An-Najah National University

Faculty of Graduate Studies

On Z-transform and Its Applications

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This thesis is submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics , Faculty of Graduate Studies, An-Najah National University, Nablus, Palestine.

2015

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Dedication

To my parents and brothers.

Acknowledgements

I would like to express my sincere gratitude to my Supervisor Dr. "Mohammad Othman" Omran for the continuous support, helpful suggestion, patience and encouragement throughout the course of this thesis. His guidance helped me at all times of research and writing of this thesis.

I would also like to thank my defense thesis committee: Dr. Mohammad Assa'd and Dr. Saed Mallak.

I want to take this opportunity to express gratitude to all my friends and to all people who help me do this work.

Last but not least, I wish to thank my parents, for the unceasing encouragement, support and attention they provide me throughout my life.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان

On Z-transform and Its Applications

اقر بأن ما اشتملت عليه هذه الرسالة إنما هي نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد وأن هذه الرسالة ككل أو أي جزء منها لم يقدم من قبل لنيل أية درجة علمية أو بحث علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualifications.

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Abstract

In this thesis we study Z-transform (the two-sided Z-transform), the one-sided Z-transform and the two-dimensional Z-transform with their properties, their inverses and some examples on them. We also present the relation between Z-transform and Laplace transform and between Z-transform and Fourier transform. Some applications of Z-transform including solutions of some kinds of linear difference equations, analysis of linear shift-invariant systems, implementation of FIR and IIR filters and design of IIR filters from analog filters are discussed. Chirp Z-transform algorithm is also presented with two applications: enhancement in poles and high resolution and narrow band frequency analysis.

Introduction

Transformation is a very powerful mathematical tool so using it in mathematical treatment of problem is arising in many applications [1].

The idea of Z-transform back to 1730 when De Moivre introduced the concept of “generating functions” to probability theory [8]. In 1947 a transform of sampled signal or sequence defined by W. Hurewicz as a tractable way to solve linear constant-coefficients difference equations. The transformation named "Z-transform" by Ragazzini and Lotfi Zadeh in the sampled-data control group at Columbia University in 1952 [21].

Z-transform is transformation for discrete data equivalent to the Laplace transform of continuous data and its a generalization of discrete Fourier transform [6].

Z-transform is used in many areas of applied mathematics as digital signal processing, control theory, economics and some other fields [8].

In this thesis, we present Z-transform, the one-sided Z-transform and the two-dimensional Z-transform with their properties, finding their inverse and some examples on them. Many applications of Z-transform are discussed as solving some kinds of linear difference equations, applications in digital signal processing. Finally chirp Z-transform is represented.

In the first chapter, some basic definitions and concepts of sequences are presented together with some theorems on integration in complex plane [1,2,5,6,10,14,19].

In the second chapter, the definition of Z-transform and one-sided Z-transform are discussed as well as some important properties and examples of them [6,8,9,13,14].

In the third chapter, methods for determining the inverse of Z-transform are represented, also we have discussed the relation between Z-transform and Laplace transform and discrete Fourier transform. The chapter is closed by describing the definition and properties of two-sided Z-transform in addition to its inverse [5,9,12,13,14,20].

In the fourth chapter, Z-transform is used to solve some kind of linear difference equations as linear difference equation of constant coefficient and Volterra difference equations of convolution type [3,4,7,18].

In the fifth chapter, applications of Z-transform in digital signal processing such as analysis of linear shift-invariant systems, implementation of finite-duration impulse response (FIR) and infinite-duration impulse response (IIR) systems and design of IIR filters from analog filters [1,6,9,11,14].

In the sixth chapter, the chirp Z-transform algorithm is studied with two applications of it such as: enhancement of poles and high resolution and narrow band frequency analysis [9,16,17,19].

Chapter One

Definitions and Concepts

In this chapter we give some basic definitions, concepts and theorems important for our thesis.

Definition 1.1: [5] The complex sequence $\{a_k\}$ is called geometric sequence if $\exists$ a constant $s \in \mathbb{C}$ s.t

$$\frac{a_{k+1}}{a_k} = s, \forall k \in \mathbb{N} \tag{1.1}$$

In that case

$$a_k = as^k \tag{1.2}$$

A geometric series is of the form

$$\sum_{k=0}^{\infty} as^k = a + as + as^2 + \cdots \tag{1.3}$$

Note that a finite geometric series is summable with

$$\sum_{k=0}^{n} as^k = \begin{cases} \dfrac{a(1-s^{n+1})}{1-s}, & s \neq 1 \\ a(n+1) \quad , & s = 1 \end{cases} \tag{1.4}$$

If $|s| < 1$, then

$$\sum_{k=0}^{\infty} as^k = \frac{a}{1-s} \tag{1.5}$$

Definition 1.2: [10]A sequence $x(n)$ is called:

a) causal if:

$$x(n) = 0, \text{for } n < 0 \tag{1.6}$$

b) anticausal if:

$$x(n) = 0, \text{for } n \geq 0 \tag{1.7}$$

Definition 1.3:[1,19]The unit step sequence or Heaviside step sequence is defined as

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{1.8}$$

And for two-dimensional space it has the form

$$u(n,m) = \begin{cases} 1, & n,m \geq 0 \\ 0, & oterwise \end{cases} \tag{1.9}$$

Definition 1.4:[1,19] The unit impulse or unit sample sequence is defined as

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases} \tag{1.10}$$

And for two-dimensional space it has the form

$$\delta(n,m) = \begin{cases} 1, & n = m = 0 \\ 0, & otherwise \end{cases} \tag{1.11}$$

Definition 1.5: [6] The convolution between two infinite sequences $x(n)$ and $y(n)$ is defined as

$$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k) \tag{1.12}$$

Example 1.1: Find the convolution of the two sequences

$$x(n) = \delta(n) - 5\delta(n-1), y(n) = 5^n u(n)$$

Solution:

$$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k)$$

$$= \sum_{k=-\infty}^{\infty} [\delta(k) - 5\delta(k-1)]5^{n-k}u(n-k)$$

For $n < k$, $u(n-k) = 0$,

For $k \neq 0$ and $k \neq 1$, $\delta(k) - 5\delta(k-1) = 0$

So

$$x(n) * y(n) = 5^n u(n) - 5.5^{n-1}u(n-1)$$

$$= 5^n \left(u(n) - u(n-1)\right)$$

Since $\delta(k) = 1$ when $k = 1$ and $\delta(k-1) = 1$ when $k = 1$.

Then we have

$$x(n) * y(n) = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases}$$

Or it can be written as

$$x(n) * y(n) = \delta(n)$$

Definition 1.6: [14] The correlation between two sequences $x(n)$ and $y(n)$ is defined as

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) = x(l) * y(-l) \qquad (1.13)$$

where l is an integer.

Definition 1.7: [14] The autocorrelation $r_{xx}(l)$ of a sequence $x(n)$ is the correlation with itself.

Definition 1.8: A function $f(z)$ is analytic at a point z_0 if it has a derivative at each point in some neighborhood of z_0.

Theorem 1.1:[2] If a function $f(z)$ is analytic at all points interior to and on a simple contour C then,

$$\oint_C f(z)dz = 0 \qquad (1.14)$$

Theorem 1.2:[2] **Laurent's Theorem**

Suppose that a function $f(z)$ is analytic throughout an annular domain $r < |z - z_0| < R$, centered at z_0, and let C be any positively oriented simple closed contour a round z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n, \quad r < |z - z_0| < R \qquad (1.15)$$

where

$$b_n = \frac{1}{2\pi j} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad n \in \mathbb{Z} \qquad (1.16)$$

The representation of $f(z)$ in Eq(1.15) is called a Laurent series.

Definition 1.9: [2] If z_0 is an isolated singular point of a function $f(z)$, then there is a positive number R such that $f(z)$ is analytic at each point z for which $0 < |z - z_0| < R$. Consequently, $f(z)$ has a Laurent series representation as in Eq(1.15). The complex number b_{-1}, which is the coefficient of $1/(z - z_0)$ in Eq(1.15) is called the residue of $f(z)$ at the isolated singular point z_0, and we shall often write

$$b_{-1} = Res[f(z), z_0]$$

Theorem 1.3: [2] An isolated singular point z_0 of a function $f(z)$ is a pole of order s if and only if $f(z)$ can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^s}$$

where $g(z)$ is analytic and nonzero at z_0.

Moreover, if $s = 1$ then,

$$Res[f(z), z_0] = Res\left[\frac{g(z)}{z - z_0}, z_0 \right] = g(z_0) \qquad (1.17)$$

And if $s \geq 2$

$$Res[f(z), z_0] = Res\left[\frac{g(z)}{(z - z_0)^s}, z_0 \right]$$
$$= \frac{1}{(s-1)!} \frac{d^{s-1}}{dz^{s-1}} g(z)\Big|_{z = z_0} \qquad (1.18)$$

Theorem 1.4: [2] **Residue Theorem or Cauchy's Residue Theorem.**

If a function $f(z)$ is analytic inside and on a simple closed contour C (described in the positive sense) except for a finite number of singular points $z_k, k = 1, 2, \dots, n$ inside C then,

$$\frac{1}{2\pi j} \oint_C f(z)dz = \sum_{k=1}^{n} Res[f(z), z_k] \qquad (1.19)$$

Theorem 1.5:[2] **Taylor's Theorem**

Suppose that a function $f(z)$ is analytic throughout a disk $|z - z_0| < R$ centered at z_0 and with radius R. Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < R \qquad (1.20)$$

where

$$a_n = \frac{f^{(n)}(z)}{n!}, \quad n = 0, 1, 2, \dots \qquad (1.21)$$

The series in Eq(1.20) is called the Taylor series of $f(z)$ about $z = z_0$ and its converges to $f(z)$ when z lies in the given open disk.

Chapter Two

The Z-transform

In this chapter we introduce the two-sided and the one-sided Z-transforms, investigate their properties and give some examples on them.

Section 2.1: Definition of Z-transform.

Definition 2.1:[9] Given an infinite complex sequence $x(n)$, we define its Z-transform $X(z)$ by the two sided infinite power series

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \qquad (2.1)$$

where z is a complex variable. This Z-transform is called a two-sided or a bilateral Z-transform.

Note: Whenever we talk about Z-transform we mean the two-sided Z-transform.

Z-transform exists only for those values of z for which the series in Eq(2.1) converges. These values of z define the region of convergence (ROC) of $X(z)$. Thus, the ROC is the domain of the Z-transform. So, whenever $X(z)$ is found, its ROC should be also defined.

The region of convergence of $X(z)$ is identical to the set of all values of z which make the sequence in Eq(2.1) absolutely summable, i.e [9]

$$\sum_{n=-\infty}^{\infty} |x(n)\, z^{-n}| < \infty \qquad (2.2)$$

Example 2.1: Determine the Z-transform of the sequence

$$x(n) = \left\{\ldots, 0, 1, 0, \underset{\uparrow}{2}, 0, 0, 7, 0, 0, \ldots\right\} = \begin{cases} 1, & n = -2 \\ 2, & n = 0 \\ 7, & n = 3 \\ 0, & otherwise \end{cases}$$

Note: The arrow in this sequence indicates the position of $x(0)$.

Solution:

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$= z^2 + 2 + 7z^{-3},$$

$X(z)$ converges for the entire z-plane except $z = 0$ and $z = \infty$, so

$$ROC: 0 < |z| < \infty$$

Example 2.2: Determine the Z-transform of the following sequences.

a) $x(n) = \left(\frac{1}{2}\right)^n$, where $n \geq 0$

b) $(n) = \left(\frac{1}{2}\right)^n$, where $n < 0$

c) $(n) = \left(\frac{1}{2}\right)^n$, where $n \in \mathbb{Z}$

Solution:

a)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n = \frac{1}{1 - (2z)^{-1}} = \frac{z}{z - \frac{1}{2}}$$

The above series is an infinite geometric series that converges only if $\left|\frac{1}{2z}\right| < 1$ or $|z| > \frac{1}{2}$. Thus, the ROC is $|z| > \frac{1}{2}$, the exterior of a circle of radius $\frac{1}{2}$ centered at the origin of the complex plane.

Note that: From **Definition 1.2 (a)** The sequence $x(n)$ is causal.

b)

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n)z^{-n} = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=-\infty}^{-1} (2z)^{-n}$$

Substitute $m = -n$ we get

$$Y(z) = \sum_{m=1}^{\infty} (2z)^m$$

$$= \frac{1}{1 - 2z} - 1 = \frac{z}{\frac{1}{2} - z}$$

Again, this is an infinite geometric series that converges only if $|2z| < 1$ or $|z| < \frac{1}{2}$. Thus, the ROC is $|z| < \frac{1}{2}$. In this case the ROC is the interior of a circle centered at the origin of the complex plane of radius $\frac{1}{2}$.

Note that: From **Definition 1.2 (b)** The sequence $y(n)$ is anticausal.

c)

$$R(z) = \sum_{n=-\infty}^{\infty} r(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=-\infty}^{-1} y(n) z^{-n} + \sum_{n=1}^{\infty} x(n) z^{-n}$$

But the first power series converges if $|z| < \frac{1}{2}$ where the second power series converges if $|z| > \frac{1}{2}$, so there is no z in the region of convergence for $R(z)$.

$\therefore ROC = \emptyset$ (the empty set).

Example 2.3: Determine the Z-transform of the sequence

$$x(n) = \alpha^n u(n), \; where \; \alpha \in \mathbb{C}$$

Solution:

$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n \;, & n \geq 0 \\ 0 \;, & n < 0 \end{cases}$$

Note that this sequence is the same as the sequence in **Example 2.2(a)** with $\alpha = \frac{1}{2}$ so,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} \alpha^{\,n} z^{-n} = \frac{z}{z - \alpha}$$

$$\therefore Z[\alpha^{\,n}\, u(n)] = \frac{z}{z - \alpha} \qquad (2.3)$$

ROC: $|z| > |\alpha|$ the exterior of a circle centered at the origin of the complex plane having radius $|\alpha|$.

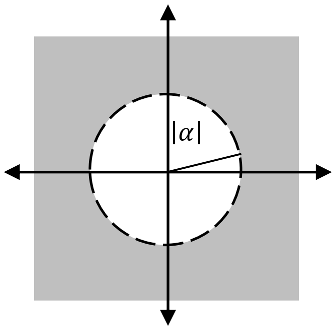

Figure 2.1 ROC for Z-transform in **Example 2.3**

Example 2.4: Determine the Z-transform of the sequence

$$x(n) = -b^{\,n}\, u(-n - 1)$$

Solution:

$$x(n) = -b^{\,n}\, u(-n - 1) = \begin{cases} 0\;, & n \geq 0 \\ -b^n, & n < 0 \end{cases}$$

Note that this sequence is the negative of the sequence in **Example 2.2 (b)** so

$$X(z) = \sum_{n=-\infty}^{-1} -b^n z^{-n} = -\sum_{m=1}^{\infty} (b^{-1} z)^m = \frac{z}{z - b}$$

$$\therefore Z[-b^{\,n}\, u(-n - 1)] = \frac{z}{z - b} \qquad (2.4)$$

ROC: $|z| < |b|$ the interior of a circle centered at the origin of the complex-plane having radius $|b|$.

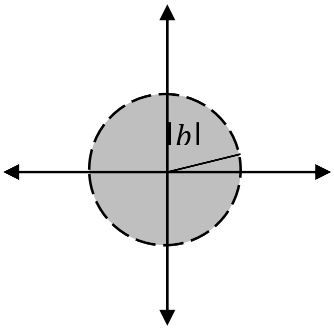

Figure 2.2 ROC for Z-transform in **Example** 2.4

Note that: if $b = \alpha$ then the function $X(z)$ in **Example 2.3** is identical to that in **Example 2.4** with different ROC. This illustrates the important fact that specifying Z-transform of a sequence requires not only the function $X(z)$ but also its region of convergence.

Example 2.5: Determine the Z-transform of the sequence

$$x(n) = \alpha^n u(n) - b^n u(-n-1)$$

Solution:

$$x(n) = \alpha^n u(n) - b^n u(-n-1) = \begin{cases} \alpha^n , & n \geq 0 \\ -b^n , & n < 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{-1} -b^n z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

$$= -\sum_{m=1}^{\infty} (b^{-1} z)^m + \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

The first power series converges if $|b^{-1} z| < 1$ or $|z| < |b|$ where the second power series converges if $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$, this gives us two cases for $X(z)$:

Case 1: If $|b| \leq |\alpha|$ then, there is no common region of convergence, i.e $X(z)$ does not exist.

Case 2: If $|b| > |\alpha|$, then there is a common region of convergence, which is $|\alpha| < |z| < |b|$, and in this case $X(z)$ will be

$$X(z) = \frac{z}{z-b} + \frac{z}{z-\alpha} = \frac{z(2z-\alpha-b)}{(z-b)(z-\alpha)}$$

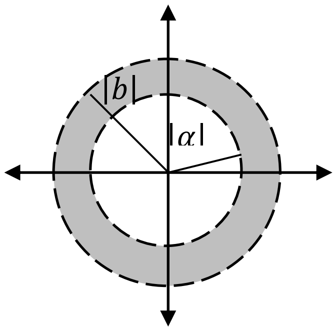

Figure 2.3 ROC for Z-transform in **Example 2.5** where $|b| >$

Note: For any sequence $x(n)$ with rational Z-transform the region of convergence cannot contain any poles and is bounded by poles or by zero or infinity.

A Note on the $\boldsymbol{ROC}$ of the Z-transform for two sided sequences [9]**.**

Let $X(z)$ be the Z-transform of the sequence $x(n)$, then

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{n_0} x(n) z^{-n} + \sum_{n=n_1}^{\infty} x(n)\, z^{-n} \qquad (2.5)$$

For the series

$$\sum_{n=n_1}^{\infty} x(n)\, z^{-n} \qquad (2.6)$$

Suppose that Eq(2.6) is a absolutely convergent for $z = z_1$, so

$$\sum_{n=n_1}^{\infty} |x(n)\, {z_1}^{-n}| < \infty \qquad (2.7)$$

We have two cases for the value of n_1 .

Case 1: If $n_1 \geq 0$**,** then for any $z \in \mathbb{C}$ such that $|z| > |z_1|$ we get,

$$\sum_{n=n_1}^{\infty} |x(n)\, {z_1}^{-n}| < \sum_{n=n_1}^{\infty} |x(n)\, z^{-n}| < \infty \qquad (2.8)$$

So the ROC in this case is $|z| > r_x$ where r_x is the smallest value of $|z|$ make Eq(2.6) convergent

Case 2: If $n_1 < 0$, then we express the series in Eq(2.6) as

$$\sum_{n=n_1}^{\infty} x(n)\, z^{-n} = \sum_{n=n_1}^{-1} x(n)\, z^{-n} + \sum_{n=0}^{\infty} x(n)\, z^{-n} \qquad (2.9)$$

The first series on the right-hand side of Eq(2.9) is finite for any finite value of z so, its convergent for all values of z except for $z = \infty$. Where the second series by (case 1) convergent for $|z| > r_x$. Thus, the series in Eq(2.6) has a region of convergence that is the exterior of a circle centered at the origin of complex-plane with radius r_x with the exception of $z = \infty$ for $n_1 < 0$.

For the series

$$\sum_{n=-\infty}^{n_0} x(n) z^{-n} \qquad (2.10)$$

If we change the index of summation through the substitution $n = -m$, we obtain the series

$$\sum_{m=-n_0}^{\infty} x(-m) z^{m} \qquad (2.11)$$

Applying the result of Eq(2.6) on Eq(2.10) (with n replaced by $-m$ and z by z^{-1}) then, the region of convergence of Eq(2.10) is the interior of a circle centered at the origin of complex-plane with radius R_x where $R_x$is the largest value of $|z|$ make Eq(2.10) convergent and not equal zero for $n_0 > 0$.

So for Eq(2.5), The first series has a region of convergence $|z| < R_x$ while the region of convergence for the second series is $r_x < |z|$. If $r_x < R_x$ then the region of convergence for $X(z)$ is $r_x < |z| < R_x$; otherwise there is no region of convergence for $X(z)$.

Section 2.2: Properties of Z-transform.

In studying discrete-time signals and systems, Z-transform is a very powerful tool due to its properties [14]. In this section, we examine some of these properties leaving the Examples to the next section.

Let $X(z)$ be the Z-transform of $x(n)$ with ROC $r_x < |z| < R_x$ and let $Y(z)$ be the Z-transform of $y(n)$ with OC $r_y < |z| < R_y$. Let $r = \max(r_x, r_y)$ and $R = \min(R_x, R_y)$, where r can be as small as 0 and R can be as large as ∞.

