An-Najah National University

Faculty of Graduate Studies

Eigenvalues of the Matrix of the Distances Reciprocals for the Complete Bipartite and Cycle Graphs

By

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Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics, Faculty of Graduate Studies, at An-Najah National University Nablus, Palestine.

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This Thesis was defended successfully on Wednesday 5/5/2004 and

approved by

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3 Dr. Mohammad Najeeb Asa'd	....

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بسم الله الرحمن الرحيم

رَبّ اشرَحْ لِي صدري(*) وَيسرْ لِيْ أمْري(*)واحللْ عقْدَةً مِنْ لِسَانِي(*) يَفقَهُوا قولي(*)

سورة طه (22)-(25)

اقْرَأْ بِاسْمِ رَبكَ الذِيْ خلَق(*)خَلَقَ الإنْسانَ مِنْ عَلَق (*)اقْرَأ وَرَبُّكَ الأكْرَم(*)ألذِيْ علّمَ بِالْقَلَمِ (*)علمَ الإنْسانَ ما لَمْ يَعْلَمْ (*)

سورة العلق (1)-(5)

وَالله أخْرَجَكُمْ مِنْ بُطُونِ أُمهاتِكُمْ لا تَعْلَمونَ شَيْئاً وَجَعَلَ لَكُمْ السمْعَ وَالأبْصارَ وَالأفْئِدَةَ لَعَلكُمْ تَشْكُرُون(*)

سورة النحل (78)

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Dedications

see me a To my parents who supported me in all of my study stages. and did their best to successful person…………

To my brothers and sisters Mohammed. Hani. Shadi. Basem. Ferial. Basma. who supported me and were beside me all the time……………

To my sons. Aymen. Mahmoud. Abdel-Kareem. and my daughters Liqa '. Fida '. Diqra. Enas who let me taste the feeling of love…………

To my best friend and wife. Um Aymen. who deserves much more than appreciation and respect for her patience and support………

To my teachers everywhere. and especially Jamal Shihadeh who did his best to see his students successful persons…………

To my family. colleagues. friends. and students everywhere ....

To whom I love…….

I dedicate my thesis……

Researcher

Riad Kamel Zaidan

5/5/2004

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Table of Contents

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All Rights Reserved - Library of University of Jordan - Center of Thesis Deposit

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Abstract

This work deals with the spectra and eigenspaces of some matrices related to the distance matrix of some connected graphs.

In particular, we investigate the (n x n) matrix B_n whose nonzero entries are the reciprocals of the corresponding nonzero entries in the distance matrix. We derive formulas for the eigenvalues and eigenvectors related to the complete bipartite graph K(r, n-r), and the cycle graphs C_n for any positive integer n.

In the beginning, we state some needed facts in graph and matrix theory.

In chapter three, we present some known results about the distance matrix and related topics.

In chapter four, the discussion was focused on the matrix B_n related to the graphs K(r, n-r), and K(2, n-2).

In chapter five we deal with the matrix B_n related to the cycles C_n for any positive integer n.

New Accomplishments

In chapter two, we state and prove a theorem for computing the eigenvalues and eigenvectors of circulant matrices, and relate them to the

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permutation matrices.

In chapter four: We state and prove a theorem for computing the eigenvalues and eigenvectors of a matrix, that appears in our main study.

Also, we state for the first time, a theorem for computing the eigenvalues and eigenvectors for the matrix B_n,whose nonzero entries are the reciprocals of the corresponding nonzero entries of the distance matrix of the graph K(r, n-r), and as a special case the graph K(2, n-2).

We will construct a table, which contains numerical values of the spectral radius (1, and those of of some complete bipartite graphs, obtained by direct calculation and by the resulting formulas.

Also, we will construct a table which contains numerical values of for K(2,n-2), K(3,n-3), K(4,n-4) as n goes bigger and bigger, then we state and prove a lemma for calculating the limit of as n approaches

In chapter five, we will find the eigenvalues and eigenvectors for B_n related to the cycle graphs C_n, and we state and prove a theorem which shows that the eigenvalues of B_n are real for any positive integer n

We will also calculate the eigenvalues and eigenvectors of B_n related to the cycle graph C_n, and those of C_n.

We will present some graphs, then we will compute the eigenvector that corresponds to the spectral radius, and will note that vertices with greater eigenvector entries are with smaller eccentricities, and tend to be in the center of the graph.

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Chapter One

General Background in Graph Theory

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[1 – 0] **Introduction:**

Graph theory is a new area of applied mathematics which is being widely used in formulating models in many problems in business, the social sciences, and the physical sciences. These applications include communications problems and the study of organizations and social structures.

Graph theory was discovered from several situations by *Leonhard Euler* (1707-1783), *Kirchoff* (1824-1887), and *Arthur Cayley* (1821-1895).

The first paper devoted exclusively to a problem in graph theory was published in 1736 by *Euler* which was about a puzzle concerning Koingsberg city. He presented the solution of the problem of the Koingsberg bridges.

Kirchoff discovered graphs while solving problems involving electrical networks and the calculations of currents.

Cayley studied a special class of graphs related to certain chemical compounds, especially the hydrocarbons. He was interested in enumeration of such graphs.

In this chapter we present a very brief introduction to the subject that includes some basic concepts and definitions which will be needed in our investigation of such situations.

[1-1]: **The Definition of a Graph**

First of all, we may consider a system of "objects" which are interrelated in some way. For example, the objects may be:

a) Countries connected by diplomatic relations

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b) Atoms connected by chemical bonds

c) Stations interconnected by rails.

In each of the previous and similar cases we can draw a diagram representing each of these cases where in each case the objects are represented by points, and the interconnections are represented by lines.

Such a diagram is called a graph. The points representing the objects are called vertices, and the lines representing the interconnections are called edges.

Next, we state the formal definition of a graph.

Definition [1-1-1]: A graph G is an ordered pair (V, E) in which the first component is a non-empty set of vertices denoted by V, and the second is a set of unordered pairs of vertices called edges and is denoted by E.

The number of vertices is called the order of the graph and the number of edges is the size of the graph.

Example 1: In Fig1 the graph G is displayed in which

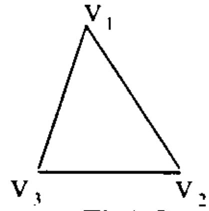

Fig1 G

$V(G) = \{v_1, v_2, v_3\}$, $E(G) = \{\{v_1, v_2\},\{v_1, v_3\},\{v_2, v_3\}\}$.

For simplicity, an edge will be denoted as v_1v_2 instead of $\{v_1, v_2\}$.

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Definition [1-1-2]: **A directed graph** abbreviated **digraph D** consists of a set of elements, called **vertices**, and a list of ordered pairs of these elements, called **arcs**. The set of vertices is called the vertex-set of **D**, denoted by V(**D**), and the list of arcs is called the arc list of **D**, denoted by A(**D**). If v_1 and v_2 are vertices of **D**, then an arc of the form v_1v_2 is said to be directed from v_1 to v_2.

Example 2: In Fig2 the digraph D is displayed

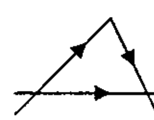

Fig2 D

where V(D)= $\{v_1, v_2, v_3\}$, and A(D)=$\{v_1v_3, v_2v_1, v_2v_3\}$.

Definitions [1-1-3]: Let G = (V, E) be a graph:

A loop is an edge of the form e = vv joining a vertex to itself.

Multiple edges are two or more edges joining any two vertices.

A simple graph is a graph which has neither loops nor multiple edges.

A walk of length k in a graph G is a succession of k edges of G of the form *uv, vw, wx, …, yz* and this walk is said to be a walk between u and z and is denoted by *u v w x…y z*

A trail is a walk in which no edge is repeated.

A path is a walk in which no vertex is repeated.

The length of a path is the number of edges included in it.

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A closed walk is a walk of the form *uv, vw, wx, …, yz, zu*

A cycle is a closed path.

Here are some examples of these concepts.

Example 3: Consider the graph G displayed in Fig 3

$v_5\ v_1\ v_2\ v_3\ v_1\ v_5\ v_4$ is a walk

$v_1\ v_2\ v_3\ v_1\ v_5$ is a trail

$v_4\ v_3\ v_2\ v_1\ v_5$ is a path

$v_1\ v_2\ v_3\ v_4\ v_5\ v_1$ is a cycle

$v_6 v_6$ is a loop.

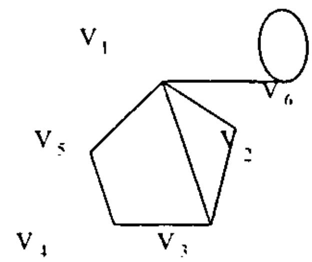

Fig 3 G

Definition [1-1-4]: Let G be a graph with no loops, and v be a vertex of G. The degree of the vertex v, denoted by **deg** (v), is the number of edges meeting at v. The degree in a simple graph is simply the number of vertices connected to vertex v by edges.

Note: A loop at a vertex v contributes 2 to the degree of v, while an edge e = uv contributes to one in the degree of vertex u and to one in the degree of vertex v.

Definitions [1-1-5]:

An isolated vertex is a vertex with degree zero.

An end vertex or a pendant vertex is a vertex with degree one.

Example 4: In the following graph, we note that

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$\deg(v_2) = 3$

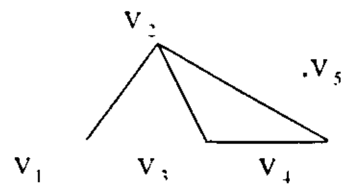

Fig 4

v_1 is an end vertex since $\deg(v_1) = 1$.
v_5 is an isolated vertex since $\deg(v_5) = 0$.

Definition [1-1-6]: Two vertices v_1, v_2 of a graph G are adjacent if there is an edge joining them, and we say that v_1, v_2 are incident with that edge. Two distinct edges are said to be adjacent if they are incident with the same vertex.

Example 5: Consider the following graph

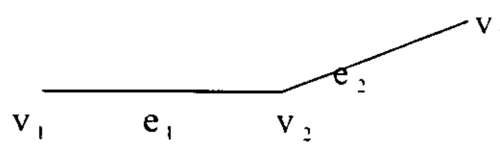

Fig 5 G

v_1, v_2 are adjacent vertices and e_1, e_2 are adjacent edges

while edge e_1 is incident with both v_1 and v_2.

Definition [1-1-7]: Let G be a graph with vertex set V(G), and edge set E(G), similarly let H be a graph with vertex set V(H) and edge set E(H). We say that H is a **subgraph** of G if the following are satisfied:

V(H) is a subset of V(G), and E(H) is a subset of E(G).

Moreover, if V(H) = V(G), then H is called a **spanning** subgraph of G.

Also, if V(H) is a subset of V(G), then H = (V(H), E(H)) is the subgraph of G **induced** by V(H) provided that every edge in E(G) having end vertices in V(H) also belongs to E(H).

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For example, in Fig 6 the graph G_1 is a subgraph of G while G_2 is a spanning subgraph of G, and G_3 is the subgraph induced by $\{v_1, v_3, v_4\}$.

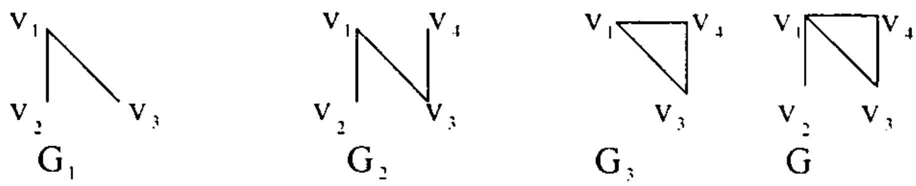

Fig 6

Definition [1-1-8]: Let G_1 and G_2 be two graphs and let f be a function from the vertex set of G_1 to the vertex set of G_2. The two graphs are said to be isomorphic and the map f is in isomorphism if it satisfies the following two conditions:

first: the map f is one-to-one and onto and

second: the map f preserves adjacency.

i.e $f(v_1)$ is adjacent to $f(v_2)$ in G_2 if and only if v_1 is adjacent to v_2 in G_1 Then we say that the function f is an isomorphism and that the two graphs are isomorphic. In short terms, one says that two graphs G_1 and G_2 are isomorphic if there is a one-to-one correspondence between the vertices of G_1 and those of G_2 that preserves adjacency.

Example 6: Consider the following two graphs G_1 and G_2

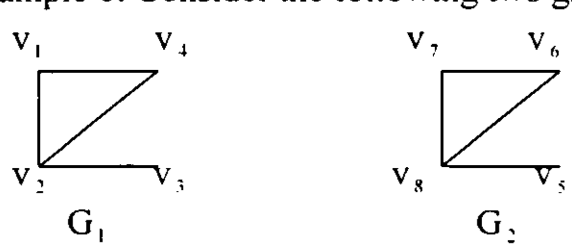

Fig 7

Note that $f = \{(v_1, v_7), (v_2, v_8), (v_3, v_5), (v_4, v_6)\}$ is a one-to-one correspondence between the set of vertices of G_1 and the set of those of G_2 which shows that these two graphs are isomorphic.

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If two graphs are isomorphic, then they can be considered as two copies of the same graph although the locations of the vertices may be different, or the shapes of these two graphs may be different. From a topological view two isomorphic graphs are just the same object.

For example, the following two graphs G_1 and G_2 are isomorphic

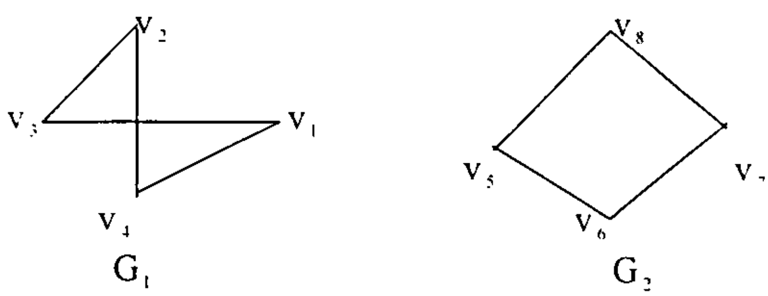

Fig 8

where $f = \{(v_1,v_8), (v_2,v_6), (v_3,v_5), (v_4,v_7)\}$ is a one-to-one correspondence between the vertices of G_1 and those of G_2 although the two graphs have different representations.

[1 - 2] Special Graphs:

We now introduce several classes of simple graphs. These graphs are often used as examples and arise in many applications.

Null Graphs:

A null graph is a graph in which every vertex is isolated and with degree equals to zero.

Notes: 1) The null graph on n vertices is denoted by N_n.

2) Every vertex in a null graph is isolated.

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Here are two examples of null graphs:

N_1 N_3

Regular Graphs:

A regular graph on n vertices is a graph in which all vertices have the same degree.

If deg $(v_i) = r$ for all i = 1, 2, …, n then the graph is called r -regular.

