a zero-divisor but is not in the center since there is a vertex E such that $d(y, E) > 2$.

Case 2. $n = 0$ and $m \geq 3$. Then $A \cong F_1 \times ... \times F_m$. Then $(0,1, ... ,1)$ is a zero-divisor but is not in the center.

Case 3. $n \geq 2$ and $m = 0$. For each $i = 1, ... , n$, choose $x_i \neq 0$ in M_i. Let $z = (1, x_1 , ... , x_n)$. Then z is a zero-divisor but is not in the center.

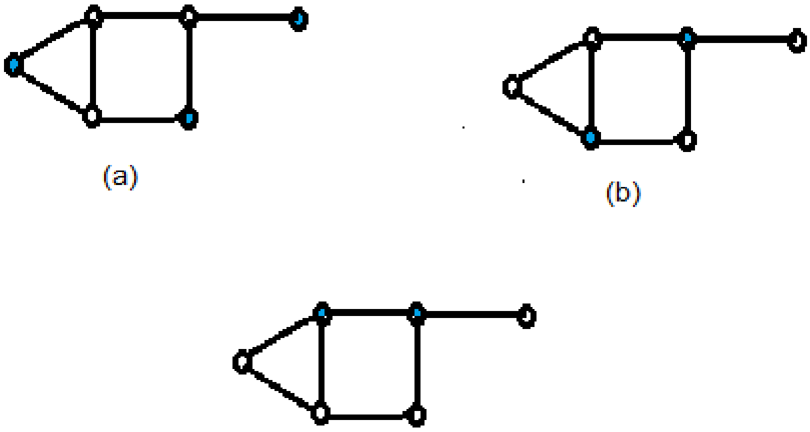
Hence, in all these remaining cases, the center is not the entire vertex set of $\Gamma(A)$. Therefore, the diameter is greater than the radius, which means that the diameter of $\Gamma(A)$ is 3.∎

3.3 MULTIPLE DOMINATION AND 2-PACKING OF $\boldsymbol{\Gamma(A)}$

This sections about domination set and the k-dominating set, it describes k-tuple and 2-packing set and cardinality of each one. And it shows the relationship between the radius and the domination number and the relationship between 2-packing number and domination number.

Recall that for a connected graph G, the dominating set of a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one vertex of D. The number of vertices in a smallest dominating set for G called a domination number of G, $\gamma(G)$.

Figures (a)–(c), there is an examples of dominating sets for a graph G. In every example, each white vertex is adjacent to at least one blue vertex, and it is said that the white vertex is dominated by blue vertex. There is a dominating set with 2 vertices in examples (b) and (c). And in example (a) dominating with 3 vertices, and we note that there is no dominating set with only 1 vertex for this graph.



(a) (b)

(c)

Definition 3.3.1: A set B is a k-dominating set for a graph A, if each vertex in $V(A)\backslash B$ is dominated by at least k vertices in B, the minimum cardinality of k-dominating set is denoted by $\gamma\kappa(A)$.

Definition 3.3.2: The set B is a k-tuple dominating set for a graph A if each element in $V(A)$ is dominated by at least k vertices in B. The minimum cardinality of a k-tuple dominating set is denoted by $\gamma \times \kappa(A)$.

Definition 3.3.3: A subset E of a vertex set $V(A)$ of a graph A is a $2-$ packing set if every $x,y \in E$, $N[x] \cap N[y] = \emptyset$. The maximum cardinality of 2-packing denoted by $\rho(A)$.

Theorem 3.3.4: Let A be a commutative Artinian ring with identity that is not a domain. If the radius of A is at most one, then the domination number of A is one. If the radius is two, then the domination number is equal to the number of factors in the Artinian decomposition of A. [22]

Proof: It is clear that if the radius is zero, then the domination number is one since the graph has exactly one vertex.

If the radius of $\Gamma(A)$ is 1, then there is a vertex which is adjacent to every other vertex. These vertices is an element of the center and forms a dominating set. Hence, the domination number is 1. Suppose the radius of $\Gamma(A)$ is 2. Let $A = \mathrm{A}_1 \times \mathrm{A}_2 \times \cdots \times \mathrm{A}_\mathrm{n} \times \mathrm{F}_1 \times \mathrm{F}_2 \times \cdots \times \mathrm{F}_\mathrm{m}$ be the Artinian decomposition of A. For each $i = 1,2, \dots, n$. Define $y_i = (0, \dots, 0, x_i, 0, \dots, 0)$ where x_i in the center of $\Gamma(\mathrm{A_i})$. For each $j = 1, 2, \dots, m$. Define $z_j = (0, \dots, 0, 1, 0, \dots, 0)$, where the entry in coordinate $n + j$ is the identity of F_j. Let $S = \{y_1, \dots, y_n, z_1, \dots z_m\}$. Note that all the elements of S are adjacent. Suppose that $w = (a_1, \dots, a_n, b_1, \dots, b_m)$ is a vertex of $\Gamma(A)$. Then some coordinate of w is a zero-divisor of the respective coordinate ring. If a_i is a zero-divisor of $\mathrm{A_i}$ for some $1 \leq i \leq n$ such that $a_i x_i = 0$, then w is adjacent to y_i. If $b_j = 0$ for some $1 \leq j \leq m$, then w is adjacent to z_j. Thus, any element of $V(A)$ is adjacent to some element of S. Hence, S is dominating set of $\Gamma(A)$. Again, suppose the radius of $\Gamma(A)$ is 2, and assume B is a dominating set for $\Gamma(A)$. Then $|B| \geq 2$ since the radius is 2 then $\Gamma(A)$ has no vertex adjacent to all others. Hence, assume $n + m \geq 3$ (if $n + m < 3$ that is a contradiction since $|B| \geq 2$). For each $k = 1, \dots, n + m$, define $t_k = (1, 1, \dots, 1, 0, 1, \dots, 1)$, where the 0 entry is in coordinate k. Every t_k is a vertex of $\Gamma(A)$, for each k, either $t_k \in B$ or there is an element of the form $(0, \dots, 0, s_k, 0, \dots, 0) \in B$ adjacent to t_k, where $s_k \in (R_k)^*$ if $1 \leq k \leq n$ and $s_k \in (F_{k-n})^*$ if $n + 1 \leq k \leq n + m$. Thus, B must contain at least $n + m$ elements.∎

The following theorem describe the k-domination number of $\Gamma(A)$ such that A is a commutative Artinian ring.

