An-Najah National University

Faculty of Graduate Studies

NORMAL COMPOSITION OPERATORS ON SUB-HARDY SPACES

By

Doua Mohammed Sarsour

Supervisor

Dr. Muath Karaki

This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics, Faculty of Graduate Studies, An-Najah National University, Nablus - Palestine.

2024

NORMAL COMPOSITION OPERATORS ON SUB-HARDY SPACES

By

Doua Mohammed Sarsour

This Thesis was Defended Successfully on 31/03/2024 and approved by

Dr. Muath Karaki Supervisor	Signature
Dr. Rania Wannan External Examiner	Signature
Dr. Mohammed Yasin Internal Examiner	Signature

Dedication

I’m dedicating this thesis to my beloved family, first of all to my father, whose wisdom and guidance shaped me into who I am today, and to my dear mother, who had endless faith in me and never failed to support me. To my dearest sisters and brothers, and a special thank you to my brother Dr. Moslem, who helped me in checking the language of the thesis.

Acknowledgements

All my success is not but through Allah. Upon him, I have to be grateful, and praise to Allah. I want to thank Dr. Muath Karaki, my supervisor professor, for his guidance on this work. He was always there to support me, and it was his drive that enabled me to finish this thesis with success, to him I am grateful. I express my gratitude to all the faculty members in the Mathematics Department of An-Najah National University they gave me the essential tools, and without them, I could not have completed this task successfully.

Declaration

I, the undersigned, declare that I submitted the thesis entitled:

NORMAL COMPOSITION OPERATORS ON SUB-HARDY SPACES

I declare that the work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's Name: **Doua Mohammed Sarsour**

Signature: Doua Mohammed Sarsour

Date: **31/03/202**

List of Contents

NORMAL COMPOSITION OPERATOR ON SUB-HARDY SPACES

By

Doua Mohammed Sadiq Sarsour

Supervisor

Dr.Muath Karaki

Abstract

Let Γ denote the open unit disk, $v(z) : \Gamma \to \Gamma$ be an analytic function, define the composition operator C_v such that $C_v f(z) = (f \circ v)(z)$. Let $\omega(z)$ be an anaytic function on the unit disk, define the weighted composition operator $W_{\omega,v} f(z) = (\omega f \circ v)(z)$ they both operators are bounded on $\mathscr{H}^2$. This thesis studies the normality of the composition operators C_v and the weighted composition operators $W_{\omega,v}$ on $\mathscr{H}^2$.

Also, if $b(z) = \lambda z^{m_0} \prod_{i=1}^{n} \left(\frac{\gamma_i - z}{1 - \overline{\gamma_i} z} \right)^{m_i}$ is the finite Blasckhe product we study the composition operator $\mathscr{C}_v f(z)$ of the model sub-space $K_b = \left\{ b\overline{zh} \ : \ h \in \mathscr{H}^2 \right\} = \mathscr{H}^2 \cap b\overline{z\mathscr{H}^2}$ and the adjoint of the composition operator $\mathscr{C}_v^* f(z)$ on K_b. Then we check the normality of $\mathscr{C}_v f(z)$, and $\mathscr{W}_{\omega,v} f(z)$ on the model space K_b.

Keywords: normal composition operators, model spaces, composition operators, Hardy spaces, operatyor theory, function spaces, spectral theory.

Chapter One

Introduction and Preliminaries

1.1 Introduction

The study of the composition operator is a link between analytic function theory and operator theory. Operator theory including the study of the composition operator of a fixed a nalytic f unction d ates b ack t o a s tudy b y E . N ordgren i n 1 960 [1], a nd o ver the years the literature has grown [2]. Two basic monographs are [Cowen books, and Shapiro books].

The study of Toeplitz, subnormal, and other operators that grew out of the theory of multiplication operators, is founded on the spectral theorem for normal operators [2]. Since the normal composition operators are extremely diverse and naturally appear in a wide range of issues, the normality of the composition operator C_v on the subspace K_b is the primary objective of this thesis.

Let $v(z) : \Gamma \to \Gamma$ be an analytic function, the composition operator is $C_v f = f \circ v$. Based on Littlewood's Theorem, the composition operator is bounded on the Hardy-Hilbert space $\mathscr{H}^2$ and $f \circ v \in \mathscr{H}^2$ for all $f \in \mathscr{H}^2$. Schwartz in [3] proved that C_v is normal if $v(z) = az, \ |a| \leq 1$, and vice versa. While Bourdon and Narayan in [4] provided an explanation of all normal weighted composition operators on $\mathscr{H}^2(\Gamma)$ with $v(z)$ fixes a point in Γ, where the weighted composition operator is $W_{\omega,v} = \omega(C_v f)$, with ω be a holomorphic function on Γ.

K_u the model subspace of $\mathscr{H}^2$ is defined to be the orthogonal complement of $u\mathscr{H}^2$ where

$u(z)$ is an inner function of the open unit disk, i.e. $K_u = \mathscr{H}^2 \ominus u\mathscr{H}^2$. With the finite Blascke product $b(z)$ as an inner function, Mashreghi and Shabankhah in [5] provided a detailed description of the bounded composition operators on K_b, showing that if $\mathcal{L}_c(K_b)$ is the classification of all bounded composition operators on K_b, the structure of $\mathcal{L}_c(K_b)$ is tightly bound to the zeros distribution of $b(z)$. In this thesis, firstly we will recall all the background needed to study the normal composition operators on $\mathscr{H}^2$.

Secondly, we studied the normal composition operators and the weighted composition operators on $\mathscr{H}^2$. Finally, we will give a brief introduction to the model subspace K_b, then we will take the case that $b(z)$ has two zeros including the origin and $b'(0) \neq 0$ in Mashreghi and Shabankhah paper to study the structure of $\mathcal{L}_c(K_b)$. Also, we will compute C_v^* on K_b with $b(z)$, and we will compute and check the normality of the composition operator $C_v\colon K_b \to K_b$.

1.2 Preliminaries

The fundamental background theory of normal composition operators on the Hardy-Hilbert space $\mathscr{H}^2$ and the normal composition operators on the model space K_u will covered in this Section. And will go over some fundamentals to define L^p and $\mathscr{H}^p$ spaces as well as normal operators. Also we will define needed operators, such as the orthogonal projection operator, the Toeplitz operator, ... etc.

1.2.1 Banach and Hilbert Spaces

This sub-section will present definitions of spaces needed before defining the Banach and Hilbert spaces. We will see that any Hilbert space is possible to write as a direct sum of a closed subset with its orthogonal complement.

One of the measures we studied in calculus is the distance function (metric function) that is defined on $\mathbb{R}$ as $d(t, s) = |t - s|$ for every pair of real points [6]. In functional analysis, there is a similar definition that is basic and useful in many applications, which is as follows:

Definition 1.2.1 [6] For a set U and a metric $d : U \times U \rightarrow \mathbb{R}$, a Pair (U, d) is called metric space if and only if for all $s, t, r \in U$

(i) d is finite real number, and nonnegative,

(ii) $d(s, t) = 0 \iff s = t$,

(iii) $d(s, t) = d(t, s)$,

(vi) $d(s, t) \leq d(s, r) + d(r, t)$.

If the elements of the set U can be added and multiplied by a constant, and the result is still an element of U, then the set is said to be a vector space [6]. Furthermore, if A is a subset of U such that $\overline{A} = U$, then we said that A is dense in U [7].

Definition 1.2.2 [8] The space V is a normed vector space if it is a vector space where a norm is defined. That is a real-valued function denoted as $\|.\|$ called a norm, and satisfying the following for all $s, t \in V$,

(i) $\|s\| \geq 0$,

$(ii)\|s\| = 0 \iff s = 0$,

(iii) $\|s + t\| \leq \|s\| + \|t\|$,

(iv) If α is a scalar, then $\|\alpha s\| = |\alpha|\|s\|$.

Definition 1.2.3 [8] A complex vector space with an inner product is called an inner product space. Where a real-valued function $\langle ., . \rangle$ define on a complex vector space V is called an inner product if and only if for all $s, t, r \in V$

(i) $\langle s, s\rangle \geq 0$,

(ii) $\langle s, s\rangle = 0 \iff s = 0$,

(iii) $\langle s, t\rangle = \overline{\langle t, s\rangle}$,

(iv) For any scalar α, $\langle \alpha s, t\rangle = \alpha\langle s, t\rangle$,

(v) $\langle s + t, r\rangle = \langle s, r\rangle + \langle t, r\rangle$.

Note that an inner product on a vector space V defines a norm on V given by $\|s\| = \langle s,\, s\rangle^{\frac{1}{2}}$. And satisfy Schwarz inequality $|\langle s,\, t\rangle| \leq \|s\|\|t\|$. If s and t were elements of an inner vector space V, we say that s and t are orthogonal if $\langle s,\, t\rangle = 0$ written as $s \perp t$. Furthermore, if U and W were subsets of V as every element in U is orthogonal to every element in W then we say that U is orthogonal to W $(U \perp W)$ [8].

Before defining the Hilbert space, we need the definition of convergent and Cauchy sequences to define the complete and Banach spaces.

Recall that a sequence s_n is convergent to s if $\forall\, \varepsilon > 0$, $\exists\, k \in \mathbb{Z}_+$ such that $\|s_n - s\| < \varepsilon$, $\forall n > k$. And a sequence s_n is Cauchy if $\forall\, \varepsilon > 0\ \exists\, k \in \mathbb{Z}_+$ such that $\|s_n - s_m\| < \varepsilon$, $\forall n, m > k$. Note that every convergent sequence is Cauchy, but the converse is not true [9].

Definition 1.2.4 [6] Let (U, d) be a metric space and let s_n be a sequence in (U, d) such that for all $\varepsilon > 0$ there exist $N = N(\varepsilon)$ where $d(s_m, s_n) < \varepsilon, \quad \forall m,\, n > N$, then s_n is called Cauchy sequence.
(U, d) is called **complete space** if every Cauchy sequences converges to a point of U.

Definition 1.2.5 [6] **Banach space** is a complete normed vector space. And the **Hilbert space** is an inner product space, that is a Banach space.

Let $\mathcal{H}$ be a Hilbert space, and $\{z_n\}$ be an orthonormal basis of $\mathcal{H}$, the Parseval's identity theorem tells us that for all $s \in \mathcal{H}$ we have

$$\|s\|^2 = \sum_{n=1}^{\infty} \langle z_n, s\rangle.$$

Examples of Hilbert spaces are ℓ^2, L^2,

$$\ell^2 = \big\{(s_1,\, s_2,\, s_3,\, ...) : \sum_{n=1}^{\infty} |s_n|^2 < \infty,\ \ \forall s_n \in \mathbb{C}\big\},$$

besides inner product defined as

$$\langle s,\, t\rangle = \sum_{n=1}^{\infty} s_n \overline{t_n}.,$$

the L^2 will be define in the next section [6].

It can be easily proved that if U is a closed subspace of the Hilbert space $\mathcal{H}$, then $\mathcal{H} = U \oplus U^{\perp}$, where $U^{\perp} = \{s \in \mathcal{H} : s \perp t,\ \forall\, t \in U\}$ is the orthogonal complement of U. Furthermore, for any subset $U \neq \varnothing$ of $\mathcal{H}$, $U^{\perp} = \{0\}$ when and only when the span of U is dense in $\mathcal{H}$ [6].

The special function $\mathscr{Q}_\beta(z)$ is said to be the reproducing kernel of a Hilbert space $\mathcal{H}$ if for all $f \in \mathcal{H}$,

$$f(\beta) = \langle f(z),\, \mathscr{Q}_\beta(z)\rangle,$$

where the family of $\mathscr{Q}_\beta$ spans a dense set of $\mathcal{H}$ [10].

1.2.2 L^p and $\mathscr{H}^p$ Spaces

This sud-section will introduce L^p and $\mathscr{H}^p$ spaces and the connection between them. We will see that for $1 < p \leq \infty$, we can identify the functions in $\mathscr{H}^p$ with their radial limits on L^p. Then we will define the Hardy- Hilbert space $\mathscr{H}^2$, and preview the basic theories needed in this thesis. Also, we will define the inner, and the outer functions, and give an example of the inner function which is the finite Blaschke product.

Definition 1 .2.6 [11] Let $(X, \mathcal{F}, \mu)$ denote the σ-finite measure space, $L^p(X, \mathcal{F}, \mu)$, $1 \leq p < \infty$ is the space of all complex-valued measurable functions on X that satisfy

$$\int_X |f(x)|^p d\mu(x) < \infty$$

and for $f \in L^p(X, \mathcal{F}, \mu)$ define the L^p norm of f as

$$\|f\|_{L^p} = \left(\int_X |f(x)|^p d\mu(x)\right)^{1/p}.$$

If $p = 1$, $L^1(X, \mathcal{F}, \mu)$ is the space of all integrable functions on X, and together with $\|.\|_{L^1}$ is a Banach space. In case $p = 2$, L^2 is a Hilbert space.

If $p = \infty$, the $L^\infty(X, \mathcal{F}, \mu)$ is defined to be the space of all essentially bounded functions, that is for a measurable function $f \in X$, and $0 < M < \infty$ we have

$$|f(x)| < M \qquad a.e.;\ x\ with\ respect\ to\ \mu.$$

The infimum of all possible such M is the norm $\|f(x)\|_{L^\infty}$ of f, note that L^∞ is also a Banach space. By Holder inequality it can be easily proved that for $p \leq q$ and X has a finite positive measure then $L^q(x) \subseteq L^p(x)$.

Definition 1.2.7 [12] Let $h : \Gamma \to \mathbb{C}$ be a measurable function, and $1 \leq p < \infty$. The Hardy space $\mathscr{H}^p$ denotes the space of all analytic functions in the unit disk for which $\|h\|_p$ is finite. Where

$$\|h\|_p \equiv \sup_{r<1} M_p(r, h),$$

and for $1 \leq p < \infty$, $M_p(r, h)$ is defined as

$$M_p(r, h) \equiv \left(\frac{1}{2\pi} \int_0^{2\pi} |h(re^{it})|^p dt \right)^{1/p}.$$

Note that for $0 < r < 1$, $M_p(r, h)$ is an increasing function of r, and it is the L^p norm of h_r, that defined on the unit circle $\mathbb{Y}$ as $h_r(e^{it}) = h(re^{it})$. Therefore $\|.\|_p$ is a norm on $\mathscr{H}^p$. Hence $\mathscr{H}^p$ is a Banach space, and for $1 \leq r \leq p$, $\mathscr{H}^p \subseteq \mathscr{H}^r \subseteq \mathscr{H}^1$. In particular, $\mathscr{H}^\infty$ is the space of all bounded analytic functions on Γ and $\mathscr{H}^\infty \subseteq \mathscr{H}^p$ for all p, where the norm

$$\|h\|_\infty = sup_{|r|<1}|M_\infty(r, h)| \equiv sup_{|r|<1}\big\{sup\{|h(re^{it)})| : 0 \leq t \leq 2\pi\}\big\}.$$

For $1 \leq p \leq \infty$ one can show that if $h(z)$ is an analytic function on Γ, , then $\lim\limits_{r \to 1-} M_p(r, h)$ exists and $\|h\|_p = \lim\limits_{r \to 1-} M_p(r, h)$. Since $M_p(r, h)$ is an increasing function of r, the Maximum Modulus Theorem tells us that $M_\infty(r, h)$ is also an increasing function of r.

