An-Najah National University

Faculty of Graduate Studies

BEST APPROXIMATION AND BEST CO - APPROXIMATION IN CONE - NORMED SPACE

By

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This thesis was defended successfully on 18/05/2017 and approved by:

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Dedication

Every challenging work needs self-efforts as well as guidance of elders especially those who were very close to our heart.

My humble effort I dedicate to my sweet and loving husband, daughter, parents, sisters & brother. Whose affection, love, encouragement and prays of day and night make me able to get such success and honor.

Along with all hard working and respected Teachers especially Dr. Abdallah A. Hakawati.

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والشكر موصول إلى زوجي وابنتي ووالدي ووالدتي وأسرتي الذين لم يبخلوا علي بالجهد والرعاية لإعداد هذه الرسالة. كما وأتقدم بالشكر والتقدير إلى جميع أساتذتي وأصدقائي وزملائي في برنامج الرياضيات في كلية الدراسات العليا في جامعة النجاح الوطنية على الجهد الذي بذلوه من أجلي.

الاقرار

انا الموقعة ادناه مقدمة الرسالة التي تحمل العنوان:

BEST APPROXIMATION AND BEST CO - APPROXIMATION IN CONE - NORMED SPACE

أقر بأن ما اشتملت عليه هذه الرسالة إنما هي نتاج جهدي الخاص، باستثناء ما تمت الاشارة إليه حيثما ورد، وأن هذه الرسالة ككل، أو أي جزء منها لم يقدم من قبل لنيل أي درجة علمية أو بحث لدى أية مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's name: **اسم الطالبة :**

Signature: **التوقيع:**

Date: **التاريخ:**

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BEST APPROXIMATION
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Abstract

In this thesis, the concept of Best Approximation and Best Co-Approximation in cone – normed spaces are studied.

We noticed similarities between Best Approximation and Best Co–Approximation in cone–normed spaces.

We obtained new results among which, we proved that:

1. We can apply Best Approximation and Best Co-Approximation theorems in normed space to cone-normed space.
2. Best Co-Approximation in cone-normed space is a counter copy of Best Approximation in cone-normed space.

The concept of Proximinal additivity and ϕ_c-Summand in cone–normed space are introduced. Also we have answered some questions about them.

Introduction

The theory of Best Approximation is an important topic in functional analysis.

The meaning of Best Approximation for a given point x, and a given set G in a cone-normed space $(X, \| . \|_c)$, is the existence of a point g_0 in G which is closest to x among all points of G, that is:

$\| x - g_0\|_c \leq \| x - g \|_c \forall g \in G$.

The meaning of Best Co-Approximation for a given point x, and a given set G in a cone-normed space $(X, \| . \|_c)$ is a point g_0 in G which is nearest to g of all points of G from x, that is:

$\| g_0 - g \|_c \leq \| x - g \|_c \forall g \in G$.

In our study, for $x \in X$ we will denote by $P_{cG}(x)$ the set of all elements $g_0 \in G$, that are best approximation.

i.e $P_{cG}(x) = \{ g_0 \in G : \| x - g_0\|_c \leq \| x - g \|_c \forall g \in G\}$.

And $R_{cG}(x)$ for the set of all elements $g_0 \in G$ that are best co-proximation.

i.e $R_{cG}(x) =\{ g_0 \in G : \| g_0 - g \|_c \leq \| x - g \|_c \forall g \in G \}$.

If $P_{cG}(x) \neq \emptyset$ then G is called a proximinal set and if $P_{cG}(x)$ is a singelton then G is called Chebychev.

If $R_{cG}(x) \neq \emptyset$ then G is called a co-proximinal set and if $R_{cG}(x)$ is a singelton then G is called co- Chebychev set.

The problem of Best Approximation began in 1853 by P. L Chebychev, and the problem of Best Co-Approximation began in 1972 by Franchetti and Furi.

This thesis consists of four chapters; each chapter is divided into sections and contains: definitions, theorems, corollaries, lemmas and other results. At the end we list all references used in our work.

In chapter one, we introduced basic definitions and concepts which will be needed in the next chapters.

We defined metric, vector space, normed and Banach space, cone, cone metric space and cone-normed space. Despite other orthogonalities, we referred to Birkhoff orthogonality, which will be used in next chapters.

Finally, we basically answered two questions in this chapter:

Q1. Is every metric coming from a norm?

Q2. Is every cone-metric coming from a cone-normed?

Chapter two has two purposes: First, we studied Best Approximation in normed space and proximinal set. Second, we studied another kind of Best Approximation (Best Co-Approximation) in normed space and co-proximinal set.

In chapter three we answered the questions:

Q1. What do we mean by Best Approximation in cone -normed linear spaces?

Q2. When do we mean by G is a co-proximinal set in X?

Q3. What do we mean by proximinal additivity and by Φ_c-Summand in cone – Banach space?

Q4. Does G being (a ϕ_c-Summandof a Banach space X) imply that G is proximinal**,** Chebychev and proximinally additivite in X?

In chapter four, we defined the concept of Best Co-Approximation in cone-normed linear spaces and co-proximinal set, and we applied theorems in section 2.2 to them.

This thesis is wished to become a ground for more researches & studies about Best Approximation & Best Co-Approximation in cone-normed spaces.

Chapter One

Preliminaries

Chapter One

Preliminaries

This chapter contains some basic definitions, theorems and results about metric spaces, vector spaces, Normed and Banach spaces, cone, cone metric spaces and cone-normed spaces which we will use in the next chapter.

1.1 Metric space:

Definition 1.1.1 [1,p4]:

Let X be any nonempty set. A metric on X is a mapping $d : X \times X \to R$ which Satisfies the following axioms: For all $x , y , z \in X$:

i. $d(x, y) \geq 0$.

ii. $d(x, y) = 0$ if and only if $x = y$.

iii. $d(x, y) = d(y, x)$.

iv. $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X , d) is called a metric space.

Definition 1.1.2 [1,p4]: (Distances between sets and distances between points and sets)

Let (X , d) be a metric space and let A , B two be non-empty subsets of X:

1. The distance between a point $x \in X$, and the set A is defined as:

$d(x,A) = \inf \{d(x , y) : y \in A\}$

2. The distance between the sets A,B denoted by $d(A,B)$ is defined as:

$d(A,B) = \inf \{d(x, y) : x \in A, y \in B\}$

Definition 1.1.3 [13,p331]: (strictly convex)

A subset G of a linear space X is convex if, for all $g_1 , g_2 \in G$

$g_1 \neq g_2$, the points $\{ \beta g_1 + (1 -\beta) g_2 : 0 <\beta< 1 \}$ are interior points of G.

Theorem 1.1.4 [1,p6]:

Let G be a nonempty subspace of a metric space X. Then:

i. $d(x + y,G) = d(x,G)$ for every $x \in X$ and $y \in G$.

ii. $d(\alpha x,G) = |\alpha|\, d(x,G)$ for every $x \in X$ and $\alpha \in R$.

iii. $d(x + y,G) \leq d(x,G) + d(y,G)$ for every x and y in X.

1.2 Vector space:

Definition 1.2.1 [2,p1]:

Let K = R or C. A vector space over K, is a set X together with two functions, $+ : X \times X \to X$, called vector addition, and $\cdot : K \times X \to X$, called scalar multiplication that satisfy the following:

1. $\forall x_1, x_2, x_3 \in X$, $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$.
2. $\exists$ an element, denoted by 0 (called the zero vector) such that for all $x \in X$, $x + 0 = 0 + x = x$.
3. $\forall x \in X$, there exists an element, denoted by $-x$, such that $x + (-x) = (-x) + x = 0$.
4. $\forall x_1, x_2$ in X, $x_1 + x_2 = x_2 + x_1$.
5. $\forall x \in X$, $1 \cdot x = x$
6. $\forall x \in X$ and all $\alpha, \beta \in K$, $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$.
7. $\forall x \in X$ and all $\alpha, \beta \in K$, $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$.
8. $\forall x_1, x_2 \in X$ and all $\alpha \in K$, $\alpha \cdot (x_1 + x_2) = \alpha \cdot x_1 + \alpha \cdot x_2$.

1.3 Normed and Banach space:

Normed space:

Definition 1.3.1 [2,p3]:

Let X be a vector space over R or C. A norm on X is a function

$\| \cdot \| : X \rightarrow [0, +\infty)$ such that:

1. For all $x \in X$, $\|x\| \geq 0$. (Positive definiteness)
2. If $x \in X$, then $\|x\| = 0$ iff $x = 0$.
3. For all $\alpha \in R$ or C and for all $x \in X$, $\|\alpha x\| = |\alpha|\|x\|$.
4. For all $x, y \in X$, $\| x + y \| \leq \| x \| + \| y \|$. (Triangle inequality)

A normed space is a vector space equipped with a norm.

If $x , y \in X$ then the number $\| x - y \|$ is the distance between x and y.

Thus $\| x \| = \| x - 0 \|$ is the distance of x from the zero vector in X.

Theorem 1.3.2 [1,p6,7]:

We can define the distance function $d(x,y) = \| x - y \|$.

1. Every normed space is a metric space with respect to the metric

$d(x , y) = \| x - y \|$.

To see this:

$\forall\, x , y , z \in X$

a. $d(x , y) = 0$ iff $\| x - y \| = 0$ iff $x - y = 0$ iff $x = y$.

b. $d(x , y) = \| x - y \| = \| -(y - x) \| = |-1| \| y - x \|$

$\| y - x \| = d(y , x)$.

c. $d(x , y) = \| x - y \| = \| x - z + z - y \|$

$\| x - z + z - y \| \leq \| x - z \| + \| z - y \| = d(x , z) + d(z , y)$.

2. For all x, y in a normed space X, $| \| x \| - \| y \| | \leq \| x - y \|$.

Question: Is every metric coming from a norm?