Theorem 3.3.5. Let A be a commutative Artinian ring with unity that is not a domain, $A = \mathrm{A_1} \times \mathrm{A_2} \times \cdots \times \mathrm{A_n} \times \mathrm{F_1} \times \mathrm{F_2} \times \cdots \times \mathrm{F_m}$. Suppose that if $n \geq 1$, then $k \leq |center(\Gamma(A_j))|, j = 1, \ldots, n$, and if $m \geq 1$, then $k \leq |F_j{*}|$, $j = 1, \ldots, m$. Then the k-domination number is equal to $k(m + n)$.[13 Thm3.1]

Proof: If A is local and $k \leq |\, center(\Gamma(A))|$, then the radius of $\Gamma(A)$ equal 1 since the graph has a vertex which is adjacent to every other vertices. Hence each vertex in the center of $\Gamma(A)$ dominates all other vertices, then we have $\gamma k(\Gamma(A)) = k$.

If A is not local, let $A = \mathrm{A_1} \times \mathrm{A_2} \times \cdots \times \mathrm{A_n} \times \mathrm{F_1} \times \mathrm{F_2} \times \cdots \times \mathrm{F_m}$. Let $Y_j = \{y_{jt}\}_{t=1}^{k}$. Where $y_{jt} = (0, 0, \ldots, 0, x_{jt}, 0, \ldots, 0)$ such that $x_{jt} \in center(\Gamma(\mathrm{A_j}))$ and $j = 1, \ldots, n$. And $Z_s = \{z_{st}\}_{t=1}^{k}$, where $z_{st} = (0, 0, \ldots, 0, u_{st}, 0, \ldots, 0)$ such that $u_{st} \in F_s{*}$ and $s = 1, \ldots, m$. Let $D = Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_m$. Suppose that $w = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$ is a vertex of $\Gamma(A)$. If a_i is a zero-divisor for some $1 \leq i \leq n$ such that $a_i x_i = 0$, then w is adjacent to Y_i, if $b_j = 0$ for some $1 \leq j \leq m$, then w adjacent to z_j. Thus, any element of $V(\Gamma(A))$ is adjacent to k element of D. Hence, D is a k-dominating set of $\Gamma(A)$ So, $\gamma k(\Gamma(A)) \leq k(m + n)$.

Now, let $\mathrm{r_j} = (1, 1, \ldots, 1, 0, 1, \ldots, 1)$ where 0 is in the ith position. Then $\mathrm{N(r_i)} = \{(0, 0, \ldots, 0, x_i, 0, \ldots, 0)$, where $x_i \in \mathrm{A_i}{*}$, if $1 \leq \mathrm{i} \leq \mathrm{n}$ and $x_i \in$

F_i* otherwise}. Assume D' is any k-dominating set of $\Gamma(A)$. Since D' contain at least k vertices adjacent r_i for each i. Note that $N[r_j] \cap N[r_l] = \emptyset$, for $i \neq j$. Thus, if D' is a k-dominating set, then $|D'| \geq k(m+n)$, hence equality holds.▮

Corollary 3.3.6: Let A be a commutative Artinian ring with unity that is not a domain, then the $2-packing$ number $\rho(\Gamma(A)) = \gamma(\Gamma(A))$.[13, coro.3.3]

Proof: $\gamma(\Gamma(A)) = m + n$ (by the above theorem 3.3.5 since $K = 1$).

If A is local then $\Gamma(A)$ has avertex which is adjacent to all other vertices. Any set with two elements is not 2-paking set so the only 2-paking set are the singleton. Let $S = \{x, y\}$ where x and y any two vertices in A, then $N[x] \cap N[y]$ is not empty. So it is not a 2-paking set, so the maximal 2-paking set have one element, which implies that $\rho(\Gamma(A)) = 1$. Now, suppose that A is not local.

Let $A = A_1 \times A_2 \times \cdots \times A_n \times F_1 \times F_2 \times \cdots \times F_m$ where $n + m \geq 2$ is the Artinian decomposition of A. Let $r_i = (1, 1, \ldots, 1, 0, 1, \ldots, 1)$ where 0 is in the ith position. $N(r_i) = \{(0, \ldots, 0, x_i, 0, \ldots, 0)$ where $x_i \in A_i$* if $1 \leq i \leq n$, and $x_i \in F_i$* if $1 \leq i \leq m$. Since $N[r_j] \cap N[r_i] = \emptyset$ for every $j \neq i$. Thus, the set $R = \{r_i : i = 1, 2, \ldots, n + m\}$ is a 2-packing set with $\rho(\Gamma(A)) = m + n$ vertices. Thus $\rho(\Gamma(A)) = \gamma(\Gamma(A))$.▮

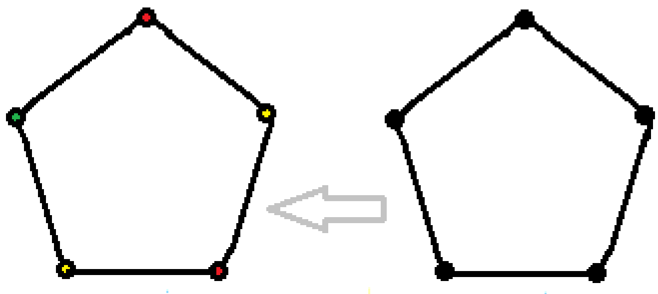
3.4 PERFECT ZERO DIVISOR GRAPH

In this section, we discuss when the graph is perfect graph. And display several situation of imperfect graph. We start by recalling the definition of clique number and the chromatic number and then the definition of perfect graph.