Example 7: The graph G in Fig 9 is a 2-regular graph

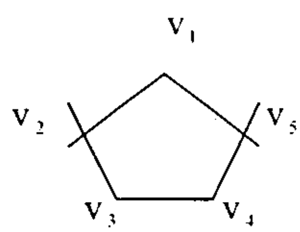

Fig 9 G

Complete Graphs:

A complete graph on n vertices denoted by K_n is a simple graph in which each pair of distinct vertices are adjacent.

Note that K_n is an (n - 1) - regular and has a size of $\frac{n(n-1)}{2}$ edges.

Example 8: The following graph represents K_4

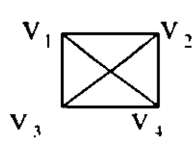

Fig 10 K_4

Next, we define another type of graphs; namely cycle graphs, which will be studied in more detail.

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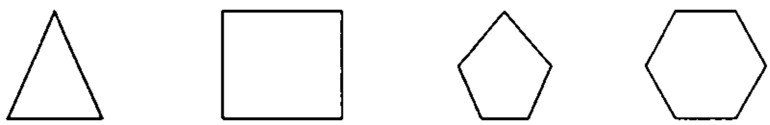
Cycle Graphs:

A cycle graph on n vertices, C_n, $n \geq 3$ is a connected graph with n vertices $v_1, v_2, \ldots, v_n$, each of which has degree 2, with v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$ and v_n is adjacent to v_1.

Example 9: The cycles C_3, C_4, C_5 and C_6 are displayed in Fig 11



C_3 C_4 C_5 C_6

Fig 11

Note: (1) Any cycle graph is a 2-regular graph.

(2) Any graph which contains no cycle is called acyclic graph.

Next, we will introduce a graph which is related to the cycle graphs.

Path Graphs:

A path graph on n vertices denoted by P_n, is a graph obtained from C_n by removing an edge. If edge $e = uv$ is deleted from C_n, then we get the path P_n and it is denoted by $C_n - e$.

Note that any path graph contains only two end vertices and any other vertex has degree equals to two.

Example 10: The following graph represents P_4

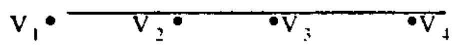


Fig 12 P_4

Next, we introduce a special type of graphs which will play an important role in our future study.

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Connected Graphs:

A graph G is connected if for any two vertices u, v ∈ V(G) there is a u-v path, otherwise it is said to be disconnected.

Example 11: The following graph G is connected

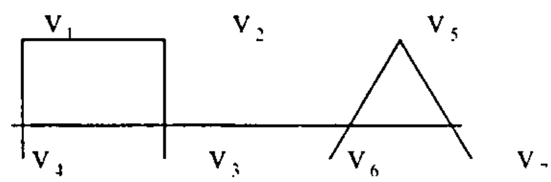

Fig 13 G

Note: Every disconnected graph G can be split up into a number of connected subgraphs, each of which is called a component of G.

Here are some definitions concerning connected graphs.

Definition [1-2-1]:

A graph G is called irreducible if for every ordered pair of vertices v_1 and v_2 there is a path in G starting at v_1 and terminating at v_2.

Note that an irreducible graph is a connected graph.

A bridge is a single edge whose removal disconnects a graph.

A cutset of a connected graph G is a set S of edges with the following properties:

a. The removal of all edges in S disconnects G.

b. The removal of some (but not all) of the edges in S does not disconnect G.

Example 12: The graph in Fig14 (a) is a graph with three components.

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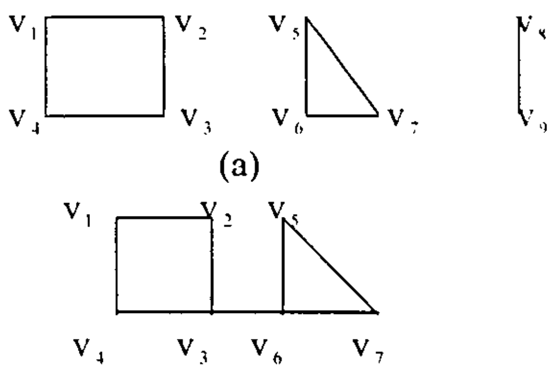

Fig 14

In Fig 14 (b), v_3v_6 is a bridge, while $\{v_1v_2, v_3v_4\}$ is a cutset.

Some graphs have the property that the set of vertices is made up of two disjoint subsets, such that each edge connects a vertex in one of these subsets to a vertex in the other subset.

Bipartite Graphs:

A bipartite graph G is a graph in which the vertex set can be split into two disjoint sets A and B so that any edge in G joins a vertex in A and a vertex in B. i.e a graph in which its vertices can be colored black and white in such a way that each edge joins a black vertex (in A) and a white vertex (in B).

Example 13: The following graph is a bipartite graph

where $A = \{v_1, v_2\}$

$B = \{v_3, v_4, v_5\}$

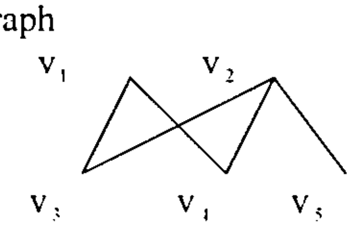

Fig 15

Complete Bipartite Graphs:

A complete bipartite graph is a bipartite graph in which every vertex in A is joined to every vertex in B.

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Note: A complete bipartite graph with r black vertices and s white vertices is denoted by K(r, s) and it has r + s vertices and r×s edges

Example 14: The graph K (3, 4) is displayed in Fig 16

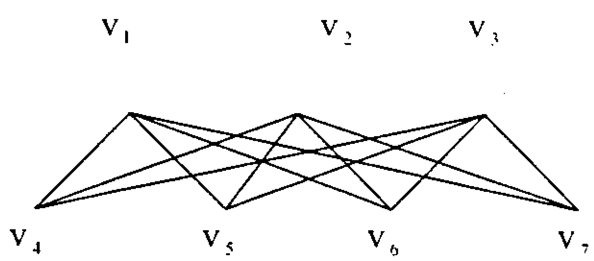

Fig16 K (3, 4)

Star Graphs

A star graph is a complete bipartite graph of the form K(1, n - 1) and is denoted by S_n.

Cube Graphs:

A k-cube graph denoted by Q_k is a graph whose vertices correspond to the sequence $(a_1, a_2, \ldots, a_k)$, where each $a_i = 0$ or 1, and whose edges join these sequences that differ in just one place.
Example 15: The following graph is the 3-cube graph Q_3

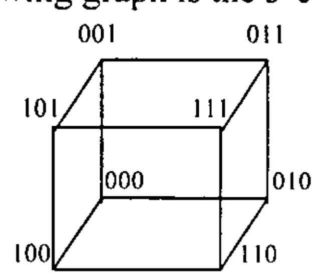

Fig 17 Q_3

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Note: a k-cube graph Q_k has an order of 2^k, a size of $k\,2^{k-1}$, and it is k-regular graph.

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In the following, we define another type of graphs-namely a tree-which is usually used in representing some chemical molecules.

Note: Any graph that contains no cycles is called acyclic graph.

Trees:

A tree is a connected acyclic graph.

Example 16: The following graph represents a tree with 6 vertices

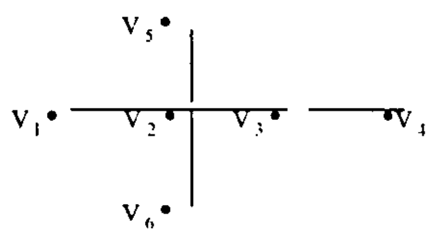

Fig 18

Trees have a lot of interesting properties, some of which are:

1) Every tree with n vertices has exactly n-1 edges.

2) Any two vertices in a tree are connected by exactly one path.

3) Any edge of a tree is a bridge.

Next, we define the complement of a given graph G.

The Complement of a Simple Graph:

If G is a simple graph with vertex set V (G), then its complement denoted by $\bar{G}$ is the simple graph with vertex set V (G) in which two vertices are adjacent if and only if they are not adjacent in G.

Example 17: Following is a graph G and its complement $\bar{G}$.

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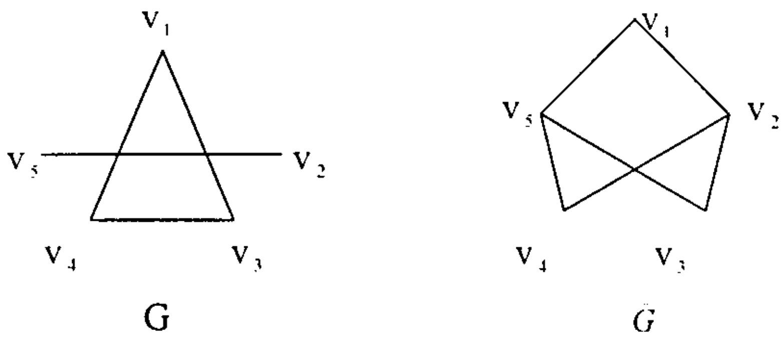

Fig 19

Planar Graphs and Coloring of Graphs:

A graph can be drawn in many different ways, for example each of the following drawings represents the same graph G.

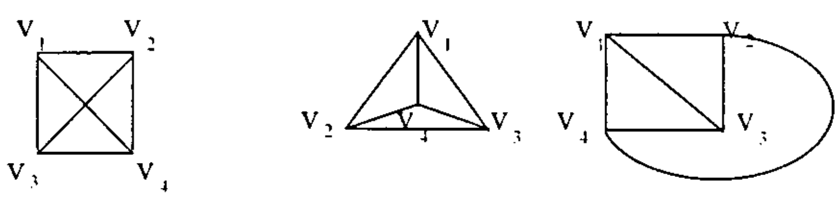

Fig 20 G

Definition: A graph G is **planar** if it can be drawn in the plane in such a way that no two edges meet each other except at a vertex to which they are both incident. Any such drawing is called a **plane drawing** of G.

If there is no plane drawing of a graph, then it is called a non-planar graph. As an example of a non-planar graph is the complete bipartite graph K (3, 3).

So the graph K_4 in Fig 20 is a planar graph.

Definition [1-2-2]: Let G be a graph without loops. A **k-coloring** of G is an assignment of k colors to the vertices of G such that adjacent vertices are assigned different colors. If G has a k-coloring, then G is said to be **k-colorable**. The **chromatic number** of G, denoted by $\chi(G)$, is the smallest number k for which G is k-colorable.

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A lot of work has been conducted on graph coloring of planar graphs. The most important of which is the following theorem.

Theorem [1-2-3]: (The Four Color Theorem).

Every planar graph is 4-colorable. (For the proof see [22]).

[1-3] **Operations on Graphs:**

There are several ways that can be used to combine two graphs in order to get new ones. The simplest of these are the **union**, **sum**, and **deletion** of graphs.

[1-3-1] **Union:**

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $V(G_1)$, $V(G_2)$ are disjoint, then the union of G_1 and G_2 denoted by $G_1 U G_2$,is the graph whose vertex-set is $V_1 U V_2$ and edge-set is $E_1 U E_2$. For example, the null graph N_n is the union of n copies of N_1.

Example 18: Consider the graph G shown in Fig 21

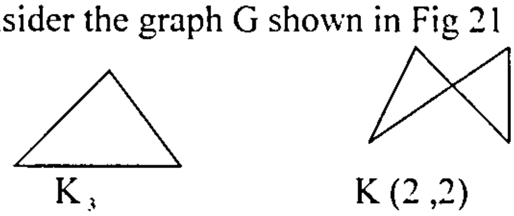

Fig 21 G

It is clear that $G = K_3 \ U \ K(2, 2)$.

Any disconnected graph is the union of two or more connected subgraphs, each pair of which has no vertex in common. These disjoint connected subgraphs are called components of the graph.

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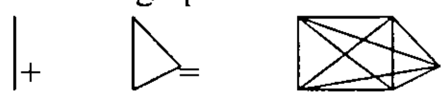
[1-3-2] **The Sum of Graphs:**

Let G_1 and G_2 be two graphs, then the sum of G_1 and G_2 denoted by (G_1+G_2) is the graph resulting by first forming $G_1 \cup G_2$ and then making every vertex of G_1 adjacent to every vertex of G_2.

Example 19: The addition of two graphs is illustrated below



$K_2 + K_3 = K_5$

Fig. 22

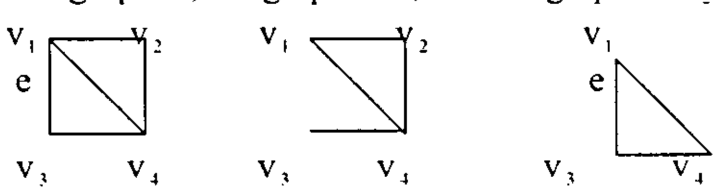
[1-3-3] **Deletion of Graphs:**

Let **e** be an edge of a graph G, then **G-e** is a subgraph of G obtained from G by deleting the edge **e**.

In general, if **F** is any set of edges of G, then G-**F** is a subgraph of G resulting by deleting all edges of F from G.

Also, if **v** is a vertex of the graph G, then G-**v** is the subgraph obtained from G by removing the vertex **v** and all incident edges.

Example 20: A graph G, the graph G-**e**, and the graph G-$\mathbf{v}_2$ are displayed.



G　　G-e　　G-v_2

Fig 23

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Chapter Two

General Background in Matrix Theory

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[2-0] **Introduction**:

In this chapter, we will make a general revision of some matrices by means of some definitions and theorems concerning special types of matrices.

First, we will provide a review of some important properties of matrices especially symmetric matrices.

[2-1] **Definitions and Theorems:**

Here we define a symmetric matrix, which will be the focus of our attention during the coming study.

Definition [2-1-1]: A matrix $A = [a_{ij}]$ is called symmetric iff $A^{t} = A$ where $A^{t} = [a^{t}_{ij}]$ is the transpose of A. i. e $a^{t}_{ij} = a_{ji}$ for every possible i, j.

Next, we define the term "eigenvalue" which is related to a square matrix.

Definition [2-1-2]: let A be an $(n\times n)$ matrix. The number λ is called an eigenvalue of A if there exists a non-zero vector **x** such that

$$A\,\mathbf{x} = \lambda\,\mathbf{x} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

and every nonzero vector **x** satisfying (1) is called an eigenvector of A associated with the eigenvalue λ.

Next, we will define the characteristic polynomial of a matrix, a polynomial of great importance of our work.

Definition [2-1-3]: Let $A = [a_{ij}]$ be an $(n\times n)$ matrix. The function

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$$f(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \dots & \dots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \dots & \dots & -a_{2n} \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ -a_{n1} & -a_{n2} & \dots & \dots & \lambda - a_{nn} \end{vmatrix}$$

is called the characteristic polynomial of A.