1. Linearity.

For any two complex numbers α, β we have

$$Z[\alpha\, x(n) + \beta\, y(n)] = \alpha\, X(z) + \beta\, Y(z), \quad r < |z| < R \tag{2.12}$$

Proof:

Let

$$w(n) = \alpha\, x(n) + \beta\, y(n)$$

then

$$W(z) = \sum_{n=-\infty}^{\infty} w(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [\alpha\, x(n) + \beta\, y(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \alpha\, x(n) z^{-n} + \sum_{n=-\infty}^{\infty} \beta\, y(n)\, z^{-n}$$

$$= \alpha \sum_{n=-\infty}^{\infty} x(n) z^{-n} + \beta \sum_{n=-\infty}^{\infty} y(n)\, z^{-n}$$

$$= \alpha\, X(z) + \beta\, Y(z)$$

ROC of $W(z) = \alpha\, X(z) + \beta\, Y(z)$ contains all z such that

$$z \in ROC(X(z)) \cap ROC(Y(z))$$

then

$$\{r_x < |z| < R_x\} \cap \{r_y < |z| < R_y\}$$

$$\therefore \max(r_x, r_y) < |z| < \min(R_x, R_y)$$

If the linear combination canceled some poles the region of convergence may be larger. For example, the sequences $c^{\,n} u(n)$ and $c^{\,n} u(n-1)$ both have a region of convergence defined by $|z| > |c|$, but the sequence corresponding to the difference $[c^{\,n}\, u(n) - c^{\,n}\, u(n - 1)] = \delta(n)$ has the entire z-plane as its ROC.

2. Shifting

If k is any integer then,

$$Z[x(n+k)] = z^k\, X(z), \qquad r_x < |z| < R_x \tag{2.13}$$

Proof:

$$Z[x(n+k)] = \sum_{n=-\infty}^{\infty} x(n+k) z^{-n} \tag{2.14}$$

Substitute $m = n + k$ in Eq(2.14) ,we get

$$Z[x(n+k)] = \sum_{m=-\infty}^{\infty} x(m) z^{-(m-k)}$$

$$= z^k \sum_{m=-\infty}^{\infty} x(m) z^{-m} = z^k X(z)$$

The ROC of $X(z)$, and $Z[x(n+k)]$ are identical, with the possible exception of $z = 0$ or $z = \infty$.

3. Multiplication by Exponential.

If α is any complex number then,

$$Z[\alpha^n x(n)] = X(\alpha^{-1} z), \quad |\alpha|\, r_x < |z| < |\alpha|\, R_x \tag{2.15}$$

Proof:

$$Z[\alpha^n x(n)] = \sum_{n=-\infty}^{\infty} \alpha^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)(\alpha^{-1} z)^{-n} = X(\alpha^{-1} z)$$

where $r_x < |\alpha^{-1} z| < R_x$ or $|\alpha| r_x < |z| < |\alpha| R_x$

4. Time Reversal.

In discrete-time signal the variable n in the sequence $x(n)$ refer to time.

$$Z[x(-n)] = X(z^{-1}), \qquad \frac{1}{R_x} < |z| < \frac{1}{r_x} \tag{2.16}$$

Proof:

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n} \tag{2.17}$$

Let $m = -n$, then Eq(2.17) become

$$\sum_{m=-\infty}^{\infty} x(m)(z)^m = \sum_{m=-\infty}^{\infty} x(m)(z^{-1})^{-m} = X(z^{-1})$$

$$ROC\ r_x < |z^{-1}| < R_x\ or\ \frac{1}{R_x} < |z| < \frac{1}{r_x}$$

The above means that if z_0 belongs to the ROC of $X(z)$ then $1/z_0$ is in the ROC of $X(z^{-1})$.

5. Conjugation.

$$Z[x^*(n)] = X^*(z^*), \qquad r_x < |z| < R_x \qquad (2.18)$$

where $x^*(n)$ is the complex conjugate of $x(n)$.

Proof:

$$Z[x^*(n)] = \sum_{n=-\infty}^{\infty} x^*(n)z^{-n} = \sum_{n=-\infty}^{\infty} [x(n)(z^*)^{-n}]^*$$
$$= \left[\sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n}\right]^* = X^*(z^*)$$

With ROC identical to the ROC of $X(z)$.

6. Multiplication by n or Differentiation of the Transform

$$Z[n\,x(n)] = -z\frac{dX(z)}{dz}, \quad r_x < |z| < R_x \qquad (2.19)$$

Proof:

From the definition of Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Differentiate both sides of the previous equation with respect to z ,we obtain

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n)\,(-n)z^{-n-1}$$
$$= -z^{-1}\sum_{n=-\infty}^{\infty} [nx(n)]\,z^{-n}$$
$$= -z^{-1}Z[n\,x(n)]$$
$$\therefore Z[n\,x(n)] = -z\frac{dX(z)}{dz}$$

With the same ROC of $X(z)$ which is $r_x < |z| < R_x$

7. Convolution of Two Sequences.

The convolution property is one of the most powerful properties of Z-transform.

$$Z[x(n) * y(n)] = X(z)Y(z) \tag{2.20}$$

Where the ROC is, at least, the intersection of ROC of $X(z)$ and ROC of $Y(z)$. However, the ROC may be larger if there is a pole-zero cancelation in the product $X(z)Y(z)$.

Proof:

Let $r(n)$ be the convolution of $x(n)$ and $y(n)$, then $r(n)$ is defined as

$$r(n) = x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k) \tag{2.21}$$

Taking the Z-transform of Eq(2.21), we obtain

$$\begin{aligned} R(z) = Z[r(n)] &= \sum_{n=-\infty}^{\infty} r(n)\, z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)y(n-k) \right] z^{-n} \end{aligned} \tag{2.22}$$

By interchanging the order of the summations of Eq(2.22) and applying the shifting property, we obtain

$$\begin{aligned} R(z) &= \sum_{k=-\infty}^{\infty} x(k) \left[\sum_{n=-\infty}^{\infty} y(n-k) z^{-n} \right] \\ &= Y(z) \sum_{k=-\infty}^{\infty} x(k) z^{-k} = Y(z)\, X(z) \end{aligned}$$

The ROC of the convolution is, at least, the intersection of ROC of $X(z)$ and ROC of $Y(z)$.

8. Correlation of Two Sequences [14].

$$Z\left[r_{xy}(l)\right] = R_{xy}(z) = X(z)Y(z^{-1}) \tag{2.23}$$

The ROC is, at least, the intersection of ROC of $X(z)$ and ROC of $Y(z^{-1})$.

Proof:

We know from **Definition 1.6** that

$$r_{xy}(l) = x(l) * y(-l) \tag{2.24}$$

Taking Z-transform of both sides of Eq(2.24), and use the convolution and time reversal properties, we get

$$R_{xy}(z) = Z[x(l)]Z[y(-l)] = X(z)Y(z^{-1}).$$

The proof of the following property and theorem will be delayed till the introduction of the inverse Z-transform in Chapter 3.

9. Multiplication of Two Sequences

$$Z\big(x(n)y(n)\big) = \frac{1}{2\pi j}\oint_C X(v)Y\left(\frac{z}{v}\right)\frac{1}{v}dv, \tag{2.25}$$

With ROC: $r_x r_y < |z| < R_x R_y$

where C is a counterclockwise closed contour that encloses the origin and lies within the common region of convergence of $X(v)$ and $Y\left(\frac{z}{v}\right)$.

Parseval's Theorem

If

$$r_x r_y < |z| = 1 < R_x R_y$$

then we have

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi j}\oint_C X(z)\mathrm{Y}^*\left(\frac{1}{z^*}\right)\frac{1}{z}dz \tag{2.26}$$

where C is a counterclockwise closed contour that encloses the origin and lies within the common region of convergence of $X(z)$ and $\mathrm{Y}^*(1/\mathrm{z}^*)$.

Note: If $y(n)$ is real sequences, then **Parseval's theorem** will be

$$\sum_{n=-\infty}^{\infty} x(n)y(n) = \frac{1}{2\pi j}\oint_C X(z)Y\left(\frac{1}{z}\right)\frac{1}{z}dz \tag{2.27}$$

A table of properties of Z-transform is given in Appendix C.

Section 2.3: Examples on the Properties of Z-transform.

Example 2.6: Determine the Z-transform of the sequence

$$x(n) = cos(nw)\, u(n), w \in \mathbb{C}$$

Solution:

By using Euler's identity

$$cos(nw) = \frac{e^{nwj} + e^{-nwj}}{2}$$

$X(z)$ will be

$$X(z) = Z[cos(nw)u(n)] = Z\left[\frac{e^{nwj} + e^{-nwj}}{2} u(n)\right]$$

By using linear property, we get

$$X(z) = \frac{1}{2}\left[Z\left[e^{nwj}u(n)\right] + Z\left[e^{-nwj}u(n)\right]\right]$$

By **Example 2.3**, with $\alpha = e^{\pm wj}$ ($|\alpha| = \left|e^{+wj}\right| = 1$), we get

$$X(z) = \frac{1}{2}\left[\frac{z}{z - e^{wj}} + \frac{z}{z - e^{-wj}}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - (e^{-wj} + e^{wj})z}{(z^2 - z(e^{wj} + e^{-wj}) + 1}\right]$$

$$= \frac{z^2 - zcos(w)}{z^2 - 2zcos(w) + 1}$$

Since the ROC of $Z\left[e^{nwj}u(n)\right]$ is $|z| > 1$and the ROC of $Z\left[e^{-nwj}u(n)\right]$ is also $|z| > 1$ and there is no pole-zero cancelation, then the ROC of $X(z)$ is the intersection of the two regions, which is $|z| > 1$.

Example 2.7: Determine the Z-transform of the sequence

$$x(n) = n3^n u(-n)$$

Solution:

Using **Example 2.3**, with $\alpha = \frac{1}{3}$ ($|\alpha| = \left|\frac{1}{3}\right| = \frac{1}{3}$) we get

$$Z\left[\frac{1}{3}^{n}\ u(n)\right] = \frac{z}{z-\frac{1}{3}}, |z| > \frac{1}{3}$$

Then by time reversal property, we obtain

$$Z[3^n u(-n)] = \frac{z^{-1}}{z^{-1}-\frac{1}{3}} = \frac{3}{3-z}, |z| < 3$$

Finally, using multiplication by n property, it follows that

$$Z[n3\ ^n u(-n)] = -z\frac{d}{dz}\frac{3}{3-z}$$

$$= \frac{-3z}{(3-z)^2}, |z| < 3$$

Example 2.8: Determine the Z-transform of the convolution of

$$x(n) = \delta(n) -\ 5\delta(n-1) \text{ and } y(n) = 5^n u(n)$$

Solution:

$$Z[x(n)] = Z[\delta(n)\ ] - 5Z[\delta(n-1)]$$

Using shifting property

$$Z[x(n)] = Z[\delta(n)\ ] - 5z^{-1}Z[\delta(n)]$$

$$= 1 - 5z^{-1}\cdot 1 = \frac{z-5}{z}, |z| > 0$$

$$Z[y(n)] = \frac{z}{z-5}, |z| > 5$$

However, Z-transform of the convolution of $x(n)$ and $y(n)$ is

$$Z[x(n) * y(n)] = X(z)Y(z) = \frac{z-5}{z}.\frac{z}{z-5} = 1$$

Which due to the pole-zero cancelation (see property 7), has a region of convergence that is the entire z-plane.

Note: From **Example 1.1** we find that

$$x(n) * y(n) = [\delta(n) -\ 5\delta(n-1)] *\ 5^n u(n) = \delta(n)$$

So

$$Z[x(n) * y(n)] = Z[\delta(n)] = 1$$

ROC is the entire z-plane.

Example 2.9: Determine the Z-transform of the autocorrelation of the sequence

$$x(n) = (0.1)^n\, u(n)$$

Solution:

Since the autocorrelation of the sequence is its correlation with itself, then

$$r_{xx}(l) = x(l) * x(-l)$$

From correlation property

$$R_{xx}(z) = Z[r_{xx}(l)] = X(z)X(z^{-1})$$

$$X(z) = \frac{z}{z - 0.1} \qquad |z| > 0.1$$

$$X(z^{-1}) = \frac{1}{1 - 0.1z} \qquad |z| < 10$$

$$R_{xx}(z) = \frac{z}{z - 0.1} \cdot \frac{1}{1 - 0.1z}$$

$$= \frac{z}{-0.1z^2 + 1.01z - 0.1}$$

$$= \frac{-10z}{z^2 - 10.1z + 1}$$

ROC $0.1 < |z| < 10$

Example 2.10: Determine the Z-transform of the sequence $w(n) = x(n) \cdot y(n)$ where $x(n) = 2^{\,n}\, u(n)$ and $y(n) = 3^{\,n}\, u(n)$.

Solution:

$$X(z) = \frac{z}{z - 2}, |z| > 2$$

$$Y(z) = \frac{z}{z - 3}, |z| > 3$$

Using multiplication of two sequences property

$$W(z) = \frac{1}{2\pi j} \oint_C X(v)Y\left(\frac{z}{v}\right)\frac{1}{v}\,dv$$

$$= \frac{1}{2\pi j} \oint_C \frac{v}{v-2} \frac{(z/v)}{[(z/v)-3]} \frac{1}{v} dv$$

$$= \frac{1}{2\pi j} \oint_C \frac{1}{v-2} \frac{z}{(z-3v)} \; dv$$

The integral has two poles, one located at $v = 2$ and the second at $v = z/3$. The contour of integration must be within the region of convergence of $X(v)$, and consequently will enclose the pole at $v = 2$. To determine whether it encloses the pole at $v = z/3$, we consider that the Z-transform $Y(z)$ is only valid for $|z| > 3$. Therefore, the corresponding expression for Y(z/v) is only valid for $|z/v| > 3$. Thus if

$$\left|\frac{z}{v}\right| > 3$$

Then

$$\left|\frac{z}{3}\right| > |v|$$

For the ROC of $Z[x(n) \cdot y(n)]$ we get

$$|v| > 2 \text{ and } \left|\frac{z}{3}\right| > |v|$$

So

$$\left|\frac{z}{3}\right| > |v| > 2$$

$$\therefore ROC \text{ of } Z[x(n) \cdot y(n)] \text{ is } |z| > 6$$

Consequently, the pole $\left|\frac{z}{3}\right|$ lie outside the contour of integration in v.

Using Cauchy residue theorem to evaluate $W(z)$, we obtain

$$W(z) = Res\left[\frac{1}{v-2} \frac{z}{(z-3v)} ,2\right]$$

$$= \frac{z}{(z-6)}, \quad |z| > 6$$

Example 2.11: Use Parseval's Theorem to find

$$\sum_{n=-\infty}^{\infty} x(n)y(n)$$

where $x(n) = \left(\frac{1}{2}\right)^n u(n)$ and $y(n) = \left(\frac{1}{3}\right)^n u(n)$.

Solution:

From Parseval's theorem, since $y(n)$ is real we use the expression in Eq(2.27) we obtain,

$$\sum_{n=-\infty}^{\infty} x(n)y(n) = \frac{1}{2\pi j} \oint_C X(z)Y\left(\frac{1}{z}\right)\frac{1}{z}dz$$

$$X(z) = \frac{z}{z-\frac{1}{2}}, |z| > \frac{1}{2}$$

$$Y\left(\frac{1}{z}\right) = \frac{3}{3-z} = \frac{-3}{z-3}, |z| < 3$$

So, the ROC of $X(z)$ and $Y\left(\frac{1}{z}\right)$ is $\frac{1}{2} < |z| < 3$

$$\sum_{n=-\infty}^{\infty} x(n)y(n) = \frac{1}{2\pi j} \oint_C \frac{z}{z-\frac{1}{2}} \frac{-3}{z-3} \frac{1}{z} dz$$

$$= \frac{1}{2\pi j} \oint_C \frac{-3}{\left(z-\frac{1}{2}\right)(z-3)} dz$$

where C is any circle with radius r, where $\frac{1}{2} < r < 3$.

$$\sum_{n=-\infty}^{\infty} x(n)y(n) = Res\left[\frac{-3}{\left(z-\frac{1}{2}\right)(z-3)}, \frac{1}{2}\right] = \frac{6}{5}$$

Example 2.12: If $Y(z) = Z[(1+j)^n u(n)] = \frac{z}{z-(1+j)}, |z| > \sqrt{2}$. Determine the Z-transform of $x(n) = (1-j)^n u(n)$.

Solution:

Note that $x(n) = [(1+j)^n u(n)]^*$

Using conjugation property

$$Z[x(n)] = Y^*(z^*) = \left[\frac{z^*}{z^* - (1+j)}\right]^*, |z| > \sqrt{2}$$
$$= \frac{z}{z - (1-j)}, \qquad |z| > \sqrt{2}$$

Example 2.13: Determine the Z-transform of the sequence

$$x(n) = \left(\frac{-1}{2}\right)^{|n|}$$

Solution:

We can write $x(n)$ as

$$x(n) = \left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{-n} u(-n) - \delta(n)$$

Using linearity and time reversal properties, we get

$$X(z) = \frac{z}{z + \frac{1}{2}} + \frac{z^{-1}}{z^{-1} + \frac{1}{2}} - 1, \quad \left|\frac{-1}{2}\right| < |z| < \frac{1}{\left|\frac{-1}{2}\right|}$$
$$= \frac{3z}{(2z+1)(z+2)}, \qquad \frac{1}{2} < |z| < 2$$

2.4: Definition and Properties of the One-Sided Z-transform.

Definition 2.2:[6] The one-sided or unilateral Z-transform of a sequence is

$$X^+(z) = Z^+[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n} \tag{2.28}$$

where z is a complex variable.

The one-sided Z-transform differs from the two-sided Z- transform in the lower limit of the summation, which is always zero whether or not the sequence $x(n)$ is zero for $n < 0$ (i.e., causal). So, it does not contain any information about the sequence $x(n)$ for the negative values of n.

Note: The one-sided Z-transform $X^+(z)$ of $x(n)$ is identical to the two-sided Z- transform of the sequence $x(n)u(n)$. Since $x(n)u(n)$ is causal, the region of convergence of its transform is always the exterior of a circle centered at the origin of z-plane.

Example 2.14: Determine the one-sided Z-transform of the sequences.

a) $x(n) = \left\{ \underset{\uparrow}{7}, 3,0,1,2,6 \right\}$

b) $y(n) = \left\{ 2, \underset{\uparrow}{3}, -4,0,5 \right\}$

c) $r(n) = \left\{ 7,3,2, \underset{\uparrow}{3}, -4,0,5 \right\}$

d) $w(n) = 4^n$

e) $h(n) = u(-n-1)$

Solution:

a)

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = 7 + 3\, z^{-1} + 1\, z^{-3} + 2\, z^{-4} + 6z^{-5}$$

ROC: entire z-plane except $z = 0$

b)

$$Y^+(z) = 3 - 4\, z^{-1} + 5\, z^{-3}, \quad |z| > 0$$

c)

$$R^+(z) = 3 - 4\, z^{-1} + 5\, z^{-3}, \quad |z| > 0$$

d)

$$W^+(z) = \frac{z}{z-4}, \quad |z| > 4$$

e)

$$H^+(z) = 0, \quad ROC \text{ is all } \mathrm{z-plane}$$

Note: The one-sided Z-transform is not unique for noncausal sequence. for example, in the previous example $Y^+(z) = R^+(z)$ but $y(n) \neq r(n)$. Also it's not unique for anticausal sequences it's always equal zero.

Most of the properties of the one-sided Z-transform are the same as those for the two-sided Z-transform except the shifting property.

