An-Najah National University

Faculty of Graduate Studies

A Confined Hydrogen Atom in A Spherical Cavity in N Dimensions “excited states”

By

Manar Abdul-Rahim Mosleh

Supervisor

Prof. Sami Al-Jaber

This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Physics, Faculty of Graduate Studies, An-Najah National University - Nablus, Palestine.

2017

A Confined Hydrogen Atom in A Spherical Cavity in N Dimensions
“excited states”

By

Manar Abdul-Rahim Mosleh

This thesis was defended successfully on 6 /4 /2017, and approved by:

<u>Defense Committee Members</u>	**<u>Signature</u>**
– **Prof. Sami Al-Jaber / Supervisor**	**………………..**
– **Prof. Atef Qasrawi / External Examiner**	**……………….**
– **Prof. Mohammed Elsaid/ Internal Examiner**	**……………….**

Dedication

After the almighty Allah, this work is dedicated to:

Generally, to every strong woman in this planet. Particularly, to my parents, my brothers: Noor, Hikmat and Hassan and my only sister: Sarah. To all my friends especially the closest ones: Manar Ishqer and Fatima Abu Saleemah.

Acknowledgment

I would like to express my sincere thanks to my supervisor Prof. Sami Al-Jaber for his endless support. At First, he was so patient with me during my work in this research. I have delayed my duties several times due to my eyes surgery. He has never complained. On the contrary, he encouraged, supported and guided me in a fruitful way. I really got benefit from him not only as a researcher but also as physics teacher. So from the bottom of my heart: Thank you.

Also, I would warmly thank all my doctors during my graduate and undergraduate studies at An-Najah National University especially, Prof. Sami Al Jaber, Prof. Hazem Abu Sara, Prof. Mohammed Elsaid, Prof. Mohammed S. Abu-Jafar, Prof. Issam Rashid. I can’t forget to thank my colleague Ayham Shaer for his first touch in Mathematica program.

Finally, I am grateful for anyone who helps me to reach this point of my studies journey.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان

A Confined Hydrogen Atom in A Spherical Cavity in N Dimensions

“excited states”

أقر بأن ما شملت عليه الرسالة هو نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد، وأن هذه الرسالة ككل أو أي جزء منها لم يقدم من قبل لنيل أي درجة أو لقب علمي أو بحثي لدى أي مؤسسة علمية أو بحثية

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degrees or qualifications.

Student's Name: **اسم الطالب:**

Signature **التوقيع:**

Date **التاريخ:**

Table of Contents

List of Tables

List of Figures

A Confined Hydrogen Atom in A Spherical Cavity in N Dimensions
“excited states”

By
Manar Abdul-Rahim Mosleh
Supervisor
Prof. Sami Al-Jaber

Abstract

The energy eigenvalues for the excited states of a confined Hydrogen atom in an impenetrable spherical cavity in N-dimensional space had been calculated by applying the boundary conditions to the radial wave function of this quantum problem which had been investigated by Shaqqur and AL-Jaber [1].

The energy eigenvalues for the excited states depend on both the space dimension and the cavity size. So, we had investigated the dependence of these energies on the dimension N and on the radius of the cavity S. Also, we deduced the limits of the cavity size at which the H-atom still has bound states. For certain N, when the cavity radius increases enough the H-atom become free atom while when it decreases enough there will be no bound states beyond; these critical values of the cavity radius had been obtained for different dimensions N.

Moreover, we had derived a simple equation which shows the relation between higher excited states with respect to higher dimensions started from N=3.

Finally, we had examined the behavior of the pressure exerted by the confined H-atom on the cavity as the radius S is varied in different space dimensions. We found that: for a fixed N, the pressure gradually increases as the cavity radius decreases up to a maximum value, suddenly after it, the value of pressure diminishes to zero.

Chapter One

Introduction

1.1 Schrödinger equation

1.2 Confined H-Atom

1.3 Higher Dimensions

1.4 Hydrogen atom in N-dimensions

1.5 Objectives

1.6 Thesis Outlines

Introduction

1.1 Schrödinger equation

Schrödinger equation is the rock-solid core on to which all of quantum mechanics was built. Once this wave equation is solved; all the information of a quantum system will be known; eigenstates, eigenvalues and any other relevant quantities. Hydrogen atom which is the simplest possible atomic system represents one of the few directly solvable problems in quantum mechanics. The exact results concerning the Hydrogen atom serve as the basis of all approximate calculations relating to more complex atoms (have several electrons) and that is why it has been the subject of numerous studies from the early days of quantum physics.

1.2 Confined H-Atom

Confinement of a particle in a potential is certainly a fundamental problem of quantum mechanics, in which different potentials are used for different ranges. Quantum confinement means confinement on a scale comparable to the atomic size [2]. The same atom can be seen in two different forms, as a free atom with certain observable properties and as a confined atom, with modified properties where confinement phenomena affects the physical properties of a quantum system significantly such as the density of states and energy spectra [2-15]. There are two different types of confinement. The first is confinement by a repulsive wall or shell (which may be penetrable or impenetrable). The second is confinement by an attractive shell, which is always penetrable. Confinement by a repulsive wall is encountered for atoms under extreme pressure (for atoms in cavities), where impenetrable cavity here refers to being unaffected by strong electric field or high temperature or flat band conditions. [3,16-20].

Considerable theoretical efforts have recently been made in performing more accurate computations on simple model systems like the hydrogen [21-24] and helium atom [24-26] which could also serve as a benchmark for the approximate methods.

The hydrogen atom confined to bounded regions has been widely discussed using different methods. This model was firstly proposed almost seventy years ago, by Michels and de Boer [28] and Sommerfeld and Welker [29-31] and focused on the determination of the ground state. In deed Michels came up with the idea of a spherically confined atom by enclosing the hydrogen atom at the center of an impenetrable cavity and calculated the effects of pressure on kinetic energy and polarizability. Higher states and the splitting of degenerated energy levels were then described by De Groot and Seldam [27]. Later on, steady flow of theoretical studies involving confinement models, both hard and soft confining boundaries with a variety of shapes and sizes have been considered [5,14,15,30-32]. Particularly, H-atom confined in a spherical or cylindrical cavity has been investigated using various methods [33-42]. The importance of those model studies can be related to many applications such as: the description of hydrogenic acceptor or donor impurities in semiconductors and nanostructures or simply to understand the effect of pressure on the atom [27,43-46].

As far as the specific case of the confinement in spherical cavities is concerned the majority of theoretical approaches have focused on the consideration that the confined atom is located at the geometric center of the cavity. A more general case is to assume that the atom is free to move inside the sphere and thus can be located at any position within the cavity [47]. The complexity of the off-center problem relies on the breaking of the spherical symmetry when the enclosed atom is displaced from the center. Consequently, the simplifications appearing from this symmetry are removed and the dimensionality of the Schrödinger equation often increases. Some theorists solved this problem using various approximate methods [48-51].

1.3 Higher Dimensions

The idea of multiple dimensions has been around for ages in terms of just mathematical concepts but in terms of physics it was more recent after Einstein developed his theory of general relativity where it works for any number of dimensions [52-56]. For whatever reason, we are not physiologically designed to observe those dimensions, in our everyday lives, we distinguish three spatial dimensions and a fourth dimension of time, but this does not mean that higher dimensions do not exist. How you look at things actually matters in terms of how many dimensions you describe it. Suppose a person walking on a tightrope, he can only move backward and forward; but not up and down, nor left and right, so he only sees one dimension. Ants living on a much smaller scale could move around the cable, in what would appear like an extra dimension to the tightrope-walker.

Einstein’s theory says that space can expand, contract, and bend. If one dimension were to contract to a size smaller than an atom, it would be hidden from our view. But if we could look on a small enough scale, that hidden dimension might become visible again. In 1926 Oskar Klein proposed this intuitive idea of compactified dimensions or curled up dimensions [57, 58]. Physicists believe that extra dimensions can illuminate connections resolving puzzles of the three plus one world. If extra dimensions exist, they could explain why the universe is expanding faster than expected, and why gravity is weaker than the other fundamental forces of nature. Randel and Sundrum recently answered this question based on string theory which is a theory candidate to higher dimensions and the fact that gravity is related to the space time geometry. Their idea is that our four-dimensional world exists on a “brane” floating in an inaccessible higher dimensional world. So, in the brane picture: the three forces electromagnetism, the weak and strong nuclear forces are localized on the brane, but gravity has no such constraint and propagates on the full spacetime. Which means that: we don’t feel the full effect of gravity because part of it spreads to extra dimensions so unification of the four fundamental forces can only be in higher dimensions. Science needs evidences for those dimensions, some theorists proposed a heavier version of standard particles in other dimensions, there would be partner particles called "Kaluza-Kline" particles – would have exactly the same properties as standard particles but with a greater mass. Such heavy particles can only be revealed at the high energies reached by the Large

Hadron Collider .So, this might suggest the presence of extra dimensions [59-61].

1.4 N-dimensional Hydrogen atom

Gathering the main physical ideas mentioned above, the quantum problem of this thesis could be formed as:

Hydrogen atom enclosed within an impenetrable spherical cavity in N dimensional space [1, 62-65]. Recently, Shaqqur and AL-Jaber had solved Schrödinger equation for this quantum system for $N \geq 3$ exactly by using series solution [1]. They had calculated the ground state energies and then, obtained their dependence on the cavity size, precisely on the cavity radius S, and the space dimension N. For a fixed N, as S increases the ground state energies decreases until the cavity size is large enough to let the confined atom be free of cavity. On the Contrary, when S decreases, the ground state energy increases until it approaches zero at a minimum value of S, which is called critical cage radius (S_c). However, for the space dimensionality influence, they had showed that for a given S, the ground state energy eigenvalues increase with N. Obviously, all of their calculations are for ground states; in this work, we will continue studying the same problem but for higher states.

1.5 Objectives:

This research project has three objectives which can be summarized as follow:

1- Obtaining the excited state energy eigenvalues for a confined H-atom in an impenetrable spherical cavity in N dimensional space then, investigate their dependence on the cavity size and space dimensionality.

2- The eigenenergies obtained above will be used to study the dependence of pressure on the cavity as the radius S is varied.

3- Obtaining the critical cage radius for excited states.

1.6 Thesis Outline

In this work, the energy eigenvalues for excited states of N-dimensional Hydrogen atom confined in an impenetrable spherical cavity have been calculated. Since these energies are found to be dimension dependent, we have investigated their dependence on the space dimensionality and the cavity size. Moreover, these energies are used to theoretically display the behavior of the pressure exerted by the confined H-atom on the cavity wall as a function of the space dimension N and the radius of the cavity S.

The rest of this thesis is organized as follow: A summary to what Shaqqur and Al-Jaber had published in 2009, the method that used to calculate the energy spectrum of the confined H-atom, and the pressure exerted on the cavity wall are presented in chapter 2. In chapter 3, the results of the energy eigenvalues and the pressure had been displayed and discussed, while the final chapter is devoted for the conclusion.

Chapter Two

Theoretical Background

2.1 Solution of Schrodinger equation for unconfined and confined N-dimensional H-atom

2.2 Mythology:

- **Energy eigenvalues for excited states.**
- **Examine the behavior of pressure exerted on the confined H-atom.**

2.1 The *N*-dimensional hydrogen atom

- Unconfined Case:

Solving Schrodinger equation is the first step in breaking the mystery of any quantum system such as simple atoms. It would give a series of pictures that show what atom looks like. A great deal about three-dimensional hydrogen atom had been revealed by solving it; the same procedure would be for N dimensional H-atom. Theorist built a great potential in studying H-atom in N-dimensional space weather it is unconfined or confined to bounded geometries. Al-Jaber had derived N-dimensional Schrodinger equation for free H-atom [64]. He started his derivation from the eigenvalue equation for a hydrogen atom:

$$\frac{-\hbar}{2\,\mu}\,\nabla^2\,\Psi(\mathrm{r})-\frac{Z\,e^2}{r}\psi\,(r)=E\,\psi(r) \qquad 2.1.1$$

Where: μ is the reduced mass and Ze is the nuclear charge. In the N-dimensional space, the Laplacian operator in polar coordinates $(\mathrm{r},\theta_1\,,\theta_2\,,\theta_3\,,\ldots\ldots\ldots..,\theta_{N-2},\phi\,)$ of R^N is

$$\nabla^2 = r^{1-N}\,\frac{\partial}{\partial r}\left(r^{N-1}\,\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\,\Lambda^2 \qquad 2.1.2$$

where r is the radial coordinate and Λ^2 is the Laplace operator on the unit hyper-sphere S^{N-1}. Using separation of variables, the Schrödinger equation splits into radial and angular differential equations with the latter having solutions that are independent of the form of the central potential. The eigenvalues of Λ^2 are $-l\,(l+N-2)$, with l being a non-negative integer. and thus, the radial equation of R(r) becomes:

$$r^{N-1}\,\frac{d}{dr}\left(r^{N-1}\,\frac{dR}{dr}\right)-\frac{\mathrm{l}\,(\mathrm{l}+N-2)}{r^2}\,\mathrm{R}+\frac{2\mu}{\hbar^2}\left(\frac{e^2}{r}-E\right)=0 \qquad 2.1.3$$

Considering $E < 0$ for bound states and defining λ and ρ as

$$\rho = qr = (\sqrt{8\,\mu\,(-E)}/\hbar\,)\,r \qquad 2.1.4$$

$$\lambda = \left(\frac{e^4\,\mu}{2\hbar^2\,(4\pi\varepsilon_o)^2(-E)}\right)^{1/2} \qquad 2.1.5$$

Substitution of Eq.2.1.4 and Eq.2.1.5 into Eq.2.1.3 yields:

$$\frac{d^2R}{d\rho^2} + \frac{N-1}{\rho}\frac{dR}{d\rho} - \frac{\mathrm{l}\,(\mathrm{l}+N-2)}{\rho^2}\;R + \left(\frac{\lambda}{\rho} - \frac{1}{4}\right)\mathrm{R} = 0 \qquad 2.1.6$$

Examining the behavior of the solutions as ($\rho \longrightarrow \infty$) and as ($\rho \longrightarrow 0$), the solution of R($\rho$) can be written as:

$$\mathrm{R}(\rho) = \rho^{\mathrm{l}}\,e^{-\rho/2}\mathcal{V}(\rho) \qquad 2.1.7$$

The substitution Eq.2.1.7 into Eq.2.1.6 yields:

$$\frac{d^2\nu}{d\rho^2} + \left(\frac{2\mathrm{l}+N-1}{\rho} - 1\right)\frac{d\nu}{d\rho} - \frac{\lambda - \mathrm{l} - (N-1/2)}{\rho^2}\;\nu = 0 \qquad 2.1.8$$

The solution of this equation can be expressed as a power series in ρ.

$$\nu(\rho) = \sum_{j=0}^{\infty} a_j\,\rho^j \qquad 2.1.9$$

But in order the function in Eq.2.1.7 have an acceptable asymptotic behavior, the series expansion of Eq.2.1.9 must be terminate, which leads to introduce the principle quantum number as the case of three-dimensional H-atom.

$$\lambda = n + \frac{N-3}{2} \qquad 2.1.10$$

Where: λ is the principle quantum number for free H-atom in N-dimensions and n is the principle quantum number for free H-atom in three dimensions.

The differential equation 2.1.8 is just the Kummer-Laplace differential equation whose standard form is given by [66]

$$\frac{d^2w}{dz^2} + \left(\frac{b}{z} - 1\right)\frac{dw}{dz} - \frac{a}{z}w(z) = 0 \qquad 2.1.11$$

and whose solution, regular at the origin, is the confluent hypergeometric function, ${}_1F_1$ (a, b, z) given by

$${}_1F_1\,(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n\, z^n}{(b)_n\, n!} \qquad 2.1.12$$

where the Pochhammer symbol $a_n$is defined by [66] :

$$(a)_n = a(a+1)(a+2)(a+3)......(a+n-1).$$

Comparing Eq.2.1.8 with Eq. 2.1.11 and with the identification

$$a = \lambda - l - \frac{N-1}{2} \qquad 2.1.13$$

$$b = 2l + N - 1 \qquad 2.1.14$$

Taking into account that the Associated Laguerre Polynomials $L_a^b(z)$ are related to the Confluent hyper geometric polynomials as:

$$L_a^b(z) = \frac{(a+b)!}{a!b!}\ {}_1F_1(-a; b+1; z) \qquad 2.1.16$$

the final form of radial solution can be expressed in terms of these two functions as follow:

$$R_{\lambda l}(\rho) = A^{\circ}\, \rho^{l}\, e^{\frac{-\rho}{2}}\ {}_1F_1\left(l + \frac{N-1}{2} - \lambda; 2l + N - 1;\ \rho\right) \qquad 2.1.17$$

Or

$$R_{\lambda l}(\rho) = A\, \rho^{l}\, e^{\frac{-\rho}{2}}\, L_{\lambda - l - \frac{N-1}{2}}^{(2l+N-2)}(\rho) \qquad 2.1.18$$

Where A = $A^{\circ}$is the normalization constant [1]

$$A = A^{\circ} = \frac{(\lambda+l+\frac{N-3}{2})!}{\left(\lambda-l-\frac{N-1}{2}\right)!\,(2l+N-2)!} \qquad 2.1.19$$

Finally, the energy eigen value can be introduced as:

$$E_{n\ \text{free}} = \frac{E_0}{(n+\frac{(N-3)}{2})^2} \qquad 2.1.20$$

Where E$_o$ is the value of the ground-state energy of free H-atom in three dimensions which is:

$$E_0 = -\frac{\hbar^2}{2\mu a_0^2} = -13.6eV \qquad 2.1.21$$

Where μ is the reduced mass and $a_\circ$ is Boher radius.

- **Confined Case:**

The above derivations are for unconfined H-atom in N dimensions and are used as basis for the quantum problem of this research:

confined H-atom in an impenetrable spherical cavity in N dimensional space for N $\geq$ 3. The potential of this system can be expressed as [1]:

$$V(r) = \left\{ \begin{array}{ll} \frac{-\mathrm{z\ e2}}{4\pi\varepsilon_\circ \mathrm{r}} & ,0 \leq r \leq S \\ \infty & ,\text{elsewhere} \end{array} \right\} \qquad 2.1.22$$

Where S is the cavity radius.

Obviously, Schrodinger equation with these boundary conditions does not differ from that of free H-atom in N dimensions Eq. 2.1.3. Therefore, their wave functions almost the same namely Eq. 2.1.17

Since the cavity is impenetrable the wave function must vanish at the cavity wall so it would stay continuous function inside the cavity. This indicates that even the free and confined H-atom have similar wave functions, they certainly would have different energy eigenvalues due to the confinement effect.

The boundary condition is:

$$R_{\lambda l}(\rho)\,\rfloor_{r=S} = R_{\lambda l}(qS) = \text{Zero} \qquad 2.1.23$$

Equation 2.1.17 indicates that the zeros of the radial wave function are the zeros of the confluent hypergeometric polynomials and cannot be found analytically as the three-dimensional case, so Mathematica software had been used to find them.

The energy eigenvalues depend on both space dimensions and the radius of the cavity and can be found by these formulas:

$$E_\lambda = \frac{E_\circ}{\lambda^2} \qquad 2.1.24$$

$$\rho = \frac{2S}{\lambda a^\circ} \qquad 2.1.25$$

$$\lambda = \frac{2S}{x_n a_\circ} \qquad 2.1.26$$

Where: λ is the principle quantum number for the confined H-atom in N-dimension, and x_n is the nth root of the Confluent Hypergeometric Polynomials.

It is worth notifying that the real numbers which appear for λ arise from applying the selection rule of the angular momentum l.

Shaqqur and Al-Jaber devoted their calculations for the ground state energies where the angular momentum is zero. They computed the ground state energies for different spatial dimensions N and then, they found the dependence of these energies on the cavity size and the space dimensionality [1].

2.2 Energy eigenvalues for excited states.

In the present work we will firstly, calculate the energy eigenvalues for excited states for different values of dimension N.

By substituting equations 2.1.25 and 2.1.26 in equation 2.1.17 and taking into account that the radius of the cavity is in terms of Bohr radius, then the radial wave function will be:

$$R_{\lambda l} = A'\ \rho^{l}\ e^{\frac{-\rho}{2}}\ {}_1F_1\left(l + \frac{N+1}{2} - \frac{2\,S}{x_n}; 2l + N - 1;\ \rho\right) \tag{2.2.1}$$

The previous equation has three variable parameters; the angular momentum l, the dimension N and the cavity radius S. As mentioned before, we need to find numerically the roots of the confluent hypergeometric polynomials in order to calculate the energy eigenvalues using equations 2.1.24 and 2.1.26. So, for specific values of dimension N and angular momentum l we suggested values of the cavity radius S, then by using Mathematica we had determined the roots.

See the examples below;

For $N = 3, l = 2,$

$${}_1F_1\left(l + \frac{N-1}{2} - \lambda; 2l + N - 1;\ \rho\right) = {}_1F_1\left(3 - \frac{2S}{x_n}; 6;\ \rho\right) \tag{2.2.2}$$

Now, for $N = 5, l = 1,$

$${}_1F_1\left(l + \frac{N-1}{2} - \lambda; 2l + N - 1;\ \rho\right) = {}_1F_1\left(3 - \frac{2S}{x_n}; 6;\ \rho\right) \tag{2.2.3}$$

One should notice that the arguments of Eqn. 2.2.1 and Eqn. 2.2.3 is the same.