The characteristic polynomial of an (n x n) matrix A is a polynomial of degree n. So

$f(\lambda)=(\lambda)^n + a_{n-1}(\lambda)^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_1(\lambda) + a_0.$

Definition [2-1-4]: If A is an (n x n) matrix, the **minor** M_{ij} is the determinant of the ((n-1) x (n-1)) submatrix of A obtained by deleting the i^{th} row and the j^{th} column of the matrix A.

In the next theorem, the coefficients of $f(\lambda)$ are determined in terms of principal minors.

Theorem [2-1-5]: Let A be an (n × n) matrix, and $f(\lambda) = \sum_{i=0}^{n} a_i \lambda^i$ denotes its characteristic polynomial. Then the scalar a_r, $0 \le r \le n$ is equal to the sum of all principal minors of A of order (n - r) multiplied by $(-1)^{n-r}$. In particular, the coefficients of $(\lambda)^n$, $(\lambda)^{n-1}$, and $(\lambda)^0$ are respectively equal to 1, $a_{n-1} = -tr(A)$, and $a_0 = (-1)^n \det(A)$. (For the proof see [21]).

Note: The equation $f(\lambda) = \det(\lambda I - A) = 0$ is called the characteristic equation of A.

Next, we define the term 'orthogonal matrix.'

Definition [2-1-6]: An (nxn) non–singular matrix A is called orthogonal, if and only if $A^{-1} = A^{t}$.

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i.e $A^T A = I$, where I is the identity matrix.

In the following theorem, some properties of real symmetric matrices will be stated.

Theorem [2-1-7]: If A is an $(n \times n)$ real and symmetric matrix then:

(i) All eigenvalues of A are real.

(ii)There is a set of orthonormal eigenvectors for the matrix A.

(iii) $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$ where Q is a matrix whose columns are the orthonormal eigenvectors.
$$\Lambda = \begin{vmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \dots & \lambda_n \end{vmatrix}$$

In the following theorem, we state the relation between the eigenvalues of a matrix, and the trace and determinant of that matrix.

Theorem [2-1-8]: Let A be an $(n\times n)$ matrix, then

(i) $\quad Tr(A) = \sum_i \lambda_i = \sum_i a_{ii}$

(ii) $\quad Det(A) = \prod_i \lambda_i$ (the product of the eigenvalues).

(For the proof see [21]).

Notes:

(i) If $\lambda = 0$ is an A, then A is singular.

(ii) If A is symmetric then so is A^n, $n \in Z^+$.

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(iii) AB may be symmetric although A and B are not.

We now state the following theorem, which is useful in a variety of applications and which applies to arbitrary real or complex matrices.

Theorem [2-1-9]: If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues, distinct or not, of a matrix A of order n, and if p(A) is any polynomial function of A, then the eigenvalues of p(A) are $p(\lambda_1), p(\lambda_2), \ldots, p(\lambda_n)$.

(For the proof see [21]).

Example 1: Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

then the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$.

Now, let $p(A) = A^2 + 3A$ be a polynomial function of A. then

$$P(A) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + 3 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 7 \\ 7 & 11 \end{bmatrix}$$

which has 4, and 18 as an eigenvalues.

Note that these values agree with the theorem, where

$p(\lambda_1) = p(1) = 1^2 + 3(1) = 4$ and,

$p(\lambda_2) = p(3) = 3^2 + 3(3) = 18.$

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Since $p(A)=A^m$, where m is a positive integer, is a polynomial function of A, then we have the following special case:

Theorem [2-1-10]: If the eigenvalues of the $(n\times n)$ matrix A are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the eigenvalues of A^m, where m is any positive integer, are $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$.

[2-2] **Permutation Matrices:**

In this section, we will consider a new type of matrices, namely permutation matrices which will play a great role in circulant matrices, which will be used in representing cycle graphs.

Definition [2-2-1]: A permutation matrix is an $(n\times n)$ matrix that has exactly one entry equals to one in every row (column) and the rest of the entries in that row (column) are zeros. i.e ; a permutation matrix is an (nxn) matrix whose entries are zeros and ones, where any row or column contains exactly one entry equals to one.

We will consider one kind of permutation matrices denoted by P.

This matrix P is defined as

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & 1 \\ 1 & 0 & 0 & 0 & . & . & 0 \end{bmatrix}$$

in which $p_{i,i+1} = 1$, $i = 1, 2, \ldots, n-1$

$p_{n1} = 1$

$p_{ij} = 0$ for every other i, j.

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Next, we will find the powers of permutation matrices.

[2-2-2] Powers of Permutation Matrices:

The powers of the permutation matrix P, and which will be used later, appear in the following example.

Example 2: Let

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

be a (4 x 4) permutation matrix then

$$P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$P^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

while

$$P^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

and $P^5 = P$.

If P is the $(n \times n)$ permutation matrix P, then $P^n = I$.

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[2-2-3] **Eigenvalues of Permutation Matrices:**

In this section, we will find the eigenvalues and the eigenvectors of permutation matrices which will be used in calculating eigenvalues and eigenvectors of circulant matrices.

Let λ_k be an eigenvalue of the $(n\times n)$ permutation matrix P, then by definition, $\det(\lambda_k I - P) = 0$ which, by calculations amounts to

$$\lambda_k^n - 1 = 0.$$

The solutions of this equation are

$$\lambda_k = e^{\frac{2\pi ki}{n}} \quad \text{where } k = 1, 2, \ldots, n \text{ and } i^2 = -1.$$

Next, we show as an example, that if λ_k is an eigenvalue of the $(n\times n)$ permutation matrix P, then λ_k^3 is the corresponding eigenvalue of the matrix P^3.

Let v_k be an eigenvector of P corresponding to λ_k, then

$$\begin{aligned}
P^3 v_k &= P^2 (P\, v_k) \\
&= P^2 (\lambda_k v_k) \\
&= \lambda_k (P^2\, v_k) \\
&= \lambda_k (P(P\, v_k) \\
&= \lambda_k (P((\lambda_k v_k) \\
&= \lambda_k (\lambda_k (P\, v_k) \\
&= \lambda_k^2 (\lambda_k v_k) \\
&= \lambda_k^3\, v_k .
\end{aligned}$$

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As a special case of theorem [2-1-10], we state and prove the following theorem concerning permutation matrices.

Theorem [2-2-3-a]: If λ_k is an eigenvalue of the (n×n) permutation matrix P, then λ_k^j is the corresponding eigenvalue of P^j for j = 1, 2, ..., n.

Proof:

Let λ_k be an eigenvalue of the (n×n) permutation matrix P and v_k be an eigenvector of P corresponding to λ_k, then using mathematical induction:

1) We first show that it is true for j=1:

$P^1 v_k = \lambda_k^1 v_k$ (by definition of the eigenvalue),

so it is true for j = 1.

(2) Assume that it is true for j = m (i.e $P^m v_k = \lambda_k^m v_k$)

$P^{m+1} v_k = P(P^m v_k)$

$= P(\lambda_k^m v_k)$ by assumption

$= \lambda_k^m (P v_k)$

$= \lambda_k^m (\lambda_k v_k)$

$= \lambda_k^{m+1} v_k$.

So it is true for j = $m+1$ (Q.E.D).

Note that this proof is valid for theorem [2-1-10].

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As a result of this theorem, it is clear that if v_k is an eigenvector of P corresponding to λ_k, then the same vector v_k is also a corresponding eigenvector of P' corresponding to λ'_k.

Now, we will find the eigenvectors of the permutation matrix P.

[2-2-4] Eigenvectors of Permutation Matrices:

Let $\mathbf{v}_k$ be an eigenvector of the (n×n) permutation matrix P corresponding to the eigenvalue λ_k, then we have

$$P\,\mathbf{v}_k = \lambda_k \mathbf{v}_k \quad (\text{where } \mathbf{v}_k = [v_{k_1}, v_{k_2}, \dots, v_{k_n}]',$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & 1 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & 1 \\ 1 & 0 & 0 & . & . & . & . & 0 \end{bmatrix} \begin{bmatrix} v_{k_1} \\ v_{k_2} \\ . \\ . \\ . \\ . \\ . \\ v_{k_n} \end{bmatrix} = e^{\left(\frac{2\pi k i}{n}\right)} \begin{bmatrix} v_{k_1} \\ v_{k_2} \\ . \\ . \\ . \\ . \\ . \\ v_{k_n} \end{bmatrix}$$

or

$$\begin{bmatrix} v_{k_2} \\ v_{k_3} \\ . \\ . \\ . \\ . \\ . \\ v_{k_1} \end{bmatrix} = e^{\frac{2\pi k i}{n}} \begin{bmatrix} v_{k_1} \\ v_{k_2} \\ . \\ . \\ . \\ . \\ . \\ v_{k_n} \end{bmatrix}$$

One of the solutions for this equation is

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$$\mathbf{v}_k = a[e^{\frac{2\pi kni}{n}}, e^{\frac{2\pi k(n-1)i}{n}}, \ldots, e^{\frac{2\pi k(2)i}{n}}, e^{\frac{2\pi ki}{n}}]^t$$

$$= a[1, e^{\frac{2\pi k(n-1)i}{n}}, \ldots, e^{\frac{2\pi k(2)i}{n}}, e^{\frac{2\pi ki}{n}}]^t$$

$$= a[1, e^{\frac{2\pi k(n-1)i}{n}}, \ldots, e^{\frac{2\pi k(2)i}{n}}, e^{\frac{2\pi ki}{n}}]^*$$

where (a) is any arbitrary constant, and the symbol(^{t}) denotes the transpose of vectors, and (*) denotes the conjugate transpose of vectors or matrices.

Next, we consider another type of matrices called irreducible matrices, which will be used to find the eigenvectors of some matrices related to special graphs.

Definition [2-2-5]: An $(n\times n)$ non-negative matrix A is said to be irreducible if there is no permutation matrix of coordinates such that

$$P^t AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where P is an $(n\times n)$ permutation matrix , A_{11} is $(r\times r)$ matrix, and A_{22} is an $((n-r)\times(n-r))$ matrix.

Irreducible matrices can be related to its powers as in the following theorem.

Theorem [2-2-6]: An $(n\times n)$ non-negative matrix A is irreducible if and only if $(I_n + A)^{n-1} > 0$.

(For the proof see [42]).

Therefore, an $(n\times n)$ non-negative matrix is irreducible if and only if the matrix $(I_n + A)^{n-1}$ has positive entries.

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Example 3: Consider the following matrix A:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

so

$$I_4 + A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$(I_4 + A)^2 = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$$

while

$$(I_4 + A)^3 = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 3 & 1 & 1 & 3 \\ 1 & 3 & 1 & 3 \\ 3 & 1 & 3 & 1 \end{bmatrix}$$

which has only positive entries, so A is irreducible.

It is easy to check that the matrix

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

is not irreducible since by direct computation we find

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$$(I_4 + M)^3 = \begin{bmatrix} 4 & 0 & 4 & 0 \\ 6 & 4 & 6 & 4 \\ 4 & 0 & 4 & 0 \\ 6 & 4 & 6 & 4 \end{bmatrix}$$

which has zero entries.

At this point, it seems appropriate to finally state the following important theorem:

Theorem [2-2-7]:**(Perron-Frobenius Theorem for Irreducible Matrices)**

If A is an $(n \times n)$ non-negative, irreducible matrix, then:

1) One of its eigenvalues is positive and its magnitude is greater than or equal to any of that of the other eigenvalues.

2) There is an eigenvector with positive entries corresponding to that eigenvalue.

3) That eigenvalue is a simple root of the characteristic equation of A.

(For the proof see [33]).

[2-3] **Circulant Matrices:**

In the following, we will consider another type of matrices, namely circulant matrices, which will be of a great importance in our future study.

Definition [2-3-1]: An $(n \times n)$ circulant matrix A is a matrix of the form

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$$A=\begin{bmatrix} a_0 & a_1 & a_2 & . & . & a_{n-1} \\ a_{n-1} & a_0 & a_1 & . & . & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & a_1 \\ a_1 & a_2 & . & . & a_{n-1} & a_0 \end{bmatrix}$$

in general, any circulant $(n\times n)$ matrix A whose first row is

$(a_0, a_1, a_2, \ldots, a_{n-1})$ can be written as a polynomial function of an $(n\times n)$ permutation matrix P and its powers as

$$A = a_0 I + a_1 P + a_2 P^2 + \ldots + a_{n-1} P^{n-1}.$$

Next, we give an example of a (5×5) circulant matrix

Example 3: The matrix D shown below is a circulant matrix

$$D=\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$$

which can be written in terms of the permutation matrix P and its powers, where

$$P=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

as

$$D = I + 2P + 3P^2 + 4P^3 + 5P^4.$$

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Also if $A = \begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}$

then $A = a\,I + b\,P + c\,P^2 + d\,P^3$, where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

[2-3-2] **Eigenvalues of Circulant Matrices:**

In the following theorem, we will find the eigenvalues of circulant matrices.

Theorem [2-3-2-a]: If A is an (n×n) circulant matrix whose first row entries are $a_0,\ a_1, a_2, \ldots, a_{n-1}$, and λ_k is an eigenvalue of the (n×n) permutation matrix P, then the corresponding eigenvalue of A, α_k is given by

$$\alpha_k = \sum_{i=0}^{n-1} a_i \lambda_k^i$$

Proof:

Let A be an (n×n) circulant matrix, α_k be an eigenvalue of A,

$A = \sum_{i=0}^{n-1} a_i P^i$, so if $\mathbf{v}_k$ is an eigenvector corresponding to α_k, then

$$A\,\mathbf{v}_k = \left(\sum_{i=0}^{n-1} a_i P^i\right)\mathbf{v}_k$$

$$= (a_0 P^0 + a_1 P^1 + a_2 P^2 + \ldots + a_{n-1} P^{n-1})\,\mathbf{v}_k$$

$$= a_0\, I\, \mathbf{v}_k + a_1 P^1\, \mathbf{v}_k + a_2 P^2\, \mathbf{v}_k + \ldots + a_{n-1} P^{n-1} \mathbf{v}_k$$

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$= a_0 \lambda_k^0 \mathbf{v}_k + a_1 \lambda_k^1 \mathbf{v}_k + a_2 \lambda_k^2 \mathbf{v}_k + \ldots + a_{n-1} \lambda^{n-1} \mathbf{v}_k$

$= (a_0 \lambda_k^0 + a_1 \lambda_k^1 + a_2 \lambda_k^2 + \ldots + a_{n-1} \lambda^{n-1}) \mathbf{v}_k$

$= (\sum_{i=0}^{n-1} a_i \lambda_k^i) \mathbf{v}_k.$

$= \alpha_k \mathbf{v}_k .$

So α_k is an eigenvalue of A…….. Q.E.D.

Also note that an eigenvector of the circulant matrix $A = \sum_{i=0}^{n-1} a_i P^i$ is

$\mathbf{v}_k$, which is an eigenvector of the permutation matrix P.

