An-Najah National University

Faculty of Graduated Studies

Numerical Methods for Solving Fractional Differential Equations with Applications

By

Aya Basem Ahmed Saadeh

Supervisor

Prof. Naji Qatanani

This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics, Faculty of Graduate Studies, An-Najah National University, Nablus, Palestine.

2018

Numerical Methods for Solving Fractional Differential Equations with Applications

By

Aya Basem Ahmed Saadeh

This thesis was successfully defended on 25 /7/2018 and approved by:

Defense Committee Members	Signature
– Prof. Naji Qatanani / Supervisor	N. Qatanani
– Dr. Maher Qarawani / External Examiner	prof. ...
– Dr. Adnan Draghma / Internal Examiner	A. Daraghmeh

Dedication

I dedicate this thesis to my beloved homeland, to my parents, to my dear husband Ali, to my daughter Laila, to my sisters and my brother, to my friends, to everyone who supports and encourages me.

Acknowledgement

First of all, I thank my God for all the blessing he bestowed on me and continues to bestow on me.

I would sincerely like to thank and deeply grateful to my supervisor Prof. Dr. Naji Qatanani who without his support, kind supervision, helpful suggestions and valuable remarks, my work would have been more difficult. My thanks also to my external examiner Dr. Maher kerawani and to my internal examiner Dr. Adnan Draghma for their useful and valuable comments. Also, my great thanks are due to my family for their support, encouragement and great efforts for me.

Finally, I would also like to acknowledge to all my teachers in An-Najah National University-Mathematical Department.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان

Numerical Methods for Solving Fractional Differential Equations with Applications

أقر بأن ما شملت عليهالرسالة هو نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد، وأن هذه الرسالة ككل أو أي جزء منها لم يقدم من قبل لنيل أي درجة أو لقب علمي أو بحثي لدى أي مؤسسة علمية أو بحثية.

<u>Declaration</u>

is the ، unless otherwise referenced،The work provided in this thesis and has not been submitted elsewhere for any other ،researcher's own work degrees or qualifications.

Student's Name: **اسم الطالبة:**

Signature: **التوقيع:**

Date: **التاريخ:**

Table of Contents

List of Figures

List of Tables

Numerical Methods for Solving Fractional Differential Equations with Applications

By

Aya Basem Ahmed Saadeh

Supervisor

Prof. Naji Qatanani

Abstract

Fractional differential equations have a wide range of applications in Engineering, Physics and Technology including fractional differential harmonic oscillator, fractional wave equation and fractional diffusion equation.

After introducing some definitions in fractional derivatives and fractional integrals including Grunwald-Letnikov, Riemann-Liouville and the Caputo fractional derivative, we focus our attention mainly on the numerical methods for solving fractional differential equation. These methods are: the Adomian decomposition method, Homotopy perturbation method, Variational iteration method and Matrix approach method.

The mathematical framework of these numerical methods together with their convergance properties will be presented. These numerical methods will be illustrated by some numerical examples. Comparisons between these methods will be drawn. Numerical results show clearly that the matrix approach method is one of the most powerful numerical techniques for solving linear fractional differential equation in a comparison with other numerical techniques.

Introduction

Historical background

The question that led to the birth of fractional calculus was from a letter which is written by L`Hopital asking Leibniz about the n^{th} derivative of the linear function $f(t) = t$, and what will happen when $n = \frac{1}{2}$ [19]. In general, what would the result be when n is fraction, then Leibniz replied and wrote " this is an apparent paradox from which, one day, useful consequences will be drawn", and from this inquestion between L`Hopital and leibniz the new field of mathematics was called fractional calculus, but in fact, the order of differentiation or intigration can be any positive real number.

After L`Hopital and Leibniz first inquestion, the field of fractional calculus has motivated many famous mathematicians, such as Fourier (1820-1822), Lacorix (1819), Riemann (1826-1866), Liouville (1809-1882), Laplace (1812), Caputo, Euler (1730), and many others, and each of them made an effort to make progress in this field of mathematics. Next we list some major contributions to fractional calculus by famous mathematicians [32], [41].

In 1812, Laplace defined the fractional derivative by means of an integral.

In 1819, Lacorix was the first mathematician to define the m^{th} fractional derivative using the gamma function. He applied his defintion on the function $f(t) = t^n$ in a paper published in 1819. The m^{th} derivative of $f(t) = t^n$ is :

$$\frac{d^m f(t)}{dt^m} = \frac{n!}{(n-m)!} t^{n-m}, \quad n \geq m, \quad m \text{ is an integer} \qquad (1)$$

2

$$= \frac{\Gamma(n+1)}{\Gamma(n-m+1)} t^{n-m}$$

Moreover, Lacorix applied this definition when $m = \frac{1}{2}$, $n = 1$, and obtained:

$$\frac{d^{1/2}(t)}{dt^{1/2}} = 2\frac{\sqrt{t}}{\sqrt{\pi}}$$

In 1823, Abel applied fractional calculus in the solution of an integral equation that arised from the formulation of the tautochrone problem.

The tautochrone problem is the problem of determining the shape of the curve such that the time of a discent that a frictionless point mass needs to slide down the curve under the work of gravity is independent of the starting point.

Over the next ten years (1823 to 1832), no significant progress has been made. In 1832, Liouville was successful in applying his definition of fractional calculus to problems in potential theory.

After that, many mathematicians made very important work in fractional calculus such as Riemann-Liouville, Grunwald-Letnikove, Caputo in 1967, K.S.Miller, B.Ross in 1993 and many others.

While fractional derivatives can be defined in different ways, we will a dapt the Riemann-Liouville and Caputo definitions.

Fractional differential equations have recently gained importance and attention. The study of fractional differential equations ranges from theoretical aspects of existence and uniqueness of solutions to the analytical and numerical methods for finding solutions.

Applications of Fractional Differential Equations

Fractional differential equations appear frequently in a number of fields such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex medium, viscoelasticity, capacity theory, electrical circuits and control theory. An excellent account in the study of fractional differential equations can be found in [24,30,36]. One of the most common fractional differential equation is the time-fractional heat conduction equation with the Caputo fractional derivative, i.e.

$$\frac{d^{\alpha}T}{dt^{\alpha}} = a\,\Delta\,T\,, \qquad 0 \leq \alpha \leq 2 \tag{2}$$

where a is the thermal diffusivity coefficient. Further examples of a fractional differential equation is the logistic equation of fractional order [5];

$$D^{\alpha}x(t) = rx(t)\big(1 - x(t)\big), \tag{3}$$

where D^{α} is the Caputo fractional derivative of order $0 < \alpha \leq 1$. Moreover, the fractional Rosenau-haynam equation which written as [33];

$$D_t^{\alpha}u = uD_{xxx}(u) + uD_x(u) + 3D_x(u)D_{xx}(u), \qquad t > 0 \tag{4}$$

where $u = u(x,t)$, α is a parameter describing the order of the fractional derivative $(0 < \alpha \leq 1)$, t is the time, and x is the spatial coordinate.

The exictence and uniqueness of solutions for different types of fractional differential equations have been discussed in [16].

In general, most of the fractional differentail equations do not have exact solutions. Therefore, several numerical methods are used to approximate solutions of fractional differential equations. Some of these methods include

Adomian Decomposition Method [2], Homotopy Perturbation Method [18], Variational Iteration Method [25], Laplace Transform Method [42], and Matrix Approach Method [11].

The fractional derivative of order $\alpha > 0$ has several definitions. These are: Riemann-Liouville and the Caputo's definitions are the most commonly used for the derivative of this order. For the fractional derivative,the Caputo's definition is used, which is a modification of the Riemann-Liouville definition; because it has an advantage of dealing properly with the initial value problem since the initial condition is given in terms of the field variables and their integer order. This case is widely used in physical applications. This thesis is organized as follows:

In chapter one, we introduce some definitions for fractional and integral derivatives. chapter two, we present some important definitions and theorems involving Grunwald-Letnikove, Riemann Liouville and Caputo fractional derivative and focus on some fractional differential equations such as fractional harmonic oscillator, fractional wave equation and fractional diffusion equation. In chapter three, we present some numerical methods; namely: the Adomian decomposition method, Homotopy perturbation method, Variational iteration method, Matrix approach method. In chapter four, we implement the aforementioned numerical methods to solve linear fractional differential equations using Maple and Matlap softwares, and draw a comparison between analytical and numerical solutions for some numerical examples.

Chapter One

Fractional Calculus

1.1 Fractional Derivatives

Definition(1.1)[13] : Let A denote the class of functions $g(x)$in the form $g(x) = x + \sum_{n=2}^{\infty} a_n \; x^n$ which are analytic in U={x∈C:|x|<1}.

The fractional derivative of order α, for $g(x) \in A$,is defined as the following:

$$D_x^\alpha g(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \left\{ \int_0^x \frac{g(\theta)}{(x-\theta)^\alpha} d\theta \right\} (0 < \alpha < 1) \qquad (1.1)$$

where the multiplicity of $(x-\theta)^{-\alpha}$ is deleted by requiring $\log(x-\theta)$ to be real when $x - \theta > 0$.

where $\Gamma(\alpha)$ is the so-called gamma function.

Definition (1.2) [38]: Gamma function: The gamma function denoted by $\Gamma(p)$ is given by the integral:

$$\Gamma(p) = \int_0^\infty x^{p-1} \, e^{-x} \, dx, \qquad p > 0. \qquad (1.2)$$

The above integral converges only for $p > 0$.

Also that the gamma function is a generalization of the factorial function, since for any positive integer p,

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx = (p-1)!.$$

Below is a list of some properties of the Gamma function where $p \in Z, n \in N$

1. $\Gamma(p+1) = p\Gamma(p)$
2. $\Gamma(\mathrm{p})\Gamma(\mathrm{p}-1) = \pi/\sin(\pi p)$
3. $\Gamma(n+1) = n\Gamma(n) = n!$
4. The Binomial coefficient is defined by the Gamma function

$$(a+b)^{\alpha} = \sum_{i=0}^{\infty} \binom{\alpha}{i} a^{\alpha-i} b^{i},$$

where $\binom{\alpha}{i} = \frac{\Gamma(\alpha+1)}{\Gamma(i+1)\Gamma(\alpha-i+1)}$.

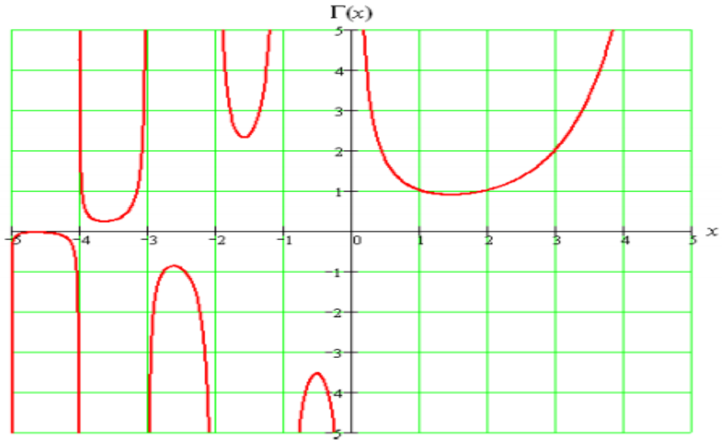

Figure 1.1: The gamma function.

Definition (1.3) [38]: Beta function: The beta function denoted by $B(u,v)$ is given by the integral:

$$B(u,v) = \int_0^1 x^{u-1}\,(1-x)^{v-1}dx, \qquad u,v > 0. \qquad (1.3)$$

The integral converges for $u,v > 0$.

The Beta function is related to the Gamma function through the relation:

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

Definition (1.4)[13]: Let $g(x) \in A$. the fractional dreivative of order $n + \alpha$ is defined as :

$$D_x^{n+\alpha} g(x) = \frac{d^n}{dx^n}\left(D_x^{\alpha} g(x)\right) \qquad (0 < \alpha < 1, n = 0,1,2 \dots) \qquad (1.4)$$

Definition (1.5)[13]: Let $m - 1 < \alpha < m, m \in N, \alpha, \lambda \in C$ and the functions $u(x)$ and $y(x)$ be such that both $D^{\alpha}u(x)$ and $D^{\alpha}y(x)$ exists. The fractional derivative is a linear operator, i.e.,

$$D^{\alpha}\left(\lambda\, u(x) + y(x)\right) = \lambda\, D^{\alpha}u(x) + D^{\alpha}y(x). \qquad (1.5)$$

Definition(1.6)[51]: Let $u(x) \in C_{\alpha}[a, b]$, the modified fractional derivative of function $u(x)$ is defined by:

$$D^{\alpha}u(x) = \lim_{c \to 0} \frac{\Delta^{\alpha}u(x)}{c^{\alpha}}, \qquad (1.6)$$

where $\Delta^{\alpha}u(x) = \sum_{i=0}^{\infty}(-1)^i \binom{\alpha}{i} u(x + (\alpha - i)c)$

Remark (1.1)[27][43] : we note that

$$\Omega^0 g(t) = g(t) = t + \sum_{n=2}^{\infty} a_n t^n,$$

$\Omega^1 g(t) = \Omega g(t) = t g'(t) = t + \sum_{n=2}^{\infty} n a_n t^n,$
and $\Omega^j g(t) = \Omega\left(\Omega^{j-1} g(t)\right) = t + \sum_{n=2}^{\infty} n^j a_n t^n \qquad (j = 1,2,3, \dots)$

which called Sălăgean derivative operator introduced by Sălăgean [44]. Also,

$\Omega^{-1} g(t) = \frac{2}{t}\int_0^t g(z)dz = t + \sum_{n=2}^{\infty} \frac{2}{n+1} a_n t^n$
and , $\Omega^{-j} g(t) = \Omega^{-1}\left(\Omega^{-j+1} g(t)\right) = t + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^j a_n t^n \; (j = 1,2,3, \dots)$

which called Libera integral operator defined by libera [27].

Definition (1.7)[15]: For $t \in R, \ \ 0 < \alpha \leq 1$, the sine and cosine functions in fractal space are given by:

$$sin_\alpha t^\alpha = \sum_{i=0}^{\infty} \frac{(-1)^i t^{\alpha(2i+1)}}{\Gamma(1+\alpha(2i+1))} \tag{1.7}$$

$$cos_\alpha t^\alpha = \sum_{i=0}^{\infty} \frac{(-1)^i t^{2\alpha i}}{\Gamma(1+2i\alpha)} \tag{1.8}$$

Here are some fractal functions which can be expressed by Mittage Leffler function

$$sin_\alpha t^\alpha = \frac{E_\alpha(i^\alpha t^\alpha) - E_\alpha(-i^\alpha t^\alpha)}{2i^\alpha}$$

$$cos_\alpha t^\alpha = \frac{E_\alpha(i^\alpha t^\alpha) + E_\alpha(-i^\alpha t^\alpha)}{2}$$

$$sinh_\alpha t^\alpha = \frac{E_\alpha(t)^\alpha + E_\alpha(-t)^\alpha}{2}$$

$$cosh_\alpha t^\alpha = \frac{E_\alpha(t)^\alpha - E_\alpha(-t)^\alpha}{2},$$

where E_α is the one parameter Mittage Leffler function.

Example 1.2: (Modified Fractional Derivative of Exponential Function)

$$\begin{aligned} D^\alpha e^{ax} &= \lim_{c\to 0} \frac{1}{c^\alpha} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} f(x+(\alpha-i)c) \\ &= \lim_{c\to 0} \frac{1}{c^\alpha} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} e^{a(x+(\alpha-i)c)} \\ &= e^{ax} \lim_{c\to 0} \frac{1}{c^\alpha} [\sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} e^{ac(\alpha-i)}]. \end{aligned}$$

Using the Binomial expression

$$(e^{ac}-1)^\alpha = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} e^{ac(\alpha-i)}.$$

Then,

$$D^{\alpha}e^{ax} = e^{ax}\lim_{c\to 0}\frac{1}{c^{\alpha}}(e^{ac}-1)^{\alpha}$$

$$= e^{ax}\left(\lim_{c\to 0}\left(\frac{e^{ac}-1}{c}\right)\right)^{\alpha}$$

$$= e^{ax}(f'(0))^{\alpha} = a^{\alpha}e^{ax}$$

So , $D^{\alpha}e^{ax} = a^{\alpha}e^{ax}$.