Answer: No.

To prove, let X be a vector space over field K(R or C).

A norm $\| . \| : X \to K$.

Satisfies the condition:

$\| \alpha x \| = | \alpha | \| x \|$. $\forall \alpha \in K$ and $x \in X$. So the metric

$d : X \times X \longrightarrow K$ defined as:

$d(x , y) = \| x - y \|$ must satisfy:

$d(\alpha x , \alpha y) = \| \alpha x - \alpha y \| = | \alpha | \| x - y \| = | \alpha | d(x , y)$

$\forall \alpha \in K$ and $x , y \in X$. But this property is not satisfied by general metrics.

For example, let d be the discrete metric,

$d(x , y) = 1$ if $x \neq y$, $d(x , y) = 0$ if $x = y$.

We want to examine the conditions of a metric:

1. $d(x , y) = 0$ iff $x = y$.
2. $d(x , y) \geq 0$.
3. $d(x , y) = d (y , x)$.
4. $d(x, y) \leq d(x, z) + d(z, y)$.

So, this is a metric but there exists no norm which induces it.

(Because $\| \alpha x - \alpha y \| \neq | \alpha | \| x - y \|$. For example take $\alpha = 2$ then for $x \neq y$, $\| 2x - 2y \| = 1$ but $| 2 | \| x - y \| = 2$).

Then d is not coming from a norm.

Definition 1.3.3 [1,p15]: (Birkhoff orthogonality)

Two vectors x and y in a normed linear space are said to be Birkhoff orthogonal if $\| x \| \leq \| x + \alpha y \|$ for all scalar α.

Symbolically, $x \perp_B y$ if and only if $\| x \| \leq \| x + \alpha y \|$ for all scalar α.

Banach space:

In a normed space, we have the notation of distance between vectors so we can define convergent sequences and Cauchy sequences.

Definitions 1.3.4 [2, p7,8]:

1. Let $(x_n)_{n\in N}$ be a sequence in X and let $x \in X$. The sequence $(x_n)_{n\in N}$ converge to x if $\forall\, \epsilon > 0$, $\exists\, M \in N$ such that for all $n \in N$ satisfying $n \geq M$, $\| x_n - x \| < \epsilon$.

This means that:

(a sequence (x_n) in X is said to be convergent if and only if there exists an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$).

2. The sequence $(x_n)_{n\in N}$ is called a Cauchy sequence if $\forall\, \epsilon > 0$, $\exists\, M \in N$ such that for all $m, n \in N$ satisfying $m, n \geq M$, $\|x_m - x_n\| < \epsilon$

3. Every convergent sequence is a Cauchy sequence, since $\forall\, \epsilon > 0$, $\|x_m - x_n\| \leq \|x_m - x\| + \|x - x_n\|$.

$< \epsilon/2 + \epsilon/2 = \epsilon$ for sufficiently large m and n.

4. A normed space $(X, \|\cdot\|)$ is called complete if every Cauchy sequence is convergent.

Definition 1.3.6 [2 ,p8]:

A Banach space (over R or C) is a complete normed vector space.

In other words:

A normed space $(X , \|.\|)$ is called a Banach space if every Cauchy sequence in X is convergent in X.

1.4 Cone:

Definition 1.4.1 [3,p1468]:

Let E be a real Banach space with norm $\|.\|$ and Ᵽ a subset of E. Ᵽ is called a cone in E if:

1. P is closed, nonempty, and $P \neq \{0\}$.
2. If $a , b \in R$, $a , b \geq 0$, $x , y \in P$ then $ax + by \in P$.
3. If $x \in P$ and $-x \in P$ then $x = 0$.

Example 1.4.2:

Let $E = R^2$ and $P = \{ (w , z) : w \geq 0 , z \geq 0 \}$ The set P is a cone in E.

Definition 1.4.3 [3,p 1469]:

Let P be a cone in a real Banach space E. We define:

1. A partial ordering $\leq$ with respect to P on E by $x \leq y$ iff $y - x \in P$.
2. $x < y$ if $x \leq y$ but $x \neq y$.
3. $x \ll y$ if $y - x \in P^0$ (this is the interior of P).

($x \ll y$ is pronounced: x is way behind y)

Type of cones:

Definition 1.4.4 [3,p1469]:

The cone P is called normal if $\exists\ k > 0$ s .t $k \in R$.

$\forall x , y \in E$. $0 \leq x \leq y$ then $\|x\| \leq k\, \|y\|$.

The smallest k is called the normal constant of P.

Remark 1.4.5 [7,p68]:

The normal constant of a normal cone P must be ≥ 1.

Proof:

Suppose that E is a Banach space and P is normal cone with $K < 1$

Take $x \in P$, $x \neq 0$ and a real number ε s .t $0 < \varepsilon < 1$, where $K < (1-\varepsilon)$.

Then, $(1-\varepsilon)\, x \leq x$.

but (1-ε) $\| x \| > K\| x \|$, which is a contradiction with the definition of the cone normal constant.

Example 1.4.6 [7,p68]:

Let E = $C^1[0,1]$ with norm $\| f \| = \| f \|_\infty + \| f' \|_\infty$ ($C^1[0,1]$ as a set is the space of all real – valued function on [0,1] with continues derivatives).

Where $\| f \|_\infty = \max \{f(x): x\in [0,1] \}$ and $\| f` \|_\infty = \max\{f`(x): x \in [0, 1]\}$, and consider the cone P = { f ∈ E : f ≥ 0}. This cone is non – normal.

To see this:

For each k ≥ 1, put f(x) =x and g (x) = x^{2k}.

Then 0 ≤ g ≤ f ,

$\| f \| = \max \{x: x \in [0, 1]\} + \max \{ 1: x\in [0,1] \}$

$= 1 + 1 = 2.$

$\| g \| = \max \{x^{2k}: x\in [0,1]\} + \max \{ 2k\, x^{2k-1}: x \in [0,1] \}$

$= 1 + 2k$

$\| f \| = 2$, and $\| g \| = 2k + 1$. Since $k \| f \| \le \| g \|$, k is not a normal constant of P and hence P is a non - normal cone.

Definition 1.4.7 [3,p1469]:

The cone P is called regular if every increasing sequence in E, which is bounded above in E is convergent.

In other words:

If $\{x_n\}$ is sequence s.t $x_1 \le x_2 \le x_3 \le \ldots\ldots \le x_n \le \ldots\ldots \le y$ for some y ∈ E, then∃ x ∈ E s .t $\|x_n – x \| \to 0$ when $n \to \infty$.

Equivalently, the cone Ᵽ is called regular iff every decreasing sequence which is bounded below in E is convergent.

Definition 1.4.8: [5,p2]

A cone Ᵽ in a real Banach space E is called minihedral if sup $\{x, y\}$ exists for every $x, y \in E$.

Definition 1.4.9:[5,p740]

Ᵽ is called strongly minihedral if every set which is bounded above has a supremum.

Example 1.4.10: [12]

Let $E = C^1_R[0,1]$, Ᵽ $= \{ g \in E : g(t) \geq 0 \}$, and $\| g \| = \| g \|_\infty + \| g` \|_\infty$.

Let $f(x) = \sin x$ and $g(x) = \cos x$. Both f and g are in E.

but $h = \sup \{f, g\} \notin C^1_R[0,1]$,since h is not differentiable at $\frac{\pi}{4}$.

So, Ᵽ is not minihedral.

Definitions 1.4.11 :[5,p740]

1. The norm $\| . \|$ is called monotonic if $\forall\ x, y \in E$, $0 \leq x \leq y \Longrightarrow \|x\| \leq \|y\|$.
2. The norm $\| . \|$ is called semi-monotonic if $\forall x, y \in E$, $\exists\ k \geq 0$, such that $0 \leq x \leq y \Longrightarrow \| x \| \leq k \| y \|$.

1.5 Cone metric space and Cone - Normed space:

Cone metric space:

We will always suppose that E is a Banach space, Ᵽ is a cone in E with $Ᵽ^0 \neq \emptyset$ and $\leq$ is the partial ordering with respect to Ᵽ.

Definition 1.5.1: [3,p1469]:

Let X be a nonempty set, and suppose the mapping $d_c : X \times X \rightarrow E$

Satisfies:

1. $d_c(x , y) \geq 0 \ \forall x , y \in X$, 0 vector in E .
2. $d_c(x , y) = 0$ iff $x = y$.
3. $d_c(x , y) = d_c(y , x) \ \forall x , y \in X$.
4. $d_c(x , y) \leq d_c(x , z) + d_c(z , y) \ \forall x , y ,z \in X$.

Then d_c is called a cone metric on X, and (X,d_c) is called a cone metric space.

Example 1.5.2 [8, p1638]:

Let $E = R^2$ and $P = \{ (x , y) : x \geq 0 , y \geq 0 \}$

$X = R$ and $d_c : X \times X \rightarrow E$ defined as: $d_c(x , y) = \{ |x - y| , \alpha| x - y|\}$ where $\alpha \geq 0$ is a constant then (X ,d_c) is a cone metric space.

Definition 1.5.3 [3,p1470]:

Let (X , d_c) be a cone metric space. let (x_n) be a sequence in X and $x \in X$. If $\forall s \in E$ with $s \gg 0 \ \exists N$ s .t $\forall n > N$, $d_c(x_n , x) \ll s$ then (x_n) is said to converge to x with limit x.

We denote that by:

$$\lim_{n\to\infty} x_n = x$$

Or $x_n \rightarrow x \quad (n \rightarrow \infty)$.

Lemma 1.5.4 [3 ,p1470]:

Let (X , d_c) be a cone metric space and let P be is a normal cone with normal constant k. Let (x_n) be a sequence in X. If (x_n) converge to x and (x_n) converge to y then $x = y$.

That is, a limit of (x_n) is unique.