Let $\mathbb{Y}$ be the unit circle, with $d\mu = |dt|/2\pi$ be the normalized Lebesgue measure and $h(e^{it}) \in L^1(\mathbb{T})$ [12], then we can write the sequence of the Fourier coefficients of $h(e^{it})$

as

$$\hat{h}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{it})e^{-int}dt, \quad n \in \mathbb{Z}.$$

The next two theorems shows that for $1 \leq p \leq \infty$ the classical hardy spaces $\mathscr{H}^p$ can be defined equivalently as

$$\mathscr{H}^p := \{h \in L^p : \hat{h}(n) = 0 \quad \forall n < 0\}.$$

Theorem 1.2.8 *[12] For $1 \leq p \leq \infty$, let $f \in \mathscr{H}^p$, then $g(z) = \lim_{r\to 1-} f_r(z)$ defines a function $g \in L^p$ with $\hat{g}(n) = 0$ for $n < 0$ and $\sum_0^\infty \hat{g}(n)z^n = f$.*

Theorem 1.2.9 *[13] For $1 \leq p \leq \infty$, let $\widetilde{\mathscr{H}^p} \equiv \{f \in L^p : \hat{f}(n) = 0, \ \forall\ n < 0\}$ then it is a closed subspace of L^p, and the map*

$$T : \mathscr{H}^p \to \widetilde{\mathscr{H}^p}$$

$$f \to \partial(f)$$

is an isometry map of $\mathscr{H}^p$ on to $\widetilde{\mathscr{H}^p}$.

If $p = 2$, $\mathscr{H}^p$ is a Hilbert space with an orthonormal basis $\{1, z, z^2, ...\}$, defined to be the space of all analytic functions on the open unit disc that have square summable Maclaurin coefficients [13], that is

$$\mathscr{H}^2 = \left\{ f : f = \sum_{n=0}^{\infty} \hat{f}(n)z^n and \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty \right\}$$

with an inner product defined as

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

where

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n, \quad and \quad g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n$$

and a norm

$$\|f(z)\| = \left(\sum_{n=0}^{\infty} |\hat{f}(n)|^2\right)^{1/2}.$$

We have to remark that in theorem 1.2.8 Fatou's lemma guarantees that the function $g \in L^p$, for all $f \in \mathscr{H}^p$. And theorem 1.2.9 tells us that $\widetilde{\mathscr{H}^2}$ is a closed subspace of L^2, and there is an isometry between $\widetilde{\mathscr{H}^2}$ and $\mathscr{H}^2$, which can be identify by taking the function $\widetilde{f} \in \widetilde{\mathscr{H}^2}$ that having the Fourier series $\sum_{n=0}^{\infty} a_n e^{int}$, onto the analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If $f \in \mathscr{H}^2$ with $\partial f \in L^{\infty}$, then $f \in \mathscr{H}^{\infty}$.

Note that the Hardy-Hilbert space is a reproducing kernel Hilbert space so that:

Definition 1.2.10 [2] For all $f \in \mathscr{H}^2$, $\beta \in \Gamma$ there exists a unique function $\mathscr{Q}_{\beta}(z)$ such that $f(\beta) = \langle f, \mathscr{Q}_{\beta}(z)\rangle$, this function is

$$\mathscr{Q}_{\beta}(z) = \frac{1}{1 - \overline{\beta}z},$$

with

$$\|\mathscr{Q}_{\beta}(z)\| = \frac{1}{\sqrt{1 - |\beta|^2}}.$$

If $f(z) \neq 0$ in $\mathscr{H}^p$, then it can be factorized uniquely as a product of inner and outer functions [14], they can be defined as follows:

Definition 1.2.11 [14] A function u in $\mathscr{H}^{\infty}$ is called an inner function if $|\lim_{r\to 1} u(re^{it})| = 1$ a.e.; on $\mathbb{Y}$.

Definition 1.2.12 [14] Let $\theta(z)$ in $\mathscr{H}^1$ such that

$$\theta(z) = \alpha\, exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} log|f(e^{it})|dt\right)$$

then $\theta(z)$ is said to be an outer function.

1.2.3 Linear and Normal Operators

This sub-section will define linear operators, the adjoint of an operator, and normal oper-ators, recall some of their properties, and give examples. Then introduce the well-known

Spectral Theorem for the normal operators. Also, it will define the orthogonal projection of L^2 onto $\mathscr{H}^2$.

Definition 1.2.13 [15] Let V and W be vector spaces over the same field, the operator $A : V \to W$ is said to be a linear operator if and only if for every $s, t \in V$, then $A(\alpha s + \beta t) = \alpha As + \beta At$, where α and β are real or complex numbers.

If $\mathcal{M}$ is a subspace, and A is a linear operator such that for every f in $\mathcal{M}$ we have Af in $\mathcal{M}$. Then $\mathcal{M}$ is said to be invariant under A [15].

Examples of linear operators are the identity, the zero, and the multiplication operators. The identity operator $I : V \to V$ is defined by $I(s) = s$ for all s in V. However, the zero operator is defined on V such that $0(s) = 0$ for all s in V. And for $f(e^{it}) \in L^\infty$ the multiplication operator $M_f : L^2 \to L^2$ is defined by $M_f \mathrm{g} = f\mathrm{g}$ for every $\mathrm{g}(e^{it})$ in L^2, and hence $\|M_f\| = \|f\|_\infty$ [6].

It is straightforward to see that the set of all linear operators on a vector space is itself a vector space.

Next we will define the product of two operators by operating the right operator on the left one as follows:

Definition 1.2.14 [15] Let A, and B be two linear operators on a vector space V, we can define their product $P = AB$ by $P(s) = A(B(s))$.

Definition 1.2.15 [15] An operator A is said to be invertible if it meets these two conditions

(i) Whenever $As_1 = As_2$, then $s_1 = s_2$.

(ii) For all $t \in V$, there is at least one vector $s \in V$ such that $As = t$.

If A is invertible then by (ii) for every t_0 there is a vector s_0 such that $As_0 = t_0$, and this vector is unique by (i), then we can define the inverse operator of A as $A^{-1}t_0 = s_0$. Clearly, A^{-1} is a linear operator and $AA^{-1} = A^{-1}A = I$ [15].

If V and W were normed vector spaces, the norm of A can defined to be

$$\|A\| = \sup_{\substack{s \in V \\ s \neq 0}} \frac{\|As\|}{\|s\|}.$$

Definition 1.2.16 [6] Let V, W be normed vector spaces, and $A : V \to W$ be a linear operator. A is regarded as bounded if for all $s \in V$ there exist a real number c such that

$$\|As\| \leq c\|s\|.$$

If V is of finite-dimension then, every linear operator on V will be bounded. We will let $\mathcal{B}(\mathcal{H})$ stand for the set of all bounded linear operators on the Hilbert space $\mathcal{H}$.

Definition 1.2.17 [6] Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator, the unique bounded linear operator $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ that satisfy

$$\langle As, t \rangle_{\mathcal{H}_2} = \langle s, A^* t \rangle_{\mathcal{H}_1}$$

is called the adjoint operator of A.

Keep in mind that the adjoint operator of a bounded linear operator exists uniquely and $\|A^*\| = \|A\|$. Moreover, if $A : \mathcal{H}_1 \to \mathcal{H}_2$, $B : \mathcal{H}_1 \to \mathcal{H}_2$ are bounded linear operators, and α is a scalar then we have

(i) $\langle A^* t, s \rangle = \langle t, As \rangle$; $(s \in \mathcal{H}_1,\ t \in \mathcal{H}_2)$,

(ii) $(A + B)^* = A^* + B^*$,

(iii) $(A^*)^* = A$,

(iv) $(\alpha A)^* = \overline{\alpha} A^*$,

(v) $\|AA^*\| = \|A^* A\| = \|A\|^2$,

(vi) $A^* A = 0 \iff A = 0$,

(vii) Assuming that $\mathcal{H}_1 = \mathcal{H}_2 \implies (AB)^* = B^* A^*$.

Now we are ready to define the main definition of this thesis, the normal operator which can be defined as follows:

Definition 1.2.18 [6] A bounded linear operator A on a Hilbert space $\mathcal{H}$ is said to be
(1) self-adjoint or Hermitian if $A = A^*$,
(2) unitary if A is bijective and $A^*A = AA^* = I$, i.e. $A^* = A^{-1}$,
(3) normal if $AA^* = A^*A$.

It is worth saying that if $A \in \mathcal{B}(\mathcal{H})$ is normal, A^* is normal too, and $\|As\| = \|A^*s\|, \forall s \in \mathcal{H}$ [6]. Furthermore, A and B are unitarily equivalent if there is a unitary operator U such that $A = U^*BU$ [6]. If A is a normal operator whenever A and B are unitarily equivalent, then B is likewise a normal operator.

For $A \in \mathcal{B}(\mathcal{H})$, the spectrum of A is $\sigma(A) := \{\alpha \in \mathbb{C} : \ A - \alpha I$ is not invertible in $\mathcal{B}(\mathcal{H})\}$. And the point spectrum of A is $\sigma_p(A) := \{\alpha \in \mathbb{C} : ker(A - \alpha I) \neq \{0\}\}$, that is the set of all eigenvalues of A. Moreover, if there exist $s \in \mathcal{H}$ such that $\{s, As, A^2s, A^3s, ...\}$ is dense in $\mathcal{H}$, then A is said to be cyclic operator, and s its cyclic vector [6].

Proposition 1.2.19 *[16] Let A be a linear bounded operator and $\alpha \in \mathbb{C}$ is an eigenvalue of A, then $\overline{\alpha}$ is an eigenvalue of A^* with identical eigenvector.*

The factorization of functions in $\mathscr{H}^2$ follows directly from the study of the invariant subspaces of the shift operator which we will introduce next.

Definition 1.2.20 [13] The unilateral (forward) shift operator $S : \mathscr{H}^2 \to \mathscr{H}^2$ is of the form

$$Sf(z) = zf(z)$$

or, considering the Taylor coefficients of $f(z) = \sum_0^\infty a_n z^n \in \mathscr{H}^2$, is of the form

$$S(a_0, a_1, a_2, ...) = (0, a_1, a_2, ...)$$

where $(a_0, a_1, a_2, ...) \in \ell^2$.

Proposition 1.2.21 *[17] Suppose that $S : \mathscr{H}^2 \to \mathscr{H}^2$ is the unilateral (forward) shift operator then:*

1. $\|Sf\| = \|f\|$, *therefore S is an isometry.*

2. *The adjoint of the unilateral (forward) shift operator also known as the backward unilateral shift operator is*

$$S^*(f(z)) = \frac{f(z) - f(0)}{z}, \;\; f(z) \in H^2$$

or,

$$S^*(a_0, a_1, a_2, ...) = (a_1, a_2, ...), \;\; (a_0, a_1, a_2, ...) \in \ell^2.$$

By using the fact that $|u| = 1$ almost everywhere on $\mathbb{Y}$ for any inner function u, if $f \in \mathscr{H}^2$ we can see that $\|uf\| = \|f\|$, therefore $u\mathscr{H}^2$ is a closed subspace of $\mathscr{H}^2$. Furthermore, $S(u\mathscr{H}^2) = zu\mathscr{H}^2 = u(z\mathscr{H}^2) \subset u\mathscr{H}^2$, hence $u\mathscr{H}^2$ is a non zero subspace that is invariant under S. According to Beurling's Theorem, these are all the $\mathscr{H}^2$ subspaces that are non-zero S-invariant.

Theorem 1.2.22 ***Beurling's Theorem*** *[13] Each non zero subspace of $\mathscr{H}^2$ that is invariant under the forward shift operator S is of the form of $u\mathscr{H}^2$, for u as an inner function.*

Now let's recall different forms of the Spectral Theorem.

Theorem 1.2.23 ***(Spectral Theorem)*** *[17] [18] Let $A \in \mathcal{B}(\mathcal{H})$ is cyclic, and let $M_+(\sigma(A))$ be the set of all positive Borel measure on $\sigma(A)$ then;*

(i) If A is normal then there exist a measure $\mu \in M_+(\sigma(A))$ so that A is unitarily equivalent to the operator $f \mapsto zf$ in $L^2(\mu)$.

(ii) If A is self-adjoint, then $\sigma(A) \subset \mathbb{R}$, and there exist a $v \in M_+(\sigma(A))$ such that A is unitarily equivalent to the operator $f \mapsto xf$ in $L^2(v)$.

(iii) If A is unitary, then $\sigma(A) \subset \mathbb{Y}$, and there exist a $\eta \in M_+(\sigma(A))$ such that A is unitarily equivalent to the operator $f \mapsto \zeta f$ in $L^2(\eta)$.

Where a subset of $\mathbb{Y}$ is called Borel if it is contained in the smallest σ-algebra of subsets of $\mathbb{Y}$ that contains all the open arcs of $\mathbb{Y}$. A countably additive function on $\mathbb{Y}$ is a Borel measure if assigned a complex number for each Borel subset of $\mathbb{Y}$.

An example of a bounded self-adjoint operator onto a subspace U of a Hilbert space $\mathcal{H}$ is the projection operator that defined as follows:

Definition 1.2.24 [13] Let U be a subspace of a Hilbert space $\mathcal{H}$ then the operator

$$Pf = g$$

where $f = g + h$ with $g \in U$, $h \in U^{\perp}$, and $\mathcal{H} = U \oplus U^{\perp}$, known as the orthogonal projection operator onto U.

Definition 1.2.25 (The Riesz projection) [13] Let P be the orthogonal projection of L^2 onto $\mathscr{H}^2$, if $f(e^{it}) = \sum_{n=-\infty}^{\infty} a_n e^{int}$ then

$$Pf = \sum_{n=0}^{\infty} a_n z^n$$

Definition 1.2.26 [19] For $\mathrm{g} \in L^{\infty}$, the Toeplitz operator T_{g} defined by

$$T_{\mathrm{g}} : \mathscr{H}^2 \to \mathscr{H}^2$$

$$f \to P(\mathrm{g}f)$$

where P is the orthogonal projection of L^2 onto $\mathscr{H}^2$.

Note that if $\mathrm{g} \in \mathscr{H}^{\infty}$, then $T_{\mathrm{g}}f = \mathrm{g}f$ be the multiplication operator, and if $\mathrm{g} = z$, then $T_{\mathrm{g}}f = zf$ be the shift operator. It is straightforwards to prove $T_{\mathrm{g}}^* = T_{\mathrm{g}}$ where g denote the conjugate of g.

Corollary 1.2.26.1 *[18] Let $f(z) \in \mathscr{H}^2$, then*

$$Sf(z) = T_z f(z)$$

and

$$S^* f(z) = T_z^* f(z)$$

1.2.4 Linear Fractional Transformation

This sub-section contains a review of the definition of linear fractional transformations and some basics.