Example 1.3 (Fractional Derivative of sine and cosine)

$$D^{\alpha}\left(\sin_{\alpha}(nt)^{\alpha}\right) = \sum_{i=0}^{\infty}\frac{(-1)^{i}n^{\alpha(2i+1)}}{\Gamma\left(1+\alpha(2i+1)\right)}\frac{\Gamma(\alpha(2i+1)+1)}{\Gamma(\alpha(2i+1)-\alpha+1)}t^{\alpha(2i+1)-\alpha}$$

$$= \sum_{i=0}^{\infty}\frac{(-1)^{i}n^{\alpha(2i+1)}}{\Gamma(1+\alpha(2i+1)-\alpha)}t^{\alpha(2i)}$$

$$= \sum_{i=0}^{\infty}\frac{(-1)^{i}n^{2\alpha i}n^{\alpha}}{\Gamma(2\alpha i+1)}t^{2\alpha i}$$

$$= n^{\alpha}\sum_{i=0}^{\infty}\frac{(-1)^{i}(nt)^{2\alpha i}}{\Gamma(1+2\alpha i)}$$

$$= n^{\alpha}\cos_{\alpha}nt^{\alpha}$$

Hence,

$$D^{\alpha}\left(\sin_{\alpha}(nt)^{\alpha}\right) = n^{\alpha}\cos_{\alpha}(nt)^{\alpha}$$

Similarly, since

$\cos_{\alpha}(nt)^{\alpha} = \sin_{\alpha}\left(\frac{\pi}{2}-(nt)^{\alpha}\right)$, we get

$$D^{\alpha}\cos_{\alpha}(nt)^{\alpha} = -n^{\alpha}\sin_{\alpha}(nt)^{\alpha} .$$

1.2 Fractional Integrals

1.2.1 Basic Definitions

Definition (1.8)[26]: The Riemann-Liouville fractional integral operator of order $p > 0, m-1 < p \leq m, m \in N$, of $u(x)$ is defined as the following:

$$J^p u(x) = \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} u(t) \quad dt \quad , x > 0, \tag{1.9}$$

1.2.2 Properties of Fractional Integrals

Definition (1.9)[45]: The identity operator: If we set $p = 0$ in the Riemann-Liouville fractional integral operator of order p (J^p) then:

$$J^0 g(x) = g(x), \quad i.e., J^0 = I \text{ (the identity operator).} \tag{1.10}$$

the semi- group property of the two operators J^α, J^β is defined by:

$$J^\alpha J^\beta = J^{\alpha+\beta} \tag{1.11}$$

Definition (1.11)[45]: The commutative property: Suppose that $\alpha, \beta \geq 0$, the commutative property of the two operators J^α, J^β, is defined by:

$$J^\alpha J^\beta = J^\beta J^\alpha. \tag{1.12}$$

From (1.11) and (1.12), we conclude that $J^\alpha J^\beta = J^{\alpha+\beta} = J^{\beta+\alpha} = J^\beta J^\alpha$.

Definition (1.12)[26]: Let $m-1 < \alpha < m, m \in N, \alpha, \lambda \in C$ and the functions $u(x)$ and $y(x)$ be such that both $J^\alpha u(x)$ and $J^\alpha y(x)$ exists. The fractional integral is a linear operator, i.e.,

$$J^\alpha\big(\lambda\, u(x) + y(x)\big) = \lambda\, J^\alpha u(x) + J^\alpha y(x) \tag{1.13}$$

1.3 Euler Lagrange Equation

Definition (1.13) [6] : If $u(x)$ is a curve in $C^2[c,d]$ that minimizes the functional

$$G[u(x)] = \int_c^d g\big(x, u(x), u'(x)\big)\, dx, \tag{1.14}$$

Then the differential equation must satisfy:

$$\frac{dg}{dx} - \frac{d}{dx}\left(\frac{dg}{dx'}\right) = 0.$$

This equation is called the Euler Lagrange equation.

Lemma (1.1)[49]: Let G(x) $is\ a$ continuous function on $[c,d]$, assume for any continuous function u(x) such that u$(c) = u(d) = 0$ we have $\int_c^d G(x)u(x)\,dx = 0$

Then G(x) is identically zero on $[c,d\ ]$.

The solution of the Euler-Lagrange equation are called critical curves.

1.4 Differential Operators

Definition (1.14) [42]: Differential operators are a generalization of the operation of differentiation

$$Du(x)=u'(x)$$

Double D allows to the second derivative:

$$D^2u(x) = D\big(Du(x)\big) = Du'(x) = u''(x)$$

n^{th} power of D allows to the n^{th}derivative:

$$D^n u(x) = u^{(n)}(x) \tag{1.15}$$

Definition (1.15)[10]: The linear differential equation of the n^{th}order is written as the following:

$u^{(n)}(x) + a_1(x)u^{(n-1)}(x) + \cdots + a_{n-1}(x)u'(x) + a_n(x)u(x) = g(x)$(1.16)

Using the differential operator D, this equation (1.16) can be written as

$G(D)u(x) = g(x)$

where $G(D)$ is a differential polynomial equal to

$G(D) = D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)$

Therefore the operator $G(D)$ is an algebraic polynomial.

1.5 Hermitian Matrix

Definition (1.16) [3]: If square matrix is self-adjiont then it is Hermitian matrix. So, a Hermitian matrix C=(c_{ij}) is defined as C=C^* where C^* that conjugate transpose. Is equivalent to the condition.

$$c_{ij} = \overline{c_{ji}}$$

Definition (1.17) [4]: Hermitian matrices have real eigenvalues whose eigenvectors form a unitary basis. Any matrix D that is not Hermitian can be written as the sum of a Hermitian matrix and an anti-Hermitian matrix

$$D = \frac{1}{2}(D + D^H) + \frac{1}{2}(D - D^H).$$

Chapter Two

Fractional Differential Equations and Exact Solutions

In this chapter we present some important definitions and theorems involving Grunwald-Letnikove, Riemann Liouville and Caputo fractional derivatives that used to find exact solutions to some fractional differential equations such as fractional differential harmonic oscillator, fractional wave equation and fractional diffusion equation.

2.1 Grunwald Letnikov Fractional Derivatives

Definition (2.1) [39]: The Grumwald-Letnikov fractional derivative with fractional order p if $u(t) \in C^n[0,t]$, is defined as :

$$ {}_aD_t^p u(t) = \lim_{\substack{h\to 0\\ mh=t-a}} h^{-p} \sum_{i=0}^{m} (-1)^i \binom{p}{i} u(t-ih) \qquad (2.1) $$

where $\binom{p}{i} = \dfrac{\Gamma(p+1)}{i!\,(\Gamma(p-i+1)}$.

Property 2.1 [7]:

(1) When 0≤ $p < 1$ $\rho\left[{}_{GL}D_{0,t}^p u(t)\right](s) = s^p u(s),$

where $u(s) = \rho[u](s)$

(2) If $p > 1$, the laplace transform of the Grumwald-letnikov fractional derivative does not exist in the classical sense .

(3) $D_{0,t}^p c = ct^{-p}/\Gamma(1-p), where\ c\ is\ a\ costant$.

Definiton (2.2) [37]: Grunwald- Letnikov composition with fractional derivatives if $u^{(i)}(b) = 0$, $(i = 0,1, \ldots, c-1)$, $c = \max(n, m)$ can be shown as

$$ {}_{b}D_{t}^{p}\left({}_{b}D_{t}^{\alpha}u(t)\right) = {}_{b}D_{t}^{\alpha}\left({}_{b}D_{t}^{p}u(t)\right) = {}_{b}D_{t}^{\alpha+p}u(t), \qquad (2.2) $$

where $0 \le m < \alpha < m+1$, $0 \le n < p < n+1$.

Definition(2.3)[37]: Grunwald–Letnikov fractional derivative of the power function $g(t) = (t-c)^{p}$ is given as:

$$ {}_{c}D_{t}^{\alpha}(t-c)^{p} = \frac{\Gamma(p+1)}{\Gamma(-\alpha+p+1)}\,(t-c)^{p-\alpha} \qquad (2.3) $$

Definition (2.4)[19]: The derivative of an integer order $\alpha - n > 0$, $n-1 < \alpha < n$ can be shown as :

$$ {}_{a}D_{t}^{\alpha-n}g(t) = \frac{1}{\Gamma(n)}\frac{d^{\alpha}}{dt^{\alpha}}\int_{a}^{t}(t-\tau)^{n-1}g(\tau)\,d\tau, \qquad (0 < n \le 1) \quad (2.4) $$

Theorem (2.1)[49]: Suppose $g(t)$ is $(n-1)$ times continuously differentiable function and $g^{(n)}$ is bounded. If $g(t)$ is a non constant periodic function with period M, then ${}_{a}D_{t}^{\alpha}g(t)$, where $0 < \alpha \notin N$ and n is the first integer greater than α, cannot be periodic functions with period M .

Corollary (2.1) [50]: A differential equation of fractional order of the form

$$ {}_{a}D_{t}^{\alpha}u(t) = f\big(u(t)\big) $$

where $0 < \alpha \notin N$, cannot have any non- constant smooth periodic solution.

2.2 Riemann-Liouville Fractional Derivatives

2.2.1 Basic Definition

Definition (2.5)[32]:(Riemann-Liouville Derivative): Let $n-1<p<n\in Z^{+}$.The Riemann– Liouville fractional derivative of order p is defined as:

$$D_{0,t}^{p}u(t)=\frac{1}{\Gamma(n-p)}\frac{d^{n}}{dt^{n}}\int_{0}^{t}\frac{u(\tau)}{(t-\tau)^{p+1-n}}\,d\tau\,. \tag{2.5}$$

Theorem (2.2)[15,51] : Let $u(z)$ and $l(z)$ are analytic functions , then the following are valid:

1. $\frac{d^{\alpha}(u(z)\pm l(z))}{dz^{\alpha}}=\frac{d^{\alpha}u(z)}{dz^{\alpha}}\pm\frac{d^{\alpha}l(z)}{dz^{\alpha}}$
2. $\frac{d^{\alpha}(u(z).l(z))}{dz^{\alpha}}=l(z)\frac{d^{\alpha}u(z)}{dz^{\alpha}}+u(z)\frac{d^{\alpha}l(z)}{dz^{\alpha}}$
3. $\frac{d^{\alpha}\frac{u(z)}{l(z)}}{dz^{\alpha}}=\frac{l(z)\frac{d^{\alpha}u(z)}{dz^{\alpha}}+u(z)\frac{d^{\alpha}l(z)}{dz^{\alpha}}}{(l(z))^{2}}$
4. $\frac{d^{\alpha}(cu(z))}{dz^{\alpha}}=c\frac{d^{\alpha}u(z)}{dz^{\alpha}},\ where\ c\ is\ a\ constant$
5. If $y(z)=(uof)(z), then\ \frac{d^{\alpha}y(z)}{dz^{\alpha}}=u^{\alpha}\big(f(z)\big)\big(f^{(1)}(z)\big)^{\alpha}$

Theorem (2.3)[38]: Leibniz rule for Riemann- Liouville fractional derivative: Let $t>0,\alpha\in R,m>\alpha>m-1,$ and $m\in N$. If $u(t),g(t),$ and their derivatives are continuous on $[0,t]$, then the following holds:

$$D^{\alpha}\big(u(t)g(t)\big)=\sum_{k=0}^{\infty}\binom{\alpha}{k}[D^{k}g(t)][D^{\alpha-k}u(t)], \tag{2.6}$$

Proof: See [38] for more details.

Theorem (2.4) [38]: Fractional derivative of constant function: Let $t > 0, p \in R, m > p > m-1,$ and $m \in N$. The Riemann-Liouville fractional derivative of constant function is in the form:

$$D^p c = \frac{c}{\Gamma(1-p)} t^{-p} \quad \text{, c is a constant} \tag{2.7}$$

Proof: See [38] for more details.

Theorem (2.5)[21]: Fractional derivative of exponential function:

Suppose that $\lambda \in C, \alpha \in R, m-1 < \alpha < m,$ and $m \in N$. The fractional derivative of exponential function as the form:

$$D^\alpha\left(e^{\lambda t}\right) = t^\alpha E_{1,1-p}(\lambda t) \tag{2.8}$$

Proof: see[21] for more details.

Theorem (2.6)[21]: Fractional derivative of cosine and sine funtions:

Suppose that $\lambda \in C$ $, \alpha \in R, m > \alpha > m-1,$ and $m \in N$. The fractional derivative of cosine and sine function as the form:

$$D^\alpha(\cos \lambda\, t) = \frac{1}{2} t^{-\alpha}\left(\left(E_{1,1-\alpha}(i\lambda\, t\right) + \left(E_{1,1-\alpha}(-i\lambda\, t\right)\right) \tag{2.9}$$

$$D^\alpha(\sin \lambda\, t) = -\frac{1}{2} t^{-\alpha}\left(\left(E_{1,1-\alpha}(i\lambda\, t\right) - \left(E_{1,1-\alpha}(-i\lambda\, t\right)\right) \tag{2.10}$$

where $E_{\rho,\sigma}(f)$ is the Mittage-Leffler function.

Proof: See [21] for moe details.

2.2.2 Properties of Riemann-Liouville Fractional Derivative [21] [38] [43]:

Definition (2.6): Comutativity: The Riemann-Liouville derivative is commutitive, if $u^{(i)}(b) = 0, (i = 0,1,2, \ldots, c-1) \quad c = \max(n, m)$

$${}_bD_t^\alpha\left({}_bD_t^p u(t)\right) = {}_bD_t^p\left({}_bD_t^\alpha u(t)\right) = {}_bD_t^{\alpha+p} u(t), \tag{2.11}$$

where $0 \leq m < p < m+1,\ 0 \leq n < \alpha < n+1.$

Definition (2.7): Linearity: The Riemann-Liouville fractional derivative of a linear combination of functions is in the form:

$$D_t^p\big(\alpha\, u(t) + \beta\, g(t)\big) = \alpha D_t^p u(t) + \beta D_t^p g(t) \qquad (2.12)$$

Definition (2.8): Interpolation: Let $m-1 < p < m, m \in N, p \in R$ and $u(t)$ be such that $D^p u(t)$ exists, then the following properties for the Riemann-Liouville operator hold:

a. $\lim_{p \to k} D_t^p u(t) = u^{(k)}(t)$

b. $\lim_{p \to k-1} D_t^p u(t) = u^{(k-1)}(t).$

Defintion(2.9): Compisition with integer order derivative:

Let $m-1 < p < m, m \in N, p \in R$, then:

$$D^p D^m u(t) = D^{p+m} u(t) \neq D^m D^p u(t). \qquad (2.13)$$

Definition (2.10): Composition with fractional derivative:

Let $m-1 \leq p \leq m$ and $n-1 \leq q \leq n$, then:

$${}_aD_t^p\left({}_aD_t^q u(t)\right) = {}_aD_t^{p+q} u(t) - \sum_{i=1}^{n} \left[D_t^{q-i}\right]_{t=a} \frac{(t-a)^{-p-i}}{\Gamma(1-p-i)} \qquad (2.14)$$

2.3 The Caputo Fractional Derivatives

2.3.1 Basic Definitions

Definition (2.11)[29]: The Caputo fractional derivative of function $u(t)$ is in the following :

$$D_*^{\alpha}u(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t \frac{u^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau, & n-1<\alpha<n \\ \dfrac{d^{(n)}u(t)}{dt^{(n)}} & \alpha = n \in \mathbb{N} \end{cases}$$

Theorem (2.7) [10]: The Caputo fractional derivative of the power function satisfies:

$$D_*^{\alpha}t^{C} = \begin{cases} \dfrac{\Gamma(C+1)}{\Gamma(C-\alpha+1)}t^{C-\alpha} = D^{\alpha}t^{C}, n-1<\alpha<n, C>n-1, C\in R, \\ 0, \qquad \mathrm{n}-1<\alpha<n, C\le n-1, C\in\mathbb{N}. \end{cases} \qquad (2.15)$$

Proof: The proof of the second case

($D_*^{\alpha}t^{C} = 0,\ n-1<\alpha<n,\ C\le n-1,\ C\in\mathbb{N}$) follows of the differentiation of the constant function,$(t^{C})^{(n)} = 0$ for $C\le n-1$, $C,\ n\in\mathbb{N}$.

The first case is proved by the definition of the Caputo fractional derivative and the properties of the beta and gamma functions:

Let $n-1<\alpha<n,\ C>n-1,\ C\in R$:

$$D_*^{\alpha}t^{C} = \frac{1}{\Gamma(n-\alpha)}\int_0^{\mathrm{t}} \frac{(\tau^{C})^{(n)}}{(t-\tau)^{\alpha+1-n}}\, d\tau$$

$$D_*^{\alpha}t^{C} = \frac{1}{\Gamma(n-\alpha)}\int_0^{\mathrm{t}} \frac{\Gamma(C+1)}{\Gamma(C-n+1)}(\tau)^{C-n}\,(t-\tau)^{n-\alpha-1}\, d\tau$$

and using substitution $\tau = \lambda t,\ 0 \le \lambda \le 1$

$$D_*^{\alpha} t^{\mathrm{C}} = \frac{\Gamma(\mathrm{C}+1)}{\Gamma(n-\alpha)\Gamma(\mathrm{C}-n+1)} \int_0^1 (\lambda t)^{\mathrm{C}-n} ((1-\lambda)t)^{n-\alpha-1} t \, d\lambda$$

$$D_*^{\alpha} t^{\mathrm{C}} = \frac{\Gamma(\mathrm{C}+1)}{\Gamma(n-\alpha)\Gamma(\mathrm{C}-n+1)} t^{\mathrm{C}-\alpha} \int_0^1 (\lambda)^{\mathrm{C}-n} (1-\lambda)^{n-\alpha-1} \, d\lambda$$

$$D_*^{\alpha} t^{\mathrm{C}} = \frac{\Gamma(\mathrm{C}+1)}{\Gamma(n-\alpha)\Gamma(\mathrm{C}-n+1)} t^{\mathrm{C}-\alpha} \ B(\mathrm{C}-n+1, n-\alpha)$$

$$D_*^{\alpha} t^{\mathrm{C}} = \frac{\Gamma(\mathrm{C}+1)}{\Gamma(n-\alpha)\Gamma(\mathrm{C}-n+1)} t^{\mathrm{C}-\alpha} \ \frac{\Gamma(\mathrm{C}-n+1)\Gamma(n-\alpha)}{\Gamma(\mathrm{C}-\alpha+1)}$$

$$D_*^{\alpha} t^{\mathrm{C}} = \frac{\Gamma(\mathrm{C}+1)}{\Gamma(\mathrm{C}-\alpha+1)} t^{\mathrm{C}-\alpha}.$$

Theorem (2.8)[21]: Leibniz rule for Caputo fractional derivative: Let $t > 0, p \in R, m > p > m-1$, and $m \in N$. If $u(t), g(t)$, and their derivatives are continuous on $[0, t]$, then the following holds

$$D_*^{p}\big(u(t)g(t)\big) = \sum_{i=0}^{\infty} \binom{p}{i} \left[D_*^{i} g(t)\right]\left[D_*^{p-i} u(t)\right] - \sum_{i=0}^{m-1} \frac{t^{i-p}}{\Gamma(i+1-p)} \big(u(t)g(t)\big)^{(i)}(0) \qquad (2.16)$$

Proof: See [21] for more details.