Proof:

For any $s \in E$ with $s \gg 0$, $\exists N$ s .t $\forall n > N$, $d_c(x_n , x) \ll s$ and

$d_c(x_n , y) \ll s$.

we have: $d_c(x , y) \leq d_c(x_n , x) + d_c(x_n , y) \leq 2s$.

hence $\| d_c(x , y) \| \leq 2k \| s \|$.

since s is arbitrary, $d_c(x , y) = 0$ therefore $x = y$.

Definition 1.5.5 [3 ,p1470]:

Let (X , d_c) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$. If $\forall s \in E$ with $s \gg 0$ $\exists N$ s .t $\forall n , m > N$, $d_c(x_n , x_m) \ll s$ then(x_n) is called a Cauchy sequence in X .

Consequently, let (X , d_c) be a cone metric space, P a normal cone with normal constant k and (x_n)be a convergent sequence in X. Then (x_n) is Cauchy.

Proof:

We need to prove that every convergent sequence is Cauchy.

Let $s \in E$ with $s \gg 0$ $\exists N$ s .t $\forall n , m > N$, $d_c(x_n , x) \ll s/2$ and $d_c(x_m , x) \ll s/2$, hence $d_c(x_n ,x_m) \leq d_c(x_n , x) + d_c(x_m , x) \ll s$.

therefore $\{x_n\}$ is called Cauchy sequence in X.

If every Cauchy sequence is convergent in X, then X is called a complete cone metric space

Cone - Normed space:

Definition 1.5.6[5,741]:

Let E be a real Banach space ordered by the strongly minihedral cone Ᵽ, then a cone normed space is an ordered pair $(X , \| . \|_c)$ where X a real vector space and

$\| . \|_c : X \to E$. such that:

A. $\| x \|_c \geq 0 \ \forall x \in X$.

B. $\| x \|_c = 0$ iff $x=0$.

C. $\| r x \|_c = |r| \| x \|_c \forall r \in R$ and all $x \in X$.

D. $\| x + y \|_c \leq \| x \|_c + \| y \|_c$. $\forall x,y \in X$.

Example 1.5.7: [7,69]:

Let $E = X = R^2$, $Ᵽ = \{ (x , y) : x \geq 0 , y \geq 0 \} \subset R^2$, and $\| (x , y) \|_c = (\alpha |x| , \beta |y|)$ $\alpha \geq 0$, $\beta \geq 0$ then $(X , \| .\|_c)$ is cone - normed space over R^2 .

Proposition 1.5.8 [6,p6]:

We observe that every cone normed space is cone metric space.

Specifically, take the cone metric $d_c(x, y) = \| x - y \|_c$.

To see this:

$\forall x,y ,z \in X$

1. $d_c(x , y)=0$ iff $\| x - y \|_c$ iff $x - y = 0$ iff $x = y$.
2. $d_c(x , y) = \| x - y \|_c = \| - (y - x) \|_c =|-1| \| y - x \|_c$ by (C)

$\| y - x \|_c =d_c(y , x)$.

3. $d_c(x , y) = \| x - y \|_c = \| x - z + z - y \|_c$ by (D)

$\| x - z + z - y \|_c \leq \| x - z \|_c + \| z - y \|_c =d_c(x , z)+ d_c(z , y)$.

The following example shows that cone metric spaces are not necessarily cone - normed linear space.

Take $X=\{0 ,1\}$ and define the cone metric d_c on the set X as follows:

Let E = R.

d_c: $X \times X \to E$.

$d_c(x, y) = 1$ if $x \neq y$, $d_c(x, y) = 0$ if $x = y$.

We want to examine E the condition of cone metric

1. $d_c(x, y) = 0$ if $x=y$.
2. $d_c(x, y) \geq 0$.
3. $d_c(x, y) = d_c(y, x)$.
4. $d_c(x, y) \leq d_c(x, z) + d_c(z, y)$.

So, this is a cone metric but there exist no cone – norm which induces this distance. (Because $\| \alpha x - \alpha y \|_c \neq | \alpha | \| x - y \|_c$, for example take $\alpha = 2$ then $\| 2x - 2y \|_c = 1$ but $| 2 | \| x - y \|_c = 2$).

Then, cone metric is not coming from a cone - normed.

Definitions 1.5.9 [5,p 741]:

In a cone normed space $(X, \|.\|_c)$ over $(E, P, \|.\|)$ the sequence $\{x_n\}$ is said to be:

1. Convergent if $\exists x \in X$ s.t $\forall s \in E$ with $s >> 0$, $\exists\, n_0 \in N$ such that $\forall\, n > n_0$, $\| x_n - x \|_c << s$.

2. Cauchy if for each $s >> 0$ $\exists\, n_0 \in N$ s.t for $m, n \geq n_0$ we have $\| x_n - x_m \|_c << s$.

Definitions 1.5.10 [5, 741]:

A cone normed space $(X, \|.\|_c)$ is called a cone Banach space if every Cauchy sequence in X is convergent in X.

Chapter Two

Best Approximation & Best Co-Approximation

In Normed Space

Chapter Two

Best Approximation & Best Co-Approximation In Normed Space

In this chapter we focus on what we mean by Best approximation in a normed space and best co- approximation in normed space. This new concept is employed to improve various characterizations of closest elements and Chebyshev sets.

2.1 Best Approximation In Normed space

Definition 2.1.1.[9,p15]

Let G be a nonempty subset of a normed linear space $(X , \|.\|)$.

An element $g_0 \in G$ is called a best approximation from G if for every $g \in G$, we have $\| x - g_0\| \leq \| x - g \|$.

The set of all such elements $g_0 \in G$ which are best approximation is denoted by $P_G(x)$. Thus

$P_G(x) = \{g_0 \in G : \| x - g_0\| \leq \| x - g \| \text{ for all } g \in G.\}$

Hence P_G defines a mapping from X into the power set of G and is called the metric projection on G.

Remark 2.1.2[9,p15]

The set $P_G(x)$ of all best approximation can be written as:

$P_G(x) = \{g_0 \in G : \| x - g_0\| = d(x,G)\}$,

where $d(x,G) = \inf \{ \| x - g\| : g \in G \}$.

Theorem 2.1.3. [9,p16]

Let G be a subspace of a normed space X, then:

If $x \in G$, then $P_G(x) = \{x\}$.

Proof:

Let $x \in G$, Then $d(x, x) = 0$, (from the definition of metric space) and so $d(x,G) = 0$ Hence,

$P_G(x) = \{y \in G: \|x-y\| = d(x,G)\} = \{y \in G: \|x-y\| = 0\} = \{x\}$.

($\|x-y\| = 0$ iff $x - y = 0$ then $x = y$) .

Theorem 2.1.4 [9,p16]:

Let X be a normed space, $x \in X$, and let G be a subspace of X, then $P_G(x)$ is a convex set.

Proof:

Let ϑ be the distance from x to G. The statement holds if $P_G(x)$ is empty or has just one point. Now suppose that $y , z \in P_G(x)$ such that $y \neq z$.

So $\|x-y\| = \|x-z\| = \vartheta$

For $0 \leq \beta \leq 1$, let $w = \beta y + (1-\beta) z$, then :

$$\begin{aligned}
\|x-w\| &= \|x-(\beta y+(1-\beta)z\| \\
&= \|x-\beta y-(1-\beta)z+\beta x-\beta x\| \\
&= \|\beta(x-y)+(1-\beta)(x-z)\| \\
&\leq \beta\|x-y\| + (1-\beta)\|x-z\| \\
&= \beta\vartheta + (1-\beta)\vartheta \\
&= \beta\vartheta + 1\vartheta - \beta\vartheta \\
&= \vartheta
\end{aligned}$$

Therefore $\|x-w\| \leq \vartheta$ ... ... (1)

Now, $w \in G$, since G is subspace , so $\vartheta \leq \|x-w\|$……(2)

From (1) and (2) we get that $\|x-w\| = \vartheta$, so $w \in P_G(x)$

Since $y,z \in P_G(x)$ were arbitrary, $P_G(x)$ is convex.

Theorem 2.1.5[9,p17] :

Let G be a subspace of a normed space X, then for $x \in X$, $P_G(x)$ is a bounded set.

Proof:

Let $g_0 \in P_G(x)$.

$\| g_0 \| = \| g_0 - x + x \|$.

$\leq \| g_0 - x \| + \| x \|$.

$\leq \| x \| + \| x \|$ since $0 \in G$.

$= 2 \|x\|$.

Then $P_G(x)$ is bounded.

Theorem 2.1.6 [9,p18] :

Let G be a subspace of a normed space X. For $x \in X$:

1. If $z \in P_G(x)$ then $\alpha z \in P_G(\alpha x)$for any scalar α .
2. If $z \in P_G(x)$ then $z+g \in P_G(x+g)$ for any $g \in G$.

Proof:

For (1):

We need to prove that $\| \alpha x - \alpha z \| \leq \| \alpha x - g \|$.

Suppose $z \in P_G(x)$ and let $g \in G$ be arbitrary.

$\| \alpha x - g \| = |\alpha| \| x - (1/\alpha) g \| \geq |\alpha| \| x - z \| = \| \alpha x - \alpha z \|$.

Thus $\alpha z \in P_G(\alpha x)$.

For (2):

We need to prove that $\| x + g - (z + g) \| \leq \| x + g - g' \|$ for all $g' \in G$.

if $g' \in G$ we have $\| x + g - g' \| \geq \| x - z \| = \| x + g - (z + g) \|$.

Then $z + g \in P_G(x+g)$.

Notation 2.1.7[1,p21]:

For a subset G of X, put

$P^{-1}_G (0) = \{x \in X : \| x \| = d(x ,G)\} = \{ x \in X : \| x \| \leq \| x- g\| \ \forall \ g \in G\}$.