Shifting Property for the One-Sided Z-transform

If $X^+(z)$ is the one-sided Z-transform of the sequence $x(n)$ with $ROC \ |z| > r_x$, and k is any positive integer then,

a) $$Z^+[x(n-k)] = z^{-k}\left[X^+(z) + \sum_{n=1}^{k} x(-n)z^n\right], |z| > r_x \qquad (2.29)$$

in case $x(n)$ is casual

$$Z^+[x(n-k)] = z^{-k}X^+(z)$$

b) $$Z^+[x(n+k)] = z^{k}\left[X^+(z) - \sum_{n=0}^{k-1} x(n)z^{-n}\right], |z| > r_x \qquad (2.30)$$

Proof:

a)

$$Z^+[x(n-k)] = \sum_{n=0}^{\infty} x(n-k)z^{-n} \qquad (2.31)$$

Substitute $m = n - k$ in Eq(2.31), we get

$$Z^+[x(n-k)] = \sum_{m=-k}^{\infty} x(m)z^{-(m+k)}$$

$$= z^{-k}\left[\sum_{m=-k}^{-1} x(m)z^{-m} + \sum_{m=0}^{\infty} x(m)z^{-m}\right]$$

$$= z^{-k}\left[\sum_{m=-k}^{-1} x(m)z^{-m} + X^+(z)\right] \qquad (2.32)$$

Substitute $n = -m$ in Eq(2.32), we obtain

$$Z^+[x(n-k)] = z^{-k}\left[X^+(z) + \sum_{n=1}^{k} x(-n)z^{n}\right]$$

b)

$$Z^+[x(n+k)] = \sum_{n=0}^{\infty} x(n+k)z^{-n} \qquad (2.33)$$

Changing the index of summation of Eq(2.33) from n to $m = n + k$

$$Z^+[x(n+k)] = \sum_{m=k}^{\infty} x(m)z^{-(m-k)}$$

$$= z^{k}\sum_{m=k}^{\infty} x(m)z^{-m)}$$

$$= z^{k}\left[\sum_{m=0}^{\infty} x(m)z^{-m} - \sum_{m=0}^{k-1} x(m)z^{-m}\right]$$

$$= z^{k}\left[X^+(z) - \sum_{m=0}^{k-1} x(m)z^{-m}\right] \qquad (2.34)$$

Changing the index of summation of Eq(2.34) from m to $n = m$

$$Z^+[x(n+k)] = z^{k}\left[X^+(z) - \sum_{n=0}^{k-1} x(n)z^{-n}\right]$$

Example 2.15: Determine the one-sided Z-transform of the sequences.

a) $x(n) = 3^n, n \in \mathbb{Z}$

b) $y(n) = x(n-2)$

c) $w(n) = x(n+3)$

Solution:

a)

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty} 3^n z^{-n} = \frac{z}{z-3}, |z| > 3$$

b)

Using shifting property in Eq(2.29) for the one-sided Z-transform with $k = 2$

$$Z^+[x(n-2)] = z^{-2}\left[X^+(z) + \sum_{n=1}^{2} x(-n)z^n\right]$$

$$= z^{-2}[X^+(z) + x(-1)z^1 + x(-2)z^2]$$

$$= z^{-2}X^+(z) + x(-1)z^{-1} + x(-2)$$

since $x(-1) = 3^{-1} = \frac{1}{3}$, $x(-2) = 3^{-2} = \frac{1}{9}$, we obtain ,

$$Z^+[x(n-2)] = \frac{1}{z^2 - 3z} + \frac{1}{3z} + \frac{1}{9}, \quad |z| > 3$$

c)

Using shifting property in Eq(2.30) for the one-sided Z-transform with $k = 3$

$$Z^+[x(n+3)] = z^3\left[X^+(z) - \sum_{n=0}^{3-1} x(n)z^{-n}\right]$$

$$= z^3[X^+(z) - (x(0)z^0 + x(1)z^{-1} + x(2)z^{-2})]$$

$$= z^3X^+(z) - x(0)z^3 - x(1)z^2 + x(2)z^1$$

since $x(0) = 3^0 = 1$, $x(1) = 3^1 = 3$ and $x(2) = 3^2 = 9$, we obtain,

$$Z^+[x(n+3)] = z^3\frac{z}{z-3} - z^3 - 3z^2 + 9z$$

$$= \frac{z^4}{z-3} - z^3 - 3z^2 + 9z, \quad |z| > 3$$

Now, we will talk about two important theorems in the analysis of sequences: the **Initial value theorem** and **Final value theorem**. They are useful if we are interested in asymptotic behavior of a sequences $x(n)$ and we know its one-sided Z-transform but not the sequences.

Initial Value Theorem[6]

Let $x(n)$ be a sequence, the initial value, $x(0)$, can be found from $X^{+}(z)$ as follows:

$$x(0) = \lim_{z\to\infty} X^{+}(z) \tag{2.35}$$

Proof:

$$X^{+}(z) = \sum_{n=o}^{\infty} x(n)\, z^{-n} = x(0) \;+ x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

If we let $z \to \infty$,each term in $X^{+}(z)$ goes to zero except the first one.

$$\therefore \lim_{z\to\infty} X^{+}(z) = x(0)$$

Final Value Theorem[8]

If $X^{+}(z)$ is the one-sided Z-transform of the sequence $x(n)$,then

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)\, X^{+}(z)\,, if\ \lim_{n\to\infty} x(n)\ \text{exist} \tag{2.36}$$

Proof:

$$Z^{+}[x(k+1) - x(k)] = \lim_{n\to\infty} \sum_{k=0}^{n} [x(k+1) - x(k)]\, z^{-k}$$

$$zX^{+}(z) - zx(0) - X^{+}(z) = \lim_{n\to\infty} \sum_{k=0}^{n} [x(k+1) - x(k)]\, z^{-k}$$

$$(z-1)X^{+}(z) - zx(0) = \lim_{n\to\infty} \sum_{k=0}^{n} [x(k+1) - x(k)]\, z^{-k}$$

By taking the limit as $z \to 1$,the above equation become

$$\lim_{z\to 1}(z-1)X^{+}(z) - x(0)$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} [x(k+1) - x(k)]$$

$$= \lim_{n\to\infty} [x(1) - x(0) + x(2) - x(1) + \cdots$$

$$x(n) - x(n-1) + x(n+1) - x(n)]$$

$$= \lim_{n\to\infty} [-x(0) + x(n+1)]$$

$$= -x(0) + \lim_{n\to\infty} x(n)$$

$$\therefore \lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)\, X^{+}(z)$$

Example 2. 16: If $X^{+}(z) = z^2/(z-1)(z-e^{-1})$, $|z| > 1$,is the one-sided Z-transform of the sequence $x(n)$. Find the value of $x(0)$ and $\lim_{n\to\infty} x(n)$.

Solution:

From **initial value theorem**

$$x(0) = \lim_{z\to\infty} X^{+}(z) = \lim_{z\to\infty} \frac{z^2}{(z-1)(z-e^{-1})} = 1$$

And from **final value theorem**

$$\begin{aligned}\lim_{n\to\infty} x(n) &= \lim_{z\to 1}(z-1)\, X^{+}(z) \\ &= \lim_{z\to 1}(z-1)\frac{z^2}{(z-1)(z-e^{-1})} \\ &= \frac{1}{(1-e^{-1})}\end{aligned}$$

Example 2.17: Generalize the Initial value theorem to find the value of $x(1)$ for a causal sequence $x(n)$ and find it for

$$X(z) = \frac{4z^3 + 5z^2}{8z^3 - 2z + 3}$$

Solution:

If $x(n)$ is causal then

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots \tag{2.37}$$

Subtract $x(0)$ from both sides of Eq(2.37)

$$X(z) - x(0) = x(1)z^{-1} + x(2)z^{-2} + \cdots \tag{2.38}$$

Multiply both sides of Eq(2.38) with z

$$z[X(z) - x(0)] = z[x(1)z^{-1} + x(2)z^{-2} + \cdots] \tag{2.39}$$

If we take the limit as $z \to \infty$ for Eq(2.39)

$$\lim_{z\to\infty} z[X(z) - x(0)] = x(1) \tag{2.40}$$

Eq(2.40) is a generalize the Initial value theorem to find the value of $x(1)$ for a causal sequence $x(n)$.

For the given Z-transform

$$x(0) = \lim_{z\to\infty} X(z) = \frac{1}{2}$$

$$x(1) = \lim_{z\to\infty} z[X(z) - x(0)]$$

$$z[X(z) - x(0)] = z\left[\frac{4z^3 + 5z^2}{8z^3 - 2z + 3} - \frac{1}{2}\right] = \frac{10z^3 + 2z^2 - 3z}{16z^3 - 4z + 6}$$

Therefore

$$x(1) = \lim_{z\to\infty} \frac{10z^3 + 2z^2 - 3z}{16z^3 - 4z + 6} = \frac{5}{8}$$

Example 2.18: Let $x(n)$ be a left-sided sequence that is equal zero for $n > 0$. If

$$X(z) = \frac{5z + 4}{9z^2 - 3z + 2}$$

Find $x(0)$.

Solution:

For a left-sided sequence that is equal zero for $n > 0$, the Z-transform is

$$X(z) = x(0) + x(-1)z + x(-2)z^2 + \cdots$$

If we take the limit as $z \to 0$, we get

$$\lim_{z\to 0} X(z) = x(0)$$

For our example

$$x(0) = \lim_{z\to 0} X(z) = \lim_{z\to 0} \frac{5z + 4}{9z^2 - 3z + 2} = 2$$

Chapter Three

The Inverse Z-transform

In this chapter we investigate methods for finding the inverse of the Z-transform, clarify the relation between Z-transform and discrete Fourier-transform and its relation with Laplace transform, then we introduce the definition of two-dimensional Z-transform, investigate its properties and how to find its inverse.

Section 3.1: The Inverse Z-transform

Just as important as technique for finding the Z-transform of a sequence are methods that may be used to invert the Z-transform and recover the sequence $x(n)$ from $X(z)$. Three methods are often used for the evaluation of the inverse of Z-transform.

1. Integration.
2. Power Series.
3. Partial-Fraction.

1. Integration Method.[9]

Integration method relies on Cauchy integral formula, which state that if C is a closed contour that encircles the origin in a counterclockwise direction then,

$$\frac{1}{2\pi j}\oint_C z^{k-1}dz = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \tag{3.1}$$

The Z-transform of a sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{3.2}$$

Multiply both sides of Eq(3.2) by $\frac{1}{2\pi j}\ z^{k-1}$ and integrating over a contour C that encloses the origin counterclockwise and lies entirely in the region of convergence of $X(z)$, we obtain

$$\frac{1}{2\pi j}\oint_C X(z)\ z^{k-1}dz = \frac{1}{2\pi j}\oint_C \sum_{n=-\infty}^{\infty} x(n)\ z^{-n+k-1}dz \qquad (3.3)$$

Interchanging the order of integration and summation on the right-hand side of Eq(3.3) (valid if the series is convergent) we get

$$\frac{1}{2\pi j}\oint_C X(z)\ z^{k-1}dz = \sum_{n=-\infty}^{\infty} x(n)\frac{1}{2\pi j}\oint_C z^{-n+k-1}dz \qquad (3.4)$$

Applying Cauchy integral formula on the integral in the right hand side of Eq(3.4), we get

$$\frac{1}{2\pi j}\oint_C z^{-n+k-1}dz = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases} \qquad (3.5)$$

So Eq(3.4) becomes

$$\frac{1}{2\pi j}\oint_C X(z)\ z^{k-1}dz = x(k) \qquad (3.6)$$

Therefore, the inverse of Z-transform is given by the integral

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)\ z^{n-1}dz \qquad (3.7)$$

where C is a counterclockwise closed contour in the region of convergence of $X(z)$ and encircling the origin of the z-plane and $n \in \mathbb{Z}$.

Note: For a rational Z-transform $X(z)$, $x(n)$ is often evaluated using the residue theorem, i.e.,

$$x(n) = \sum [residue\ of\ X(z)\ z^{n-1} at\ the\ poles\ inside\ C\ ] \qquad (3.8)$$

Example 3.1: Find the inverse Z-transform of

$$X(z) = \frac{z}{z+3}\ , \quad |z| > 3$$

Solution:

From Eq(3.7) $x(n)$, will be

$$x(n) = \frac{1}{2\pi j}\oint_C \frac{z}{z+3}\; z^{n-1}dz = \frac{1}{2\pi j}\oint_C \frac{z^n}{z+3}dz$$

where the contour of integration, *C* is a circle of radius greater than 3.

For $n \geq 0$, the contour of integration encloses only one pole at $z = -3$. So

$$x(n) = \; Res\left[\frac{z^n}{z+3}, -3\;\right] = \; (-3)^n$$

For $n < 0$, in an addition to the pole at $z = -3$ there is a multiple-order pole at $z = 0$ whose order depends on n.

For $n = -1$,

$$x(-1) = \frac{1}{2\pi j}\oint_C \frac{1}{z(z+3)}dz$$

$$= Res\left[\frac{1}{z(z+3)}, 0\;\right] + Res\left[\frac{1}{z(z+3)}, -3\;\right] = \frac{1}{3} + \frac{-1}{3} = 0$$

For $n = -2$,

$$\begin{aligned} x(-2) &= \frac{1}{2\pi j}\oint_C \frac{1}{z^2(z+3)}dz \\ &= Res\left[\frac{1}{z^2(z+3)}, 0\;\right] + Res\left[\frac{1}{z^2(z+3)}, -3\;\right] \\ &= \frac{1}{(1)!}\frac{d}{dz}\frac{1}{(z+3)}\Big|_{z\,=\,0} + \frac{1}{9} = \frac{-1}{9} + \frac{1}{9} = 0 \end{aligned}$$

Continuing this procedure it can be verified that $x(n) = 0, n < 0$. So

$$x(n) = \begin{cases} (-3)^n \;, & n \geq 0 \\ \quad 0 \quad\;, & n < 0 \end{cases}$$

or

$$x(n) = (-3)^n u(n)$$

Example 3.2: Find the inverse Z-transform of

$$X(z) = z^2 + 6 + 7z^{-3}, \;\; 0 < |z| < \infty$$

Solution:

From Eq(3.7) $x(n)$, will be

$$x(n) = \frac{1}{2\pi j} \oint_C (z^2 + 6 + 7z^{-3})\, z^{n-1} dz$$

where C is the unit circle taken counterclockwise .

$$x(n) = \frac{1}{2\pi j} \left[\oint_C z^{n+1} dz + 6 \oint_C z^{n-1} dz + 7 \oint_C z^{n-4} dz \right]$$

From Cauchy integral formula in Eq(3.1) we get,

$$x(-2) = \frac{1}{2\pi j}\, [2\pi j \cdot 1 + 0 + 0] = 1$$

$$x(0) = \frac{1}{2\pi j}\, [0 + 6 \cdot 2\pi j + 0] = 6$$

$$x(3) = \frac{1}{2\pi j}\, [0 + 0 + 7 \cdot 2\pi j] = 7$$

For $n \neq -2, 0, 3$

$$x(n) = 0$$

So

$$x(n) = \begin{cases} 1\,, & n = -2 \\ 6\,, & n = 0 \\ 7\,, & n = 3 \\ 0\,, & otherwise \end{cases}$$

Note: Integration method is particularly useful if only a specific values of *x(n)* are needed.

We now prove the **Multiplication of Two Sequences Property** and **Parseval's Theorem**.

Multiplication of Two Sequences Property:

If $X(z)$ is the Z-transform of $x(n)$ with $ROC\ r_x < |z| < R_x$ and $Y(z)$ is the Z-transform of $y(n)$ with $ROC\ \ r_y < |z| < R_y$, then

$$Z\big(x(n)y(n)\big) = \frac{1}{2\pi j}\oint_C X(v)Y\left(\frac{z}{v}\right)\frac{1}{v}dv, \tag{3.9}$$

$ROC: r_x r_y < |z| < R_x R_y$

where C is a counterclockwise closed contour that encloses the origin and lies within the common region of convergence of $X(v)$ and $Y\left(\frac{z}{v}\right)$.

Proof: [14]

Let C be as given in the above theorem and

$$w(n) = x(n)y(n) \tag{3.10}$$

Then, the Z-transform of $w(n)$ is

$$W(z) = \sum_{n=-\infty}^{\infty} x(n)y(n)z^{-n} \tag{3.11}$$

But

$$x(n) = \frac{1}{2\pi j}\oint_C X(v)\, v^{n-1}dv \tag{3.12}$$

Substituting $x(n)$ in Eq(3.11) and interchanging the order of summation and integration we obtain

$$W(z) = \frac{1}{2\pi j}\oint_C X(v)\left[\sum_{n=-\infty}^{\infty} y(n)\left(\frac{z}{v}\right)^{-n}\right]\frac{1}{v}\, dv \tag{3.13}$$

The sum in the brackets of Eq(3.13) is simply the transform $Y(z)$ evaluated at z/v. Therefore,

$$W(z) = \frac{1}{2\pi j}\oint_C X(v)Y\left(\frac{z}{v}\right)\frac{1}{v}dv$$

To obtain the ROC of $W(z)$ we note that if $X(v)$ converges for $r_x < |v| < R_x$ and $Y(z)$ converges for $r_y < |z| < R_y$, then the ROC of $Y(z/v)$ is

$$r_y < \left|\frac{z}{v}\right| < R_y$$

Hence the ROC for $W(z)$ is at least

$$r_x r_y < |z| < R_x R_y$$

Parseval's Theorem

Let $X(z)$ be the Z-transform of $x(n)$ with $ROC\ \ r_x < |z| < R_x$

and let $Y(z)$ be the Z-transform of $y(n)$ with $ROC\ \ r_y < |z| < R_y$, with

$$r_x r_y < |z| = 1 < R_x R_y$$

then we have

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi j}\oint_C X(z)\mathrm{Y}^*\left(\frac{1}{z^*}\right)\frac{1}{z}dz \qquad (3.14)$$

where C is a counterclockwise closed contour that encloses the origin and lies within the common region of convergence of $X(z)$ and $\mathrm{Y}^*(1/\mathrm{z}^*)$.

Proof: [9]

Let

$$w(n) = x(n)y^*(n) \qquad (3.15)$$

Noting that

$$\sum_{n=-\infty}^{\infty} w(n) = W(z)|_{z=1} \qquad (3.16)$$

By the multiplication of two sequences and the conjugation of Z-transform properties, Eq(3.16) becomes

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi j}\oint_C X(v)Y^*\left(\frac{1}{v^*}\right)\frac{1}{v}dv \qquad (3.17)$$

Replace the dummy variable v with z in Eq(3.17) we obtain

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi j}\oint_C X(z)Y^*\left(\frac{1}{z^*}\right)\frac{1}{z}dz$$

where C is a counterclockwise closed contour that encloses the origin and lies within the common region of convergence of $X(z)$ and $Y^*(1/\mathrm{z}^*)$.

2. Power Series Method

The idea of this method is to write $X(z)$ as a power series of the form

$$X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n} \tag{3.18}$$

which convergence in the ROC. Then by uniqueness of $X(z)$ we can say that $x(n) = c_n$ for all n.

Example 3.3: Find the inverse Z-transform of

$$X(z) = z^2 + 6 + 7z^{-3}, \;\; 0 < |z| < \infty$$

Solution:

Since $X(z)$ is a finite-order integer power function, $x(n)$ is a finite-length sequence. Therefore, $x(n)$ is the coefficient that multiplies z^{-n} in $X(z)$. Thus, $x(3) = 7$, $x(0) = 6$ and $x(-2) = 1$.

$$\therefore x(n) = \begin{cases} 1\,, & n = -2 \\ 6\,, & n = 0 \\ 7\,, & n = 3 \\ 0\,, & otherwise \end{cases}$$

Note: Comparing **Example 3.3**with **Example 3.2** we note that the power series method is easier than integration method in such cases.