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Chapter Three

Matrix Representation of Graphs

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[3-0] **Introduction:**

In chapter 1, we noticed that a graph can be used to represent the relationships between objects, we simply represent the objects by vertices, and the relationships by edges joining these vertices.

In order to investigate these relationships more closely, we need to study the theory of graphs in greater detail. We will introduce some useful terminology which will be needed in the following study.

Matrices will provide a convenient way of describing a graph, and since matrices lend themselves well to computer use, they make it possible to use the computer for extensive computational work in graph theory.

There are various types of matrices that can be used to specify a given graph. Here we describe the most important ones- the *adjacency matrix*, the *incidence matrix*, and the *distance matrix*.

For simplicity, we restrict our attention to graphs without loops.

[3-1]**The Adjacency Matrix**

Here we define the adjacency matrix of a connected graph.

Definition [3-1-1]: Let G be a graph, with n vertices labeled as

$v_1, v_2, \ldots, v_n$, then the *adjacency matrix*, denoted by $A(G) = [a_{ij}]$ is the $(n \times n)$ matrix in which a_{ij} is the number of edges joining vertices i and j.

For a simple graph, the adjacency matrix is a symmetric (0, 1)-matrix in which $a_{ij}=1$ if v_i is adjacent to v_j and zero otherwise.

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In the following example, the graph G and its adjacency matrix A(G) are displayed.

Example 1: The graph G shown in Fig 24 is represented by the following adjacency matrix

$$A\,(G) = \begin{bmatrix} 0 & 1 & 0 & 3 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \end{bmatrix}$$

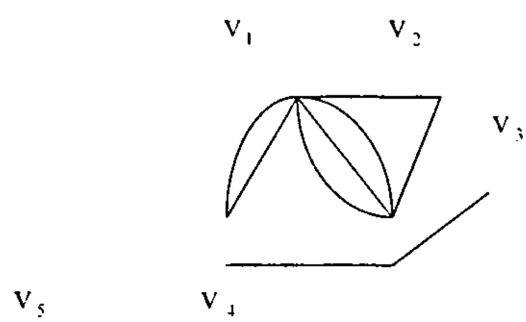

Fig 24: G

Next, we define a graph which is used to represent chemical molecules. In particular it will represent the skeleton structure of an organic compound with vertices representing usually the carbon atoms and edges representing the chemical bonds between the atoms.

Definition [3-1-2]: Let G be a simple connected graph, and the set

$\{v_1, v_2, \ldots, v_n\}$ be the set of vertices of G, and $\deg(v_i)$ be the degree of vertex v_i. Then G is said to be a chemical graph if $\deg(v_i) \le 4$ for all i = 1, 2, …, n.

In the following example, we see the graph and the matrix representation of Ethane molecule.

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Example 2: The graph shown in Fig 25 represents the chemical graph for the Ethane molecule C_2H_6, the corresponding graph, and its adjacency matrix.

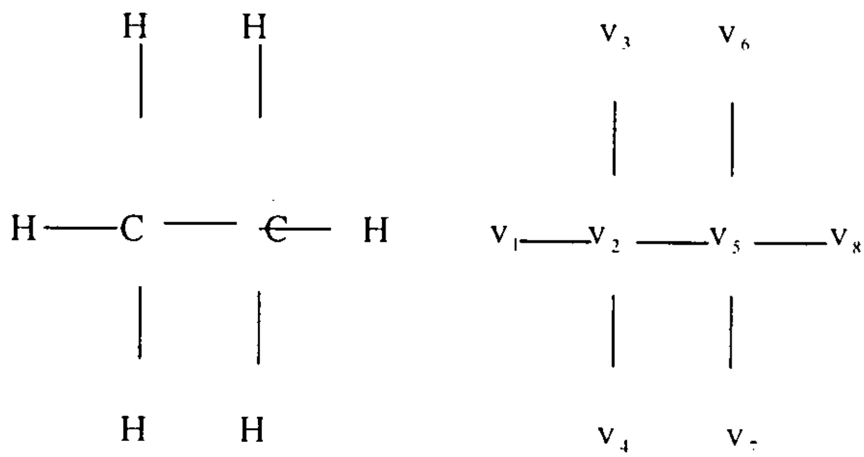

Fig 25

$$A(G)=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

This graph is a tree as it is a simple connected acyclic graph.

[3-1-a] **Properties of adjacency matrix A (G):**

1) A (G) is symmetric.

2) The sum of the numbers in any row or column of A(G) is the degree of the corresponding vertex.

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3) If G has no loops then all the entries on the main diagonal are zeros.

4) If G has no multiple edges then the entries of A(G) are either zero or one.

5) For any $(n \times n)$ symmetric matrix A with non-negative integer entries, we can associate a graph G with A as its adjacency matrix.

6) If G is a graph with vertex set $V=\{v_1, v_2, \ldots, v_n\}$, and if

$A(G) = [a_{ij}]$ is its adjacency matrix, and for any positive integer m, let

$A^m = [u_{ij}]$ denotes the matrix multiplication of m copies. then for each i and j, u_{ij} is the number of different walks of length m from v_i to v_j.

7) If A(G) is the adjacency matrix for a graph G. and $A^3 = [c_{ij}]$, then the number of triangles in G is $(\frac{1}{6})(\mathrm{Trace}(A^3))$, and the number of triangles in G having v_i as a vertex is $(\frac{1}{2})(c_{ii})$.

[3-1-b] **The Adjacency Matrix of a Disconnected Graph:**

Recall that a graph G is connected if for any two of its vertices there is a path between them, otherwise it is called disconnected.

A disconnected graph can be regarded as the union of connected graphs called components.

The vertices of any disconnected graph G can be labeled so

that its adjacency matrix A(G) has a block-diagonal form.

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The following example shows a connected graph and two disconnected graphs with a number of components for each graph.

Example 3: Consider the graph shown below

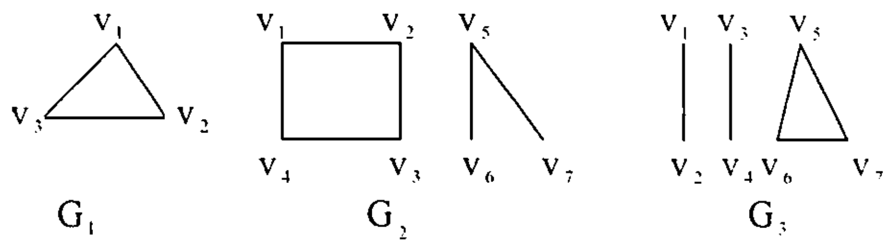

Fig 26

G_1 is connected, while G_2, G_3 are disconnected and have two and three components respectively.

The adjacency matrix of G_2 is:

$$A(G_2) = \begin{bmatrix} 0 & 1 & 0 & 1 & \vdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \vdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \vdots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 \end{bmatrix}$$

which is a block-diagonal matrix. Note also that the number of components of any disconnected graph G is the same as the number of blocks on the diagonal of A(G), and each block is a square $(k \times k)$ matrix, where k is the number of vertices in the corresponding component of G.

[3-2]**The Incidence Matrix**

Recall that the adjacency matrix of a graph represents the adjacency of vertices, but the incidence matrix represents the incidence of vertices and edges.

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Definition [3-2-1]: Let G be a graph without loops, with n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges $\{e_1, e_2, \ldots, e_m\}$. The *incidence matrix* of the graph G is the $(n \times m)$ matrix $U(G) = [u_{ij}]$ where

$$u_{ij} = \begin{cases} 1 & \text{, If } v_i \text{ is incident with } e_j. \\ 0 & \text{, otherwise.} \end{cases}$$

Example 4: Here is a graph G and its incidence matrix U(G)

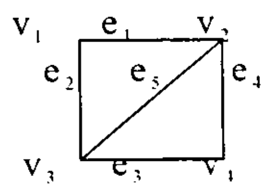

Fig 27

$$U(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Note: The sum of the i^{th} row of the incidence matrix is the degree of the vertex v_i while the sum in any column of the incidence matrix is 2.

Next, we define the line graph; a graph referred to in coloring problems as L (G).

Definition [3-2-2]: The line graph **L(G)** of a simple graph G is the graph obtained by taking the edges of G as vertices, and joining two of these vertices whenever the corresponding edges of G have a vertex in common. Simply, we can say that if e_1 and e_2 are two adjacent edges in the graph G, then e_1 and e_2 are adjacent vertices in the line graph **L(G)**.

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In the following example, we see a graph G and its corresponding line graph L(G).

Example 5: Here is a graph G and its line graph L(G):

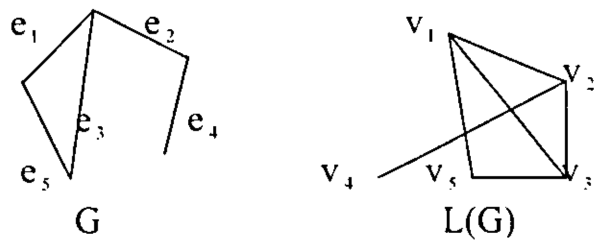

Fig 28

If G is a graph with p vertices and q edges, then the adjacency matrix of L(G), and the incidence matrix of G; U(G) are related by the following formula:

$$A\,(L(G)) = U^{T}U - 2\,I\ .$$ (Due to Kirnchoff [17]).

where U: the incidence matrix of G

U^{T}: the transpose of U

I: the identity $(q \times q)$ matrix

A(L(G)) : the adjacency matrix for the line graph.

In the next example, we apply the previous formula to a given graph G.

Example 6: Here is a graph G, and the corresponding line graph L(G), then we apply the previous formula:

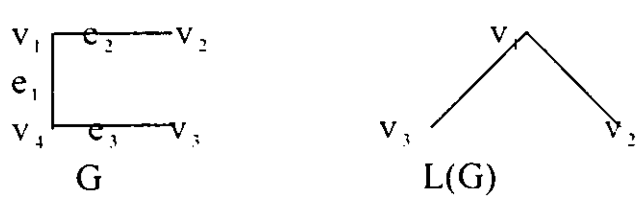

Fig 29

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$$U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} , \quad A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$A(L(G)) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$U^T U - 2I = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = A(L(G))$$

[3-3]The Distance Matrix

Given any two distinct vertices v_i and v_j in the same component of a graph G, there is at least one path between them, and there may be several of various lengths. The length of the shortest path joining them is called the *distance* between v_i , v_j and is denoted by $d(v_i, v_j)$.

Here are some basic concepts concerning connected graphs.

Definition [3-3-1] Let G (V, E) be a connected graph, and $v \in V$, then the *eccentricity* of v denoted by **e(v)** is the maximum value of d(u ,v) where u is allowed to range over all of the vertices of the graph G.

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$$\mathbf{e(v)} = \max \{ d(u, v) : u \in V, u \neq v \}.$$

Definition [3-3-2]: The *radius* of a graph G(V, E), denoted by **rad**(G), is defined to be the minimum eccentricity.

$$\mathbf{rad}\,(G) = \min \{ e(v) : v \in V \}.$$

The diameter of a connected graph is defined as the maximum eccentricity.

Definition [3-3-3]: The *center* of a graph G (V, E) is defined as

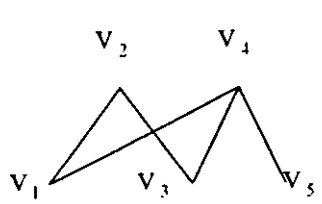
$$\text{center}\,(G) = \{ v \in V : e(v) = \text{rad}\,(G) \}.$$

Now, we introduce the last matrix representation of graphs; namely the distance matrix.

Definition [3-3-4] Let G be a connected graph on n vertices, the *distance matrix* of G denoted by $D(G) = [d_{ij}]$ is an $(n \times n)$ matrix where

$$d_{ij} = \begin{cases} d(v_i, v_j), & i \neq j \\ 0 \; , & i = j \end{cases}$$

Example 5: Here is a graph G and its distance matrix



$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 1 \\ 2 & 3 & 2 & 1 & 0 \end{bmatrix}$$

Fig 30: G

Properties of Distance Matrix:

1) D(G) is a square symmetric matrix with positive integer entries.

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2) $d_{ii} = 0$ identity relation

3) $d_{ij} = d_{ji}$ symmetry

4) $d_{ij} \le d_{ik} + d_{kj}$ triangle inequality where i , j , k = 1,2,...,n.

[3-4] Some Known results Concerning Distance Matrices

Here we state some of the known results concerning the distance matrix.

Theorem [3-4-a]: (R. Graham and H. Pollak 1971)

If T is a tree on n vertices, D = D(T) is its distance matrix .then

1) $\det(D) = (-1)^{n-1} (n-1) 2^{n-2}$ and

2) D is non-singular with one positive eigenvalue and (n-1) negative eigenvalues.

Theorem [3-4-b]: (R. Graham and H. Pollak 1973).

If G =K(m, n) is a complete bipartite graph on n + m vertices then $\det(D) = (-1)^{n+m} \; 2^{n+m-2} \; (3nm-4n-4m+4)$.

Theorem [3-4-c]: (Subhi N. Ruzieh 1989).

If T is a star S_n on n vertices, D is its distance matrix with eigenvalues:

$\delta_1 > \delta_2 \ge \delta_3 \ge \dots \ge \delta_n$,

then

$\delta_1 = n - 2 + \sqrt{n^2 - 3n + 3}$,

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$\delta_2 = n - 2 - \sqrt{n^2 - 3n + 3}$

and $\delta_i = -2$ for i=3,4,…,n.

Theorem [3-4-d]: (Graham, R. and Pollak, H. 1973).

If $G = K_n$ is the complete graph on n vertices, then

$\det(D) = (n-1)(-1)^{n-1}$.

Theorem [3-4-e]: (Ruzieh, Subhi N. 1989).

If $G = C_{2m}$ is a cycle on an even number of vertices, n = 2m, then $\delta_1(D) = m^2$ is the largest eigenvalue of D.

Lemma [3-4-f]: (Graham, R. et al. 1971)

If G is a cycle on n vertices, then the eigenvalues of D are given by

$f(u) = \sum_{i=1}^{n-1} a_i u^i$

where u = n^{th} root of unity and $(0, a_1, a_2, \ldots, a_{n-1})$ is the first row in the distance matrix of the cycle C_n.

Proposition [3-4-g]: (Ruzieh, Subhi N. 1989).

If C_n is the cycle on an odd number of vertices n=2m+1, then in the spectrum of $D(C_n)$ we have

1) $\delta_1 = m(m+1)$ and

2) the rest of the eigenvalues are all negative and are given by

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$$h(k) = -\frac{\sin^2(\frac{mk\pi}{n})}{\sin^2(\frac{k\pi}{n})}, \text{ for } k = 1, 2, \ldots, 2m.$$

Proposition [3-4-h]: (Ruzieh, Subhi N. 1989).