Theorem 2.1.8[1,p21]:

Let X be a normed linear space and G be a subspace of X, Then:

1. $g_0 \in P_G(x)$ if and only if $(x - g_0) \perp_B G$.
2. $g_0 \in P_G(x)$ if and only if $(x - g_0) \in P^{-1}_G (0) \ \forall \ x \in X$.

Proof:

For (1)

($\Longrightarrow$) Suppose $g_0 \in P_G(x)$.

we need to prove that foe $\alpha \in R$.$\| x - g_0 \| \leq \| (x-g_0) + \alpha g \|$

(by definition 1.3.3).

Since $g_0 \in P_G(x)$ this means that $\| x - g_0 \| \leq \| x - g_1 \|$,$\forall\, g_1 \in G$.

But then for any fixed $g \in G$ and $\alpha \in R$, putting $g_1 = g_0 - \alpha g$, we have so

$\|x - g_0 \| \leq \| x - (g_0 - \alpha g) \|$.

Then $\| x - g_0 \| \leq \| (x-g_0) + \alpha g \|$.

Therefore $(x - g_0) \perp_B G$.

($\Longleftarrow$) suppose $(x - g_0) \perp_B G$.

We need to prove that $\| x - g_0\| \leq \| x – g \|$, for all $g \in G$.

Then for all $\alpha \in R$ and $g_1 \in G$ we have,

$\|x - g_0 \| \leq \| x - g_0 + \alpha g_1\|$.

Let $g \in G$ be arbitrary and fixed . Take $g_1 = 1/ \alpha\, (g_0 – g)$

in the foregoing inequality to get: $\| x - g_0 \| \leq \| x – g \|$.

Therefore $g_0 \in P_G(x)$.

For (2):

$g_0 \in P_G(x)$ if and only if $x - g_0 \perp_B G$ (by (1))

this means that $\|x - g_0 \| \leq \| x - g_0 + \alpha g\|$. for all scalar α and $g \in G$.

But this holds if and only if $x - g_0 \in P^{-1}{}_G (0)$

(by definition of $P^{-1}{}_G (0)$; since G is a subspace).

We can summarize this theorem as the following remark.

Remark 2.1.9:

Let X be a normed linear space and G be a subspace of X, Then

$g_0 \in P_G(x)$ iff $x - g_0 \in P^{-1}{}_G (0)$ iff $x - g_0 \perp_B G$.

Corollary 2.1.10[1,p21]:

Let X be a normed linear space and G be a subspace of X. Then:

i. $x \in P^{-1}_G(0)$ implies that $\alpha x \in P^{-1}_G(0), \forall \alpha \in R$.

ii. $x \in P^{-1}_G(0)$ if and only if $0 \in P_G(x)$.

Proof:

For (i):

Let $x \in P^{-1}_G(0)$, then $x \perp_B G$ (By remark 2.1.9)

so that, $\| x \| \leq \| x+\gamma g \| \ \forall \gamma \in R$.

then $\forall \alpha \in R$ $\| \alpha x \| \leq \| \alpha x + \alpha\gamma g \|$,

we have, $\| \alpha x \| \leq \| \alpha x + \mu g \|$ where $\mu = \alpha\gamma$.

This implies that $\alpha x \perp_B g \ \ \forall g \in G$.

Therefore, $\alpha x \perp_B G$ then $\alpha x \in P^{-1}_G(0)$.

For (ii):

The proof of this part comes directly from theorem 2.1.8.

By taking $g_0 = 0$.

$0 \in P_G(x)$ iff $x - 0 \in P^{-1}_G(0)$.

Theorem 2.1.11[1,p22]:

Let G be a subspace of a normed linear space X.

Then $P_G(x) = G \cap (x - P^{-1}_G(0))$ for $x \neq 0$.

Proof:

Suppose that $g_0 \in G \cap (x - P^{-1}_G(0))$ if and only if $g_0 \in G$, and $g_0 \in (x - P^{-1}_G(0))$, such that $g_0 = x - g^\wedge$, where $g^\wedge \in P^{-1}_G(0)$, and $g^\wedge = x - g_0 \in P^{-1}_G(0)$ if and only if $g_0 \in P_G(x)$ (By theorem 2.1.8)

Therefore, $P_G(x) = G \cap (x - P^{-1}_G(0))$.

Definition 2.1.12 [1,p20]:

Let X be a normed linear space, if each element $x \in X$ has a unique best approximation in G, then G is called a Chebyshev set of X.

In other words, G is a Chebyshev set iff $P_G(x)$ is a singleton for each $x \in X$.

Definition 2.1.13: [9,p19]:

If $P_G(x)$ contains at least one element, then the set G is called Proximinal in X.

In other words, if $P_G(x) \neq \emptyset$ for all $x \in G$ then G is called Proximinal in X.

Theorem 2.1.14. [9,p21] .

For a linear subspace G of normed linear space X, the following statements are equivalent:

(1) G is Proximinal in X.

(2) $X = G + P^{-1}_G(0) = \{g + x : g \in G; x \in P^{-1}_G(0)\}$.

Proof:

(1) $\Longrightarrow$ (2)

If G is the Proximinal then $g_0 \in P_G(x)$ iff $(x - g_0) \in P^{-1}_G(0)$ when $g_0 \in G$ and $x \in X$ (by theorem 2.1.8)

Now, $x = g_0 + (x - g_0) \in G + P^{-1}_G(0)$,

[since $g_0 \in G$ and $x - g_0 \in P^{-1}_G(0)$].

Hence $X = G + P^{-1}_G(0)$.

(2) $\Longrightarrow$ (1)

Let $X = G + P^{-1}_G(0) = \{g + x : g \in G, x \in P^{-1}_G(0)\}$

and let $x \in X$. Then $x = g_0 + y$, for some $g_0 \in G$, and some $y \in P^{-1}_G(0)$.

then $y \in P^{-1}_G(0)$, and so ,$0 \in P_G(y)$.(by corollary 2.1.10)

But $y = x - g_0$, so $P_G(y) = P_G(x - g_0)$, this implies that $0 \in P_G(x - g_0)$,

then for all $g \in G$, $\| x - g_0 - 0 \| \leq \| x - g_0 - g \|$.

and so, $\| x - g_0 \| \leq \| x - (g_0 + g) \|$ for all $g \in G$, But $g_0 + g \in G$, then

$\| x - g_0 \| \leq \| x - g_1 \|$,$\forall g_1 = g_0 + g \in G$.

This means that $g_0 \in P_G(x)$.

Therefore G is Proximinal in X .

Definition 2.1.15:

1. $P_{G+y}(x + y) = \{ y_0 \in G + y : \| x + y - y_0 \| \leq \| x + y - (g + y) \|,$
$\forall\, g+y \in G+y\}$.

2. $P_{\alpha G}(\alpha x) = \{ y_0 \in \alpha G : \| \alpha x - y_0 \| \leq \| \alpha x - \alpha g) \| , \forall\, \alpha g \in \alpha G \}$.

The next theorem is going to be needed to prove theorem 2.1.17.

Theorem 2.1.16. [1,p22]:

Let G be a nonempty subset of a normed linear space X: Then

i. $P_{G+y}(x + y) = P_G(x) + y$ for every $x, y \in X$.

ii. $P_{\alpha G}(\alpha x) = \alpha P_G(x)$ for every $x \in X$ and $\alpha \in R$.

Theorem 2.1.17 [1,p23]

Let G be a nonempty subset of a normed space X: Then

i. G is Proximinal in X iff G + y is Proximinal in X for any given $y \in X$.

ii. G is Proximinal in X iff αG is Proximinal in X for any scalar α.

Proof:

i. G is Proximinal in X iff $P_G(x) \neq \emptyset$ iff $P_G(x) + y \neq \emptyset$ iff $P_{G+y}(x + y) \neq \emptyset$ iff G + y is Proximinal in X. [By theorem 2.1.16].

ii. G is Proximinal in X iff $P_G(x) \neq \emptyset$, iff $\alpha P_G(x) \neq \emptyset$ iff $P_{\alpha G}(\alpha x) \neq \emptyset$ iff αG is Proximinal in X. [By theorem 2.1.16)].

Now we present another kind of approximation.

2. 2 Best Co – Approximation in Normed space.

Definition 2.2.1 [1,p45].

Let G be a nonempty subset of a normed linear space X.

An element $g_0 \in G$ is called a best co- approximations from G if for every $g \in G$, $\| g_0 - g \| \leq \| x - g\|$.

Remark 2.2.2 [1,p45]:

The set $R_G(x)$ of all best co –approximation from G can be written as:

$R_G(x) = \{g_0 \in G : \| g_0 - g \| \leq \| x - g \|$ for all $g \in G\}$.

Example 2.2.3: [1 ,p52]

Suppose $X = R^2$ with norm $\| (x , y) \| = | x | + | y |$,

and $G = \{ (x , y) : x \geq 0 , y \geq 0 \}$,as a subset of X, then $(0 , 1) \in R_G(-1,1)$.

Proof:

For any $(g_1 ,g_2) \in G$.

We have $\| (0 , 1) - (g_1 ,g_2) \| = \| (- g_1 , 1-g_2) \| = | -g_1| + | 1-g_2|$

$= g_1 + | 1-g_2 | \leq | 1+ g_1 | + | 1-g_2 |$

$= | - (1+ g_1) | + | 1-g_2 |$

$= \| (-1 ,1) -(g_1 ,g_2)\|$.

This implies that $\| (0 , 1) - (g_1 ,g_2) \| \leq \| (-1 ,1) -(g_1 ,g_2)\|$. For all $g_1,g_2 \in G$

Then $g_0 = (0 , 1) \in R_G(x)$.

Theorem 2.2.4 [1,p51]:

Let X be a normed space, $x \in X$, and let G be a subspace of X, then $R_G(x)$ is a convex set.