Example 3.4: Find the inverse Z-transform of

$$X(z) = sin\,z\,, \qquad |z| \geq 0$$

Solution:

Using Taylor series for $sin\,z$ about $z = 0$ we get,

$$X(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+1}}{(2m+1)!}$$

But

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

So

$$\sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+1}}{(2m+1)!}$$

Comparing the powers of z on the both sides of the previous equation we find

$$-n = 2m + 1, m \in \mathbb{N}$$

So $-n$ is odd integer,

$$n = -1, -3, -5, \ldots$$

Thus $x(n)$ becomes

$$x(n) = \frac{(-1)^{-\left(\frac{n+1}{2}\right)}}{(-n)!}\ , n = -1, -3, -5, \ldots$$

Example 3.5: Find the inverse Z-transform of

a) $X(z) = \dfrac{z}{z+3}\ , \quad |z| > 3$

b) $X(z) = \dfrac{z}{z+3}\ , \quad |z| < 3$

Solution:

a) Since the region of convergence is the exterior of a circle, the sequence is a right-sided sequence. Furthermore, since $\lim_{z\to\infty} X(z) = 1 =$ constant, it's a causal sequence. By long division to obtain power series in z^{-1}.

$$
\begin{array}{r|l}
 & 1-3z^{-1}+9z^{-2}+\cdots \\
\hline
1+3z^{-1} & 1 \\
 & \underline{1+3z^{-1}} \\
 & -3z^{-1} \\
 & \underline{-3z^{-1}-9z^{-2}} \\
 & 9z^{-2} \\
 & \underline{9z^{-2}+27z^{-3}} \\
 & -27z^{-3} \\
 & \cdots
\end{array}
$$

$$\therefore X(z) = 1 - 3z^{-1} + 9z^{-2} + \cdots = \sum_{n=0}^{\infty} (-3)^n z^{-n}$$

So that

$$x(n) = (-3)^n u(n)$$

b) Since the region of convergence is the interior of a circle, the sequence is a left-sided sequence and since $X(0) = 0$, the sequence is anticausal. Thus we divide to obtain a series in powers of z.

$$
\begin{array}{r|l}
 & \frac{1}{3}z-\frac{1}{9}z^2+\frac{1}{27}z^3+\cdots \\
\hline
3+z & z \\
 & \underline{z+\frac{1}{3}z^2} \\
 & -\frac{1}{3}z^2 \\
 & \underline{-\frac{1}{3}z^2-\frac{1}{9}z^3} \\
 & \frac{1}{9}z^3 \\
 & \underline{\frac{1}{9}z^3+\frac{1}{27}z^4} \\
 & -\frac{1}{27}z^4 \\
 & \cdots
\end{array}
$$

$$X(z) = \frac{1}{3}z - \frac{1}{9}z^2 + \frac{1}{27}z^3 + \cdots = \sum_{n=-\infty}^{-1} -(-3)^n z^{-n}$$

$$\therefore x(n) = -(-3)^n u(-n-1)$$

3. Partial-Fraction Method.

If $X(z)$ is rational function then the partial fraction method is often useful method to find its inverse [9]. The idea of this method is to write $X(z)$ as

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + \cdots + \alpha_k X_k(z) \qquad (3.19)$$

where each $X_i(z)$ has inverse Z-transform $x_i(n)$ and $\alpha_i \in \mathbb{C}$ for $i = 1, 2, \ldots, k$.

If $X(z)$ is rational function of z then $X(z)$ can be expressed as

$$X(z) = \frac{A(z)}{B(z)} = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_M z^M}{b_0 + b_1 z + b_2 z^2 + \cdots + b_N z^N} \qquad (3.20)$$

where $M, N \in \mathbb{N}$.

If $N \neq 0$ and $M < N$ then $X(z)$ is called proper rational function, otherwise $X(z)$ is improper, and it can always be written as a sum of polynomial and proper rational function. i.e.

$$X(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_k z^k + \frac{A_1(z)}{B(z)} \qquad (3.21)$$

The inverse of the polynomial can be easily found but to find the inverse of the proper rational function we need to write it as a sum of simple functions, for this purpose we factorized the denominator into factor of poles $p_0, p_1, \ldots, p_m$ of $X(z)$.

Remark: [5] Sometimes it's better to expand $X(z)/z$ rather than $X(z)$ because most Z-transforms have the term z in their numerator.

We have two cases for the poles[5].

Case 1: Simple poles.

If all poles of $X(z)$ are simple then

$$X(z) = \sum_{i=1}^{m} \frac{A_i}{z - p_i} \tag{3.22}$$

where

$$A_i = (z - p_i)X(z)\Big|_{z\, =\, p_i} \tag{3.23}$$

Then

$$Z^{-1}\left[\frac{z}{z - p_i}\right] = \begin{cases} (p_i)^n u(n) & if \quad ROC\ |z| > |p_i| \\ -(p_i)^n u(-n-1) & if \quad ROC\ |z| < |p_i| \end{cases} \tag{3.24}$$

If $x(n)$ is causal and some poles of $X(z)$ are complex then if p is a pole then p^* is also a pole and in this case

$$x(n) = [A(p)^n + \ A^*(p^*)^n]u(n) \tag{3.25}$$

In polar form Eq(3.25) become

$$x(n) = 2|A||p|^n\, cos(\theta n + \varphi)\, u(n) \tag{3.26}$$

where θ and φ are the argument of the pole p and the argument of the partial fraction coefficient A, respectively.

Case 2: Multiple poles.

For a function $X(z)$ with a repeated pole of multiplicity r, r partial fraction coefficients are associated with this repeated pole. The partial fraction expansion of $X(z)$ will be of the form

$$X(z) = \sum_{k=1}^{r} \frac{A_{1k}}{(z - p_1)^{r+1-k}} + \sum_{k=r+1}^{m} \frac{A_k}{z - p_k} \tag{3.27}$$

where

$$A_{1k} = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (z - p_1)^r X(z)\Big|_{z\, =\, p_1}, k = 1, 2, \dots , r \tag{3.28}$$

Example 3.6: Find the inverse Z-transform of

$$X(z) = \frac{z^2 + 3z}{z^2 - 3z + 2} \text{ if } ROC \text{ is}$$

a) $|z| > 1$

b) $|z| < 2$

c) $1 < |z| < 2$

Solution:

First we write $X(z)/z$ as a partial fraction

$$\frac{X(z)}{z} = \frac{z+3}{z^2 - 3z + 2} = \frac{z+3}{(z-2)(z-1)}$$

$$= \frac{A}{z-2} + \frac{B}{z-1}$$

$$A = (z-2)\frac{X(z)}{z}\Big|_{z=2} = 5$$

$$B = (z-1)\frac{X(z)}{z}\Big|_{z=1} = -4$$

So

$$X(z) = 5\frac{z}{z-2} - 4\frac{z}{z-1}$$

For (a):

Since the ROC of $X(z)$ is $|z| > 1$, the sequence $x(n)$ is causal sequence so we obtain,

$$x(n) = (5\ (2)^n - 4)u(n)$$

b) The ROC of $X(z)$ is $|z| < 2$ so the sequence $x(n)$ is anticausal so,

$$x(n) = (-5\ (2)^n + 4)u(-n-1)$$

c) The last ROC $1 < |z| < 2$ of $X(z)$ is annular, so the sequence $x(n)$ is two-sided. Thus one of the terms is causal and the other is anticausal. The ROC is overlapping $|z| > 1$and $|z| < 2$ so the pole $p_1 = 1$ provides the causal sequence and the pole $p_2 = 2$ provides the anticausal sequence.

Thus

$$x(n) = -4u(n) - 5\,(2)^n u(-n-1)$$

Example 3.7: Find the inverse Z-transform of

$$X(z) = \frac{z+1}{z^2 - 2z + 2} \;\; ROC|z| > \sqrt{2}$$

Solution:

We write $Y(z) = \mathrm{X(z)/z}$ as a partial fraction

$$Y(z) = \frac{X(z)}{z} = \frac{z+1}{z(z^2 - 2z + 2)} = \frac{A}{z} + \frac{B}{z-(1+j)} + \frac{C}{z-(1-j)}$$

$$A = zY(z)\Big|_{z=0} = \frac{1}{2}$$

$$B = [z-(1+j)]Y(z)\Big|_{z=1+j} = \frac{2+j}{2j-2} = -\left(\frac{1+3j}{4}\right)$$

$$C = [z-(1-j)]Y(z)\Big|_{z=1-j} = \frac{2-j}{-(2+2j)} = -\left(\frac{1-3j}{4}\right)$$

$$X(z) = \frac{1}{2} - \left(\frac{1+3j}{4}\right)\frac{z}{z-(1+j)} - \left(\frac{1-3j}{4}\right)\frac{z}{z-(1-j)}$$

$$\therefore x(n) = \frac{1}{2}\delta(n) - \left(\frac{1+3j}{4}\right)(1+j)^n u(n) - \left(\frac{1-3j}{4}\right)(1-j)^n u(n)$$

Note that:

$$x(0) = \frac{1}{2} - \left(\frac{1+3j}{4}\right) - \left(\frac{1-3j}{4}\right) = 0$$

So,

$$x(n) = \begin{cases} -\left(\frac{1+3j}{4}\right)(1+j)^n - \left(\frac{1-3j}{4}\right)(1-j)^n & , n \geq 1 \\ 0 & , otherwise \end{cases}$$

For $n \geq 1$, $x(n)$ can be written in polar form as:

$$x(n) = \left(\frac{\sqrt{10}}{2}\right)\left(\sqrt{2}\right)^n \; cos\left(\frac{\pi}{4}n + \tan^{-1} 3\right)$$

$$= \sqrt{5}\left(\sqrt{2}\right)^{n-1} cos\left(\frac{\pi}{4}n + \tan^{-1} 3\right)$$

Example 3.8: Find the inverse Z-transform of

$$X(z) = \frac{z^3}{z^3 - z^2 - 5z - 3} \;\; ROC|z| > 1$$

Solution:

We write $Y(z) = X(z)/z$ as a partial fraction

$$Y(z) = \frac{X(z)}{z} = \frac{z^2}{z^3 - z^2 - 5z - 3} = \frac{z^2}{(z-3)(z+1)^2}$$

$$= \frac{A}{z-3} + \frac{B}{z+1} + \frac{C}{(z+1)^2}$$

$$A = (z-3)Y(z)\Big|_{z=3} = \frac{9}{16}$$

$$B = \frac{1}{(2-1)!}\frac{d}{dz}(z+1)^2 Y(z)\Big|_{z=-1} = \frac{7}{16}$$

$$C = (z+1)^2 Y(z)\Big|_{z=-1} = \frac{-1}{4}$$

$$X(z) = \frac{9}{16}\frac{z}{z-3} + \frac{7}{16}\frac{z}{z+1} - \frac{1}{4}\frac{z}{(z+1)^2}$$

$$\therefore x(n) = \left[\frac{9}{16}(3)^n + \frac{7}{16}(-1)^n + \frac{1}{4}n(-1)^n\right]u(n)$$

$$x(n) = \frac{1}{16}[(3)^{n+2} + (-1)^n(7+4n)]u(n)$$

Example 3.9: Find the inverse Z-transform of

$$X(z) = \frac{z^5 + 6z^3 + 7}{z^3}, \quad 0 < |z| < \infty$$

Solution:

$X(z)$ is an improper rational function so it can be written as

$$X(z) = z^2 + 6 + \frac{7}{z^3}$$

So,

$$x(n) = \delta(n+2) + 6\delta(n) + 7\delta(n-3)$$

or

$$x(n) = \begin{cases} 1\,, & n = -2 \\ 6\,, & n = 0 \\ 7\,, & n = 3 \\ 0\,, & otherwise \end{cases}$$

Section 3.2: The Relation Between Z-transform and the Discrete Fourier Transform.

Let $x(n)$ be a sequence, then the discrete Fourier transform (DFT) of $x(n)$ is [5],

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-n\omega j} \tag{3.29}$$

where ω is real.

The Z-transform of $x(n)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{3.30}$$

where z is a complex variable.

In polar form z is written as

$$z = re^{j\omega} \tag{3.31}$$

where $r = |z|$ and $\omega = arg(z)$.

Substituting Eq(3.31) in Eq(3.30) we obtain

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\left(re^{j\omega}\right)^{-n} = \sum_{n=-\infty}^{\infty} x(n)r^{-n}(e^{j\omega})^{-n} \tag{3.32}$$

which is the discrete Fourier-transform for $[x(n)r^{-n}]$. If $r = 1$, then $z = e^{j\omega}$and the Z-transform of $x(n)$ becomes the discrete Fourier-transform.

i.e

$$X(z)\Big|_{z\,=\,e^{j\omega}} = X\left(e^{j\omega}\right)$$

So, Z-transform is generalization of discrete Fourier-transform.

Section 3.3: The Relation Between Z-transform and Laplace Transform.

Let $f(t)$ be a continuous function, then we can take a sampled discrete function $f_s(t)$ which can be written as [13]

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(n-t) = \sum_{n=-\infty}^{\infty} f(n)\delta(n-t) \qquad (3.33)$$

The Laplace transform of sampled function $f_s(t)$ is:

$$X(s) = \mathcal{L}[f_s(t)] = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} f(n)\delta(t-n) \right] e^{-st} dt \qquad (3.34)$$

where $s = \sigma + j\omega$, σ and ω are real variables, by interchanging the order of the summation and the integration of Eq(3.34) we get

$$X(s) = \sum_{n=-\infty}^{\infty} f(n) \left[\int_{-\infty}^{\infty} \delta(t-n) e^{-st} dt \right] = \sum_{n=-\infty}^{\infty} f(n)\, e^{-ns}$$

$$= \sum_{n=-\infty}^{\infty} f(n)\, (e^{s})^{-n} = X(z)\Big|_{z\,=\,e^{s}} \qquad (3.35)$$

So the relation between Laplace-transform and Z-transform is

$$X(s) = X(z)\Big|_{z\,=\,e^{s}} \qquad (3.36)$$

or

$$X(z) = X(s)\Big|_{s\,=\,ln\, z} \qquad (3.37)$$

The most important correspondence between the s-plane and z-plane are:

1. The points on the jω-axis in the s-plane mapped onto the unit circle in the z-plane.
2. The points in the right half of the s-plane mapped outside the unit circle in the z-plane where the points in the left half mapped inside the unit circle.

3. The lines $s = \sigma$ parallel to the jω-axis in the s-plane mapped into circles with radius $|z| = e^{\sigma}$ in z-plane where the lines $s = ju$ parallel to the σ-axis mapped into rays of the form $arg\ z = u$ radians from $z = 0$.
4. The origin of the s-plane mapped to the point $z = 1$ in the z-plane.

Section 3.4: The Two-Dimensional Z-transform.

3.4.1: Definition of the Two-Dimensional Z-transform.

Definition 3.1:[20] The two-dimensional Z-transform $X(z_1, z_2)$ of a sequence $x(n,m)$ is defined as

$$X(z_1, z_2) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(n,m) {z_1}^{-n} {z_2}^{-m} \qquad (3.38)$$

where $(z_1, z_2) \in \mathbb{C}^2$ and the ROC is the set of all (z_1, z_2) points for which

$$\sum_{n} \sum_{m} |x(n,m)||{z_1}^{-n}||{z_2}^{-m}| < \infty$$

Example 3.10: Find the two-dimensional Z-transform of

$$x(n,m) = \begin{cases} 3, & n = 0, m = -2 \\ 1, & n = 0, m = 0 \\ 4, & n = 5, m = 3 \\ 0, & otherwise \end{cases}$$

Solution:

$$X(z_1, z_2) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(n,m) {z_1}^{-n} {z_2}^{-m}$$

$$= 3{z_2}^2 + 1 + 4{z_1}^{-5}{z_2}^{-3}$$

ROC all $\mathbb{C}^2$ except $z_1 = 0$ or $z_2 = 0, \infty$.

Example 3.11: Find the two-dimensional Z-transform of

$$x(n,m) = 2^n \delta(n-m) u(n,m)$$

Solution:

$$X(z_1,z_2)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}x(n,m){z_1}^{-n}\,{z_2}^{-m}$$
$$=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}2^n\delta(n-m)u(n,m){z_1}^{-n}\,{z_2}^{-m}$$
$$=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}2^n\delta(n-m)u(n)u(m){z_1}^{-n}\,{z_2}^{-m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}2^n\delta(n-m){z_1}^{-n}\,{z_2}^{-m}$$
$$=\sum_{n=0}^{\infty}2^n{z_1}^{-n}\,{z_2}^{-n}=\frac{z_1z_2}{z_1z_2-2},|z_1||z_2|>2$$

3.4.2: Properties of the Two-Dimensional Z-transform.

The properties of the two-dimensional Z-transform are the same as the properties of Z-transform and their proofs are parallel with an additional important property, the separable of sequences propriety.

Let $X(z_1,z_2)$ be the two-dimensional Z-transform of $x(n,m)$ with $ROC\ \ \tilde{R}_x$ and $Y(z_1,z_2)$ the Z-transform of $y(n,m)$ with $C\ \tilde{R}_y$. Then the properties are

1. Linearity.

For any complex numbers α, β we have

$$Z[\alpha x(n,m)\ +\ \beta y(n,m)]=\alpha X(z_1,z_2)\ +\ \beta Y(z_1,z_2)\quad(3.39)$$

ROC: at least $\tilde{R}_x\cap\tilde{R}_y$

2. Shifting.

For $n_1,m_1\in\mathbb{Z}$

$$Z[x(n+n_1,m+m_1)]={z_1}^{n_1}\,{z_2}^{m_1}X(z_1,z_2)\qquad(3.40)$$

ROC: $\tilde{R}_x$with possible exceptions $|z_1|=0,\infty$ and $|z_2|=0,\infty$

3. Multiplication by Exponential.

For $a, b \in \mathbb{C}$

$$Z[a^n b^m x(n,m)] = X(a^{-1}z_1, b^{-1}z_2) \qquad (3.41)$$

ROC: $\tilde{R}_x$scaled by |a| in the $z_1$variable and by $|b|$ in the z_2 variable.

4. Time Reversal.

$$Z[x(-n,-m)\,] = X({z_1}^{-1}, {z_2}^{-1}) \qquad (3.42)$$

ROC: ${z_1}^{-1}, {z_2}^{-1}$ in $\tilde{R}_x$

5. Conjugation.

$$Z[x^*(n,m)] = X^*({z_1}^*, {z_2}^*) \qquad (3.43)$$

ROC: $\tilde{R}_x$

6. Multiplication by n or Differentiation of the Transform

$$Z[n\,m\,x(n,m)\,] = z_1 z_2 \frac{\partial^2 X(z_1, z_2)}{\partial z_1 \partial z_2} \qquad (3.44)$$

ROC: $\tilde{R}_x$

7. Convolution of Two Sequences.

$$Z[x(n,m) * y(n,m)] = X(z_1, z_2)\, Y(z_1, z_2) \qquad (3.45)$$

ROC: at least $\tilde{R}_x \cap \tilde{R}_y$

8. Multiplication of Two Sequences

$$Z[x(n,m))y(n,m)] = \left(\frac{1}{2\pi j}\right)^2 \oint_{C_2} \oint_{C_1} X(z_1, z_2) Y\left(\frac{z_1}{v_1}, \frac{z_2}{v_2}\right) \frac{1}{v_1} \frac{1}{v_2} dv_1 dv_2 \qquad (3.46)$$

9. Separable of Sequences Property

If $X_1(z_1)$, $X_2(z_2)$ are the Z-transform of $x_1(n)$, $x_2(m)$ respectively and

If $x(n,m) = x_1(n) \cdot x_2(m)$ then

$$X(z_1, z_2) = X_1(z_1) \cdot X_2(z_2) \qquad (3.47)$$

with ROC: $z_1 \in ROC\ X_1(z_1)$ and $z_2 \in ROC\ X_2(z_2)$

Example 3.11: Find the two-dimensional Z-transform of

$$x(n,m) = \frac{3^{n-m}}{n!}u(n,m)$$

Solution:

$$x(n,m) = \frac{3^{n-m}}{n!}u(n,m) = \frac{3^n}{n!}u(n) \cdot \frac{1}{3^m}u(m) = x_1(n) \cdot x_2(n)$$

Using separable of sequences propriety we get,

$$X(z_1,z_2) = X_1(z_1) \cdot X_2(z_2)$$
$$= e^{3/z_1} \cdot \frac{z_2}{z_2 - \frac{1}{3}}$$

$ROC \ \ |z_1| > 0, |z_2| > \frac{1}{3}$

3.4.3: The Inverse of the Two-Dimensional Z-transform

Definition 3.2:[12] The inverse of the two-dimensional Z-transform $X(z_1,z_2)$ is

$$x(n,m) = \left(\frac{1}{2\pi j}\right)^2 \oint_{C_2}\oint_{C_1} X(z_1,z_2)\, {z_1}^{n-1}\, {z_2}^{m-1} dz_1\, dz_2 \qquad (3.48)$$

where C_1 and C_2 are a counterclockwise closed contours encircling the origin and within the ROC of $X(z_1,z_2)$.

Example 3.12: Find the inverse of the two-dimensional Z-transform

$$X(z) = \frac{3z_1}{3z_1 - z_2}, \quad |z_1| > \frac{|z_2|}{3}$$

Solution:

$$x(n,m) = \left(\frac{1}{2\pi j}\right)^2 \oint_{C_2}\oint_{C_1} X(z_1,z_2)\, {z_1}^{n-1}\, {z_2}^{m-1} dz_1\, dz_2$$
$$= \left(\frac{1}{2\pi j}\right)^2 \oint_{C_2}\oint_{C_1} \frac{3z_1}{3z_1 - z_2}\, {z_1}^{n-1}\, {z_2}^{m-1} dz_1\, dz_2$$

where C_1 and C_2 are a counterclockwise closed contours encircling the origin and within the ROC of $X(z_1,z_2)$.

$$= \frac{1}{2\pi j} \oint_{C_2} z_2{}^{m-1} \left[\frac{1}{2\pi j} \oint_{C_1} \frac{z_1{}^n}{z_1 - \frac{z_2}{3}} dz_1 \right] dz_2$$

By using residue theorem and since $|z_1| > \frac{|z_2|}{3}$

$$\frac{1}{2\pi j} \oint_{C_1} \frac{z_1{}^n}{z_1 - \frac{z_2}{3}} dz_1 = Res \left[\frac{z_1{}^n}{z_1 - \frac{z_2}{3}}, \frac{z_2}{3} \right] = \left(\frac{z_2}{3}\right)^n u(n)$$

So

$$x(n,m) = \frac{1}{2\pi j} \oint_{C_2} z_2{}^{m-1} \left(\frac{z_2}{3}\right)^n u(n) dz_2$$

$$= \frac{1}{2\pi j} \oint_{C_2} z_2{}^{m+n-1} \left(\frac{1}{3}\right)^n u(n) dz_2$$

$$= \left(\frac{1}{3}\right)^n \delta(n+m)$$

Chapter Four

Z-transform and Solution of Some Difference Equations

One of the most important applications of Z-transform is solving some linear difference equations. Z-transform is also one of the most effective methods for solving Volterra difference equations of convolution type[4].