If C_n is the cycle on an even number of vertices n = 2m, then in the distance spectrum of C_n we have

(1) the distance spectral radius is $\delta_1 = m^2$,

(2) $\delta_{k+1} = 0$ for k = 2, 4, …, 2m-2 and

(3) $\delta_{k+1} = \frac{-1}{\sin^2(\frac{k\pi}{n})}$ for k = 1, 3, …, 2m - 1.

Theorem [3-4-i]: (Ruzieh, Subhi N. 1989).

If C_n is a cycle on n = 2m + 1 vertices, then

(1) the distance matrix $D(C_n)$ is non-singular,

(2) its spectral radius is $\frac{n^2 - 1}{4}$ and

(3) the remaining (n - 1) eigenvalues are all negative.

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Chapter Four

Eigenvalues and Eigenvectors related to Special Matrices

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[4-0] **Introduction:**

In this chapter, we will investigate the eigenvalues and eigenvectors of the matrix B_n, whose nonzero entries are the reciprocals of the corresponding nonzero entries of the distance matrix of the graph G. The work will concentrate on the matrix B_n related to the graph K(r, n-r).

Some of related results will be proved first, and those will be utilized for our goal.

[4-1] **Eigenvalues of Special Matrices:**

In this section, we will **state and prove** a theorem to find the eigenvalues of a matrix closely related to our work and whose results will form the corner stone for reaching some main results in this work.

Theorem [4-1-1]: Consider the following matrix:

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & 1 & 1 & \cdots & 1 \\ \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & 1 & 1 & \cdots & 1 \\ . & . & \cdots & . & . & . & \cdots & 1 \\ \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ . & . & \cdots & . & . & . & \cdots & . \\ . & . & \cdots & . & . & . & \cdots & . \\ 1 & 1 & \cdots & 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} \text{r rows} \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} \text{(n-r) rows} \end{matrix}$$

If $P_C(\alpha)$ is the characteristic polynomial of C, then

$P_C(\alpha) = \alpha^n + a_{n-1}\alpha^{n-1} + a_{n-2}\alpha^{n-2}$ where the coefficients a_{n-1}, a_{n-2} are given by:

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$a_{n-1} = -\frac{1}{2}\, n$

$a_{n-2} = -\frac{3}{4}\, r\,(n-r).$

Proof:

Since the characteristic polynomial of C can be written as

$P_C(\alpha) = \alpha^{n} + a_{n-1}\alpha^{n-1} + a_{n-2}\alpha^{n-2} + a_{n-3}\alpha^{n-3} + \ldots + a_0$ where

a_r = the sum of all principal minor determinants of order (n-r) multiplied by $(-1)^r$. Therefore $a_{n-3} = a_{n-4} = \ldots = a_0 = 0$, so the matrix has rank equals to two. Thus the matrix C has exactly two nonzero eigenvalues.

Therefore the characteristic polynomial takes the form

$P_C(\alpha) = \alpha^{n} + a_{n-1}\alpha^{n-1} + a_{n-2}\alpha^{n-2}$.

Next, we find a_{n-1} and a_{n-2}:

Now each (2×2) principal minor is the determinant of a (2×2) principal square sub-matrix consisting of the elements on the intersections of rows and columns i , j where:

$i = 1, 2, \ldots, r$

$j = r+1, r+2, \ldots, n.$

Any non-zero principal has the value = $\begin{vmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{vmatrix} = -\frac{3}{4}$

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The number of choices of such principals = r(n - r). Therefore, the sum of all (2×2) principal minors = $-\frac{3}{4}$(r) (n - r). But a_{n-2} = sum of all (2×2) principal minors, then

$$a_{n-2} = -\frac{3}{4}\, r\,(n - r)$$

$$a_{n-1} = \text{-trace}\,(C) = -\left(\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2}\right) = -\left(\frac{1}{2}\right) n \ \ldots \text{Q.E.D.}$$

This completes the proof of this theorem.

[4-1-2] **Evaluation of eigenvalues of a special matrix**

A related matrix to the distance matrix is the matrix B_{ij} whose entries are the reciprocals of the distances from v_i to v_j . This matrix models reasonably some physical situations and it is expected to be related to some physical properties of some organic compounds, so its eigenvalues and eigenvectors are of a great interest.

Definition [4-1-3]: Recall the definition of the distance matrix D(G)of a graph G.

Define the matrix $B_{ij} = [b_{ij}]$ where

$$b_{ij} = \begin{cases} \frac{1}{d_{ij}}, & \text{if } d_{ij} \neq 0 \\ 0, & \text{if } d_{ij} = 0. \end{cases}$$

Example 1: Here is the complete bipartite graph K (2, 3) and the corresponding matrix B_{ij} defined above.

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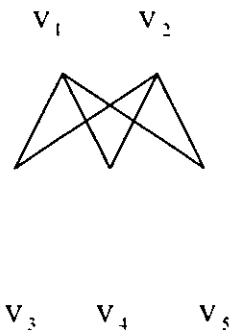

$$B_n = \begin{bmatrix} 0 & 0.5 & 1 & 1 & 1 \\ 0.5 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0.5 & 0.5 \\ 1 & 1 & 0.5 & 0 & 0.5 \\ 1 & 1 & 0.5 & 0.5 & 0 \end{bmatrix}$$

Fig 31 K (2, 3)

Now, we state a known result concerning the spectrum of B_n related to the complete bipartite graph K(1, n-1), n>2.

Theorem [4-1- 4]: (Al-shelleh, Mukhtar M. 1999)

If B_n is the matrix defined as before that corresponds to the complete bipartite graph K(1,n-1) , (n>2) , then the spectrum of B_n contains exactly three distinct eigenvalues $\lambda_1 > 0$, -0.5 with multiplicity n-2, and the last is $\lambda_n < 0$.

The values of λ_1, λ_n are given by

$$\lambda_1 = \frac{n-2+\sqrt{n^2+12n-12}}{4} \quad \text{and}$$

$$\lambda_n = \frac{n-2+\sqrt{n^2+12n-12}}{4}$$ (For the proof see [44])

[4-2] The Spectrum of B_n related to Some Complete Bipartite Graphs

In this section, we will deal with some special cases of the matrix B_n, namely the matrix for K(r, n - r).

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Here we state and prove the following theorem about the spectrum of B_n and its eigenvectors.

Theorem [4-2-1]: Let B_n be the matrix defined in [4-1-3] which corresponds to the complete bipartite graph k(r, n - r), n > r then the spectrum of B_n contains exactly three distinct eigenvalues, $\lambda_1 > 0$, $-(\frac{1}{2})$ with multiplicity (n - 2), $\lambda_n < 0$

where the values of λ_1 and λ_n are given by:

$$\lambda_1 = \frac{n-2+\sqrt{n^2+12rn-12r^2}}{4}$$

$$\lambda_n = \frac{n-2-\sqrt{n^2+12rn-12r^2}}{4}$$

Proof:

Let D be the distance matrix corresponding to K(r, n - r), then

$$D = \begin{bmatrix} 0 & 2 & 2 & \cdots & 2 & 1 & 1 & 1 & \cdots & 1 \\ 2 & 0 & 2 & \cdots & 2 & 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & 0 & \cdots & \vdots & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & 2 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & \cdots & 2 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 2 & 2 & \cdots & 2 \\ 1 & 1 & 1 & \cdots & 1 & 2 & 0 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & 2 & 2 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & 2 \\ 1 & 1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & 0 \end{bmatrix}$$

and

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$$B_n = \begin{bmatrix} 0 & 0.5 & 0.5 & \cdots & 0.5 & 1 & 1 & 1 & \cdots & 1 \\ 0.5 & 0 & 0.5 & \cdots & 0.5 & 1 & 1 & 1 & \cdots & 1 \\ 0.5 & 0.5 & 0 & \cdots & \vdots & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & 0.5 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.5 & 0.5 & \cdots & 0.5 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0.5 & 0.5 & \cdots & 0.5 \\ 1 & 1 & 1 & \cdots & 1 & 0.5 & 0 & 0.5 & \cdots & 0.5 \\ \vdots & \vdots & \vdots & \cdots & \vdots & 0.5 & 0.5 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0.5 \\ 1 & 1 & 1 & \cdots & 1 & 0.5 & 0.5 & \cdots & 0.5 & 0 \end{bmatrix}$$

Consider the matrix $C = B_n + \frac{1}{2}I$, then

$$C = \begin{bmatrix} 0.5 & 0.5 & 0.5 & \cdots & 0.5 & 1 & 1 & 1 & \cdots & 1 \\ 0.5 & 0.5 & 0.5 & \cdots & 0.5 & 1 & 1 & 1 & \cdots & 1 \\ 0.5 & 0.5 & 0.5 & \cdots & \vdots & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & 0.5 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.5 & 0.5 & \cdots & 0.5 & 0.5 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0.5 & 0.5 & 0.5 & \cdots & 0.5 \\ 1 & 1 & 1 & \cdots & 1 & 0.5 & 0.5 & 0.5 & \cdots & 0.5 \\ \vdots & \vdots & \vdots & \cdots & \vdots & 0.5 & 0.5 & 0.5 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0.5 \\ 1 & 1 & 1 & \cdots & 1 & 0.5 & 0.5 & \cdots & 0.5 & 0.5 \end{bmatrix}$$

Let spectrum $(B_n) = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\}$ and

spectrum $(C) = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$.

It is clear that $\alpha_i = \lambda_i + \frac{1}{2}$, and we see that C has only (2) independent rows; rank(C) =2 so it has 2 nonzero eigenvalues.

So spectrum(C) = $\{\alpha_1, 0, \alpha_n\}$, 0 with multiplicity (n-2), therefore, let

$$P_C(\alpha) = \alpha^n + a_{n-1}\alpha^{n-1} + a_{n-2}\alpha^{n-2}$$

be the characteristic polynomial of C then

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$a_{n-1} = -\ Tr\,(C) = -\frac{1}{2}n$, and

a_{n-2} = sum of all (2×2) principal minors

$= -\ (\frac{3}{4})\ r\,(n - r)$ (see theorem [4-1-1]).

Hence, $P_C\,(\alpha) = \alpha^n + (-\frac{1}{2})\ n\ \alpha^{n-1} + (-\frac{3}{4})\ r\,(n - r)\ \alpha^{n-2}$

$= \alpha^{n-2}(\alpha^2 - \frac{1}{2}\ n\ \alpha - \frac{3}{4}\ r\,(n - r))$

Setting $P_C\,(\alpha) = 0$ we get the following equation:

$\alpha^{n-2}(\alpha^2 - \frac{1}{2}\ n\ \alpha - \frac{3}{4}\ r\,(n - r)) = 0$, which has the solutions

$\alpha = 0$ with multiplicity (n - 2) and

$$\alpha_1 = \frac{1}{2}(\frac{1}{2}n + \sqrt{\frac{1}{4}n^2 + 4(\frac{3}{4})r(n-r)})$$

$$\alpha_n = \frac{1}{2}(\frac{1}{2}n - \sqrt{\frac{1}{4}n^2 + 4(\frac{3}{4})r(n-r)})$$

So for C we have

$$\alpha_1 = \frac{n + \sqrt{n^2 + 12rn - 12r^2}}{4}$$

$$\alpha_n = \frac{n - \sqrt{n^2 + 12rn - 12r^2}}{4}$$

$\alpha_2 = \alpha_3 = \ldots = \alpha_{n-1} = 0.$

Now, for B_n:

Since $\alpha_i = \lambda_i + \frac{1}{2}$ we have

$\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1} = -(\frac{1}{2})$ while

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$$\lambda_1 = \frac{n-2+\sqrt{n^2+12rn-12r^2}}{4}$$

$$\lambda_n = \frac{n-2-\sqrt{n^2+12rn-12r^2}}{4} \quad \ldots \text{ Q.E.D}$$

Here we find the eigenvectors of B_n:

Let **x** be an $(n\times 1)$ non-zero vector which is an eigenvector corresponds to an eigenvalue α of $C = B_n + \frac{1}{2}I$.

Since spectrum (C) = $\{\alpha_1, 0, \alpha_n\}$, 0 with multiplicity (n - 2) then for $\alpha = 0$ we have

$C\,\mathbf{x} = \alpha\,\mathbf{x} \Rightarrow C\,\mathbf{x} = \underline{0}$

$$\begin{bmatrix} 0.5 & 0.5 & 0.5 & \cdots & 0.5 & 1 & 1 & 1 & \cdots & 1 \\ 0.5 & 0.5 & 0.5 & \cdots & 0.5 & 1 & 1 & 1 & \cdots & 1 \\ 0.5 & 0.5 & 0.5 & \cdots & \vdots & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & 0.5 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.5 & 0.5 & \cdots & 0.5 & 0.5 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0.5 & 0.5 & 0.5 & \cdots & 0.5 \\ 1 & 1 & 1 & \cdots & 1 & 0.5 & 0.5 & 0.5 & \cdots & 0.5 \\ \vdots & \vdots & \vdots & \cdots & \vdots & 0.5 & 0.5 & 0.5 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0.5 \\ 1 & 1 & 1 & \cdots & 1 & 0.5 & 0.5 & \cdots & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

Then E_i: $\frac{1}{2}x_1 + \frac{1}{2}x_2 + \ldots + \frac{1}{2}x_r + x_{r+1} + x_{r+2} + \ldots + x_n = 0$

$\forall\ i = 1, 2, \ldots, r.$ (1)

E_k: $x_1 + x_2 + \ldots + x_r + \frac{1}{2}x_{r+1} + \frac{1}{2}x_{r+2} + \ldots + \frac{1}{2}x_n = 0$

$\forall\ k = r+1, r+2, \ldots, n.$ (2)

From (1) we get the following equation:

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$$x_1 + x_2 + \ldots + x_r + 2x_{r+1} + 2x_{r+2} + \ldots + 2x_n = 0$$

Therefore, we get the following (r - 1) independent orthonormal vectors:

$$\frac{1}{\sqrt{2}}(1,-1,0,0,\ldots,\ 0,0,\ldots,\ 0)^T$$

$$\frac{1}{\sqrt{2+(-2)^2}}(1,1,-2,0,0,\ldots,\ 0,0,\ldots,\ 0)^T$$

$$\frac{1}{\sqrt{3+(-3)^2}}(1,1,1,-3,0,0,\ldots,\ 0,0,\ldots,\ 0)^T$$

$\vdots$

$\vdots$

$$\frac{1}{\sqrt{(r-1)+(1-r)^2}}(1,1,\ldots,1-r,0,0,\ldots,0)^T$$

Also from (2) we get the following equation:

$$2x_1 + 2x_2 + \ldots + 2x_r + x_{r+1} + x_{r+2} + \ldots + x_n = 0$$

Therefore, we get will the following (n-r-1) orthonormal independent vectors:

$$\frac{1}{\sqrt{1+(-1)^2}}(0,0,\ldots,\ 0,1,-1,0,0,\ldots,\ 0)^T$$

$$\frac{1}{\sqrt{2+(-2)^2}}(0,0,\ldots\ 0,1,1,-2,0,0,\ldots,\ 0)^T$$

$$\frac{1}{\sqrt{3+(-3)^2}}(0,0,\ldots,\ 0,1,1,1,-3,0,0,\ldots,\ 0)^T \quad .$$

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⋮

⋮

$$\frac{1}{\sqrt{(n-r-1)+(1-(n-r))^2}}(0,0,\ldots,0,1,1,\ldots,1-(n-r))^t$$

So we have (n - 2) independent orthonormal eigenvectors which correspond to $\alpha_i = 0$, for i = 2 , 3 ,... , n - 1.