Another example:

For $N = 4, l = 2,$

$$ {}_1F_1\left(l + \frac{N-1}{2} - \lambda; 2l + N - 1;\ \rho\right) = {}_1F_1\left(3.5 - \frac{2S}{x_n}; 7;\ \rho\right) \qquad 2.2.4 $$

Now, for $N = 6, l = 1,$

$$ {}_1F_1\left(l + \frac{N-1}{2} - \lambda; 2l + N - 1;\ \rho\right) = {}_1F_1\left(3.5 - \frac{2S}{x_n}; 7;\ \rho\right) \qquad 2.2.5 $$

Again, the arguments for these two cases are the same.

One way to think about this, similar confluent hypergeometric polynomials arguments for different values of N and l; mean similar roots, so same energy eigenvalues for those cases. As a result, accidental degeneracy in these energies shows up and by repeating the calculations for different values of N and l, we have noticed precisely that the degeneracy is caused by the correlations occur between higher odd dimensions with lower odd dimensions for different angular momentums l ≥ 1, or even higher dimension with lower even dimensions for different angular momentums l ≥ 1 too. Equation (2.2.6) represents a notation for these correlations. It indicates that the energy eigenvalues of confined H-atom in a spherical cavity for any higher dimension are related to the calculated ones either for N=3 if that dimension is odd or N=4 if that dimension is even, sure for different values of l.

A notation for N-L correlation:

$$F(N, l) = F(N-2, l+1) \ldots\ldots \text{For } N > 4, l \geq 1 \qquad 2.2.6$$

Where: F stands for confluent hypegeometric polynomials and the subscript are dropped for simplicity.

Now, using Eq. 2.2.6 one can represent some states:

$$F(N+1, l) = F(N-1, l+1) \qquad 2.2.7$$

$$F(N, l+1) = F(N-2, l+2) \qquad 2.2.7$$

$$F(N, l+2) = F(N-2, l+3) \qquad 2.2.8$$

Therefore, we can conclude that only the data for N=3 and N=4 are needed to study the H-atom in extra dimensions ($N \geq 3$), so we have just mentioned them in this study.

2.3 Examining the behavior of pressure exerted on the confined H-atom.

Confinement effects simulate the high-pressure environment. Both cause changes in physical, chemical, and structural characteristics of matter. For insight, physics of nanostructure systems such as artificial atoms or quantum dots. Confined atoms in an impenetrable cavity were firstly proposed for atoms subjected to very high pressure [27,43-46]. Pressure thus serves as a versatile tool in materials researches.

It is known in physics that pressure may be considered to be a measure of energy per unit volume or energy density.

$$P = \frac{-dE}{dV} \qquad 2.3.1$$

In previous section, we determined the energy formula Eq.2.2.5 which can be expressed as:

$$E_\lambda = \frac{x_n^2 \; a_\circ^2}{4\, S^2}\, E_\circ, \qquad 2.3.2$$

and taking into account that dV= A dS, where A is the area of hyper sphere which is equal to [66]:

$$A_N(S) = \frac{2\,(\pi)^{N/2}}{(\frac{N}{2}-1)!}\, S^{N-1}\,, \qquad 2.3.3$$

so Eq. 2.3.1 gives:

$$P = \frac{-1}{A}\,\frac{dE_n}{dS} \qquad 2.3.4$$

Shaqqur and Al-Jaber had done the rest derivations, so the final relation of the pressure in any dimension N is [66]:

$$P_{N=} \frac{\left(\frac{N}{2}-1\right)!}{\pi^{N/2}}\,\frac{x_n^2\, a_\circ^2}{4S^{N+2}}\, E_\circ \qquad 2.3.5$$

Chapter Three

Results and Discussion

3.1 Results of the energy eigenvalues for the excited states

- **N=3 Data, calculations and graphs.**
- **N=4 Data, calculations and graphs.**

The data and graphs and their observations of this study are arranged in three parts: Part I represents results of the energy eigenvalues for excited states. Part II represents results of critical cage radius. Part III represents results of the pressure.

3.1 Results for excited states energy eigenvalues.

In this part of our study we will examine the dependence for excited states energy eigenvalues on the cavity size and the space dimension. For a given N, energy eigenvalues will be calculated for different values of angular momentum $\mathrm{l} > 0$. At First, we find the roots of the radial wave function which is the roots of the confluent hypergeometric function numerically by using Mathematica software.

Then, the following formulas which represented the radius of the cavity in terms of Bohr radius, are needed to investigate the calculations [1].

$$R_{\lambda \mathrm{l}}(\rho) = A' \ \rho^{l} \ e^{\frac{-\rho}{2}} \ {}_1F_1\left(l + \frac{N+1}{2} - \frac{2\,S}{x_n}; 2l + N - 1; \ \rho\right) \tag{3.1.1}$$

$$\rho = \frac{2S}{\lambda a^{\circ}} \tag{3.1.2}$$

$$\lambda = \frac{2\,S}{\rho\ a_{\circ}} = \frac{2S}{x_n\ a_{\circ}} \tag{3.1.3}$$

$$E_{\lambda} = \frac{-13.6}{\lambda^{2}} \tag{3.1.4}$$

As mentioned in the previous chapter, only the data of space dimension N=3 and N=4 are displayed due to the accidental degeneracies.

- **N=3 Calculations:**

For the dimension, N=3, the suggested values of the cavity radius S are listed in columns as well as the calculated energy eigenvalues for excited states where l is from 1 to 15. The chosen cavity radii are in terms of Bohr radius $a_\circ$ and expressed in meters, while the energies expressed in (eV).

Table (3. 1): The energy eigenvalues for different values of S, and specific values of l, when N=3.

$l = 1$		$l = 2$		$l = 3$	
S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)
23.500000	-3.4000000000	48.00000000	-1.5111111111	61.00000000	-0.8500000000
20.000000	-3.3998640014	40.00000000	-1.5111035556	50.00000000	-0.8498844039
16.000000	-3.3969831695	30.00000000	-1.5103707574	40.00000000	-0.8474688871
12.000000	-3.3525103302	20.00000000	-1.4679193487	30.00000000	-0.8165177163
9.000000	-3.0933853623	15.00000000	-1.2686793333	20.00000000	-0.5440000000
7.000000	-2.3794309542	12.00000000	-0.8500000000	15.40000000	0.0000003077
5.500000	-0.8037515410	9.80000000	-0.0957111351	15.36430000	-0.0001649648
5.090000	-0.0038250990	9.61790000	-0.0002900598	15.36381000	-0.0000698686
5.089000	-0.0015646910	9.61739000	-0.0000016072	15.36370000	-0.0000485190
5.088300	-0.0000368462	9.61736700	-0.0000005206	15.36345010	-0.0000000145
5.883100	-0.0000030002	9.61736690	-0.0000004663	15.36345004	-0.0000000028

Table (3. 2): The energy eigenvalues for different values of S, and specific values of l, when N=3.

$l = 4$		$l = 5$		$l = 6$	
S×$a_\circ$ (m)	E(ev)	S×$a_\circ$ (m)	E(ev)	S×$a_\circ$ (m)	E(ev)
86.000000	-0.5440000000	110.0000	-0.3777764040	157.500000	-0.2775510204
60.000000	-0.5432069559	100.0000	-0.3777656890	130.000000	-0.2775456829
40.000000	-0.5089120110	80.00000	-0.3772235923	110.000000	-0.2774137758
23.000000	-0.0576735110	60.00000	-0.3663434125	100.000000	-0.2769776189
22.500000	-0.0176684933	40.00000	-0.2520868989	80.000000	-0.2710867772
22.292900	-0.0000633820	31.00000	-0.0265032283	60.000000	-0.2320493909
22.292700	-0.0000461112	30.40000	-0.0008699169	40.000000	-0.0096248962
22.292200	-0.0000027995	30.38050	-0.0000035750	39.611450	-0.0000015981
22.292190	-0.0000019340	30.38044	-0.0000009060	39.611402	-0.0000003865
22.292169	-0.0000001164	30.38042	-0.0000000163	39.611400	-0.0000003362

Table (3. 3): The energy eigenvalues for different values of S, and specific values of l, when N=3.

$l = 7$		$l = 8$		$l = 9$	
S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)
180.00000	-0.2125000000	225.000000	-0.1679012346	250.0000	-0.1360000000
160.00000	-0.2124957500	200.000000	-0.1679001432	200.0000	-0.1359510444
140.00000	-0.2124332195	180.000000	-0.1678852843	160.0000	-0.1347068387
120.00000	-0.2117880140	160.000000	-0.1677571852	140.0000	-0.1313331942
100.00000	-0.2075078688	140.000000	-0.1669493881	120.0000	-0.1218394114
80.00000	-0.1876032374	120.000000	-0.1633032752	100.0000	-0.0973580872
60.00000	-0.1083802074	100.000000	-0.1504945758	80.0000	-0.0344078559
50.00000	-0.0004285624	80.000000	-0.1106823477	74.0500	-0.0000416494
49.97300	-0.0000129888	61.452500	-0.0000006000	74.0450	-0.0000080658
49.97218	-0.0000003540	61.452448	-0.0000000826	74.0440	-0.0000013480
49.97216	-0.0000000458	61.452440	-0.0000000030	74.0438	-0.0000000044

Table (3. 4): The energy eigenvalues for different values of S, and specific values of l, when N=3.

$l = 10$		$l = 11$		$l = 12$	
S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)
313.50000	-0.1123966942	336.00000	-0.0944444444	390.00000	-0.0804733728
300.00000	-0.1123964694	300.00000	-0.0944410445	340.00000	-0.0804697042
280.00000	-0.1123962928	280.00000	-0.0944271754	300.00000	-0.0804191697
240.00000	-0.1123716405	240.00000	-0.0941759411	260.00000	-0.0799707990
200.00000	-0.1118845814	200.00000	-0.0921760754	220.00000	-0.0775934249
160.00000	-0.1074126366	160.00000	-0.0824251277	180.00000	-0.0686375535
100.00000	-0.0446776844	140.00000	-0.0693877551	140.00000	-0.0390727767
87.75000	-0.0000509285	102.60000	-0.0002286704	118.41880	-0.0000003599
87.73930	-0.0000006153	102.53260	-0.0000002551	118.41870	-0.0000001088
87.73917	-0.0000000039	102.53253	-0.0000000176	118.41866	-0.0000000084

Table (3. 5): The energy eigenvalues for different values of S, and specific values of l, when N=3.

l = 13		l = 14		l = 15	
S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)
455.00000	-0.0693877551	498.000000	-0.0604444444	540.00000	-0.0531250000
440.00000	-0.0693876605	440.000000	-0.0604428647	500.00000	-0.0531241500
400.00000	-0.0693866449	400.000000	-0.0604278234	460.00000	-0.0531176090
340.00000	-0.0693402530	360.000000	-0.0603223580	420.00000	-0.0530742145
300.00000	-0.0690305914	340.000000	-0.0601548366	380.00000	-0.0528649506
280.00000	-0.0685510479	300.000000	-0.0591763924	340.00000	-0.0521284677
240.00000	-0.0658352278	260.000000	-0.0561752627	300.00000	-0.0500771490
200.00000	-0.0574648621	220.000000	-0.0482688565	260.00000	-0.0451134015
160.00000	-0.0337069592	180.000000	-0.0284210765	200.00000	-0.0235291102
135.50000	-0.0002028555	153.460000	-0.0000123744	173.00000	-0.0004668804
135.39400	-0.0000018882	153.451700	-0.0000002320	172.59160	-0.0000010653
135.39340	-0.0000007489	153.451550	-0.0000000126	172.59090	-0.0000002631
135.39301	-0.0000000084	153.451544	-0.0000000038	172.59068	-0.0000000110

The following graphs represent these results.

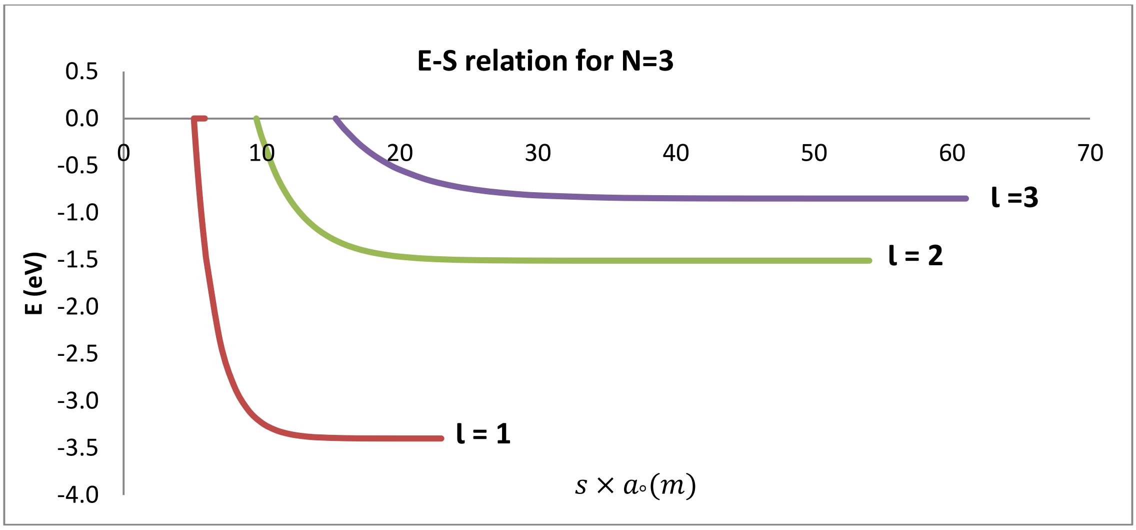

Fig (3. 1): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given angular momentums when the space dimension N=3.

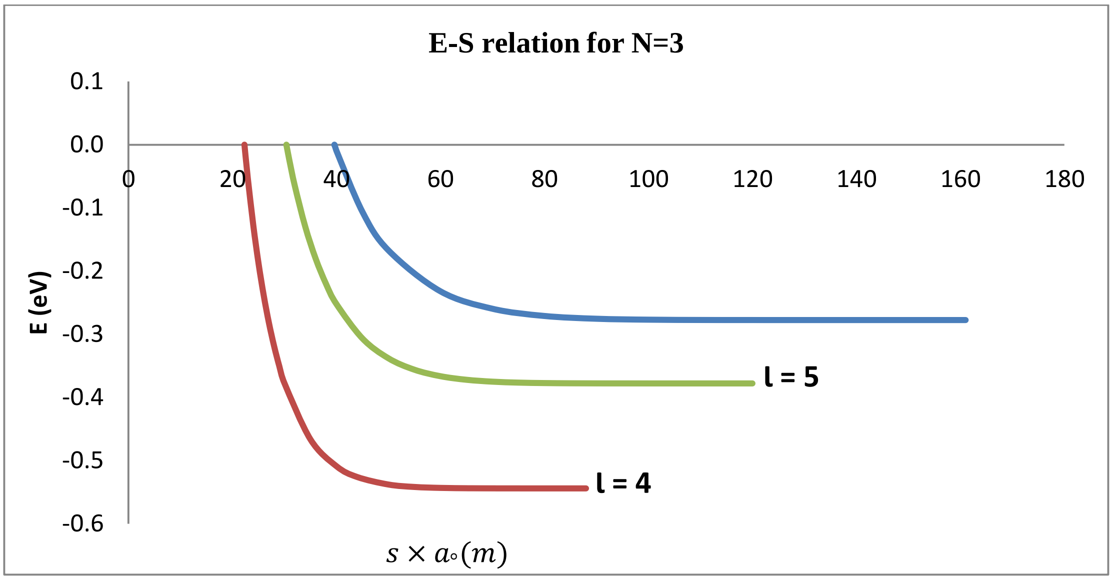

Fig (3. 2): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given angular momentums when the space dimension N=3.

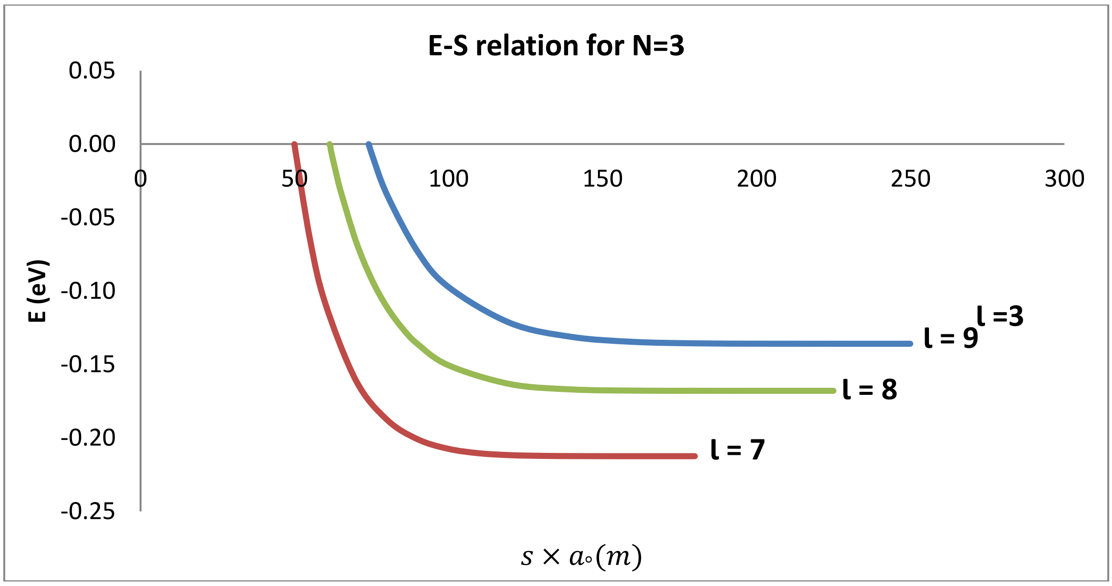

Fig (3. 3): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given angular momentums when the space dimension N=3

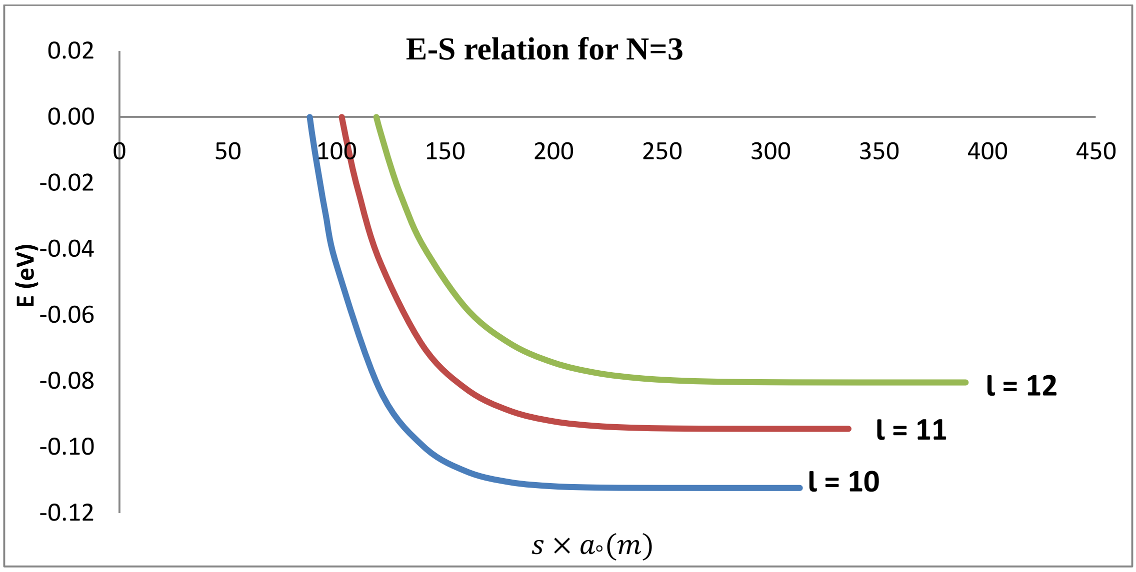

Fig (3. 4): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given angular momentums when the space dimension N=3.

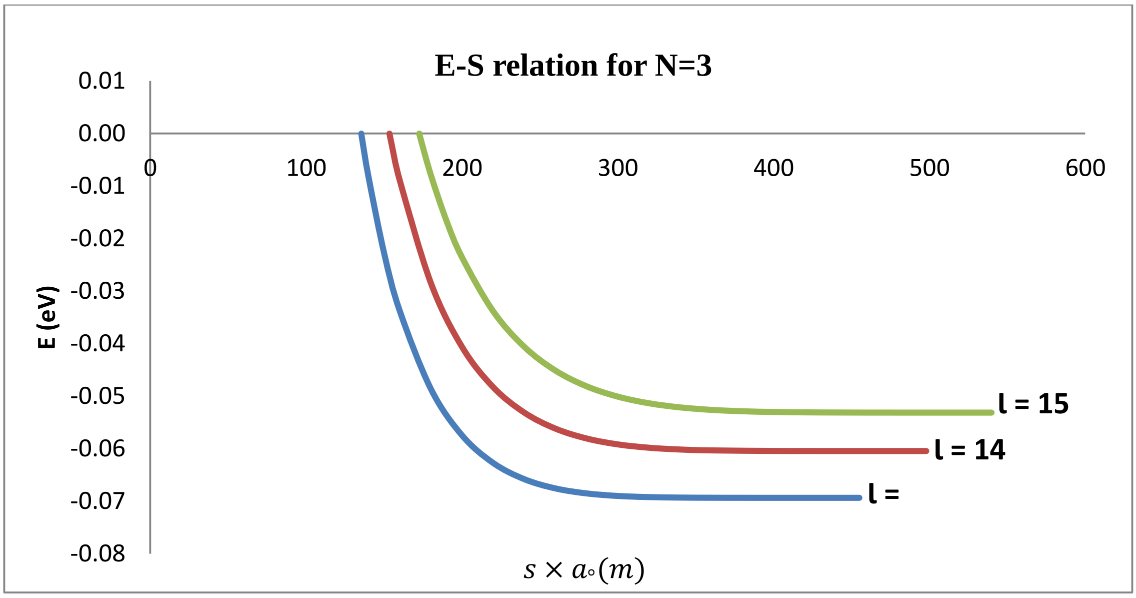

Fig (3. 5): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given angular momentums when the space dimension N=3.