Section 4.1: Linear Difference Equations with Constant Coefficients.

Definition 4.1: [3] A linear difference equation with constant coefficients has the form,

$$y(n+k) + a_1 y(n+k-1) + a_2 y(n+k-2) + \cdots + a_k y(n) = x(n) \quad (4.1)$$

where a_i's are real or complex constants for all $i = 1, 2, \ldots, k$. If $x(n) = 0$ then the equation is called homogeneous otherwise its non homogeneous.

The order of linear difference equation is the difference between the largest and smallest indices of the unknown sequence $y(n)$.

If the initial values $y(0),\ y(1), y(2),\ \ldots, y(k-1)$ are all given, then Eq(4.1) is called initial value problem (IVP).

Theorem 4.1: [7] **On the Existence of a Unique Solution**

Consider the general kth IVP,

$$y(n+k) = g(n, y(n), y(n+1), \ldots, y(n+k-1), k = 0, 1, 2, \ldots \quad (4.2)$$

where $y(0) = y_0,\ y(1) = y_1, y(2) = y_2,\ \ldots, y(k-1) = y_{k-1}$, are given and the function g is defined for all its arguments $n, y(n), y(n+1)$, $y(n+2), \ldots, y(n+k-1)$. Then the IVP has a unique solution corresponding to each arbitrary set of the k initial values $y(0), y(1), y(2), \dots, y(k-1)$.

Proof: see [7].

Method of Solution:

For solving linear difference equation by using Z-transform method we take Z-transform of the difference equation which transforms the unknown sequence $y(n)$ into an a algebraic equation on Z-transform $Y(z)$ then we obtain the sequence $y(n)$ from $Y(z)$ by taking the inverse Z-transform of $Y(z)$.[4]

In order to solve linear difference equations with constant coefficients with non zero initial conditions we use the one sided Z-transform**.**

Example 4.1: Use the Z-transform method to solve the linear difference equation

$$y(n+2) = y(n+1) + \ y(n)$$

where $y(0) = 0$ and $y(1) = 1$.

Solution:

Take the one-sided Z-transform for both sides of the linear difference equation , we get,

$$Z^{+}[y(n+2)] = Z^{+}[y(n+1)] + \ Z^{+}[y(n)]$$

$$z^2Y^{+}(z) - z^2y(0) - z\,y(1) \ = zY^{+}(z) - zy(0) + Y^{+}(z)$$

$$Y^{+}(z)[z^2 - z - 1] = z^2y(0) - zy(0) + z\,y(1)$$

Substitute $y(0) = 0$ and $y(1) = 1$ in the previous equation we get,

$$Y^{+}(z)[z^2 - z - 1] = z$$

So, let

$$W(z)=\frac{Y^+(z)}{z}=\frac{1}{z^2-z-1}=\frac{1}{\left[z-\left(\frac{1+\sqrt{5}}{2}\right)\right]\left[z+\left(\frac{1-\sqrt{5}}{2}\right)\right]}$$

Using partial fraction method we get,

$$W(z)=\frac{A}{z-\left(\frac{1+\sqrt{5}}{2}\right)}+\frac{B}{z+\left(\frac{1-\sqrt{5}}{2}\right)}$$

$$=\frac{1}{\sqrt{5}}\frac{1}{z-\left(\frac{1+\sqrt{5}}{2}\right)}-\frac{1}{\sqrt{5}}\frac{1}{z+\left(\frac{1-\sqrt{5}}{2}\right)}$$

So

$$Y^+(z)=\frac{1}{\sqrt{5}}\frac{z}{z-\left(\frac{1+\sqrt{5}}{2}\right)}-\frac{1}{\sqrt{5}}\frac{z}{z+\left(\frac{1-\sqrt{5}}{2}\right)}$$

Taking the inverse Z-transform we get,

$$y(n)=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Using binomial theorem we get,

$$y(n)=\frac{1}{2^n\sqrt{5}}\left[\sum_{r=0}^{n}\binom{n}{r}\left(\sqrt{5}\right)^r-\sum_{r=0}^{n}\binom{n}{r}\left(-\sqrt{5}\right)^r\right]$$

$$=\frac{2}{2^n\sqrt{5}}\left[\sum_{r=0}^{L}\binom{n}{2r+1}\left(\sqrt{5}\right)^{2r+1}\right], L=\left\lfloor\frac{n-1}{2}\right\rfloor$$

$$=\frac{1}{2^{n-1}}\left[\sum_{r=0}^{L}\binom{n}{2r+1}5^r\right],\ \ n>0$$

Remark: The sequence in our example is called Fibonacci sequence.

Example 4.2: Solve the linear difference equation

$$y(n)-\frac{1}{3}y(n-1)=u(n), y(-1)=2$$

Solution:

We take the one sided Z-transform of the linear difference equation, to get

$$Y^+(z)-\frac{1}{3}[z^{-1}Y^+(z)+y(-1)]=\frac{z}{z-1}$$

Substitute $y(-1)=2$ in the previous equation we get,

$$Y^+(z)\left[1-\frac{1}{3}z^{-1}\right]-\frac{2}{3}=\frac{z}{z-1}$$

By some mathematical operation we obtain,

$$Y^+(z)=\frac{z(5z-2)}{(3z-1)(z-1)}$$

Then, write as $Y^+(z)/\mathrm{z}$ as a partial fraction

$$\frac{Y^+(z)}{z}=\frac{(5z-2)}{(3z-1)(z-1)}=\frac{A}{3z-1}+\frac{B}{z-1}$$

Using partial fraction method we get,

$$A=\frac{Y^+(z)}{z}(3z-1)\bigg|_{z=\frac{1}{3}}=\frac{1}{2}$$

$$B=\frac{Y^+(z)}{z}(z-1)\bigg|_{z=1}=\frac{3}{2}$$

Then

$$Y^+(z)=\frac{1}{2}\frac{z}{3z-1}+\frac{3}{2}\frac{z}{z-1}$$
$$=\frac{1}{6}\frac{z}{z-\frac{1}{3}}+\frac{3}{2}\frac{z}{z-1}$$

Take the inverse Z-transform we obtain

$$y(n)=\left[\frac{1}{6}\left(\frac{1}{3}\right)^n+\frac{3}{2}\right]u(n)$$

$$=\frac{1}{2}\left[\left(\frac{1}{3}\right)^{n+1}+3\right]u(n)$$

Section 4.2: Volterra Difference Equations of Convolution Type.

In this section we will use Z-transform to solve Volterra difference equations of convolution type.

Definition 4.2: [18] Volterra difference equation of convolution type of the first kind is of the form

$$x(n) = \sum_{m=0}^{n} K(n-m)y(m) \tag{4.3}$$

where $n = 0, 1, 2, \ldots$ $K(n)$ and $y(n)$ are sequences and $K(n)$ is called the kernel.

For solving Eq(4.3), take the Z-transform for both sides of Eq(4.3) we get,

$$X(z) = K(z)Y(z)$$

So

$$Y(z) = \frac{X(z)}{K(z)}$$

Thus

$$y(n) = Z^{-1}\left[\frac{X(z)}{K(z)}\right]$$

Definition 4.3: [18] Volterra difference equation of convolution type of the second kind is of the form

$$y(n) = f(n) + \sum_{m=0}^{n} K(n-m)y(m) \tag{4.4}$$

where $n = 0, 1, 2, \ldots.$ $f(n), K(n), y(n)$ are sequences and $K(n)$ is called the kernel.

For solving Eq(4.4), take the Z-transform for both sides of Eq(4.4) we get,

$$Y(z) = F(z) + K(z)Y(z)$$

So

$$Y(z) = \frac{F(z)}{1 - K(z)}$$

Thus

$$y(n) = Z^{-1}\left[\frac{F(z)}{1 - K(z)}\right]$$

Example 4.3: Solve Volterra difference equation

$$n^2 = \sum_{m=0}^{n} (n-m)u(n-m)y(m)$$

Solution:

We can write the above equation as:

$$n^2 = nu(n) * y(n)$$

Take the Z-transform for both sides we get,

$$Z[n^2] = Z[nu(n) * y(n)\ ]$$
$$\frac{z(z+1)}{(z-1)^3} = \frac{z}{(z-1)^2}Y(z)$$

Simplify, we get

$$\begin{aligned} Y(z) &= \frac{z+1}{z-1} \\ &= \frac{z}{z-1} + \frac{1}{z-1} \\ &= \frac{z}{z-1} + z^{-1}\frac{z}{z-1} \end{aligned}$$

Taking the inverse Z-transform we get

$$y(n) = u(n) + u(n-1) = 2u(n) - \delta(n)$$

Chapter Five

Z-transform and Digital Signal Processing

In this chapter we study some applications of Z-transform in digital signal processing such as analysis of linear shift invariant systems, realization of finite-duration impulse response (**FIR**) and infinite-duration impulse response (**IIR**) systems and design of IIR filters from analog filters.

Section 5.1: Introduction

In this section we introduce some necessary definitions and theorems in digital signal processing.

A signal is any physical quantity that varies with time, space or any independent variable or variables. Mathematically, we describe a signal as a function of one or more independent variables. In this chapter, time will be the independent variable whether it's continuous or discrete. If it's continuous then the signal is called **continuous-time signal** and it's represented by a function -of continuous variable $f(t)$. But, if the time is in discrete form the signal is called **discrete-time signal** and is represented by a sequence of numbers $\{a_n\}$ where $\in \mathbb{Z}$ [9,14].

Also, the range of the signal (amplitude) can be continuous or discrete.

Analog time signals are signals with continuous time and amplitude, where **digital signals** are signals with discrete time and amplitude and they are represented by a function of an integer independent variable n and its values are taken from a finite set [9,14].

Definition 5.1:[14] A system is a physical device that performs an operation on a signal. These operations are usually referred to as signal processing. Continuous-time systems are systems for which the input and output are continuous-time signals, where the discrete-time systems are systems with discrete-time input and output signals, and its represented by the notation $T[.]$ which transforms the input signal $x(n)$ into the output signal $y(n)$. We write

$$T[x(n)] = y(n)$$

Note that any sequence $x(n)$ can be expressed as a sum of scaled and delayed unit samples. i.e

$$x(n) = \cdots + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + \cdots \quad (5.1)$$

Eq(5.1) can be written as

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \tag{5.2}$$

In discrete-time system, the output of $x(n)$ is

$$T[x(n)] = y(n) = T\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] \tag{5.3}$$

Definition 5.2:[6] A system is said to be linear if

$$T[ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]$$

for any two inputs $x_1(n)$ and $x_2(n)$ and for any complex constants a and b. If the system is linear, then Eq(5.3) become

$$y(n) = \sum_{k=-\infty}^{\infty} T\,[x(k)\delta(n-k)] \tag{5.4}$$

But $x(k)$ is constant with respect to n, so

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)T[\delta(n-k)] \tag{5.5}$$

Let $h_k(n)$ be the response of the system to at unit sample at $n = k$, then Eq(5.5) become

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h_k(n) \tag{5.6}$$

Definition 5.3:[6] Let $y(n)$ be the response of a system to an arbitrary input $x(n)$. The system is said to be shift-invariant if, for any delay n_0, the response to $x(n - n_0)$ is $y(n - n_0)$. Otherwise, the system is shift-varying.

Definition 5.4: [6] A system is called a linear shift-invariant (**LSI**) system if it's linear and shift-invariant.

If we have LSI system Eq(5.6) become

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{5.7}$$

$$= x(n) * h(n) \tag{5.8}$$

where $h(n)$ is the unit sample response (response of $\delta(n)$).

An **LSI** system are classified as finite-duration impulse response (FIR) and infinite-duration impulse response (IIR) according to whether $h(n)$ has finite or infinite duration, respectively.

Definition 5.5: [11]: A system is said to be causal if the output of the system at n_o depends on the input at $n \le n_o$.

Theorem 5.1:[14] A LSI system is causal if and only if the unit sample response $h(n)$ equals zero for $n < 0$.

Proof: see [14]

Definition 5.6:[11] A system is said to be bounded input–bounded output (BIBO) stable if the response of any bounded input remains bounded.

Theorem 5.2:[9] A LSI system is (BIBO) stable if and only if the unit sample response $h(n)$ is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Proof: see [9]

Definition 5.7:[6]: A system whose output $y(n)$ depends on any number of past outputs is called recursive. In contrast, if the output of the system depends only on the present and past inputs is called non recursive.

Section 5.2: Analysis of Linear Shift-Invariant (LSI) Systems and Z-transform.

If we have an **LSI** system with input $x(n)$ and output $y(n)$, then as in Eq(5.8)

$$y(n) = x(n) * h(n)$$

where $h(n)$ is the sample response.

Taking Z-transform for both sides of Eq(5.8) we get,

$$Y(z) = X(z)H(z) \tag{5.9}$$

where $H(z)$ is the Z-transform of $h(n)$ and is called the transfer function.

Theorem 5.3:[9] The LSI system is causal if and only if the ROC of transfer function is the exterior of a circle centered at the origin.

Proof:

The LSI system is causal if and only if the unit sample response is causal if and only if the ROC of the Z-transform of the sample response (transfer function) is the exterior of a circle centered at the origin.

Theorem 5.4:[14] The LSI system is BIBO stable if and only if the ROC of transfer function contains the unit circle.

Proof: **a)**

Suppose the **LSI** system is BIBO stable

The transfer function of LSI system is

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

Then

$$|H(z)| = \left| \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} |h(n) z^{-n}| = \sum_{n=-\infty}^{\infty} |h(n)|\, |z^{-n}|$$

If $|z| = 1$, then

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)|$$

But the LSI system is BIBO stable so from Theorem (5.2)

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Therefore

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

So $h(n)z^{-n}$ is summable at each value of z of magnitude 1, so

$$\{|z| = 1\} \subseteq ROC \text{ of } H(z)$$

b) Suppose $\{|z| = 1\} \subseteq\ ROC$ of $H(z)$

Since $\{|z| = 1\} \subseteq ROC$, then $h(n)z^{-n}$ is summable for each value of z of magnitude 1, which gives

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

So, from Theorem (5.3) the system is BIBO stable.

There are many methods for analyzing the behavior or response of linear systems. One of these methods is to solve the input-output equation of the system which is for an LSI system with input $x(n)$ and output $y(n)$ is a linear difference equation with constant-coefficient of the form

$$\sum_{k=0}^{N} a(k)y(n-k) = \sum_{r=o}^{M} b(r)x(n-r) \qquad (5.10)$$

where M, N are integers and N is the order of the difference equation.

Example 5.1: An LSI system is represented by the linear difference equation

$$y(n) - y(n-1) + \frac{1}{4}y(n-2) = x(n) - \frac{1}{4}x(n-1)$$

a) Find the unit sample response of the system.

b) Is the system stable?

c) Find the response of the input signal $x(n) = \left(\frac{1}{4}\right)^n u(n)$.

Solution:

a) We take the Z-transform of the linear difference equation and get

$$Y(z) - z^{-1}Y(z) + \frac{1}{4}z^{-2}Y(z) = X(z) - \frac{1}{4}z^{-1}X(z)$$

$$Y(z)\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right) = X(z)\left(1 - \frac{1}{4}z^{-1}\right)$$

So the transfer function is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{4}z^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}} = \frac{z\left(z - \frac{1}{4}\right)}{z^2 - z + \frac{1}{4}}$$

$$\therefore \frac{H(z)}{z} = \frac{z - \frac{1}{4}}{z^2 - z + \frac{1}{4}} = \frac{z - \frac{1}{4}}{\left(z - \frac{1}{2}\right)^2} = \frac{A}{z - \frac{1}{2}} + \frac{B}{\left(z - \frac{1}{2}\right)^2}$$

$$A\left(z - \frac{1}{2}\right) + B = z - \frac{1}{4} \qquad (5.11)$$

If $z = \frac{1}{2}$ then, $B = \frac{1}{4}$

Differentiate both sides of Eq(5.11) with respect to z we get $A = 1$, then the transfer function can be written as,

$$H(z) = \frac{z}{z - \frac{1}{2}} + \frac{1}{4} \frac{z}{\left(z - \frac{1}{2}\right)^2}$$

Since the linear difference equation is causal, the ROC of $H(z)$ is $|z| > \frac{1}{2}$, so the inverse of the transfer function (sample response of the system) is,

$$h(n) = \left(\frac{1}{2}\right)^n u(n) + \frac{1}{2} n \left(\frac{1}{2}\right)^n u(n)$$
$$= \left[1 + \frac{n}{2}\right] \left(\frac{1}{2}\right)^n u(n)$$

b) Since the ROC of the transfer function contains the unit circle the system is stable.

c) To find the response $y(n)$ of the input signal $x(n)$we first find the Z-transform of $y(n)$

$$Y(z) = X(z)H(z) = \frac{z}{z - \frac{1}{4}} \frac{z\left(z - \frac{1}{4}\right)}{z^2 - z + \frac{1}{4}}$$
$$= \frac{z^2}{z^2 - z + \frac{1}{4}} = \frac{z^2}{\left(z - \frac{1}{2}\right)^2}$$
$$= \frac{z}{z - \frac{1}{2}} + \frac{1}{2} \frac{z}{\left(z - \frac{1}{2}\right)^2}$$

The ROC of $H(z)$ can be either $|z| > \frac{1}{2}$ or $|z| < \frac{1}{2}$ but since the ROC of $X(z)$ is $|z| > \frac{1}{4}$, and by convolution property of Z-transform the ROC of $Y(z)$ is at least the intersection of ROC of $X(z)$ and ROC of $H(z)$, the ROC of $H(z)$ is $|z| > \frac{1}{2}$ so the response of $y(n)$ is,

$$y(n) = \left(\frac{1}{2}\right)^n u(n) + n \left(\frac{1}{2}\right)^n u(n)$$

$$= [1+n]\left(\frac{1}{2}\right)^n u(n).$$

Consider Eq(5.10) again with $a(0) = 1$ we get,

$$y(n) = -\sum_{k=1}^{N} a(k)y(n-k) + \sum_{r=o}^{M} b(r)x(n-r) \qquad (5.12)$$

To find the transfer function for the **LSI** system we take Z-transform of both sides of Eq(5.12), we get

$$Y(z) + \sum_{k=1}^{N} a(k)z^{-k}Y(z) = \sum_{r=o}^{M} b(r)z^{-r}X(z) \qquad (5.13)$$

So the transfer function is of the form

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{r=o}^{M} b(r)z^{-r}}{1+\sum_{k=1}^{N} a(k)z^{-k}} \qquad (5.14)$$

So, an **LSI** system represented by a linear difference equation with constant coefficient has a rational transfer function as in Eq(5.14). There are many cases for the values of $b(r)$ and $a(k)$.

Case 1: $a(k) = 0$ for $1 \le k \le N$ then, Eq(5.14) becomes,

$$H(z) = \sum_{r=o}^{M} b(r)z^{-r} = \frac{1}{z^M}\sum_{r=o}^{M} b(r)z^{M-r} \qquad (5.15)$$

From Eq(5.15) there is a multiple-pole of order M at $z = 0$ and M zeros for $H(z)$. If the transfer function in Eq(5.15) contains M nontrivial zeros then the system is called all-zero system, and has finite-duration impulse response (FIR).

Case 2: if $b(r) = 0$ for $1 \le r \le M$ then, Eq(5.14) become,

$$H(z) = \frac{b(0)}{1+\sum_{k=1}^{N} a(k)z^{-k}} \qquad (5.16)$$

$$= \frac{b(0)z^N}{\sum_{k=0}^{N} a(k)z^{N-k}}, \quad a(0) = 1 \qquad (5.17)$$

In Eq(5.17) $H(z)$ has N poles and a multiple-zero of order N at $z = 0$. If the transfer function in Eq(5.17) has N nontrivial poles then the system is called all-pole system. Due to the presence of poles, the impulse response of such systems is infinite in duration, and hence it is an infinite-duration impulse response (IIR). But the general case of $H(z)$ is to have both zeros and poles, such system is called a pole-zero system and due to the presence of poles it is an IIR system.