For $\alpha_1, \alpha_n \neq 0$:

Let **x** be a non-zero vector with the property that

$\mathbf{Cx} = \alpha\,\mathbf{x}$, $\alpha = \alpha_1, \alpha_n$. So

$$\begin{bmatrix} 0.5 & 0.5 & 0.5 & \ldots & 0.5 & 1 & 1 & 1 & \ldots & 1 \\ 0.5 & 0.5 & 0.5 & \ldots & 0.5 & 1 & 1 & 1 & \ldots & 1 \\ 0.5 & 0.5 & 0.5 & \ldots & \vdots & 1 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ldots & 0.5 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.5 & 0.5 & \ldots & 0.5 & 0.5 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 & 0.5 & 0.5 & 0.5 & \ldots & 0.5 \\ 1 & 1 & 1 & \ldots & 1 & 0.5 & 0.5 & 0.5 & \ldots & 0.5 \\ \vdots & \vdots & \vdots & \ldots & \vdots & 0.5 & 0.5 & 0.5 & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \ldots & 0.5 \\ 1 & 1 & 1 & \ldots & 1 & 0.5 & 0.5 & \ldots & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha\,x_1 \\ \alpha\,x_2 \\ \alpha\,x_3 \\ \vdots \\ \alpha\,x_r \\ \alpha\,x_{r+1} \\ \vdots \\ \vdots \\ \vdots \\ \alpha\,x_n \end{bmatrix}$$

which gives the following equations:

$$E_1 : \sum_{i=1}^{r} x_i + 2\sum_{i=r+1}^{n} x_i = \alpha x_1$$

$$E_2 : \sum_{i=1}^{r} x_i + 2\sum_{i=r+1}^{n} x_i = \alpha x_2$$

⋮

⋮

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$E_r: \sum_{i=1}^{r} x_i + 2\sum_{i=r+1}^{n} x_i = \alpha x_r$

$E_{r+1}: 2\sum_{i=1}^{r} x_i + \sum_{i=r+1}^{n} x_i = \alpha x_{r+1}$

$\vdots$

$\vdots$

$E_n : 2\sum_{i=1}^{r} x_i + \sum_{i=r+1}^{n} x_i = \alpha x_n$

Subtract E_1 from E_2 we get $\alpha\ (x_2 - x_1) = 0 \Rightarrow x_2 = x_1$ since $\alpha \neq 0$, and similarly we get $x_1 = x_2 = \ldots = x_r$.

By the same principle we get $x_{r+1} = x_{r+2} = \ldots = x_n$, and so the vector is of the form

$$(\underbrace{a, a, \ldots, a}_{r\ times}, \underbrace{b, b, \ldots, b}_{(n-r)\ times})^T .$$

Since $\alpha_1 > 0$, then by **Perron-Frobenius Theorem**, there is an eigenvector $\mathbf{x}^{(1)}$ such that

$C\,\mathbf{x}^{(1)} = \alpha_1\ \mathbf{x}^{(1)}$ and $\mathbf{x}^{(1)}$ has all positive entries, so a vector corresponding to α_1 is of the form $(\underbrace{a, a, \ldots, a}_{r\ times}, \underbrace{b, b, \ldots, b}_{(n-r)\ times})^T$ where both a and b are positive,

thus the vector

$$\mathbf{x}^{(1)} = \frac{1}{\sqrt{n}}(\underbrace{1, 1, \ldots, 1}_{r\ times}\ \underbrace{1, 1, \ldots, 1}_{(n-r)\ times})^T$$

is an eigenvector corresponds to $\alpha_1 > 0$.

For $\alpha = \alpha_n < 0$, there is an eigenvector

$\mathbf{x}^{(n)}$: $C\,\mathbf{x}^{(n)} = \alpha_n \mathbf{x}^{(n)}$, $\mathbf{x}^{(n)}$ contains both positive and negative entries.

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So $\mathbf{x}^{(n)}$ is of the form

$(\underbrace{a, a, \ldots, a}_{r\ times}, \underbrace{b, b, \ldots, b}_{(n-r)\ times})^T$ where $a > 0$, $b < 0$. So

$$\mathbf{x}^{(n)} = \sqrt{\frac{n-r}{nr}}(\underbrace{-1, -1, \ldots, -1}_{r\ times}, \underbrace{\frac{r}{n-r}, \frac{r}{n-r}, \ldots, \frac{r}{n-r}}_{(n-r) times})^T$$

is an eigenvector for C corresponds to $\alpha_n < 0$.

Recall that the eigenvectors of B are the same as of C.

As a special case, when r = 2, we have the following results:

$$\lambda_1 = \frac{n-2+\sqrt{n^2+24n-48}}{4}$$

$$\lambda_n = \frac{n-2-\sqrt{n^2+24n-48}}{4}$$

$$\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1} = -(\frac{1}{2})$$

The following (n-2) independent orthonormal eigenvectors correspond to $\lambda_i = -\frac{1}{2}$ for i =2, 3, ..., n-1:

$$\frac{1}{\sqrt{1+(-1)^2}}(0, 0, 1, -1, 0, 0, \ldots, 0)^T.$$

$$\frac{1}{\sqrt{2+(-2)^2}}(0, 0, 1, 1, -2, 0, \ldots, 0)^T.$$

$$\frac{1}{\sqrt{3+(-3)^2}}(0, 0, 1, 1, 1, -3, 0, \ldots, 0)^T.$$

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$$\frac{1}{\sqrt{(n-3)+(3-n)^2}}(0,0,1,1,1,\dots,1,3-n)^T.$$

For λ_1 , we have the following eigenvector:

$$\mathbf{x}^{(1)} = \frac{1}{\sqrt{n}}(1,1,1,\dots,1)^T$$

For λ_n , we have the following eigenvector:

$$\mathbf{x}^{(n)}=\sqrt{\frac{n-2}{2n}}(-1,-1,\underbrace{\frac{2}{n-2},\frac{2}{n-2},\dots,\frac{2}{n-2}}_{(n-2)times})^T \quad .$$

Note: If the complete bipartite graph is of the form K(m, n), then we get the following forms:

$$\lambda_1 = \frac{m+n-2+\sqrt{m^2+14mn+n^2}}{4}$$

$$\lambda_n = \frac{m+n-2-\sqrt{m^2+14mn+n^2}}{4}$$

$$\lambda_2 = \lambda_3 = \dots = \lambda_{m+n-1} = -(\frac{1}{2})$$

For $\lambda_2 = \lambda_3 = \dots = \lambda_{m+n-1} = -(\frac{1}{2})$ we have the following (m+n-2) independent orthonormal eigenvectors:

$$\frac{1}{\sqrt{2}}(1,-1,0,0,\dots,0,0,\dots,0)^T$$

$$\frac{1}{\sqrt{2+(-2)^2}}(1,1,-2,0,0,\dots,0,0,\dots,0)^T$$

$$\frac{1}{\sqrt{3+(-3)^2}}(1,1,1,-3,0,0,\dots,0,0,\dots,0)^T$$

$\vdots$

$\vdots$

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$$\frac{1}{\sqrt{(m-1)+(1-m)^2}}(1,1,\dots,1-m,0,0,\dots,0)^t$$

$$\frac{1}{\sqrt{1+(-1)^2}}(0,0,\dots,0,1,-1,0,0,\dots,0)^t$$

$$\frac{1}{\sqrt{2+(-2)^2}}(0,0,\dots,0,1,1,-2,0,0,\dots,0)^T$$

$$\frac{1}{\sqrt{3+(-3)^2}}(0,0,\dots,0,1,1,1,-3,0,0,\dots,0)^T$$

$\vdots$

$\vdots$

$$\frac{1}{\sqrt{(n-1)+(1-n)^2}}(0,0,\dots,0,1,1,\dots,1-n))^T$$

For λ_1 we have the following eigenvector

$$\mathbf{x}^{(1)} = \frac{1}{\sqrt{m+n}}(\underbrace{1,1,1,\dots,1}_{(m+n)\ times})^T .$$

For λ_n we have the following eigenvector

$$\mathbf{x}^{(n)} = \sqrt{\frac{n}{m(m+n)}}(\underbrace{-1,-1,\dots,-1}_{m\ times},\underbrace{\frac{m}{n},\frac{m}{n},\dots,\frac{m}{n}}_{(n)times})^T .$$

Corollary [4-2-3]: If B_n is the matrix mentioned in theorem [4-2-1], then $\det(B_n) = -(\frac{1}{2})^n (1-n-3rn+3r^2)$.

Proof:

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$$B_n = \begin{bmatrix} 0 & 0.5 & 0.5 & \dots & 0.5 & 1 & 1 & 1 & \dots & \vdots \\ 0.5 & 0 & 0.5 & \dots & 0.5 & 1 & 1 & 1 & \dots & 1 \\ 0.5 & 0.5 & 0 & \dots & \vdots & 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \dots & 0.5 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.5 & 0.5 & \dots & 0.5 & 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 & 0.5 & 0.5 & \dots & 0.5 \\ 1 & 1 & 1 & \dots & 1 & 0.5 & 0 & 0.5 & \dots & 0.5 \\ \vdots & \vdots & \vdots & \dots & \vdots & 0.5 & 0.5 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & 0.5 \\ 1 & 1 & 1 & \dots & 1 & 0.5 & 0.5 & \dots & 0.5 & 0 \end{bmatrix}$$

From theorem [4-2-1] we have

$\lambda_2 = \lambda_3 = \dots = \lambda_{n-1} = -(\frac{1}{2})$ while

$$\lambda_1 = \frac{n-2+\sqrt{n^2+12rn-12r^2}}{4}$$

$$\lambda_n = \frac{n-2-\sqrt{n^2+12rn-12r^2}}{4},$$

but $\det(B_n) = \prod_{i=1}^{n} \lambda_i = (-\frac{1}{2})^{n-2} \lambda_1 \lambda_n$

$$= (-\frac{1}{2})^{n-2} (\frac{1}{16})((n-2)^2-(n^2+12rn-12r^2))$$

$$= (-\frac{1}{2})^{n-2} (\frac{1}{16}) (n^2-4n+4-n^2-12rn+12r^2)$$

$$= (-\frac{1}{2})^{n-2} (\frac{1}{16}) 4 (1-n-3rn+3r^2)$$

$$= (-\frac{1}{2})^{n-2}(1-n-3rn+3r^2) \qquad \text{Q.E.D}$$

The following table contains numerical values of the spectral radius λ_1 and those of λ_n of the complete bipartite graph on n vertices K(r, n-r), obtained both by direct calculation using QR algorithm, and by the resulting formula in theorem [4-2-1].

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Graph	λ_1		λ_n	
	By QR algorithm	By Formula	By QR algorithm	By Formula
K(2.1)	1.686141	1.6861406	-1.186141	-1.1861406
K(3.1)	2.302776	2.3027756	-1.302776	-1.3027756
K(4.1)	2.886001	2.8860009	-1.386001	-1.3860009
K(5.1)	3.4494940	3.4494897	-1.449490	-1.4494897
K(2.2)	2.5	2.5	-1.5	-1.5
K(3.2)	3.212214	3.212214	-1.712214	-1.712214
K(4.2)	3.872281	3.872281	-1.872281	-1.872281
K(5.2)	4.5	4.5	-2	-2
K(6.2)	5.105552	5.105551	-2.105551	-2.105551
K(3.3)	4	4	-2	-2
K(4.3)	4.723112	4.72311099	-2.223111	-2.2231109
K(5.3)	5.405126	5.4051248	-2.405125	-2.4051248
K(6.3)	6.058422	6.0584129	-2.558423	-2.5584129
K(6.4)	6.924429	6.9244289	-2.924429	-2.9244289
K(6.5)	7.732929	7.732928	-3.232929	-3.232928

Note: As we proved before, if we consider the complete bipartite graph K(r, n - r), spectrum (B) = { λ_1, -($\frac{1}{2}$), λ_n }, where

$$\lambda_1 = \frac{n-2+\sqrt{n^2+12rn-12r^2}}{4}$$

$$\lambda_n = \frac{n-2-\sqrt{n^2+12rn-12r^2}}{4}$$

If r is fixed, then $\lim_{n\to\infty} \lambda_1 = \infty$.

So λ_1 increases as n increases.

In the following table, we find the eigenvalues λ_n from the resulting formula for several graphs as n increases rabidly:

Complete bipartite graph K(2.n-2)	λ_n	Complete bipartite graph K(3,n-3)	λ_n	Complete bipartite graph K(4.n-4)	λ_n
K(2.4)	-1.872	K(3.3)	-1.712	K(4.4)	-1.872
K(2.6)	-2.106	K(3.5)	-2.223	K(4.6)	-2.500
K(2.8)	-2.272	K(3.7)	-2.558	K(4.8)	-2.924
K(2.10)	-2.399	K(3.9)	-2.806	K(4.10)	-3.245

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Complete bipartite graph K(2,n-2)	λ_n	Complete bipartite graph K(3,n-3)	λ_n	Complete bipartite graph K(4,n-4)	λ_n
K(2,20)	-2.762	K(3,21)	-3.585	K(4,22)	-4.298
K(2,30)	-2.940	K(3,31)	-3.893	K(4,32)	-4.738
K(2,40)	-3.048	K(3,41)	-4.087	K(4,42)	-5.024
K(2,50)	-3.120	K(3,51)	-4.222	K(4,52)	-5.228
K(2,100)	-3.289	K(3,100)	-4.550	K(4,100)	-5.741
K(2,200)	-3.388	K(3,200)	-4.755	K(4,200)	-6.077
K(2,300)	-3.424	K(3,300)	-4.832	K(4,300)	-6.207
K(2,400)	-3.442	K(3,400)	-4.872	K(4,402)	-6.275
K(2,500)	-3.453	K(3,500)	-4.896	K(4,502)	-6.318
K(2,600)	-3.461	K(3,600)	-4.913	K(4,600)	-6.347
K(2,1000)	-3.476	K(3,1000)	-4.947	K(4,1000)	-6.407
K(2,2000)	-3.488	K(3,2000)	-4.973	K(4,2000)	-6.453
K(2,3000)	-3.492	K(3,3000)	-4.982	K(4,3000)	-6.468
K(2,4000)	-3.494	K(3,4000)	-4.987	K(4,4000)	-6.476
K(2,5000)	-3.495	K(3,5000)	-4.989	K(4,5000)	-6.481
K(2,10000)	-3.498	K(3,10000)	-4.995	K(4,10000)	-6.490
K(2,11000)	-3.498	K(3,11000)	-4.995	K(4,11000)	-6.491
K(2,12000)	-3.492	K(3,12000)	-4.996	K(4,12000)	-6.492
K(2,20000)	-3.499	K(3,20000)	-4.997	K(4,20000)	-6.495
K(2,30000)	-3.499	K(3,30000)	-4.998	K(4,30000)	-6.497
K(2,40000)	-3.499	K(3,40000)	-4.999	K(4,40000)	-6.499

From the table above, we see that the eigenvalue λ_n has a limit as $n \to \infty$. Next we prove the following lemma which deals with the asymptotic values of λ_n.