- **N=4 Calculations:**

For the dimension, N=4, the suggested values of the cavity radius S are listed in columns as well as the calculated energy eigenvalues for excited states l from 1 to 15. The chosen cavity radii are in terms of Bohr radius $a_\circ$ and expressed in meters, while the energies expressed in (eV).

Table (3. 6): The energy eigenvalues for different values of S, and specific values of l, when N=4.

$l = 1$		$l = 2$		$l = 3$	
S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)
31.00000	-2.1760000000	63.00000	-1.1102040816	81.000000	-0.6716049383
25.00000	-2.1756518539	50.00000	-1.1102018612	70.000000	-0.6716001411
18.00000	-2.1602224814	40.00000	-1.1099765015	60.000000	-0.6715209903
10.00000	-1.5624922524	30.00000	-1.1024001569	50.000000	-0.6703710956
8.00000	-0.6794151930	20.00000	-0.9680421795	40.000000	-0.6593400192
7.40000	-0.1985420565	14.00000	-0.4086156699	30.000000	-0.5854239085
7.00100	-0.0024435833	12.35000	-0.0028851656	20.000000	-0.1466573443
7.19850	-0.0006547368	12.35330	-0.0039162600	18.681615	-0.0000006206
7.19801	-0.0001474384	12.34080	-0.0000049975	18.681611	-0.0000001005
7.19790	-0.0000335372	12.34079	-0.0000018624	18.681610	-0.0000000117

Table (3. 7): The energy eigenvalues for different values of S, and specific values of l, when N=4.

$l = 4$		$l = 5$		$l = 6$	
S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)
110.000000	-0.4495867769	130.000	-0.3218934911	165.000000	-0.2417777778
80.000000	-0.4495092265	100.000	-0.3217840566	140.000000	-0.2417682795
60.000000	-0.4457834827	80.000	-0.3195964676	120.000000	-0.2416010101
40.000000	-0.3764623693	60.000	-0.2966845961	100.000000	-0.2397986623
28.000000	-0.0971562471	40.000	-0.1307210458	80.000000	-0.2278839097
26.400000	-0.0124963281	38.000	-0.0882351484	60.000000	-0.1697869938
26.192600	-0.0000051972	36.000	-0.0356900854	45.000000	-0.0067212889
26.192518	-0.0000001825	34.890	-0.0011932104	43.655000	-0.0000769582
26.192516	-0.0000000541	34.855	-0.0000350192	44.651245	-0.0000000816
26.192515	-0.0000000113	34.854	-0.0000018617	44.651244	-0.0000000728

Table (3. 8): The energy eigenvalues for different values of S, and specific values of l, when N=4.

$l = 7$		$l = 8$		$l = 9$	
S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)
221.000000	-0.1882352941	285.000000	-0.1506925208	315.000000	-0.1233560091
200.000000	-0.1882351059	240.000000	-0.1506924266	280.000000	-0.1233558549
180.000000	-0.1882331503	200.000000	-0.1506827636	240.000000	-0.1233506894
160.000000	-0.1882047071	160.000000	-0.1501980434	200.000000	-0.1231751430
140.000000	-0.1879362533	140.000000	-0.1483651969	140.000000	-0.1152552272
120.000000	-0.1861836602	120.000000	-0.1420868910	100.000000	-0.0710783619
100.000000	-0.1781608361	100.000000	-0.1237251810	85.000000	-0.0215060891
80.000000	-0.1488211152	80.000000	-0.0726581844	82.000000	-0.0067701730
60.000000	-0.0469272685	67.700000	-0.0007326566	80.753900	-0.0000001069
55.572925	-0.0000000206	67.609710	-0.0000000469	80.753890	-0.0000000510
55.572923	-0.0000000009	67.609705	-0.0000000062	80.753881	-0.0000000006

Table (3. 9): The energy eigenvalues for different values of S, and specific values of l, when N=4.

$l = 10$		$l = 11$		$l = 12$	
S$\times a_\circ$ (m)	E(eV)	S$\times a_\circ$ (m)	E(eV)	S$\times a_\circ$ (m)	E(eV)
356.5000000	-0.1028355388	360.000000	-0.0870400000	418.500000	-0.0746227709
300.0000000	-0.1028350931	340.000000	-0.0870393600	360.000000	-0.0746190398
260.0000000	-0.1028155659	300.000000	-0.0870244060	340.000000	-0.0746083409
220.0000000	-0.1024758593	260.000000	-0.0868275513	300.000000	-0.0744699099
180.0000000	-0.0995673890	220.000000	-0.0853828292	280.000000	-0.0742071428
140.0000000	-0.0845012108	180.000000	-0.0787294103	240.000000	-0.0723933440
100.0000000	-0.0180099525	120.000000	-0.0238601533	200.000000	-0.0659165502
95.0000000	-0.0000042149	110.500000	-0.0004664129	160.000000	-0.0237275559
94.9991000	-0.0000006318	110.350000	-0.0000311592	128.000000	-0.0026287342
94.9989440	-0.0000000108	110.339295	-0.0000000078	126.770069	-0.0000000042
94.9989415	-0.0000000008	110.339292	-0.0000000003	126.770067	-0.0000000001

Table (3. 10): The energy eigenvalues for different values of S, and specific values of l, when N=4.

$l = 13$		$l = 14$		$l = 15$	
S$\times a_\circ$ (m)	E(eV)	S$\times a_\circ$ (m)	E(eV)	S$\times a_\circ$ (m)	E(eV)
478.500000	-0.0646848989	527.000000	-0.0566077003	577.500000	-0.0499540863
420.000000	-0.0646837053	460.000000	-0.0566057191	500.000000	-0.0499508543
380.000000	-0.0646693670	420.000000	-0.0565896345	460.000000	-0.0499315552
340.000000	-0.0645570953	380.000000	-0.0564880234	420.000000	-0.0498325043
280.000000	-0.0632002317	340.000000	-0.0560383835	380.000000	-0.0494583777
240.000000	-0.0594351158	300.000000	-0.0545669034	340.000000	-0.0483615997
200.000000	-0.0490387757	260.000000	-0.0506228316	300.000000	-0.0456613090
160.000000	-0.0213424624	220.000000	-0.0410489305	260.000000	-0.0396152421
145.000000	-0.0011744504	180.000000	-0.0181823607	220.000000	-0.0264905974
144.500000	-0.0003532562	163.000000	-0.0001468155	183.000000	-0.0004420843
144.286990	-0.0000000099	162.886250	-0.0000000111	182.564440	-0.0000000044
144.286985	-0.0000000016	162.886242	-0.0000000008	182.564436	-0.0000000003

The following graphs represent these results.

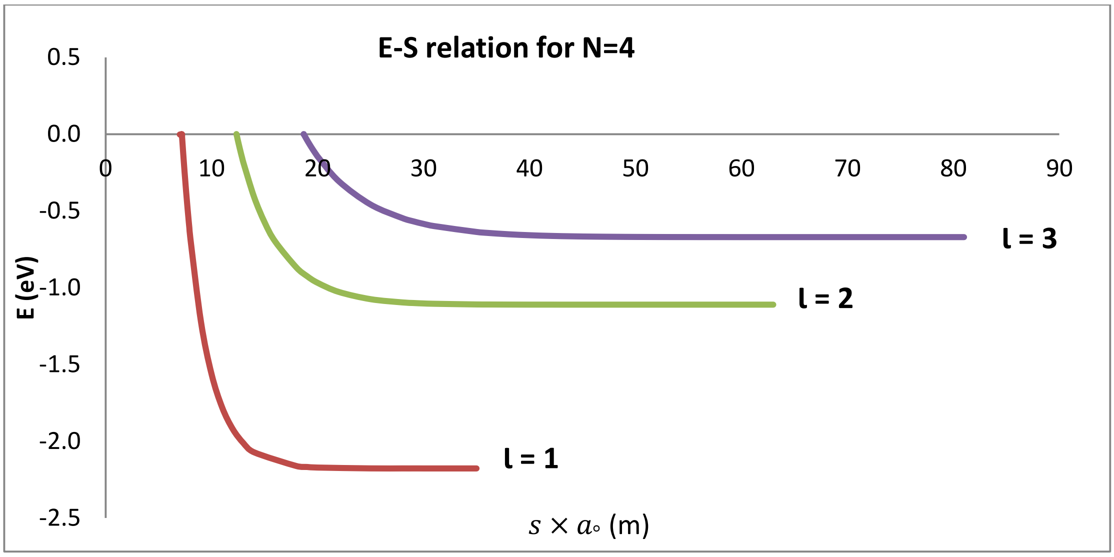

Fig (3. 6): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given angular momentums when the space dimension N=4.

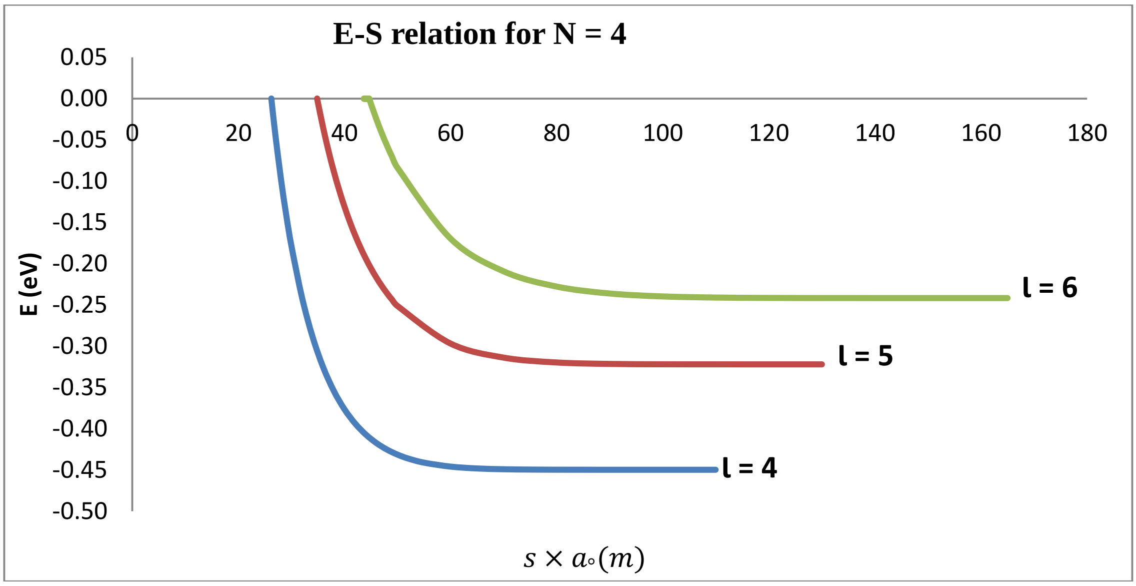

Fig (3. 7): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given angular momentum when the space dimension N=4.

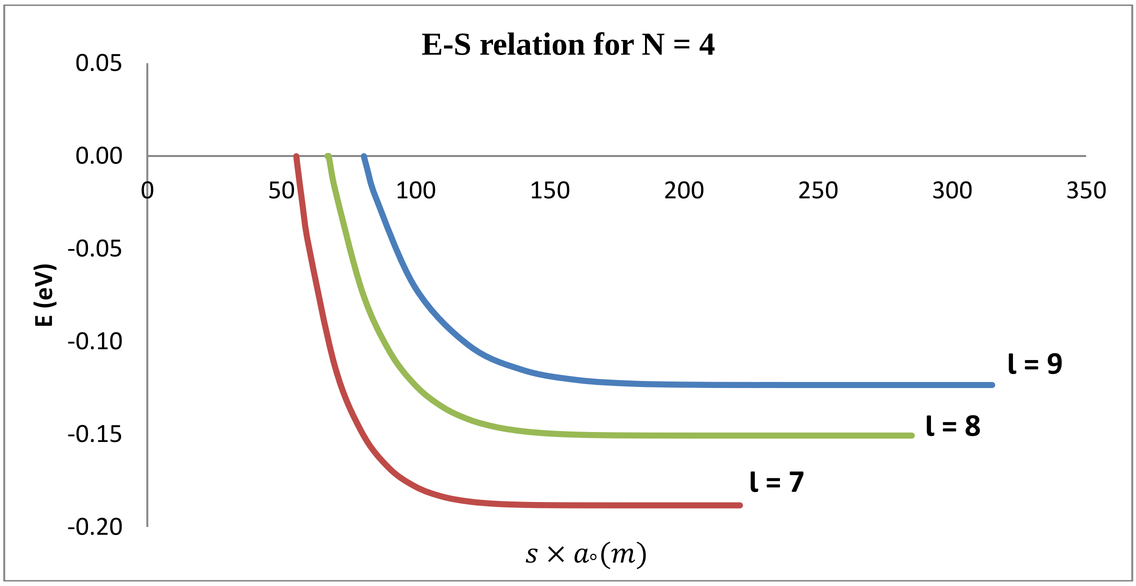

Fig (3. 8): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given angular momentum when the space dimension N=4.

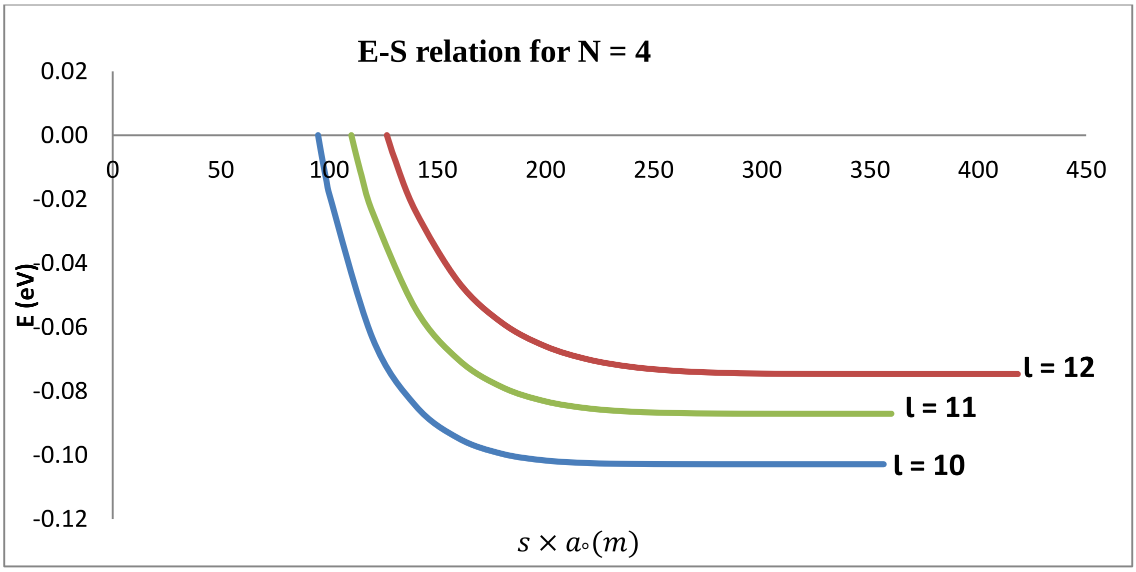

Fig (3. 9): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given angular momentum when the space dimension N=4.

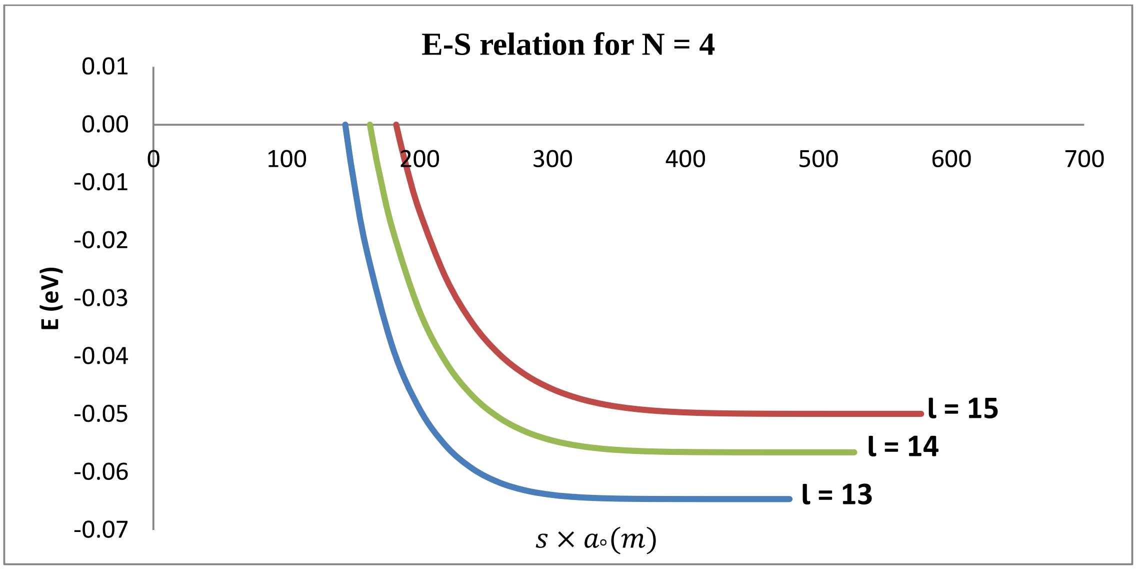

Fig (3. 10): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given angular momentum when the space dimension N=4.

- In this section, we will see how the energy eigenvalues of a specific excited state vary with space dimension N. To do that, we choose the first two excited state (ɭ =1, 2) and then, calculate the corresponding energies.

Table (3. 11): The energy eigenvalues for different values of S and specific values of space dimension N when ɭ =1.

N=3, ɭ=1		N=4, ɭ=1		N=5, ɭ=1	
S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)
23.00000	-3.400000000	35.00000	-2.176000000	48.00000	-1.511111111
20.00000	-3.399864001	30.00000	-2.175981867	44.00000	-1.511107677
16.00000	-3.396983170	25.00000	-2.175651854	40.00000	-1.511103556
12.00000	-3.352510330	20.00000	-2.170346077	35.00000	-1.511042033
9.00000	-3.093385362	14.00000	-2.070606334	30.00000	-1.510370757
5.50000	-0.803751541	9.00000	-1.222676625	20.00000	-1.467919349
5.09000	-0.003825099	7.50000	-0.289273135	14.00000	-1.172686745
5.08839	-0.0001849993	7.19810	-0.000240626	12.00000	-0.850000000
5.88310	-0.0000030002	7.19790	-0.0000335372	9.61739	-0.000013016

Table (3. 12): The energy eigenvalues for different values of S and specific values of space dimension N when l =1.

N=6, l=1		N=7, l=1		N=8, l=1	
S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)
63.00	-1.110204082	61.0000000	-0.850000000	81.00000	-0.6716049380
55.00	-1.110206100	55.0000000	-0.849975273	75.00000	-0.6716035950
50.00	-1.110201861	50.0000000	-0.849884404	70.00000	-0.6716001410
40.00	-1.109976501	40.0000000	-0.847468887	65.00000	-0.6715868570
36.00	-1.109197417	35.0000000	-0.840207226	60.00000	-0.6715209900
30.00	-1.102400157	30.0000000	-0.816517716	55.00000	-0.6712673470
25.00	-1.074542307	22.0000000	-0.651067963	50.00000	-0.6703710960
22.00	-1.027429279	15.5000000	-0.026056976	40.00000	-0.6593400190
12.90	-0.161027039	15.3639000	-0.000087336	30.00000	-0.5854239090
12.40	-0.018394948	15.3634501	-0.0000000048	18.70000	-0.0023284780
12.35	-0.002885166	15.3634500	-0.0000000028	18.68162	-0.0000125488

Table (3. 13): The energy eigenvalues for different values of S and specific values of space dimension N when l=1.

N=9, l=1		N=10, l=1	
S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)
86.00000	-0.5440000000	110.000000	-0.4495867770
80.00000	-0.5439932000	105.000000	-0.4495863490
74.00000	-0.5439705950	100.000000	-0.4495858780
70.00000	-0.5439222880	95.000000	-0.4495853570
65.00000	-0.5437447680	90.000000	-0.4495792840
60.00000	-0.5432069560	85.000000	-0.4495608600
55.00000	-0.5417126640	80.000000	-0.4495092260
50.00000	-0.5378863940	75.000000	-0.4493584160
44.00000	-0.5260076470	70.000000	-0.4489672030
40.00000	-0.5089120110	65.000000	-0.4480032070
35.00000	-0.4665461070	60.000000	-0.4457834830
30.00000	-0.3777777780	55.000000	-0.4410005970
25.00000	-0.1879011740	50.000000	-0.4312801280
22.30000	-0.0006774470	46.000000	-0.4171832360
22.29220	-0.0000027995	40.000000	-0.3764623690
22.29217	-0.0000001163	30.000000	-0.1791964340
		26.500000	-0.0183840620
		26.192515	-0.0000000112

Table (3. 14): The energy eigenvalues for different values of S and specific values of space dimension N when l=1.