Definition 1 .2.27 [11] The mapping of the form

$$T(z) = \frac{az + b}{cz + d}$$

with $ad - bc \neq 0$, is known as a linear fractional (Mobius) transformation. And $T(z)$ is said to be in standard form if $ad - bc = 1$.

The notation $LFT(\hat{C})$ represents the collection of all such maps, which forms a group under composition. We meant to draw attention to the fact that each linear fractional transformation can be viewed as a one-to-one holomorphic mapping of the Riemann Sphere $\hat{C}$ onto itself, with the obvious rules concerning the point at infinity.

It is clear that the linear fractional transformation $T(z) = \frac{az+b}{cz+d}$ has a fixed point at ∞ when and only when $c = 0$, and it would be the only one when and only when $a = d$ and $b \neq 0$. If not, the solutions of the quadratic equation $cz^2 + (d - a)z - b = 0$ are the fixed points [11].

Here, what interests us is the subgroup $LFT(\Gamma)$ of $LFT(\hat{C})$ that maps Γ into itself called automorphism mapping. It should be noted that each self-conformal mapping of Γ is a linear fractional transformation. And the unique fixed point z_0 of the transformation $f(z)$ in the closed disk such that $|f'(z_0)| \leq 1$ is called the Denjoy-Wolff point. The Schwarz-Pick Lemma implies that $T(z)$ has not more than one fixed point in Γ, and if it does exist it must be the Denjoy point [20].

Corollary 1.2.27.1 *[2] If $T(z)$ is a univalent (one-to-one) inner function then $T(z)$ is an automorphism of the unit disk Γ.*

The following definition will present the non-identity Mobius transformations geometry classifications on the unit disk.

Definition 1.2.28 [21] Let $T(z) \in LFT(\Gamma)$ be a non-identity automorphism. then

1. $T(z)$ is said to be of parabolic type when and only when it has a single fixed point on the unit circle.

2. $T(z)$ is said to be hyperbolic when and only when it has two fixed points on the unit circle.

3. $T(z)$ is said to be elliptic when and only when it has a fixed point in the unit disk.

Chapter Two

Normal Weighted Composition Operators on Hardy-Hilbert space

This chapter will introduce the composition, and the weighted composition operators, and it discuss some of their properties. We will see Schwartz's result that answers the question when the composition operator is normal on $\mathscr{H}^2(\Gamma)$. Also, we will study and look over Bordon and Narayan's characterization of the unitary weighted composition operators, and their description of normal weighted composition operators that are induced by a function $\upsilon(z)$ Aixes a Aoint An (Γ).

2.1 Composition Operators

For a domain YAn Ahe Aiemann Aphere $\mathbb{C}$, Aet $\upsilon(z)$ be a Aelf-map analytic Aunction Af Y, Aet $H(Y)$ be Ahe Aet Af all Aolomorphic Aunctions defined An Y, and Aet $X \subseteq H(Y)$ be a Banach Apace, Ahe Ainear Aperator

$$C_\upsilon : X \to X$$

$$f \to f \circ \upsilon$$

is called the composition operator induced by $\upsilon(z)$ [2].

If $\upsilon(z)$ defined on the open unit disk Γ, LittleWood's Theorem tells us that $C_\upsilon f(z)$ is bounded on the Hardy space $\mathscr{H}^2(\Gamma)$, and $(f \circ \upsilon)(z) \in \mathscr{H}^2(\Gamma)$ for every $f(z) \in \mathscr{H}^2(\Gamma)$

[22]. Moreover,

$$\left(\frac{1}{1-|\upsilon(0)|^2}\right)^{\frac{1}{2}} \leq \|C_\upsilon\| \leq \left(\frac{1+|\upsilon(0)|}{1-|\upsilon(0)|}\right)^{\frac{1}{2}}.$$

It is clear that the product of two composition operators is a composition operator [2]. That is, if C_υ and C_ξ are composition operators then

$$\begin{aligned} C_\upsilon C_\xi f(z) &= C_\upsilon(f \circ \xi)(z) \\ &= (f \circ \xi \circ \upsilon)(z) \\ &= C_{\xi \circ \upsilon} f(z) \end{aligned}$$

Our discussion is about the normality of the composition operator which includes the computation of its adjoint, since C_υ is bounded on $\mathscr{H}^2(\Gamma)$, it has an adjoint C_υ^* also defined on $\mathscr{H}^2(\Gamma)$. The computation of C_υ^* is not that easy for all the functions in $\mathscr{H}^2(\Gamma)$ still, we always have its action on a special set of functions, and is it the family of reproducing kernels [22].

Lemma 2.1.1 *[22] For all $f(z) \in \mathscr{H}^2(\Gamma)$, $\beta \in \Gamma$, we have*

$$C_\upsilon^* \mathscr{Q}_\beta(z) = \mathscr{Q}_{\upsilon(\beta)}(z)$$

Proof. By definition 1.2.17 $\langle f, C_\upsilon^* \mathscr{Q}_\beta \rangle = \langle C_\upsilon f, \mathscr{Q}_\beta \rangle$, since $\mathscr{Q}_\beta(z)$ is the reproducing kernel of $\mathscr{H}^2(\Gamma)$ we have

$$\begin{aligned} \langle C_\upsilon f, \mathscr{Q}_\beta \rangle &= C_\upsilon f(\beta) \\ &= f\big(\upsilon(\beta)\big) \\ &= \langle f, \mathscr{Q}_{\upsilon(\beta)} \rangle \end{aligned}$$

therefore $\langle f, C_\upsilon^* \mathscr{Q}_\beta \rangle = \langle f, \mathscr{Q}_{\upsilon(\beta)} \rangle$, hence $C_\upsilon^* \mathscr{Q}_\beta(z) = \mathscr{Q}_{\upsilon(\beta)}(z)$. □

Schwartz 1969 proved that the adjoint of the composition operator C_υ is a composition operator when and only when $\upsilon(z) = \alpha z, \quad |\alpha| \leq 1$, and the composition operator C_υ is normal when and only when $\upsilon(z) = \alpha z, \quad |\alpha| \leq 1$, as follows:

Theorem 2.1.2 *[3] If $|\alpha| \leq 1$, then $C^*_{\alpha z} = C_{\overline{\alpha} z}$.*

Proof. Here we use a different proof from Schwartz. Since $\mathscr{Q}_\beta$ span a dense set of $\mathscr{H}^2$, it is enough to show that
$C^*_{\alpha z}\mathscr{Q}_\beta(z) = C_{\overline{\alpha} z}\mathscr{Q}_\beta(z)$.
By lemma 2.1.1, and definition 1.2.10 $\mathscr{Q}_\beta(z) = \frac{1}{1-\overline{\beta} z}$ we get

$$\begin{aligned}
C^*_{\alpha z}\mathscr{Q}_\beta(z) &= \mathscr{Q}_{\alpha\beta}(z) \\
&= \frac{1}{1-\overline{\alpha\beta} z} \\
&= \frac{1}{1-\overline{\beta}(\overline{\alpha} z)} \\
&= \mathscr{Q}_\beta(\overline{\alpha} z) \\
&= C_{\overline{\alpha} z}\mathscr{Q}_\beta(z).
\end{aligned}$$

□

Theorem 2.1.3 *[3] The composition operator C_υ is normal when and only when $\upsilon(z) = \alpha z, \quad |\alpha| \leq 1$.*

Proof. Let C_υ be normal, and $\upsilon(z) = \sum_{n=0}^{\infty} a_n z^n$, an orthonormal basis of $\mathscr{H}^2$ is the sequence $\{1, z, z^2, ...\}$, therefore by Parseval's identity

$$\begin{aligned}
\|C_v^* 1\|^2 &= \sum_{n=0}^{\infty} |\langle C_v^* 1, z^n\rangle|^2 \\
&= \sum_{n=0}^{\infty} |\langle 1, C_v z^n\rangle|^2 \\
&= \sum_{n=0}^{\infty} |\langle 1, v^n\rangle|^2 \\
&= |\langle 1, v^0\rangle|^2 + |\langle 1, v^1\rangle|^2 + |\langle 1, v^2\rangle|^2 + ... \\
&= 1 + |a_0|^2 + |(a_0)^2|^2 + ... \\
&= \sum_{n=0}^{\infty} |a_0|^{2n}.
\end{aligned}$$

On the other hand

$$\|C_v 1\|^2 = \|1\|^2 = 1$$

but given that C_v is normal operator, so that $\|C_v^* 1\| = \|C_v 1\|$, this implies

$$\sum_{n=0}^{\infty} |a_0|^{2n} = 1,$$

therefore

$$|a_0|^2 + |(a_0)|^4 + |(a_0)|^6 + ... = 0,$$

and hence $a_0 = 0$, i.e. $v(z) = \sum_{n=1}^{\infty} a_n z^n$.

Similarly, if $f(z) = z$ we have

$$\|C_\upsilon^* z\|^2 = \sum_{n=0}^{\infty} |\langle C_\upsilon^* z, z^n\rangle|^2$$

$$= \sum_{n=0}^{\infty} |\langle z, C_\upsilon z^n\rangle|^2$$

$$= \sum_{n=0}^{\infty} |\langle z, \upsilon^n\rangle|^2 = |\langle z, \upsilon^0\rangle|^2 + |\langle z, \upsilon^1\rangle|^2 + |\langle z, \upsilon^2\rangle|^2 + |\langle z, \upsilon^3\rangle|^2 ...$$

$$= 0 + |\langle z, \upsilon\rangle|^2 + 0 + 0 + ... = |\langle z, \upsilon\rangle|^2 = |a_1|^2$$

while

$$\|C_\upsilon z\|^2 = \|\upsilon\|^2$$

$$= \sum_{n=1}^{\infty} |a_n|^2$$

$$= |a_1|^2 + |a_2|^2 + |a_3|^2 + ...$$

but $\|C_\upsilon^*\| = \|C_\upsilon\|$, this implies

$$|a_1|^2 = |a_1|^2 + |a_2|^2 + |a_3|^2 + ...,$$

so that

$$|a_2|^2 + |a_3|^2 + ... = 0,$$

and hence $|a_n| = 0$ for all $n \geq 2$. Therefore, $\upsilon(z) = a_1 z$, and $|a_1| < 1$.

Conversely, Let $f(z) \in \mathscr{H}^2$, and suppose $\upsilon(z) = \alpha z$, by lemma 2.1.2 we get

$$\begin{aligned} C_\upsilon^* C_\upsilon f(z) &= C_{\alpha z}^* C_{\alpha z} f(z) \\ &= C_{\overline{\alpha} z} C_{\alpha z} f(z) \\ &= C_{\overline{\alpha} z} f(\alpha z) \\ &= f(\alpha \overline{\alpha} z) \\ &= f(\overline{\alpha} \alpha z) \\ &= C_{\alpha z} f(\overline{\alpha} z) \\ &= C_{\alpha z} C_{\overline{\alpha} z} f(z) = C_{\alpha z} C_{\alpha z}^* f(z) \end{aligned}$$

□

Cowen obtained that if $\upsilon(z)$ is a linear fractional transformation on the Γ, then the adjoint C_υ^* is a product of Toepletz operators and a composition operator. Before proving the theorem we need to prove the following lemma:

Lemma 2.1.4 *[2] If* $\mathrm{g} \in \mathscr{H}^\infty$*, then*

$$T_{\mathrm{g}}^* \mathscr{Q}_\beta(z) = \overline{\mathrm{g}(\beta)} \mathscr{Q}_\beta(z).$$

Proof.

$$\begin{aligned} \langle f, T_{\mathrm{g}}^* \mathscr{Q}_\beta \rangle &= \langle T_{\mathrm{g}} f, \mathscr{Q}_\beta \rangle \\ &= \mathrm{g}(\beta) f(\beta) \\ &= \mathrm{g}(\beta) \langle f, \mathscr{Q}_\beta \rangle = \langle f, \overline{\mathrm{g}(\beta)} \mathscr{Q}_\beta \rangle. \end{aligned}$$

□

Theorem 2.1.5 *[23] Let* $\upsilon(z) = \frac{az+b}{cz+d}$ *where* $ad - bc \neq 0$*, then* $\tau(z) = \frac{\overline{a}z - \overline{c}}{\overline{d} - \overline{b}z}$ *maps the unit* *disk into itself,* $j(z) = \frac{1}{\overline{d} - \overline{b}z}$*, and* $\rho(z) = cz + d$ *are in* $\mathscr{H}^\infty$*, and we have* $C_v^* = T_j C_\tau T_\rho^*$*.*

Proof. Clearly $\rho(z) = cz + d \in \mathscr{H}^\infty$. Since $v(z)$ is a linear fractional mapping that maps Γ into itself, it is a one-to-one mapping on the Riemann sphere, hence $\overline{v(\overline{z})}$ and its inverse also maps Γ into itself. Thus,

$$\tau(z) = \frac{1}{\overline{v}^{-1}\left(\frac{1}{\overline{z}}\right)}$$

maps Γ into itself. The function $j(z)$ has the same denominator of $\tau(z)$, therefore, $j(z) \in \mathscr{H}^\infty$.

Now, to prove $C_v^* = T_j C_\tau T_\rho^*$, it is enough to show that $C_v^* \mathscr{Q}_\beta(z) = T_j C_\tau T_\rho^*(\mathscr{Q}_\beta(z))$:

$$\begin{aligned}
T_j C_\tau T_\rho^*(\mathscr{Q}_\beta(z)) &= T_j C_\tau \overline{\rho(\beta)} \mathscr{Q}_\beta(z) \\
&= \overline{\rho(\beta)}.j(z).\mathscr{Q}_\beta(\tau(z)) \\
&= \left(\overline{c\beta} + \overline{d}\right)\left(\frac{1}{\overline{d} - \overline{b}z}\right)\left(\frac{\overline{d} - \overline{b}z}{\overline{d} - \overline{b}z - \overline{a\beta}z + \overline{c\beta}}\right) \\
&= \frac{\overline{c\beta} + \overline{d}}{(\overline{d} + \overline{c\beta}) - z(\overline{b} - \overline{a\beta})} \\
&= \frac{\frac{\overline{c\beta}+\overline{d}}{\overline{c\beta}+\overline{d}}}{\frac{(\overline{d}+\overline{c\beta})-z(\overline{b}-\overline{a\beta})}{\overline{c\beta}+\overline{d}}} \\
&= \frac{1}{1 - \overline{v(\beta)}z} \\
&= \mathscr{Q}_{v(\beta)}(z) = C_v^* \mathscr{Q}_\beta(z).
\end{aligned}$$

□

In [24] Hammond rewrote Cowen's formula using the fact in corollary (1.2.26.1) as

$$C_v^* f(z) = \left(\frac{(\overline{ad} - \overline{bc})z}{(\overline{a}z - \overline{c})(\overline{d} - \overline{b}z)}\right) f(\tau(z)) - \frac{\overline{c}}{\overline{a}z - \overline{c}} f(0). \tag{2.1}$$

Proof.

$$\begin{aligned}
C_\upsilon^* f(z) &= T_j C_\tau T_\rho^* f(z) \\
&= j(z) C_\tau \overline{\rho} f(z) \\
&= j(z) C_\tau \Big(\overline{cz+d}\Big) f(z) \\
&= j(z) C_\tau \left[\overline{cz} f(z) + \overline{d} f(z)\right] \\
&= j(z) C_\tau \left[\overline{c}\Big(\overline{z} f(z)\Big) + \overline{d} f(z)\right] \\
&= j(z) C_\tau \left[\overline{c}\Big(\frac{f(z)-f(0)}{z}\Big) + \overline{d} f(z)\right] \\
&= j(z) \left(\overline{c}\frac{f\big(\tau(z)\big)-f(0)}{\tau(z)} + \overline{d} f\big(\tau(z)\big)\right) \\
&= \left(\frac{\overline{c}}{\overline{a}z-\overline{c}} + \frac{\overline{d}}{\overline{d}-\overline{b}z}\right) f(\tau(z)) - \frac{\overline{c}}{\overline{a}z-\overline{c}} f(0)
\end{aligned}$$

2.2 Weighted Composition Operators

This section studies the weighted composition operator, its definition, and some properties.