Lemma: If $\lambda_n = \dfrac{n-2-\sqrt{n^2+12rn-12r^2}}{4}$, then

$$\lim_{n\to\infty} \lambda_n = -(\frac{1}{2}) - (\frac{3}{2})r.$$

Proof:

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \frac{n-2-\sqrt{n^2+12rn-12r^2}}{4}$$

$$= \lim_{n\to\infty} \frac{n-2-\sqrt{n^2+12rn-12r^2}}{4} \cdot \frac{(n-2+\sqrt{n^2+12rn-12r^2})}{(n-2+\sqrt{n^2+12rn-12r^2})}$$

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$$= \lim_{n\to\infty} \left(\frac{1}{4}\right)\frac{(n-2)^2-(n^2+12rn-12r^2)}{(n-2+\sqrt{n^2+12rn-12r^2})}$$

$$= \lim_{n\to\infty} \left(\frac{1}{4}\right)\left(\frac{(n^2-4n+4-n^2-12rn+12r^2)}{(n-2+\sqrt{n^2+12rn-12r^2})}\right.$$

$$= \lim_{n\to\infty} \frac{(-n+1-3rn+3r^2)}{(n-2+\sqrt{n^2+12rn-12r^2})}$$

$$= \lim_{n\to\infty} \frac{(-1+(\frac{1}{n})-3r+(\frac{3r^2}{n})}{1-\frac{2}{n}+\sqrt{1+(\frac{12r}{n})-(\frac{12r^2}{n^2})}}$$

$$= \frac{-1-3r}{2}$$

$$= -\left(\frac{1}{2}\right)-\left(\frac{3}{2}\right)r.$$

This agrees with the previous table where as $n \to \infty$

$\lambda_n \to -3.5$ for r = 2

$\lambda_n \to -5$ for r = 3

$\lambda_n \to -6.5$ for r = 4.

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Chapter Five

Eigenvalues and Eigenvectors related to Cycle Graphs

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[5-0]: **Introduction**

In this section, we deal with the matrix representation of cycle graphs, and then we find the eigenvalues and eigenvectors of the matrix B_n related to cycle graphs for even and odd cases.

Recall that a cycle graph is a connected graph in which each of its vertices has degree 2.

[5-1] **Matrix Representation of Cycle Graphs**:

If C_n is a cycle graph then its distance matrix is a matrix of the form

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & . & . & . & 1 \\ 1 & 0 & 1 & 2 & . & . & . & 2 \\ 2 & 3 & 0 & 1 & 2 & . & . & 3 \\ . & . & . & 0 & . & . & . & . \\ . & . & . & . & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & 1 \\ 1 & 2 & 3 & . & . & . & 1 & 0 \end{bmatrix}.$$

This matrix is a circulant matrix of the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & \cdots & a_0 \end{bmatrix}$$

where <u>for even n</u> $a_i = i$ for $0 \le i \le (\frac{1}{2})n$ and

$$a_j = a_k \quad \text{whenever} \quad j + k = n$$

while <u>for odd</u> values of n we have

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$a_i = i \quad \text{for } 0 \le i \le \frac{(n-1)}{2}$ and

$a_j = a_k , j + k = n$

[5-2] **Eigenvalues and Eigenvectors for B_n related to C_n:**

Here we find the eigenvalues and eigenvectors for B_n that is related to C_n, and we will also show that the eigenvalues of this matrix are real because of the symmetry of B_n.

Let P be an (n x n) permutation matrix, then

spectrum (P)={ $\lambda_1, \lambda_2, \ldots, \lambda_n$} where all the eigenvalues are the roots of unity;

$\lambda^n = 1 \Rightarrow \lambda = \sqrt[n]{1} = e^{(\frac{2\pi k}{n})i}$ where $k = 0, 1, 2, \ldots, n - 1$

and $i^2 = -1$.

So if $B_n = a_0 I + a_1 P + a_2 P^2 + \ldots + a_{n-1} P^{n-1} = \sum_{k=0}^{n-1} a_k P^k$

then an eigenvalue α_j of B_n that corresponds to λ_j is given by

$\alpha_j = \sum_{k=0}^{n-1} a_k \lambda_j^k$, j = 1, 2,...n. (Theorem [2-1-9]).

Theorem [5-2-1]: Let P be an (n x n) permutation matrix, and λ_k is an eigenvalue of P, then the corresponding eigenvalue of B_n related to C_n; α_k is a real number.

Proof:

Let λ_k be an eigenvalue of P, α_k the corresponding eigenvalue of B_n.

We consider the following two cases:

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1) n is odd : Let n = 2p+1

$$\alpha_k = \sum_{l=0}^{n-1} a_l \lambda_k^l = \sum_{l=0}^{2p} a_l \lambda_k^l$$

$$= a_0 \lambda_k^0 + a_1 \lambda_k^1 + a_2 \lambda_k^2 + \ldots + a_{p+1} \lambda_k^{p+1} + a_{p+2} \lambda_k^{p+2} + \ldots + a_{2p} \lambda_k^{2p}$$

$$= a_0 + (a_1 \lambda_k^1 + a_{2p} \lambda_k^{2p}) + (a_2 \lambda_k^2 + a_{2p-1} \lambda_k^{2p-1}) + \ldots + (a_p \lambda_k^p + a_{p+1} \lambda_k^{p+1})$$

Now for $a_m \lambda_k^m + a_l \lambda_k^l$ where $m+l = 2p + 1 = n$, we have $a_m = a_l$ and,

$$a_m \lambda_k^m + a_l \lambda_k^l = a_m (\lambda_k^m + \lambda_k^l)$$

$$= a_m \left(\cos\left(\frac{2\pi\, km}{n}\right) + i \sin\left(\frac{2\pi\, km}{n}\right) + \cos\left(\frac{2\pi\, kl}{n}\right) + i \sin\left(\frac{2\pi\, kl}{n}\right)\right)$$

$$= a_m \left(\cos\left(\frac{2\pi\, km}{n}\right) + \cos\left(\frac{2\pi\, kl}{n}\right) + i\left(\sin\left(\frac{2\pi\, km}{n}\right) + \sin\left(\frac{2\pi\, kl}{n}\right)\right)\right).$$

Note that $\frac{2\pi\, km}{n} + \frac{2\pi\, kl}{n} = \frac{2\pi\, k(m+l)}{n} = \frac{2\pi\, kn}{n} = 2\pi k.$

So, $\sin\left(\frac{2\pi\, km}{n}\right) = -\sin\left(\frac{2\pi\, kl}{n}\right)$, then the imaginary part of the corresponding terms from the sum cancels and the remaining terms are real. So α_k is a real number.

The second part deals with even values of n.

2) n is even: Let n = 2p

$$\alpha_k = \sum_{l=0}^{n-1} a_l \lambda_k^l = \sum_{l=0}^{2p-1} a_l \lambda_k^l$$

$$= a_0 \lambda_k^0 + a_1 \lambda_k^1 + \ldots + a_{p-1} \lambda_k^{p-1} + a_p \lambda_k^p + a_{p+1} \lambda_k^{p+1} + \ldots + a_{2p-1} \lambda_k^{2p-1}$$

$$= a_0 + (a_1 \lambda_k^1 + a_{2p-1} \lambda_k^{2p-1}) + (a_2 \lambda_k^2 + a_{2p-2} \lambda \lambda_k^{2p-2}) + \ldots +$$
$$(a_{p-1} \lambda_k^{p-1} + a_{p+1} \lambda_k^{p+1}) + a_p \lambda_k^p$$

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Now for $a_r \lambda_k^r + a_m \lambda_k^m$ where $r + m = 2p = n$, we have $a_r = a_m$ and

$$a_r \lambda_k^r + a_m \lambda_k^m = a_m (\lambda_k^r + \lambda_k^m)$$

$$= a_r \left(\cos\left(\frac{2\pi\, kr}{2p}\right) + i \sin\left(\frac{2\pi\, kr}{2p}\right)\right) + \cos\left(\frac{2\pi\, km}{2p}\right) + i \sin\left(\frac{2\pi\, km}{2p}\right))$$

$$= a_r \left(\cos\left(\frac{2\pi\, kr}{2p}\right) + \cos\left(\frac{2\pi\, km}{2p}\right)\right) + i \left(\sin\left(\frac{2\pi\, kr}{2p}\right) + \sin\left(\frac{2\pi\, km}{2p}\right)\right).$$

Note that $\left(\frac{2\pi\, kr}{2p}\right) + \left(\frac{2\pi\, km}{2p}\right) = \frac{2\pi\, k(r+m)}{n} = 2\pi k$

since $\left(\frac{2\pi\, kr}{n}\right) + \left(\frac{2\pi\, km}{n}\right) = 2\pi k \Rightarrow \sin\left(\frac{2\pi\, kr}{n}\right) = -\sin\left(\frac{2\pi\, km}{n}\right)$, so the complex part cancels.

For $a_p \lambda_k^p = a_p\left(\cos\left(\frac{2\pi\, kp}{2p}\right) + i \sin\left(\frac{2\pi\, kp}{2p}\right)\right)$

$$= a_p(\cos(\pi k)\pi k + i \sin(\pi k)).$$

So, $\sin(\pi k) = 0 \Rightarrow$ the complex part cancels so α_k is real. Q.E.D.

[5-3] **Examples:**

In the following two examples we will compute the eigenvalues of the matrix B_n of C_4 and C_5.

Example 1: In this example we compute the eigenvalues and the corresponding eigenvectors for the matrix B_4 of the cycle graph C_4.

C_4 is the graph shown below

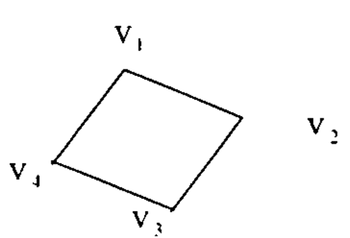

Fig 32 C_4

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which has the following distance matrix

$$D = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

and

$$B_4(D) = \begin{bmatrix} 0 & 1 & \frac{1}{2} & 1 \\ 1 & 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 & 1 \\ 1 & \frac{1}{2} & 1 & 0 \end{bmatrix}$$

which can be written in terms of the permutation matrix P as

$B_4(D) = 0 I_4 + P + \frac{1}{2} P^2 + P^3.$

Now, since spectrum (P) = $\{ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \}$ where

$\lambda^4 = 1$ which has the following solutions:

$\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = i,\ \lambda_4 = -i.$

So

$$\alpha_i = \sum_{k=0}^{3} a_k \lambda_i^k \quad i = 1, 2, 3, 4.$$

By direct calculation we found that

$\alpha_1 = 2.5,\ \alpha_2 = -1.5,\ \alpha_3 = -0.5,\ \alpha_4 = -0.5$

Then spectrum ($B_4(D)$) = $\{2.5, -(\frac{1}{2}), -1.5\}$, $-(\frac{1}{2})$ with multiplicity 2.

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Note: The graph C_4 can be regarded as K (2, 2) which has real eigenvalues as calculated before and which were shown to be

$\alpha_1 = 2.5,\ \alpha_2 = -1.5,\ \alpha_3 = \alpha_4 = -(\frac{1}{2})$.

Here we find the eigenvectors of B_4 for C_4 corresponding to these eigenvalues:

Let α be an eigenvalue of B_4, and $\mathbf{v} \in C^n : \mathbf{v} = \mathbf{x} + i\,\mathbf{y},\ \mathbf{x}, \mathbf{y} \in R^n$ be the corresponding eigenvector.

So $B_4\mathbf{v} = \alpha\ \mathbf{v}$, $\alpha \in R$ implies $B_4(\mathbf{x} + i\,\mathbf{y}) = \alpha\ (\mathbf{x} + i\,\mathbf{y})$

$B_4\mathbf{x} + B_4(i\,\mathbf{y}) = \alpha\ \mathbf{x} + \alpha\ (i\,\mathbf{y})$ where

x: is a pure real vector

i**y**: is a pure imaginary vector. So

$B_4\mathbf{x} = \alpha\ \mathbf{x}$: gives a real vector

$B_4(i\,\mathbf{y}) = \alpha\ (i\,\mathbf{y})$: gives a complex vector.

So corresponding to $\alpha \in R$:

There is a pure real vector which is wanted, and a pure complex vector. Hence for the real part we have

$B_4\mathbf{x} = \alpha\ \mathbf{x}$ which is equivalent to the following equation

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$$\begin{bmatrix} 0 & 1 & \frac{1}{2} & 1 \\ 1 & 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 & 1 \\ 1 & \frac{1}{2} & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

By solving this system, we have

$x_1(½ + \alpha) = x_3(½ + \alpha)$ i.e $x_1 = x_3$ when $\alpha \neq -\frac{1}{2}$.

Similarly $x_4 = x_2$ and $x_2 = \frac{1}{2}(\alpha - \frac{1}{2})x_1$.

So for $\alpha = 2.5$, $x_1 = 1$ we have the corresponding eigenvector:

$v_1 = [1, 1, 1, 1]^T$.

For $\alpha = -1.5$, we have the corresponding eigenvector

$v_4 = [1, -1, 1, -1]^T$.

For $\alpha = -(\frac{1}{2})$, we will have the following two independent equations:

$x_1 + 2x_2 + x_3 + 2x_4 = 0.$

$2x_1 + x_2 + 2x_3 + x_4 = 0.$

By solving these equations we will have the following eigenvectors:

$v_2 = [0, 1, 0, -1]^T$

$v_3 = [1, 0, -1, 0]^T$.

So the set $\{(\frac{1}{2})v_1, (\frac{1}{\sqrt{2}})v_2, (\frac{1}{\sqrt{2}})v_3, (\frac{1}{2})v_4\}$ is an orthonormal set of eigenvectors.

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Example 2: In this example we compute the eigenvalues and the corresponding eigenvectors for the matrix B_5 of the cycle graph C_5.