N=20, l=1		N=25, l=1		N=30, l=1	
S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)	S×$a_\circ$ (m)	E(eV)
315.00000	-0.1233560090	390.00000	-0.080473373	527.00000	-0.056607700
300.00000	-0.1233561940	360.00000	-0.080472747	460.00000	-0.056605719
260.00000	-0.1233554630	340.00000	-0.080469704	440.00000	-0.056601532
240.00000	-0.1233506890	300.00000	-0.080419170	400.00000	-0.056559226
200.00000	-0.1231751430	260.00000	-0.079970799	360.00000	-0.056336239
160.00000	-0.1206118530	240.00000	-0.079205253	340.00000	-0.056038384
120.00000	-0.1021118260	200.00000	-0.074454480	300.00000	-0.054566903
80.90000	-0.0008151050	160.00000	-0.058110466	260.00000	-0.050622832
80.75500	-0.0000062646	120.00000	-0.003868931	240.00000	-0.046856062
80.75390	-0.0000001069	118.45000	-0.000078664	200.00000	-0.032099196
80.75389	-0.0000000509	118.41880	-0.000000359	162.90000	-0.000017778
80.75388	-0.0000000005	118.41870	-0.000000108	162.88650	-0.0000003342
		118.41866	-0.000000008	162.88625	-0.0000000111

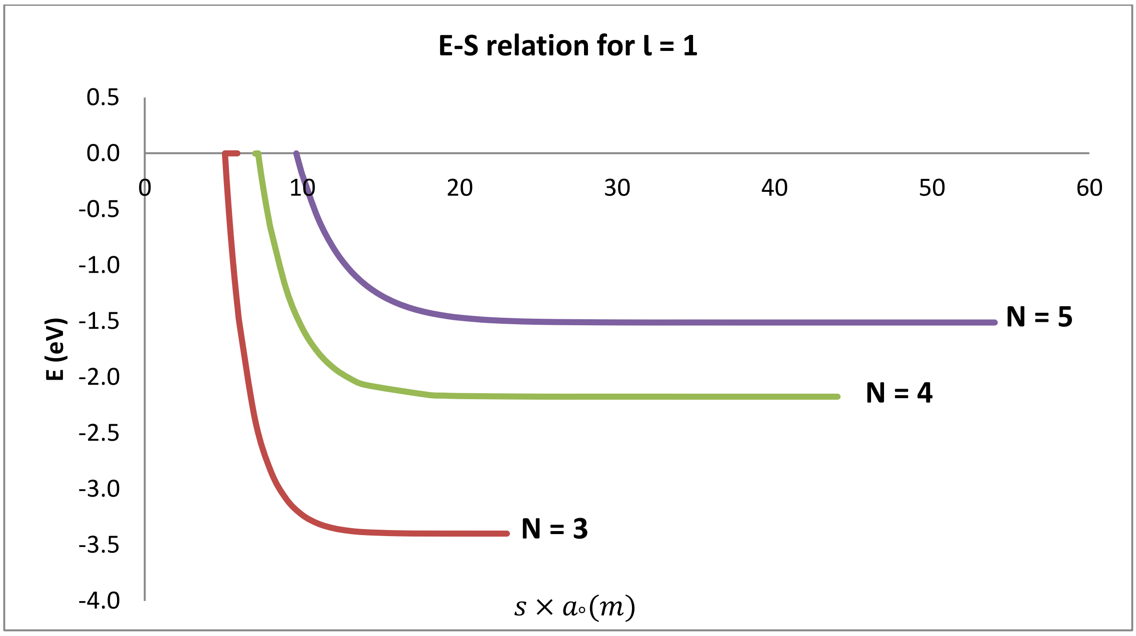

Fig (3. 11): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given space dimensions when the angular momentum l = 1(m).

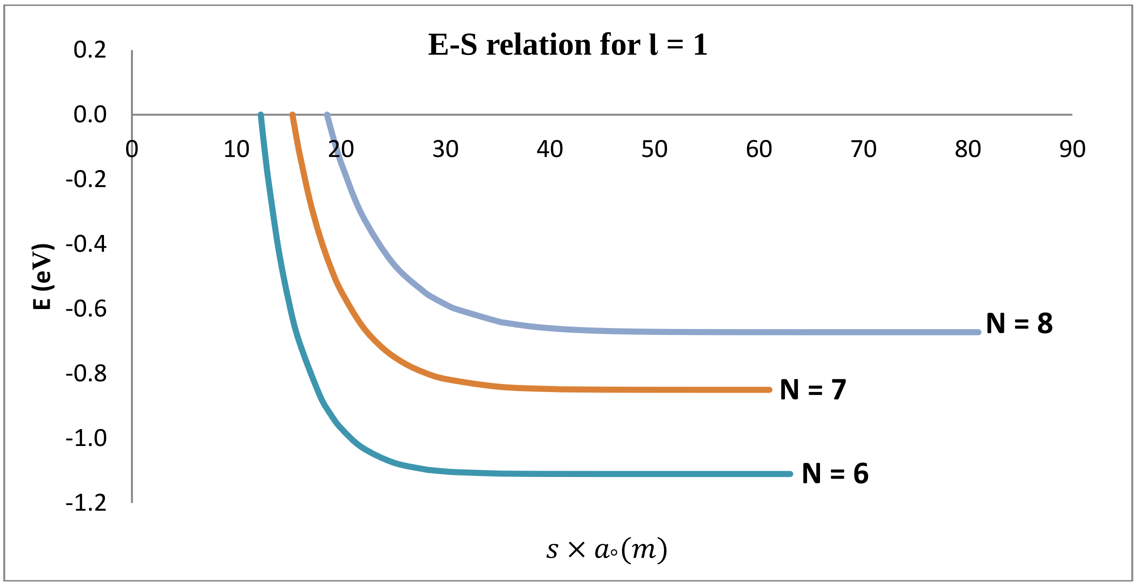

Fig (3. 12): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given space dimensions when the angular momentum l = 1(m).

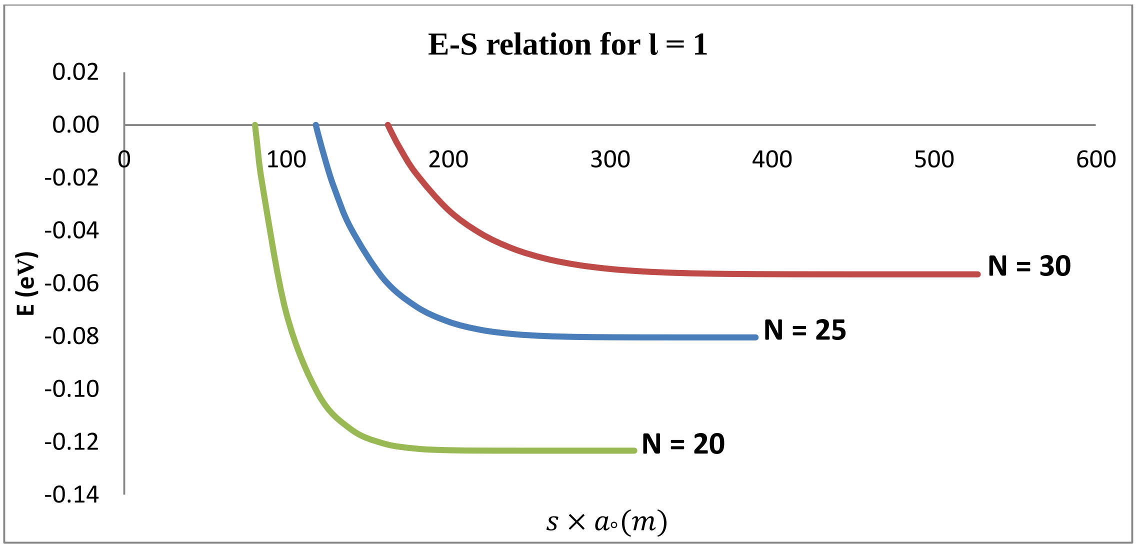

Fig (3. 13): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given space dimension when the angular momentum l = 1(m).

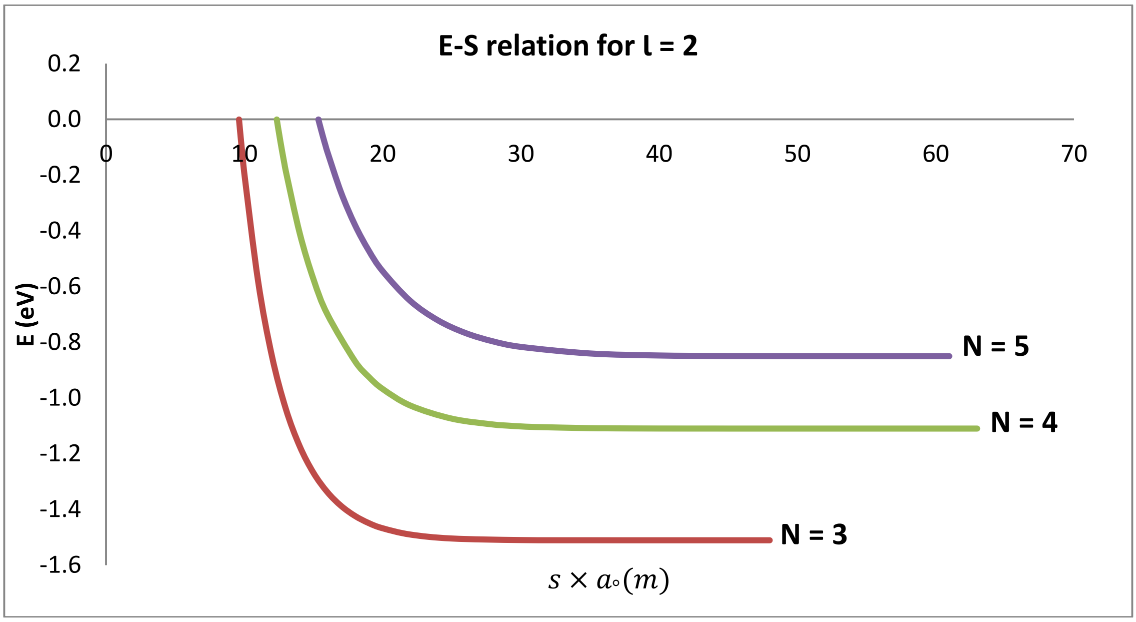

Fig (3. 14): Relation between energy eigenvalues of a confined H-atom and the radius of the cavity for a given space dimension when the angular momentum ɭ = 2(m).

By studying the above data and the graphs, one can conclude the dependence of excited energy eigenvalues on the cavity radii and the space dimensions. In a specific dimension, N; it is obvious that the energy for any excited states (ɭ > 0) of a confined H-atom decreases as the cavity radius increases until it becomes equal to the energy of a free H-atom in that dimension. This means that the confinement effect becomes negligible at this limit. Conversely, as the cavity radius decreases the excited energy increases until it approaches almost zero where the kinetic and potential energy are equal at a value of the cavity radius is called critical cage radius (S_c). This is simply because the cavity is impenetrable and as it becomes smaller the confined H-atom will experience more collisions with the cavity wall. In other words, its kinetic energy (positive energy) increases while the confined energy (negative energy) reduces and hence the total energy increases.

For a specific angular momentum (ɭ > 0), it is found that the excited energy eigenvalues are dimension dependent, they increase as N increases, so the energies of excited states increase even with l when N is fixed or with N when l is fixed, and this is due to the increase in the value of repulsive term in the effective potential which causes the total energy to be less negative or increase. See V_{eff} formula [64]:

$$V_{eff} = V(r) + \left[l(l + N - 2) + \frac{(N-1)(N-3)}{4} \right] \frac{1}{r^2} \qquad 3.1.1$$

Also, it is noticed that for a given N as l increases or for a given l as N increases, the curves shifted to the right of x-axes which indicates that critical cage radius increases for higher states or higher space dimensions. [Discussed later in part 2].

- **Specific values of S**

This section is an extension to the previous one. Precisely, we chose three radii S= 22× $a_\circ$, 50× $a_\circ$ and 100× $a_\circ$. For each radius, we have calculated all the bound excited state energies in different space dimensions. The number of bound states is equal to the number of the roots of the confluent hypergeometric polynomials. Once we locate the zeros, the bound state energies can be easily determined using equation 2.1.22. To that end, we have graphed these energies for the first time with the corresponding angular momentums for a given N; [Fig3.15 - Fig3.19] and then with the corresponding space dimensions for a given angular momentum l [Fig3.20 - Fig3.25].

Table (3. 15): Bound excited states energies for the confined H-atom in different space dimensions when the radius S = 22 × $a_\circ$ (m).

S = 22 × $a_\circ$	N = 3	N = 4	N = 5
l	E(eV)	E(eV)	E(eV)
1	-3.399969091	-2.174071700	-1.490479794
2	-1.490479794	-1.027429279	-0.651067963
3	-0.651067963	-0.307511462	
S = 22 × $a_\circ$	N= 6	N= 7	N = 8
l	E(eV)	E(eV)	E(eV)
1	-1.027429279	-0.651067963	-0.307511462
2	-0.307511462		

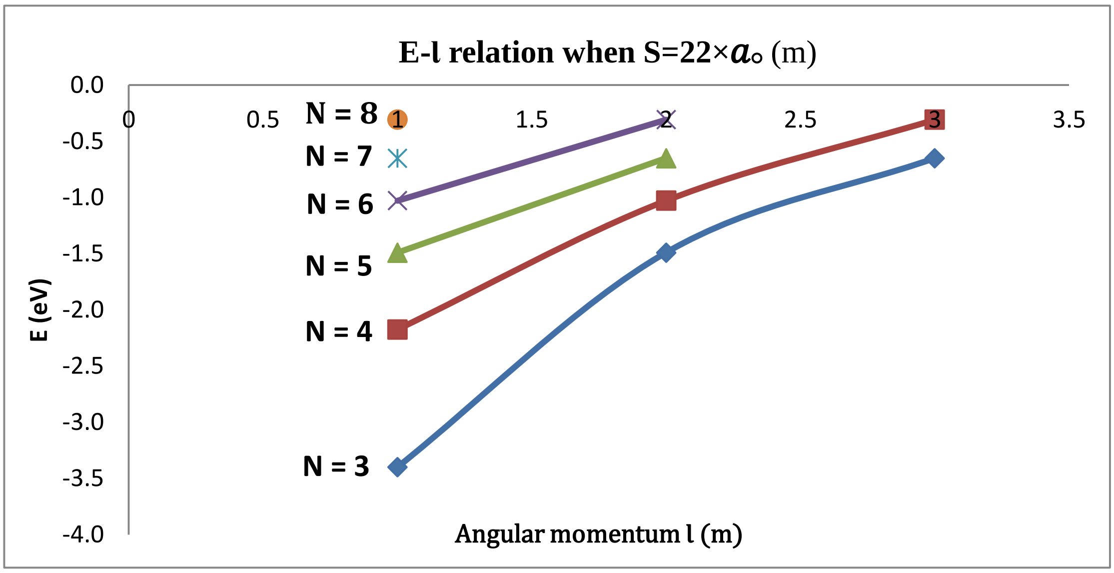

Fig (3. 15): Relation between energy eigenvalues of a confined H-atom and angular momentum for S=22× $a_\circ$ in different space dimensions.

Table (3. 16): Bound excited states energies for the confined H-atom in different space dimensions when the radius S = 50 × $a_\circ$ (m)

S = 50 × $a_\circ$	N = 3	N = 4	N = 5	N= 6	N= 7
l	E(eV)	E(eV)	E(eV)	E(eV)	E(eV)
1	-3.40000000	-2.167000000	-1.51110809	-1.11020186	-0.84988440
2	-1.51110809	-1.110201861	-0.84988440	-0.67037110	-0.53788639
3	-0.84988440	-0.670371100	-0.53788639	-0.43128013	-0.33799504
4	-0.53788639	-0.431280128	-0.33799504	-0.25119060	-0.16720952
5	-0.33799504	-0.251190600	-0.16720952	-0.08402004	-0.00042856
6	-0.16720952	-0.084020040	-0.00042856		
7	-0.00042856				
S = 50 × $a_\circ$	N = 8	N= 9	N= 10	N = 11	N= 12
l	E(eV)	E(eV)	E(eV)	E(eV)	E(eV)
1	-0.67037110	-0.53788639	-0.431280128	-0.33799504	-0.25119060
2	-0.43128013	-0.33799504	-0.251190600	-0.16720952	-0.08402004
3	-0.25119060	-0.16720952	-0.084020040	-0.00042856	
4	-0.08402004	-0.00042856			
S = 50 × $a_\circ$	N = 13	N=14	N=15		
l	E(eV)	E(eV)	E(eV)		
1	-0.16720952	-0.08402004	-0.00042856		
2	-0.00042856				

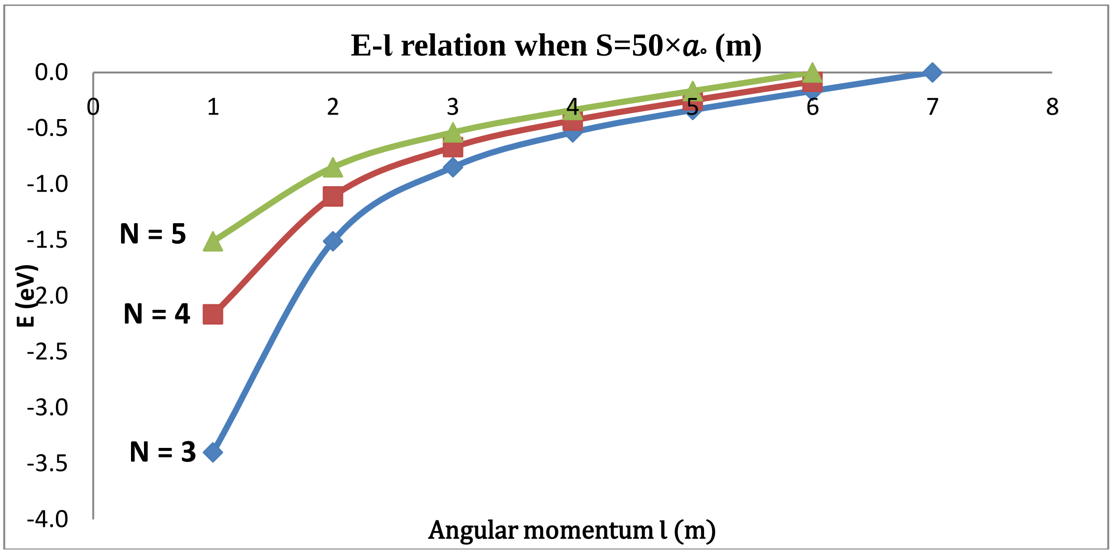

Fig (3. 16): Relation between energy eigenvalues of a confined H-atom and angular momentum for S=50 × $a_\circ$ in different space dimensions.

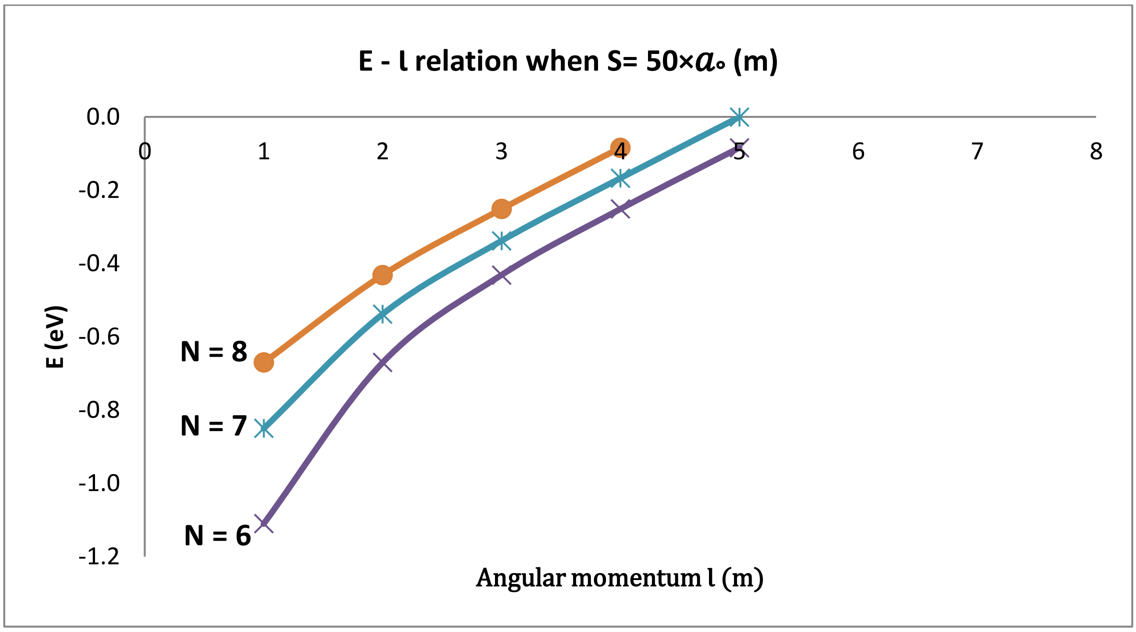

Fig (3. 17): Relation between energy eigenvalues of a confined H-atom and angular momentum for S=50 × $a_\circ$ in different space dimensions.