Let Γ denote the open unit disk, let $\upsilon(z)$ be an analytic self map function on Γ, and let $\omega(z)$ be a holomorphic function on Γ, we can see in [2] that the weighted composition operator can defined to be

$$W_{\omega,\upsilon} f(z) = \omega(z) C_\upsilon f(z) = \omega(z)(f \circ \upsilon)(z).$$

The boundedness of $W_{\omega,\upsilon}$ on $\mathscr{H}^2(\Gamma)$ it follows obviously if the weighted function $\omega(z)$ is bounded on $H(\Gamma)$ (the set of all holomorphic function on Γ) but it's not necessarily, i.e.,

$W_{\omega,\upsilon}$ still can be bounded while ω is not. It is clear that if $\omega(z) \in \mathscr{H}^\infty$, then in [2] we can see that the weighted composition operator can be rewritten as

$$W_{\omega,\upsilon} f(z) = \omega(z) C_\upsilon f(z) = T_\omega C_\upsilon f(z).$$

Corollary 2.2.0.1 *[2] Let $\omega_1(z)$, and $\omega_2(z)$ be in $\mathscr{H}^\infty$, then*

$$W_{\omega_1,\upsilon_1} W_{\omega_2,\upsilon_2} f(z) = W_{\omega_1(\omega_2 \circ \upsilon_1),(\varphi_2 \circ \upsilon_1)} f(z).$$

Proof.

$$\begin{aligned}
W_{\omega_1,\upsilon_1} W_{\omega_2,\upsilon_2} f(z) &= T_{\omega_1} C_{\upsilon_1} T_{\omega_2} C_{\upsilon_2} f(z) \\
&= \omega_1 C_{\upsilon_1} (\omega_2 f \circ \upsilon_2)(z) \\
&= \omega_1 (\omega_2 \circ \upsilon_1)(f \circ \upsilon_2 \circ \upsilon_1)(z) \\
&= W_{\omega_1(\omega_2 \circ \upsilon_1),(\upsilon_2 \circ \upsilon_1)} f(z).
\end{aligned}$$

□

The following lemma is similar to the lemma 2.1.1.

Lemma 2.2.1 *[4] Let $\beta \in \Gamma$ and suppose that $W_{\omega,\upsilon} : \mathscr{H}^2(\Gamma) \to \mathscr{H}^2(\Gamma)$ is bounded. Then*

$$W^*_{\omega,\upsilon} \mathscr{Q}_\beta(z) = \overline{\omega(\beta)} \mathscr{Q}_{\upsilon(\beta)}(z).$$

Proof. Let $f(z) \in \mathscr{H}^2(\Gamma)$, by definition 1.2.17, and since $\mathscr{Q}_\beta(z)$ is the reproducing kernel of $\mathscr{H}^2(\Gamma)$, we get

$$\begin{aligned}
\langle f, W^*_{\omega,\upsilon} \mathscr{Q}_\beta \rangle &= \langle \omega f \circ \upsilon, \mathscr{Q}_\beta \rangle \\
&= \omega(\beta) f(\upsilon(\beta)) = \langle f, \overline{\omega(\beta)} \mathscr{Q}_{\upsilon(\beta)} \rangle
\end{aligned}$$

□

The next two propositions are two necessary conditions for normal weighted composition operators.

Lemma 2.2.2 *[4] If $W_{\omega,\upsilon}$ is normal then either $\omega(z) \equiv 0$ or $\omega(z)$ never vanish for all $z \in \Gamma$.*

Proof. Suppose $W_{\omega,\upsilon}$ is normal and $\omega(\beta) = 0$ for some β in Γ, then by lemma 2.2.1

$$W^*_{\omega,\upsilon}\mathscr{Q}_\beta(z) = \overline{\omega(\beta)}\mathscr{Q}_{\upsilon(\beta)}(z) \equiv 0$$

therefore,

$$\|W_{\omega,\upsilon}\mathscr{Q}_\beta(z)\| = \|W^*_{\omega,\upsilon}\mathscr{Q}_\beta(z)\| = 0$$

but

$$0 = \|W_{\omega,\upsilon}\mathscr{Q}_\beta(z)\| = \|(\omega\mathscr{Q}_\beta \circ \upsilon)(z)\| = \left\|\omega(z)\frac{1}{1-\bar{\beta}\upsilon(z)}\right\|$$

which implies

$$\frac{\omega(z)}{1-\bar{\beta}\upsilon(z)} = 0, \ \forall z \in \Gamma \implies \omega(z) \equiv 0, \forall z \in \Gamma.$$

Thus, if $W_{\omega,\upsilon}$ is normal then either $\omega \equiv 0$ or ω is non zero at each point in Γ. □

Proposition 2.2.3 *[4] If $W_{\omega,\upsilon}$ is normal, $\upsilon(z)$ is not a constant function, and $\omega(z) \not\equiv 0$. Then $\upsilon(z)$ is univalent.*

Proof. Suppose $\upsilon(z)$ is a non-constant non-univalent function on Γ. So, there are two different points a and b in Γ while $\upsilon(a) = \upsilon(b)$, and since $\omega(z) \not\equiv 0$, lemma 2.2.2 implies that $\omega(a) \neq 0$, and $\omega(b) \neq 0$.

Let

$$q(z) = \frac{\mathscr{Q}_a(z)}{\overline{\omega(a)}} - \frac{\mathscr{Q}_b(z)}{\overline{\omega(b)}}$$

notice that $q(z)$ is well defined non zero function on $\mathscr{H}^2(\Gamma)$. Applying linearity and lemma 2.2.1 we get

$$
\begin{aligned}
W^*_{\omega,\upsilon} q(z) &= W^*_{\omega,\upsilon}\left(\frac{\mathscr{Q}_a(z)}{\overline{\omega(a)}} - \frac{\mathscr{Q}_b(z)}{\overline{\omega(b)}}\right) \\
&= W^*_{\omega,\upsilon}\frac{\mathscr{Q}_a(z)}{\overline{\omega(a)}} - W^*_{\omega,\upsilon}\frac{\mathscr{Q}_b(z)}{\overline{\omega(b)}} \\
&= \overline{\omega(a)}\frac{\mathscr{Q}_{\upsilon(a)}(z)}{\overline{\omega(a)}} - \overline{\omega(b)}\frac{\mathscr{Q}_{\upsilon(b)}(z)}{\overline{\omega(b)}} \\
&= \mathscr{Q}_{\upsilon(a)}(z) - \mathscr{Q}_{\upsilon(b)}(z) = 0 \qquad\qquad (we\, assumed\ above\ \upsilon(a) = \upsilon(b))
\end{aligned}
$$

given $W_{\omega,\upsilon}$ is normal, so that

$$
\begin{aligned}
0 &= \|W^*_{\omega,\upsilon} q(z)\| \\
&= \|W_{\omega,\upsilon} q(z)\| \\
&= \|\omega(z) q(\upsilon(z))\|
\end{aligned}
$$

since $\omega(z) \neq 0$ this implies that $q(\upsilon(z)) = 0$. But, $\upsilon(z)$ is non-constant, then $q(z)$ must vanish on a nonempty open subset of Γ. However, Γ is connected, so $q(z)$ must be the zero function, a contradiction. Hence $\upsilon(z)$ is univalent. □

2.3 Unitary Weighted Composition Operators

Here we will see Bordon and Narayan's characterization of the unitary weighted composition operator on $\mathscr{H}^2(\Gamma)$, and we will show that every automorphism of Γ induces a unitary weighted composition operator.

Let $\upsilon(z)$ be the constant function, i.e. there exist $c \in \Gamma$ such that

$$
\upsilon(z) = c, \ \forall z \in \Gamma
$$

then $W_{\omega,\upsilon}$ cant be unitary, since

$$W_{\omega,\upsilon} f(z) = \omega(z) f(c)$$

this implies that $\|W_{\omega,\upsilon} f(z)\| = 0$, $\forall f \; s.t \; f(c) = 0$, while $\|f(z)\| \neq 0$, hence $W_{\omega,\upsilon}$ can't be norm preserving [4].

Proposition 2.3.1 *[4] If* $W_{\omega,\upsilon} : \mathscr{H}^2(\Gamma) \to \mathscr{H}^2(\Gamma)$ *is unitary operator. Then* $\upsilon(z)$ *is an automorphism of* Γ.

Proof. To prove that $\upsilon(z)$ must be an automorphism of Γ, it is enough to show that $\upsilon(z)$ is a univalent inner function by corollary 1.2.27.1.

Suppose $W_{\omega,\upsilon}$ is unitary, then it is a norm preserving, therefore,

$$\begin{aligned} 1 &= \|W_{\omega,\upsilon} 1\| \\ &= \|(\omega(z))(1)\| \\ &= \|\omega(z)\|, \end{aligned}$$

and if $f(z) = z$ then

$$\begin{aligned} 1 &= \|z\| \\ &= \|f(z)\| \\ &= \|W_{\omega,\upsilon} f(z)\| \\ &= \|(\omega.\upsilon)(z)\| \end{aligned}$$

which implies that $\|\omega(z)\|^2 = \|(\omega.\upsilon)(z)\|^2$, i.e.

$$\frac{1}{2\pi}\int_0^{2\pi} |\omega(e^{it})\upsilon(e^{it})|^2 dt = \frac{1}{2\pi}\int_0^{2\pi} |\omega(e^{it})|^2 dt.$$

Now, since $|\upsilon(e^{it})| \leq 1$ for almost everywhere $t \in [0, 2\pi]$, and both $\omega(z)$, $\upsilon(z)$ are bounded we get

$$0 = \int_0^{2\pi} \left(|\omega(e^{it})|^2 - |\omega(e^{it})\upsilon(e^{it})|^2 \right) dt$$

$$= |\omega(e^{it})|^2 - |\omega(e^{it})\upsilon(e^{it})|^2$$

$$= |\omega(e^{it})|^2 \left(1 - |\upsilon(e^{it})|^2 \right)$$

but we proved that $\|\omega(z)\| \neq 0$, so $\omega(z) \not\equiv 0$, which implies that $1 - |\upsilon(e^{it})|^2 = 0$, hence $|\upsilon(e^{it})|^2 = 1$, and thus $\upsilon(z)$ is a non constant inner function. And proposition 2.2.3 tells us that $\upsilon(z)$ is also univalent which completes the proof. □

In the following theorem, we will show that if $W_{\omega,\upsilon}$ is induced by an automorphism of Γ, i.e. $\upsilon(z) = e^{i\theta}\frac{a-z}{1-\bar{a}z}$ for some $\theta \in \mathbb{R}$, $a \in \Gamma$, and $\omega(z)$ is a multiple of the normalized reproducing kernel, then it is unitary, and vice versa.

Proposition 2.3.2 *[4] If $W_{\omega,\upsilon}$ is unitary, and it is inducing a map $\upsilon(z)$ such that $\upsilon(\beta) = 0$ for some $\beta \in \Gamma$. Then*

$$\omega(z) = c\frac{\mathscr{Q}_\beta(z)}{\|\mathscr{Q}_\beta(z)\|},$$

where $|c| = 1$.

Proof. Suppose that $W_{\omega,\upsilon}$ is unitary, this implies $WW^* = W^*W = I$, so that

$$W_{\omega,\upsilon}W^*_{\omega,\upsilon}\mathscr{Q}_\beta(z) = \mathscr{Q}_\beta(z),$$

since $\upsilon(\beta) = 0$ lemma 2.2.1 tell us that

$$\begin{aligned}
\mathscr{Q}_\beta(z) &= W_{\omega,\upsilon}\overline{\omega(\beta)}\mathscr{Q}_0(z) \\
&= W_{\omega,\upsilon}\overline{\omega(\beta)}.\frac{1}{1-(0)(z)} \\
&= W_{\omega,\upsilon}\overline{\omega(\beta)} \\
&= \omega(z)\overline{\omega(\beta)}.
\end{aligned}$$

If we substitute β, and note that

$$\mathscr{Q}_\beta(\beta) = \Big\langle \mathscr{Q}_\beta(z), \mathscr{Q}_\beta(z) \Big\rangle = \big\|\mathscr{Q}_\beta(z)\big\|^2$$

then

$$\big\|\mathscr{Q}_\beta(z)\big\|^2 = \omega(\beta)\overline{\omega(\beta)} = |\omega(\beta)|^2,$$

since the$\|.\|$ and $|.|$ both are positive we get

$$\big\|\mathscr{Q}_\beta(z)\big\| = \big|\omega(\beta)\big|,$$

and hence

$$\begin{aligned}
\omega(z) &= \frac{\mathscr{Q}_\beta(z)}{\overline{\omega(\beta)}} \\
&= \frac{\omega(\beta)\mathscr{Q}_\beta(z)}{\big|\omega(\beta)\big|^2} \\
&= \frac{\omega(\beta)}{\big|\omega(\beta)\big|}\,\frac{\mathscr{Q}_\beta(z)}{\big\|\mathscr{Q}_\beta(z)\big\|}
\end{aligned}$$

let $c = \frac{\omega(\beta)}{\big|\omega(\beta)\big|}$, where $\left|\frac{\omega(\beta)}{\big|\omega(\beta)\big|}\right| = 1$ and the result follows. □

Theorem 2.3.3 *[4] $W_{\omega,\upsilon}$ is unitary on $\mathscr{H}^2(\Gamma)$ when and only when $\upsilon(z)$ is an automorphism of Γ and $\omega(z) = c\frac{\mathscr{Q}_\beta(z)}{\|\mathscr{Q}_\beta(z)\|}$ where $\upsilon(\beta) = 0$ and $|c| = 1$.*

Proof. Suppose $W_{\psi,\upsilon}$ is unitary on $\mathscr{H}^2(\Gamma)$, then proposition 2.3.1 shows that $\upsilon(z)$ must be an automorphism. And by proposition 2.3.2 $\omega(z)$ has the form as claimed.