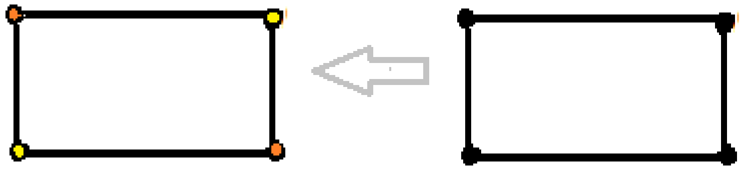
The definition of a perfect graph connecting chromatic number and clique number. Recall that the chromatic number of a graph G is the minimum number of colors required to color the vertices of G such that any two adjacent vertices have not the same color. The clique number of graph G is the size of the largest complete subgraph of G.

Definition 3.4.1: A perfect graph is a graph G for which every induced subgraph H has chromatic number equal to its clique number. For every $H \subseteq G,\ \omega(H) = \chi(H)$. Otherwise, G is called an imperfect graph.

Example 3.4.2: Take the graph of a 5-cycle, The chromatic number of the graph is 3, while the clique number is 2. Hence, the graph of a 5-cycle is not perfect (imperfect).



Now, in the graph of a 4-cycle, the chromatic number and the clique number of the graphs are 2. Thus, the chromatic number and the clique number are equal. Hence the graph of a 4-cycle is perfect.



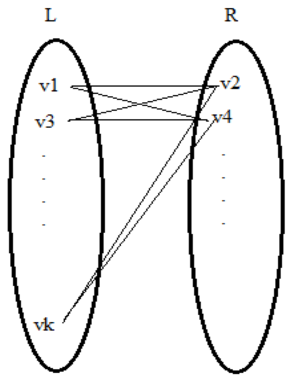
Lemma 3.4.3: The graph $G = (V, E)$ is bipartite graph if and only if G has no cycles of odd length.[11Thm 2.5]

Proof: Suppose $G = (V, E)$ is bipartite and let $v_1, \ldots, v_{k-1}, v_k, v_1$ be a cycle of odd length in G. Suppose $v_1 \in L$. Then $v_2 \in R$, since there is an edge between $\{v_1, v_2\}$. Then $v_3 \in L$, since $\{v_3, v_2\}$ are adjacent vertices. Continuing this way, we see that if i is odd, then $v_i \in L$, and if i is even then $v_i \in R$, see the below sketch. Thus, since $v_k \in L$, then v_k and v_1, are adjacent containing in L, which is a contradiction. Hence G has no cycle of odd length.

Suppose G has no cycles of odd length. We may assume that G is connected. Choose any vertex $u_0 \in V$. For every vertex $v \in V$, let p_v be any path from u_0 to v, and let d_v be its length. And p_u be any path from u_0 to u, and let d_u be its length, the Set $L = \{v \in V \mid d_v \text{ is even}\}$ and let $R = \{v \in V \mid d_v$ is odd$\}$. Clearly $V = L \cup R$ is a partition of V. We now show that G is bipartite.

If not, then there are two vertices u and v adjacent such that both $u, v \in L$ or both $u, v \in R$. In either case, there is a closed path (cycle) in G given by p_u, $\{u, v\}$, p_v (from u_0 to u, then u to v, then v to u_0), whose total length is

$d_u + d_v + 1$ which is odd (because u and v in the same set), then G also has a cycle of odd length. This is a contradiction.$\blacksquare$



The following result is from reference [13] where the authors didn't proof it here I am providing a proof of it.

Proposition 3.4.4: No cycle of odd length at least 5 is perfect.[13]

Proof: If the graph has odd cycle of length 3, then the graph is perfect since the graph is complete, hence the chromatic number and clique number are equal 3. Suppose that G has no cycle of odd length, (By theorem 3.4.3) G is bipartite graph. In bipartite graph there is two disjoint vertex sets V_1 and V_2 such that every edge connects a vertex in V_1 to a vertex in V_2, then chromatic number and clique number are equal 2. Hence G is perfect.$\blacksquare$

Definition 3.4.5: A graph is called slightly triangulated if it contains no induced odd cycle of length at least five and every induced subgraph H contains a vertex whose neighborhood in H does not contain a P_4.[13]

Definition 3.4.6: If the graph G contains no C_5, P_6 or a complement of P_6 as an induced subgraph, then the graph G is called a murky graph.[13]

This results are proved in [18,24]. I mentioned them here since they will be used in proving more than one result. Namely, Theorem 3.4.8 and Theorem 3.4.10.

Theorem 3.4.7 :

(1) If G is slightly triangulated graph, then G is perfect.[24]

(2) If G is murky graph, then G is perfect.[18]

The following theorem describes that the product of three integral domain is perfect.

Theorem 3.4.8:If $A = A_1 \times A_2 \times A_3$, where A_1, A_2 and A_3 are integral domains, then $\Gamma(A)$ is a perfect graph.[13 Thm5.6]

Proof: Suppose that $A = A_1 \times A_2 \times A_3$. Where A_1,A_2 and A_3 are integral domains. Then it easy to check that $\Gamma(A)$ is a slightly triangulated graph.