Schür-Cohn Stability Test [14].

Schür-Cohn stability test is a test used to determine if the causal linear shift invariant system is stable or not depends on the theorem which say a linear shift invariant system is causal and stable if and only if all its poles lies inside the unit circle.

Before set Schür-Cohn stability test we need to define some important notations.

We know that the transfer function of LSI system is rational, i.e.

$$H(z) = \frac{A(z)}{B(z)}$$

Where

$$B(z) = 1 + b(1)z^{-1} + b(2)z^{-2} + \cdots + b(N)z^{-N} = \sum_{k=0}^{N} b(k)z^{-k}$$

Let $B_m(z)$ be a polynomial in z^{-1} of degree m of the form,

$$B_m(z) = \sum_{k=0}^{m} b_m(k)z^{-k}, b_m(0) = 1$$

where $m = 0, 1, \dots, N$

And $C_m(z)$ is the reverse polynomial in z^{-1} of degree m defined as,

$$C_m(z) = z^{-m} B_m(z^{-1}) = z^{-m} \sum_{k=0}^{m} b_m(k) z^k$$

$$= \sum_{h=0}^{m} b_m(m-h) z^{-h}, h = m - k$$

We define the reflection coefficients K_1, K_2, …, K_N taken from $B_m(z)$. First note that

$$B(z) = B_N(z)$$

Let

$$K_N = b_N(N)$$

For computing a lower degree of $B_m(z)$, $m = N, N-1, N-2, \dots, 1$ we use the recursive equation,

$$B_{m-1}(z) = \frac{B_m(z) - K_m C_m(z)}{1 - {K_m}^2} \tag{5.18}$$

where $K_m$is defined as,

$$K_m = b_m(m)$$

From Schür-Cohn stability test the causal LSI system will be stable if and only if $|K_m| < 1$, for all $m = 1, 2, \dots, N$.

Example 5.2: A LSI system has the transfer function

$$H(z) = \frac{-z}{12z^2 - 7z + 3}$$

a) Find the difference equation that represented the system.

b) Is the system stable?

Solution:

a) We write $H(z)$ in term of z^{-1}

$$H(z) = \frac{-z^{-1}}{12 - 7z^{-1} + 3z^{-2}}$$

So

$$\frac{Y(z)}{X(z)} = \frac{-z^{-1}}{12 - 7z^{-1} + 3z^{-2}} \tag{5.19}$$

By cross multiplication of Eq(5.19) we obtain,

$$12Y(z) - 7z^{-1}Y(z) + 3z^{-2}Y(z) = -z^{-1}X(z) \tag{5.20}$$

We take the inverse Z-transform for both sides of Eq(5.20) and do some operations we get the difference equation which represented the system,

$$y(n) = \frac{7}{12}y(n-1) - \frac{3}{12}y(n-2) - \frac{1}{12}x(n-1)$$

b) Since the system is causal LSI system we can use Schür-Cohn stability test to determine the stability of the system.

First we write $H(z)$ as,

$$H(z) = \frac{-\frac{1}{12}z^{-1}}{1 - \frac{7}{12}z^{-1} + \frac{3}{12}z^{-2}}$$

where

$$B_2(z) = 1 - \frac{7}{12}z^{-1} + \frac{3}{12}z^{-2}$$

Hence

$$K_2 = b_2(2) = \frac{3}{12}$$

$$|K_2| = \left|\frac{3}{12}\right| = \frac{3}{12} < 1$$

Now

$$C_2(z) = z^{-2}B_2(z^{-1}) = \frac{3}{12} - \frac{7}{12}z^{-1} + z^{-2}$$

And

$$B_1(z) = \frac{B_2(z) - K_2C_2(z)}{1 - {K_2}^2} = 1 - \frac{63}{135}z^{-1}$$

So

$$K_1 = b_1(1) = \frac{-63}{135},$$

$$|K_1| = \left|\frac{-36}{135}\right| = \frac{63}{135} < 1$$

Since $|K_m| < 1$, for $m = 1, 2$, the system is stable.

Example 5.3: Is the LSI system represented by the following linear difference equation stable?

$$y(n) = 2.5y(n-1) - y(n-2) + x(n) - 5x(n-1) - 6x(n-2)$$

Solution:

Since the system is a causal **LSI** system we can use Schür-Cohn stability test. First, we take Z-transform for both sides of the linear difference equation we get,

$$Y(z) = 2.5z^{-1}Y(z) - z^{-2}Y(z) + X(z) - 5z^{-1}X(z) - 6z^{-2}X(z)$$

Then, the transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 5z^{-1} - 6z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$$

So

$$B_2(z) = 1 - 2.5z^{-1} + z^{-2}$$

From $B_2(z)$ we get $K_2 = 1$

Since $|K_2| = 1 \nless 1$, then by Schür-Cohn stability test the system is unstable.

Section 5.3: Realization of FIR Systems.

There are several methods for implementing an FIR system. In this section we will present some of these methods like direct-form, cascade-form, and lattice realization for an FIR system.

In general an FIR system is described by the linear difference equation

$$y(n) = \sum_{k=0}^{M-1} b(k)x(n-k) \tag{5.21}$$

or by the transfer function

$$H(z) = \sum_{k=0}^{M-1} b(k)z^{-k} \tag{5.22}$$

where the unit sample response of FIR system is,

$$h(n) = \begin{cases} b(n), & 0 \le n \le M-1 \\ 0 \;, & otherwise \end{cases}$$

1. Direct-Form Realization.

Direct-form realization is the most common way to implement the FIR systems. It follows directly from the non recursive difference equation given by Eq(5.21) or equivalently by,

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \tag{5.23}$$

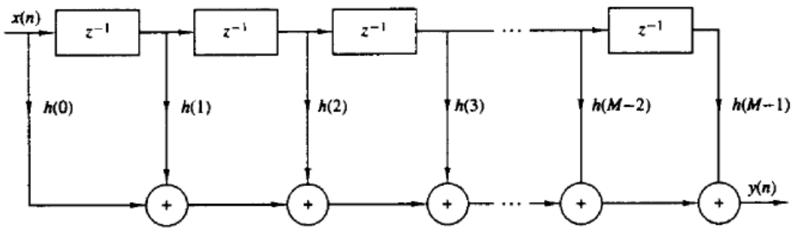

Figure (**5.1**)**:** Direct-form realization of FIR systems. [14]

2. Cascade-Form Realization.

Cascade form is alternative to the direct form by factoring the transfer function into second-order FIR systems as,

$$H(z) = G\prod_{k=1}^{K}(1 + b_k(1)z^{-1} + b_k(2)z^{-2}) \tag{5.24}$$

where K is the integer part of $(M+1)/2$ and G is called the gain parameter.

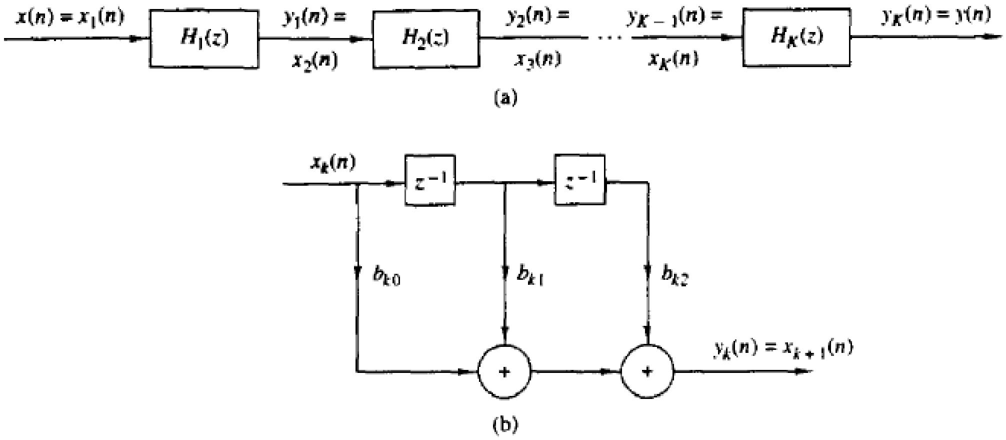

Figure (5.2)**:** Cascade-form realization of FIR systems. [14]

3. Lattice Realization.

Let us start by defining a sequence of FIR filters with transfer function

$$H_m(z) = A_m(z) \tag{5.25}$$

where $A_m(z)$ is a polynomial of degree m defined by,

$$A_m(z) = 1 + \sum_{k=1}^{m} a_m(k) z^{-k}\,, \;\; m \geq 1 \tag{5.26}$$

and $A_0(z) = 1$.

So, the unit sample response $h_m(k)$ is

$$h_m(k) = \begin{cases} 1 \qquad\;\;, & k = 0 \\ a_m(k), & 1 \leq k \leq m \end{cases}$$

Lattice realization or FIR lattice filter is a cascade of two-port networks each one of them has two inputs $f_{m-1}(n)$ and $g_{m-1}(n)$ related to the two outputs $f_m(n)$, $g_m(n)$ by the two difference equations give us a relation between m-order direct-form FIR filter and m-stage lattice filter,

$$f_m(n) = f_{m-1}(n) + K_m g_{m-1}(n-1), \;\; m = 1, 2, \ldots, M-1 \tag{5.27}$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1), \;\; m = 1, 2, \ldots, M-1 \tag{5.28}$$

For $m = 0$

$$g_0(n) = f_0(n) = x(n)$$

And the output of $(M-1)$-stage is

$$y(n) = f_{M-1}(n)$$

Where K_m is the reflection coefficient which is the same reflection coefficient of Schür-Cohn stability test.

For each $m = 0, 1, \dots, M-1$, we will define the transfer function as, the transfer function in Eq(5.25), the relation between the input $x(n)$ and the intermediate output $f_m(n)$ is given by,

$$f_m(n) = \sum_{k=0}^{m} a_m(k) x(n-k) \tag{5.29}$$

where $a_m(0) = 1$.

By taking the Z-transform of both sides of Eq(5.29) we get,

$$F_m(z) = A_m(z) X(z) \tag{5.30}$$

From Eq(5.27) and Eq(5.28) we can see that for each $m = 1, 2, \dots, M-1$ the coefficients of $g_m(n)$ are the same as those of $f_m(n)$ but in reverse order, i.e

$$\begin{aligned} g_m(n) &= \sum_{k=0}^{m} a_m(m-k) x(n-k) \\ &= \sum_{k=0}^{m} b_m(k) x(n-k),\ b_m(k) = a_m(m-k) \end{aligned}$$

By taking the Z-transform the previous equation we get,

$$G_m(z) = B_m(z) X(z)$$

where

$$\begin{aligned} B_m(z) = \sum_{k=0}^{m} b_m(k) z^{-k} &= \sum_{k=0}^{m} a_m(m-k) z^{-k} \\ &= \sum_{l=0}^{m} a_m(l) z^{l-m}, l = m-k \end{aligned}$$

$$= \sum_{k=0}^{m} a_m(l) z^{l-m} = z^{-m} \sum_{k=0}^{m} a_m(l) z^{l}$$

$$= z^{-m} A_m(z^{-1}) \qquad (5.31)$$

So

$$G_m(z) = z^{-m} A_m(z^{-1}) X(z) \qquad (5.31)$$

Taking the Z-transform for both sides of Eq(5.27) and Eq(5.28) and divide them with $X(z)$ then solve the result equations with Eq(5.30) and Eq(5.31) to get the recurrence relation

$$A_m(z) = A_{m-1}(z) + K_m z^{-m} A_{m-1}(z^{-1}), \quad m = 1,2,\ldots,M-1 \qquad (5.32)$$

and $A_0(z) = 1$.

This recurrence relation called step-up recurrence and it used to converse the lattice reflection coefficient to direct-form filter coefficient.

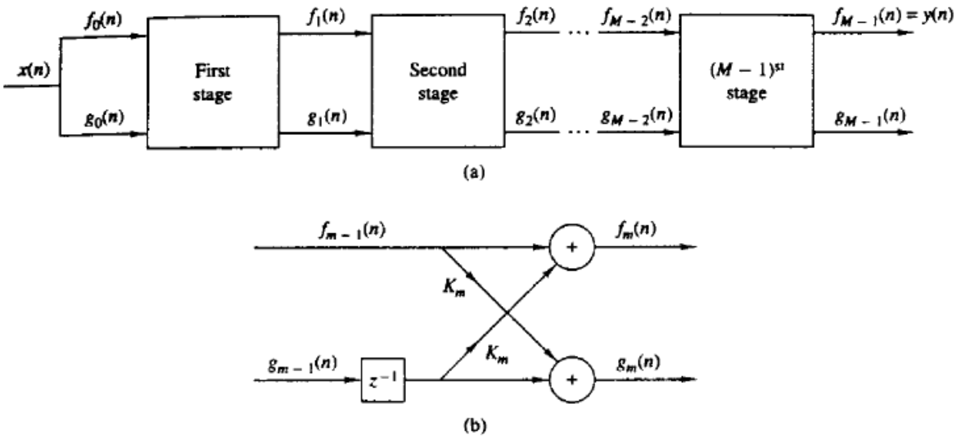

Figure (5.3): An $(M-1)$-stage lattice filter with typical stage [14]

Example 5.4: Given a three-stage FIR lattice filter with reflection coefficients $K_1 = \frac{1}{2}, K_2 = \frac{-1}{3}, K_3 = \frac{1}{4}$, determine the FIR filter coefficients for the direct-form structure.

Solution:

First,

$$A_0(z) = A_0(z^{-1}) = 1$$

From the step-up recurrence in Eq(5.32) with $m = 1$ we get the single-stage lattice

$$\begin{aligned} A_1(z) &= A_0(z) + K_1 z^{-1} A_0(z^{-1}) \\ &= 1 + \frac{1}{2} z^{-1} \end{aligned}$$

So, the coefficients of an FIR filter corresponding to single-stage lattice are $a_1(0) = 1, a_1(1) = \frac{1}{2}$.

Next, for $m = 2$ we get the second-stage of lattice filter,

$$\begin{aligned} A_2(z) &= A_1(z) + K_2 z^{-2} A_1(z^{-1}) \\ &= 1 + \frac{1}{2} z^{-1} + \frac{-1}{3} \; z^{-2} \left(1 + \frac{1}{2} z\right) \\ &= 1 + \frac{1}{3} z^{-1} - \frac{1}{3} z^{-2} \end{aligned}$$

Hence, the coefficients of an FIR filter corresponding to second-stage lattice are $a_2(0) = 1, a_2(1) = \frac{1}{3}. \, a_2(2) = \frac{-1}{3}$.

Finally, for $m = 3$ we get the result of the third-stage of lattice filter,

$$\begin{aligned} A_3(z) &= A_2(z) + K_3 z^{-3} A_2(z^{-1}) \\ &= 1 + \frac{1}{3} z^{-1} - \frac{1}{3} z^{-2} + \frac{1}{4} \; z^{-3} \left(1 + \frac{1}{3} z - \frac{1}{3} z^2\right) \\ &= 1 + \frac{1}{4} z^{-1} - \frac{1}{4} z^{-2} + \frac{1}{4} z^{-3} \end{aligned}$$

Hence, direct-form of an FIR filter is characterized the coefficients,
$a_3(0) = 1, a_3(1) = \dfrac{1}{4}, a_3(2) = \dfrac{-1}{4}, a_3(3) = \dfrac{1}{4}$.

To find the lattice reflection coefficients from direct-form FIR filter coefficients we use the step-down recurrence relation determined from step-up recurrence in Eq(5.32),

$$A_m(z) = A_{m-1}(z) + K_m z^{-m} A_{m-1}(z^{-1}), \quad m = 1,2,\dots,M-1$$

If we substitute z^{-1} instead of z in Eq(5.32) we get,

$$A_m(z^{-1}) = A_{m-1}(z^{-1}) + K_m z^{m} A_{m-1}(z), \quad m = 1,2,\dots,M-1 \qquad (5.33)$$

If we solve Eq(5.33) for $A_{m-1}(z^{-1})$ we obtain,

$$A_{m-1}(z^{-1}) = A_m(z^{-1}) - K_m z^{m} A_{m-1}(z), \quad m = 1,2,\dots,M-1 \qquad (5.34)$$

Substitute Eq(5.34) in Eq(5.32),

$$A_m(z) = A_{m-1}(z) + K_m z^{-m}\left(A_m(z^{-1}) - K_m z^{m} A_{m-1}(z)\right)$$

$$m = 1,2,\dots,M-1 \qquad (5.35)$$

If we solve Eq(5.35) for $A_{m-1}(z)$ we get,

$$A_{m-1}(z) = \frac{1}{1-K_m{}^2}\left(A_m(z) - K_m z^{-m} A_m(z^{-1})\right) \qquad (5.36)$$

where $m = M-1, M-2, \dots, 1$

Observe that the step-down recurrence works if $|K_m| \neq 1, \quad m = 1,2,\dots,M-1$

Example 5.5: Determine the lattice coefficients corresponding to the third-order FIR filter with transfer function.

$$H(z) = A_3(z) = 1 + \frac{1}{4}z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{4}z^{-3}$$

Solution:

$$K_3 = a_3(3) = \frac{1}{4}$$

From the step-down relation Eq(5.36) with $m = 3$ we get,

$$A_2(z) = \frac{1}{1-K_3{}^2}\left(A_3(z) - K_3 z^{-3} A_3(z^{-1})\right)$$

$$= \frac{1}{1-\left(\frac{1}{4}\right)^2}\left[1 + \frac{1}{4}(z^{-1} - z^{-2} + z^{-3}) - \frac{1}{4}z^{-3}\left(1 + \frac{1}{4}(z - z^2 + z^3)\right)\right]$$

$$= \frac{16}{15}\left(\frac{15}{16} + \frac{5}{16}z^{-1} - \frac{5}{16}z^{-2}\right)$$

$$= 1 + \frac{1}{3}z^{-1} - \frac{1}{3}z^{-2}$$

So,

$$K_2 = a_2(2) = \frac{-1}{3}$$

Now, by repeating the step-down relation in Eq(5.36) with $m = 1$ we get,

$$A_1(z) = \frac{1}{1 - {K_2}^2}\left(A_2(z) - K_2 z^{-2} A_2(z^{-1})\right)$$

$$= \frac{1}{1 - \left(\frac{-1}{3}\right)^2}\left[1 + \frac{1}{3}z^{-1} - \frac{1}{3}z^{-2} - \frac{-1}{3}z^{-2}\left(1 + \frac{1}{3}z - \frac{1}{3}z^2\right)\right]$$

$$= \frac{9}{8}\left(\frac{8}{9} + \frac{4}{9}z^{-1}\right)$$

$$= 1 + \frac{1}{2}z^{-1}$$

Hence,

$$K_1 = a_1(1) = \frac{1}{2}$$

Section 5.4: Realization of IIR Systems.

As in FIR systems, IIR systems have many types of realization, including direct form, cascade form, parallel form and lattice form.

IIR systems are described by the difference equation in Eq(5.12) or by the transfer function in Eq(5.14) for $N \neq 0$.

1. Direct-Form Realization.

In direct-form realization of IIR systems the transfer function in Eq(5.14) can be written as a cascade of two transfer functions, as

$$H(z) = H_1(z)H_2(z) \tag{5.37}$$

where

$$H_1(z) = \sum_{r=o}^{M} b(r)z^{-r} \tag{5.38}$$

and

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a(k) z^{-k}} \tag{5.39}$$

$H_1(z)$ consists of the zeros of $H(z)$ where $H_2(z)$ consists of the poles of $H(z)$. There are two direct forms to realize IIR systems depending whether $H_1(z)$ precedes $H_2(z)$ or vice versa. If all-zero filter $H_1(z)$ placed before all-pole filter $H_2(z)$, then we get direct-form I realization which need $M + N + 1$ multiplications, $M + N$ additions and $M + N$ delays and is depicted in Fig(5.4). But if the first filter is $H_2(z)$, then we obtain direct-form II and the system becomes a cascade of the recursive system

$$w(n) = -\sum_{k=1}^{N} a(k) w(n-k) + x(n) \tag{5.40}$$

with input $x(n)$ and output $w(n)$, followed by the non recursive system

$$y(n) = -\sum_{k=1}^{M} b(k) w(n-k) \tag{5.41}$$

which has $w(n)$ as input and $y(n)$ output.