Theorem (2.9) [38]: The Caputo fractional derivative of constant function is equal to zero

Proof: Let $m-1<p<m, m \in N$, and applying the definition of Caputo derivative and since the m^{th} derivative of a constant is equal to zero, it follows that:

$$D_*^p c = \frac{1}{\Gamma(m-p)} \int_a^t \frac{c^{(m)}}{(t-\tau)^{p+1-k}} d\tau$$

$$= 0. \qquad (2.17)$$

Theorem (2.10)[21]: Fractional derivative of exponential function:

Suppose that $\lambda \in C, \alpha \in R, m-1<\alpha<m$, and $m \in N$. The fractional derivative of exponential function as the form:

$$D_*^{\alpha}\left(e^{\lambda t}\right) = \sum_{i=0}^{\infty} \frac{\lambda^{i+m} t^{i+m-\alpha}}{\Gamma(i+1+m-\alpha)} = \lambda^m t^{m-p} E_{1,m-p+1}(\lambda t) \qquad (2.18)$$

Proof: see[21] for more details.

Theorem(2.11)[21]: The fractional derivative of cosine and sine:

Suppose that $\lambda \in C, p \in R, m>p>m-1$, and $m \in N$. The fractional derivative of cosine and sine function is in the form:

$$D_*^p(\cos \lambda\, t) = \frac{1}{2}(i\lambda)^m t^{m-p}\left(\left(E_{1,m-p+1}(i\lambda\, t)\right.\right. + (-1)^m\left(E_{1,m-p+1}(-i\lambda\, t)\right)\Big)$$

$$D_*^P(\sin \lambda\, t) = -\frac{1}{2} i(i\lambda)^m t^{m-p}\left(\left(E_{1,m-p+1}(i\lambda\, t)\right) - (-1)^m\left(E_{1,m-p+1}(-i\lambda\, t)\right)\right)$$

where $E_{\rho,\sigma}(f)$ is the Mittage-Leffler function.

Proof: The proof of the Caputo fractional derivative of cosine function:
Recall that :$\cos(t) = \frac{e^{it}+e^{-it}}{2}$, t$\in C$

$$D_*^p(\cos\lambda\, t) = D_*^p\left(\frac{e^{i\lambda t}+e^{-i\lambda t}}{2}\right)$$

$$= \frac{1}{2}\Big(\big(D_*^p(e^{i\lambda t}\big) + D_*^p(e^{-i\lambda t})\Big)$$

$$= \frac{1}{2}\Big((i\lambda)^m t^{m-p}\big(E_{1,m-p+1}(i\lambda\, t\big)$$

$$+ (-i\lambda)^m t^{m-p}\big(E_{1,m-p+1}(-i\lambda\, t)\big)\Big)$$

$$= \frac{1}{2}(i\lambda)^m t^{m-p}\Big(\big(E_{1,m-p+1}(i\lambda\, t\big) + (-1)^m\big(E_{1,m-p+1}(-i\lambda\, t)\big)\Big)$$

2.3.2 Properties of Caputo Fractional Derivative [21][38]:

Definition (2.12): Interpolation: Let $m-1<p<m, m\in N, p\in R$ and $u(t)$ be such that $D_*^p u(t)$ exists, then the following properties for the Caputo operator hold:

a. $\lim_{p\to m} D_*^p u(t) = u^{(m)}(t)$

b. $\lim_{p\to m-1} D_*^p u(t) = u^{(m-1)}(t) - u^{(m-1)}(0)$.

Definition (2.13): Linearity: The Caputo fractional derivative of a linear combination of functions is in the form:

$$D_*^p\big(\alpha\, u(t) + \beta\, g(t)\big) = \alpha D_*^p u(t) + \beta D_*^p g(t) \qquad (2.19)$$

Definition(2.14): Composition with integer order: Let $k-1<p<k$, $k, m\in N, p\in R$, then:

$$D_*^p D_*^m u(t) = D_*^{p+m} u(t) \neq D_*^m D_*^p u(t).$$

2.4 Relation between Riemann-Liouville and Caputo Operator

Theorem(2.12)[15]: Let $t > 0$, $\alpha \in R$, and m-1$< \alpha < m \in N$, then the following relation between the Riemann-Liouville and the Caputo operators holds:

$$D_*^{\alpha}u(t) = D^{\alpha}u(t) - \sum_{i=0}^{m-1} \frac{t^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)}(0) \qquad (2.20)$$

Proof: we will prove this theorem using the taylor series expansion about the point 0 which is:

$$u(t) = u(0) + tu'(0) + \frac{t^2}{2!}u''(0) + \cdots + \frac{t^{m-1}}{(m-1)!}u^{(m-1)}(0) + R_{m-1}(M_{m-1})$$

where the remainder term $R_{m-1}(M_{m-1})$, where M_{m-1} between 0 and t

$$= \sum_{i=0}^{m-1} \frac{t^i}{\Gamma(i+1)} u^{(i)}(0) + R_{m-1}(M_{m-1})$$

where the remainder term $R_{m-1}(M_{m-1})$ is given by:

$$R_{m-1}(M_{m-1}) = \int_0^t \frac{u^{(m)}(\tau)(t-\tau)^{m-1}}{(m-1)!} d\tau$$

$$= \frac{1}{\Gamma(m)} \int_0^t u^{(m)}(\tau)(t-\tau)^{m-1} d\tau$$

$$= D^{-m}u^{(m)}(t)$$

Now, by using some properties of the Riemann-Liouville fractional derivatives, and the Riemann-Liouville fractional derivative of the power function, then we will have:

$$D^{\alpha}u(t) = D^{\alpha}\left(\sum_{i=0}^{m-1}\frac{t^i}{\Gamma(i+1)}u^{(i)}(0) + R_{m-1}\right)$$

$$= \sum_{i=0}^{m-1}\frac{D^{\alpha}t^i}{\Gamma(i+1)}u^{(i)}(0) + D^{\alpha}R_{m-1}$$

$$= \sum_{i=0}^{m-1}\frac{\Gamma(i+1)}{\Gamma(i-\alpha+1)}\frac{t^{i-\alpha}}{\Gamma(i+1)}u^{(i)}(0) + D^{\alpha}D^{-m}u^{(m)}(t)$$

$$= \sum_{i=0}^{m-1}\frac{t^{i-\alpha}}{\Gamma(i-\alpha+1)}u^{(i)}(0) + D^{-m+\alpha}u^{(m)}(t)$$

$$= \sum_{i=0}^{m-1}\frac{t^{i-\alpha}}{\Gamma(i-\alpha+1)}u^{(i)}(0) + D_*^{\alpha}u(t).$$

This mean that:

$$D_*^{\alpha}u(t) = D^{\alpha}u(t) - \sum_{i=0}^{m-1}\frac{t^{i-\alpha}}{\Gamma(i-\alpha+1)}u^{(i)}(0).$$

Remark(2.1)[39]: This formula implies that the Caputo and Riemann-Liouville fractional operator conicide if $u(t)$ together with its first $m-1$ derivatives vanish at $t=0$.

2.5 Applications of Fractional Differntial Equations and Exact Solutions

Fractional differential equation plays very important role in Engineering and Technology due to it has many applications.

In this section we present some applications of fractional differential equations. These applications are: Fractional harmonic oscillator, Fractional wave equation, Fractional diffusion equation.

2.5.1 The Fractional Harmonic Oscillator

The fractional differential harmonic oscillator equation has a wide range of applications, it is important in theoritical physics, in the field of classical mechanics where it describes free oscillations and is a harmonic approximation for an arbitrary potential mininum. This equation has the fractional form as following [52]:

$$\left(n\frac{d^{2p}}{dx^{2p}}+r\right)y(x)=0\,, \tag{2.21}$$

where p is fractional derivative and we are free to adjust the meaning and the dimensions of the parametes n, r or both.

With the settings $\left(\frac{d^{2p}}{dx^{2p}}+\frac{r}{n}\right)y(x)=0$

in units of nrs $-$system $[r/n]$ is given as $1/s^{2p}$.

The general form of the solution of equation (2.21) is given as :

$y(x)=c_0\cos(ux)+c_1\sin(ux)$,

where c_0, c_1 are constants, and u is the angular frequency of the oscillation.

2.5.1.1 The Harmonic Oscillator According to Riemann-Liouville

We will use Riemann-Liouville fractional derivative definition to solve the harmonic oscillator differential equation using:

$$D^p x^{mp} = \frac{\Gamma(1+\mathrm{mp})}{\Gamma(1+(m-1)p)} x^{(m-1)p}$$

We get two different linearly independent solutions [48]:

$$\sin(p,x) = x^{p-1} \sum_{m=0}^{\infty} (-1)^m \frac{x^{(2m+1)p}}{\Gamma((2m+2)p)} \quad = \tag{2.22}$$

$$\cos(p,x) = x^{p-1} \sum_{m=0}^{\infty} (-1)^m \frac{x^{2mp}}{\Gamma((2m+1)p)} \tag{2.23}$$

With the property

$$D^p \sin(p,ux) = u^p \cos(p,ux) \tag{2.24}$$

$$D^p \cos(p,ux) = -u^p \sin(p,ux) \tag{2.25}$$

These functions are related to the Mittage-Liffler function:

$$\sin(p,x) = x^{2p-1} E(2p,2p,-x^{2p}) \tag{2.26}$$

$$\cos(p,x) = x^{p-1} E(2p,p,-x^{2p}) \tag{2.27}$$

When $p < 1$, $\lim_{x\to 0} \cos(p,ux) = \infty$, then we cannot give a non singular solution.

2.5.1.2 The Harmonic Oscillator According to Caputo

We will use Caputo fractional derivative definition to solve the harmonic oscillator differential equation using:

$$D^p x^{mp} = \begin{cases} \frac{\Gamma(1+\mathrm{mp})}{\Gamma(1+(m-1)p)} x^{(m-1)p} & m > 0 \\ 0 & m = 0 \end{cases}$$

We get two different linearly independent solutions [47]:

$$\sin(p,x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{(2m+1)p}}{\Gamma(1+(2m+1)p)} \tag{2.28}$$

$$\cos(p,x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2mp}}{\Gamma(1+2mp)} \tag{2.29}$$

With the property

$$D^p \sin(p, ux) = u^p \cos(p, ux) \tag{2.30}$$

$$D^p \cos(p, ux) = -u^p \sin(p, ux) \tag{2.31}$$

These functions are related to the Mittage-Liffler function:

$$\sin(p, x) = x^p E(2p, 1 + p, -x^{2p}) \tag{2.32}$$

$$\cos(p, x) = E(2p, -x^{2p}) \tag{2.33}$$

2.5.2The Fractional Damped Simple Harmonic Oscillator

The fractional harmonic oscillator behaves like a damped harmonic oscillator for $p < 1$. The simple harmonic oscillator is written as the following [46]:

$$y''(x) + u^2 y(x) = 0\ , \tag{2.34}$$

where u is the angular frequency of the oscillation.

When we add a damping force proportional to $y'(x)$ then equation (2.34) becomes as:

$$y''(x) + ay'(x) + u^2 y(x) = 0. \tag{2.35}$$

When there is an external force $g(x)$ then equation (2.35) becomes as:

$$y''(x) + ay'(x) + u^2 y(x) = g(x).$$

2.5.2.1 The Fractional Damped Simple Harmonic Oscillator According to Laplace Transform Method

The fractional damped simple harmonic oscillator is obtaind from the calssical one, but we replace $y'(x)$ in the classical case by caputo fractional derivative of order p and is formed as the following [46] :

$$y''(x) + aD^p y(x) + u^p y(x) = g(x), \qquad 0 < p < 1 \qquad (2.36)$$

with two initial conditions :

$$y(0) = c_0 \qquad y'(0) = c_1 \quad ,$$

where a, u, c_0 and c_1 are constants.

The solution of (2.36) is :

$$y(x) = c_0 y_0(x) - \frac{c_1}{u^2} y_0'(x) - \frac{1}{u^2} \int_0^x y_0'(x-\varepsilon) g(\varepsilon) d\varepsilon,$$

where $y_0(x) = L^{-1}\left\{\frac{s+as^{p-1}}{s^2+as^p+u^2}; x\right\}$.

2.5.3 Fractional Wave Equation

Fractional wave equation used for the determination of the eigen frequencies and eigen functions of vibrating systems.

The classical three dimensional wave equation with fractional case is given as [26]:

$$\left(\frac{d^{2p}}{dx^{2p}}+\frac{d^{2p}}{dy^{2p}}+\frac{d^{2p}}{dz^{2p}}-\rho\frac{d^2}{dt^2}\right)\emptyset(x,y,z,t)=0\,, \qquad (2.37)$$

where the dimension of the parameter ρ is determined in the nrs $-$system by $\left[\frac{s}{n^p}\right]$ and $\emptyset(x,y,z,t)=X(x)Y(y)Z(z)T(t)$

The partial differential equation (2.37) is changed into a system of ordinary differential equations:

$$\frac{d^{2p}}{dx^{2p}}X = -c_x^2\ X \qquad (2.38)$$

$$\frac{d^{2p}}{dx^{2p}}Y = -c_y^2\ Y \qquad (2.39)$$

$$\frac{d^{2p}}{dx^{2p}}Z = -c_z^2\ Z \qquad (2.40)$$

$$\frac{d^2T}{dx^2} = \frac{u^2}{\rho^2}\ T\,. \qquad (2.41)$$

The costants u, c_x, c_y, c_z satisfy the condition: $u=\rho\sqrt{c_x^2+c_y^2+c_z^2}$.

And the solution of this system determined if there is a convenient set of boundary conditions (Dirichlet, Neumann, mixed).

The simple case of a vibrating string with leangth equals $2p$, which is explained classicly by a one dimensional wave equation for the oscillation $X(x)$ and it has just the Dirichlet boundary conditions :

$$X(-b)=X(b)=0.$$

And the differential equation for a vibrating string is equivalent to the differential equation for the harmonic oscillator.

2.5.4 Fractional Diffusion Equation

The fractional diffusion equations are the most important application of fractional order derivatives, and the order of the resulting equation is related to the so-called fractal dimension of the porous material.

To describe the transfer processes in fractals we get the following equation:

$$ {}_0D_t^{\frac{1}{d}-1} L(t) = hy(t), \tag{2.42} $$

where $L(t)$ is the macroscopic flow a cross the fractal interface, $y(t)$ is the local driving force, h is a constant, and d is the fractal dimension.

The first type of fractional diffusion equation is formed as [37]:

$$ {}_0D_t^{\frac{1}{d}} \rho(s,t) = -B\left(\frac{\partial \rho(s,t)}{\partial s} + \frac{k}{s}\rho(s,t)\right), \tag{2.43} $$

where $\rho(s,t)$ is the average probability density of random walks on fractals, B and k are constants, and d is the anomalous diffusion exponent, which depends on the fractal dimension of media.

The second type of fractional diffusion equation is formed as [31]:

$$ {}_0D_t^{\frac{2}{d_u}} \rho(s,t) = \frac{1}{s^{d_r-1}} \frac{\partial}{\partial s}\left(s^{d_r-1} \frac{\partial}{\partial s}\rho(s,t)\right), \tag{2.44} $$

where d_u and d_r depend on the fractal dimension of the media.

The simple case of one dimensional differential equation is formed as [35]:

$$ {}_0D_t^{p} \rho(y,t) = \frac{\partial^2 \rho(y,t)}{\partial y^2}. \tag{2.45} $$

If the order p be an arbitrary real order, including $p = 1$ and $p = 2$, then equation (2.45) is called the fractional diffusion-wave equation, for $p = 1$ equation (2.45) becomes the classical diffusion equation and for $p = 2$ it becomes the classical wave equation[28].

Chapter Three

Numerical Methods for Solving Fractional Differential Equation

In this chapter we introduce some important numerical techniques for solving fractional differential equations; namely: The Adomian decomposition method, Variational iteration method, Homotopy perturbation method and Matrix approach method.

3.1 Theoretical Frameworks

Definition (3.1)[40]: (The Riemann-Liouville integral): The Riemann-Liouville integral of order $p > 0$ is in the form:

$$ {}_cD_t^{-p} g(t) = \frac{1}{\Gamma(p)} \int_c^t (t-\varepsilon)^{p-1} g(\varepsilon)\, d\varepsilon, \qquad p > 0. \tag{3.1}$$

From a bove definition (3.1), we note that:

$$ {}_cD_t^{-p}[(t-c)^m] = \frac{\Gamma(1+m)}{\Gamma(1+m+p)}(t-c)^{m+p} \tag{3.2}$$

$$ {}_cD_t^{p}[(t-\tau)^m] = \frac{\Gamma(1+m)}{\Gamma(1+m-p)}(t-c)^{m-p} \tag{3.3}$$

Lemma (3.1)[40]: If $u(t)$ is continuous function, then

$$ {}_aD_t^{-q}\left({}_aD_t^{p}\right)u(t) = {}_aD_t^{p-q}u(t) - \sum_{i=0}^{n-1} u^{(i)}(a)\frac{(t-a)^{i-p+q}}{\Gamma(i-p+q+1)}, \tag{3.4}$$

where $n-1 < p < n$ and $p \leq q$.

Proposition (3.1)[20]:

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} a_{i_1,i_2,\dots,i_{n-1},i_n} = \sum_{m=0}^{\infty} \sum_{i_1,i_2,\dots,i_{n-1},i_n \geq 0} a_{i_1,i_2,\dots,i_{n-1},i_n}. \qquad (3.5)$$

where $i_1 + i_2 + \cdots + i_{n-1} + i_n = m$.