For the converse assume $\upsilon(z)$ is an automorphism of Γ and $\omega = c\frac{\mathscr{Q}_\beta(z)}{\|\mathscr{Q}_\beta(z)\|}$ where $\upsilon(\beta) = 0$ *and* $|c| = 1$. Therefore, we can write $\upsilon(z) = \eta\frac{\beta - z}{1-\overline{\beta}z}$, where $|\eta| = 1$, and the Cowen auxiliary functions of $\upsilon(z)$ as

$$j(z) = \frac{1}{1 - \overline{\beta\eta}z}, \quad \tau(z) = \frac{\beta - \overline{\eta}z}{1 - \overline{\eta\beta}z}, \quad and \ \rho(z) = 1 - \overline{\beta}z.$$

Notice that $W_{\omega,\upsilon}f(z) = T_\omega C_\upsilon f(z)$, and $\tau(z) = \upsilon^{-1}(z)$ we can see that

$$\begin{aligned} j \circ \upsilon(z) &= \frac{1}{1 - \overline{\beta\eta}.\eta\frac{\beta - z}{1-\overline{\beta}z}} \\ &= \frac{1 - \overline{\beta}z}{1 - \overline{\beta}z - |\beta|^2|\eta|^2 + \overline{\beta}|\eta|^2 z} \\ &= \frac{1 - \overline{\beta}z}{1 - |\beta|^2} \qquad\qquad (since \ |\eta| = 1) \\ &= \|\mathscr{Q}_\beta\|^2 / K_\beta \end{aligned}$$

$$\begin{aligned} \omega \circ \tau(z) &= c\frac{\mathscr{Q}_\beta(\tau(z))}{\|\mathscr{Q}_\beta(z)\|} \\ &= c\frac{\frac{1}{1-\overline{\beta}\frac{\beta - \overline{\eta}z}{1-\overline{\eta\beta}z}}}{\|\mathscr{Q}(z)_\beta\|} \\ &= c\left(\frac{1 - \overline{\eta\beta}z}{1 - \overline{\eta\beta}z - |\beta|^2 + \overline{\eta\beta}z}\right)\left(\frac{1}{\|\mathscr{Q}_\beta(z)\|}\right) \\ &= c\left(\frac{1}{1 - |\beta|^2}\right)\left(\frac{1 - \overline{\eta\beta}z}{\|\mathscr{Q}_\beta(z)\|}\right) \\ &= c\frac{\|\mathscr{Q}_\beta(z)\|}{j(z)} \end{aligned}$$

and

$$\begin{aligned} T_\rho^* T_\omega^* &= \overline{\rho\omega} \\ &= (1-\beta\overline{z}) \,.\, \overline{c}\, \frac{\mathscr{Q}_\beta}{\|\mathscr{Q}_\beta\|} \\ &= \frac{\overline{c}}{\|\mathscr{Q}_\beta\|}. \end{aligned}$$

On the one hand, we have

$$\begin{aligned} W_{\omega,\upsilon} W_{\omega,\upsilon}^* f(z) &= T_\omega C_\upsilon \big(T_\omega C_\upsilon\big)^* f(z) \\ &= T_\omega C_\upsilon C_\upsilon^* T_\omega^* f(z) \\ &= T_\omega C_\upsilon T_j C_\tau T_\rho^* T_\omega^* f(z) \\ &= \omega(z) \,.\, \Big(j \circ \upsilon\Big)(z) \,.\, C_{\tau\circ\upsilon}\bigg(\frac{\overline{c}}{\|\mathscr{Q}_\beta(z)\|} \,.\, f(z)\bigg) \;\; (by\ corollary\ 2.2.0.1) \\ &= \bigg(c\frac{\mathscr{Q}_\beta(z)}{\|\mathscr{Q}_\beta(z)\|}\bigg).\bigg(\frac{\|\mathscr{Q}_\beta(z)\|^2}{\mathscr{Q}_\beta(z)}\bigg).\bigg(\frac{\overline{c}}{\|\mathscr{Q}_\beta(z)\|}\bigg).f(I(z)) \\ &= f(z) \end{aligned}$$

on the other hand

$$
\begin{aligned}
W^*_{\omega,\upsilon} W_{\omega,\upsilon} f(z) &= \big(T_\omega C_\upsilon\big)^* T_\omega C_\upsilon f(z) \\
&= C^*_\upsilon T^*_\omega T_\omega C_\upsilon f(z) \\
&= T_j C_\tau T^*_\rho T^*_\omega T_\omega C_\upsilon f(z) \\
&= j(z)\ .\ \Big(\omega\circ\tau\Big)(z)\ .\ C_{\upsilon\circ\tau}\Big(\frac{\bar{c}}{\|\mathscr{Q}_\beta(z)\|}\ .\ f(z)\Big) \\
&= \big(j(z)\big)\Big(c\frac{\|\mathscr{Q}_\beta(z)\|}{j(z)}\Big).\Big(\frac{\bar{c}}{\|\mathscr{Q}_\beta(z)\|}\Big).\Big(f(I(z))\Big) \\
&= f(z).
\end{aligned}
$$

Thus $W_{\omega,\upsilon} f(z)$ is unitary on $H^2(\Gamma)$. $\square$

2.4 Normal weighted composition operators

This section is the main object of this thesis. Firstly, we will see the form of the weighted function $\omega(z)$ while the $W_{\omega,\upsilon}$ is normal. Next, we will examine the forms of $\upsilon(z)$ that imply $W_{\omega,\upsilon}$ to be normal.

Proposition 2.4.1 *[4] Assume that $W_{\omega,\upsilon} f(z)$ is normal and $p_0 \in \Gamma$ is a fixed point of $\upsilon(z)$ then*

$$\omega(z) = \omega(p_0)\frac{\mathscr{Q}_{p_0}(z)}{\mathscr{Q}_{p_0}\big(\upsilon(z)\big)}.$$

Proof. Since $W_{\omega,\upsilon}$ is bounded and $p_0 \in \Gamma$, lemma 2.1.1 implies that $W^*_{\omega,\upsilon}\mathscr{Q}_{p_0} = \overline{\omega(p_0)}\mathscr{Q}_{\upsilon(p_0)=p_0}$, so that $\mathscr{Q}_{p_0}$ is the eigenvector for $W^*_{\omega,\upsilon}$ with $\overline{\omega(p_0)}$ as an eigenvalue. But proposition 1.2.19 tells us that an eigenvector of a normal operator is an eigenvector of its adjoint operator with the corresponding eigenvalue that is the conjugate of its eigenvalue, so $W_{\omega,\upsilon}$ has $\mathscr{Q}_{p_0}$

as an eigenvector with $\omega(p_0)$ as an eigenvalue so that

$$\begin{aligned}\omega(p_0)\mathscr{Q}_{p_0}(z) &= W_{\omega,\upsilon}\mathscr{Q}_{p_0}(z)\\ &= \omega(z)\mathscr{Q}_{p_0}\big(\upsilon(z)\big).\end{aligned}$$

□

Corollary 2.4.1.1 *[4] Let $\upsilon(0) = 0$. Then $W_{\omega,\upsilon}f(z)$ is normal $\iff$ $\omega(z)$ is constant and $C_\upsilon f(z)$ is normal for all $f(z) \in \mathscr{H}^2(\Gamma)$.*

Proof. Let $\upsilon(0) = 0$. On the one hand, proposition 2.4.1 showed that if $W_{\omega,\upsilon}f(z)$ is normal then

$$\omega(z) = \omega(0)\frac{\mathscr{Q}_0(z)}{\mathscr{Q}_0(\upsilon(z))} = \omega(0),$$

which is constant. On the other hand, if there exists $q_0 \in \Gamma$ such that $\omega(z) = q_0$, then $W_{\omega,\upsilon}f(z)$ is normal when and only when

$W_{\omega,\upsilon}W^*_{\omega,\upsilon}f(z) = W^*_{\omega,\upsilon}W_{\omega,\upsilon}f(z)$

$\iff T_\omega C_\upsilon C^*_\upsilon T^*_\omega f(z) = C^*_\upsilon T^*_\omega T_\omega C_\upsilon f(z)$

$\iff (q_0)C_\upsilon C^*_\upsilon(\overline{q_0})f(z) = C^*_\upsilon(\overline{q_0})(q_0)C_\upsilon f(z),$

$\iff (q_0)(\overline{q_0})C_\upsilon C^*_\upsilon f(z) = (\overline{q_0})(q_0)C^*_\upsilon C_\upsilon f(z),$

when and only when $C_\upsilon f(z)$ is normal operator for all $f(z) \in \mathscr{H}^2(\Gamma)$. □

Theorem 2.4.2 *[4] Suppose $\upsilon(z)$ has a fixed point p_0 in Γ. Then $W_{\omega,\upsilon}f(z)$ is normal when and only when*

$$\omega(z) = q_0\frac{\mathscr{Q}_{p_0}(z)}{\big(\mathscr{Q}_{p_0}\circ\upsilon\big)(z)} \;\; \textit{and} \;\; \upsilon(z) = \big(\alpha\circ(q_1\alpha)\big)(z)$$

where $\alpha(z) = (p_0 - z)/(1 - \overline{p_0}z)$, and q_1, q_0 are constants with $|q_1| \leq 1$.

Proof. Assume $W_{\omega,\upsilon}f(z)$ is normal, and there is $p_0 \in \Gamma$ such that $\upsilon(p_0) = p_0$, then proposition 2.4.1 implies that

$$\omega(z) = \omega(p_0)\frac{\mathscr{Q}_{p_0}(z)}{\mathscr{Q}_{p_0} \circ \upsilon(z)},$$

let $q_0 = \omega(p_0)$ that is constant, and the first part done.

It remains to show that $\upsilon(z) = \big(\alpha \circ (q_1\alpha)\big)(z)$, to prove that let

$$\delta(z) = \frac{\mathscr{Q}_{p_0}(z)}{\|\mathscr{Q}_{p_0}(z)\|},$$

since $\alpha(z)$ is an automorphism with $\alpha(p_0) = \frac{p_0 - p_0}{1 - \overline{p_0}p_0} = 0$, theorem 2.3.3 implies that $W_{\delta,\alpha}f(z)$ is unitary, and hence the operator L which defined as

$$Lf(z) := \big(W^*_{\delta,\alpha}\big)\big(W_{\omega,\upsilon}\big)(W_{\delta,\alpha})(f(z))$$

is normal.

Let $j(z)$, $\rho(z)$, and $\tau(z)$ be the Cowen auxiliary functions for $\alpha(z)$, then

$$\tau(z) = \alpha(z) = \alpha^{-1}(z),\ j(z) = \frac{1}{1 - \overline{p_0}z},\ and\, \rho(z) = 1 - \overline{p_0}z.$$

Observe that

$$C^*_\alpha f(z) = T_j C_{\tau=\alpha} T^*_\rho f(z)$$

and

$$\begin{aligned}(T_\delta T_\rho)^* &= T^*_\rho T^*_\delta \\ &= T_{\overline{h}} T_{\overline{\delta}} \\ &= \overline{(1 - \overline{p_0}z)}\overline{\left(\frac{K_{p_0}(z)}{\|\mathscr{Q}_{p_0}(z)\|}\right)} \\ &= \frac{1}{\|\mathscr{Q}_{p_0}(z)\|}.\end{aligned}$$

Therefore,

$$\begin{aligned}
L &= W^*_{\delta,\alpha} W_{\omega,\upsilon} W_{\delta,\alpha}(f(z)) \\
&= C^*_\alpha T^*_\delta T_\omega C_\upsilon T_\delta C_\alpha(f(z)) \\
&= T_j C_\alpha T^*_\rho T^*_\delta T_\omega C_\upsilon T_\delta C_\alpha(f(z)) \\
&= \frac{1}{\|\mathscr{Q}_{p_0}\|} T_j C_\alpha T_\omega C_\upsilon T_\delta C_\alpha(f(z)) \\
&= \frac{1}{\|\mathscr{Q}_{p_0}\|} T_{j.(\omega\circ\alpha)} C_{\upsilon\circ\alpha} T_\delta C_\alpha(f(z)) \qquad (by\ corollary(2.2.0.1)) \\
&= \frac{1}{\|\mathscr{Q}_{p_0}\|} T_{j.(\omega\circ\alpha).(\delta\circ\upsilon\circ\alpha)} C_{\alpha\circ\upsilon\circ\alpha}(f(z)),
\end{aligned}$$

and since

$$\begin{aligned}
(\alpha\circ\upsilon\circ\alpha)(0) &= \alpha\Big(\upsilon\big(\alpha(0)\big)\Big) \\
&= \alpha\big(\upsilon(p_0)\big) \\
&= \alpha(p_0) = 0
\end{aligned}$$

we get L is a weighted composition operator with a weighted function $j.(\omega\circ\alpha).(\delta\circ\upsilon\circ\alpha) \in \mathscr{H}^\infty$ inducing a function $\alpha\circ\upsilon\circ\alpha$ fixes 0.

Since L is normal corollary 2.4.1.1 implies that $j.(\omega\circ\alpha).(\delta\circ\upsilon\circ\alpha)$ is constant, and $C_{\alpha\circ\upsilon\circ\alpha}$ is normal. Schwartz's theorem 2.1.3 showed that since $C_{\alpha\circ\upsilon\circ\alpha}$ is normal on $\mathscr{H}^2(\Gamma)$, there must be a constant q_1 with $|q_1| \le 1$ such that $\alpha\circ\upsilon\circ\alpha = q_1 z$

$\implies \alpha^{-1}\big(\alpha\circ\upsilon\circ\alpha(z)\big)(z) = \alpha^{-1}\big(q_1 z\big)$

$\implies \upsilon\circ\alpha(z) = \alpha\big(q_1 z\big)$, take $z = \alpha(z)$

$\implies \upsilon(z) = \alpha(q_1\alpha(z)) = \alpha\circ(q_1\alpha)(z).$

To show that $j.(\omega \circ \alpha).(\delta \circ \upsilon \circ \alpha)$ is constant, this can be yielded by

$$
\begin{aligned}
j.(\omega \circ \alpha).(\delta \circ \upsilon \circ \alpha)(z) &\equiv \left(\frac{1}{1-\overline{p_0}z}\right)\left(\omega(p_0)\frac{\mathscr{Q}_{p_0}\big(\alpha(z)\big)}{\mathscr{Q}_{p_0}\big(\upsilon \circ \alpha\big)(z)}\right)\left(\frac{\mathscr{Q}_{p_0}\big(\upsilon \circ \alpha\big)(z)}{\big\|\mathscr{Q}_{p_0}\big(\upsilon \circ \alpha\big)(z)\big\|}\right) \\
&\equiv \left(\frac{1}{1-\overline{p_0}z}\right)\left(\omega(p_0)\frac{1-\overline{p_0}z}{1-\overline{p_0}z-|p_0|^2+\overline{p_0}z}\right)\left(\frac{1}{\big\|\mathscr{Q}_{p_0}(z)\big\|}\right) \\
&\equiv \omega(p_0)\big\|\mathscr{Q}_{p_0}(z)\big\|^2\frac{1}{\big\|\mathscr{Q}_{p_0}(z)\big\|} \equiv \omega(p_0)\big\|\mathscr{Q}_{p_0}(z)\big\|
\end{aligned}
$$