Proof:

Let $y, w \in R_G(x)$ and $y \neq w$ and $0 \leq \beta \leq 1$. $m = \beta y + (1 - \beta) w$.

$\| g - y \| \leq \| x - g \|$, for all $g \in G$.

$\| g - w \| \leq \| x - g \|$, for all $g \in G$.

$\| g - m \| = \| g - \beta y + (1 - \beta) w \|$.

$= \| g - \beta y + w - \beta w \|$.

$= \| g - \beta y + w - \beta w + \beta g - \beta g \|$.

$= \| \beta (g - y) + (1 - \beta) (g - w) \|$.

$\leq \beta \|(g - y) \| + (1 - \beta) \| (g - w) \|$.

$\leq \beta \|(x - g) \| + (1 - \beta) \| (x - g) \|$.

$= \beta \|(x - g) \| + \| (x - g) \| - \beta \|(x - g) \|$.

$= \| x - g \|$.

Then $\| g - m \| \leq \| x - g \|$.

Then $m \in R_G(x)$.

Therefore, $R_G(x)$ is convex.

Notation 2.2.5 [1 ,p 52]

For a subset G of X, put

$R^{-1}{}_G (0) = \{x \in X : \| g \| \leq \| x - g \| \ \forall \ g \in G\} = \{ x \in X : G \perp_B x \}$.

Theorem 2.2.6. [1,p 48 , 52]

Let G be a subspace of a normed linear space X. Then for all $x \in X$:

1. $g_0 \in R_G(x)$ iff $G \perp_B (x - g_0)$.
2. $g_0 \in R_G(x)$ iff $(x - g_0) \in R^{-1}{}_{cG} (0)$.

Proof:

For (1):

($\Longrightarrow$) Suppose $g_0 \in R_G(x)$ we need to prove that $\| g \| \leq \| g + \alpha (x - g_0) \|$

for all $g \in G$ and for all $\alpha \in R$ with $\alpha \neq 0$.

(because this definition of Birkhoff orthogonality).

Since $g_0 \in R_G(x)$ then $\| g_0 - g_1 \| \leq \| x - g_1 \|$, $\forall \ g \in G$.

Put $g_1 = g_0 - (1/\alpha) g$ when $g \in G$ and $\forall \alpha \in R$ where $\alpha \neq 0$.

Therefore,

$\| (1/\alpha) g \| \leq \| x - g_0 + (1/\alpha) g \|$ (multiply by α)

$\| g \| \leq \| g + \alpha (x - g_0) \|$.

Thus, $g \perp_B (x - g_0)$ and so $G \perp_B (x - g_0)$.

($\Longleftarrow$) Let $G \perp_B (x - g_0)$ Then for all $\alpha \in R$ and $g_1 \in G$ we have

$\| g_1 \| \leq \| g_1 + \alpha(x - g_0) \|$.

Let $g \in G$ .By putting $g_1 = g_0 - g$ and $\alpha = 1$, it follows that

$\| g - g_0 \| \leq \| x - g \|$.

Therefore $g_0 \in R_G(x)$.

For (2):

$g_0 \in R_G(x)$ iff $G \perp_B (x - g_0)$ by (1)

this mean $\| g \| \leq \| g + \alpha (x - g_0) \|$.

by definition of $R^{-1}_G(0)$ then $(x - g_0) \in R^{-1}_{cG}(0)$.

This theorem now reads as:

Corollary 2.2.7:

Let G be a subspace of a normed linear space X. Then for all $x \in X, g_0 \in R_G(x)$ iff $G \perp_B (x - g_0)$ iff $(x - g_0) \in R^{-1}_G(0)$.

Corollary 2.2.8: [1 ,p 53]

Let X be a normed linear space X and G be a subspace of X.

Then $R_G(x) = G \cap (X - R^{-1}_G(0))$.

Proof:

Suppose that $g_0 \in G \cap (X - R^{-1}_G(0))$ iff $g_0 \in G$,and $g_0 \in (X - R^{-1}_G(0))$ such that $g_0 = x - g^{\cdot}$, where $g^{\cdot} \in R^{-1}_G(0)$ and $g^{\cdot} = x - g_0 \in R^{-1}_G(0)$ iff $g_0 \in R_G(x)$ (by theorem 2.2.6)

Therefore, $R_G(x) = G \cap (X - R^{-1}_G(0))$.

Definition 2.2.9. [1,p46]:

Let X be a normed linear space, if each element $x \in X$ has a unique best co –approximation in G, then G is called a Co-chebyshev set of X.

In other words, G is a co-Chebyshev set iff $R_G(x)$ is a singleton For each $x \in X$.

Definition 2.2.10. [1,p46]

If for every $x \in X$, $R_G(x)$ contains at least one element, then the subset G of the normed space X is called co – proximinal in X.

In other words, if for every $x \in X$, $R_G(x) \neq \emptyset$ then G is called co – proximinal in X.

Definition 2.2.11: [1 ,p46]

1. $R_{G+y}(x + y) = \{g_0 \in G + y : \| g_0 - (g + y) \| \leq \| x + y - (g + y) \|$ for all $g + y \in G + y\}$.
2. $R_{\alpha G}(\alpha x) = \{g_0 \in G : \| g_0 - g \| \leq \| \alpha x - g \|$ for all $g \in G\}$.

Proposition 2.2.12 [1,p46]

Let G be a subspace of a normed linear space X, then:

1. $R_{G+y}(x + y) = R_G(x) + y$ for every $x, y \in X$.
2. $R_{\alpha G}(\alpha x) = \alpha R_G(x)$ for any scalar α, and every $x \in X$.

Corollary 2.2.13 [1,p 46]

Let G be a subspace of a normed linear space X. Then:

1. G is co – proximinal in X iff G + y is co – proximinal in X for every $y \in X$.
2. G is co – proximinal in X iff αG is co – proximinal in X for any scalar α.

Proof:

For (1):

G is co – proximinal in X iff $R_G(x) \neq \emptyset$ iff $R_G(x) + y \neq \emptyset$ iff $R_{G+y}(x + y) \neq \emptyset$.

(by proposition 2.2.12)

Hence, G + y is co – proximinal in X.

For (2):

G is co – proximinal in X iff $R_G(x) \neq \emptyset$ iff $\alpha R_G(x) \neq \emptyset$ iff $R_{\alpha G}(\alpha x) \neq \emptyset$

(by proposition 2.2.12)

Hence, αG is co – proximinal in X.

Chapter Three

Best Approximation In Cone-Normed Space

Chapter Three

Best Approximation In Cone-Normed Space

In this chapter we will discuss best approximation in Cone-Normed Space, where we will propose a definition for best approximation in this new setting. We'll also define the Proximity and Chebychevness for Cone-Normed Space, and implant all the theorems of section 2.1in Cone-Normed Space. We will also define Proximinal additivity and ϕ_c-Summands of cone-Banach spaces and apply some results.

3.1 Introduction:

Definition 3.1.1 [10,p104]:

Let $(X , \|.\|_c)$ be a cone –normed space, G a nonempty set in X, and $x \in X$. We say that $g_0 \in G$ is a best approximation of x if $\|x - g_0\|_c \leq \|x - g\|_c, \forall\, g \in G$.

We denote the set of best approximation in G by $P_{cG}(x)$.

Definition 3.1.2 [10,p104]:

For $x \in X$, we define the cone distance as:

$d_c(x,G) = \inf \{ \|x-g\|_c : g \in G\}$.

Remark 3.1.3:

The set $P_{cG}(x)$ can be written as:

$P_{cG}(x)=\{g_0 \in G: \|x-g_0\|_c = d_c(x,G)\}$.

(In this chapter, G is assumed to be a subspace of a cone – normed space

$(X, \|.\|_c)$.

Theorem 3.1.4 [10.p107]:

Let G be a subspace of a cone – normed space X. Then:

If $x \in G$, then $P_{cG}(x) = \{x\}$.

Proof:

Let $x \in G$,then $d_c(x,x) = 0$ (from the definition of cone – metric space, and so $d_c(x,G) = 0$.

Hence,$P_{cG}(x) = \{ y \in G: \|x - y\|_c = d_c(x,G)\} = \{ y \in G : \|x - y\|_c = 0\}$

and $\|x - y\|_c = 0$ iff $x – y =0$ iff $x = y$.

so,$P_{cG}(x) =\{ y \in G : \|x - y\|_c = 0\}= \{x\}$.

Theorem 3.1.5: [10,p107]:

Let $(X, \|.\|_c)$ be a cone – normed space and let G be a subspace of X, then $P_{cG}(x)$ is a convex set.

Proof:

Let ϑ be the distance from x to G, so

$\vartheta = d_c(x,G)$.

The statement holds if $P_{cG}(x)$ is empty or is a singleton.

Now suppose that $y,z \in P_{cG}(x)$ and $y \neq z$.

For $0 \leq \beta \leq 1$, let $w = \beta y+(1-\beta)z$.

Then $\|x-w\|_c= \|x-(\beta y+(1-\beta)z)+ \beta x-\beta x\|_c$

$= \|x-\beta y -(1-\beta)z+\beta x-\beta x\|_c$

$= \| \beta (x-y)+(1-\beta)(x-z)\|_c$

$\leq \beta \|x-y\|_c+(1-\beta)\|x-z\|_c$

$=\beta\vartheta+(1-\beta)\ \vartheta$

$=\vartheta.$

Since G is a subspace, $w \in G$, this implies that $\vartheta \leq \|x - w\|_c$ because

$d_c(x,G) = \inf \{ \|x - w\|_c : w \in G \}$.

Therefore $\|x - w\|_c=\vartheta$.

Theorem 3.1.6: [10.p108]:

Let G be a subspace of a cone – normed space X. Then for $x \in X$, the set $P_{cG}(x)$ is bounded.

Proof:

Let $g_0 \in P_{cG}(x)$.