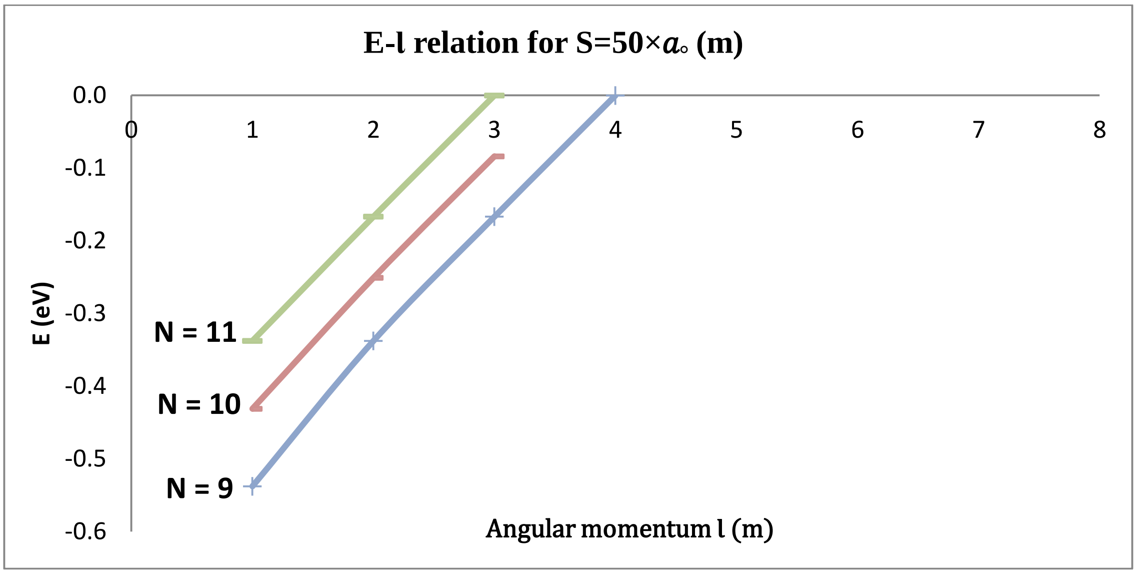

Fig (3. 18): Relation between energy eigenvalues of a confined H-atom and angular momentum for S=50 × $a_\circ$ in different space dimensions.

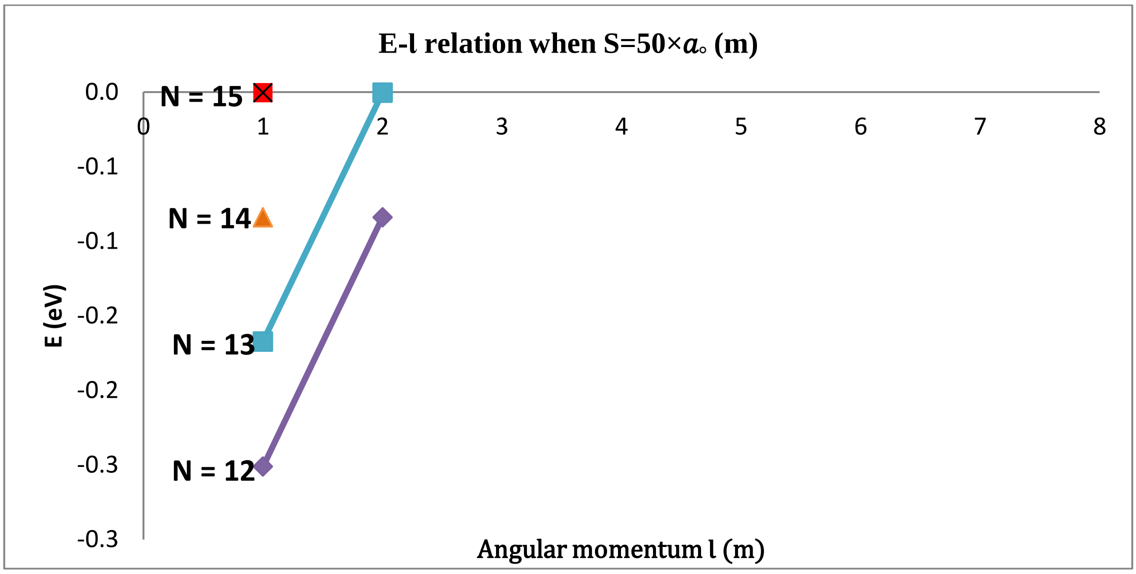

Fig (3. 19): Relation between energy eigenvalues of a confined H-atom and angular momentum for S=50 × $a_\circ$ in different space dimensions.

Table (3.17): Bound excited states energies for the confined H-atom in different space dimensions when the radius S = 100 × $a_\circ$.

S = 100 × $a_\circ$	N = 3	N = 4	N = 5	N= 6	N= 7
l	E(eV)	E(eV)	E(eV)	E(eV)	E(eV)
1	-3.400000000	-2.176000000	-1.511112622	-1.110205747	-0.850000000
2	-1.511112622	-1.110205747	-0.850000000	-0.671603595	-0.544000000
3	-0.850000000	-0.671603595	-0.544000000	-0.449585878	-0.377765689
4	-0.544000000	-0.449585878	-0.377765689	-0.321784057	-0.276977619
5	-0.377765689	-0.321784057	-0.276977619	-0.239798662	-0.207507869
6	-0.276977619	-0.239798662	-0.207507869	-0.178160836	-0.150494576
7	-0.207507869	-0.178160836	-0.150494576	-0.123725181	-0.097358087
8	-0.150494576	-0.123725181	-0.097358087	-0.071078362	-0.044677684
9	-0.097358087	-0.071078362	-0.044677684	-0.018009952	
10	-0.044677684	-0.018009952			
S = 100 × $a_\circ$	N = 8	N= 9	N= 10	N = 11	N= 12
l	E(eV)	E(eV)	E(eV)	E(eV)	E(eV)
1	-0.671603595	-0.544000000	-0.449585878	-0.377765689	-0.321784057
2	-0.449585878	-0.377765689	-0.321784057	-0.276977619	-0.239798662
3	-0.321784057	-0.276977619	-0.239798662	-0.207507869	-0.178160836
4	-0.239798662	-0.207507869	-0.178160836	-0.150494576	-0.123725181
5	-0.178160836	-0.150494576	-0.123725181	-0.097358087	-0.071078362
6	-0.123725181	-0.097358087	-0.071078362	-0.044677684	-0.018009952
7	-0.071078362	-0.044677684	-0.018009952		
8	-0.018009952				
S = 100 × $a_\circ$	N = 13	N= 14	N= 15	N = 16	N= 17
l	E(eV)	E(eV)	E(eV)	E(eV)	E(eV)
1	-0.276977619	-0.239798662	-0.207507869	-0.178160836	-0.150494576
2	-0.207507869	-0.178160836	-0.150494576	-0.123725181	-0.097358087
3	-0.150494576	-0.123725181	-0.097358087	-0.071078362	-0.044677684
4	-0.097358087	-0.071078362	-0.044677684	-0.018009952	
5	-0.044677684	-0.018009952			
S = 100 × $a_\circ$	N = 18	N= 19	N= 20	N = 21	N= 22
l	E(eV)	E(eV)	E(eV)	E(eV)	E(eV)
1	-0.123725181	-0.097358087	-0.071078362	-0.044677684	-0.018009952
2	-0.071078362	-0.044677684	-0.018009952		
3	-0.018009952				

Table (3.17): Bound excited states energies for the confined H-atom in the allowed space dimensions when the radius S = 22 × $a_{\circ}$.

S = 22 × $a_{\circ}$	ɩ = 1	ɩ = 2	ɩ = 3
N	E(eV)	E(eV)	E(eV)
3	-3.399969091	-1.490479794	-0.651067963
4	-2.17407170	-1.027429279	-0.307511462
5	-1.490479794	-0.651067963	
6	-1.027429279	-0.307511462	
7	-0.651067960		
8	-0.307511462		

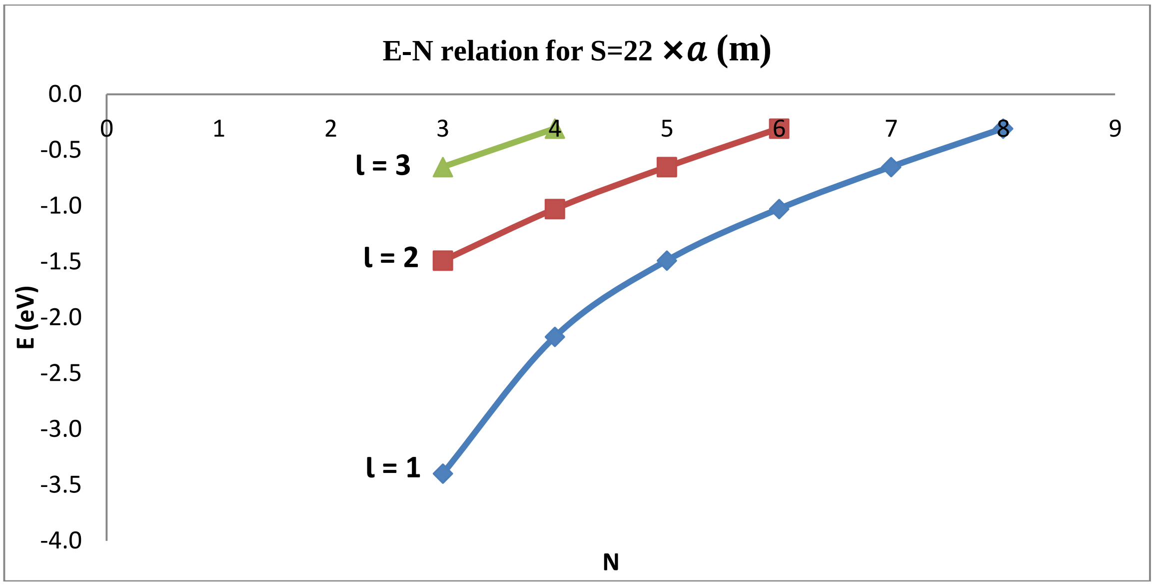

Fig (3. 20): Relation between energy eigenvalues of a confined H-atom and space dimensions N for S=22 × $a_{\circ}$ in different angular momentums.

Table (3. 17): Bound excited states energies for the confined H-atom in the allowed space dimensions when the radius S = 50 × $a_\circ$.

S = 50 × $a_\circ$	ℓ = 1	ℓ = 2	ℓ = 3
N	E(eV)	E(eV)	E(eV)
3	-3.400000000	-1.511108089	-0.849884404
4	-2.176000000	-1.110201861	-0.670371096
5	-1.511108089	-0.849884404	-0.537886394
6	-1.110201861	-0.670371096	-0.431280128
7	-0.849884404	-0.537886394	-0.337995042
8	-0.670371100	-0.431280130	-0.251190600
9	-0.537886394	-0.337995042	-0.167209524
10	-0.431280128	-0.251190602	-0.084020042
S = 50 × $a_\circ$	ℓ = 5	ℓ = 6	ℓ = 7
N	E(eV)	E(eV)	E(eV)
3	-0.337995042	-0.167209524	-0.000428562
4	-0.251190602	-0.084020042	
5	-0.167209524	-0.000428562	
6	-0.084020042		
7	-0.000428562		
8	-0.337995042		
9	-0.251190602		
10	-0.167209524		

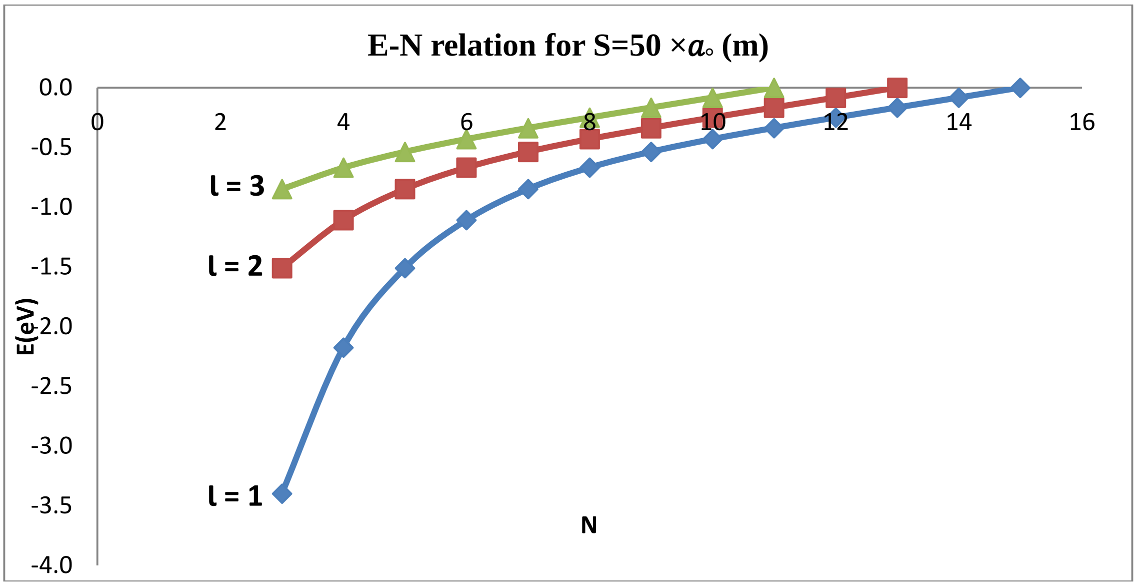

Fig (3. 21): Relation between energy eigenvalues of a confined H-atom and space dimensions N for S=50 × $a_\circ$. in different angular momentums .

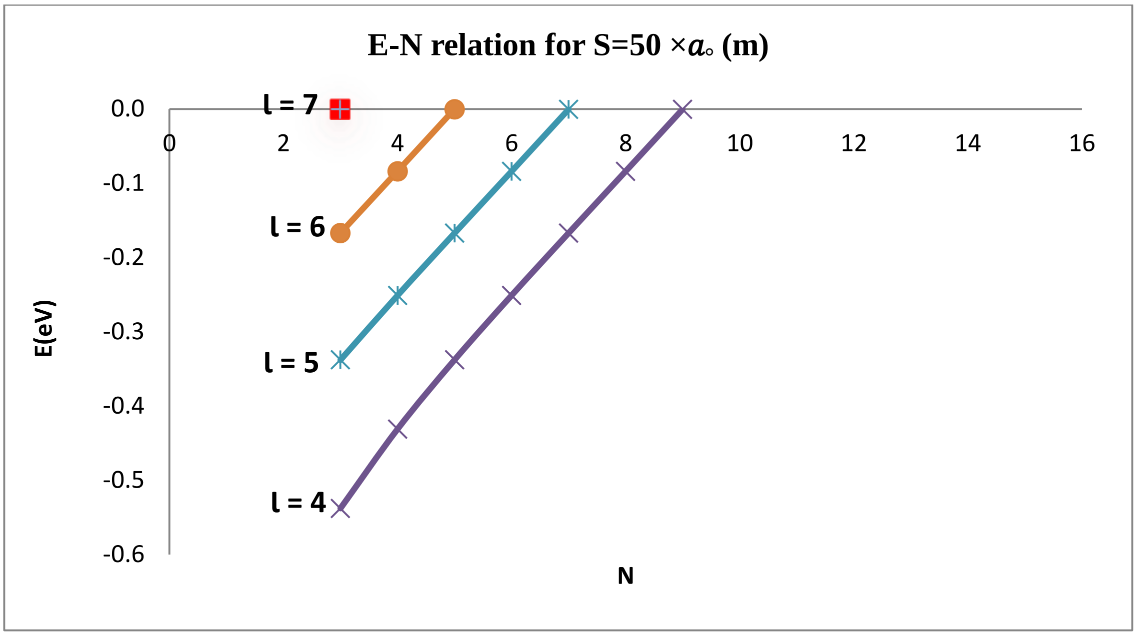

Fig (3. 22): Relation between energy eigenvalues of a confined H-atom and space dimensions N for S=50 × $a_\circ$ in different angular momentums.

.

Table (3. 18): Bound excited states energies for the confined H-atom in the allowed space dimensions when the radius $S = 100 \times a_\circ$.

$S = 100 \times a_\circ$	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 5$
N	E(eV)	E(eV)	E(eV)	E(eV)	E(eV)
3	-3.400000000	-1.511112622	-0.850000000	-0.544000000	-0.377765689
4	-2.176000000	-1.110205747	-0.671603595	-0.449585878	-0.321784057
5	-1.511112622	-0.850000000	-0.544000000	-0.377765689	-0.276977619
6	-1.110205747	-0.671603595	-0.449585878	-0.321784057	-0.239798662
7	-0.850000000	-0.544000000	-0.377765689	-0.276977619	-0.207507869
8	-0.671603595	-0.449585878	-0.321784057	-0.239798662	-0.178160836
9	-0.544000000	-0.377765689	-0.276977619	-0.207507869	-0.150494576
10	-0.449585878	-0.321784057	-0.239798662	-0.178160836	-0.123725181
11	-0.377765689	-0.276977619	-0.207507869	-0.150494576	-0.097358087
12	-0.321784057	-0.239798662	-0.178160836	-0.123725181	-0.071078362
13	-0.276977619	-0.207507869	-0.150494576	-0.097358087	-0.044677684
14	-0.239798662	-0.178160836	-0.123725181	-0.071078362	-0.018009952
15	-0.207507869	-0.150494576	-0.097358087	-0.044677684	
16	-0.178160836	-0.123725181	-0.071078362	-0.018009952	
17	-0.150494576	-0.097358087	-0.044677684		
18	-0.123725181	-0.071078362	-0.018009952		
19	-0.097358087	-0.044677684			
20	-0.071078362	-0.018009952			
21	-0.044677684				
22	-0.018009952				
$S = 100 \times a_\circ$	$l = 6$	$l = 7$	$l = 8$	$l = 9$	$l = 10$
N	E(eV)	E(eV)	E(eV)	E(eV)	E(eV)
3	-0.276977619	-0.207507869	-0.150494576	-0.097358087	-0.044677684
4	-0.239798662	-0.178160836	-0.123725181	-0.071078362	-0.018009952
5	-0.207507869	-0.150494576	-0.097358087	-0.044677684	
6	-0.178160836	-0.123725181	-0.071078362	-0.018009952	
7	-0.150494576	-0.097358087	-0.044677684		
8	-0.123725181	-0.071078362	-0.018009952		
9	-0.097358087	-0.044677684			
10	-0.071078362	-0.018009952			
11	-0.044677684				
12	-0.018009952				

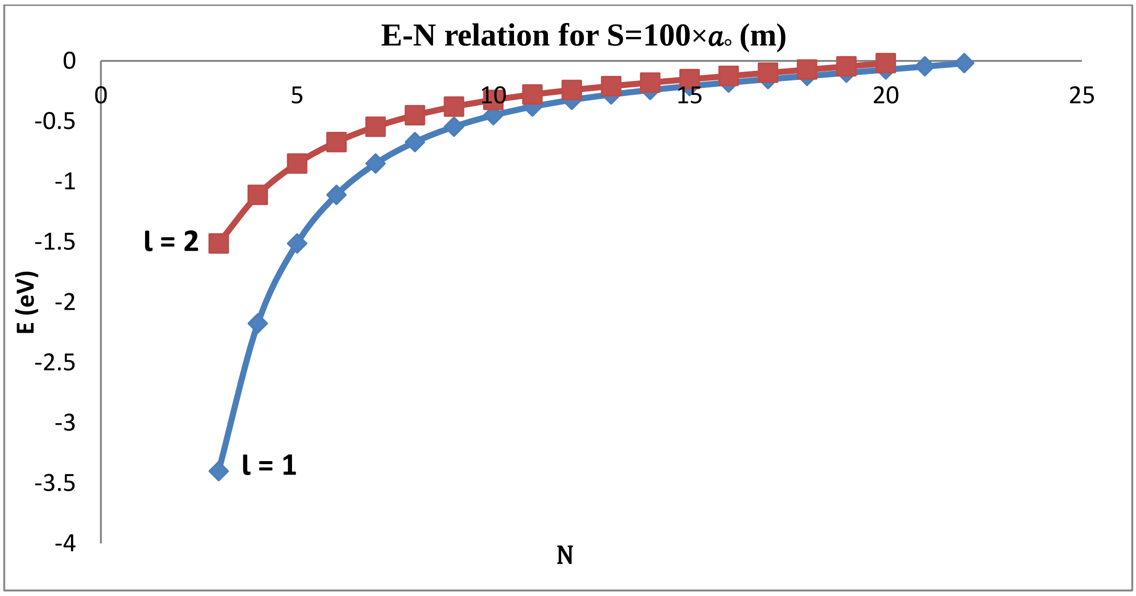

Fig (3. 23): Relation between energy eigenvalues of a confined H-atom and space dimensions N for S=100 × $a_\circ$ in different angular momentums.

.