Conversely, suppose that $\upsilon(z)$ has a fixed point $p_0 \in \Gamma$, and let

$$
\omega = q_0\frac{\mathscr{Q}_{p_0}(z)}{(\mathscr{Q}_{p_0} \circ \upsilon)(z)} \quad \textit{and} \quad \upsilon(z) = (\alpha \circ q_1\alpha)(z)
$$

where $\alpha(z) = (p_0 - z)/(1 - \overline{p_0}z)$, and q_0 and q_1 are constants, with $|q_1| \leq 1$, we want to prove that $W_{\omega,\upsilon}$ is normal. We have

$$
\begin{aligned}
\upsilon(z) &= \alpha\big(q_1\alpha(z)\big) \\
&= \frac{p_0 - q_1\alpha(z)}{1-\overline{p_0}q_1\alpha(z)} \\
&= \frac{p_0 - q_1\frac{p_0-z}{1-\overline{p_0}z}}{1-\overline{p_0}q_1\frac{p_0-z}{1-\overline{p_0}z}} \\
&= \frac{p_0 - |p_0|^2z - q_1p_0 + q_1z}{1-\overline{p_0}z - q_1|p_0|^2 + q_1\overline{p_0}z} \\
&= \frac{p_0(1-q_1) + (q_1 - |p_0|^2)z}{1-q_1|p_0|^2 + (q_1-1)\overline{p_0}z}.
\end{aligned}
$$

Since $\omega(z) = q_0 \frac{\mathscr{Q}_{p_0}(z)}{\big(\mathscr{Q}_{p_0} \circ \upsilon\big)(z)}$ W.L.O.G let $q_0 = 1$, we get

$$\begin{aligned}
\omega(z) &= \frac{\mathscr{Q}_{p_0}(z)}{\big(\mathscr{Q}_{p_0} \circ \upsilon\big)(z)} \\
&= \frac{\frac{1}{1-\overline{p_0}z}}{\frac{1}{1-\overline{p_0}\upsilon(z)}} \\
&= \frac{1-\overline{p_0}\upsilon(z)}{1-\overline{p_0}z} \\
&= \frac{1-\overline{p_0}\frac{p_0(1-q_1)+(q_1-|p_0|^2)z}{1-q_1|p_0|^2+(q_1-1)\overline{p_0}z}}{1-\overline{p_0}z} \\
&= \frac{1-q_1|p_0|^2+q_1\overline{p_0}z-\overline{p_0}z-|p_0|^2+|p_0|^2q_1-\overline{p_0}q_1z+\overline{p_0}|p_0|^2z}{\big(1-q_1|p_0|^2+(q_1-1)\overline{p_0}z\big)\big(1-p_0z\big)} \\
&= \frac{\big(1-|p_0|^2\big)\big(1-\overline{p_0}z\big)}{\big(1-|p_0|^2q_1-\overline{p_0}z(1-q_1)\big)\big(1-\overline{p_0}z\big)} \\
&= \frac{1-|p_0|^2}{1-|p_0|^2q_1-\overline{p_0}z(1-q_1)}
\end{aligned}$$

By Schwartz's theorem, we have C_{q_1z}, $|q_1| \leq 1$ is normal. To show that $W_{\omega,\upsilon}$ is normal, it is enough to show $W_{\omega,\upsilon} = W_{\delta,\alpha}C_{q_1z}W^*_{\delta,\alpha}$, that is unitarily equivalent to a normal operator, and hence is normal. Observe that,

$$\begin{aligned}
W^*_{\delta,\alpha}f(z) &= \big(T_\delta C_\alpha\big)^* f(z) \\
&= C^*_\alpha T^*_\delta f(z) \\
&= T_j C_\alpha T^*_\rho T^*_\delta f(z) \\
&= \frac{1}{\|\mathscr{Q}_{p_0}(z)\|} T_j C_\alpha f(z)
\end{aligned}$$

Now we can see that

$$\begin{aligned} W_{\delta,\alpha} C_{q_1 z} W^*_{\delta,\alpha}(f(z)) &= T_\delta C_\alpha C_{q_1 z} T_j C_\alpha T^*_\rho T^*_\delta f((z)) \\ &= \frac{1}{\|\mathscr{Q}_{p_0}(z)\|} T_\delta C_\alpha C_{q_1 z} T_j C_\alpha (f(z)) \\ &= \frac{1}{\|\mathscr{Q}_{p_0}(z)\|} T_\delta C_{q_1 z \circ \alpha} T_j C_\alpha (f(z)) \\ &= W_{\frac{1}{\|\mathscr{Q}_{p_0}(z)\|}\delta.(j\circ q_1 z\circ\alpha),\,(\alpha\circ q_1 z\circ\alpha)}(f(z)) \qquad (by\ corollary\ 2.2.0.1) \end{aligned}$$

It remain to show that $\omega(z) = \frac{1}{\|\mathscr{Q}_{p_0}(z)\|}\delta.(j \circ q_1 z \circ \alpha)(z)$ and $\upsilon = (\alpha \circ q_1 z \circ \alpha)(z)$ as following:

$$\begin{aligned} \frac{1}{\|\mathscr{Q}_{p_0}(z)\|}\delta.(j \circ q_1 z \circ \alpha)(z) &= \frac{1}{\|\mathscr{Q}_{p_0}(z)\|} \frac{K_p}{\|\mathscr{Q}_{p_0}(z)\|} j(q_1 \alpha) \\ &= \mathscr{Q}_{p_0}(z)(1 - |p_0|^2) \frac{1}{1 - \overline{p_0} q_1 \frac{p_0 - z}{1 - \overline{p_0} z}} \\ &= \frac{1}{1 - \overline{p_0} z} . (1 - |p_0|^2) . \frac{1 - \overline{p_0} z}{1 - \overline{p_0} z - |p_0|^2 q_1 + \overline{p_0} q_1 z} \\ &= \frac{1 - |p_0|^2}{1 - \overline{p_0} z - |p_0|^2 q_1 + \overline{p_0} q_1 z} = \omega(z). \end{aligned}$$

Clearly,

$$\begin{aligned} \alpha \circ q_1 z \circ \alpha(z) &= \alpha \circ (q_1 \alpha(z)) \\ &= \upsilon(z), \end{aligned}$$

then

$$W_{\omega,\upsilon}(f(z)) = W_{\delta,\alpha} C_{q_1 z} W^*_{\delta,\alpha} f(z),$$

because if an operator was normal and another operator was unitarily equivalent to it, then the second one will also be normal, hence $W_{\omega,\upsilon} f(z)$ is also a normal operator on $\mathscr{H}^2$. □

Proposition 2.4.3 *[4] Suppose $\upsilon : \Gamma \to \Gamma$ is a linear fractional transformation such that*

$\upsilon(z) = \frac{az+b}{cz+d}$, *and let* $\omega(z) = \mathscr{Q}_{\tau(0)}(z)$ *where* $\tau(z) = \frac{\overline{a}z-\overline{c}}{\overline{d}-\overline{b}z}$. *Then* $W_{\omega,\upsilon}f(z)$ *is normal when and only when*

$$\frac{|d|^2}{|d|^2-|b|^2-(\overline{b}a-\overline{d}c)z}C_{\tau\circ\upsilon} = \frac{|d|^2}{|d|^2-|c|^2-(\overline{b}d-c\overline{a})z}C_{\upsilon\circ\tau}.$$

Proof. ($\Longrightarrow$) Suppose $W_{\omega,\upsilon}$ is normal this implies that $W_{\omega,\upsilon}W^*_{\omega,\upsilon} = W^*_{\omega,\upsilon}W_{\omega,\upsilon}$. For the right-hand side, we have

$$\begin{aligned}
W_{\omega,\upsilon}W^*_{\omega,\upsilon}f(z) &= T_\omega C_\upsilon \big(T_\omega C_\upsilon\big)^* f(z) \\
&= T_\omega C_\upsilon C^*_\upsilon T^*_\omega f(z) \\
&= T_\omega C_\upsilon T_j C_\tau T^*_\rho T^*_\omega f(z) \\
&= \overline{d}\ T_\omega C_\upsilon T_j C_\tau f(z) \\
&= \overline{d}\ \omega\ j\circ\upsilon C_{\tau\circ\upsilon} f(z) \\
&= \overline{d}\left(\frac{d}{d+cz}\right)\left(\frac{1}{\overline{d}-\overline{b}\frac{az+b}{cz+d}}\right)C_{\tau\circ\upsilon}f(z) \\
&= \overline{d}\left(\frac{d}{d+cz}\right)\left(\frac{cz+d}{\overline{d}cz+|d|^2-a\overline{b}z-|b|^2}\right)C_{\tau\circ\upsilon}f(z) \\
&= \frac{|d|^2}{|d|^2-|b|^2-(a\overline{b}-c\overline{d})z}C_{\tau\circ\upsilon}f(z).
\end{aligned}$$

For the left-hand side, we have

$$
\begin{aligned}
W^*_{\omega,\upsilon} W_{\omega,\upsilon} f(z) &= T_j C_\tau T^*_\rho T^*_\omega T_\omega C_\upsilon f(z) \\
&= \overline{d} T_j C_\tau T_{\mathscr{Q}_{\tau(0)}} C_\upsilon f(z) \\
&= \overline{d}\, j \left(\mathscr{Q}_{\tau(0)} \circ \tau\right) C_{\upsilon\circ\tau} f(z) \\
&= \overline{d}\left(\frac{1}{\overline{d} - \overline{b}z}\right)\left(\frac{d}{d + (c)(\tau(z))}\right) C_{\upsilon\circ\tau} f(z) \\
&= \overline{d}\left(\frac{1}{\overline{d} - \overline{b}z}\right)\left(\frac{d}{d + c\frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}}\right) C_{\upsilon\circ\tau} f(z) \\
&= \overline{d}\left(\frac{1}{\overline{d} - \overline{b}z}\right)\left(\frac{d(\overline{d} - \overline{b}z)}{-d\overline{b}z + |d|^2 + c\overline{a}z - |c|^2}\right) C_{\upsilon\circ\tau} f(z) \\
&= \frac{|d|^2}{|d|^2 - |c|^2 - (\overline{b}d - c\overline{a})z} C_{\upsilon\circ\tau} f(z)
\end{aligned}
$$

Since $W^*_{\omega,\upsilon} W_{\omega,\upsilon} = W_{\omega,\upsilon} W^*_{\omega,\upsilon}$ this implies

$$
\frac{|d|^2}{|d|^2 - |b|^2 - (\overline{b}a - \overline{d}c)z} C_{\tau\circ\upsilon} = \frac{|d|^2}{|d|^2 - |c|^2 - (\overline{b}d - c\overline{a})z} C_{\upsilon\circ\tau}.
$$

Where τ as given, $\rho(z) = cz + d$, and $j(z) = 1/(-\overline{b}z + \overline{d})$ and since

$$
\begin{aligned}
\omega(z) &= \mathscr{Q}_{\tau(0)}(z) \\
&= \frac{1}{1 - \overline{\left(\frac{-\overline{c}}{\overline{d}}\right)} z} \qquad\qquad \left(\tau(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}\right) \\
&= \frac{d}{d + cz}.
\end{aligned}
$$

then

$$\begin{aligned} T_\rho^* T_\omega^* &= \left(T_\omega T_\rho\right)^* \\ &= T_\omega^* \rho \\ &= T_{\overline{\omega\rho}} \\ &= \overline{\omega\rho} = \overline{\left(\frac{d}{d+cz}\right)} \overline{cz+d} = \overline{d}. \end{aligned}$$

($\Longleftarrow$) Conversely, suppose

$$\frac{|d|^2}{|d|^2 - |b|^2 - (\overline{b}a - \overline{d}c)z} C_{\tau\circ\upsilon} = \frac{|d|^2}{|d|^2 - |c|^2 - (\overline{b}d - c\overline{a})z} C_{\upsilon\circ\tau}$$

then $W_{\omega,\upsilon} W_{\omega,\upsilon}^* = W_{\omega,\upsilon}^* W_{\omega,\upsilon}$, as we have seen in the first direction

$$\frac{|d|^2}{|d|^2 - |c|^2 - (\overline{b}d - c\overline{a})z} C_{\upsilon\circ\tau} f(z) = W_{\omega,\upsilon}^* W_{\omega,\upsilon} f(z)$$

and

$$\frac{|d|^2}{|d|^2 - |b|^2 - (\overline{b}a - \overline{d}c)z} C_{\tau\circ\upsilon} f(z) = W_{\omega,\upsilon} W_{\omega,\upsilon}^* f(z).$$

Hence $W_{\omega,\upsilon}$ is normal. □

Proposition 2.4.4 *[4] Suppose that $\upsilon : \Gamma \to \Gamma$ is a linear-fractional transformation of parabolic type and let $\omega(z) = \mathscr{Q}_{\tau(0)}(z)$ then $W_{\omega,\upsilon} f(z)$ is normal.*

Chapter Three

Composition operator on K_b

In this chapter, we will define the model subspace K_b and study the conditions for C_v to map K_b into itself, then we will find the adjoint of C_v on K_b, and prove C_v is normal on K_b Af and Anly Af C_v As Aormal An $\mathscr{H}$.

3.1 Introduction to Model Subspace

Beurling's Aheorem A949 characterizes Ahe closed Aubspaces Af $\mathscr{H}$Ahat are Anvariant un-der Ahe Aorward Ahift Aperator S as $u\mathscr{H}^2$, Ahere u As an Anner Aunction An Γ [18]. The Arthogonal complement Af $u\mathscr{H}^2$ As Anvariant Ander Ahe backward Ahift S^* and Ahe corre-sponding Aodel Aubspace denoted as

$$K_u = \left\{ f \in \mathscr{H}^2 : \langle f, uh \rangle = 0, \forall h \in \mathscr{H}^2 \right\}.$$

Proposition 3.1.1 *[18] Let u be an inner function, then we can write the model sub-space K_u*

$$K_u = \left\{ u\overline{zh} : h \in \mathscr{H}^2 \right\} = \mathscr{H}^2 \cap u\overline{z\mathscr{H}^2}$$

almost everywhere on $\mathbb{Y}$.

Proposition 3.1.2 *[18] Let $u(z)$ be an inner function, and $v(z) \in \mathscr{H}^\infty$, then $T_{\overline{v}}K_u$ is a proper subset of K_u.*

Note that K_u is a reproducing kernel Hilbert space, which means that there exists a function $\kappa_\gamma^u(z)$ such that $f(\gamma) = \langle f, \kappa_\gamma^u \rangle$ [18], for every $f \in K_u$ and $\gamma \in \Gamma$ that is

$$\kappa_\gamma^u(z) = \frac{1 - \overline{u(\gamma)}u(z)}{1 - \overline{\gamma}z} \qquad z \in \Gamma. \tag{3.1}$$

The Blaschke product is an example of inner functions defined as follows:

Definition 3.1.3 [14] A finite Blaschke product is defined as

$$b(z) = \lambda z^{m_0} \prod_{i=1}^{n} \left(\frac{\gamma_i - z}{1 - \overline{\gamma_i} z} \right)^{m_i}$$

where $\left(\gamma_i\right)_{1 \le i \le n}$ are n different points in $\Gamma \setminus \{0\}$, $\left(m_i\right)_{0 \le i \le n}$ are non-negative finite sequence of integers, and $|\lambda| = 1$.