$\| g_0 \|_c = \| g_0 - x + x\|_c$

$\leq \| g_0 - x \|_c + \| x \|_c$

$\leq \| 0 - x \|_c + \| x \|_c$, since $0 \in G$.

$= 2 \| x \|_c$.

Thus $P_{cG}(x)$ is bounded.

Theorem 3.1.7:

Let G be a subspace of cone –normed space X. For $x \in X$.

Then:

1. If $z \in P_{cG}(x)$ then $\alpha z \in P_{cG}(\alpha x)$ for any scalar α.
2. If $z \in P_{cG}(x)$ then $z+g \in P_{cG}(x+g)$ for all $g \in G$.

Proof:

For (1):

$\|\alpha x - g\|_c = |\alpha|\ \| x - (1/\alpha)g\|_c \geq |\alpha|\ \|x - z\|_c = \|\alpha x - \alpha z\|_c$ for all $g \in G$.

Thus $\alpha z \in P_{cG}(\alpha x)$.

For (2):

If $g' \in G$ we have $\|x + g - g'\|_c \geq \|x - z\|_c = \|x + g - (z+g)\|_c$.

Then $z+g \in P_{cG}(x+g)$ for all $g' \in G$.

The next definition has been again quoted from Birkhoff orthogonality.

Definition 3.1.8:

Let (X , $\|.\|_c$) be a cone – normed space, G a subspace of X, and $x \in X$. We say that x is Birkhoff orthogonal to G iff

$\|x\|_c \leq \|x+ \alpha g\|_c$ for all scalars α and all $g \in G$.

Symbolically, $(x \perp_B^c G)$ iff $\| x \|_c \leq \| x+ \alpha g\|_c$ for all scalar $\alpha \in R$ and all $g \in G$

Notation 3.1.9:

For a subset G of X, put

$P^{-1}_{cG}(0) = \{ x \in X : \|x\|_c = d_c(x,G) \} = \{ x \in X : \|x\|_c \leq \|x - g\|_c , \forall g \in G\}$

Theorem 3.1.10:

Let (X , $\|.\|_c$) be a cone –normed space, G be a subspace of X, then

1. $g_0 \in P_{cG}(x)$ iff $(x-g_0) \perp_B^c G$.
2. $g_0 \in P_{cG}(x)$ iff $(x-g_0) \in P^{-1}_{cG}(0)$.

Proof:

For (1):

($\Longrightarrow$) Suppose $g_0 \in P_{cG}(x)$,put $g_1 = g_0 - \alpha g$. For any fixed $g \in G$ and $\alpha \in R$,

since $g_0 \in P_{cG}(x)$ and $g_1 \in G$, $\| x - g_0 \|_c \leq \| x - g_1 \|_c$ and so

$\|x - g_0\|_c \leq \|x - (g_0 - \alpha g)\|_c$

Then $\|x - g_0\|_c \leq \|(x - g_0) + \alpha g\|_c$

Therefore $(x - g_0) \perp_B{}^c G$.

($\Longleftarrow$) Let $(x - g_0) \perp_B{}^c G$. Then for all $\alpha \in R$ and $g_1 \in G$.

we have $\| x - g_0\|_c \leq \| x - g_0 + \alpha g_1\|_c$

Let $g \in G$ be arbitrary and fixed.

Take $g_1 = 1/\alpha\, (g_0 - g)$.

$\| x - g_0\|_c \leq \| x - g_0 + \alpha\, (1/\alpha\, (g_0 - g)\|_c$.

Then $\|x - g_0\|_c \leq \| x - g\|_c$.

Therefore $g_0 \in P_{cG}(x)$.

For (2):

$g_0 \in P_{cG}(x)$ iff $x-g_0 \perp_B{}^c G$. This means that $\|x - g_0\|_c \leq \| x - g_0 + \alpha g\|_c$ by (1).

Thus $x-g_0 \in P^{-1}{}_{cG}(0)$ (by definition of $P^{-1}{}_{cG}(0)$, since G is a subspace).

Theorem 3.1. 10 can now be stated as:

Fact 3.1.11:

Let $(X,\|.\|_c)$ be a cone –normed space, G be a subspace of X, then $g_0 \in P_{cG}(x)$ iff $x-g_0 \in P^{-1}_{cG}(0)$ iff $x - g_0 \perp_B^c G$.

Corollary 3.1.12:

Let $(X , \|.\|_c)$ be a cone – normed space, G subspace of X, then:

1. If $x \in P^{-1}_{cG}(0)$ then $\alpha x \in P^{-1}_{cG}(0)$. $(\forall \alpha \in R)$.
2. $x \in P^{-1}_{cG}(0)$ iff $0 \in P_{cG}(x)$.

Proof:

For (1):

Let $x \in P^{-1}_{cG}(0)$ then $x \perp_B G$ (by Fact 3.1.11)

So that $\|x \|_c \leq \| x +\gamma g\|_c \forall \gamma \in R$

then $\forall \alpha \in R$ $\|\alpha x \|_c \leq \|\alpha x +\alpha\gamma g\|_c$

let $\nu = \alpha\gamma$, then we have $\|\alpha x \|_c \leq \|\alpha x +\nu g\|_c, \forall \nu \in R$, this implies that $\alpha x \perp_B^c g \quad \forall g \in G$.

Therefore $\alpha x \perp_B^c G$ and so $\alpha x \in P^{-1}_{cG}(0)$.

For (2):

$0 \in P_{cG}(x)$ iff $x \in P^{-1}_{cG}(0)$

then $x \in P^{-1}_{cG}(0)$ (by Fact3.1.11)

taking $g_0 = 0$ gives the conclusion.

Theorem 3.1.13:

Let G be a subspace of a cone – normed space X and $P_{cG}(x)$ be the set of all best approximation, then for $x \neq 0$, $P_{cG}(x) = G \cap (X - P^{-1}_{cG}(0))$.

Proof:

$g_0 \in G \cap (X - P^{-1}_{cG}(0))$ iff $g_0 \in G$ and $g_0 \in (X - P^{-1}_{cG}(0))$ iff $g_0 \in G$ and

$g_0 = x - \breve{g}$, for some $\breve{g} \in P^{-1}_{cG}(0)$ iff $g_0 \in G$ and $\breve{g} = x - g_0 \in P^{-1}_{cG}(0)$ iff

$g_0 \in P_{cG}(x)$ (by theorem 3.1.10)

therefore $P_{cG}(x) = G \cap (X - P^{-1}_{cG}(0))$.

Definition 3.1.14[10,p104]:

If each $x \in X$ has a unique best approximation in a G then G is called

a Chebychev set .

In other words, G is called a Chebychev set iff $P_{\mathbf{cG}}(x)$ is a singleton.

3.2 Proximinal Set

Definition 3.2.1 [10,p104]:

If $P_{\mathbf{cG}}(x)$ contains at least one element, then the subset G is called

a proximinal set in X.

In other words, if $P_{\mathbf{cG}}(x) \neq \emptyset$, then G called a proximinal in X.

Theorem 3.2.2:

For a subspace G of a cone –normed space X, the following statements are equivalent:

1. G is a proximinal.
2. $X = G + P^{-1}_{cG}(0) = \{ g + x : g \in G , x \in P^{-1}_{cG}(0)\}$.

Proof:

(1) $\Longrightarrow$ (2) Suppose that G is a proximinal

Let $x \in X$, $g_0 \in G$ and $g_0 \in P_{cG}(x)$ then $(x - g_0) \in P^{-1}_{cG}(0)$ (by theorem 3.1.10)

Now $x = g_0 + x - g_0 \in G + P^{-1}_{cG}(0)$ (since $g_0 \in G$, $x - g_0 \in P^{-1}_{cG}(0)$)

Hence $X = G + P^{-1}_{cG}(0)$.

(2) $\Longrightarrow$ (1) Let $X = G + P^{-1}_{cG}(0) = \{ g + x : g \in G , x \in P^{-1}_{cG}(0)\}$ and let $x \in X$ then $x = g_0 + y$, for $g_0 \in G$ and $y \in P^{-1}_{cG}(0)$ i.e $y \in P^{-1}_{cG}(0)$ and so $0 \in P_{cG}(y)$ (by corollary 3.1.12)

But $y = x - g_0$ so $P_{cG}(y) = P_{cG}(x - g_0)$ which implies that $0 \in P_{cG}(x - g_0)$.

Now, $\forall g \in G$,

$\|x - g_0 - 0\|_c \leq \| x - g_0 - g\|_c$ and so

$\|x - g_0\|_c \leq \| x -(g_0+g)\|_c \forall g \in G$.

But $g_0+g \in G$ then $\|x - g_0\|_c \leq \| x - g_1\|_c \forall g_1 = g_0+g \in G$.

Definitions 3.2.3:

1. $P_{cG+y}(x + y) = \{ y_0 \in G +y : \| x + y - y_0 \|_c \leq \| x + y - (g + y) \|_c,$
$\forall g + y \in G +y \}$.
2. $P_{\alpha cG} (\alpha x) = \{ y_0 \in \alpha G : \| \alpha x - y_0 \|_c \leq \| \alpha x - \alpha g\|_c, \forall g \in G \}$.

Theorem 3.2.4:

Let G be a nonempty subspace of a cone –normed space X, then:

1. $P_{cG+y}(x + y) = P_{cG}(x) + y ,\forall x , y \in X$.
2. $P_{c\alpha G} (\alpha x) = \alpha P_{cG}(x) ,\forall x \in X$ and $\alpha \in R$.

Proof:

For (1)

$y_0 \in P_{cG+y}(x + y)$ iff $y_0 \in G + y$ and $\| x + y - y_0 \|_c \leq \| x + y - (g + y) \|_c \; \forall g + y \in G + y$ iff $y_0 - y \in G$ and $\| x + (y - y_0)\|_c \leq \| x - g \|_c$, $\forall g \in G$ iff $y_0 - y \in P_{cG}(x)$ iff $y_0 \in P_{cG}(x) + y$.