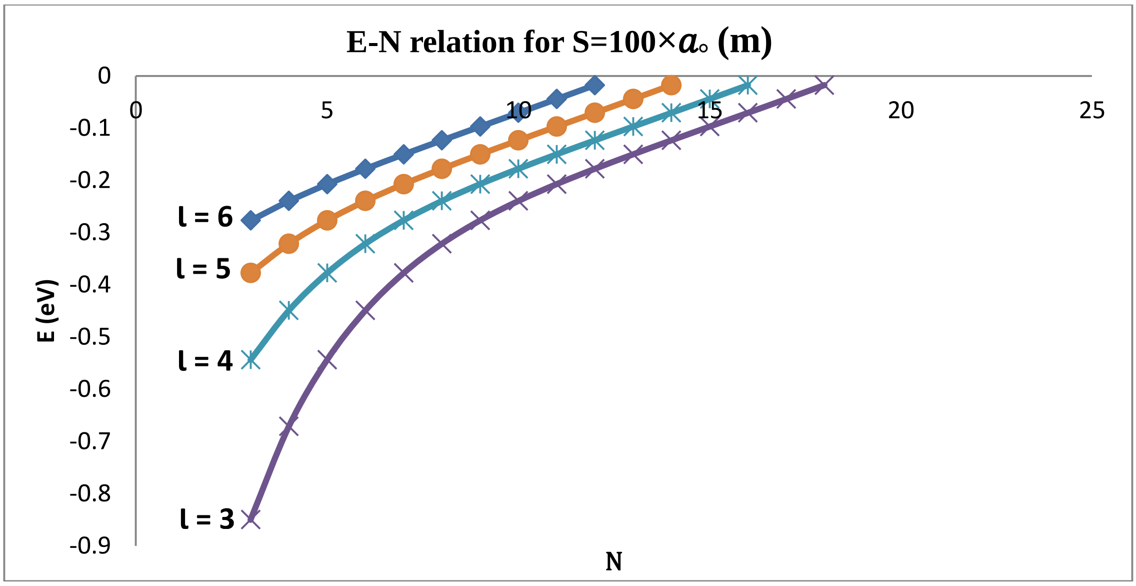

Fig (3. 24): Relation between energy eigenvalues of a confined H-atom and space dimensions N for S=100 × $a_\circ$ in different angular momentums.

.

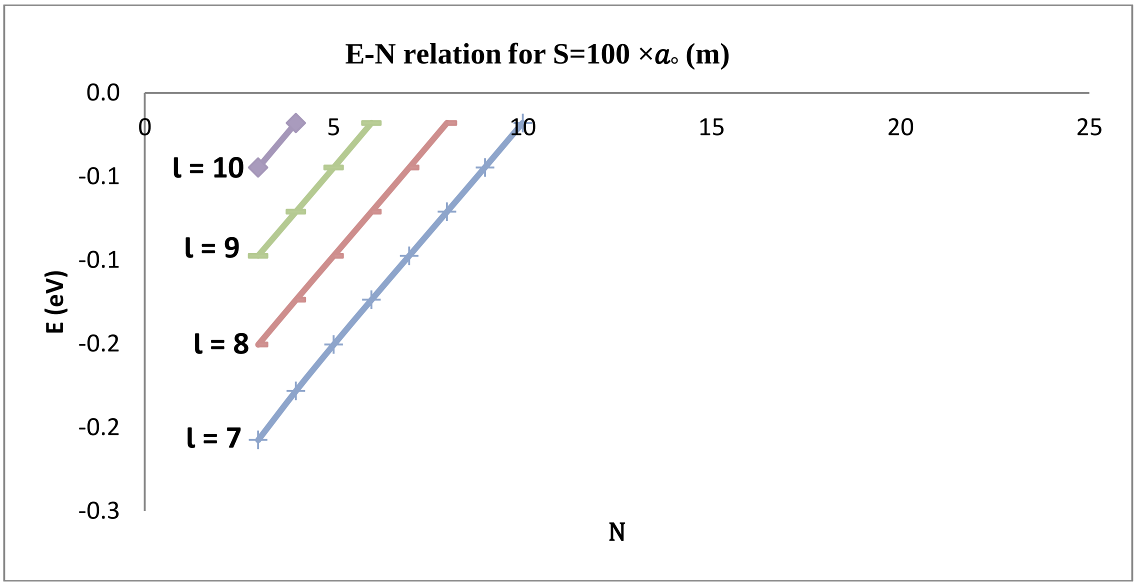

Fig (3. 25): Relation between energy eigenvalues of a confined H-atom and space dimensions N for S=100 × $a_\circ$ in different angular momentums.

The above graphs and data for the three chosen radii clarified the behavior of the H-atom when it is trapped in an impenetrable spherical cavity. Being in a higher state means being less bounded to the nucleus so; the total energy keeps increasing until there are no other bound states. For a specific S, there are limited higher states the electron is allowed to be in; the larger cavity radius allowed more additional bound states in a given dimension. For example, N=3, S=22× $a_\circ$ (m) only three higher states are allowed while ten higher states when S=100× $a_\circ$ (m) in the same dimension. This is absolutely reasonable, as the electron excited to higher states the atomic shell become larger due to the increase in the value of the repulsive term in the effective potential.

For a specific S, there are limited number of higher dimensions still have bound states; larger the cavity radius larger the allowed higher dimensions, S=22× $a_\circ$ (m) is allowed to be in five higher dimensions started from three while S=100× $a_\circ$ (m) can be in nineteen higher dimensions started from three too. This is simply because the cavity is not empty, equation 3.1.1

illustrates that as the dimension increases the electron is repelled away from the nucleus due to the rise in the value of repulsive term in the effective potential.

For a given S, it is noticed that there is a shift in the energy eigenvalues for higher states and dimensions; the second excited energy in a lower dimension will be the first excited energy in the next higher dimension, and the third excited energy in that lower dimension will be the second excited energy for that higher dimension and so on, it is an alternative sequence for odd dimensions alone or even dimensions alone. So, the number of bound states in each dimension or the number of the allowed dimensions can be easily predicted by relation [2.2.6]. For deep understanding imagine a constant horizontally line drawn between these degenerated values. This make the plot somehow looks like step function. So researchers in this domain just need the information about N=3 and N=4, they give the indications about any higher dimension.

On the other hand, it is noticed that for different values of S there are similarities in energy eigenvalues for the same angular momentum and dimension. For example, for N=3 and $\ell=1$, E = -3.399 eV, -3.400 eV, -3.400 eV, for S= $22\times a_\circ$, $50\times a_\circ$, $100\times a_\circ$ respectively. This is an accidental degeneracy occurs because the three radii are almost equal or higher than S=$24\times a_\circ$ at which the H-atom act as free (unconfined) one for N=3 and $\ell=1$.The same mythology explains all the accidental degeneracies for different angular momentums or space dimensions.

Part 2: Critical Cage Radius ($S_c \times a_\circ$):

Our results show that the binding energy for the confined H-atom diminishes as the cavity radius decreases, so it is very important to detect the minimum cavity size that still allows a bound state which means finding the critical value. This critical value is called critical cage radius ($S_c \times a_\circ$); it is the minimum cavity radius at which the total energy almost approaches zero and so after it there is no more bound states. The behavior of the confined H-atom when the radius of the cavity is greater than critical cage radius vanishes is out of our interests in this study.

For the same reason mentioned before, the critical cage radii for N=3 and N=4 are only obtained for different higher states.

The data and graphs [Fig3.26 – Fig3.28] show that the critical cage value increases with angular momentum for a specific dimension, and the graph [fig3.29] indicates that critical cage value increases with space dimension for a specific angular momentum. This is due to the increase in repulsion potential.

Critical cage radius of the cavity $S_c \times a_\circ$ in different angular momentums when the space dimension N=3.

N = 3	
l	$S_c \times a_\circ$
1	5.883100000
2	9.617366900
3	15.36345004
4	22.29216900
5	30.38042000
6	39.61140001
7	49.97216000
8	61.45244000
9	74.04380000
10	87.73917000
11	102.5325300
12	118.4186600
13	135.3930100
14	153.4515440
15	172.5906800

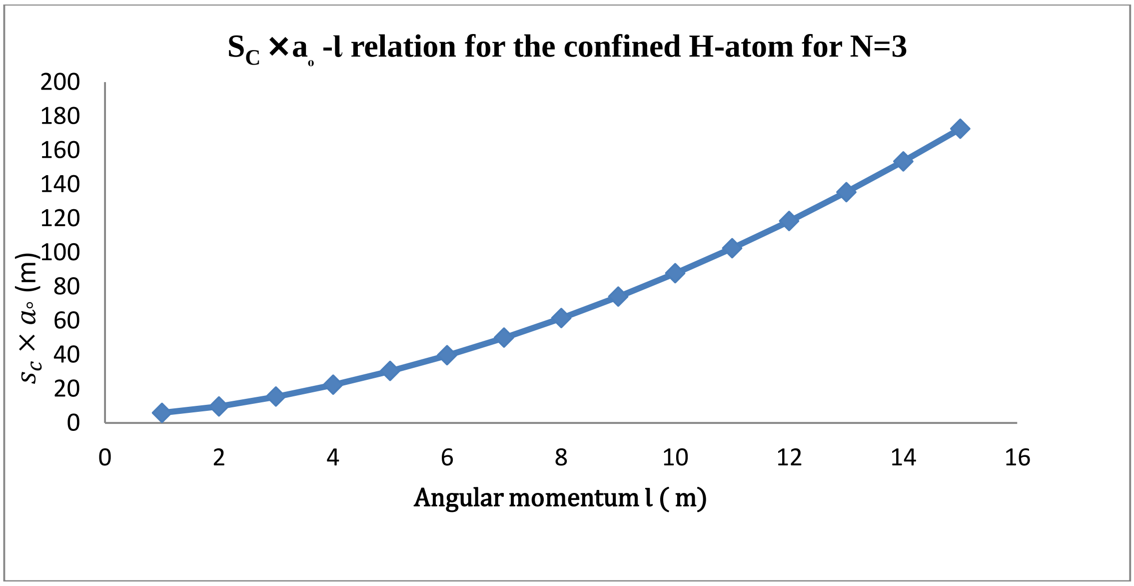

Fig (3. 26): Relation between the critical cage radius of the cavity and the angular momentums when the space dimension N=3.

Table (3. 19): critical cage radius of the cavity $S_c \times a_o$ in different angular momentums when the space dimension N=4.

N = 4	
l	$S_c \times a_\circ$
1	7.197900000
2	12.34079000
3	18.68161020
4	26.19251520
5	34.85400000
6	44.65124405
7	55.57292340
8	67.60970500
9	80.75388100
10	94.99894130
11	110.3392923
12	126.7700671
13	144.2869850
14	162.8862420
15	182.5644360

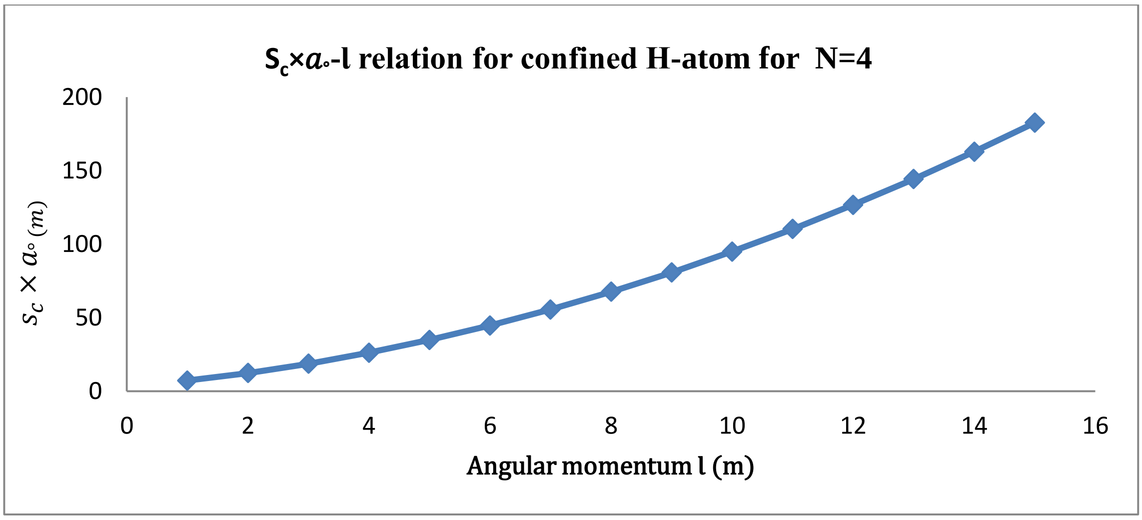

Fig (3. 27): Relation between the critical cage radius of the cavity and the angular momentums when the space dimension N=4.

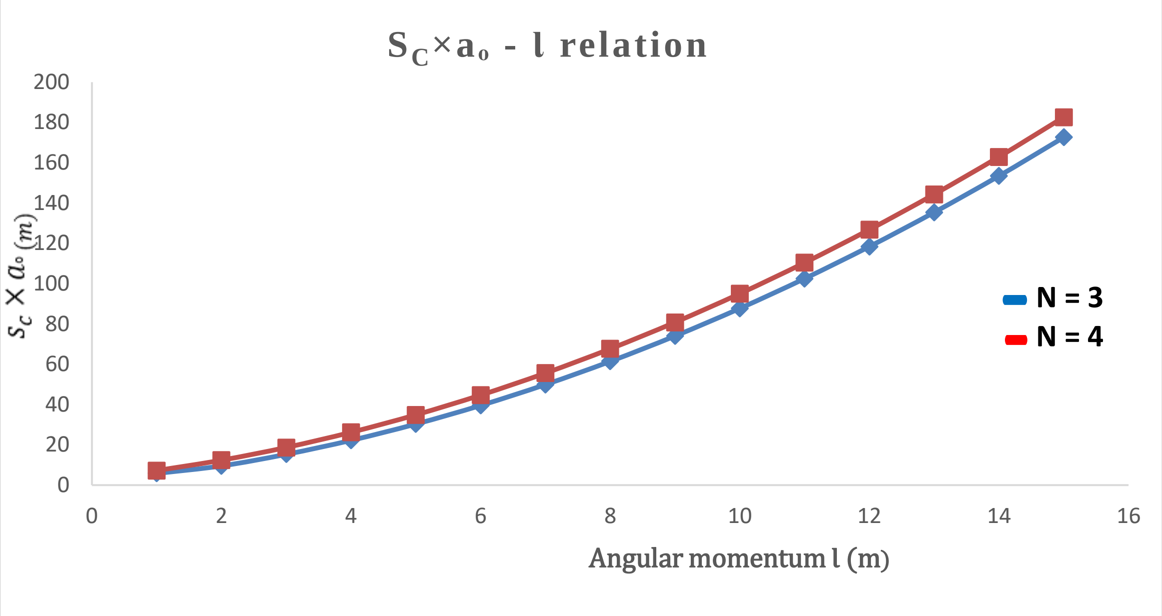

Fig (3. 28): Relation between the critical cage radius of the cavity and the angular momentums in two dimensions N=3 and N=4.

Table (3. 20): critical cage radius of the cavity $S_c \times a_\circ$ in different space dimensions for the first excited states when angular momentums ɭ =1.

ɭ = 1	
N	$S_c \times a_\circ$
3	5.883100
4	7.197900
5	9.617367
6	12.34079
7	15.36345
8	18.68161
9	22.29217
10	26.19252
20	80.75388
25	118.4187

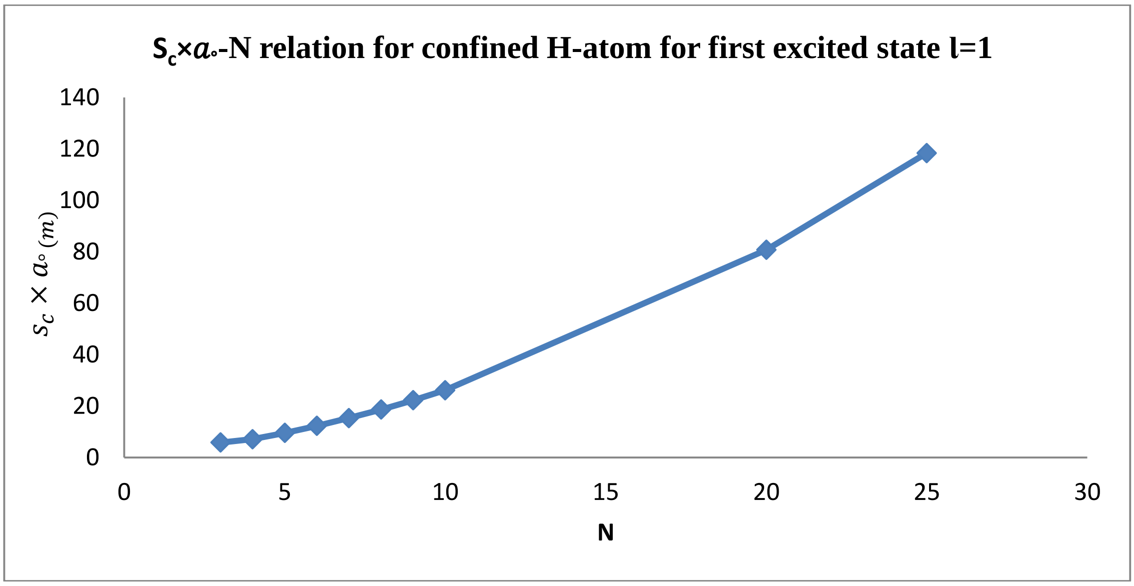

Fig (3. 29): Relation between the critical cage radius of the cavity and the space dimensions when the angular momentum ɩ =1.

- **Pressure**

  The pressure exerted on a confined H-atom is a function of the radius of the cavity S and the spatial dimension N [66]. Using equation 2.3.5 and taking into account the correlations shown in equation 2.2.6 we will calculate the pressure in different space dimensions.

$$P_N = \frac{\left(\frac{N}{2}-1\right)!}{\pi^{N/2}} \frac{X_n^2 a_\circ^2}{4S^{N+2}} E_\circ \qquad 3.3.1$$

For N =3, $P_{N=3} = \frac{13.6\, X_n^2}{8\pi S^5}\left(\frac{eV}{a_\circ^3}\right)$ 3.3.2

For N =4, $P_{N=4} = \frac{13.6\, X_n^2}{4\pi^2 S^6}\left(\frac{eV}{a_\circ^3}\right)$ 3.3.3

The calculations are presented in Tables and graphs:

Table (3. 21): Pressures exerted on a confined H- atom and the radius of the cavity for N=3 and ꞁ=1,2,3.

$l = 1$		$l = 2$		$l = 3$	
S$\times a_\circ$ (m)	P $(ev/a_\circ^3)$	S$\times a_\circ$ (m)	P $(ev/a_\circ^3)$	S$\times a_\circ$ (m)	P $(ev/a_\circ^3)$
23.00000	0.0000444750	48.00000	0.0000021747	61.00000	0.0000005960
22.00000	0.0000508191	44.00000	0.0000028233	55.00000	0.0000008131
20.00000	0.0000676381	40.00000	0.0000037578	50.00000	0.0000010821
18.00000	0.0000927672	35.00000	0.0000056091	44.00000	0.0000015867
16.00000	0.0001319940	30.00000	0.0000089031	40.00000	0.0000021075
14.00000	0.0001964770	25.00000	0.0000153272	35.00000	0.0000031189
12.00000	0.0003087780	20.00000	0.0000292033	30.00000	0.0000048131
10.00000	0.0005145450	18.00000	0.0000388476	24.00000	0.0000082998
8.00000	0.0008831380	16.00000	0.0000519802	19.00000	0.0000109095
6.00000	0.0011134300	14.00000	0.0000680171	16.00000	0.0000044402
5.90000	0.0010759530	12.00000	0.0000782880	15.37000	0.0000000557
5.80000	0.0010254420	10.00000	0.0000307047	15.36500	0.0000000132
5.70000	0.0009594530	9.900000	0.0000238280	15.36490	0.0000000123
5.60000	0.0008750720	9.800000	0.0000161847	15.36410	0.0000000055
5.50000	0.0007688720	9.650000	0.0000031197	15.36390	0.0000000038
5.40000	0.0006367630	9.619000	0.0000001587	15.36385	0.0000000034
5.08900	0.0000018895	9.617390	0.0000000023	15.36381	0.0000000031
5.08850	0.0000005240	9.617370	0.0000000004	15.36370	0.0000000021
5.88330	0.0000000287	9.617369	0.0000000003	15.36410	0.0000000055
5.88310	0.0000000023	9.617367	0.0000000001	15.36346	0.0000000001

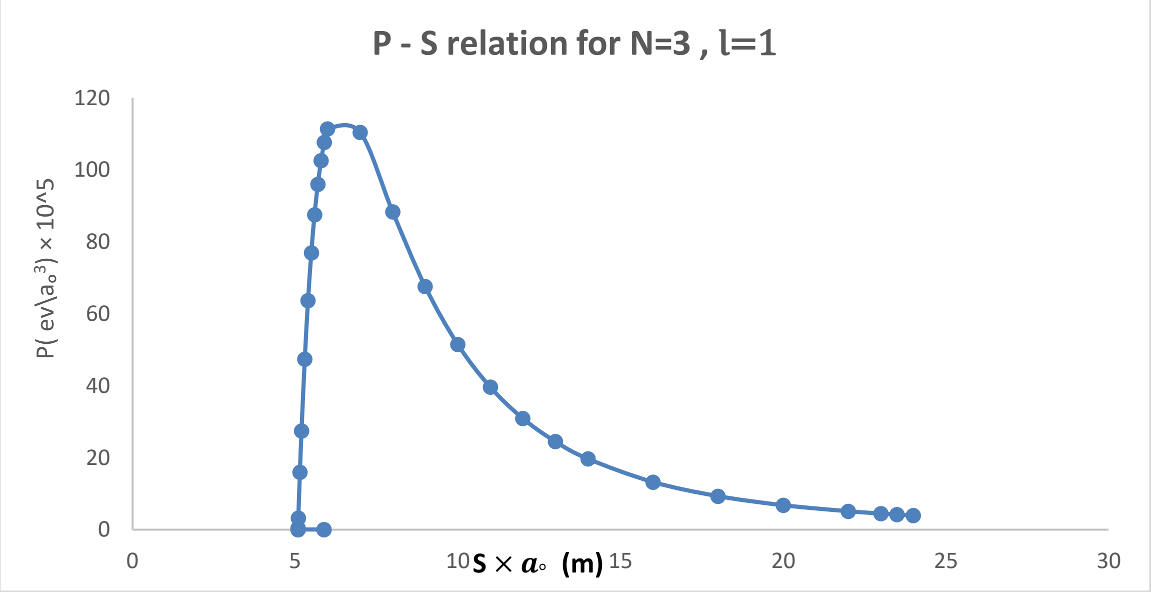

Fig (3. 30): Pressure vs. cavity radius when N=3 and ꞁ=1.