Now, any P_4 path of $\Gamma(A)$ is one of the following:

$(x_1, 0, x_3) - (0, y_2, 0) - (z_1, 0, 0) - (0, w_2, w_3)$

$(x_1, x_2, 0) - (0, 0, y_3) - (0, z_2, 0) - (w_1, 0, w_3)$

Or $(x_1, x_2, 0) - (0, 0, y_3) - (z_1, 0, 0) - (0, w_2, w_3)$

Where $x_i, y_i, z_i, w_i \in A_i{}^*$. Hence $\Gamma(A)$ has no induced odd cycle C_n, of length at least 5 and there is no vertex $v \in \Gamma(A)$ such that $N(v)$ contains a P_4. Which implies that $\Gamma(A)$ is a slightly triangulated graph and thus $\Gamma(A)$ is perfect.$\blacksquare$

The following is a direct consequence of Theorem 3.4.8 and the Chinese remainder theorem in definition 3.1.6.

***Corollary 3.4.9*:** If $n = a_1a_2a_3$, where a_1, a_2, a_3 are primes, Then $\Gamma(\mathbb{Z}_n)$ is perfect.

Proof: If $n = a_1a_2a_3$**,** then $\mathbb{Z}_n = \mathbb{Z}_{a_1a_2a_3} \cong \mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{a_3}$ (By Chinese Remainder Theorem), it is clear that every $\mathbb{Z}_{a_i}$ is an integral domain since a_i is a prime, hence (by theorem 3.4.8) $\Gamma(\mathbb{Z}_n)$ is perfect.$\blacksquare$

Note that, if $A = A_1 \times A_2 \times A_3 \times A_4$ where A_i is an integral domains for every i, then $\Gamma(A)$ is not a slightly triangulated graph since $N((1,0,0,0))$ contains $(0,1,0,0) - (0,0,1,0) - (0,0,0,1) - (0,1,1,0)$ P_4 as an induced subgraph. Then the next theorem insures that $\Gamma(A)$ is perfect.

Theorem 3.4.10: If $A = A_1 \times A_2 \times A_3 \times A_4$ where A_i is an integral domain for i = 1, 2, 3 and 4, then $\Gamma(A)$ is perfect.[13. Thm5.8]

***Proof*:** Let $x = (x_i)$, $y = (y_i)$, $z = (z_i)$, $w = (w_i)$, $u = (u_i)$, $v = (v_i) \in Z^*(A)$. Suppose that $x - y - z - w - u$ is an induced P_5 subgraph of $\Gamma(A)$. Then x has at least two non-zero components (if x has one non-zero component $x = (x_1, 0,0,0)$ that implies $wu \neq 0$ since $xw \neq 0$ and $xu \neq 0$ hence $w_1 \neq 0$ and $u_1 \neq 0$ then $wu \neq 0$, a contradiction). So, we have two cases:

Case 1: x has exactly two non-zero components, $x = (x_1, x_2, 0,0)$. Then $y = (0,0,y_3,0)$, $y = (0,0,0,y_4)$ or $y = (0,0,y_3,y_4)$. Clearly, $yw \neq 0$ and $yu \neq 0$. If $y = (0,0,y_3,0)$ then $w_3 \neq 0$ and $u_3 \neq 0$, then $uw \neq 0$, which is a contradiction. Similarly if $y = (0,0,0,y_4)$. While if $y = (0,0,y_3,y_4)$, then $z = (z_1, 0, 0, 0)$ or $z = (0, z_2, 0, 0)$ or $z = (z_1, z_2, 0, 0)$. If $z = (z_1, 0, 0, 0)$ then $w_1 = 0$, since $zw = 0$. If $z = (0, z_2, 0, 0)$ then $w_2 = 0$, since $zw = 0$. If $z = (z_1, z_2, 0, 0)$ then $w_1 = w_2 = 0$. Which implies that $w = (0, 0, w_3, w_4)$. Now, it is clear that $wu = 0$ then $u_3 = u_4 = 0$, implies that $u = (u_1, u_2, 0, 0)$. Hence $uy = 0$, which is a contradiction..

Case 2: x has exactly three non-zero components, $x = (x_1, x_2, x_3, 0)$. Then $y = (0, 0, 0, y_4)$. Clearly, $yw \neq 0$ and $yu \neq 0$, we have $w = (0,0,0, w_4)$ and $u = (0,0,0, u_4)$ hence $wu \neq 0$,which is a contradiction. Now, it is easy to see that $\Gamma(A)$ has no C_5 or P_6 as an induced subgraph. Moreover the complement of $\Gamma(A)$ has no induced P_6 path. Let $x - y - z - w - u - v$ be an induced path of the complement of $\Gamma(A)$. Then we have three cases:

Case1: x has exactly one non-zero component x_1. Then $y_1 \neq 0$ (in a complement graph x and y adjacent if $xy \neq 0$). We have one of the following:

1) If y has only one non zero components $y = (y_1, 0, 0, 0)$, it is clear that $yz \neq 0$, then $z_1 \neq 0$. But $xz = 0$, which is acontradiction.

2) If y has only two non zero components $y = (y_1, y_2, 0, 0)$, then $z = (0, z_2, z_3, z_4)$ or $z = (0, z_2, z_3, 0)$ or $z = (0, z_2, 0, z_4)$, ($z_1 = 0$ since $x_1 \neq 0$). Now, if $z = (0, z_2, z_3, z_4)$, it clear that $yu = uz = 0$(not adjacent vertices) , then $u = (0,0,0,0)$, contradiction. And if $z = (0, z_2, z_3, 0)$, by above path $yu = 0$ and $zu = 0$ implies that $u_1 = u_2 = u_3 = 0$, then $u = (0, 0, 0, u_4)$ and $wy = 0$, $zw \neq 0$, $uw \neq 0$ implies that $w = (0, 0, w_3, w_4)$ and $vw = 0$, $vy = 0$ then $v = (0,0,0,0)$, which is a contradiction. If $z =$

$(0, z_2, 0, z_4)$, $wy = 0$ then $w = (0, 0, w_3, w_4)$ and $wv = yv = 0$ implies that $v = (0,0,0,0)$, which is a contradiction.