Direct-form II require $M + N + 1$ multiplications, $M + N$ additions but $max\{M, N\}$ delays and its depicted in Fig(5.5). Since $H_2(z)$ need less number of delays to realize $H(z)$ it's called canonic.

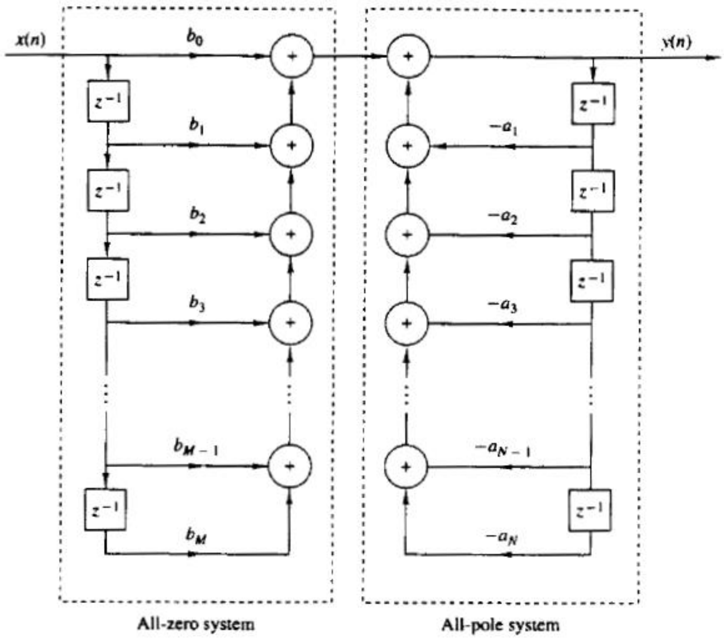

Figure $(\mathbf{5.4})$**:** Direct-form I realization of IIR systems [14].

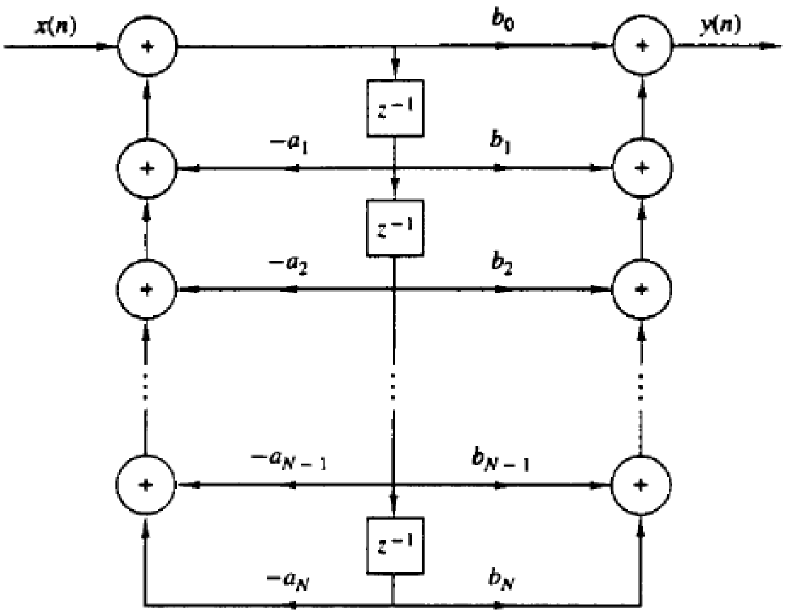

Figure $(\mathbf{5.5})$**:** Direct-form II realization of IIR systems for $M = N$ [14].

2. Cascade-Form Realization.

Cascade-form realization is derived by factoring the transfer function in Eq(5.14) into a cascade of second-order sections. We start by considering $N \geq M$ then $H(z)$ factored as

$$H(z) = G \prod_{k=1}^{K} H_k(z) \tag{5.42}$$

where is K the integer part of $(N+1)/2$ and $H_k(z)$ has the general form

$$H_k(z) = \frac{1 + b_k(1)z^{-1} + b_k(2)z^{-2}}{1 + a_k(1)z^{-1} + a_k(2)z^{-2}} \tag{5.43}$$

Where G is the gain parameter and equal $b(0)$, the coefficients $a_k(i)$, $b_k(i)$ are real coefficients for $i = 1,2$ and $k = 1, \ldots, K$.The chosen of the numerator and denominator of $H_k(z)$ are arbitrary. If $N < M$, then some coefficients of the denominator of the second-order sections will equal zero.

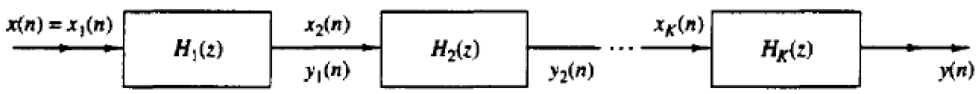

Figure (5.6)**:** Cascade-form realization of IIR system

3. Parallel-Form Realization.

Parallel-form realization can be obtained by expanding the transfer function $H(z)$ in Eq(5.14) using partial fraction method. Without loss of generality, we assume that the poles of $H(z)$ are distinct. If $N > M$, then $H(z)$ expressed as

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \tag{5.44}$$

where A_k's are the coefficients of $H(z)$ and p_k's are the poles.

In general, the poles of $H(z)$ are complex so there corresponding coefficients are complex. To avoid multiplication of complex numbers we combine the complex-conjugate poles to have second-order section $H_k(z)$ of the form

$$H_k(z) = \frac{b_k(0) + b_k(1)z^{-1}}{1 + a_k(1)z^{-1} + a_k(2)z^{-2}} \tag{5.45}$$

where the coefficients are real, then $H(z)$ become

$$H(z) = \sum_{k=1}^{K} H_k(z) \tag{5.46}$$

where K is the integer part of $(N+1)/2$.

If $N \leq M$, then the partial fraction expansion will contain a term of the form

$$C = c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)}$$

which is FIR filter placed in parallel with the other terms of expansion of $H(z)$.

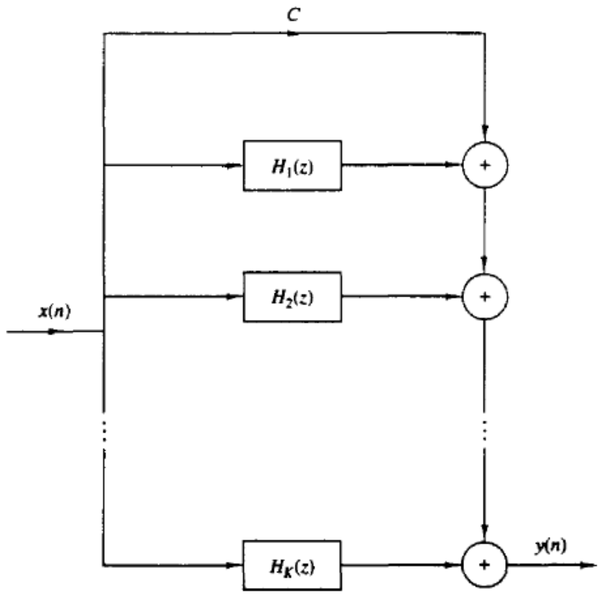

Figure (5.7): Parallel -form realization of IIR systems. [14]

Example 5.6: Determine the transfer function for cascade and parallel realizations for the system described by the transfer function

$$H(z)=\frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)}{\left(1-\left(\frac{1}{2}+\frac{1}{2}j\right)z^{-1}\right)\left(1-\left(\frac{1}{2}-\frac{1}{2}j\right)z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

Solution:

Since $N=3>M=2$ then $K=$ integer part of $(3+1)/2=2$

For cascade realization,

$$H(z)=\prod_{k=1}^{2}H_k(z)$$

A possible paring for poles and zeros is

$$H_1(z)=\frac{1-\frac{1}{2}z^{-1}}{\left(1-\left(\frac{1}{2}+\frac{1}{2}j\right)z^{-1}\right)\left(1-\left(\frac{1}{2}-\frac{1}{2}j\right)z^{-1}\right)}$$
$$=\frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$
$$H_2(z)=\frac{1+\frac{3}{4}z^{-1}}{1-\frac{1}{4}z^{-1}}$$

So

$$H(z)=\frac{1+\frac{3}{4}z^{-1}}{1-\frac{1}{4}z^{-1}}\,\frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

For parallel-form realization we need to expand $H(z)$ using partial fraction method,

$$H(z)=\sum_{k=1}^{3}\frac{A_k}{1-p_kz^{-1}}$$
$$=\frac{A_1}{1-\frac{1}{4}z^{-1}}+\frac{A_2}{1-\left(\frac{1}{2}+\frac{1}{2}j\right)z^{-1}}+\frac{A_3}{1-\left(\frac{1}{2}-\frac{1}{2}j\right)z^{-1}}$$

Where the coefficients A_1, A_2, A_3 are

$$A_1 = H(z)\left(1-\frac{1}{4}z^{-1}\right)\Big|_{z^{-1}=4} = \frac{-4}{5}$$

$$A_2 = H(z)\left[1-\left(\frac{1}{2}+\frac{1}{2}j\right)z^{-1}\right]\Big|_{z^{-1}=1-j} = \frac{9-8j}{10}$$

$$A_3 = H(z)\left[1-\left(\frac{1}{2}-\frac{1}{2}j\right)z^{-1}\right]\Big|_{z^{-1}=1+j} = \frac{9+8j}{10}$$

So, the transfer function become

$$H(z) = \frac{-4}{5}\frac{1}{1-\frac{1}{4}z^{-1}} + \frac{1}{10}\frac{9-8j}{1-\left(\frac{1}{2}+\frac{1}{2}j\right)z^{-1}} + \frac{1}{10}\frac{9+8j}{1-\left(\frac{1}{2}-\frac{1}{2}j\right)z^{-1}}$$

or

$$H(z) = \frac{-4}{5}\frac{1}{1-\frac{1}{4}z^{-1}} + \frac{1}{10}\frac{18-z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

4. Lattice-Form Realization.

In this section we will develop a lattice filter structure equivalent to IIR systems, and our notations is the same as in the Lattice realization of FIR filters.

Let us start with all pole system which has the transfer function

$$H(z) = \frac{1}{1+\sum_{k=1}^{N} a_N(k)z^{-k}} = \frac{1}{A_N(z)} \qquad (5.47)$$

The direct form realization of the previous system represented by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_N(k)y(n-k) + x(n) \qquad (5.48)$$

If we interchange the input signal $x(n)$ with the output signal $y(n)$ and the output $y(n)$ with the input $x(n)$ we get

$$x(n) = -\sum_{k=1}^{N} a_N(k)x(n-k) + y(n) \qquad (5.49)$$

Which can be written as

$$y(n) = x(n) + \sum_{k=1}^{N} a_N(k)x(n-k) \tag{5.50}$$

Which is the linear difference equation describe an FIR system having the transfer function $H(z) = A_N(z)$.

So, we can obtain the IIR lattice filter from an FIR by interchanging the input and the output. Therefore, the definition of IIR lattice filter is the opposite of FIR lattice filter.

By solving Eq(5.27) for $f_{m-1}(n)$ and let Eq(5.28) stay the same for $g_m(n)$ we get the following rules for IIR lattice filter

$$f_{m-1}(n) = f_m(n) - K_m g_{m-1}(n-1), \;\; m = N, N-1, \dots, 1 \tag{5.51}$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1), \;\; m = N, N-1, \dots, 1 \tag{5.52}$$

with

$$g_0(n) = f_0(n) = y(n)$$

$$f_N(n) = x(n)$$

In general, the transfer function for all-pole IIR system is

$$H_a(z) = \frac{Y(z)}{X(z)} = \frac{F_0(z)}{F_m(z)} = \frac{1}{A_m(z)} \tag{5.53}$$

And for all-zero FIR system is

$$H_b(z) = \frac{G_m(z)}{Y(z)} = \frac{G_m(z)}{G_0(z)} = \;\; z^{-m}A_m(z^{-1}) \tag{5.54}$$

So, for the IIR system contains both poles and zeros has the transfer function

$$H(z) = \frac{\sum_{k=1}^{M} c_M(k)z^{-k}}{1 + \sum_{k=1}^{N} a_N(k)z^{-k}} = \frac{C_M(z)}{A_N(z)} \tag{5.55}$$

Without loss of generality, if $M \leq N$, then $H(z)$ consists of two components the first is an all-pole lattice with parameters K_m, $1 \leq m \leq N$, and the other is a tapped delay line with coefficients $c_M(k)$.

If $M = N$ then

$$C_m(z) = C_{m-1}(z) + c_m(m)z^{-m}A_m(z^{-1}), m = 1,2,\dots,M \qquad (5.56)$$

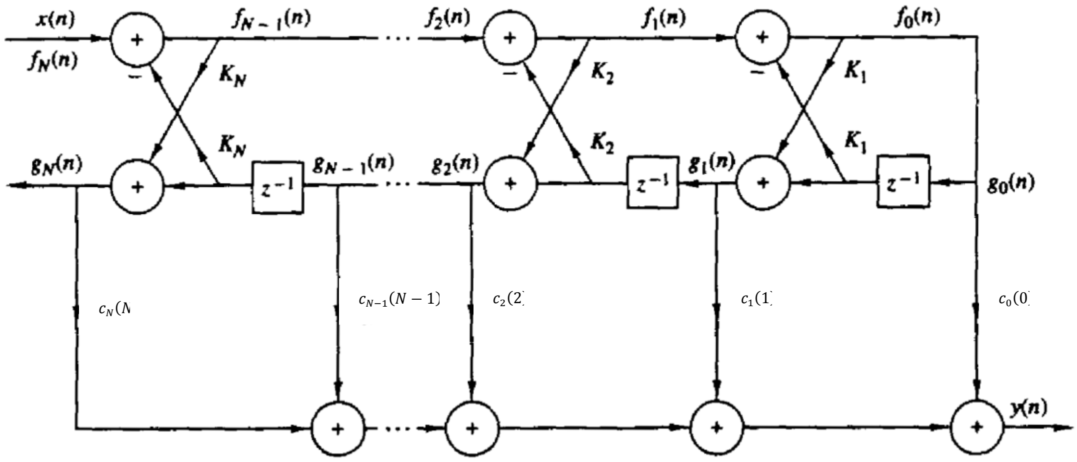

Figure (5.8)**:** Lattice-form realization of IIR systems

Section 5.5: Design of IIR Filters From Analog Filters.

Before talking about designing IIR filter from analog filter we need to sample the analog signal $x_a(t)$ by using uniform sample described by the relation

$$x(n) = x_a(nT) \qquad (5.57)$$

where T is the sample period and $x(n)$ is the discrete signal obtained by sampling the analog signal $x_a(t)$.

Note: The subscript a will denote analog signal.

An analog filter may be described by the transfer function

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} b(k)s^k}{\sum_{k=0}^{N} a(k)s^k} \qquad (5.58)$$

where $b(k)$'s, $a(k)$'s are the filter coefficients.

Or by its impulse response related to $H_a(s)$ by Laplace transform

$$H_a(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt \qquad (5.59)$$

Also, such filters can be described by the differential equation

$$\sum_{k=0}^{N} a(k)\frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b(k)\frac{d^k x(t)}{dt^k} \qquad (5.60)$$

where $x(t)$ is the input signal and $y(t)$ is the output signal.

If all poles of the transfer function are in the left half of s-plane then the system is stable. For effective conversion from analog signal to digital signal the mapping which we use should map the points on the $j\Omega$-axis in the s-plane into the unit circle in the z-plane and the points in the right half of the s-plane into outside the unit circle in the z-plane where the points in the left half mapped inside the unit circle.

In this section we will design IIR filter from analog filter by using two methods impulse invariant and bilinear transform.

1. Impulse Invariant Method.

In impulse invariant method the digital IIR filter is designed by sampling the impulse response of the analog filter.

$$h(n) = h_a(nT) \qquad (5.61)$$

where T is the sampling interval.

From sampling theorem, if we have a continuous time signal $x_a(t)$ with spectrum $X_a(F)$ sampled at a rate $F_s = 1/T$ sample per second, the spectrum of the sampled signal $X(f)$ is the periodic repetition of the scaled spectrum $F_s X_a(F)$ with period F_s. Specially the relation is

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a[(f-k)F_s] \qquad (5.62)$$

where $f = F/F_s$ is the normalized frequency.

If we apply Eq(5.62) for impulse response of an analog filter with frequency response $H_a(F)$, then the unit sample response $h(n) = h_a(nT)$ for digital filter has the frequency response.

$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a[(f-k)F_s] \tag{5.63}$$

or, equivalent to

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a[(\omega - 2\pi k)F_s] \tag{5.64}$$

or

$$H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right) \tag{5.65}$$

To obtain the mapping between the z-plane and the s-plane implied by sampling process, with a generalization of Eq(5.65) which relates the Z-transform of $h(n)$ to Laplace transform of $h_a(nT)$ by the relation

$$H(\mathrm{z})\Big|_{z\,=\,e^{sT}} = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\mathrm{s} - j\frac{2\pi k}{T}\right) \tag{5.66}$$

where

$$H(\mathrm{z}) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

So

$$H(\mathrm{z})\Big|_{z\,=\,e^{sT}} = \sum_{n=0}^{\infty} h(n) e^{-sT} \tag{5.67}$$

The general characteristic of our mapping is

$$z = e^{sT} \tag{5.68}$$

where

$$s = \sigma + \Omega j \tag{5.69}$$

Substitute Eq(5.69) in Eq(5.68), then express the result equation in polar form we get

$$re^{j\omega} = e^{\sigma T} e^{\Omega \mathrm{T} j}$$

Clearly

$$r = e^{\sigma T}$$

$$\omega = \Omega \mathrm{T}$$

To see how poles and zeros of the analog filter mapped using impulse invariant method we express the transfer function of the analog filter in partial fraction form. With assumption that the poles of the transfer function of the analog filter are distinct we get

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \tag{5.70}$$

where N is a positive integer, $p_k, k = 1, 2, \ldots, N$ are the poles of the transfer function of the analog filter and $c_k, k = 1, 2, \ldots, N$ are the coefficients of the partial fraction expansion. So, the unit sample response is

$$h_a(t) = \sum_{k=1}^{N} c_k e^{p_k \mathrm{t}}, t \geq 0 \tag{5.71}$$

If we sampled $h_a(t)$ periodically at $t = nT$ we have

$$h(n) = h_a(nT) = \sum_{k=1}^{N} c_k e^{p_k \mathrm{Tn}} \tag{5.72}$$

Now, substitute the unit sample response in Eq(5.72) in the transfer function of digital IIR filter we get

$$H(\mathrm{z}) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} c_k e^{p_k \mathrm{Tn}} \right) z^{-n}$$

$$= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} [e^{p_k \mathrm{T}} z^{-1}]^n \qquad (5.73)$$

The inner summation of Eq(5.73) is converges because $p_k < 0$, so

$$H(\mathrm{z}) = \sum_{k=1}^{N} c_k \frac{1}{1 - e^{p_k \mathrm{T}} z^{-1}} \qquad (5.74)$$

Therefore, the transfer function of the digital IIR filter has poles at

$$z = e^{p_k \mathrm{T}}, k = 1, 2, \dots, N \qquad (5.75)$$

If some poles are complex we can combined them to form two-pole filter section.

The impulse invariant method is appropriate only for low pass filters and a limited class of band pass filters.

Example 5.7: Convert the analog filter with transfer function

$$H_a(s) = \frac{s+1}{(s+1)^2 + 9}$$

into a digital IIR filter by impulse invariant method.

Solution:

$H_a(s)$ has two poles at $p = -1 + 3j, -1 - 3j$, so the transfer function can be written as

$$H_a(s) = \frac{s+1}{[s - (-1+3j)][s - (-1-3j)]}$$

By partial fraction expansion

$$H_a(s) = \frac{1}{2} \frac{1}{[s - (-1+3j)]} + \frac{1}{2} \frac{1}{[s - (-1-3j)]}$$

Then

$$H(z) = \frac{1}{2}\frac{1}{[1 - e^{(-1+3j)T}z^{-1}]} + \frac{1}{2}\frac{1}{[1 - e^{(-1-3j)T}z^{-1}]}$$
$$= \frac{1 - e^{-T}\cos 3T\ z^{-1}}{1 - 2e^{-T}\cos 3T\ z^{-1} + e^{-2T}z^{-2}}$$

2. Bilinear Transformation Method.

The bilinear transform overcome the limitations of impulse invariant. The bilinear transformation transform the $j\Omega$-axis in the s-plane into the unit circle in the z-plane only once. The bilinear transformation linked to the trapezoidal formula for numerical integration. For example, consider an analog filter has the transfer function.

$$H(s) = \frac{b}{s+a} \tag{5.76}$$

The transfer function in Eq(5.76) can be represented by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \tag{5.77}$$

Integrate the derivative in Eq(5.77)

$$y(t) = \int_{t_0}^{t} y'(\tau)d\tau + y(t_0) \tag{5.78}$$

where $y'(t)$ is the derivative of y(t).