Proposition (3.2)[20]: Moreover,

$$\sum_{m=0}^{\infty} \sum_{i_1,i_2,\dots,i_{n-1},i_n \geq 0} a_{i_1,i_2,\dots,i_{n-1},i_n} = \sum_{s=0}^{\infty} \sum_{a_{i_1,i_2,\dots,i_{n-1},i_n} \geq 0} \sum_{i_n=0}^{\infty} a_{i_1,i_2,\dots,i_{n-1},i_n}. \quad (3.6)$$

Definition (3.2)[38]: The Riemann-Liouville fractional integral operator(Ω^{α}) of order $\alpha \geq 0$, of a function $f \in C_{\lambda}, \lambda \geq -1$, can be defined as

$$\Omega^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\varepsilon)\, d\varepsilon \quad (\alpha > 0), \qquad (3.7)$$

$\Omega^{0} f(t) = f(t)$,

Some properties of the operator Ω^{α}:

Fo $\lambda, \mathrm{B} \geq 0$ and $\delta \geq -1$:

(1) $\Omega^{\lambda}\Omega^{\beta} f(t) = \Omega^{\beta}\Omega^{\lambda} f(t) = \Omega^{\lambda+\beta} f(t)$,

(2) $\Omega^{\lambda} t^{\delta} = \dfrac{\Gamma(\delta+1)}{\Gamma(\lambda+\delta+1)} t^{\lambda+\delta}$.

Definition (3.3)[22]:(The Mittage-Leffler function): The Mittage-Leffler function is defined in two ways:

(1) The one-parameter Mittage –Leffler function is defined as:

$$E_\rho(f) = \sum_{j=0}^{\infty} \frac{f^j}{\Gamma(\rho j + 1)}, \qquad \rho > 0, \quad \rho \in R, \quad f \in \mathbb{C} \qquad (3.8)$$

(2) The two-parameter Mittage-Leffler function is defined as:

$$E_{\rho,\sigma}(f) = \sum_{j=0}^{\infty} \frac{f^j}{\Gamma(\rho j + \sigma)}, \quad \rho,\sigma > 0, \quad \rho,\sigma \in R, \quad f \in \mathbb{C} \qquad (3.9)$$

3.2 The Adomian Decomposition Method

Let the n-term linear fractional differential equations with constant coefficients:

$$c_n[\,{}^aD^{\alpha_n}]f(t) + c_{n-1}[\,{}^aD^{\alpha_{n-1}}]f(t) + \cdots + c_1[\,{}^aD^{\alpha_1}] + c_0[\,{}^aD^{\alpha_0}]f(t) = g(t), (3.10)$$

$$f^j(0) = a_{kj}\ , \quad k = 0,1,\dots,n\ , \qquad j = 1,2,\dots,l_k\ , \quad l_k - 1 \le \alpha < l_k\ ,$$

where $n + 1 > \alpha_n \ge n > \alpha_{n-1} \dots > \alpha_1 > \alpha_0$, c_k *and* a_{kj} are real constants, ${}^aD^p = {}_0^aD_t^p$ denotes Caputo fractional derivative of order α.

take ${}^aD^{-\alpha_n}$ to both sides of (3.10) and using lemma (3.1), then we get

$$f(t) + \frac{c_{n-1}}{c_n}\ {}^aD^{\alpha_{n-1}-\alpha_n}f(t) + \cdots + \frac{c_0}{c_n}\ {}^aD^{\alpha_0-\alpha_n}f(t)$$

$$= \frac{1}{c_n}\ {}^aD^{-\alpha_n}g(t) + \frac{1}{c_n}\sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} f^{(j)}(0)\frac{t^{\alpha_n-\alpha_k+j}}{\Gamma(1+\alpha_n-\alpha_k+j)}. \quad (3.11)$$

we obtain the recursive relationship by the Adomian decomposition method, as follows:

$$f_0(t) = \frac{1}{c_n}\ {}^{a}D^{-\alpha_n} g(t) + \frac{1}{c_n}\sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} f^{(j)}(0)\frac{t^{\alpha_n-\alpha_k+j}}{\Gamma(1+\alpha_n-\alpha_k+j)},$$

$$f_1(t) = -\left(\frac{c_{n-1}}{c_n}\ {}^{a}D^{\alpha_{n-1}-\alpha_n} + \cdots + \frac{c_0}{c_n}\ {}^{a}D^{\alpha_0-\alpha_n}\right) f_0(t),$$

$$f_2(t) = (-1)^2\left(\frac{c_{n-1}}{c_n}\ {}^{a}D^{\alpha_{n-1}-\alpha_n} + \cdots + \frac{c_0}{c_n}\ {}^{a}D^{\alpha_0-\alpha_n}\right)^2 f_0(t), \quad (3.12)$$

$$\vdots$$

$$f_s(t) = (-1)^s\left(\frac{c_{n-1}}{c_n}\ {}^{a}D^{\alpha_{n-1}-\alpha_n} + \cdots + \frac{c_0}{c_n}\ {}^{a}D^{\alpha_0-\alpha_n}\right)^s f_0(t),$$

adding all terms of the recursion, we get the solution of (3.10) as

$$f(t) = \sum_{s=0}^{\infty} f_s(t)$$

$$= \sum_{s=0}^{\infty}(-1)^s\left(\frac{c_{n-1}}{c_n}\ {}^{a}D^{\alpha_{n-1}-\alpha_n} + \cdots + \frac{c_0}{c_n}\ {}^{a}D^{\alpha_0-\alpha_n}\right)^s f_0(t) \qquad (3.13)$$

$$= \frac{1}{c_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{c_{n-1}}{c_n}\ {}^{a}D^{\alpha_{n-1}-\alpha_n} + \cdots + \frac{c_0}{c_n}\ {}^{a}D^{\alpha_0-\alpha_n}\right)^s D^{-\alpha_n} g(t)$$

$$+\frac{1}{c_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{c_{n-1}}{c_n}\ {}^{a}D^{\alpha_{n-1}-\alpha_n} + \cdots\right.$$

$$\left. + \frac{c_0}{c_n}\ {}^{a}D^{\alpha_0-\alpha_n}\right)^s \sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} f^{(j)}(0)\frac{t^{\alpha_n-\alpha_k+j}}{\Gamma(1+\alpha_n-\alpha_k+j)},$$

Let

$$I_1 = \frac{1}{c_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{c_{n-1}}{c_n}\ {}^{a}D^{\alpha_{n-1}-\alpha_n} + \cdots + \frac{c_0}{c_n}\ {}^{a}D^{\alpha_0-\alpha_n}\right)^s D^{-\alpha_n} g(t)$$

$$I_2 = \frac{1}{c_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{c_{n-1}}{c_n}\ {}_aD^{\alpha_{n-1}-\alpha_n} + \cdots + \frac{c_0}{c_n}\ {}_aD^{\alpha_0-\alpha_n}\right)^s \sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} f^{(j)}(0)\frac{t^{\alpha_n-\alpha_k+j}}{\Gamma(1+\alpha_n-\alpha_k+j)}. \qquad (3.14)$$

Then,

$$f(t) = I_1 + I_2. \qquad (3.15)$$

Then, we appreciate I_1 and I_2:

For I_1, by [27] we obtain

$$\frac{1}{c_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{c_{n-1}}{c_n}\ {}_aD^{\alpha_{n-1}-\alpha_n} + \cdots + \frac{c_0}{c_n}\ {}_aD^{\alpha_0-\alpha_n}\right)^s D^{-\alpha_n}g(t),$$

$$= \int_0^t \frac{1}{c_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{i_0,i_1,\ldots,i_{n-2}\geq 0}(n; i_0, i_1, \ldots, i_{n-2})$$

$$\times \prod_{p=0}^{n-2}\left(\frac{c_p}{c_n}\right)^{i_p} (t-\tau)^{(\alpha_n-\alpha_{n-1})m+\alpha_n+\sum_{j=0}^{n-2}(\alpha_{n-1}-\alpha_j)i_j-1} \qquad (3.16)$$

$$\times E^{(m)}_{\alpha_n-\alpha_{n-1},\alpha_n+\sum_{j=0}^{n-2}(\alpha_{n-1}-\alpha_j)i_j}\left(-\frac{c_{n-1}}{c_n}D^{\alpha_n-\alpha_{n-1}}\right)g(\tau)d\tau$$

where $i_0 + i_1 + \cdots + i_{n-2} = m$.

For I_2, using lemma (3.1) then we obtain

$$\frac{1}{c_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{c_{n-1}}{c_n}\ {}_aD^{\alpha_{n-1}-\alpha_n} + \cdots + \frac{c_0}{c_n}\ {}_aD^{\alpha_0-\alpha_n}\right)^s \sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} f^{(j)}(0)\frac{t^{\alpha_n-\alpha_k+j}}{\Gamma(1+\alpha_n-\alpha_k+j)}$$

$$= \frac{1}{c_n}\sum_{s=0}^{\infty}(-1)^s \sum_{i_0,i_1,\dots,i_{n-2}\geq 0} \frac{s!}{i_0!\, i_1! \dots i_{n-2}!}\left(\frac{c_{n-1}}{c_n}\right)^{i_{n-1}} \left(\frac{c_{n-2}}{c_n}\right)^{i_{n-2}} \dots \left(\frac{c_0}{c_n}\right)^{i_0} \quad (3.17)$$

$$\times D^{i_{n-1}(\alpha_{n-1}-\alpha_n)+i_{n-2}(\alpha_{n-2}-\alpha_n)+\dots+i_0(\alpha_0-\alpha_n)} \sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} a_{ij} \frac{t^{\alpha_n-\alpha_{k+j}}}{\Gamma(1+\alpha_n-\alpha_k+j)}.$$

We can rewrite the above expession as the form:

$$\frac{1}{c_n}\sum_{s=0}^{\infty}(-1)^s \sum_{i_0,i_1,\dots,i_{n-2}\geq 0} \frac{s!}{i_0!\, i_1! \dots i_{n-2}!}\left(\frac{c_{n-1}}{c_n}\right)^{i_{n-1}} \left(\frac{c_{n-2}}{c_n}\right)^{i_{n-2}} \dots \left(\frac{c_0}{c_n}\right)^{i_0}$$

$$\times \sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} a_{kj} \frac{\Gamma(\mathcal{B})t^{\alpha_n-\alpha_i+j+i_{n-1}(\alpha_n-\alpha_{n-1})+i_{n-2}(\alpha_n-\alpha_{n-2})+\dots+i_0(\alpha_n-\alpha_0)}}{\Gamma(\mathcal{B}).\Gamma\big(\mathcal{B}+i_{n-1}(\alpha_n-\alpha_{n-1})+i_{n-2}(\alpha_n-\alpha_{n-2})+\dots+i_0(\alpha_n-\alpha_0)\big)}$$

$$= \frac{1}{c_n}\sum_{s=0}^{\infty}(-1)^s \sum_{i_0,i_1,\dots,i_{n-2}\geq 0} \frac{s!}{i_0!\, i_1! \dots i_{n-2}!}\left(\frac{c_{n-1}}{c_n}\right)^{i_{n-1}} \left(\frac{c_{n-2}}{c_n}\right)^{i_{n-2}} \dots \left(\frac{c_0}{c_n}\right)^{i_0}$$

$$\times \sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} a_{kj} \frac{t^{\alpha_n-\alpha_k+j+i_{n-1}(\alpha_n-\alpha_{n-1})+i_{n-2}(\alpha_n-\alpha_{n-2})+\dots+i_0(\alpha_n-\alpha_0)}}{\Gamma\big(\mathcal{B}+i_{n-1}(\alpha_n-\alpha_{n-1})+i_{n-2}(\alpha_n-\alpha_{n-2})+\dots+i_0(\alpha_n-\alpha_0)\big)}$$

$$= \sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} a_{kj}\, D^{(\alpha_k-j-1)} \left[\begin{array}{c} \frac{1}{c_n}\sum_{s=0}^{\infty}(-1)^s \sum_{i_0,i_1,\dots,i_{n-2}\geq 0} \frac{s!}{i_0!\, i_1! \dots i_{n-2}!} \\ \times \left(\frac{c_{n-1}}{c_n}\right)^{i_{n-1}} \left(\frac{c_{n-2}}{c_n}\right)^{i_{n-2}} \dots \left(\frac{c_0}{c_n}\right)^{i_0} \\ \times \frac{t^{\alpha_n-\alpha_k+j+i_{n-1}(\alpha_n-\alpha_{n-1})+i_{n-2}(\alpha_n-\alpha_{n-2})+\dots+i_0(\alpha_n-\alpha_0)}}{\Gamma\big(\alpha_n+i_{n-1}(\alpha_n-\alpha_{n-1})+i_{n-2}(\alpha_n-\alpha_{n-2})+\dots+i_0(\alpha_n-\alpha_0)\big)} \end{array}\right] \quad (3.18)$$

where β denotes $1+\alpha_n-\alpha_k+j$.

The above solution is equivalent to the following form:

$$\sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} a_{kj}\, D^{(\alpha_k-j-1)} \left[\begin{array}{c} \frac{1}{c_n}\sum_{i_{n-1}=0}^{\infty}\sum_{m=0}^{\infty} \sum_{i_0,i_1,\dots,i_{n-2}\geq 0} (-1)^{i_{n-1}+m} \frac{(i_{n-1}+m)!}{i_0!\, i_1! \dots i_{n-1}!} \\ \times \left(\frac{c_{n-1}}{c_n}\right)^{i_{n-1}} \left(\frac{c_{n-2}}{c_n}\right)^{i_{n-2}} \dots \left(\frac{c_0}{c_n}\right)^{i_0} \\ \times \frac{t^{\alpha_n-\alpha_k+j+i_{n-1}(\alpha_n-\alpha_{n-1})+i_{n-2}(\alpha_n-\alpha_{n-2})+\dots+i_0(\alpha_n-\alpha_0)}}{\Gamma\big(\alpha_n+i_{n-1}(\alpha_n-\alpha_{n-1})+i_{n-2}(\alpha_n-\alpha_{n-2})+\dots+i_0(\alpha_n-\alpha_0)\big)} \end{array}\right]$$

$$= \sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} a_{kj}\, D^{(\alpha_k-j-1)} \left[\begin{array}{c} \frac{1}{c_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{i_0,i_1,\dots,i_{n-2}\geq 0}\frac{m!}{i_0!\,i_1!\dots i_{n-2}!} \\ \times \prod_{r=0}^{n-2}\left(\frac{c_r}{c_n}\right)^{i_r} t^{(\alpha_n-\alpha_{n-1})m+\sum_{k=0}^{n-2}(\alpha_{n-1}-\alpha_k)i_k+\alpha_{n-1}} \\ \times \sum_{i_{n-1}=0}^{\infty}(-1)^{i_{n-1}}\left(\frac{c_{n-1}}{c_n}\right)^{i_{n-1}}\frac{(i_{n-1}+m)!}{i_{n-1}!} \\ \times \frac{t^{i_{n-1}(\alpha_n-\alpha_{n-1})}}{\Gamma(i_{n-1}(\alpha_n-\alpha_{n-1})+(\alpha_n-\alpha_{n-1})m+\sum_{k=0}^{n-2}(\alpha_{n-1}-\alpha_k)i_k+\alpha_n)} \end{array} \right]$$

$$= \sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} a_{kj}\, D^{(\alpha_k-j-1)} \left[\begin{array}{c} \frac{1}{c_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{i_0,i_1,\dots,i_{n-2}\geq 0}(m; i_0, i_1, \dots, i_{n-2}) \\ \times \prod_{r=0}^{n-2}\left(\frac{c_r}{c_n}\right)^{i_r} t^{(\alpha_n-\alpha_{n-1})m+\sum_{k=0}^{n-2}(\alpha_{n-1}-\alpha_k)i_k+\alpha_{n-1}} \\ \times E^{(m)}_{\alpha_n-\alpha_{n-1},\sum_{k=0}^{n-2}(\alpha_{n-1}-\alpha_k)i_k+\alpha_n}\left(-\frac{c_{n-1}}{c_n}t^{\alpha_n-\alpha_{n-1}}\right) \end{array} \right]. \quad (3.19)$$

So, by [2] we get the solution $f(t) = I_1 + I_2$

$$= \int_0^t \frac{1}{c_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{i_0,i_1,\dots,i_{n-2}\geq 0}(m; i_0, i_1, \dots, i_{n-2})$$

$$\times \prod_{p=0}^{n-2}\left(\frac{c_p}{c_n}\right)^{i_p}(t-\tau)^{(\alpha_n-\alpha_{n-1})m+\alpha_n+\sum_{j=0}^{n-2}(\alpha_{n-1}-\alpha_j)i_j-1} \quad (3.20)$$

$$\times E^{(m)}_{\alpha_n-\alpha_{n-1},\alpha_n+\sum_{j=0}^{n-2}(\alpha_{n-1}-\alpha_j)i_j}\left(-\frac{c_{n-1}}{c_n}D^{\alpha_n-\alpha_{n-1}}\right)g(\tau)d\tau$$

$$+ \sum_{k=0}^{n} c_k \sum_{j=0}^{l_k-1} a_{kj}\, D^{(\alpha_k-j-1)} \left[\begin{array}{l} \frac{1}{c_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{i_0,i_1,\dots,i_{n-2}\geq 0}(m; i_0, i_1, \dots, i_{n-2}) \\ \times \prod_{r=0}^{n-2}\left(\frac{c_r}{c_n}\right)^{i_r} t^{(\alpha_n-\alpha_{n-1})m+\sum_{k=0}^{n-2}(\alpha_{n-1}-\alpha_k)i_k+\alpha_{n-1}} \\ \times E^{(m)}_{\alpha_n-\alpha_{n-1},\sum_{k=0}^{n-2}(\alpha_{n-1}-\alpha_k)i_k+\alpha_n}\left(-\frac{c_{n-1}}{c_n}t^{\alpha_n-\alpha_{n-1}}\right) \end{array} \right]$$

where

$$(m;\ i_0, i_1, \dots, i_{n-2}) = \frac{m!}{i_0!i_1!\dots i_{n-2}!},$$

and $E^{(k)}_{\rho,\sigma}(f)$ is the Mittag-Leffler function

$$E_{\rho,\sigma}^{(k)}(f) = \frac{d^k}{df^k} E_{\rho,\sigma}$$

$$= \sum_{j=0}^{\infty} \frac{(k+j)!\, f^k}{j!\, \Gamma(\rho j + \rho k + \sigma)}. \quad (3.21)$$