3) If y has three non-zero components, say $\mathrm{y} = (\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, 0)$, then $\mathrm{yw} = 0$ then $\mathrm{w} = (0,0,0,\mathrm{w}_4)$ and $\mathrm{zw} \neq 0$ then $\mathrm{z} = (0,0,0,\mathrm{z}_4)$. But $uz = uy = 0$ implies that $u = (0,0,0,0)$, a contradiction.

Case 2: x has exactly two non-zero components $= (x_1, x_2, 0,0)$. Then $xu = 0$ hence $u_1 = u_2 = 0$, and $zx = 0$ then $z = (0,0, z_3, z_4)$, and $zu = 0$ then $u_3 = u_4 = 0$. Hence $u = (0,0,0,0)$, a contradiction.

Case 3: x has exactly three non zero components $x = (x_1, x_2, x_3, 0)$. Then $xz = 0$ then $z_1 = z_2 = z_3 = 0$ implies that $z = (0, 0, 0, z_4)$. And $xw = 0$ then $w = (0, 0, 0, w_4)$ but $uw \neq 0$, then $u = (0,0,0, u_4)$. Which implies that $uz \neq 0$, a contradiction. So, $\Gamma(A)$ is a murky graph and hence it is perfect.$\blacksquare$

Corollary 3.4.11: If $n = t_1 t_2 t_3 t_4$, where t_1, t_2, t_3, t_4 are primes, then $\Gamma(\mathbb{Z}_n)$ is perfect.

Proof: The proof is a direct consequence of the Chinese remainder theorem and the previous theorem (3.4.10).$\blacksquare$

The following theorem shows that if $A = A_1 \times A_2 \times A_3$, where A_1 and A_2 are non-integral domains, then the graph is imperfect graph.

***Theorem 3.4.12*:** If $A = A_1 \times A_2 \times A_3$, where A_1 and A_2 are non-integral domains, then $\Gamma(A)$ is imperfect.[13]

***Proof*:** Let $x, y \in Z^*(A_1)$ such that $xy = 0$ in A_1. And $a, b \in Z^*(A_2)$ such that $ab = 0$ in A_2. Then we have an induced C_5 subgraph of $\Gamma(A)$ $(1,0,0) - (0,a,1) - (x,b,0) - (y,0,1) - (0,1,0) - (1,0,0)$. Which implies that $\Gamma(A)$ is imperfect, ($\Gamma(A)$ has a cycle of length 5).■

The following theorem shows that the product of two imperfect graph is imperfect.

Theorem 3.4.13: If $A = A_1 \times A_2$ and $\Gamma(A_j)$ is an imperfect graph for $j =$ 1 or 2, then $\Gamma(A)$ is imperfect.[13]

Proof: Assume that $A = A_1 \times A_2$ and $\Gamma(A_2)$ is imperfect graph. Then $\Gamma(A_2)$ has an induced odd cycle of length at least five. If n is an odd integer, then $u_1 - u_2 - u_3 - \cdots - u_n$ is a cycle of length n of $\Gamma(A_2)$ if and only if $(0, u_1) - (0, u_2) - (0, u_3) - \cdots - (0, u_n)$ is a cycle of length n of $\Gamma(A)$. Hence $\Gamma(A)$ is imperfect.■

***Theorem 3.4.14*:** If $A = \prod_{i=1}^{n} A_i$, $n \geq 5$, then $\Gamma(A)$ is imperfect.[13]

***Proof*:** By induction.

If $n = 5$, then $\Gamma(A)$ has odd cycle of length at least five $(1,0,1,1,0) - (0,1,0,0,1) - (1,0,0,1,0) - (0,0,0,0,1) - (0,1,0,0,0) - (1,0,1,1,0)$.

Hence, $\Gamma(A)$ is imperfect graph .

Assume that $\prod_{i=1}^{k} A_i$, $k \geq 5$ is imperfect graph, then (by theorem 3.4.13) $\prod_{i=1}^{k+1} A_i \cong (\prod_{i=1}^{k} A_i) \times A_{k+1}$ is imperfect.■

Chapter Four

The Zero divisor Graph of Some Special Rings

Preview

In this chapter we investigate the zero divisor graph of $Boolean$, $k-$ Boolean, and nilpotent rings. The effect of these notions on some basic graph theory properties such as the completeness, the diameter, and having a vertex adjacent to all other vertices for the zero divisor graph are displayed in this chapter.

4.1 The zero Divisor Graph of Boolean and k-Boolean rings

In this section, we discuss the zero divisor graph of Boolean and k-Boolean rings. We start by recalling the definition of Boolean ring.

Definition 4.1.1: Let A be a ring. Then A is called a Boolean ring if $x^2 = x$ for every $x \in A$.

Remark 4.1.2: It is clear that if A is Boolean, then $x(x-1) = 0$ for any $x \in A$. Which implies that if $x \neq 1$, then x and $(x-1)$ are zero divisors. i.e., $V(\Gamma\ (A))\ =\ A \setminus \{0, 1\}$.

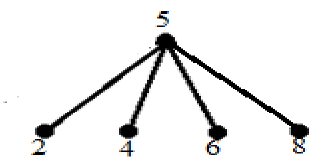
Example 4.1.3: Consider the ring $A = \mathbb{Z}_2 \times \mathbb{Z}_2$. It is clear that if x is an element in A, then $x^2 = x$. So that A is Boolean and the vertices of A are $V\big(\Gamma\ (A)\big) = A\setminus\{(0,0), (1,1)\} = \{(0,1), (1,0)\}$.