By using the trapezoidal formula at $t = nT$ and $t_0 = nT - T$ to approximate the integral in Eq(5.78), we get

$$y(nT) = \frac{T}{2}[y'(nT) + y'(nT - T)] + y(nT - T) \tag{5.79}$$

The derivatives of $y(t)$ at $t = nT$ and $t = nT - T$ according to Eq(5.77) are

$$y'(nT) = -ay(nT) + bx(nT) \tag{5.80}$$

$$y'(nT - T) = -ay(nT - T) + bx(nT - T) \tag{5.81}$$

Substitute Eq(5.80) and Eq(5.81) in Eq(5.79), we obtain the linear difference equation. With $y(n) = y(nT)$ and $x(n) = x(nT)$ we get

$$\left(1+\frac{aT}{2}\right)y(n)-\left(1-\frac{aT}{2}\right)y(n-1)=\frac{bT}{2}[x(n)+x(n-1)] \quad (5.82)$$

Take the Z-transform of the difference equation in Eq(5.82), we get

$$\left(1+\frac{aT}{2}\right)Y(z)-\left(1-\frac{aT}{2}\right)z^{-1}Y(z)=\frac{bT}{2}(1+z^{-1})X(z) \quad (5.83)$$

So, the transfer function is

$$H(z)=\frac{Y(z)}{X(z)}=\frac{(bT\,/\,2)(1+z^{-1})}{1+aT/2-\left(1-\frac{aT}{2}\right)z^{-1}} \quad (5.84)$$

or, equivalent to

$$H(z)=\frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a} \quad (5.85)$$

Therefore, the mapping from the s-plane to the z-plane is

$$s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad (5.86)$$

Which called the bilinear transformation.

Our previous work was for first-order differential equation. In general for Nth order differential equation let.

$$z = re^{j\omega}$$

$$s = \sigma + \Omega j$$

Then Eq(5.86) can be expressed as

$$\begin{aligned} s &= \frac{2}{T}\frac{z-1}{z+1} \\ &= \frac{2}{T}\frac{re^{j\omega}-1}{re^{j\omega}+1} \\ &= \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}+j\frac{2r\ \sin\omega}{1+r^2+2r\cos\omega}\right] \quad (5.87) \end{aligned}$$

So

$$\sigma = \frac{2}{T}\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} \tag{5.88}$$

$$\Omega = \frac{2}{T}\frac{2r\ \sin\omega}{1 + r^2 + 2r\cos\omega} \tag{5.89}$$

Note, if $r < 1$, then $\sigma < 0$ and if $r > 1$, then $\sigma > 0$. But, if $r = 1$ then $\sigma = 0$ and

$$\begin{aligned}\Omega &= \frac{2}{T}\frac{\sin\omega}{1 + \cos\omega} \\ &= \frac{2}{T}\tan\frac{\omega}{2}\end{aligned} \tag{5.90}$$

or

$$\omega = 2\tan^{-1}\frac{\Omega T}{2} \tag{5.91}$$

Chapter Six

The Chirp Z-transform Algorithm and Its Applications

The chirp Z-transform algorithm is a computational algorithm for numerical evaluation of Z-transform for a sequence of N points. By using it we can easily find Z-transform of M points in the z-plane lying on circular or spiral contour beginning at any an arbitrary point in the z-plane. The algorithm based on the fact that the value of Z-transform on a circular or spiral contour can be expressed as a discrete convolution [16].

We will restrict our work to the Z-transform of sequences of a finite number N of non zero points. Therefore Z-transform of these sequences can be written without loss of generality as,

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} \qquad (6.1)$$

Note that $X(z)$ converges for all z except $z = 0$.

If we have a train of impulses with equally space T of magnitude $x(n)$, it will be of the form

$$\sum_{n} x(n)\, \delta(t - nT)$$

Its Laplace transform will be

$$\sum_{n} x(n)\, e^{-snT}$$

which is the same of Eq(6.1) if we set

$$z = e^{sT} \qquad (6.2)$$

Since we can compute Z-transform for a finite set of samples, we can evaluate it for a finite number of points $z_k$where $0 \leq k \leq M - 1$ in z-plane, so Z-transform of $x(n)$ evaluated at z_k will be written as,

$$X(k) = X(z_k) = \sum_{n=0}^{N-1} x(n)[z_k]^{-n} \tag{6.3}$$

We are interested in the set of points equally spaced around the unit circle, these points are of the form

$$z_k = exp\left(j\frac{2\pi}{N}k\right), \quad \mathrm{k} = 0, 1, \dots, \mathrm{N} - 1 \tag{6.4}$$

If we substitute Eq(6.4) in Eq (6.3) we get

$$X(k) = \sum_{n=0}^{N-1} x(n)exp\left(-j\frac{2\pi}{N}nk\right), \quad k = 0, 1, \dots, N - 1 \tag{6.5}$$

Eq(6.5) is the discrete Fourier transform with $\left(\omega = \frac{2\pi k}{N}\right)$ which has many applications but finding it requires N^2additions and multiplications so we use fast Fourier transform to compute it which requires $N \log_2 N$ operation if N is a power of 2 and $N \sum_i m_i$operation if N is not a power of 2 where m_i is the prime factored of N.

Because fast Fourier transform has many limitations we use chirp Z-transform which eliminates some of these limitations, for example the number of samples of the sequence $x(n)$ need not to be equal to the number of samples in the z-plane. Neither N nor M need be composite integers, they can be primes. The angular spacing of z_k is arbitrary and the contour need not to be a circle, it can be a spiral in or out with respect to the origin [9,17].

Let's compute Z-transform on the more general contour of the form

$z_k = AW^{-k}, \quad k = 0, 1, \dots, M - 1$ (6.6)

where M is arbitrary integer and A, W are arbitrary complex numbers of the form

$$A = A_0 exp(j2\pi\theta_0)$$

and

$$W = W_0 exp(j2\pi\varphi_0)$$

The case where $A = 1$, $M = N$ and $W = exp(-j2\pi/N)$ correspond to the discrete Fourier transform.

The equivalent s-plane contour of Eq(6.6) begins with the point

$$s_0 = \sigma_0 + jw_0 = \frac{1}{T}\ln A \qquad (6.7)$$

And the form of general contour on the s-plane is

$$\begin{aligned} s_k &= s_0 + k(\Delta\sigma + j\Delta w) \\ &= \frac{1}{T}(\ln A - k\ \ln W), \quad k = 0, 1, \dots, M-1 \end{aligned} \qquad (6.8)$$

Note that the z-plane contour maps into arbitrary finite length straight line in the s-plane.

To compute Z-transform a long the contour described by Eq(6.6) we need MN multiplications and additions.

Now we will derive the chirp Z-transform algorithm [16,19].

If we find the Z-transform for the point along the contour represented in Eq(6.6) we get,

$$X(k) = \sum_{n=0}^{N-1} x(n)A^{-n}W^{nk}, k = 0, 1, \dots, M-1 \qquad (6.9)$$

If we use the Bluestein's ingenious substitution

$$nk = \frac{n^2 + k^2 - (k-n)^2}{2} \qquad (6.10)$$

in Eq(6.9) we get

$$X(k) = \sum_{n=0}^{N-1} x(n)A^{-n}W^{(n^2/2)}W^{(k^2/2)}W^{-(k-n)^2/2}$$

$$k = 0, 1, \dots, M-1 \qquad (6.11)$$

Eq(6.11) can be divided into three steps,

1. Forming a sequence $y(n)$ of the form

$$y(n) = x(n)A^{-n}W^{(n^2/2)}, \quad n = 0, 1, \dots, N-1 \tag{6.12}$$

2. Define the sequence $v(n)$ as

$$v(n) = W^{-n^2/2} \tag{6.13}$$

To get the sequence $g(k)$

$$g(k) = \sum_{n=0}^{N-1} y(n)v(k-n), \quad k = 0, 1, \dots, M-1 \tag{6.14}$$

which is a convolution of $y(n)$ and $v(n)$ so it can be expressed as

$$g(n) = y(n) * v(n) \tag{6.15}$$

3. Multiply $g(k)$ with $W^{(k^2/2)}$ to give $X(k)$

$$X(k) = g(k)W^{(k^2/2)}, \quad k = 0, 1, \dots, M-1 \tag{6.16}$$

The steps 1 and 3 need N, M multiplications respectively where step 2 which is the main step in computation needs the most time.

In summary the chirp Z-transform algorithm consist of the following steps

1. Choose L to be the smallest integer greater than or equal to $N + M - 1$ to be compatible with discrete Fourier transform which mean for most users that L is a power of 2.
2. Form the sequence y(n) of L points by the equation

$$y(n) = \begin{cases} x(n)A^{-n}W^{(n^2/2)} & , n = 0, 1, \dots, N-1 \\ 0 & , n = N, N+1, \dots, L-1 \end{cases} \tag{6.17}$$

3. Use the fast Fourier transform to compute the discrete Fourier transform of $y(n)$ and call them $Y(r)$, $r = 0, 1, \dots, L-1$.
4. Define the sequence $v(n)$ as

$$v(n) = \begin{cases} W^{-n^2/2} & ,0 \le n \le M-1 \\ 0 & ,M-1 < n < L-N+1 \;\; if\; L > N+M-1 \\ W^{-(L-n)^2/2} & ,L-N+1 \le n < L \end{cases} \qquad (6.18)$$

Note that if $L = N + M - 1$, then there is no value of $v(n)$ equal zero.

5. Compute the L point discrete Fourier transform of $v(n)$ and call them $V(r)$, $r = 0, 1, \dots , L-1$.
6. Multiply $Y(r)$ and $V(r)$ to get $G(r)$

$$G(r) = Y(r)V(r) \;\; , r = 0, 1, \dots , L-1$$

7. Compute the L point of $g(r)$ by taking the inverse discrete Fourier transform of $G(r)$.
8. Multiply $g(k)$ and $W^{(k^2/2)}$ to get $X(k)$

$$X(k) = g(k)W^{(k^2/2)}, \qquad k = 0, 1, \dots, M-1$$

where $g(k)$ for $k \ge M$ are discarded.

The computational time for the chirp Z-transform is proportional to $L \log_2 L$. Therefore the direct method for computing $X(k)$ is most efficient for small values of N, M where chirp Z-transform will be efficient for large values or even for relatively modest values of M and N of the order of 50 [9,16].

Chirp Z-transform algorithm has many applications as**:**

1. Enhancement of Poles.

Evaluating Z-transform at points outside and inside the unit circle is one of the advantages of Z-transform over fast Fourier transform, this advantage help us to find the poles and zeros for systems whose transfer function is of polynomial form by making the contour which we use closer to the poles and zeros.

2. High Resolution, Narrow Band Frequency Analysis.

The ability to evaluate high resolution, narrow band frequency is an important application of chirp Z-transform algorithm because it allows us to chose the initial frequency and frequency space independent of the number of time sample. Where by using the fast Fourier transform, if the frequency resolution $\leq \Delta F$ and sampling frequency $1/T$, then we require

$N \geq 1/(T\Delta F)$ samples which will be very large for small values of ΔF.

Conclusion

For finding the inverse of Z-transform three methods are used: integration method is useful for finding a few values of $x(n)$, where the power series method is efficient when we find the inverse of finite-order integer power function with partial fraction method is suitable for finding the inverse of rational Z-transform. Z-transform is a transform for discrete data equivalent to Laplace transform for continuous data and it's a generalization of discrete Fourier transform. It's an efficient method for solving linear difference equations with constant coefficients and Volterra difference equations of convolution type. Also, it has many important applications in digital signal processing as analysis of linear shift-invariant systems, implementation of FIR and IIR filters and design of IIR filters from analog filters. Chirp Z-transform algorithm is an important algorithm that overcomes the limits of fast Fourier transform and has many applications such as enhancement of poles and high resolution, narrow band frequency analysis.

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Appendix A

Some Maple Commands on Z-transform.

Example 2.3:

$$ztrans(a^n, n, z)$$

Example 2.4:

$$sum(-(b * z^{-1})^n, n = -\infty..-1)$$

Example 2.14:

$$ztrans(7charfcn[0](n) + 3charfcn[1](n) + 0charfcn[2](n)$$

$$+1charfcn[3](n) + 2charfcn[4](n) + 6charfcn[5](n), n, z)$$

Example 3.6: (a)

$$invztrans\left(\frac{z^2 + 3z}{z^2 - 3z + 2}, z, n\right)$$

Example 3.8:

$$invztrans\left(\frac{z^3}{z^3 - z^2 - 5z - 3}, z, n\right)$$

Example 4.1:

$$rsolve(\{y(0) = 0, y(1) = 1, y(n + 2) = y(n + 1) + y(n)\}, y)$$

Example 4.2:

$$rsolve\left(\left\{y(-1) = 2, y(n) = \frac{1}{3}y(n - 1) + 1\right\}, y\right)$$

Appendix B

Example on using Chirp Z-transform Algorithm.

Example: Use chirp Z-transform algorithm to find the Z-transform of the sequence

$$x(n) = \begin{cases} 1\,, & n = 0 \\ 2\,, & n = 1 \\ 0\,, & otherwise \end{cases}$$

at the given points

$$z_k = AW^{-k}, \quad k = 0, 1, 2$$

where

$$A = 1 \text{ and } W = \frac{1}{2} \cdot e^{\frac{-\pi j}{2}}$$

Solution:

1. For choosing L

$$N + M - 1 = 2 + 3 - 1 = 4$$

So $L = 4$ because 4 is a power of 2.

2.

$$y(n) = \begin{cases} x(n)1^{-n} \left[\frac{1}{2} \cdot e^{\frac{-\pi j}{2}}\right]^{(n^2/2)} & , n = 0, 1 \\ 0 & , n = 2, 3 \end{cases}$$

$$= \begin{cases} 1 & , n = 0 \\ 1 - j & , n = 1 \\ 0 & , n = 2, 3 \end{cases}$$

3.

$$Y(r) = \sum_{n=0}^{L-1} y(n) e^{\frac{-2\pi jnr}{L}}, \quad r = 0, 1, \ldots, L - 1$$

$$Y(r) = \sum_{n=0}^{3} y(n) e^{\frac{-\pi jnr}{2}}, \quad r = 0, 1, 2, 3$$

$$=\begin{cases} 2-j & ,r=0 \\ -j & ,r=1 \\ j & ,r=2 \\ 2+j & ,r=\ 3 \end{cases}$$

4.

$$v(n)=\begin{cases} \left[\frac{1}{2}\cdot e^{\frac{-\pi j}{2}}\right]^{-n^2/2} & ,0\le n\le 2 \\ \left[\frac{1}{2}\cdot e^{\frac{-\pi j}{2}}\right]^{-(4-n)^2/2} & ,3\le n<4 \end{cases}$$

$$=\begin{cases} 1 & ,n=0 \\ 1+j & ,n=1 \\ -4 & ,n=2 \\ 1+j & ,n=\ 3 \end{cases}$$

5.

$$V(r)=\sum_{n=0}^{3} v(n)e^{\frac{-\pi jnr}{2}},\quad r=0,1,2{,}3$$

$$=\begin{cases} -1+2j & ,r=0 \\ 5 & ,r=1 \\ -5-2j & ,r=2 \\ 5 & ,r=\ 3 \end{cases}$$

6.

$$G(r)=Y(r)V(r)\ \ ,r=0,1,2\ ,3$$

$$=\begin{cases} 5j & ,r=0 \\ -5j & ,r=1 \\ 2-5j & ,r=2 \\ 10+5j & ,r=\ 3 \end{cases}$$

7.

$$g(r)=\frac{1}{L}\sum_{n=0}^{L-1} G(n)e^{\frac{2\pi jnr}{L}},\quad r=0,1,\dots,L-1$$

$$g(r)=\frac{1}{4}\sum_{n=0}^{3} G(n)e^{\frac{\pi jnr}{2}},\quad r=0,1,2{,}3$$

$$= \begin{cases} 3 & , r = 0 \\ 2 & , r = 1 \\ -2 & , r = 2 \\ -3 + 5j & , r = 3 \end{cases}$$

8.

$$X(k) = g(k)\left[\frac{1}{2} \cdot e^{\frac{-\pi j}{2}}\right]^{(k^2/2)}, \quad k = 0, 1, 2$$

$$= \begin{cases} 3 & , k = 0 \\ 1 - j & , k = 1 \\ \frac{1}{2} & , k = 2 \end{cases}$$

Appendix C

A Table of Properties of Z-transform

Property	**Sequence**	**Z-transform**	***ROC***
Linearity	$\alpha x(n)\ +\ \beta y(n)$	$\alpha\, X(z)\ +\ \beta\, Y(z)$	$r < \lvert z\rvert < R$
Shifting	$x(n+k)$	$z^k\, X(z)$	$r_x < \lvert z\rvert < R_x$
Multiplication by Exponential	$\alpha^n x(n)$	$X(\alpha^{-1}z)$	$\lvert\alpha\rvert r_x < \lvert z\rvert < \lvert\alpha\rvert R_x$
Time Reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{R_x} < \lvert z\rvert < \frac{1}{r_x}$
Conjugation	$x^*(n)$	$X^*(z^*)$	$r_x < \lvert z\rvert < R_x$
Multiplication by n	$n\, x(n)$	$-z\frac{dX(z)}{dz}$	$r_x < \lvert z\rvert < R_x$
Convolution of Two Sequences	$x(n) * y(n)$	$X(z)Y(z)$	at least, ROC of $X(z) \cap ROC$ of $Y(z)$
Correlation of Two Sequences	$r_{xy}(l)$	$X(z)Y(z^{-1})$	at least, ROC of $X(z) \cap ROC$ of $Y(z^{-1})$
Multiplication of Two Sequences	$x(n)y(n)$	$\frac{1}{2\pi j}\oint_C X(v)Y\left(\frac{z}{v}\right)\frac{1}{v}$	$r_x r_y < \lvert z\rvert < R_x R_y$

Note: $X(z)$ is the Z-transform of $x(n)$ with $ROC\ r_x < |z| < R_x$ and $Y(z)$ is the Z-transform of $y(n)$ with $C\ \ r_y < |z| < R_y$, $r = \max(r_x, r_y)$ and $R = \min(R_x, R_y)$.

Appendix D

A Table of Common Z-transform Pair.

Sequence	**Z-transform**	***ROC***
$\delta(n)$	1	all z
$u(n)$	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
$n\,u(n)$	$\dfrac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$n^2\,u(n)$	$\dfrac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
$\alpha^n\,u(n)$	$\dfrac{z}{z-\alpha}$	$\lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n\,u(n)$	$\dfrac{\alpha z}{(z-\alpha)^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-a^n\,u(-n-1)$	$\dfrac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
$-n\,a^n\,u(-n-1)$	$\dfrac{\alpha z}{(z-\alpha)^2}$	$\lvert z\rvert < \lvert a\rvert$
$cos(n\alpha)u(n)$	$\dfrac{z^2-zcos(\alpha)}{z^2-2zcos(\alpha)+1}$	$\lvert z\rvert > 1$
$sin(n\alpha)u(n)$	$\dfrac{zsin(\alpha)}{z^2-2zcos(w)+1}$	$\lvert z\rvert > 1$

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2015

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الملخص

تم في هذه الأطروحة دراسة تحويل زد (ثنائي الجانب) وتحويل زد احادي الجانب بالإضافة لتحويل زد ثنائي الأبعاد مع خصائصهم وطريقة ايجاد المعكوس لهم وتم حل امثلة على كل مما سبق. ثم تم تقديم العلاقة بين تحويل زد وتحويلي لابلاس و فورييه.

تم استخدام تحويل زد في حل معادلات فرق خطية معاملاتها ثوابت وأيضا حل معادلتي فروقات فولتيرا من صنف الالتفاف من الدرجتين الاولى والثانية.

ثم تم مناقشة بعض التطبيقات لتحويل زد في مجال معالجة الإشارات الرقمية مثل: تحليل النظم السببية ذات الزمن الثابت وتحليل المرشح الرقمي ذو الاستجابة المنتهية والمرشح الرقمي ذو الاستجابة غير المنتهية وتصميم مرشح رقمي باستجابة غير منتهية من مرشح متصل.