C_5 is the graph shown below

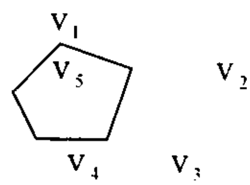

Fig 33 C_5

Let D be the distance matrix corresponding to C_5 then

$$B_5(D) = 0\,I + P + (\frac{1}{2})P^2 + (\frac{1}{2})P^3 + P^4$$

where P is a (5×5) permutation matrix then

spectrum(P) $=\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ and each value of λ_k satisfies

$$\lambda_{k+1}^5 = 1 \Rightarrow \lambda_{k+1} = \sqrt[5]{1} = e^{(\frac{2\pi ki}{5})} \quad \text{where } k = 0.1.2.3.4 .$$

so $\lambda_{k+1} = \cos(\frac{2\pi k}{5}) + i\sin(\frac{2\pi k}{5})$ where k = 0.1.2.3.4 .

By direct computation we will get the following results:

$\lambda_1 = 1$

$\lambda_2 = \cos(\frac{2\pi}{5}) + i\ \sin(\frac{2\pi}{5}) = 0.309016 + 0.9510565\, i$

$\lambda_3 = \cos(\frac{4\pi}{5}) + i\ \sin(\frac{4\pi}{5}) = -0.8090169 + 0.587785\, i$

$\lambda_4 = \cos(\frac{6\pi}{5}) + i\ \sin(\frac{6\pi}{5}) = -0.8090169 - 0.587785\, i$

$\lambda_5 = \cos(\frac{8\pi}{5}) + i\ \sin(\frac{8\pi}{5}) = 0.309016 - 0.9510565\, i.$

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So by using the formula

$$\alpha_i = \sum_{k=0}^{4} a_k \lambda_i^k \quad i = 1,2,3,4,5.$$

we get $\alpha_1 = 3$

$\alpha_2 = -0.1909830056$

$\alpha_3 = -1.3090169943$

$\alpha_4 = -1.3090169943$

$\alpha_5 = -0.1909830056$

By direct calculation we get the following normalized eigenvectors:

$v_1 = [0.447216, 0.447216, 0.447216, 0.447216, 0.447216]'$

$v_2 = [-0.583731, -0.411899, 0.329163, 0.615333, 0.051133]'$

$v_3 = [0.019128, -0.387053, 0.607137, -0.595315, 0.356103]'$

$v_4 = [-0.632166, 0.500190, -0.177158, -0.213542, 0.522677]'$

$v_5 = [0.243430, -0.479937, -0.540048, 0.146169, 0.630385]'$

[5-4] Applications:

The subject of eigenvalues and eigenvectors has a lot of applications in graph theory as well as in all of sciences. What makes it more applicable in graph theory is that most of the matrices we deal with are symmetric.

The eigenvalues of the adjacency matrix of a graph play an important role in graph coloring. In particular, the spectral radius of the adjacency matrix provides a very accurate bound on the chromatic number of a graph. They play a good role also in solving differential equations when a graphical method is applied.

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The eigenvalues of the distance matrix of a connected graph are widely involved in the study of chemical applications of graph theory. They reflect and reveal many of the physical properties of the compound as melting, freezing or boiling points besides some other properties of the compound. In fact, it has been shown that the eigenvector entries of the vector associated with the spectral radius of the distance matrix are smallest in the center of the graph and tend to increase as we move away from the center to assume their maximum values on the boundary of the graph.

In our work, we are examining the matrix whose nonzero entries are the reciprocals of the nonzero distances of a connected graph. In this case we note that the eigenvector entries are maximum in the center of graph and tend to decrease as we move away from the center and to assume their minimum values on the boundary of the graph. These phenomena make it easy when planning in the network. If one is looking for the network consisting of the cities and towns in a certain country, then the eigenvector entries place more attention on the cities in the center of the country which have minimum eccentricities or cities with more connections with other cities. This helps, for example, when assigning a budget of each city for the purpose of development and underground work.

This is a small part of the story. For more on this subject one may refer to books dealing with applications on this subject.

In the following graphs we will compare the eccentricity of a vertex and the corresponding entry of the eigenvector which corresponds to the spectral radius of the matrix B_n.

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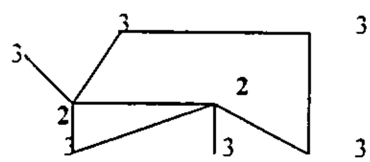
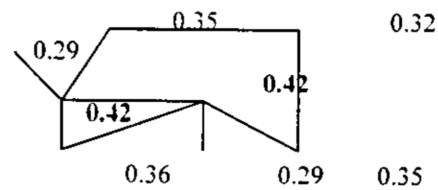
The graph with eccentricities on vertices.

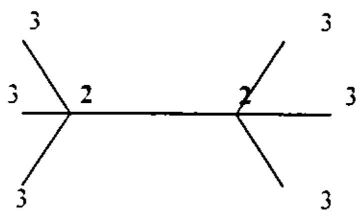
The graph with entries of the eigenvector corresponding to the spectral radius.

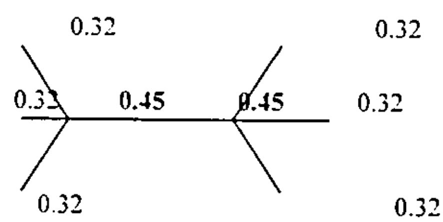
**********************************   **********************************



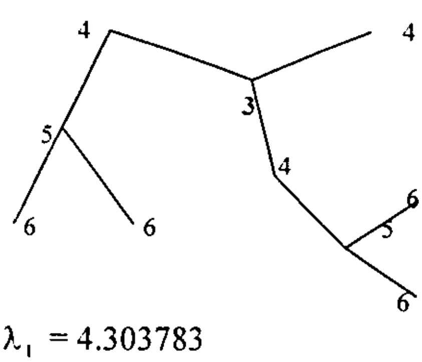


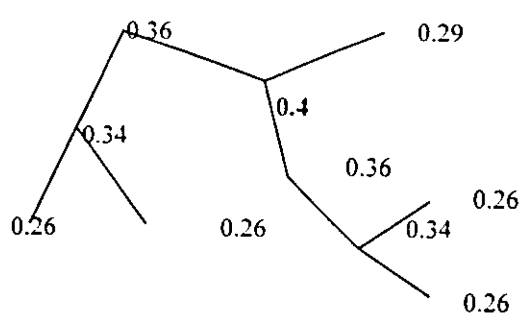
$\lambda_1 = 4.474814$



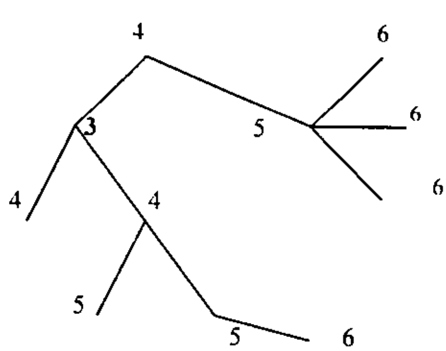


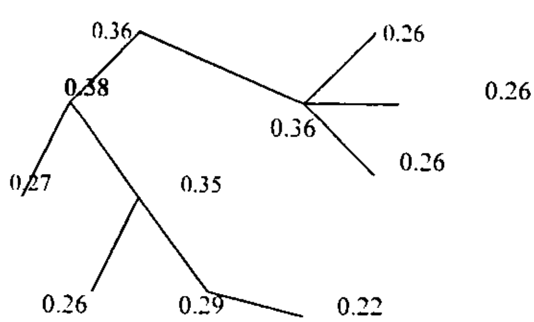
$\lambda_1 = 4.145751$





$\lambda_1 = 4.303783$





$\lambda_1 = 4.711060$

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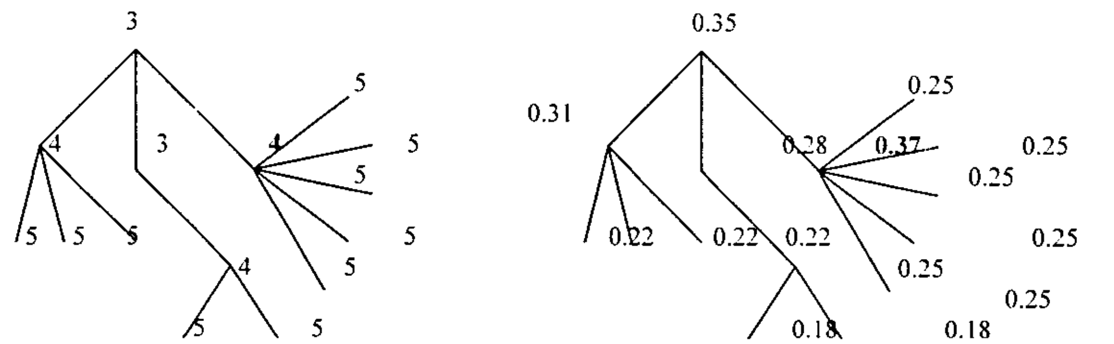
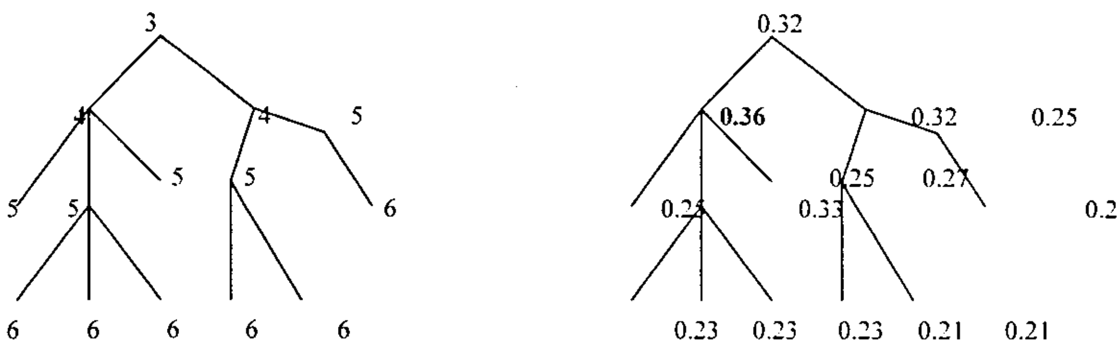
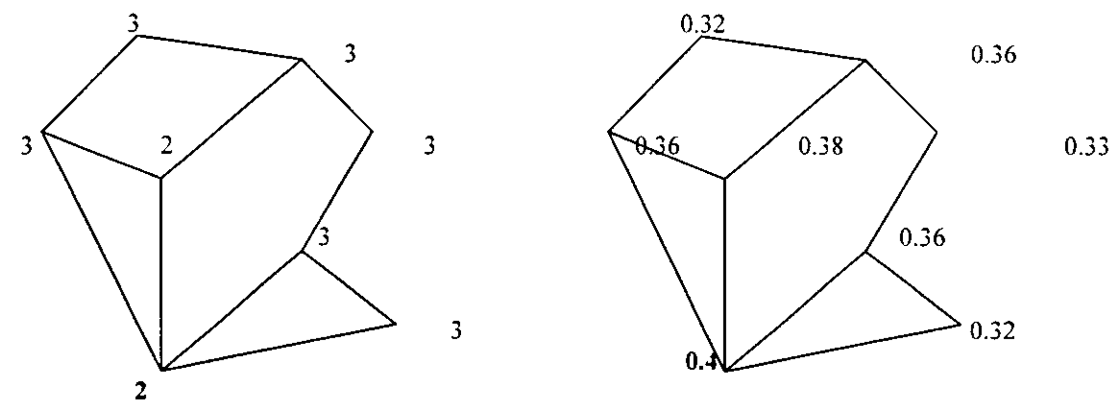


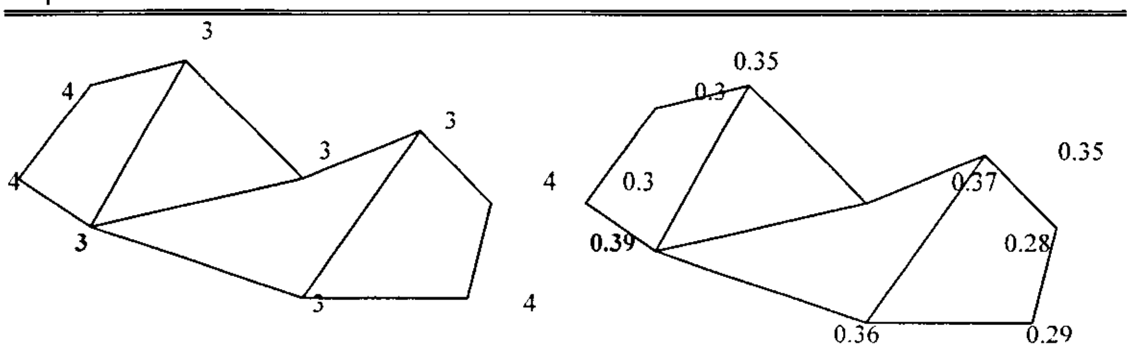
$\lambda_1 = 6.197865$



$\lambda_1 = 5.58233$



$\lambda_1 = 4.742977$



$\lambda_1 = 4.938878$

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We just note at this stage, that vertices with greater eigenvector entries are with smaller eccentricities, and tend to be in the center of the graph. This could be investigated later.

Conclusion

In this work, some results were derived explicitly giving the eigenvalues and eigenvectors of the matrix B_n, whose nonzero entries are the reciprocals of the corresponding nonzero entries in the distance matrix of a connected graph G. The discussion was focused on the complete bipartite graph K(r, n - r) and the cycle graphs C_n for any integer $n \geq 3$.

The spectra of the matrix B_n and the eigenvectors are explicitly stated.

We hope that, in the future, the work will be continued from this point on, and the matrices related to the other graphs like the path graph P_n, branching cycles, and other graphs will be discussed.

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القيم المميزة لمصفوفة مقلوبات المسافات

للرسومات ذات الجزأين والرسومات الدائرية

اعداد

رياض كامل حسن زيدان

اشراف

د. صبحي رزية

الملخص

في هذا البحث، تم اشتقاق بعض الصيغ لحساب القيم المميزة والمتجهات المميزة للمصفوفات التي مدخلاتها غير الصفرية هي مقلوبات نظيراتها غير الصفرية في مصفوفة المسافات بين رؤوس الرسومات، وقد تم التركيز على أنواع خاصة من الرسومات مثل الرسومات ذات ألجزأين(Complete Bipartite Graphs) التي يرمز لها بالرمز K(m,n)، وكذلك الرسومات الدائرية (Cycles) والتي يرمز لها بالرمز (C_n).

اننا نأمل أن يستمر العمل – في المستقبل– ويتم تناول المصفوفات المرتبطة ببعض الرسومات الأخرى مثل الممرات(Paths)، وكذلك الرسومات الدائرية المتفرعة (Branching Cycles)، ورسومات أخرى تتم مناقشتها.

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القيم المميزة لمصفوفة مقلوبات المسافات للرسومات ذات الجزأين والرسومات الدائرية

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