Substituting the Green function into the above expression,

$$G_n(t) = \frac{1}{c_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{i_0, i_1, \dots, i_{n-2} \geq 0} (m; i_0, i_1, \dots, i_{n-2})$$

$$\times \prod_{p=0}^{n-2} \left(\frac{c_p}{c_n}\right)^{i_p} (t - \tau)^{(\alpha_n - \alpha_{n-1})m + \alpha_n + \sum_{j=0}^{n-2} (\alpha_{n-1} - \alpha_j) i_j - 1}$$

$$\times E^{(m)}_{\alpha_n - \alpha_{n-1}, \alpha_n + \sum_{j=0}^{n-2} (\alpha_{n-1} - \alpha_j) i_j} \left(-\frac{c_{n-1}}{c_n} D^{\alpha_n - \alpha_{n-1}}\right) \quad (3.22)$$

So the analytical general solution of (3.10) is in the form:

$$f(t) = \int_0^t G_n(t-\tau) g(\tau) d\tau + \sum_{k=0}^{n} c_k \sum_{j=0}^{l_k - 1} a_{kj} G_n^{(\alpha_k - j - 1)}(t)\ . \quad (3.23)$$

3.3 Variational Iteration Method

The variational iteration method is greatly successful method that was proposed in 1998 to solve fractional differential equations. Consider the fractional differential equation [25] :

$$\frac{D^p f}{Dt^p} + g = 0\ . \quad (3.24)$$

where $\frac{D^p f(t)}{Dt^p}$ is the Caputo's fractional derivatives defined as

$$\frac{D^p f(t)}{Dt^p} = \frac{1}{\Gamma(m+1-p)} \int_a^t \frac{f^{(m+1)}(\tau)}{(t-\tau)^{p-m}}\, d\tau \;,\;\; m < p < m+1, \qquad (3.25)$$

To show the basic notion of the variational iteration method, let the following general linear equation: $Lf(t) = g(t)$,

where L is the linear operator and $g(t)$ is the inhomogeneous term. By [25] the variational iteration method is as the form:

$$f_{n+1}(t) = f_n(t) + \int_0^t \lambda(\epsilon)\big(Lf_n(t) - g(t)\big)\, d\epsilon \;. \qquad (3.26)$$

The variational iteration algorithms for three cases of the order of p by [12] are given as:

In the case $0 < p < 1$, we rewrite equation (3.24) in the form

$$\frac{df}{dt} + \frac{D^p f}{Dt^p} - \frac{df}{dt} + g = 0 \;. \qquad (3.27)$$

The variational iteration algorithms are given as follows

$$\begin{cases} f_{n+1}(t) = f_n(t) - \int_0^t \left(\frac{D^p f_n}{Dt^p} + g_n\right) ds \\ f_{n+1}(t) = f_0(t) - \int_0^t \left(\frac{D^p f_n}{Dt^p} - \frac{df_n}{dt} + g_n\right) ds \\ f_{n+1}(t) = f_0 - \left\{\int_0^t \left\{\left(\frac{D^p f_n}{Dt^p} - \frac{df_n}{dt} + g_n\right) - \left(\frac{D^p f_{n-1}}{Dt^p} - \frac{df_{n-1}}{dt} + g_{n-1}\right)\right\} ds. \right. \end{cases} \qquad (3.28)$$

The above iteration formulas are also valid when $1 < p < 2$. We can rewrite equation (3.24) in the form

$$\frac{d^2 f}{dt^2} + \frac{D^p f}{Dt^p} - \frac{d^2 f}{dt^2} + g = 0\,. \qquad (3.29)$$

And the following iteration formulas are valid

$$\begin{cases} f_{n+1}(t) = f_n(t) + \int_0^t (s-t)(\frac{D^p f_n}{Dt^p} + g_n) \\ f_{n+1}(t) = f_0(t) + \int_0^t (s-t)\left(\frac{D^p f_n}{Dt^p} - \frac{d^2 f_n}{dt^2} + g_n\right) ds \\ f_{n+1}(t) = f_n(t) + \int_0^t (s-t)\left\{\left(\frac{D^p f_n}{Dt^p} - \frac{d^2 f_n}{dt^2} + g_n\right) - \left(\frac{D^p f_{n-1}}{Dt^p} - \frac{d^2 f_{n-1}}{dt^2} + g_{n-1}\right)\right\} ds. \end{cases} \qquad (3.30)$$

when p is close to 1, equation (3.30) is valid for p approaching 2.

In the case $N < p < N+1$, where N is a natural number, by [12] the iteration formulas as form:

$$\begin{cases} \mathrm{f}_{n+1}(t) = \mathrm{f}_n(t) + (-1)^N \int_0^t \frac{1}{(N-1)!}(s-t)^{N-1}\left(\frac{D^p f_n}{Dt^p} + g_n\right) ds \\ \mathrm{f}_{n+1}(t) = \mathrm{f}_0(t) + (-1)^N \int_0^t \frac{1}{(N-1)!}(s-t)^{N-1}\left(\frac{D^p f_n}{Dt^p} - \frac{d^N f_n}{dt^N} + g_n\right) ds \\ \mathrm{f}_{n+1}(t) = \mathrm{f}_n(t) + (-1)^N \int_0^t \frac{1}{(N-1)!}(s-t)^{N-1}\left\{\left(\begin{matrix} \left(\frac{D^p f_n}{Dt^p} - \frac{d^N f_n}{dt^N} + g_n\right) - \\ \left(\frac{D^p f_{n-1}}{Dt^p} - \frac{d^N f_{n-1}}{dt^N} + g_n\right) \end{matrix}\right)\right\} ds\,. \end{cases} \qquad (3.31)$$

or

$$\begin{cases} f_{n+1}(t) = f_n(t) + (-1)^{N+1} \int_0^t \frac{1}{N!}(s-t)^N\left(\frac{D^p f_n}{Dt^p} + g_n\right) ds \\ f_{n+1}(t) = f_0(t) + (-1)^{N+1} \int_0^t \frac{1}{N!}(s-t)^N\left(\frac{D^p f_n}{Dt^p} - \frac{d^{N+1} f_n}{dt^{N+1}} + g_n\right) ds \\ f_{n+1}(t) = f_n(t) + (-1)^{N+1} \int_0^t \frac{1}{N!}(s-t)^N\left\{\left(\begin{matrix} \left(\frac{D^p f_n}{Dt^p} - \frac{d^{N+1} f_n}{dt^{N+1}} + g_n\right) - \\ \left(\frac{D^p f_{n-1}}{Dt^p} - \frac{d^{N+1} f_{n-1}}{dt^{N+1}} + g_n\right) ds \end{matrix}\right)\right\} ds\,. \end{cases} \qquad (3.32)$$

Equation (3.31) is valid when p is close to N and when p is close to N+1 then equation (3.32) is valid and is more effectively.

3.4 Homotopy Perturbation Method

The fractional initial value problem in the operator form is:

$$D^{\alpha}f(t) + Lf(t) = g(t), \quad (3.33)$$

$$f^{(i)}(0) = c_i, \qquad i = 0,1,\dots,n-1, \quad (3.34)$$

where c_i is the initial conditions, L is the linear operator which might include other fractional derivative operators $D^{\beta}(\beta < \alpha)$, while the function g, the source function is assumed to be in C_{-1}if α is an integer, and in C_{-1}^{1} if α is not an integer. The solution $f(t)$ is to be determined in C_{-1}^{n} .

In view of HPM, the follwing homotopy as the form [18]:

$$(1-p)D^{\alpha}f + p[D^{\alpha}f + Lf(t) - g(t)] = 0. \quad (3.35)$$

or

$$D^{\alpha}f + p[Lf(t) - g(t)] = 0, \quad (3.36)$$

where $p \in [0,1]$ is an embedding parameter. If $p = 0$, equations.(3.35) and (3.36) become

$$D^{\alpha}f = 0, \quad (3.37)$$

and when $p = 1$, both (3.35) and (3.36) turn out to be the original FDE (3.33).

The solution of equation (3.33) is:

$$f(t) = f_0(t) + pf_1(t) + p^2 f_2(t) + p^3 f_3(t) + \cdots \tag{3.38}$$

Substituting $p = 1$ in equation (3.38) then we get the solution of equation (3.33) as the form:

$$f(t) = f_0(t) + f_1(t) + f_2(t) + f_3(t) + \cdots. \tag{3.39}$$

Substituting (3.38) in (3.36) and gathering all the terms with the same powers of p, we get

$$p^0 \colon D^\alpha f_0 = 0, \tag{3.40}$$

$$p^1 \colon D^\alpha f_1 = -Lf_0 + g(t), \tag{3.41}$$

$$p^2 \colon D^\alpha f_2 = -Lf_1(t), \tag{3.42}$$

$$p^3 \colon D^\alpha f_3 = -Lf_2(t), \tag{3.43}$$

and so on.

take the operator J^α, the inverse operator of D^α, which is defined by on both sides of the above linear equations, the first three terms of the HPM solution can be given as [17]:

$$f_0 = \sum_{i=0}^{n-1} f^{(i)}(0) \frac{t^i}{i!} = \sum_{i=0}^{n-1} c_i \frac{t^i}{i!},$$

$$f_1 = -\Omega^\alpha[Lf_0(t)] + \Omega^\alpha[g(t)],$$

$$f_2 = -\Omega^\alpha[Lf_1(t)],$$

$$f_3 = -\Omega^\alpha[Lf_2(t)],$$

and the general form of the HPM solution can be given as:

$f_n = -\Omega^\alpha[Lf_{n-1}(t)]$.

Then the general solution of equation (3.33) given as:

$$f(t) = f_0 + f_1 + f_3 + \cdots + f_n + \cdots. \quad (3.44)$$

3.5 Matrix Approach Method

3.5.1 Left-sided Fractional Derivative

Consider the function $g(x)$, defined in $[c,d]$, such that $g(x) \equiv 0$ for $x < c$. Of real order $m-1 \le \alpha \le m$, such as:

$$ {}_cD_x^\alpha g(x) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^m \int_c^x \frac{g(\varepsilon)\ d\varepsilon}{(x-\varepsilon)^{\alpha-m+1}} \quad ,(c < x < d). \quad (3.45)$$

Let us take equidistant nodes with the step $l: x_i = il$ (i=0,1,...N) , in the interval $[c,d]$, where $x_0 = c$ and $x_N = d$.

Using the backward fractional difference approximation for the α^{th} derivative at the points x_i,$i = 0,1,\dots,N$, we have:

$$ {}_cD_{xi}^\alpha g(x) \approx \frac{\nabla^\alpha g(x_i)}{l^\alpha} = l^{-\alpha}\sum_{j=0}^{i}(-1)^j\binom{\alpha}{j} g_{i-j} \quad ,i = 0,1,\dots,N \quad (3.46)$$

All $N+1$ formulas (3.46) is equivalent the following matrix [11] :

$$\begin{bmatrix} l^{-\alpha}\nabla^{\alpha} g(x_0) \\ l^{-\alpha}\nabla^{\alpha} g(x_1) \\ l^{-\alpha}\nabla^{\alpha} g(x_2) \\ \vdots \\ l^{-\alpha}\nabla^{\alpha} g(x_{N-1}) \\ l^{-\alpha}\nabla^{\alpha} g(x_N) \end{bmatrix} =$$

$$\frac{1}{l^{\alpha}} \begin{bmatrix} w_0^{(\alpha)} & 0 & 0 & 0 & \dots & 0 \\ w_1^{(\alpha)} & w_0^{(\alpha)} & 0 & 0 & \dots & 0 \\ w_2^{(\alpha)} & w_1^{(\alpha)} & w_0^{(\alpha)} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ w_{N-1}^{(\alpha)} & \ddots & w_2^{(\alpha)} & w_1^{(\alpha)} & w_0^{(\alpha)} & 0 \\ w_{N-2}^{(\alpha)} & w_{N-1}^{(\alpha)} & \ddots & w_2^{(\alpha)} & w_1^{(\alpha)} & w_0^{(\alpha)} \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ \\ \\ g_{N-1} \\ g_N \end{bmatrix} \tag{3.47}$$

$$w_i^{(\alpha)} = (-1)^i \binom{\alpha}{i} \qquad i = 0,1,\dots,N\,. \tag{3.48}$$

In equation (3.47) the column vector of function $g_i (i = 0,1,\dots,N)$ is multiplied by the matrix

$$A_N^{\alpha} = \frac{1}{l^{\alpha}} \begin{bmatrix} w_0^{(\alpha)} & 0 & 0 & 0 & \dots & 0 \\ w_1^{(\alpha)} & w_0^{(\alpha)} & 0 & 0 & \dots & 0 \\ w_2^{(\alpha)} & w_1^{(\alpha)} & w_0^{(\alpha)} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ w_{N-1}^{(\alpha)} & \ddots & w_2^{(\alpha)} & w_1^{(\alpha)} & w_0^{(\alpha)} & 0 \\ w_{N-2}^{(\alpha)} & w_{N-1}^{(\alpha)} & \ddots & w_2^{(\alpha)} & w_1^{(\alpha)} & w_0^{(\alpha)} \end{bmatrix}. \tag{3.49}$$

The result is the column vector of approximated values of the fractional derivative ${}_cD_{x_i}^{\alpha} g(x)$, $i = 0,1,\dots N$.

The generating function for the matrix is

$$A_{\alpha}(z) = l^{-\alpha}(1-z)^{\alpha}. \tag{3.50}$$

Since for lower triangular matrices A_N^{α} and A_N^{B} we ever have

$$A_N^{\alpha} A_N^{\beta} = A_N^{\beta} A_N^{\alpha} = A_N^{\alpha+\beta}.$$

Theorem (3.1)[38]: If ${}_cD_x^{\alpha}\left({}_cD_x^{\beta} g(x)\right) = {}_cD_x^{\beta}\left({}_cD_x^{\alpha} g(x)\right) = {}_cD_x^{\alpha+\beta} g(x),$

which holds if

$$g^{(i)}(c) = 0, \qquad i = 1,2, \dots, a-1, \text{ where } a = \max\{n, m\}. \quad (3.51)$$

Then we can treat such matrices as discrete analogues of the corresponding left-sided fractional derivatives ${}_cD_x^{\alpha}$ *and* ${}_cD_x^{\beta}$, where

$$n - 1 \le \alpha < n \;\; and \; m - 1 \le \beta < m.$$

This means that if left-sided fractional derivatives of a function $g(x)$of orders less than some integer a are considered, then they can all be replaced with their corresponding discrete analogues, if the function $g(x)$ satisfies to condition (3.51).

3.5.2 Right-sided Fractional Derivative

Consider a function $g(x)$, defined in $[c, d]$, such that $g(x) \equiv 0$ for $x > d$.

Assum that the function $g(x)$ is good enough for considering its right sided fractional derivative of real order α where $(m - 1 \le \alpha < m)$,

$${}_cD_x^{\alpha} g(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_x^d \frac{g(\varepsilon) d\varepsilon}{(x-\varepsilon)^{\alpha-m+1}} \quad , (c < x < d). \quad (3.52)$$

We get the discrete analogue of the right sided fractional differentiation with the stip $l: x_i = il \quad (i = 0,1,\dots,N)$, in the interval $[c,d]$, where $x_0 = c$ and $x_N = d$, which is represented by the matrix [11]:

$$G_N^{\alpha} = \frac{1}{l^{\alpha}} \begin{bmatrix} w_0^{(\alpha)} & w_1^{(\alpha)} & \ddots & \ddots & w_{N-1}^{(\alpha)} & w_N^{(\alpha)} \\ 0 & w_0^{(\alpha)} & w_1^{(\alpha)} & \ddots & \ddots & w_{N-1}^{(\alpha)} \\ 0 & 0 & w_0^{(\alpha)} & w_1^{(\alpha)} & \ddots & \ddots \\ \dots & \dots & \dots & \ddots & \ddots & \ddots \\ 0 & \dots & 0 & 0 & w_0^{(\alpha)} & w_1^{(\alpha)} \\ 0 & 0 & \dots & 0 & 0 & w_0^{(\alpha)} \end{bmatrix} \qquad (3.53)$$

The generating function for the matrix G_N^{α} is the same for A_N^{α} :

$$A_{\alpha}(z) = l^{-\alpha}(1-z)^{\alpha}.$$

And the transposition of the matrix A_N^{α} gives the matrix G_N^{α} and the opposite is holds:

$$(A_N^{\alpha})^T = G_N^{\alpha} \ , \quad (G_N^{\alpha})^T = A_N^{\alpha} \ . \qquad (3.54)$$

Theorem (3.2) [38]: If the function $g(x)$ satisfies the condition

$$g^{(i)}(d) = 0 \ , \ \ i = 1,2,\dots,a-1. \qquad (3.55)$$

and if right sided fractional derivatives of a function $g(x)$ of orders less than some integer a are considered, then they can all be replaced with their corresponding discrete analogues .