In [1], Ali Mohammadian proved that if A is finite Boolean ring with cardinality $|A| > 4$. Then $\Gamma(A)$ contains no vertex adjacent to all other vertices of the graph. In the following theorem, we lift their result to any ring (possibly infinite). Moreover, the following theorem characterizes the Boolean rings that have vertex which is adjacent to all other vertices.

Theorem 4.1.4: Let A be any Boolean ring. Then $\Gamma(A)$ contains a vertex adjacent to all other vertices of the graph if and only if $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof: Suppose that A is Boolean ring and a vertex x is adjacent to all other vertices of $\Gamma(A)$. Let $y \in Z(A) \setminus \{0, x\}$. Then we have $x(x + y) = x^2 + xy = x \neq 0$, ($xy = 0$ as x adjacent to all other vertices). Which implies that $x + y$ is a nonzero-divisor of A. Thus $x + y = 1$ is the identity of A, (As we remarked before example 4.1.3 all the elements of A are zero divisor or unit). But y was arbitrary, which implies that $A = \{0, 1, x, 1 - x\}$. Now, A being Boolean implies that $(x + 1)^2 = x + 1$ which tends to $x^2 + 2x + 1 = x + 2x + 1 = x + 1$. Thus $2x = 0$. The last equality insures that $x \neq 1 - x$ and consequently $|A| = 4$. Remains to show that $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. For this consider the map $f: A \to \mathbb{Z}_2 \times \mathbb{Z}_2$, $f(0) = (0{,}0), f(1) = (1{,}1), f(x) = (1{,}0)$ and $f(1 - x) = (0{,}1)$.$\blacksquare$

Example 4.1.5: This graph has a vertex adjacent to all other vertices. By theorem 4.1.4. A in not Boolean ring.



In 1992,Vasantha Kandasamy introduced the $k-Boolean$ rings as a generalization of the Boolean notion. He define the $k-Boolean$ ring as following.

Definition 4.1.6: Let A be a ring with identity. Then A is called a k−Boolean ring if $x^{2k} = x$ for all $x \in A$, Where k is a positive integer.

Remarks 4.1.7:

1. It is clear that if A is a $k-$ Boolean ring, then $(x(x^{2k-1} - 1) = 0$ for all x in A.
2. It is trivial that if $k = 1$ in definition 4.1.6, then we have the Boolean rings.[25]

Proposition 4.1.8: Every Boolean ring is k−Boolean ring.[25]

Proof: Let A be a Boolean ring. Then $x^2 = x$ for every x in A. Now $x(x^2 = x)$ implies that $x^3 = x^2 = x$, and hence $(x^3 = x)$. Similarly $x^4 = x$,$x^5 = x$,…, thus $x^n = x$ for any n. So $x^{2k} = x$ for any k. Hence A is a $k-$ Boolean ring.$\blacksquare$

The following theorem proves that if A is a $k-boolean$ ring with $|A| > 4$. Then the zero divisor graph of A contains no vertex adjacent to all other vertices. It is very clear that this theorem is a generalization of theorem 4.1.4.

Theorem 4.1.9: Let A be any $k-Boolean$ ring. Then $\Gamma(A)$ contains a vertex which is adjacent to all other vertices if and only if $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if A is $Boolean$.

Proof: Suppose that a vertex x is adjacent to all other vertices of $\Gamma(A)$. Let $y \in Z(A) \setminus \{0, x\}$. Then we have $x(x^{2k-1} + y) = x^{2k} + xy = x \neq 0$. Which implies that $x^{2k-1} + y$ is a nonzero-zero divisor of A. Thus $x^{2k-1} + y = 1$ is the identity of A. But, y was arbitrary implies that $A = \{0, 1, x, 1 - x^{2k-1}\}$. Now, A being $k - Boolean$ implies that $(x^{2k-1})^{2k} = x^{2k-1}$ which tends to $(x^{4k-2})^k = (x^{4k}x^{-2})^k = (x^2x^{-2})^k = 1$. Thus $x^{2k-1} = 1$. Which implies that $x^{2k-1} \neq 1 - x^{2k-1}$ and consequently $|A| = 4$. Remains to show that $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. For this consider the map $f: A \to \mathbb{Z}_2 \times \mathbb{Z}_2$, $f(0) = (0,0), f(1) = (1,1), f(x) = (1,0)$ and $f(1 - x^{2k-1}) = (0,1)$.$\blacksquare$

Remark 4.1.10: Observes that there is no $k - Boolean$ which is not $Boolean$ with vertex adjacent to all other vertices.

After we discussed the effect of the Booleanity and k −Booleanity of the rings. On the property of having a vertex which is adjacent to all other vertices, we would like to see the effect of these notions on another graph theory property; namely, the completeness. In the following corollary we appeal to theorem 4.1.4 and theorem 4.1.9 to show that the zero divisor graph of a $(k) - Boolean$ ring is never complete.

Corollary 4.1.11: Let A be a $(k) - Boolean$ ring with $|A| > 4$. Then $\Gamma(A)$ cannot be a complete graph.

Proof: Suppose that A is a $(k) - Boolean$ ring and $\Gamma(A)$ is a complete graph. Then (by definition of complete graph) every vertex in $\Gamma(A)$ is adjacent to all other vertices. But $\Gamma(A)$ contains no vertex adjacent to every other vertices unless $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (by theorem 4.1.4 and theorem 4.1.9). which is a contradiction.∎

4.2 The zero Divisor Graph of nilpotent ring

nilpotent rings. For example, in Corollary 4.2.4, we observe that the diameter of zero divisor graph of a nilpotent ring is mostly 2. We start by recalling the definition of nilpotent ring.

Definition 4.2.1: Let A be a ring. An element x of A is called nilpotent if there exists integer number $n \geq 0$ such that $x^n = 0$. If every element in A is nilpotent, then A is called nilpotent ring.

Remark 4.2.2: If A is nilpotent ring, then every element of A is either zero or a zero-divisor.