Therefore $P_{cG+y}(x + y) = P_{cG}(x) + y$.

For (2)

$y_0 \in P_{\alpha cG}(\alpha x)$ iff $y_0 \in \alpha G$ and $\| \alpha x - y_0 \|_c \leq \| \alpha x - \alpha g \|_c$, $\forall g \in G$

iff $|\alpha| \, \| x - \frac{1}{\alpha} y_0 \|_c \leq |\alpha| \, \| x - g \|_c$, $\forall g \in G$ iff $\frac{1}{\alpha} y_0 \in G$ and

$|\alpha| \, \| x - \frac{1}{\alpha} y_0 \|_c \leq |\alpha| \, \| x - g \|_c$ iff $\frac{1}{\alpha} y_0 \in P_{cG}(x)$ iff $y_0 \in \alpha P_{cG}(x)$.

Therefore, $P_{\alpha cG}(\alpha x) = \alpha P_{cG}(x)$.

Theorem 3.2.5:

Let G be a nonempty subspace of a cone –normed space X, then:

1. G is a proximinal in X iff G + y is a proximinal in X for any given $y \in X$.
2. G is a proximinal in X iff αG is a proximinal in X for any scalar α.

Proof:

For (1):

G is aproximinal in X iff $P_{cG}(x) \neq \emptyset$ for all $x \in X$ iff $P_{cG}(x) + y \neq \emptyset$ iff

$P_{cG+y}(x + y) \neq \emptyset$ (by theorem 3.2.4) iff G + y is aproximinal in X.

For (2):

G is aproximinal in X iff $P_{cG}(x) \neq \emptyset$ iff $\alpha P_{cG}(x) \neq \emptyset$ iff

$P_{c\alpha G}\ (\alpha x) \neq \emptyset$ (by theorem 3.2.4)

iff αG is a proximinal in X.

3.3 Proximinal additivity in cone – normed spaces:

Definition 3.3.1:

let X be a cone -Banach space, and G be a nonempty subspace of X.

the space G is said to be proximinally additivite in X if:

$z_1 \in P_{cG}(x_1)$ and $z_2 \in P_{cG}(x_2)$ imply that $z_1 + z_2 \in P_{cG}(x_1+x_2)$

$(\forall\ x_1 , x_2 \in X)$.

Example 3.3.2:

Let $X = R^2$,P =[0,∞). Set $G = \{ (x , 0) : x \in R \}$ with the Euclidean norm.

If $x_1= (d ,e)$ then $P_{cG}(x_1) = \{ (d ,0)\}$ i.e $z_1= (d ,0)$.

And if $x_2= (f,g)$ then $P_{cG}(x_2) = \{ (f ,0)\}$ i.e $z_2= (f ,0)$.

But $x_1+x_2 = (d+f ,e+g)$, so $P_{cG}(x_1+x_2) = \{(d+f,0)\}$ .i.e

$(d+f,0) = z_1 +z_2$.

Therefore, G is proximinal additivity**.**

Definition 3.3.3:

Let X be a cone - Banach space and G proximinal in X, then any map which associates with each element of X one of its best approximation in G is called aproximity map.

Theorem 3.3.4:

let X be a cone - Banach space, and G be a c-Chebychev subspace of X. There exists a linear proximity map iff G is proximinally additivite in X.

Proof:

($\Longrightarrow$)

Let F be a linear proximity map $F : X \rightarrow G$. $F(x) = z$ where z is the unique best c–approximation of x in G. We need to prove that G is proximinally additivite in X. Let $z_1 \in P_{cG}(x_1)$ and $z_2 \in P_{cG}(x_2)$.

We show that $z_1 + z_2 \in P_{cG}(x_1+x_2)$. Now, $z_1 + z_2 \in F(x_1) + F(x_2)$
$= F(x_1 + x_2) \in P_{cG}(x_1+x_2)$.

Therefore G is proximinally additivite in X.

($\Longleftarrow$)

Assume that G is proximinally additivite in X. Define $F : X \rightarrow G$ such that $F(x) \in P_{cG}(x)$, since G is a Chebychev.

Now, we need to prove that F is linear.

1. $F(x_1) + F(x_2) = F(x_1 + x_2)$.
2. $F(\alpha x) = \alpha . F(x)$

Let $x_1 , x_2 \in X$, we show that $F(x_1) + F(x_2) = F(x_1 + x_2)$.

Now , $F(x_1) \in P_{cG}(x_1)$ and $F(x_2) \in P_{cG}(x_2)$.

Since G is proximinally additivite in X, then

$F(x_1) + F(x_2) \in P_{cG}(x_1+x_2)$.

Also $F(x_1 + x_2) \in P_{cG}(x_1+x_2)$ since G is Chebychev. Then

$F(x_1) + F(x_2) = F(x_1 + x_2)$ ………….. (1)

Let $x \in X$, α scalar and $F(x)$ is the unique elements in $P_{cG}(x)$.

Then $\alpha. F(x) \in P_{cG}(\alpha x)$ (by theorem 3.1.7).

and $F(\alpha x) \in P_{cG}(\alpha x)$.

Since G is Chebychev,

$F(\alpha x) = \alpha . F(x)$ …………….. (2)

By (1) And (2) F is a linear.

Theorem 3.3.5:

Let X be a cone - Banach space and G be a nonempty subspace in X.

If G is proximinally additivite in X, then $P^{-1}_{cG}(0)$ is proximinally additivite in X.

Proof:

Suppose that G is proximinally additivite in X so, Let $P^{-1}_{cG}(0) = \breve{G}$

if $g_1 \in P_{cG}(x_1)$ and $g_2 \in P_{cG}(x_2)$ then $g_1 + g_2 \in P_{cG}(x_1+x_2) \forall x_1 , x_2 \in X$.

Now , let $z_1 \in P_{c\breve{G}}(x_1)$ and $z_2 \in P_{c\breve{G}}(x_2)$.

we need to show that : $z_1 + z_2 \in P_{c\breve{G}}(x_1 + x_2)$.

let $z = x - g$ where $x \in X$, $g \in G$.

therefore $x_1 - z_1 \in P_{cG}(x_1)$ and $x_2 - z_2 \in P_{cG}(x_2)$.

Since G is proximinally additivite in X then

$x_1 + x_2 - (z_1 + z_2) \in P_{cG}(x_1+x_2)$ which implies that $z_1 + z_2 \in P_{c\breve{G}}(x_1 + x_2)$.

3.4 ϕ_c- Summands:

Definition 3.4.1:

Let E be a Banach space and P is a cone in E.

A function ϕ_c : P→P is called a cone - modulus function if the following are satisfied:

1. ϕ_c is continuous at zero and strictly increasing.
2. $\phi_c(0) = 0$.
3. ϕ_c is subaddditive i.e $\phi_c(x + y) \leq \phi_c(x) + \phi_c(y)\forall x ,y \in P$.

Definition 3.4.2:

let X be a cone - Banach space, and G be a nonempty subspace X, G is called a ϕ_c- Summand if ∃ bounded projection.

$F : X\to G$ s.t $\forall x \in X$,

$\phi_c(\|x\|_c) = \phi_c(\|F(x)\|_c) + \phi_c(\|x - F(x)\|_c)$.

Where ϕ_c is a cone - modulus function.

Remarks 3.4.3:

i. If G is a ϕ_c- Summand of a cone - Banach space X then G is proximinal.
ii. If G is a ϕ_c- Summand of a cone - Banach space X then G is Chebychev.
iii. If G is a ϕ_c- Summand of a cone - Banach space X then G is proximinally additivite in X.

Proof:

For (i):

let $F : X\to G$ s.t $\forall x \in X$, $\forall g \in G$ we have:

$\phi_c(\|x- g\|_c) = \phi_c(\| F(x- g)\|_c) + \phi_c(\|(x- g) - F(x- g)\|_c)$

$= \phi_c (\| F(x) - F(g) \|_c) + \phi_c (\| x- g - F(x) - F(g) \|_c)$.

$= \phi_c (\|F(x) - g) \|_c) + \phi_c (\|(x- F(x) \|_c)$.

$\geq \phi_c (\|x - F(x) \|_c)$.

$\phi_c (\|x- g \|_c) \geq \phi_c (\|x - F(x) \|_c)$ ……….(*)

Since ϕ_c^{-1} exists and strictly increasing.

Multiply (*) by ϕ_c^{-1} to get that $\|x- g \|_c \geq \|x - F(x) \|_c$.

i.e $F(x) \in P_{cG}(x)$. Then G is proximinal.

For (ii):

Assume that G is a ϕ- Summand of X, so $\forall\, x \in X$, $F(x) \in P_{cG}(x)$ by (i)

Now suppose that $m \in G$ is another closet element to x:

i.e $\|x- m \|_c = \|x - F(x) \|_c$ ……..(*)

But $x - m \in E$, so:

$\phi (\| x- m \|_c) = \phi (\| F(x - m) \|_c) + \phi (\|(x - m) - F(x - m) \|_c)$.

$= \phi (\|F(x) - m) \|_c) + \phi (\| x-m - F(x) - m \|_c)$.

$= \phi (\|F(x) - m) \|_c) + \phi (\| x - F(x) \|_c)$.

By (*) we conclude that $\phi (\| F(x) - m) \|_c) = 0$.

$\| x- m \|_c - \|x - F(x) \|_c = 0$.

$\phi (\| x- m \|_c) - \phi (\|x - F(x) \|_c) = 0$

Since ϕ_c^{-1} exists and strictly increasing.

Multiply (*) by ϕ_c^{-1} to get $\| F(x) - m) \|_c = \phi_c^{-1} (0)$.

$\| F(x) - m) \|_c = 0$.