Useful Matrices: Eleminators

Eleminator, $S_{i_0,i_1,\dots,i_N}$, is obtained from the unit matrix by deleting rows $i_0, i_1, \dots, i_N$.

Example 3.1 [21]:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}; \qquad S_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BS_1^T = \begin{bmatrix} b_{12} & b_{13} \\ b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix}; \quad S_1B = \begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \;;\; S_1BS_1^T = \begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix}$$

The system for finding approximated derivatives[21]: Consider the following fractional differential equation with constant coefficients: $\sum_{j=0}^{n} p_j\, D^{\alpha_j}\, g(x) = f(x),\ \ 0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n\, , m-1 < \alpha_n < m$ (3.56)

the Riemann-Liouville and Caputo derivatives coincide if

$m-1 < \alpha_n < m$ and $g^{(i)}(x_0) = 0, i = 0,1, \dots, m-1,$

where p_j is constant coefficient , $g_N = \big(g(x_0), g(x_1), \dots, g(x_N)\big)^T$ and $F_N = \big(f(x_0), f(x_1), \dots, f(x_N)\big)^T$.

By Backward differences the approximated derivatives in the a bove condition have:

$$g(x_0) = g(x_1) = \cdots = g(x_{m-1}) = 0. \qquad (3.57)$$

The system for finding the rest approximated terms can be formed as:

$$\sum_{j=0}^{n} p_j\, A_{N-m}^{\alpha_j} \{S_{0,1,\dots,m-1} g_N\} = S_{0,1,\dots,m-1} F_N. \qquad (3.58)$$

Chapter Four

Numerical Examples and Results

In this chapter we try to apply the aforementioned numerical schemes; namely: the Adomian decomposition method, Variational iteration method, Homotopy perturbation method and Matrix approach method to find an approximate solution of some linear fractional differential equations. This can be achieved by using proper algorithms, Matlab and Maple software. A comparison between the exact and numerical solutions will be drawn.

Example 4.1

Consider the linear fractional differential equation:

$$D^{\alpha}x(t) + x(t) = \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + t^{3} \qquad (4.1)$$

with initial conditions:

$$x(0) = 0\,, x'(0) = 0.$$

The exact solution of equation (4.1) when $\alpha = 1.9$ [34] is:

$$x(t) = t^{2}.$$

4.1 The Numerical Realization of Equation (4.1) Using Adomian Decomposition Method

Using lemma (3.1) and take $D^{-\alpha}$ to both sides of equation (4.1)

$$x(t) + D^{-\alpha}x(t) = D^{-\alpha}f(t) + \sum_{i=0}^{1} x^{(i)}(0)\frac{t^i}{\Gamma(i+1)}. \qquad (4.2)$$

Using Adomian decomposition method, we get:

$$x_0(t) = D^{-\alpha}f(t) + \sum_{i=0}^{1} x^{(i)}(0)\frac{t^i}{\Gamma(i+1)}$$

$$x_1(t) = -D^{-\alpha}x_0(t),$$

$$x_2(t) = -D^{-\alpha}x_1(t) = (-1)^2 D^{-2\alpha}x_0(t),$$

$$\vdots$$

$$x_s(t) = -D^{-\alpha}x_{s-1}(t) = (-1)^s D^{-s\alpha}x_0(t), \qquad (4.3)$$

then the solution of equation (4.1) obtained by adding all of the above terms as follows:

$$x(t) = \sum_{s=0}^{\infty} x_s(t),$$

$$= \sum_{s=0}^{\infty}(-1)^s D^{-s\alpha}x_0(t),$$

$$= \sum_{s=0}^{\infty}(-1)^s D^{(-s-1)\alpha}f(t) + \sum_{s=0}^{\infty}(-1)^s D^{-s\alpha}\sum_{i=0}^{1} x^{(i)}(0)\frac{t^i}{\Gamma(i+1)},$$

$$= \sum_{s=0}^{\infty}(-1)^s \frac{1}{\Gamma((s+1)\alpha)}\int_0^t (t-\tau)^{(s+1)\alpha-1}f(\tau)d\tau$$
$$+ \sum_{i=0}^{1} x^{(i)}(0)\sum_{s=0}^{\infty}(-1)^s \frac{t^{i+s\alpha}}{\Gamma(i+s\alpha+1)}, \qquad (4.4)$$

$$= \int_0^t \sum_{s=0}^{\infty} (t-\tau)^{\alpha-1} \frac{[-(t-\tau)^{\alpha}]^s}{\Gamma(s\alpha+\alpha)} f(\tau)d\tau$$

$$+ \sum_{i=0}^{1} x^{(i)}(0) D^{(\alpha-i-1)} \left[\sum_{s=0}^{\infty} t^{\alpha-1} \frac{(-t^{\alpha})^s}{\Gamma(s\alpha+\alpha)} \right],$$

$$= \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[-(t-\tau)^{\alpha}] f(\tau)d\tau$$

$$+ \sum_{i=0}^{1} x^{(i)}(0) D^{(\alpha-i-1)} \left[t^{\alpha-1} E_{\alpha,\alpha}(-t^{\alpha}) \right],$$

$$= \int_0^t G_1(t-\tau) f(\tau)d\tau + \sum_{i=0}^{1} x^{(i)}(0) D^{(\alpha-i-1)} G_1(t),$$

where $G_1(t) = t^{\alpha-1} E_{\alpha,\alpha}(-t^{\alpha})$

so, $x(t) = \int_0^1 G_1(t-\tau) f(\tau)d\tau + x^{(0)}(0) D^{(\alpha-1)} G_1(t) + x^{(1)}(0) D^{\alpha} G_1(t),$

(4.5)

by equation (3.21), we obtain:

$E_{\alpha,\alpha}(-t^{\alpha}) = \sum_{i=0}^{\infty} \frac{(-t^{\alpha})^i}{\Gamma(\alpha i+\alpha)}$, $G_1(t) = t^{\alpha-1} \sum_{i=0}^{\infty} \frac{(-t^{\alpha})^i}{\Gamma(\alpha i+\alpha)}$,

$$G_1(t) = t^{\alpha-1} \sum_{i=0}^{\infty} \frac{(-t^{\alpha})^i}{\Gamma(\alpha i+\alpha)},$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i t^{\alpha i+\alpha-1}}{\Gamma(\alpha i+\alpha)},$$

then the solution of equation (4.1) in general form is:

$$x(t) = \int_0^t \sum_{i=0}^{\infty} (-1)^i \frac{(t-\tau)^{\alpha i+\alpha-1}}{\Gamma(\alpha i+\alpha)} . \left(\frac{2}{\Gamma(3-\alpha)} \tau^{2-\alpha} + \tau^3 \right) d\tau. \qquad (4.6)$$

$$= \sum_{i=0}^{\infty} (-1)^i \left[\int_0^t \frac{(t-\tau)^{\alpha i+\alpha-1}}{\Gamma(\alpha i+\alpha)} \frac{2}{\Gamma(3-\alpha)} \tau^{2-\alpha} + \int_0^t \frac{(t-\tau)^{\alpha i+\alpha-1}}{\Gamma(\alpha i+\alpha)} \tau^3 \right] d\tau$$

$$= \sum_{i=0}^{\infty} (-1)^i \left[D^{-(\alpha i+\alpha)} \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + D^{-(\alpha i+\alpha)} t^3 \right].$$

By definition (3.1), we get:

$$x(t) = \sum_{i=0}^{\infty} (-1)^i \left[\frac{2}{\Gamma(3-\alpha)} \frac{\Gamma(3-\alpha)}{\Gamma(3-\alpha+\alpha i+\alpha)} t^{2-\alpha+\alpha i+\alpha} + \frac{\Gamma(1+3)}{\Gamma(1+3+\alpha i+\alpha)} t^{3+\alpha i+\alpha} \right],$$

when $\alpha = 1.9$

$$x(t) = \sum_{i=0}^{\infty} (-1)^i \left[\frac{2}{\Gamma(3+1.9i)} t^{2+1.9i} + \frac{6}{\Gamma(5.9+1.9i)} t^{4.9+1.9i} \right],$$

$$= \frac{2}{\Gamma(3)} t^2 + \frac{6}{\Gamma(5.9)} t^{4.9} - \frac{2}{\Gamma(4.9)} t^{3.9} - \frac{6}{\Gamma(7.8)} t^{6.8} + \cdots.$$

$$= t^2 + 0.059247439 t^{4.9} - 0.096770806 t^{3.9} - 0.001776766299 t^{6.8} + \cdots$$

$$= t^2 - small\ terms. \approx t^2.$$

4.2 The Numerical Realization of Equation (4.1) Using Homotopy Perturbation Method

According to equation (3.36), we can write the following homotopy:

$$D^{\alpha}x(t) + px(t) - \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} - t^3 = 0,$$

the solution of equation (4.1) is:

$$x(t) = x_0(t) + px_1(t) + p^2x_2(t) + p^3x_3(t) + \cdots. \tag{4.7}$$

Substituting (4.7) into the above equation, and gathering terms with the same power of p, then we get:

$$p^0: D^{\alpha}x_0(t) = 0, \tag{4.8}$$

$$p^1: D^{\alpha}x_1(t) = -x_0(t) + f(t), \tag{4.9}$$

$$p^2: D^{\alpha}x_2(t) = -x_1(t), \tag{4.10}$$

$$p^3: D^{\alpha}x_3(t) = -x_2(t), \tag{4.11}$$

$\vdots$

Applying Ω^{α}, the inverse operator of D^{α}, on both sides of the equations from (4.8)-(4.11), then we get:

$$x_0(t) = \sum_{i=0}^{1} x^{(i)}(0)\frac{t^i}{i!}$$

$$= x(0)\frac{t^0}{0!} + x'(0)\frac{t^1}{1!}$$

$$= 0$$

$$x_1(t) = -\Omega^\alpha[x_0(t)] + \Omega^\alpha[f(t)]$$

$$= \Omega^\alpha\left[\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + t^3\right],$$

$$= \Omega^\alpha\left[\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}\right] + \Omega^\alpha[t^3],$$

by using the definition of Riemann-Liouville fractional integral operator (Ω^α) of order $\alpha \geq 0$, we obtain:

$$= \frac{2}{\Gamma(3-\alpha)}\frac{\Gamma(3-\alpha)}{\Gamma(3-\alpha+\alpha)}t^{\alpha+2-\alpha} + \frac{\Gamma(4)}{\Gamma(4+\alpha)}t^{3+\alpha},$$

$$= t^2 + \frac{\Gamma(4)}{\Gamma(4+\alpha)}t^{3+\alpha},$$

$$x_2(t) = -\Omega^\alpha[x_1(t)],$$

$$= -\Omega^\alpha\left[t^2 + \frac{\Gamma(4)}{\Gamma(4+\alpha)}t^{3+\alpha}\right],$$

$$= -\Omega^\alpha[t^2] - \Omega^\alpha\left[\frac{\Gamma(4)}{\Gamma(4+\alpha)}t^{3+\alpha}\right],$$

$$= -\frac{\Gamma(3)}{\Gamma(3+\alpha)}t^{2+\alpha} - \frac{\Gamma(4)}{\Gamma(4+\alpha)}\frac{\Gamma(4+\alpha)}{\Gamma(4+\alpha+\alpha)}t^{3+\alpha+\alpha},$$

$$= -\frac{2}{\Gamma(3+\alpha)}t^{2+\alpha} - \frac{6}{\Gamma(4+2\alpha)}t^{3+2\alpha},$$

$$x_3(t) = -\Omega^\alpha[x_2(t)],$$

$$= -\Omega^\alpha\left[-\frac{2}{\Gamma(3+\alpha)}t^{2+\alpha} - \frac{6}{\Gamma(4+2\alpha)}t^{3+2\alpha}\right],$$

$$= -\Omega^\alpha\left[-\frac{2}{\Gamma(3+\alpha)}t^{2+\alpha}\right] - \Omega^\alpha\left[-\frac{6}{\Gamma(4+2\alpha)}t^{3+2\alpha}\right],$$

$$= \frac{2}{\Gamma(3+\alpha)} \frac{\Gamma(3+\alpha)}{\Gamma(3+2\alpha)} t^{2+2\alpha} + \frac{6}{\Gamma(4+2\alpha)} \frac{\Gamma(4+2\alpha)}{\Gamma(3+3\alpha)} t^{3+3\alpha},$$

$$= \frac{2}{\Gamma(3+2\alpha)} t^{2+2\alpha} + \frac{6}{\Gamma(3+3\alpha)} t^{3+3\alpha}.$$

So the general solution of equation (4.1) is:

$$x(t) = x_0(t) + x_1(t) + x_2(t) + x_3(t) + \cdots. \tag{4.12}$$

$$= t^2 + \frac{\Gamma(4)}{\Gamma(4+\alpha)} t^{3+\alpha} - \frac{2}{\Gamma(3+\alpha)} t^{2+\alpha} - \frac{6}{\Gamma(4+2\alpha)} t^{3+2\alpha} + \cdots. \tag{4.13}$$

When $\alpha = 1.9$

$$x(t) = t^2 + \frac{6}{\Gamma(5.9)} t^{4.9} - \frac{2}{\Gamma(4.9)} t^{3.9} - \frac{6}{\Gamma(7.8)} t^{6.8} + \cdots. \tag{4.14}$$

$$= t^2 + 0.059247439 t^{4.9} - 0.096770806 t^{3.9}$$
$$- 0.001776766299 t^{6.8} + \cdots$$

$$= t^2 - small\ terms$$

$$\approx t^2.$$

4.3 The Numerical Realization of Equation (4.1) Using Variational Iteration Method

The following algorithm is applied to find an approximate solution of equation (4.1) using the variational iteration method.

If we take $\alpha = 1.9$ and $y(t) = J_0^{2-\alpha}\big(x(t)\big)$, then equation (4.1) can be written as the form:

$$y''(t) + D_0^{0.1}y(t) = \frac{2}{\Gamma(1.1)}t^{0.1} + t^3 \qquad (4.15)$$

Algorithm (4.1)

This algorithm can be explained as follows:

let we start with $y_0(t) = 0$

1. Calculate: 1) y_n'' with respect to x

2) $D^{0.1}y_n(x)$ by using definition (3.1)

3) $y_n(t)$ $, n = 1,2,3, \dots$ by using this recurance relation

$$y_{n+1}(t) = y_n(t) + \int_0^t (x-t)\left[y_n''(x) + D_x^{0.1}y_n(x) - \frac{2}{\Gamma(1.1)}t^{0.1} - t^3\right]dx \quad (4.16)$$

For more details see [20].

We will use algorithm (4.1) to solve the numerical example (4.1).

By using the above recurance relation (4.16), we obtain $y_n(t)$ $, n = 1,2,3, \dots$ by using the MAPLE package as follows:

$y_1(t) = 0.08333333333t^4 + 0.9100753299t^{2.1}$

$y_2(t) = 0.9100753299t^{2.1} - 0.003347313231t^{5.9}$

$y_3(t) = 0.9100753299t^{2.1} + 0.0007593018360t^{7.8}$

$$y_4(t) = 0.9100753299t^{2.1} - 0.000001111207366t^{9.7}$$

$$\vdots$$

$$y_n(t) = 0.9100753299t^{2.1} - small\ terms.$$

The sequence is convergent to $y(t) = \lim_{n \to \infty} y_n(t)$

$$y(t) = 0.9100753299t^{2.1}.$$

4.4 The Numerical Realization of Equation (4.1) Using Matrix Approach Method

The following algorithm is applied to find an approximation solution of equation (4.1) using the matrix approach method.

Algorithm (4.2)

1. Input: 1) a, b: $[a, b]$ is the interval for the solution function.

2) N: The number of subdivisions of $[a, b]$.

3) Alpha: The order of differential equation.

4) x intial: $x_0 = x_1 = 0$.

5) f: The function on the right hand side of the inhomogeneous fractional differential equation.

6) Exact solution $g(t)$

7) Set $x_0 = a$ and $x_N = b$.

2. Calculate: 1) $h = \frac{b-a}{N}$

2) $t_{\boldsymbol{k}} = k * h$

3) $f(t_{\boldsymbol{k}})$

3. Caculate: 1) $h^{-\alpha}$

2) $w_i^{(\alpha)} = (-1)^{\boldsymbol{i}}\binom{\alpha}{i}$

3) The matrix A_N^α

4) Adding identity matrix in the same size to A_N^α

5) The inverse of the resulting matrix

4. Calculate: Solving the linear system

$$x_n(t) = \quad (h^{-\alpha}A_N^\alpha + I)^{-1} \times f$$

For more details see [8].

This linear system has a dimension $N \times 1$. For equation (4.1) the dimension of the system is 51×1.

We use the algorithm (4.2) to solve equation (4.1) . Table (4.1) displays the exact and the numerical results using the Matrix approach method of equation (4.1) when $\alpha = 1.9$ and the resulted error.

Table (4.1): The exact and numerical solutions using the Matrix approach method where N=51.

t_k	**Exact solution** $x(t) = t^2$	**Approximation solution** $x_n(t)$	**Error** $= \lvert x(t) - x_n(t) \rvert$
0.0	0	0	0
0.1	0.1000	0.008621129726629	0.001378870273371
0.2	0.0400	0.037689907370590	0.00231009269410
0.3	0.0900	0.086539405801948	0.003460594198052
0.4	0.1600	0.154740697067752	0.005259302932248
0.5	0.2500	0.241931774005283	0.008068225994717
0.6	0.3600	0.347860012395293	0.012139987604707
0.7	0.4900	0.472440448226374	0.01755955177362
0.8	0.6400	0.615814946797499	0.024185053202501
0.9	0.8100	0.77840884576597	0.031591154203403
1	1.0000	0.960983804641099	0.039016195358901

It can be observed that the maximum error is 0.039016195358901.