K_b is the finite-dimensional subspace generated by the finite Blaschke product $b(z)$ and spanned by

$$1,\ z^2,\ z^3,\ .....,\ z^{m_0},$$

and

$$\frac{1}{1 - \overline{\gamma_i} z},\ \frac{1}{\left(1 - \overline{\gamma_i} z\right)^2},\ ......,\ \frac{1}{\left(1 - \overline{\gamma_i} z\right)^{m_i}},\quad 1 \le i \le n.$$

In other words, $f(z)$ is in the model subspace K_b if and only if it is of the form [25]

$$f(z) = \sum_{k=0}^{m_0 - 1} c_{0,k} z^k + \sum_{i=1}^{n} \sum_{k=1}^{m_i} \frac{c_{i,k}}{\left(1 - \overline{\gamma_i} z\right)^k}. \tag{3.2}$$

3.2 $\mathcal{L}_C(K_b)$ Structure

In the following, we will consider that $\mathcal{L}_c(K_b)$ is the set of all bounded composition operators on K_b. To figure out $\mathcal{L}_c(K_b)$ structure in case that $b(z)$ has two zeros including the point $\{0\}$, and $b'(0) \neq 0$ we will introduce the following theorems. $N(b)$ stands for the group of zeros of $b(z)$, multiplicity not counted.

The first theorem involves looking at the single zero Blaschke products exclude the origin that can be repeated a finite number of times, i.e. the case $N(b) = \{\gamma\}$, $b'(0) \neq 0$ [25].

Theorem 3.2.1 *[25] Suppose that*

$$b(z) = \left(\frac{\gamma - z}{1 - \overline{\gamma} z}\right)^m$$

where $\gamma \neq 0$, $m \geq 1$, *then*

$$\mathcal{L}_c(K_b) = \left\{ C_{(1-\overline{\gamma}a_1)z+a_1} \,:\, a_1 \in \mathbb{C}, a_1 \neq \frac{1}{\overline{\gamma}} \right\}.$$

Proof. Let $C_\upsilon \in \mathcal{L}_c(K_b)$, on the one hand we have $C_\upsilon k_\gamma^m \in K_b$. Therefore, by (3.1) and (3.2), there exist some constants c_k so that

$$C_\upsilon \kappa_\gamma^m = \left(\frac{1 - \overline{b(\gamma)} b(z)}{1 - \overline{\gamma}\upsilon(z)}\right)^m = \sum_{k=1}^{m} \frac{c_k}{\big(1 - \overline{\gamma} z\big)^k},$$

but $\overline{b(\gamma)} = 0$ then

$$\frac{1}{\big(1 - \overline{\gamma}\upsilon(z)\big)^m} = \sum_{k=1}^{m} \frac{c_k}{\big(1 - \overline{\gamma} z\big)^k}.$$

This representation demonstrates there is a polynomial $p(z)$ of degree at most $m-1$ such that

$$p(z)\big(1 - \overline{\gamma}\upsilon(z)\big)^m = \big(1 - \overline{\gamma} z\big)^m$$

this implies that $p(z)$ must be constant and there exist $a_2, a_2 \in \mathbb{C}$ such that $\upsilon(z) = a_2 z + a_1$. On the other hand

$$\begin{aligned} C_\upsilon \kappa_\gamma &= \frac{1}{1 - \overline{\gamma}\upsilon(z)} \\ &= \frac{1}{1 - \overline{\gamma} a_1} \times \frac{1}{1 - \overline{\left(\frac{\overline{a_2}\gamma}{1 - \overline{a_1}\gamma}\right)} z} \\ &= \frac{c_1}{1 - \overline{\left(\frac{\overline{a_2}\gamma}{1 - \overline{a_1}\gamma}\right)} z} \end{aligned}$$

so that we must have $(1 - a_1\overline{\gamma}) \neq 0$, and $\gamma = \left(\frac{\overline{a_2}\gamma}{1-\overline{a_1}\gamma}\right)$, hence $a_2 = (1 - a_1\overline{\gamma})$. $\square$

If we only look for the symbols $\upsilon : \Gamma \to \Gamma$, we come up with the same characterization plus one additional condition $\left|1 - \overline{\gamma}a_1\right| + \left|a_1\right| \leq 1$. However, this inequality only applies for $a_1 = 0$. Therefore, $\upsilon(z) = z$ is only acceptable.

The second theorem considers the finite Blaschke products that have the origin as a simple zero and another zero with a multiplicity of 1 or more at $\lambda \neq 0$, i.e. the case $N(b) = \{0, \gamma\}$, $b'(0) \neq 0$.

Theorem 3.2.2 *[25] Suppose that*

$$b(z) = z\left(\frac{\gamma - z}{1 - \overline{\gamma}z}\right)^m$$

where $\gamma \neq 0$, $m \geq 1$, *then*

$$\mathcal{L}_c(K_b) = \left\{C_{\frac{a_1z+a_2}{a_3z+1}} : a_1 \in \mathbb{C},\, a_2 \in \mathbb{C}\backslash\frac{1}{\overline{\gamma}},\, a_3 = (a_1 - 1)\overline{\gamma} + a_2\overline{\gamma}^2\right\}.$$

Proof. The proof is similar to the proof of theorem 3.2.1. Let $C_\upsilon \in \mathcal{L}_c(K_b)$, so that $C_\upsilon \kappa_\gamma^m \in K_b$. Therefore, by (3.1) and (3.2), there exist some constants c_k so that

$$C_\upsilon \kappa_\gamma^m = \left(\frac{1 - \overline{b(\gamma)}b(\upsilon(z))}{1 - \overline{\gamma}\upsilon(z)}\right)^m = c_0 + \sum_{k=1}^{m} \frac{c_k}{\left(1 - \overline{\gamma}z\right)^k},$$

but $\overline{b(\gamma)} = 0$ then

$$\frac{1}{\left(1 - \overline{\gamma}\upsilon(z)\right)^m} = c_0 + \sum_{k=1}^{m} \frac{c_k}{\left(1 - \overline{\gamma}z\right)^k}$$

this implies that by binomial theorem for some constant q

$$\left(1 - \overline{\gamma}\upsilon(z)\right)^m = \frac{\left(1 - \overline{\gamma}z\right)^m}{\left(q - \overline{\gamma}z\right)^m}.$$

This formula shows that $\upsilon(z)$ is a linear fractional transformation that is normalized as $\upsilon(z) = \frac{a_1z+a_2}{a_3z+1}$.

On the other hand,

$$\begin{aligned}
C_\upsilon \kappa_\gamma &= \frac{1}{1-\overline{\gamma}\upsilon(z)} \\
&= \frac{1}{1-\overline{\gamma}\frac{a_1 z + a_2}{a_3 z + 1}} \\
&= \frac{a_3 z + 1}{a_3 z + 1 - \overline{\gamma} a_1 z - \overline{\gamma} a_2} \\
&= \frac{\frac{a_3 z + 1}{1-\gamma\overline{a_2}}}{1-\overline{\left(\frac{\gamma\overline{a_1}-\overline{a_3}}{1-\gamma\overline{a_2}}\right)}z}
\end{aligned}$$

but equation (3.2) shows that γ must be as follows

$$\gamma = \left(\frac{\gamma\overline{a_1}-\overline{a_3}}{1-\gamma\overline{a_2}}\right).$$

Therefore, $a_2 \neq \frac{1}{\overline{\gamma}}$, and $a_3 = (a_1 - 1)\overline{\gamma} + a_2\overline{\gamma}^2$. □

Keep in mind that if $a_1 = 1 - \overline{\gamma} a_2$ that is $a_3 = 0$, we get the subclass $\upsilon(z) = (1-\overline{\gamma}a_2)z + a_2$, where $a_2 \neq \frac{1}{\overline{\gamma}}$.

Corollary 3.2.2.1 *[25] Suppose that*

$$b(z) = z\left(\frac{\gamma - z}{1-\overline{\gamma}z}\right)^m$$

where $\gamma \neq 0$, $m \geq 1$, and let $\upsilon(z)$ be a self map analytic function on Γ. Then C_υ is self map operator on K_b if and only if

$$\upsilon(z) \equiv a_2, \quad a_2 \in \Gamma,$$

or

$$\upsilon(z) = \frac{a_1 z + a_2}{a_3 z + 1},$$

where $a_3 = (a_1 - 1)\overline{\gamma} + a_2\overline{\gamma}^2$ for $a_1 \in \mathbb{C}$, and $a_2 \in \Gamma$ with the condition $\left|a_1 - a_2 a_3\right| +$

$\left|a_3 - a_1\overline{a_2}\right| \leq 1 - |\overline{a_2}|^2$.

Proof. Theorem (3.2.2) tells that $\upsilon(z) = \frac{a_1 z + a_2}{a_3 z + 1}$, with $a_1 \in \mathbb{C}$, $a_2 \in \mathbb{C}\{\frac{1}{\gamma}\}$, and $a_3 = (a_1 - 1)\overline{\gamma} + a_2\overline{\gamma}^2$. In case $a_2 \in \Gamma$, then we have $\upsilon(0) = a_2$, so that if $a_1 = a_2 a_3$ then $\upsilon(z) \equiv b$. Now assume that $a_1 \neq a_2 a_3$, and let the disc automorphism be $\sigma_{a_2}(z) = \frac{a_2 - z}{1 - \overline{a_2} z}$, then $\upsilon(z)$ is a self map of the unit disc if and only if so does $(\sigma_{a_2} \circ \upsilon)(z)$

$$
\begin{aligned}
(\sigma_{a_2} \circ \upsilon)(z) &= \frac{a_2 - \frac{a_1 z + a_2}{a_3 z + 1}}{1 - \overline{a_2}\frac{a_1 z + a_2}{a_3 z + 1}} \\
&= \frac{a_2 a_3 z + a_2 - a_1 z - a_2}{a_3 z + 1 - a_1 \overline{a_2} z - |a_2|^2} \\
&= \frac{(a_2 a_3 - a_1) z}{(a_3 - a_1\overline{a_2}) z + (1 - |a_2|^2)}.
\end{aligned}
$$

Schwarz's lemma tells that $(\sigma_{a_2} \circ \upsilon)(z)$ maps Γ into itself if and only if $|z| \leq 1$ implies

$$
\left| \frac{(a_2 a_3 - a_1)}{(a_3 - a_1\overline{a_2}) z + (1 - |a_2|^2)} \right| \leq 1
$$

if and only if

$$
\left|(a_1 - a_2 a_3)\right| \leq (1 - |a_2|^2) - \left|(a_3 - a_1\overline{a_2})\right|
$$

and

$$
\left|(a_3 - a_1\overline{a_2})\right| \leq 1 - |a_2|^2
$$

but the first inequality implies the second one. □

3.3 Adjoint and normality of Composition Operators on K_u

In this section, we will find the adjoint of the composition operator C_υ on the model space K_b, and we will check the normality of Certain υ's.

We can consider the model space K_u as a closed subspace of the Hilbert space L^2 over the non-tangential boundary values. Therefore, the orthogonal projection P_u of L^2 onto

K_u can given as follow:

Proposition 3.3.1 *[18] Let $f(z) \in L^2$, and $\gamma \in \Gamma$, then $P_u f(\gamma) = \langle f, \kappa_\gamma \rangle$. Furthermore, $\kappa_\gamma(z) = P_u \mathscr{Q}_\gamma(z)$, where $\mathscr{Q}_\gamma(z)$ is the reproducing kernel on $\mathscr{H}^2$.*

Proof. We know that $\kappa_\gamma(z) \in K_u$, so that $P_u \kappa_\gamma(z) = \kappa_\gamma(z)$, and

$$\begin{aligned} \langle f, \kappa_\gamma \rangle &= \langle f, P_u \kappa_\gamma \rangle \\ &= \langle P_u f, \kappa_\gamma \rangle \\ &= P_u f(\gamma). \end{aligned}$$

To show that $\kappa_\gamma(z) = P_u \mathscr{Q}_\gamma(z)$, where $\mathscr{Q}_\gamma(z)$ is the reproducing kernel on $\mathscr{H}^2$, let $f(z) \in K_u$ then

$$\begin{aligned} \langle f, \kappa_\gamma \rangle &= f(\gamma) \\ &= \langle f, \mathscr{Q}_\gamma \rangle \qquad\qquad since\ f(z) is\ also\ in\ \mathscr{H}^2 \\ &= \langle P_u f, \mathscr{Q}_\gamma \rangle \\ &= \langle f, P_u \mathscr{Q}_\gamma \rangle. \end{aligned}$$

□

The Riesz projection P, that is is the orthogonal projection of L^2 onto $\mathscr{H}^2$, and P_u have the following straightforward relationship.

Proposition 3.3.2 *([18]) Let $f(z)$ be a function in $\mathscr{H}^2$ then*

$$P_u f(z) = f(z) - uP(\overline{u} f)(z).$$

Proof. By proposition 3.3.1 for all $f(z) \in \mathscr{H}^2$ and $\gamma \in \Gamma$ we have

$$
\begin{aligned}
P_u f(\gamma) &= \langle f, \kappa_\gamma \rangle \\
&= \left\langle f, \frac{1-\overline{u(\gamma)}u}{1-\overline{\gamma}z} \right\rangle \\
&= \left\langle f, \frac{1}{1-\overline{\gamma}z} \right\rangle - \left\langle f, \frac{\overline{u(\gamma)}u}{1-\overline{\gamma}z} \right\rangle \\
&= \left\langle f, \frac{1}{1-\overline{\gamma}z} \right\rangle - u(\gamma)\left\langle f, \frac{u}{1-\overline{\gamma}z} \right\rangle \\
&= \left\langle f, \frac{1}{1-\overline{\gamma}z} \right\rangle - u(\gamma)\left\langle \overline{u}f, \frac{1}{1-\overline{\gamma}z} \right\rangle \\
&= \left\langle f, \frac{1}{1-\overline{\gamma}z} \right\rangle - u(\gamma)\left\langle \overline{u}f, P\left(\frac{1}{1-\overline{\gamma}z}\right) \right\rangle \\
&= \left\langle f, \frac{1}{1-\overline{\gamma}z} \right\rangle - u(\gamma)\left\langle P\big(\overline{u}f\big), \frac{1}{1-\overline{\gamma}z} \right\rangle \\
&= f(\gamma) - u(\gamma)P\big(\overline{u}f\big)(\gamma).
\end{aligned}
$$

□

Now we are ready to derive the adjoint of the composition operator on the model space K_b as follows:

Proposition 3.3.3 *Let $f(z) \in K_b$, and let $\mathscr{C}_v : K_b \to K_b$ be the composition operator then*

$$\mathscr{C}_v^* f(z) = (1-|b|^2)C_v^* f(z)$$

where $C_v f(z)$ is the composition operator defined on $\mathscr{H}^2$.