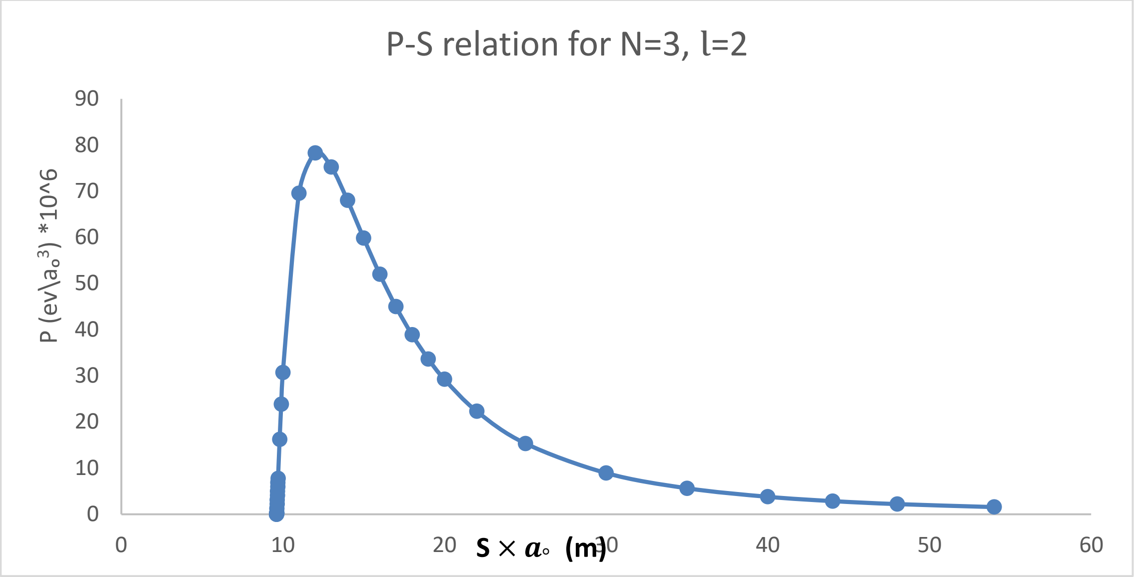

Fig (3. 31): Pressure vs. cavity radius when N=3 and l=2.

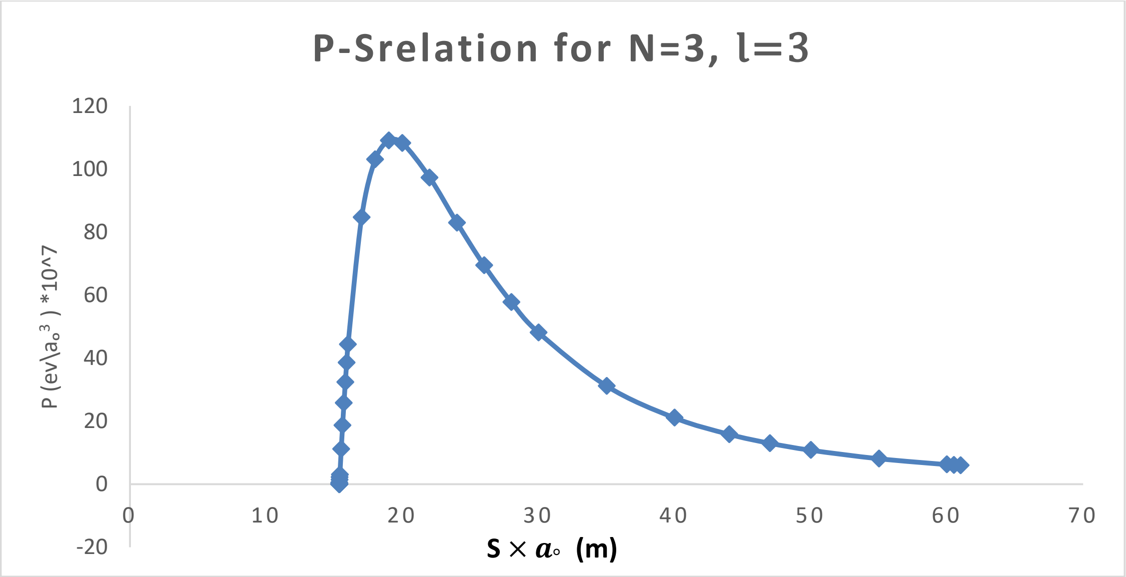

Fig (3. 32): Pressure vs. cavity radius when N=3 and l=3.

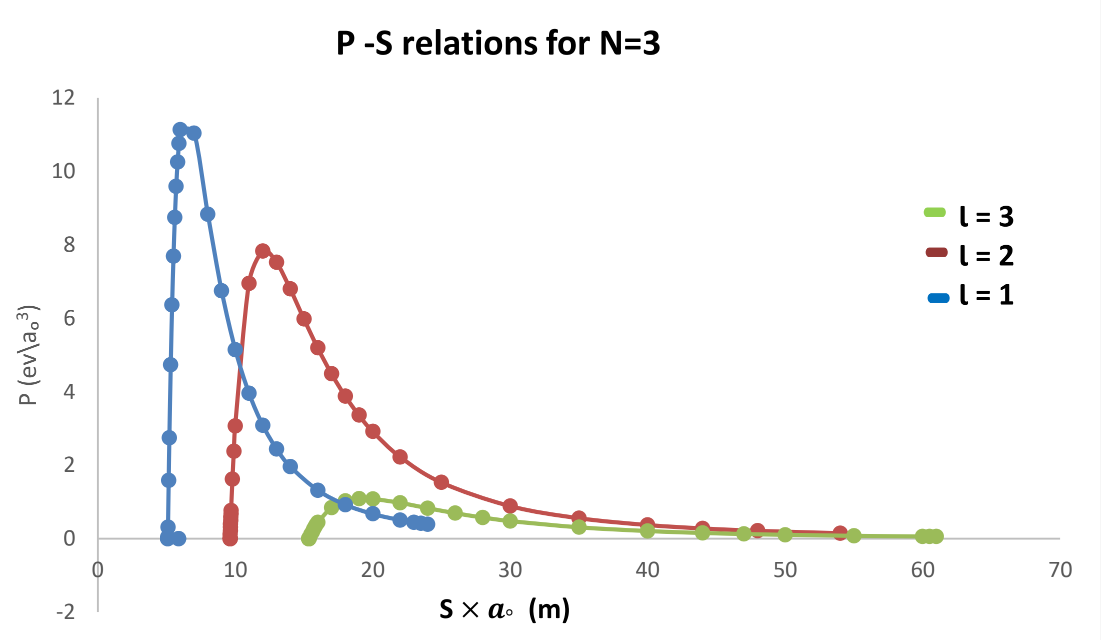

Fig (3. 33): Pressure vs. cavity radius when N=3 for different angular momentums.

Table (3. 22): Pressures exerted on a confined H- atom and the radius of the cavity for N=4 and ɭ=1,2,3.

$l = 1$		$l = 2$		$l = 3$	
S× $a_\circ$ (m)	P $(ev/a_\circ^3)$	S× $a_\circ$ (m)	P $(ev/a_\circ^3)$	S× $a_\circ$ (m)	P $(ev/a_\circ^3)$
35.0000	0.0000001469	63.0000	0.0000000071	81.0000	0.0000000016
30.0000	0.0000002387	60.0000	0.0000000076	75.0000	0.0000000022
25.0000	0.0000002722	57.0000	0.0000000081	70.0000	0.0000000028
20.0000	0.0000003117	55.0000	0.0000000087	65.0000	0.0000000038
19.0000	0.0000003587	50.0000	0.0000000123	60.0000	0.0000000052
18.0000	0.0000005643	40.0000	0.0000000300	55.0000	0.0000000074
14.0000	0.0000013744	38.0000	0.0000000361	50.0000	0.0000000109
13.0000	0.0000016844	36.0000	0.0000000439	46.0000	0.0000000151
12.0000	0.0000020850	30.0000	0.0000001379	40.0000	0.0000000261
11.0000	0.0000054612	25.0000	0.0000002787	36.0000	0.0000000388
10.0000	0.0000071344	20.0000	0.0000006130	30.0000	0.0000000732
9.0000	0.0000093804	15.0000	0.0000011497	25.0000	0.0000001191
8.0000	0.0000123140	12.9000	0.0000005892	22.0000	0.0000001330
7.5000	0.0000146470	12.8000	0.0000005066	19.0000	0.0000000304
7.0010	0.0000000832	12.7000	0.0000004151	18.9000	0.0000000215
7.1990	0.0000004312	12.6000	0.0000003139	18.8000	0.0000000120
7.1985	0.0000002772	12.5000	0.0000002021	18.7000	0.0000000019
7.1980	0.0000000442	12.4000	0.0000000788	18.6900	0.0000000009
7.1980	0.0000000247	12.3500	0.0000000126	18.6840	0.0000000003
7.1979	0.0000000091	12.3408	0.00000000002	18.6830	0.0000000001

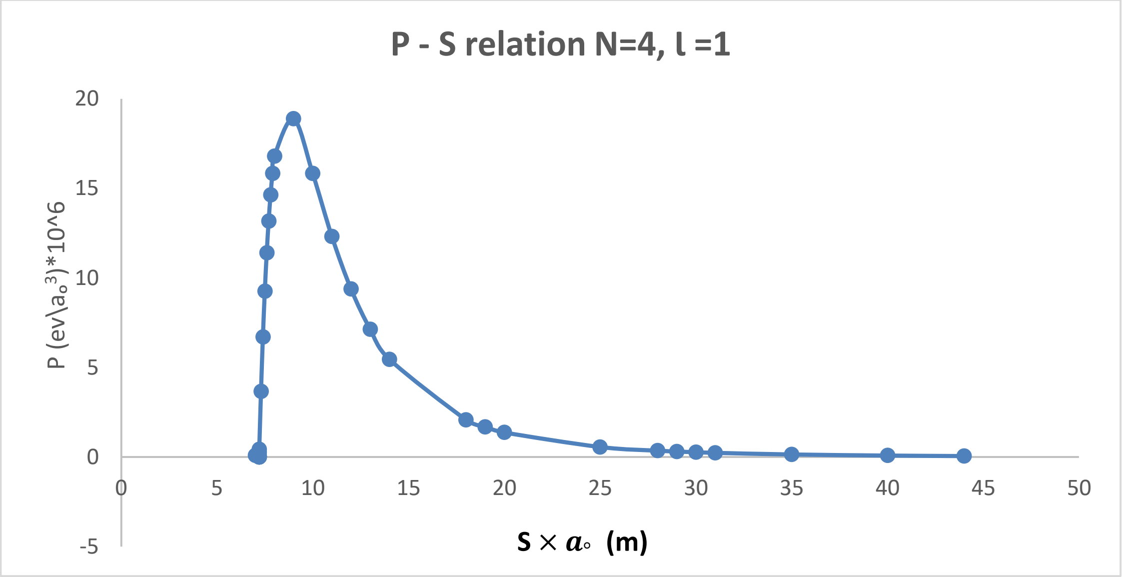

Fig (3. 34): Pressure vs. cavity radius when N=4 and ɩ =1.

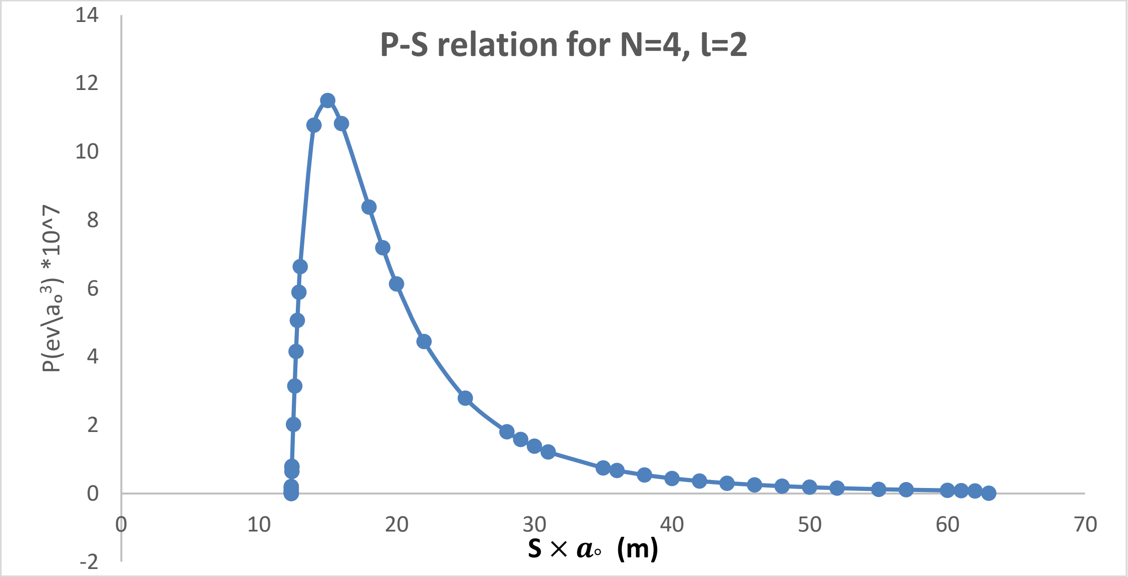

Fig (3. 35): Pressure vs. cavity radius when N=4 and ɩ =2.

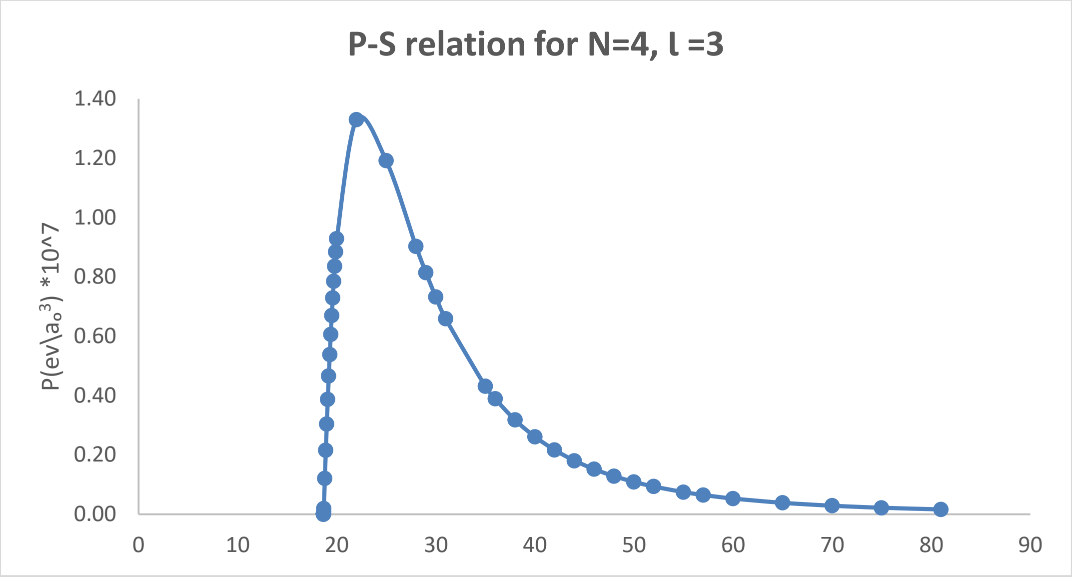

Fig (3. 36): Pressure vs. cavity radius when N=4 and l =3.

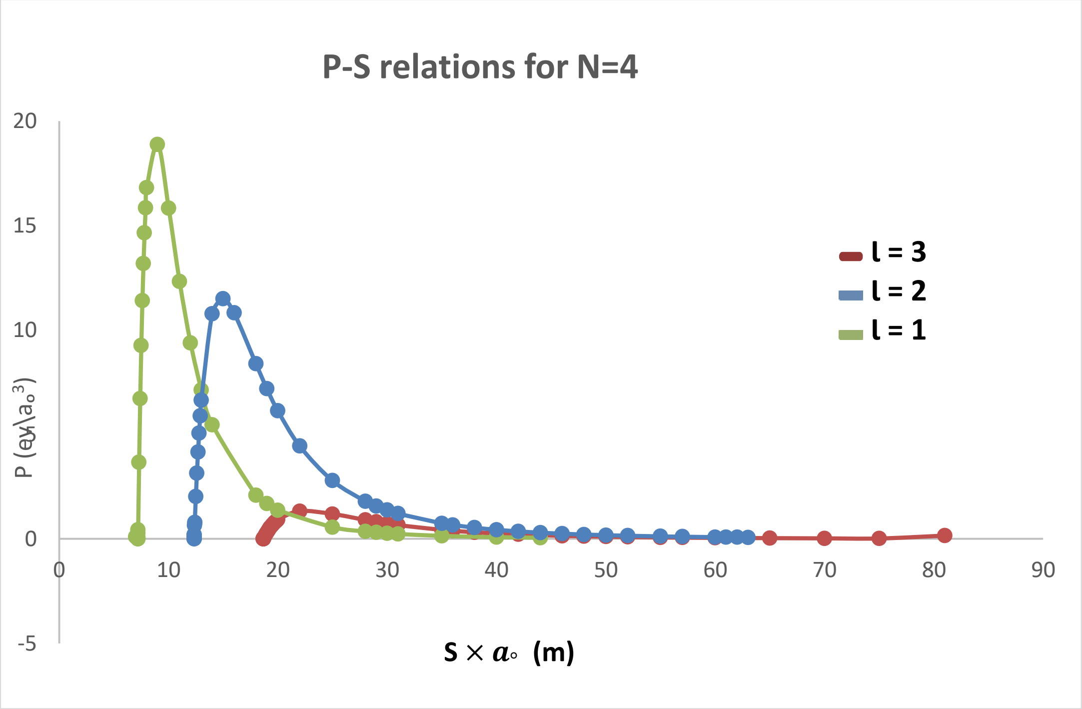

Fig (3. 37): Pressure vs. cavity radius when N=4 for different angular momentums.

The relation between the pressure on the confined H-atom and the cavity size is pretty excited. For each value of N and l, the pressure has a maximum value at certain value of S called ($S_{p\ max}$), while below or above this maximum value, the pressure decreases until it diminishes. A difference between these two beside cases have been noticed that for radii greater than $S_{P\ max}$, the pressure decreases gradually while for radii smaller than $S_{P\ max}$,it decreases rapidly within a very short range of S until it approaches zero, so the confined H-atom acts like a free one for enough large radii which is expected. But for small radii (less than $S_{P\ max}$); we proposed that the pressure is forced to decrease by the atomic structure because the cavity surface gets close to the position at which probability of finding the electron is maximum. In other words, we approached the atomic size at the same position is near to the critical cage radius where no bound states can exist. Other point of view, the pressure is the first derivative of energy with respect to the cavity volume and the energy is zero at this point. This matches with Heisenberg uncertainty principle, it states that the more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa.

Table (3. 23): Relation between the cavity radius at which the pressure is maximum ($S_{p\ max}$), and the angular momentum when N=3.

l	$S_{p\ max} \times a_{\circ}$ (m)
1	6
2	12
3	19
4	25
5	38
6	50
7	60
8	75
9	90
10	120
11	140
12	180
13	200
14	220

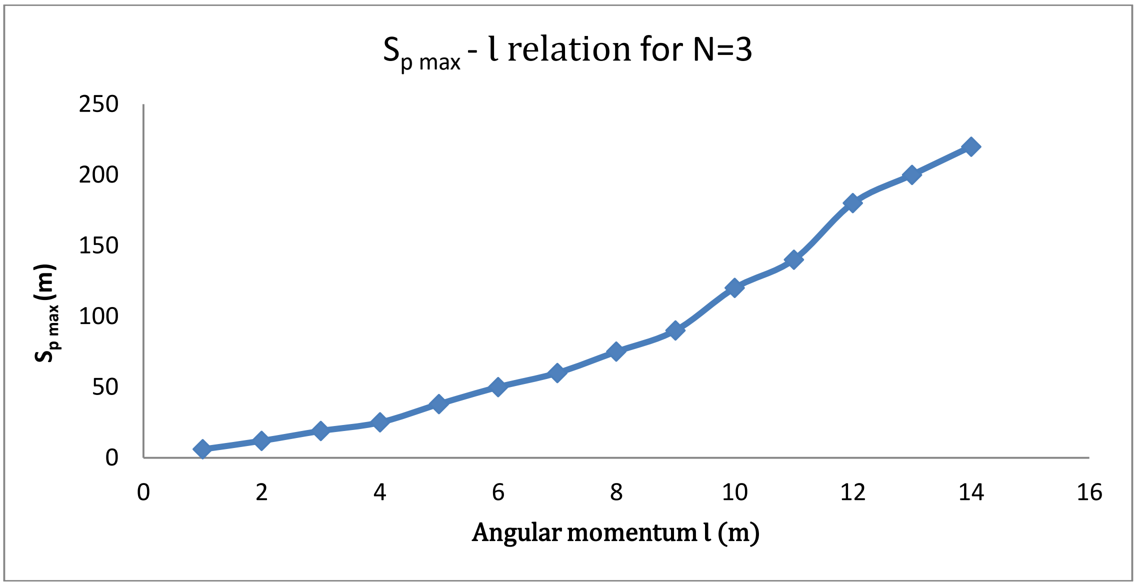

Fig (3. 38): Relation between the cavity radius at which the pressure is maximum and the angular momentums for N=3.