The exact and approximate results of equation (4.1) are shown in Fig.4.2 (a) and the resulted error is shown in Fig 4.2 (b).

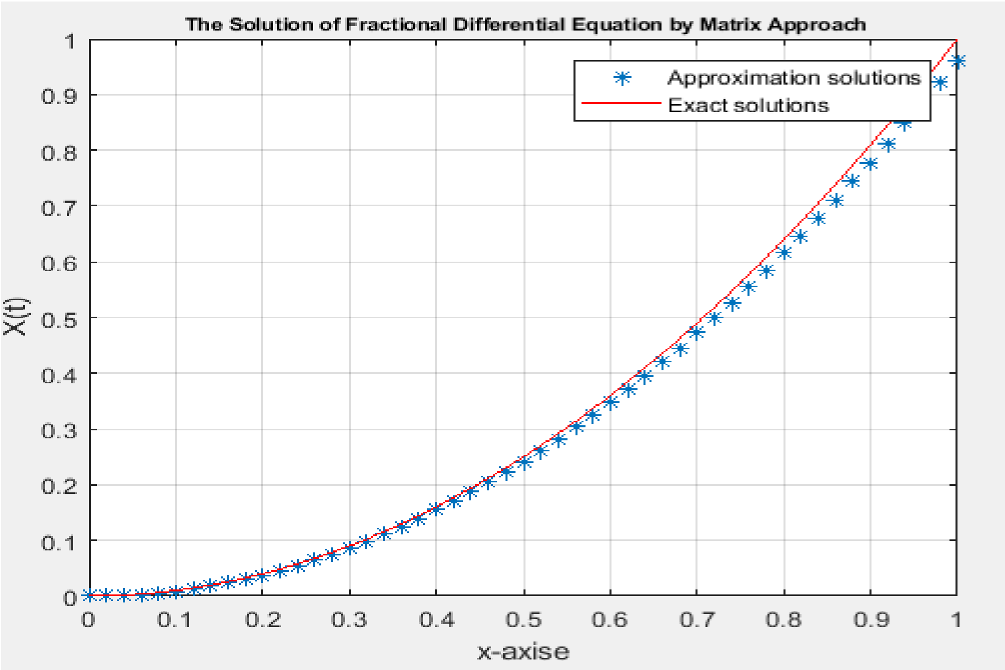

Fig. 4.1 (a) A comparison between the exact and approximate solution in example 4.1

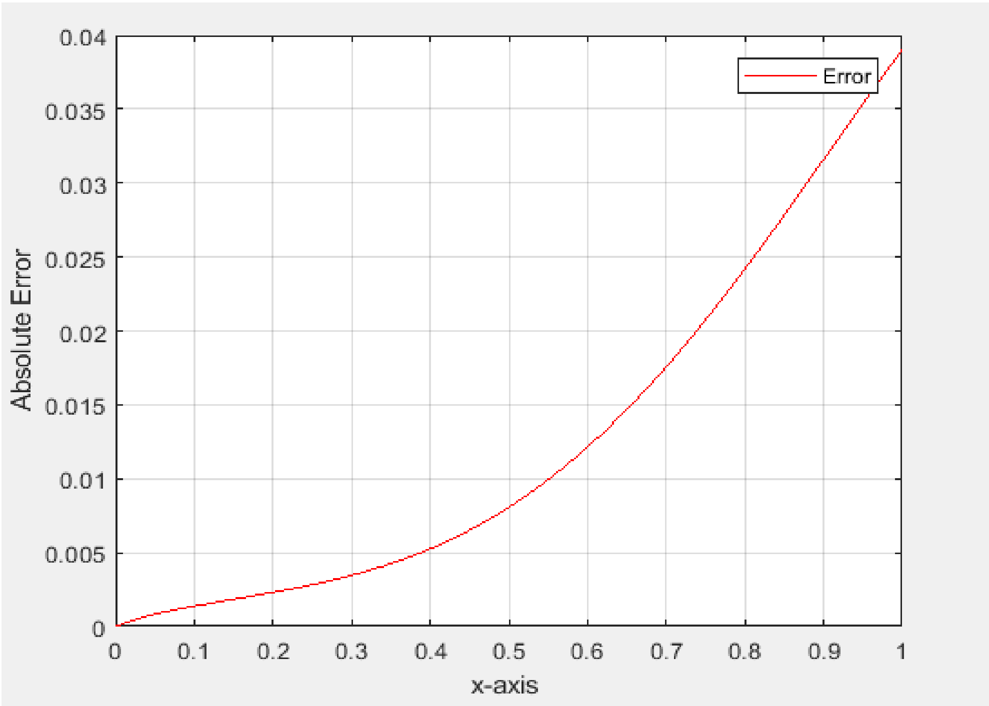

Fig. 4.1 (b) Absolute error between exact and numerical solution in example 4.1

Example 4.2

Consider the linear fractional differential equation:

$$D^{\alpha}x(t) + x(t) = 1, \quad \alpha \in (1,2) \tag{4.17}$$

with initial conditions:

$$x(0) = 0\,, x'(0) = 0.$$

The exact solution of example (4.2)[23] is:

$$\mathrm{x(t)} = t^{1.1}E_{1.1,2.1}(-t^{1.1})$$

4.5 The Numerical Realization of Equation (4.17) Using Adomian Decomposition Method

Using lemma (3.1) and take $D^{-\alpha}$ to both sides of equation (4.17)

$$x(t) + D^{-\alpha}x(t) = D^{-\alpha}f(t) + \sum_{i=0}^{1} x^{(i)}(0)\frac{t^i}{\Gamma(i+1)}. \tag{4.18}$$

Using Adomian decomposition method, we get:

$$x_0(t) = D^{-\alpha}f(t) + \sum_{i=0}^{1} x^{(i)}(0)\frac{t^i}{\Gamma(i+1)}$$

$$x_1(t) = -D^{-\alpha}x_0(t),$$

$$x_2(t) = -D^{-\alpha}x_1(t) = (-1)^2D^{-2\alpha}x_0(t),$$

$$\vdots$$

$$x_s(t) = -D^{-\alpha}x_{s-1}(t) = (-1)^s D^{-s\alpha}x_0(t), \tag{4.19}$$

$$\vdots$$

Then the solution of equation (4.17) obtained by adding all of the above terms

as follows:

$$x(t) = \sum_{s=0}^{\infty} x_s(t),$$

$$= \sum_{s=0}^{\infty} (-1)^s D^{-s\alpha} x_0(t),$$

$$= \sum_{s=0}^{\infty} (-1)^s D^{(-s-1)\alpha} f(t) + \sum_{s=0}^{\infty} (-1)^s D^{-s\alpha} \sum_{i=0}^{1} x^{(i)}(0) \frac{t^i}{\Gamma(i+1)},$$

$$= \sum_{s=0}^{\infty} (-1)^s \frac{1}{\Gamma((s+1)\alpha)} \int_0^t (t-\tau)^{(s+1)\alpha-1} f(\tau) d\tau$$

$$+ \sum_{i=0}^{1} x^{(i)}(0) \sum_{s=0}^{\infty} (-1)^s \frac{t^{i+s\alpha}}{\Gamma(i+s\alpha+1)}, \tag{4.20}$$

$$= \int_0^t \sum_{s=0}^{\infty} (t-\tau)^{\alpha-1} \frac{[-(t-\tau)^{\alpha}]^s}{\Gamma(s\alpha+\alpha)} f(\tau) d\tau$$

$$+ \sum_{i=0}^{1} x^{(i)}(0) D^{(\alpha-i-1)} \left[\sum_{s=0}^{\infty} t^{\alpha-1} \frac{(-t^{\alpha})^s}{\Gamma(s\alpha+\alpha)} \right],$$

$$= \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[-(t-\tau)^{\alpha}] f(\tau) d\tau$$

$$+\sum_{i=0}^{1} x^{(i)}(0) D^{(\alpha-i-1)}\left[t^{\alpha-1} E_{\alpha,\alpha}(-t^{\alpha})\right],$$

$$= \int_{0}^{t} G_1(t-\tau) f(\tau) d\tau + \sum_{i=0}^{1} x^{(i)}(0) D^{(\alpha-i-1)} G_1(t),$$

where $G_1(t) = t^{\alpha-1} E_{\alpha,\alpha}(-t^{\alpha})$

so, $x(t) = \int_0^t G_1(t-\tau) f(\tau) d\tau + x^{(0)}(0) D^{(\alpha-1)} G_1(t) + x^{(1)}(0) D^{\alpha} G_1(t),$ (4.21)

by equation (3.21), we obtain:

$$E_{\alpha,\alpha}(-t^{\alpha}) = \sum_{i=0}^{\infty} \frac{(-t^{\alpha})^i}{\Gamma(\alpha i + \alpha)},$$

So,

$$G_1(t) = t^{\alpha-1} \sum_{i=0}^{\infty} \frac{(-t^{\alpha})^i}{\Gamma(\alpha i + \alpha)},$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i t^{\alpha i + \alpha - 1}}{\Gamma(\alpha i + \alpha)},$$

then the solution of equation (4.17) in general form is:

$$x(t) = \int_{0}^{t} \sum_{i=0}^{\infty} (-1)^i \frac{(t-\tau)^{\alpha i + \alpha - 1}}{\Gamma(\alpha i + \alpha)} . 1 d\tau \qquad (4.22)$$

$$= \sum_{i=0}^{\infty} (-1)^i \int_{0}^{t} \frac{(t-\tau)^{\alpha i + \alpha - 1}}{\Gamma(\alpha i + \alpha)} . 1 d\tau$$

$$= \sum_{i=0}^{\infty} (-1)^i \left[D^{-(\alpha i+\alpha)} 1\right].$$

By definition (3.1), we get:

$$x(t) = \sum_{i=0}^{\infty} (-1)^i \left[\frac{\Gamma(1)}{\Gamma(1+\alpha i+\alpha)} t^{\alpha i+\alpha}\right],$$

when $\alpha = 1.1$

$$x(t) = \frac{1}{\Gamma(2.1)} t^{1.1} - \frac{1}{\Gamma(3.1)} t^{2.2} + \frac{1}{\Gamma(4.3)} t^{3.3} - \frac{1}{\Gamma(5.4)} t^{4.4} + \cdots$$

$$= \frac{1}{0.95135} t^{1.1} - \frac{1}{2.42397} t^{2.2} + \frac{1}{8.85534} t^{3.3} - \frac{1}{44.59885} t^{4.4} + \cdots$$

$$= 0.95557 t^{1.1} - 0.41255\, t^{2.2} + 0.11293\, t^{3.3} - 0.02242\, t^{4.4} + \cdots.$$

4.6 The Numerical Realization of Equation (4.17) Using Homotopy Perturbation Method

According to equation (3.36), we can write the following homotopy:

$$D^{\alpha} x(t) + p x(t) - 1 = 0,$$

the solution of equation (4.2) is in the following form:

$$x(t) = x_0(t) + p x_1(t) + p^2 x_2(t) + p^3 x_3(t) + \cdots. \quad (4.23)$$

Substituting (4.23) into the above equation, and gathering terms with the same power of p, then we get:

$$p^0 : D^{\alpha} x_0(t) = 0, \quad (4.24)$$

$$p^1: D^\alpha x_1(t) = -x_0(t) + f(t), \tag{4.25}$$

$$p^2: D^\alpha x_2(t) = -x_1(t), \tag{4.26}$$

$$p^3: D^\alpha x_3(t) = -x_2(t), \tag{4.27}$$

$\vdots$

Applying Ω^α, the inverse operator of D^α, on both sides of the equations from (4.24)-(4.25), then we get:

$$x_0(t) = \sum_{i=0}^{1} x^{(i)}(0)\frac{t^i}{i!}$$

$$= x(0)\frac{t^0}{0!} + x'(0)\frac{t^1}{1!}$$

$$= 0$$

$$x_1(t) = -\Omega^\alpha[x_0(t)] + \Omega^\alpha[f(t)]$$

$$= \Omega^\alpha[1],$$

by using the definition of Riemann-Liouville fractional integral operator (Ω^α) of order $\alpha \geq 0$, we obtain:

$$= \frac{1}{\Gamma(1+\alpha)}t^\alpha,$$

$$x_2(t) = -\Omega^\alpha[x_1(t)],$$

$$= -\Omega^\alpha\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right],$$

$$= -\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$x_3(t) = -\Omega^{\alpha}[x_2(t)],$$

$$= -\Omega^{\alpha}\left[-\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right],$$

$$= \frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$\vdots$$

therefore, the solution of equation (4.17) in general form can be written as:

$$x(t) = x_0(t) + x_1(t) + x_2(t) + x_3(t) + \cdots. \tag{4.27}$$

$$= \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots. \tag{4.28}$$

$$= \sum_{i=1}^{\infty}(-1)^{i+1}\frac{t^{\alpha i}}{\Gamma(\alpha i+1)} \tag{4.29}$$

When $\alpha = 1.1$

$$x(t) = \frac{t^{1.1}}{\Gamma(2.1)} - \frac{t^{2.2}}{\Gamma(3.2)} - \frac{t^{3.3}}{\Gamma(4.3)} - \frac{t^{4.4}}{\Gamma(5.4)} + \cdots. \tag{4.30}$$

$$= \frac{t^{1.1}}{0.95135} - \frac{t^{2.2}}{2.42397} + \frac{t^{3.3}}{8.85534} - \frac{t^{4.4}}{44.59885} + \cdots.$$

$$= 0.95557\, t^{1.1} - 0.41255\, t^{2.2} + 0.11293\, t^{3.3} - 0.02242\, t^{4.4} + \cdots$$

4.7 The Numerical Realization of Equation (4.17) Using Variational Iteration Method

The following algorithm is applied to find an approximate solution of equation (4.17) by using the variational iteration method.

If we take $\alpha = 1.1$ and $y(t) = J_0^{2-\alpha}\big(x(t)\big)$, then equation (4.17) can be written as the form:

$$y''(t) + D_0^{0.9} y(t) = 1 \qquad (4.31)$$

Using algorithm (4.1) to solve the numerical example (4.2), we obtain $y_n(t)$, $n = 1,2,3, \ldots$ by using the MAPLE package as follows:

$$y_1(t) = \frac{1}{2}t^2$$

$$y_2(t) = \frac{1}{2}t^2 - \frac{1}{6}t^3$$

$$y_3(t) = \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4$$

$$y_4(t) = \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{120}t^5$$

$\vdots$

$$y_n(t) = \sum_{i=2}^{n} \frac{t^i}{i!}$$

The sequence is convergent to $y(t) = \lim_{n\to\infty} y_n(t)$

$$y(t) = e^{-t} - 1 + t.$$

4.8 The Numerical Realization of Equation (4.17) Using Matrix Approach Method

Using algorithm (4.2) to solve equation (4.2), we obtain Table (4.2) which displays the exact and the numerical results using the Matrix approach method when $\alpha = 1.1$ and the resulted error.

Table (4.2): The exact and numerical solutions using the Matrix approach method where N=51.

t_k	Exact solution $= \sum_{i=}^{4} \frac{(-1)^{i+1} t^{1.1i}}{\Gamma(1.1i+1)}$	Approximation solution $x_n(t)$	Error $= \lvert x(t) - x_n(t) \rvert$
0.0	0	0	0
0.1	0.073357053781371	0.058092499960337	0.015264553821034
0.2	0.151282884629052	0.135430820233760	0.015852064395293
0.3	0.226984580680193	0.211095760530730	0.015888820149463
0.4	0.29890238480688	0.283237827667134	0.015664557141554
0.5	0.366411147911488	0. 351151361983237	0.015259785928251
0.6	0.429259300754372	0.414572350125656	0.014686950628716
0.7	0.487372840288318	0.473456997040934	0.013915843247384
0.8	0.540762298480572	0.52788452424550	0.012877774238022
0.9	0.589469952960841	0.578006370352886	0.011463582607954
1	0.633536032460000	0.624016649518593	0.009519382941407

It can be observed that the maximum error is 0.015888820149463 .

The exact and approximate results of equation (4.17) are shown in Fig.4.4 (a) and resulted error is shown in Fig 4.4 (b).