Proof: Suppose that A is nilpotent ring. Let $x \in A$ be nilpotent element and let n≥ 0 be the minimal integer such that $x^n = 0$. If $n = 0$ then either $x = 0$ or $1 = 0$. Which implies that A is a zero ring, hence $x = 0$. If $n = 1$, then $x = 0$. If $n > 1$ and $x \neq 0$, then $0 = x^n = x.x^{n-1}$ with $n - 1 > 0$ and $x^{n-1} \neq 0$ by minimality of n. Thus x is a zero-divisor.∎ The following lemma is an enhancement of a result was proved by D.F. Anderson and A.D. Badawi in [5]

Lemma 4.2.3: Let A be a ring with $Z(A) = Nil(A)$. Then $diam\, \Gamma(A) \leq 2$.[5 lemma2.1]

Proof: Suppose that $x, y \in Z(A)$ such that $xy \neq 0$ and $x, y \in Nil(A)^*$ and suppose the $diam\, \Gamma(A) = 3$, Then there exist two vertices x and y such that $d(x, y) = 3$. Let $n \geq 2$ be the least positive integer such that $x^n = 0$ and $m \geq 2$ be the least positive integer such that $y^m = 0$. The $d(x, y) = 3$, then we have $u \in Z(A)$ such that $xu = 0$ and $yu \neq 0$. It is clear that $yxu = 0$. then $yx = 0$ or $yx \neq 0 \in Z(A)$. But $yx \neq 0$. Thus we have this path of length 4: $y - y^{m-1} - xy - x^{n-1} - x$. Which is a contradiction with (Theorem 2.5.5). Hence $diam\, \Gamma(A) \leq 2$.$\blacksquare$

As a particular case of ***Lemma 4.2.3,*** one may deduce the following corollary.

Corollary 4.2.4: Let A be a nilpotent rings. Then $diam\, \Gamma(A) \leq 2$.

Proof: This is a direct consequence of lemma 4.2.3 and the fact that $Nil(A) = Z(A)$ in the nilpotent rings.$\blacksquare$

By theorem 2.5.5, the diameter of any ring is mostly 3. But in the case of R is a ring with $Z(A) = Nil(A)$ (particularly, when A is nilpotent ring), $diam\, \Gamma(A) \neq 3$ (by the lemma 4.2.3).

Moreover, one can go further and describes the rings A with $Nil(A) = Z(A)$ and : 1) $diam\, \Gamma(A) = 0$.

2) $diam\, \Gamma(A) = 1$.

3) $diam\, \Gamma(A) = 2$

as it is shown in the following theorem.

Theorem 4.2.5: Let A be a ring with $Z(A) = Nil(A)$. Then exactly one of the following three cases must occur.[5 ,Theorem 2.2]

(1) $|Z(A)^*| = 1$. In this case, A is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X] / (x^2)$, and $diam\ \Gamma(A) = 0$.

(2) $|Z(A)^*| \geq 2$ and $Z(A)^2 = 0$. In this case, $\Gamma(A)$ is a complete graph, and $diam\ \Gamma(A) = 1$.

(3) $Z(A)^2 \neq \{0\}$. In this case, $diam\ \Gamma(A) = 2$.

Proof:

1) If $|Z(A)^*| = 1$, then $A \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X] / (x^2)$[3]. Which implies that $diam\ \Gamma(A) = 0$, because the graph has only one vertex.
2) If $Z(A)^2 = 0$, then $xy = 0$ for every x, y in $Z(A)$. Hence $\Gamma(A)$ is a complete graph. Which implies that the $diam\ \Gamma(A) = 1$ (because the graph is complete). Since the $|Z(A)^*| \geq 2$ (if $|Z(A)^*| = 1$, then the $diam\ \Gamma(A) = 0$).
3) Suppose that $Z(A)^2 \neq \{0\}$. Then (by remark 2.4.4) the graph not complete. Hence the $diam\ \Gamma(A) = 2$ (by lemma 4.2.3).$\blacksquare$

The following Proposition proves that if A is a nilpotent ring. Then the zero divisor graph of A contains a vertex adjacent to all other vertices***Proposition 4.2.6:*** Let A be a nilpotent ring. Then $\Gamma(A)$ contains a vertex which is adjacent to all other vertices of the graph.

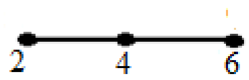
Proof: Suppose that A is nilpotent ring and $x, y \in Z^*(A)$ such that $\Gamma(A)$ has no vertex which is adjacent to all other vertices. Let n be the least positive integer such that $x^n = 0$ and m be the least positive integer such that $y^m = 0$. Thus $xx^{n-1}y^{m-1} = 0$, and we have two cases:

Case 1: If $x^{n-1}y^{m-1} = 0$, then x^{n-1} , $y^{m-1} \in Z(A)$. Hence we have this path $x - x^{n-1} - y^{m-1} - \ y$ of length 3.

Case 2: If $x^{n-1}y^{m-1} \in Z(A)$ it is clear that $x^{n-1}y^{m-1}y = 0$. Similarly, $yy^{m-2}y = 0$ implies that $yy^{m-2} = y^{m-1} \in Z(A)$. Thus we have this path $x - x^{n-1}y^{m-1} - y - \ y^{m-1}$ of length 3.

In both cases we have a contradiction with lemma 4.2.3. Hence $\Gamma(A)$ contains a vertex which is adjacent to all other vertices of the graph. ∎

Example 4.2.7: Take this ring $\mathbb{Z}_8$. The zero divisor of this ring is $\{2,4,8\}$ it is clear that $Z(A)$ is a nilpotent element and $\Gamma(\mathbb{Z}_8)$ has a vertex $\{4\}$ adjacent to every other vertex. Below are the graph.