Table (3. 24): Relation between the cavity radius at which the pressure is maximum ($S_{p\ max}$), and the angular momentum when N=4.

l	$S_{p\ max} \times a_\circ$ (m)
1	9
2	15
3	22
4	32
5	42
6	50
7	70
8	80
9	100
10	120
11	140
12	160
13	180
14	200

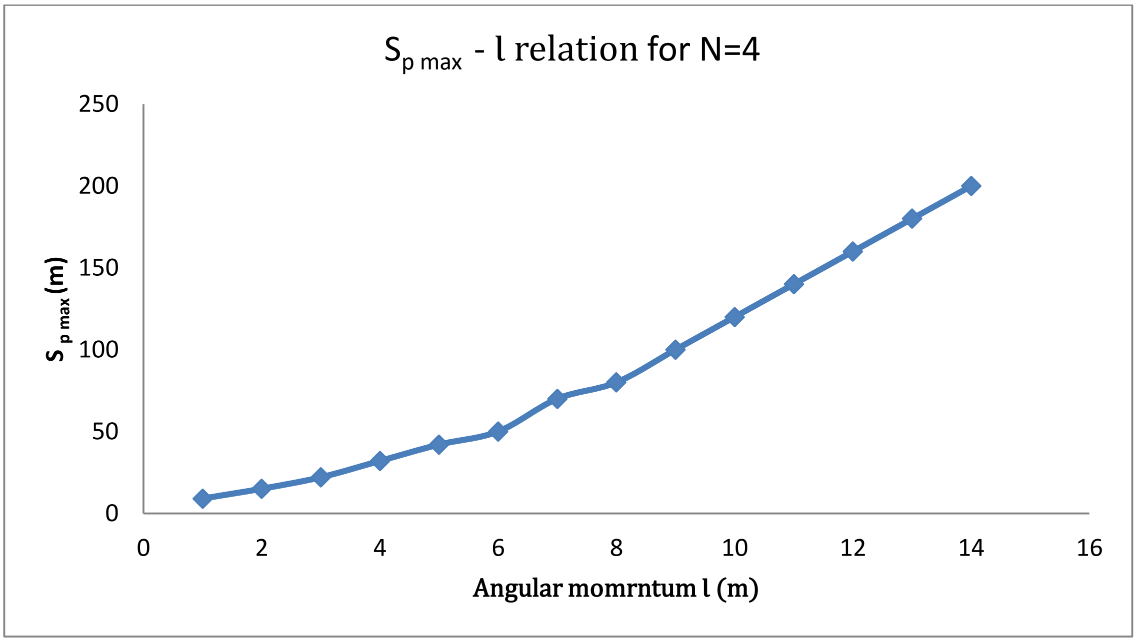

Fig (3. 39): Relation between the cavity radius at which the pressure is maximum and the angular momentum for N=4.

Table (3. 25): Relation between the cavity radius at which the pressure is maximum ($S_{p\ max}$), and space dimension N when the angular momentum ɭ=1.

N	$S_{p\ max} \times a_\circ$ (m)
3	6
4	9
5	11
6	14
7	17
8	20
9	25
10	29

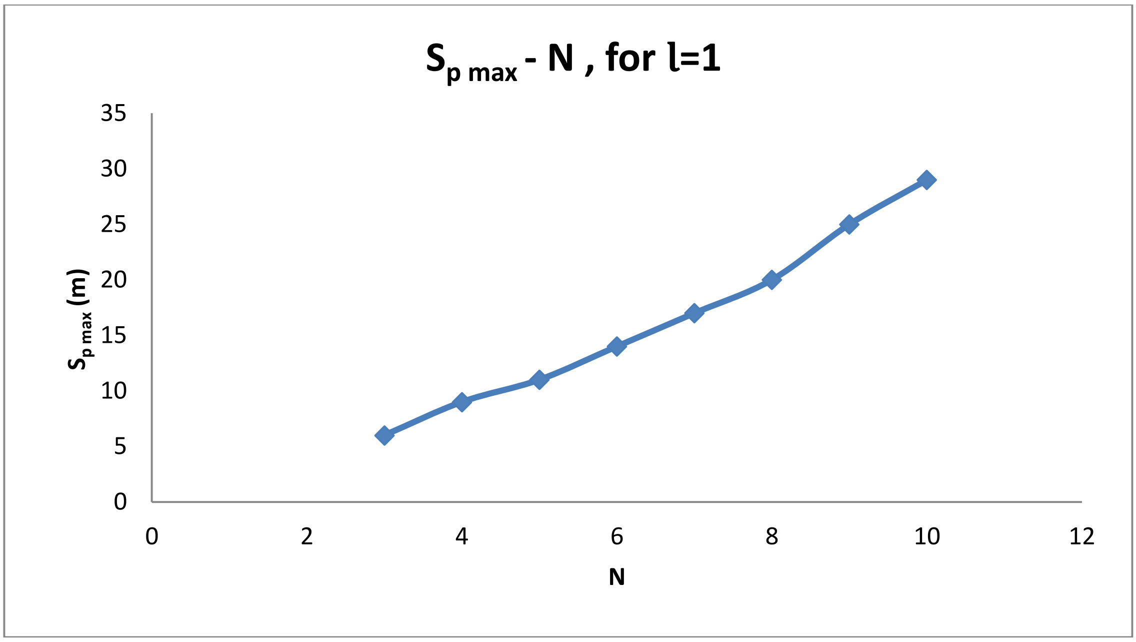

Fig (3. 40): Relation between the cavity radius at which the pressure is maximum and the space dimensions N for ɭ=1.

The above figures show that the cavity radius at which there is a maximum value of pressure increases for higher states or for higher dimensions. This is because of the additional terms in the effective potential. This is come with the critical cage radius results because it is found beyond $S_{p\ max}$, and the both increase with N and ɭ.

Chapter Four

Conclusion

Conclusion:

In this work, we had calculated the energy eigenvalues of excited state for the confined H-atom in a spherical cavity for space dimensions $N \geq 3$. We had found that these energies are quantized and depend on the dimensionality of the space and on the cavity size. For a given N, the eigenenergies increase for higher angular momentums and they also increase as N increase for a given ℓ. For the cavity size dependence in a fixed dimension N, the excited state energies decrease as the radius of the cavity increase until it become equal to the free H-atom energy in that dimension. Alternatively, the excited state energies keep increasing as the radius of the cavity decreases until it becomes zero at a value of the cavity radius that we had called critical cage radius S_c. Critical cage radius had been calculated in different dimensions and the outcomes show that its values increase as N increases.

Accidental degeneracy in the excited state energy eigenvalues in different space dimensions has been noticed. We have derived simple equation which illustrates exactly the positions where might be degenerated values.

$F(N, \ell) = F(N-2, \ell+1)$ …… For $N > 4, \ell \geq 1$ 4.1.6

Where: F stands for confluent hypergeometric polynomials and the subscribed dropped for simplicity.

Finally, the results of the energy eigenvalues have been used to study the behavior of the pressure exerted on the cavity wall due to the confined atom. For a given N, the pressure increases gradually as the radius of the cavity decreases until it had a peak which is called P_{max}. After that, a sudden decline in the pressure value occurs almost when the cavity radius is nearby the critical cage radius. Furthermore, the maximum value of pressure and its corresponding value of the cavity radius increases as N increases.

References

1. M Shaqqur and S. Al Jaber. “***Confined hydrogen atom in higher space dimensions***. **Int J Theor Phys”**. **48.** 2462–2472. (2009).
2. M. Neek-Amal, G. Tyebirad, M. Molayen, M. E. Foulaadvand, L. Esmaeili-Sereshki and A. Namiranian. ***Ground state study of simple atoms within a nano-scale box.*** Available at: http://arXiv:0804.4839 [cond-mat.other].
3. J.P. Connerade and P. Kengkan Proc. ***Confined Atoms***. Idea-Finding Symposium, Frankfurt Institute for Advanced Studies, 35–46 (2003).
4. K.D Sen**. *Electronic Structure of Quantum Confined Atoms and Molecules*.**(2014)
5. J. Sabin, E. Brãndas and S. Cruz. **Theory of Confined Quantum Systems**: Part One (ed) (Advances in Quantum Chemistry vol. **57**) (New-York: Academic Press) (2009).
6. J. Sabin, E. Brãndas and S. Cruz. **Theory of Confined Quantum Systems**: Part Two (ed) (Advances in Quantum Chemistry vol. **58**) (New-York: Academic Press) (2009).
7. P. O. Fröman, S. Yngve, and N. Fröman, ***The energy levels and corresponding normalized wave functions for a model of a compressed atom.*** J. Math. Phys. Available at https://doi.org/JMAPAQ, **28**,1813–1826 (1987).
8. J.P Connerade, V K Dolmatovd and P.A Lakshmi. ***The filling of shells in compressed atoms.*** J. Phys. B: At. Mol. Phys. **33** No.2 (2000).

9. M.V Faassen, ***Atoms in boxes: From confined atoms to electron-atom scattering.*** J. Chem. Phys. Available at:
10. http://doi.org/10.1063/1.3223281. **131**, 104108 (2009).
11. P.O Fröman, S. Yngve, and N. Fröman.. ***Spatial and Shell-Confined One Electron Atomic and Molecular Systems: Structure and Dipole Polarizability.*** J. Math. Phys. Available at: doi: http://dx.doi.org/10.1063/1.527441. **28,** 1813 (1987).
12. S.Yngve**. The energy levels and the corresponding normalized wave functions for a model of a compressed atom. II.** J. Math. Phys. Available at: http://dx.doi.org/10.1063/1.527990. **29**, 931 (1988).
13. K.R. Brownstein. ***Variational principle for confined quantum systems.*** Phys. Rev. Lett. **71** 1427 (1993).
14. W. Jaskólski. ***Confined many-electron systems.*** Phys. Rep. 271 (1996).
15. T.E. Hull, R.S. Julius. ***Enclosed quantum mechanical systems.*** **Canadian Journal of Physics**, **34** (1956).
16. V.K. Dolmatov. **Advances in Quantum Chemistry: Theory of Quantum Confined Systems.** Academic (2009).
17. J.P. Conneradec, V.K. Dolmatova, A.S. Baltenkovb, S.T. Manson. **Structure and photoionization of confined atoms.** Radiation Physics and Chemistry. **70** (2004).
18. J.P. Connerade and R. Semaoune. ***Atomic compressibility and reversible insertion of atoms into solids***. J. Phys. **B 33** 869 (2000).

19. J.P. Connerade, V.K. Dolmatov, S.T. Manson. ***Controlled strong non-dipole effects in photoionization of confined atoms***. J. Phys. **B 33** 279 (2000).
20. J.P. Connerade, P. Kengkan, P. Anantha Lakshmi and R. Semaoune. ***Scaling laws for atomic compressibility.*** J. Phys**. B 33** L847 (2000).
21. J.P. Connerade, V.K. Dolmatov, P.A. Lakshmi, S.T. Manson. ***Electron structure of endohedrally confined atoms: atomic hydrogen in an attractive shell.*** J. Phys. B: At. Mol. Phys**. 32** (1990).
22. C. Laughlin, B.L. Burrows and M. Cohen. ***A hydrogen-like atom confined within an impenetrable spherical box*.** J. Phys. B: At. Mol. Phys**. 35** No.3 (2002).
23. B L. Burrows, M. Cohen. ***Exact solutions for spherically confined hydrogen-like atoms.*** Intl. J. Quantum Chem. **106**, 478 (2006).
24. B. L. Burrows and M. Cohen. ***Confined one- and two-center systems.*** Phys. Rev. A **88**, 052511(2013).
25. T.Sako and GH.F. Diercksen. ***Confined quantum systems: a comparison of the spectral properties of the two-electron quantum dot, the negative hydrogen ion and the helium atom.*** J. Phys. B: At. Mol. Phys. **36** No.9 (2003).
26. N. Aquino, J.Garza, A. Flores-Riveros, J. F. Rivas-Silva, and K. D. Sen. ***Confined helium atom low-lying*** *SS* ***states analyzed through correlated Hylleraas wave functions and the Kohn-Sham model.*** J. Chem.Phys. Available at: http://doi.org/10.1063/1.2148948. **124**, 054311 (2006).

27. N. Aquino ***.The role of correlation in the ground state energy of confined helium atom.*** AIP Conference Proceedings. Available at: doi: http://dx.doi.org/10.1063/1.4862428, **136**, (2014).

28. S R. De Groot and C.A. Ten Seldam, ***On the energy levels of a model of the compressed hydrogen atom.*** Physica (Amsterdam) Available at https://doi.org/PHYSAG, **12**, 669–682 (1946).

29. A**.** Michels, J. De Boer, and A. Bijl. ***Remarks Concerning Molecular Interaction and their Influence on the Polarisation***. Physica (Amsterdam). **4**, 981–994, (1937).

30. A. Sommerfeld and H. Welker, Ann. Phys. **32**, 56 (1938).

31. Francisco M. Fernández and Eduardo A. Castro. ***Hypervirial theorems and enclosed quantum-mechanical systems.*** Phys. Rev. A **24**, 2344 – (1981).

32. Y. P. Varshni R. Dutt, and. A. Mukherjee, ***Supersymmetric semiclassical approach to confined quantum problems.*** Phys. Rev. A **52**, 1750 (1995).

33. S.H. Patil. ***Hydrogen atom confined to an ellipsoid.*** J. Phys. B: At. Mol. Phys. **34** No.6 (2001).

34. M. L. Glasser . ***Shooting for the stars: The spherically confined H-atom redux.*** Am. J. Phy. ***71, (***2003).

35. A. Banerjee, K. Sen, J. Garza, and R. Vargas, ***Mean excitation energy, static polarizability and hyperpolarizability of the spherically confined hydrogen atom.*** J. Chem. Phys. Available at: https://doi.org/JCPSA6,**116**, 4054–4057 (2002).

36. E. Ley-Koo and S. Rubinstein, ***The hydrogen atom within spherical boxes with penetrable walls***. J. Chem. Phys. https://doi.org/JCPSA6**71**,351–357 (1979).
37. B. Burrows. and M. Cohen. ***Exact solutions for perturbed confined hydrogen atoms: Polarizabilities and nuclear shielding factors***. Phys. Rev. **A 72,** 032508 (2005).
38. S. Sahoo and Y. K. Ho. ***Anomalous Stark effect in the ground state of the confined hydrogen atom in a spherical quantum dot.*** Phys. Rev. **B 69**, 165323 (2004).
39. M.N. Guimarães and F.V. Prudente**. *A study of the confined hydrogen atom using the finite element method.*** J. Phys. B: At. Mol. Phys. **38** (2005).
40. K. D. Sen. ***Shell-confined hydrogen atom.*** J. Chem. Phys. Available at: http://dx.doi.org/10.1063/1.1901584 **122**, 194324 (2005).
41. E. Ley-Koo and S.Rubinstein. ***The hydrogen atom within spherical boxes with penetrable walls*.** J. Chem. Phys. Available atdoi: http://dx.doi.org/10.1063/1.438077 **.71**, 351 (1979).
42. N.Aquino. ***Accurate energy eigenvalues for enclosed hydrogen atom within spherical impenetrable boxes*.** Int. J. Quantum Chem. Available at: https://doi.org/10.1002/qua.560540206 **54**, 107 (1995).
43. S.A. Ndengué, O. Motapon, R.L. Melingui Melono and A.J. Etindele. ***Electronic structure of a cylindrically confined hydrogen atom by the B-splines method: energy levels and dipole polarizability.*** J. Phys. B: At. Mol. Phys. **47** No.1 (2014).

44. Y. P.Varshmi. ***Accurate wavefunctions for the confined hydrogen atom at high pressures.*** J. Phys. B: At. Mol. Phys. **30**, L 589 (1997).
45. Y.P. Varshni. ***Scaling Relation for the Energy Levels of a Hydrogen Atom at High Pressures.*** Z. Naturforsch. **57 a**, 915–918 (2002).
46. R. Dutt, A.Mukherjee, Y.P Varshni. ***Dipole polarizability of hydrogen atom at high pressures.*** Phys. Letters. Available at: http://dx.doi.org/10.1016/S0375-9601(01)00067-6 **. A 280** 318 (2001).
47. J. McMinis, R.C. Clay, D.Lee, and M.A. Morales ***Molecular to Atomic Phase Transition in Hydrogen under High Pressure.*** Phys. Rev. Lett. **114**, 105305, 2015.
48. R.L. Melingui Melono, A.J. Etindélé, T. Tchakoua, S.A. Ndengué and O. Motapon. ***Polarizability of off-center spherically confined hydrogen atom.*** J. Phys. B: At. Mol. Phys**, 48** No.21 (2015).
49. N Aquino and R-A Rojas. ***The confined hydrogen atom: a linear variational approach.*** Eur.J. Phy.**37**, 1(2015).
50. B. L. Burrows and M. Cohen. ***Exact solutions for perturbed confined hydrogen atoms: Polarizabilities and nuclear shielding factors.*** Phys. Rev. A**.** Available at: https://doi.org/10.1103/PhysRevA.72.032508. **72**, 032508 (2005).
51. F. M. Fernández and E. A. Castro. ***Hypervirial perturbational treatment of the bounded hydrogen atom.*** J.Math.Phys. Available at: https://doi.org/10.106. 3/1.525475. **23**, 1103 (1982).
52. D. Djajaputra and B. R Cooper. ***Hydrogen atom in a spherical well: linear approximation.*** Eur. J. Phys. **21,** 261 (2002).

53. D.M Marfatia**. New Aspects of Physics with Extra Dimensions.** University of Wisconsin--Madison, (2001)
54. Ch. McMullen. **The Visual Guide to Extra Dimensions**. **Vol.1** Kindle Edition.
55. I. M. Freeman. **Why is space three-dimensional?** Am. J. Phys. **37**, 1222 (1969).
56. J. D. Barrow. ***Dimensionality.*** R. Soc. London, Ser. A **310**, 337 (1983).
57. C. G. Bollini, J. J. Giambiagi, and O. Obregon. **Are some physical theories related with a specific number of dimensions?** Mod. Phys. Lett. A**7**, 593 (1992).
58. T. Kaluza. ***On the Problem of Unity in Physics.*** Sitzungsber. Preuss. Akad. Wiss. Berlin. Math. Phys. Preuss. Akad. Wiss. **966** (1921).
59. O. Klein, ***Quantum Theory and Five-Dimensional Theory of Relativity.*** Z. Phys. **37**, 895 (1926).
60. L. Randall and R. Sundrum**.** ***Large Mass Hierarchy from a Small Extra Dimension***. Phys. Rev. Lett. **83** 3370 (1999).
61. L. Randall and R. Sundrum, ***An Alternative to Compactification***, Phys.Rev.Lett.83. (1999).
62. L. Randall. **Warped passages: unraveling the mysteries of the Universe's hidden dimensions.** (2006).
63. F. Burgbacher and A. Macias**.** ***Is there a stable hydrogen atom in higher dimensions?*** J. Math. Phys. **40**, 625 (1999).
64. L. Gurevich and V. Mostepanenko. ***On the existence of atoms in n-dimensional space*****.** Phys. Lett. A. **35**, 201 (1971).

65. S M. AL-Jaber. ***Hydrogen atom in N dimensions.*** Int. J. Theor. Phys., **37** No.4 1289-1298 (1998).

66. M A. Shaqqur and S M. Al Jaber. ***Confined hydrogen atom in higher space dimensions under pressure.*** An - Najah Univ. **25**, (2011).

67. Issam R. Abdelraziq, Zuhair A. Majjad, Mohammed S. Abu-Jafar, Bassam A.Manasrah. **Special Functions**. by the Deanship of Scientific Research, An-Najah National University. (2004)

جامعة النجاح الوطنية
كلية الدراسات العليا

ذرة الهيدروجين المحصورة في فجوة كروية في فضاء متعدد الأبعاد " المستويات العليا "

اعداد

منار عبدالرحيم صالح مصلح

اشراف

أ.د. سامي الجبر

قدمت هذه الأطروحة استكمالاً لمتطلبات الحصول على درجة الماجستير في الفيزياء بكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس– فلسطين

2017

ذرة الهيدروجين المحصورة في فجوة كروية في فضاء متعدد الأبعاد

"المستويات العليا"

اعداد

منار عبدالرحيم صالح مصلح

اشراف

أ.د. سامي الجبر

الملخص

تم حساب قيم طاقة المستويات العليا لذرة هيدروجين محصورة في فجوة كروية و في فضاء متعدد الابعاد (ن ≤ ٣) . وذلك عن طريق ايجاد اصفار الاقترانات الموجية التي تمثل ذرة الهيدروجين تحت هذه الظروف – والتي اوجدها شقور والجبر في بحثهما المنشور عام ٢٠٠٩. لقد وجدنا ان طاقة المستويات العليا تعتمد على ن (عدد الأبعاد التي تمثل الفضاء الذي يحتوي ذرة الهيدروجين) وعلى نصف قطر الفجوة التي تحتوي ذرة الهيدروجين. قمنا بدراسة هذا الاعتماد ولاحظنا ان نفس قيم الطاقة لمستويات عليا قد تتواجد في ابعاد فضاء مختلفة. قمنا باشتقاق علاقة رياضية توضح بالضبط اماكن تكرار هذه القيم. كما وضحت الدراسة اثر الضغط الناتج من ذرة الهيدروجين المحصوره على جدران الفجوة.