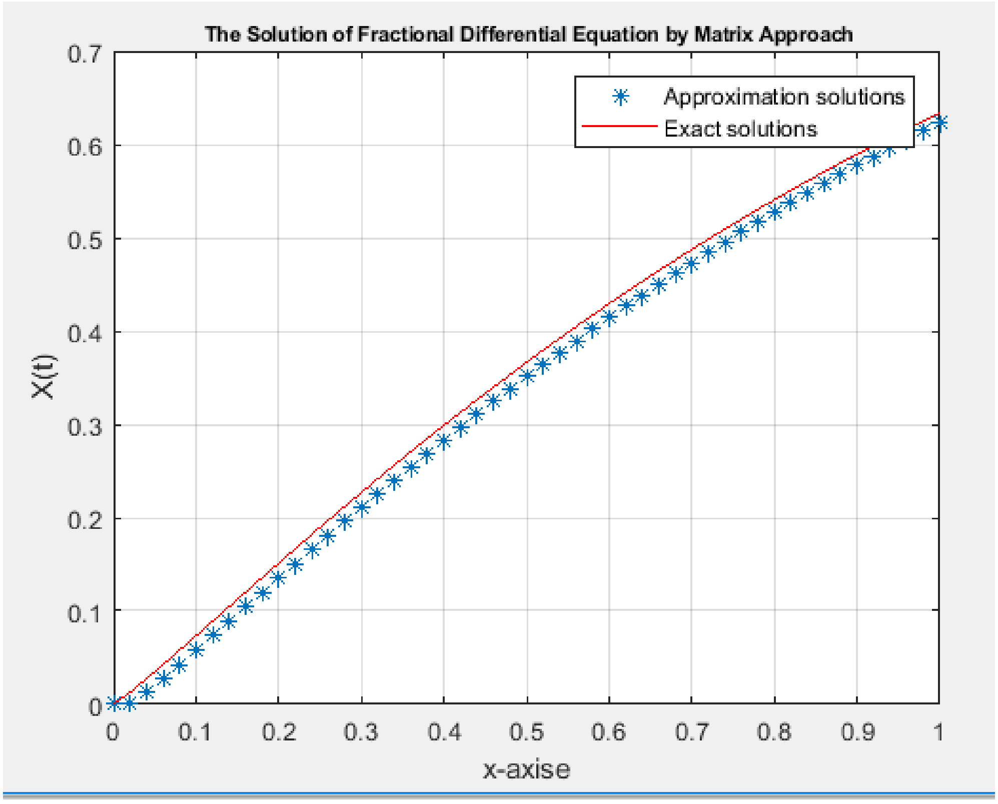

Fig. 4.2 (a) A comparison between the exact and approximate solution in example 4.2

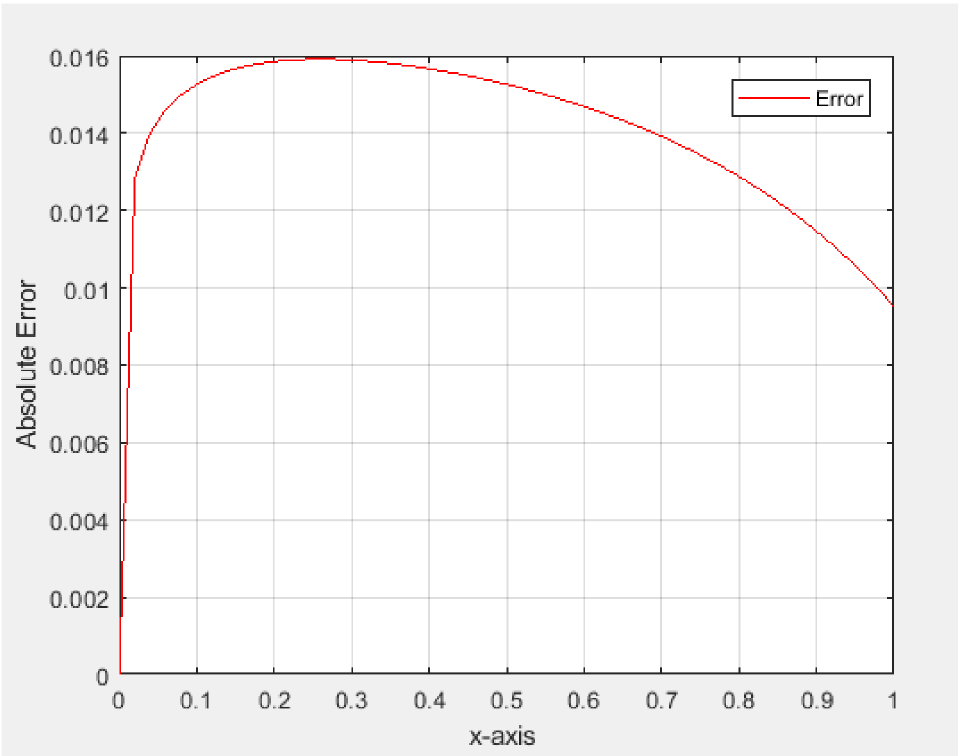

Fig. 4.2 (b) Absolute error between exact and numerical solution in example 4.2

Conclusions

Fractional differential equations are used frequently in various fields of physics and engineering.

In this thesis we have solved linear fractional differential equations using various analytical and numerical techniques, namely; The Adomian decomposition method, the Variational iteration method, the Homotopy perturbation method and Matrix approach method.

The numerical methods were implemented in a form of algorithms to solve some numerical test cases using Matlab and Maple softwares.

Numerical results have shown to be in a close agreement with the analytical ones. Moreover, the numerical results for the proposed examples show clearly that the matrix approach method is more efficient than its counterparts.

References

[1] G. Adomian, **A Review of The Decomposition Method in Applied Mathmatics, Journal of Mathematical Analysis and Application**, Vol.135,no.2, pp. 501-544, 1988.

[2] G. Adomian, Solving Frontier Problems of Physics: **The Decomposition Methd, vol. 60 of Fundamental Theories of Physics**, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994.

[3] G. Arfken **Hermitian Matrices, Unitary Matrices.**4.5 in Mathmatical Methods for Physicists, 3^{rd}ed. Orlando, FL: Academic Press, PP. 209-217, 1985.

[4] F. Jr Ayres Shaum`s Outline of **Theory and Problems of Matrices**. NewYork: Schaum, PP.13 and 117-118, 1962.

[5] S. Bhalekar and V. Daftardar, **Solving Fractional – Order Logistic Equation Using New Iterative Method**, Journal of Applied Mathematics, Vol. 2012, 25 – 37, 2012.

[6] C. Boyer, **A History of Mathematics**, John Wiley and Sons, 1991.

[7] P.L.Butzer, V.Westphal, **An Introduction to Fractional Calculus**, World Sientistic, Singapor, 2000.

[8] R. F. Cameron and S. Mckee , **The Analysis of Product Integration Methods for Abel`s Equation Using Discrete Fractional Differentiation**. I M A Journal of Numerical Analysis 5(1985), 339-353.

[9] M. **Caputo Linear Model of Dissipation Whose Q is Almost Frequancy Independent- II**, Geophys . J. R. Astr. Soc., vol. 13, 1967.

[10] R. Courant. D. Hilber, " **Methods of Mathematical Physics. Partial differential equation**", 2, Interscience (1965).

[11] L. J. Derr, **Triangular Matrices with The Isoelnial Property. Pacific Journal of Mathematics** 37, No1 (1971), 41- 43.

[12] GE. Draganescu , **Application of a Variational Method to Linear and Non Linear Viscoelastic Models with Fractional Derivatives**, Journal of Mathmematical Physics, 47 (8) (2006): Art. No. 082902.

[13] P. L. Duren, **Univalent Functions**, Springer – Verlag, NewYork, Berlin, Heidelberg, Tokyo, 1983.

[14] R. Gorenflo, F. Mainardi and I. Podlubny, **Introduction to Fractional Calculus**, R. Vilela Mendes, Joly 2008.

[15] Y. Guo, **Local Fractional Transform Space, Advances in Digital Multimeding**, Vol. 1, No. 2.2012

[16] I. Hashim, O. Abdulaziz and S. Momani, **Homotopy Analysis Method for Fractional IVPs**, Commun. Non- linear Sci. Numer.Simul.,14 (2009), 674-684.

[17] J.- H. He**, A Coupling Method of a Homotopy Technique and a Perturbation Technique for Non – Linear Problems**, Int. J. Nonlinear Mech. 35 (2000) 37 - 43.

[18] J.- H. He, **Homotopy Perturbation Technique,** Comput. Meth. Appl. Mech. Eng. 178 (1999) 257 – 262.

[19] R. Hilfer (editor); **Applications of Fractional Calculus in Physics**, world scientific Publishing Co. 2000.

[20] Y. Hu, Y. Luo, and Z. Lu, **Analytical Solution of The Linear Fractional Differential Equation by Adomian Decomposition Method**, Journal of Computational and Applied Mathematics, Vol. 215, no.1, pp. 220 – 229, 2008.

[21] M. Ishteva, **Properties and Applications of the Caputo Fractional Operator, Department of Mathematics**, Universitat Karlsruhe (TH), 2005.

[22] M. Joseph Kimeu**, Fractional Calculus: Definitions and Applications, Western Kentucky University**, 2009.

[23] P. Kumar, O. P. Agrawal, **An Approximate Method for Numerical Solution of Fractional Differential Equations**, Signal Process. 86 (2006) 2602 – 2610.

[24] C. Lederman, J. M. Roquejoffre and N. Wolanski, **Mathematical Justifcation of a Nonlinear Integrodifferential Equation for The Propagation of Spherical**, Ann. Mat. Pura Appl., 183 (2004), 173-239.

[25] G. Wu and E. W. M. Lee**, Fractional Variational Iteration Method and Its Application,** Phys. Lett, 374 (2010), 2506 – 2509.

[26] T. Lee, D. Oehme, R. And Yang, C. N. **Remarks on Possible Noninvariance Under Time Reversal and Charge Conjugation Phys**. Rev. 106, 340 – 345. 1957.

[27] R. J. Libera, **Some Classes of Regular Univalent Funations**, Proc. Amer. Math. Soc. 16 (1965), 755- 758.

[28] F. Mainardi, **Fractional Relaxation – Oscillation and Fractional Diffusion – Wave Phonemena**, Chaos, Solitons and Fractals, vol. 7, 1996, pp. 1461 – 1477.

[29] F. Mainardi, Fractional Calculus: **Some Basic Problems in Continum and Statistical Mechanics, in: Fractals and Fractional Calculus in Continm Mechanics** (Eds. A. Caroipnteri and F. Mainardi) Springers Verlag, Wien, (1997).

[30] F. C. Meral, T. J. Royston and R. Magin, **Fractional Calculus in Viscoelasticity: An Experimental Study, Commun. Nonlinear** Sci. Num. Simul., 15 (2010), 939-945.

[31] R. Metzler, W. G. Glockle, and T. F. Nonnenmacher, **Fractional Model Equation for Anomalous Diffusion, Physica** A, vol. 211, 1994, pp. 13 – 24.

[32] K. S. Miller and B. Ross, **An Introduction to The Fractional Calculus and Frational Differential Equations**, Jhon Wiley & Sons, INC 1993.

[33] R. Molliq and M. Noorani, **Solving the Fractional Rosenau – Hyman Equation via Homotopy Perturbation Method Journal of Applied Mathematics,** Vol. 2012, 1- 14, 2012.

[34] S. Momani, Z. Odibat**, Numerical Approach to Differential Equations of Fractional Order**, J. Comput. Appl. Math.207 (2007) 96-110.

[35] R. R. Nigmatullim, **To The Theoretical Explanation of The "Universal Response"**, Phys. Sta. Sol. (b), vol. 123, 1984, pp. 739 – 745.

[36] K. B. Oldham, **Fractional Differential Equations in Electrochemistry**, Adv. Eng. Soft., 41 (2010), 9-12.

[37] K. B. Oldham and J. Spanier, **The Factional Calculus**, Academic Press, NewYork – London, 1974.

[38] I. Podlubny, **Fractional Differential Equations**, Academic Press, NewYork, 1998.

[39] I. Podlubny. **Fractional Differential Equations**, Academic press, San Diego,1999.

[40] I. Podlubny, **Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Application**, vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, 1999.

[41] I. Podlubny, **An Over view of Fractional Order Calculus: History, Theory, Application**, Technical University of Kosice, Slovak Rebublic.

[42] I. Podlubny; **The Laplace Transform Method for Linear Differential Equations of the Fractional Order**, June 1994.

[43] M. Rahimy, **Applications of Fractional Differential Equations**, Applied Mathematical Sciences, Vol. 4, 2010, no. 50, 2543-244.

[44] G. S. *Sălăgean*, **Subclass of Univalent Functions,** Complex Analysis – Fifth Romanian – Finnish Seminar, Lecture Notes in Math. 2013, Springer – Verlag, Berlin (1983).

[45] S. G. Samko, A. A. Kilbas, and O. Imarichev, **Integrals and Derivatives of the Fractional Order and Some of Their Applications**. Nauka i Tekhnika, Minsk (1987) (In Russian, English transl.: Fractional Integrals and Derivatives, Gordon and Breach, Amsterdam (1993)).

[46] A. Stanislavsky. **Fractional Oscillator Phys**. **Rev**. E 70, 051103 – 051109. 2004.

[47] A. Stanislavsky. **Hamiltonian Formalism of Fractional Systems Europ**. **Phys**. J B 49, 93 – 101. (2006b)

[48] M. S. Tarazoei**, A Note on Fractional – Order Derivatives of Periodic Functions**, Automatic, 46. 945 – 948. (2010)

[49] M. S. Tarazoei, M. Haeri**, A Proof for Non Existance of Periodic Solutions in Time Invariant Fractional Order Systems**, Automatica, 45 (2009) 1886 – 1890.

[50] J. Troutman, **Variational Calculus with Elementary Convexity, Springer –Verlag**, 1980.

[51] X. J. Yang, **A Short Note on Local Fractional Calculus of Function of One Variable, Journal of Applied Library and Information Science (JALIS)**, Vol. 1, No. 1. 2012.

[52] K. Yonggang, and Z. Xiu`e, **Some Comparison of Two Fractional Oscillators Physica** B 405, 369 – 373. 2010.

Appendix

Matlab code for the Matrix approach Method for solving example 3.1

```
clc;
clear all ;
syms t

gama=0.95135;
alpha=1.9;
a=0; % begin of interval
b=1; % end of interval
N=100;
p1=1;
p2=1;
X_initial = [0 0]';
f=(2/gama)*t.^(2-alpha)+t.^3;
exact_f = t.^2;




h=(b-a)/N;
K=0:1:N;
tk=h.*K
```

```
format long
f=(eval(subs(f,t,tk)));





Bn=eye(N-2);
w=zeros(1,N-1);
for j=0:N-2
w(j+1) = (-1)^j * get_combination(alpha,j);
end

for c =1:N-1
    for r=1:N-1

     if r-c==0
         Bn(r,r)=w(1);
     elseif r>c
      Bn(r,c)=w(r-c+1);
     end
     end
end
 F=f(3:length(f));
 X=inv(h^(-1.9)*Bn+eye(N-1))*F';
```

```
 X_total = [X_initial;X]

exact_f= eval(subs(exact_f,t,tk))'

e=X_total-exact_f
abs_error = abs(e);
max_abs_error = max(abs_error)
%plot
m=[X_total',exact_f',abs_error']
plot(tk,X_total,'*',tk,exact_f,'r')
xlabel('x-axise')
ylabel('X(t)')
title('\fontname{Artial} The Solution of
Fractional Differential Equation by Matrix
Approach','FontSize',8)
legend('Approximation solutions','Exact
solutions')
grid on
plot(tk,abs_error,'r')
xlabel('x-axis ')
ylabel('Absolute Error')
legend('Error')
grid on
```

Matlab code for the Matrix approach Method for solving example 3.2

```
clc;
clear all ;
syms t

gama=0.95135;
alpha=1.1;
a=0; % begin of interval
b=1; % end of interval
N=100;
p1=1;
p2=1;
X_initial = [0 0]';
f=1;
exact_f =.9555790964*t^(11/10)-
.4125471292*t^(11/5)+.1129261690*t^(33/10)-
0.2242210374e-1*t^(22/5) ;

h=(b-a)/N;
K=0:1:N;
tk=h.*K;
format long
f=(eval(subs(f,t,tk)));
```

```
Bn=eye(N-2);
w=zeros(1,N-1);
for j=0:N-2
w(j+1) = (-1)^j * get_combination(alpha,j);
end

for c =1:N-1
    for r=1:N-1

     if r-c==0
         Bn(r,r)=w(1);
     elseif r>c
      Bn(r,c)=w(r-c+1);
     end
     end
end
 F=f(3:length(f));
 X=inv(h^(-1.1)*Bn+eye(N-1))*F';
 X_total = [X_initial;X]

exact_f= eval(subs(exact_f,t,tk))'

e=X_total-exact_f
abs_error = abs(e);
```

```
max_abs_error = max(abs_error)
%plot
m=[X_total',exact_f',abs_error']
plot(tk,X_total,'*',tk,exact_f,'r')
xlabel('x-axise')
ylabel('X(t)')
title('\fontname{Artial} The Solution of
Fractional Differential Equation by Matrix
Approach','FontSize',8)
legend('Approximation solutions','Exact
solutions')
grid on
plot(tk,abs_error,'r')
xlabel('x-axis ')
ylabel('Absolute Error')
legend('Error')
grid on
```

جامعة النجاح الوطنية

كلية الدراسات العليا

الحلول العددية لحل المعادلات التفاضلية الكسرية وتطبيقاتها

اعداد

ايه باسم احمد سعادة

اشراف

أ.د. ناجي قطناني

قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية، نابلس– فلسطين.

2018

الحلول العددية لحل المعادلات التفاضلية الكسرية وتطبيقاتها

اعداد

ايه باسم احمد سعادة

اشراف

أ.د.ناجي قطناني

الملخص

في هذه الرسالة نحن نركز على الحلول التحليلية والعددية لمعادلة تفاضلية كسرية نظرا لمداها الواسع في الهندسة، الفيزياء والتكنولوجيا مثل معادلة المذبذب التوافقي، معادلة الموجة الجزئية ومعادلة الانتشار الجزئي.

بعد أن تناولنا المفاهيم الأساسية في التكاملات الكسرية والمشتقات الكسرية، قمنا بالتركيز على الطرق العددية والتحليلية لحل معادلة تفاضلية كسرية، الطرق التحليلية هي: تعريف جرونوالد ،ريمان ليوفيل وكابوتو للمشتقة الكسرية، والطرق العددية هي: طريقة ادوميان التفككية، طريقة هوموتوبي الاضطرابية، طريقة التكرار المتغير وطريقة نهج المصفوفة الاضطرابية.

وللتحقق من كفاءة الطرق العددية قمنا بحل بعض الأمثلة العددية ورسم المقارنات بين هذه الطرق، حيث اظهرت النتائج العددية دقتها وقربها من النتائج التحليلية، وكانت طريقة نهج المصفوفة هي الأقوى والأدق في حل معادلة تفاضلية كسرية بالمقارنة مع الطرق العددية الأخرى.