Proof. Let $C_v f(z)$ be the composition operator defined on $\mathscr{H}^2$, then $C_v^* f(z) \in \mathscr{H}^2$, so that proposition 3.3.2 tells us that

$$\begin{aligned}
\mathscr{C}_\upsilon^* f(z) &= P_b C_\upsilon^* f(z) \\
&= C_\upsilon^* f(z) - bP(\bar{b}C_\upsilon^*)(f(z)) \\
&= C_\upsilon^* f(z) - T_b T_b^* C_\upsilon^*(f(z)) && (by\ definition\ 1.2.26) \\
&= C_\upsilon^* f(z) - b\bar{b}C_\upsilon^*(f(z)) && (b(z) \in \mathscr{H}^\infty \implies T_b f = bf) \\
&= (1 - |b|^2)C_\upsilon^* f(z)
\end{aligned}$$

□

Example 1 *Let* $b(z) = z\frac{\frac{3}{4}-z}{1-\frac{3}{4}z}$, $f(z) \in K_b$*, and* $\upsilon(z) = \frac{\frac{1}{3}z}{1-\frac{1}{2}z}$,

find $\mathscr{C}_\upsilon^* f(z)$.

Solution. Since $f(z) \in K_b$ then by equation (3.2)

$$f(z) = c_1 + \frac{c_0}{1 - \frac{3}{4}z}$$

that can be rewritten as

$$f(z) = \frac{(4c_1 + 4c_0) - 3c_1 z}{4 - 3z}$$

also $b(z)$ and $\upsilon(z)$ can be rewritten as

$$b(z) = z\frac{3 - 4z}{4 - 3z} \qquad \upsilon(z) = \frac{2z}{6 - 3z}$$

so that

$$\tau(z) = \frac{1}{3}z + \frac{1}{2}.$$

Corollary (3.2.2.1) tells us that $\mathscr{C}_\upsilon f(z)$ maps K_b onto itself as $\upsilon(z) = \frac{\frac{1}{3}z}{1-\frac{1}{2}z}$ maps Γ into it self, where

$$\left|a_1 - a_2 a_3\right| + \left|a_3 - a_1 \overline{a_2}\right| \leq 1 - |\overline{a_2}|^2 \quad and \quad a_3 = (a_1 - 1)\overline{\gamma} + a_2 \overline{\gamma}^2.$$

Now we can use equation (2.1) and proposition (3.3.3) to find $\mathscr{C}_\upsilon^* f(z)$ that is

$$\begin{aligned}
\mathscr{C}_\upsilon^* f(z) &= (1 - |b|^2) C_\upsilon^* f(z) \\
&= \left[1 - \left(z\frac{3-4z}{4-3z}\right)\left(\overline{z}\frac{3-4\overline{z}}{4-3\overline{z}}\right)\right]. \\
&\left[\left(\frac{12z}{(2z+3)(6)}\right)\left(\frac{(4c_1+4c_0) - 3c_1\tau(z)}{4-3\tau(z)}\right) - \left(\frac{-3}{2z+3}\frac{4c_1+4c_0}{4}\right)\right] \\
&= \left[1 - \left(z\frac{3-4z}{4-3z}\right)\left(\overline{z}\frac{3-4\overline{z}}{4-3\overline{z}}\right)\right]. \\
&\left[\frac{-4c_1 z^2 + 4c_1 z + 10c_0 z + 15c_1 + 15c_0}{(2z+3)(-2z+5)}\right].
\end{aligned}$$

Example 2 *Let* $b(z) = z\frac{0.2-z}{1-0.2z}$*,* $f(z) \in K_b$*, and* $\upsilon(z) = \frac{0.64z+0.1}{1-0.068z}$*,*

find $\mathscr{C}_\upsilon^* f(z)$*.*

Solution. Observe that

$$\tau(z) = \frac{0.64z + 0.068}{1 - 0.1z}, \quad and \;\; f(z) = \frac{(c_1 + c_0) - 0.2c_1 z}{1 - 0.2z}$$

using equation (2.1) and proposition (3.3.3) we get

$$\mathscr{C}_v^* f(z) = (1-|b|^2)C_v^* f(z)$$

$$= \left[1 - \left(z\frac{0.2-z}{1-0.2z}\right)\left(\overline{z}\frac{0.2-\overline{z}}{1-0.2\overline{z}}\right)\right].$$

$$\left[\left(\frac{0.708z}{(0.64z+0.068)(1-0.1z)}\right)\left(\frac{(c_1+c_0)-0.2c_1\tau(z)}{1-0.2\tau(z)}\right) + \left(\frac{0.068(c_1+c_0)}{0.64z+0.068}\right)\right]$$

$$= \left[1 - \left(z\frac{0.2-z}{1-0.2z}\right)\left(\overline{z}\frac{0.2-\overline{z}}{1-0.2\overline{z}}\right)\right].$$

$$\left[\frac{-0.16c_1z^2 - 0.069c_0z^2 + 0.676c_1z + 0.686c_0z + 0.067c_1 + 0.067c_0}{(1-0.1z)(0.64z_0.068)(0.986-0.228z)}\right].$$

Corollary 3.3.3.1 *If*

$$b(z) = z\left(\frac{\gamma - z}{1-\overline{\gamma}z}\right)^n, \qquad (\gamma \neq 0, n \geq 1),$$

then $\mathscr{C}_v : K_b \to K_b$ *is normal if and only if* $C_v : \mathscr{H}^2 \to \mathscr{H}^2$ *is normal.*

Proof. let $f(z) \in K_b$, and suppose that $C_v f(z)$ is normal. It is enough to show that $\mathscr{C}_v\mathscr{C}_v^* f(z) = \mathscr{C}_v^*\mathscr{C}_v f(z)$

$$\mathscr{C}_v\mathscr{C}_v^* f(z) = \mathscr{C}_v\big((1-|b|^2)C_v^* f(z)\big) \qquad (by\ proposition 3.3.3)$$

$$= (1-|b|^2)C_v\big((1-|b|^2)C_v^* f(z)\big)$$

$$= (1-|b|^2)^2 C_v C_v^* f(z)$$

$$= (1-|b|^2)^2 C_v^* C_v f(z) \qquad (C_v\, is\, normal\, on\, \mathscr{H}^2)$$

$$= (1-|b|^2)C_v^*\big((1-|b|^2)C_v f(z)\big) = \mathscr{C}_v^*\mathscr{C}_v f(z)$$

□

Corollary 3.3.3.2 *If*

$$b(z) = z\left(\frac{\gamma - z}{1 - \overline{\gamma} z}\right)^n, \qquad (\gamma \neq 0, n \geq 1),$$

then $\mathscr{W}_{\omega,\upsilon} : K_b \to K_b$ *is normal if and only if* $W_{\omega,\upsilon} : \mathscr{H}^2 \to \mathscr{H}^2$ *is normal.*

Proof. Similar to corollary (3.3.3.1). □

List of Abbreviations

Abbreviations	Meaning
$\mathbb{C}$	Set of All Complex Numbers
C_v	Composition Operator
C_v^*	Adjoint of Composition Operator
$\mathcal{H}$	Hilbert Space
$\mathscr{H}^p$	Hardy Space
$\mathscr{H}^2$	Hardy-Hilbert Space
$\mathscr{H}^\infty$	Space of All Bounded Analytic Functions
K_u	Model space
κ_γ^u	Reproducing Kernel on K_u
$\mathcal{L}_c(K_b)$	Set of All Bounded Composition Operators on K_b
L^p	Space of All p-Integrable complex Measurable Functions
L^∞	Space of All Essentially Bounded Functions
P	Orthogonal Projection Operator
$\mathscr{Q}_\beta$	Reproducing Kernel Function on $\mathscr{H}^2$
S	Unilateral Shift Operator on $\mathscr{H}^2$
S^*	Backward Shift Operator on $\mathscr{H}^2$
T_g	Toeplitz operator
$W_{\omega,v}$	Weighted Composition Operator
$\mathbb{Y}$	Unit Circle = $\{z \in \mathbb{C} : |z| = 1\}$
Γ	Unit Disk = $\{z \in \mathbb{C} : |z| < 1\}$

References

[1] Nordgren E. Composition Ooerators. Canadian Journal of Mathematics. 1968;20:442-9.

[2] Cowen C, MacCluer B. Composition Operators on Spaces of Analytic Functions. Studies in Advanced Mathematics. Taylor & Francis; 1995. Available from: https://books.google.ps/books?id=WWMI52-GX_oC.

[3] Schwartz H. Composition Operator on H^p, Thesis. The University of Toledo; 1969.

[4] Bourdon P, Narayan S. Normal Weighted Composition operators on the Hardy Space $\mathrm{H}^2(\mathbb{D})$. Journal of Mathematical Analysis and Applications. 2010;367:278-86.

[5] Mashreghi J, Shabankhah M. Composition operators on finite rank model subspaces. Glasgow Mathematical Journal. 2013;55(01):69-83.

[6] Kreyszig E. Introductory Functional Analysis With Applications. Wiley Classics Library. John Wiley & Sons; 1978. Available from: https://books.google.co.il/books?id=Va8rAAAAYAAJ.

[7] Bourbaki N. General Topology. Elements of Mathematics. Berlin New York: Springer Science & Business Media; (1995)[1971]. Available from: https://link.springer.com/book/10.1007/978-3-642-61701-0.

[8] Rudin W. Functional Analysis. 2nd ed. International series in pure and applied mathematics. McGraw-Hill; 1991. Available from: https://books.google.co.il/books?id=l7XFfDmjp5IC.

[9] Royden HL. Real Analysis. 3rd ed. Macmillan; 1988. Available from: https://books.google.ps/books/about/Real_Analysis. html?id=J4k_AQAAIAAJ&redir_esc=y.

[10] Bergman S. The Kernel Function and Conformal Mapping. Mathematical Survey. American Mathematical Society; 1950. Available from: https: //books.google.ps/books/about/The_Kernel_Function_and_Conformal_Mappin.html?id=NSKxAAAAIAAJ&redir_esc=y.

[11] Rudin W. Real and Complex Analysis. Mathematics series. McGraw-Hill; 1987. Available from: https://books.google.ps/books?id=NmW7QgAACAAJ.

[12] Duren P. Theory of H^p Spaces. Academic Press New York and London; 1970. Available from: https://books.google.ps/books/about/Theory_of_Hp_Spaces.html?id=fs4rPPcJ7HUC&redir_esc=y.

[13] Hoffman K. Banach Spaces of Analytic Functions. Prentic-Hall, Inc.,Englewood Cliffs, N.J.; 1962. Available from: https://books.google.ps/books/about/Banach_Spaces_of_Analytic_Functions.html?id=YUP_AwAAQBAJ&redir_esc=y.

[14] Koosis P. Introduction to Hp Spaces. 2nd ed. Cambridge Tracts in Mathematics. Cambridge University Press; 1998. Available from: https://books.google.ps/books/about/Introduction_to_ Hp_Spaces.html?id=tWvb6AHsCVIC&redir_esc=y.

[15] Abramovich YA, Aliprantis CD. An Invitation to Operator Theory. Graduate studies in mathematics. American Mathematical Society, Providence, R.I.; 2002. Available from: https://books.google.ps/books?id=misPCgAAQBAJ&printsec=copyright&redir_esc=y#v=onepage&q&f=false.

[16] Conway J. A Course in Functional Analysis. Springer My Copy UK; 2014. Available from: https://books.google.co.il/books?id=DBvIoAEACAAJ.

[17] Peetre J, Nikol'skii NK, Hruscev SV, Peller VV. Treatise on the Shift Operator: Spectral Function Theory. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg; 2012. Available from: https://books.google. ps/books?id=SlHoCAAAQBAJ.

[18] Mashreghi J, Garcia S, Ross W. Introduction to Model Spaces and their Operators. Cambridge Studies in Advanced Mathematics. Cambridge University Press; 2016. Available from: https://doi.org/10.1017/CBO9781316258231.

[19] Bottcher A, Silbermann B. Analysis of Toeplitz Operators. 2nd ed. Springer Berlin, Heidelberg; 2006. Available from: https://link.springer.com/book/10.1007/3-540-32436-4.

[20] Denjoy A. Sur l'iteration des fonctions analytiques'. C R Acad Sci Paris. 1926;182:255-7.

[21] Vladimir k. Geometry of Mobius Transformations: Elliptic, Parabolic and Hyperbolic Actions of $SL_2(R)$. Imperial College Press; 2012.

[22] Shapiro JH. Composition Operators: and Classical Function Theory. Universitext / Universitext: Tracts in Mathematics. Springer New York; 1993. Available from: https://books.google.ps/books?id=RyrvAAAAMAAJ.

[23] Cowen CC. Linear fractional composition operators on H^2. Integral Equations and Operator Theory. 1988;11:151-60.

[24] Hammond C. On the Norm of a Composition Operator with linear fractional symbol. Acta Scientiarum Mathematicarum. 2003;69:813-29.

[25] Mashreghi J, Shabankhah M. Composition operators on finite rank model subspaces. Glasgow Mathematical Journal. 2013;55(01):69-83.

جـامعـــة النجاح الوطنية

كليـــة الدراســـات العليـــا

المؤثرات المركبة المتعامدة على فضاء جزئي من فضاء هاردي

إعداد

دعاء محمد صادق صرصور

إشراف

معاذ كركي

قدمت هذه الرسالة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات، من كلية الدراسات العليا، في جامعة النجاح الوطنية، نابلس– فلسطين.

2024

المؤثرات المركبة المتعامدة على فضاء جزئي
من فضاء هاردي

اعداد

دعاء محمد صادق صرصور

إشراف

معاذ كركي

الملخص

لنفترض أن Γتمثل قرص الوحدة المفتوح وكان $\upsilon(z): \Gamma \to \Gamma$ تابع تحليلي، عرف المؤثر المركب $C_\upsilon f(z): f \circ \upsilon$، وكان $\omega(z)$ تابع هولومورفي على قرص الوحدة، عرف المؤثر المركب المرجح $W_{\omega.\upsilon} f(z) = \omega f \circ \upsilon$ كلاهما مركبات محدودة على H^2. هذه الرسالة تدرس المؤثرات المركبة المتعامدة C_υ والمؤثرات المركبة المرجحة المتعامدة $W_{\upsilon.\omega}$ على فضاء هاردي H^2. كما أننا ندرس المؤثرات المركبة والمؤثرات المركبة المترافقة على الفضاء الجزئي النموذجي من فضاء هاردي K_b، حيث $b(z) = \lambda z^{m_0} \prod_{i=1}^{n} (\frac{\gamma_i - z}{1-\overline{\gamma_i} z})^{m_i}$ هو تابع ضربي بلاشكي المحدود، إننا ندرس المؤثر المركب $C_\upsilon f(z)$ على الفضاء الجزئي النموذجي $K_b = \{b\overline{zh}: h\epsilon H^2\} = H^2 \cap b\overline{zH^2}$ والمؤثر المركب المرافق $C_\upsilon^* f(z)$ على K_b. ثم نفحص تعامد $C_\upsilon f(z)$ و $W_{\omega.\upsilon} f(z)$ على هذا الفضاء الجزئي النموذجي K_b.

الكلمات المفتاحية: المؤثرات المركبة المتعامدة، فضاء نموذجي، المؤثرات المركبة، فضاء هاردي، نظرية المؤثرات، فضاءات الدوال، نظرية الطيف.