References

[1] Ali Mohammadian, " on zero divisor graph of Boolean rings",Pacific Journal Of Mathematics. Vol.251.NO.2, 2011.

[2] Atiyah, M. F., MacDonald, I. G. (1969). Introduction to commutative

[3] algebra. Addison-Wesley-Longman. ISBN 978-0-201-40751-8.

[4] David F. Anderson and Philip S. Livigston, The Zero Divisors Graph of a Commutative ring, Journal of Algebra 217, 434 -447(1999).

[5] D. D. Anderson and M. Naseer, Beck's coloring of a commutative ring, J. Algebra 159 Ž .1993, 500]514.

[6] D.F. Anderson and A.D. Badawi, On the zero-divisor graph of a ring, Commun. Algebra 36 (2008) 3073–3092. doi:10.1080/00927870802110888

[7] E. Abu Osba, S. Al-Addasi and N. Abu Jaradeh, Zero divisor graph for the ring of Gaussian integers modulo n, Commun. Algebra 36 (10) (2008)3865–3877. doi:10.1080/00927870802160859.

[8] [7] E. Abu Osba, The Complement graph for Gaussian integers modulo nL<,Commun. Algebra 40 (5), (2012) 1886–1892. doi:10.1080/00927872.2011.560588.

[9] F. Harary, ‘‘Graph Theory,’’ Addison-Wesley, Reading, MA, 1972.

[10] Gravier S, Khelladi A. 1995 March. Communication On the domination number of cross products of graphs. Discrete Mathematics 145: 273-277.

[11] I. Kaplansky, ‘‘Commutative Rings,’’ rev. ed., Univ. of Chicago Press, Chicago, 1974.

[12] J.Salvatore, Bipartite Graphs and Problem Solving, university of Chicago. August 8, 2007.

[13] K. Nazzal and M. Ghanem, On the line graph of the zero divisor graph for the ring of Gaussian integers modulo n, Int. J. Comb. (2012) 13 pages.

[14] K. Nazzal and M. Ghanem, Some properties of the zero divisors graph of a commutative ring, General Algebra and Applications 34 (2014) 167-181.

[15] M. F. Atiyah and I. G. MacDonald, ‘‘Introduction to Commutative Algebra,’’ Addison]Wesley, Reading, MA, 1969.

[16] M. Ghanem and K. Nazzal, On the line graph of the complement graph for the ring of Gaussian integers modulo n, Open J. Disc. Math. 2 (1) (2012) 24–34.

[17] Ph.S. Livingston, Structure in Zero-Divisor Graphs of Commutative Rings, Masters Theses, University of Tennessee (Knoxville, 1997).

[18] P. S. Livingston, ‘‘Structure in Zero-divisor Graphs of Commutative

[19] Rings,’’ Masters Thesis, The University of Tennessee, Knoxville, TN, December 1997.

[20] R.B. Hayward, Murky graphs, J. Combin. Theory (B) 49 (1990) 200–235.

[21] R. Diestel, ‘‘Graph Theory,’’ Springer-Verlag, New York, 1997.

[22] Redmond, Shane Patrick, "Generalizations of the Zero-Divisor Graph of a Ring. " PhD diss., University of Tennessee, 2001.

[23] S. B. Mulay, Cycles and symmetries of zero-divisors, Comm. Algebra, 30(7) (2002), 3533-3558.

[24] S.P. Redmond, Central sets and Radii of the zero divisor graphs of commutative ring, Commun. Algebra 34 (2006) 2389–2401.

[25] S Skiena - section § 5.5. 2 of Implementing Discrete Mathematics,1990 - Addison-Wesley.

[26] T.F. Maire, Slightly triangulated graphs are perfect, Graphs Combin. 10 (1994) 263–268.

[27] W.B. Vasantha Kandasamy. K-Boolean rings. University of Lyon, 1992, p. 73-7

[28] Anderson, D. F., Frazier, A., Lauve, A., Livingston, P. S. (2001). The Zero-Divisor Graph of a Commutative Ring, II. Lecture Notes Pure Appl. Math. Vol. 202. New York/Basel: Marcel Dekker, pp.

جامعة النجاح الوطنية

كلية الدراسات العليا

رسوم القواسم الصفرية المتصلة لحلقات تبديلية معينة

إعداد

ليلى سفيان مصلح

إشراف
د. خالد عداربة

قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات في كلية الدراسات العليا في جامعة النجاح الوطنية في نابلس، فلسطين.

2021

رسوم القواسم الصفرية المتصلة لحلقات تبديلية معينة

إعداد

ليلى سفيان مصلح

إشراف

د. خالد عداربة

الملخص

لنفترض ان R حلقة تبادلية مع وجود 1, في عام 1998 قام David F. Anderson and Philip S. Livingston بربط الحلقة R برسم بياني $\Gamma(R)$ واطلقوا عليه الرسم البياني للمقسوم على الصفر في الحلقة R. بحيث ان رؤوس الرسم البياني هي المجموعة $Z(R)^* = Z(R) - \{0\}$ حيث يرمز $Z(R)$ الى جميع القواسم الصفرية في الحلقة R, وان أي عنصرين في القواسم الصفرية x, y متجاوران اذا وفقط اذا كان $xy = 0$.

في هذه الاطروحة نقدم دراسة عن تاثير بعض الخصائص النظرية للحلقة R على الرسم البياني للمقسوم الصفري $\Gamma(R)$, عن طريق اعادة انتاج العمل الرئيسي الذي تم انجازه في [3,12], وتوضيحه باستخدام امثلة جديدة.

واخيرا ,قمنا بالتحقيق لاول مرة في التفاعل بين الخصائص النظرية للحلقة لبعض الحلقات الخاصة مثل : Boolean, K-Boolean, and nilpotent , والخصائص النظرية للرسم البياني للرسوم البيانية للمقسوم الصفري.