Hence m = F(x).

Since x and m were arbitrary, G is Chebychev.

For (iii):

Let $z_1 \in P_{cG}(x_1)$and $z_2 \in P_{cG}(x_2)$. Since G is a ϕ_c- Summand of X, then ∃a bounded projection $F : X \rightarrow G$ s .t F(x) is the unique Best approximation of x in G, $\forall x \in X$ by (i) and (ii).

$F(x) \in P_{cG}(x)$ and $z = F(x)$.

So, $z_1 = F(x_1)$ and $z_2 = F(x_2)$.

$z_1 + z_2 = F(x_1) + F(x_2) = F(x_{1} + x_2)$ since F is linear.

This implies that $z_1 + z_2 \in P_{cG}(x_1 + x_2)$. Then G is proximinally additivite in X.

Chapter Four

Best Co - Approximation In Cone-Normed Space

Chapter Four

Best Co – Approximation In Cone-Normed Space

In this chapter we will discuss best co- approximation in Cone-Normed Space and the set of closest relatives in normed spaces and we will apply some theorems in section 3.2 on cone-normed space.

4.1 Introduction:

Definition 4.1.1:

Let G be a nonempty subset of a cone - normed linear space X.

An element $g_0 \in G$ is called a best co-approximation from Gif for every $g \in G$,

$\| g_0 - g \|_c \leq \| x - g\|_c$.

The set of all such elements g_0 which is called a set of best co-approximation is denoted by $R_{cG}(x)$.

Remark 4.1.2:

The set $R_{cG}(x)$ of all best co – approximation from G can be written as:

$R_{cG}(x) = \{g_0 \in G : \| g_0 - g \|_c \leq \| x - g \|_c$ for all $g \in G\}$.

Example 4.1.3:

Suppose $X = R^2$ and $E=R$, with cone -normed $\| (x , y) \|_c = | x | + | y |$.

And $G = \{ (x , y) : x \geq 0 , y \geq 0 \}$, G is subset of X then

$(0,1) \in R_{cG}(-1,1)$.

Proof:

For any $(g_1 ,g_2) \in G$.

We have $\| (0 , 1) - (g_1 ,g_2) \|_c = \|(- g_1 , 1-g_2) \|_c = | -g_1| + | 1-g_2|$.

$$= g_1 +|1-g_2| \leq |1+ g_1| +|1-g_2|.$$

$$= \|-(1+ g_1) ,(1-g_2)\|_c .$$

$$= \| (-1 ,1) -(g_1 ,g_2)\|_c.$$

This imply that $\| (0 , 1) - (g_1 ,g_2) \|_c \leq \| (-1 ,1) -(g_1 ,g_2)\|_c$.

Then $g_0 = (0 , 1) \in R_{cG}(x)$.

Theorem 4.1.4:

Let $(X , \|.\|_c)$ be a cone – normed space, $x \in X$, and let G be a subspace of X, then $R_{cG}(x)$ is a convex set.

Proof:

Let $y, w \in R_{cG}(x)$ and $y \neq w$ and $0\leq \beta\leq 1$. $m = \beta y + (1 - \beta) w$.

$\| g - y \|_c \leq \| x - g \|_c$, for all $g \in G$.

$\| g - w \|_c \leq \| x - g \|_c$, for all $g \in G$.

$\| g - m \|_c = \| g - \beta y + (1 - \beta) w \|_c$.

$= \| g - \beta y + w - \beta w \|_c$.

$= \| g - \beta y + w - \beta w + \beta g - \beta g \|_c$.

$= \| \beta (g - y) + (1 -\beta) (g -w) \|_c$.

$\leq \beta \|(g - y) \|_c+ (1 -\beta) \| (g -w) \|_c$.

$\leq \beta \|(x - g) \|_c+ (1 -\beta) \| (x -g) \|_c$.

$= \beta\|(x - g) \|_c+\| (x -g) \|_c - \beta\|(x - g) \|_c$.

$= \| x -g \|_c$.

Then $\| g - m \|_c \leq \| x -g \|_c$.

Then $m \in R_{cG}(x)$.

Therefore, $R_{cG}(x)$ is convex.

Notation 4.1.5:

For a subset G of X such that X is cone - normed, put

$R^{-1}_{cG} (0) = \{x \in X :\| g \|_c \leq \| x- g\|_c \forall g\in G\}$.

Theorem 4.1.6:

Let G be a subspace of a cone - normed linear space X. for all $x \in X$.

Then:

1. $g_0\in R_{cG}(x)$ iff $G \perp_B^c (x - g_0)$.
2. $g_0\in R_{cG}(x)$ iff $(x - g_0) \in R^{-1}_{cG} (0)$.

Proof:

For (1):

($\Longrightarrow$)

Suppose $g_0 \in R_{cG}(x)$. We need to prove that $\| g \|_c \leq \| g + \alpha (x - g_0) \|_c$ (because this is the definition of Birkhoff orthogonality).

Let $g \in G$: for $\alpha \in R$ and $\alpha \neq 0$,put $g_1 = g_0 - 1/\alpha\, g$.

Since $g_0 \in R_{cG}(x)$ then $\| g_0 - g_1 \|_c \leq \| x - g_1 \|_c$.

put $g_1 = g_0 - 1/\alpha\, g$ when $g \in G$ and $\forall \alpha \in R$ where $\alpha \neq 0$.

Therefore,

$\| 1/\alpha\, g \|_c \leq \| x - g_0 + 1/\alpha\, g \|_c$ (multiple by α)

$\| g \|_c \leq \| g + \alpha (x - g_0) \|_c$.

Thus, $g \perp_B (x - g_0)$ and so $G \perp_B^c (x - g_0)$.

($\Longleftarrow$)

Let $G \perp_B^c (x - g_0)$.

Then for all $\alpha \in R$ and $g_1 \in G$ we have

$\| g_1 \|_c \leq \| g_1 + \alpha(x - g_0) \|_c$.

Let $g \in G$, by putting $g_1 = g_0 - g$ and $\alpha = 1$, it follows that

$\| g - g_0 \|_c \leq \| x - g \|_c$.

Therefore $g_0 \in R_{cG}(x)$.

For (2):

$g_0 \in R_{cG}(x)$ iff $G \perp_B^c (x - g_0)$ by (1)

this means that $\| g \|_c \leq \| g + \alpha (x - g_0) \|_c$.

by definition of Ģ then $(x - g_0) \in R^{-1}_{cG}(0)$.

This theorem can be written as:

Fact 4.1.7:

Let G be a subspace of a cone - normed linear space X. for all $x \in X$.

Then:

$g_0 \in R_{cG}(x)$ if and only if $G \perp_B^c (x - g_0)$ if and only if

$(x – g_0) \in R^{-1}_{cG} (0)$.

Corollary 4.1.8:

Let $(X , \|.\|_c)$ be a cone - normed linear space X, and G a subspace of X. then $R_{cG}(x) = G \cap (X – R^{-1}_{cG} (0))$.

Proof:

$g_0 \in G \cap (X - R^{-1}_{cG} (0))$ iff $g_0 \in G$ and $g_0 \in (X - R^{-1}_{cG} (0))$, so one has:

$g_0 = x – \acute{g}$, where $\acute{g} \in R^{-1}_{cG} (0)$ and $\acute{g} = x - g_0 \in R^{-1}_{cG} (0)$ iff $g_0 \in R_{cG}(x)$ (by theorem 4.1.6)

Therefore, $R_{cG}(x) = G \cap (X - R^{-1}_{cG} (0))$.

Definition 4.1.9:

Let X be a cone - normed linear space, if each element $x \in X$ has a unique best co-approximation in G, then G is called a co-Chebyshev subset of X. In other words, G is a co-Chebyshev set iff $R_{cG}(x)$ is a singleton.

4.2 Co-proximinal:

Definition 4.2.1:

If for every $x \in X$, $R_{cG}(x)$ contains at least one element, then the subset G of the cone - normed space X is called co - proximinal.

In other words, if for every $x \in X$, $R_{cG}(x) \neq \emptyset$ then G is called a co-proximinal.

Definition 4.2.2:

1. $R_{cG+y}(x + y) = \{g_0 \in G + y : \| g_0 - (g + y) \|_c \leq \| x + y - (g + y) \|_c$ for all $g + y \in G+y\}$.

2. $R_{c\alpha G}(\alpha x) = \{g_0 \in G : \| g_0 - g \|_c \leq \| \alpha x - g \|_c$ for all $g \in G\}$.

Proposition 4.2.3:

Let G be a subspace of a cone - normed linear space X, then:

1. $R_{cG+y}(x + y) = R_{cG}(x) + y$ for every $x, y \in X$.
2. $R_{c\alpha G}(\alpha x) = \alpha R_{cG}(x)$ for any scalar α, and every $x \in X$.

Proof:

For (1)

$g_0 \in R_{cG+y}(x + y)$ iff $\| g_0 - (g + y) \|_c \leq \| x + y - (g + y) \|_c$
$\forall g + y \in G+y$ iff $\| (g_0 - y) - g \|_c \leq \| x - g \|_c$, $\forall g \in G$ iff
$g_0 - y \in R_{cG}(x)$ iff $g_0 \in R_{cG}(x) + y$.

Therefore, $R_{cG+y}(x + y) = R_{cG}(x) + y$.

For (2)

$g_0 \in R_{c\alpha G}(\alpha x)$ iff $\| g_0 - g \|_c \leq \| \alpha x - g \|_c$, $\forall g \in G$ iff
$\| \frac{1}{\alpha}g_0 - \frac{1}{\alpha}g \|_c \leq \| x - \frac{1}{\alpha}g \|_c$. $\forall \frac{1}{\alpha}g \in G$ iff $\| \frac{1}{\alpha}g_0 - g_1 \|_c \leq \| x - g_1\|_c$
$\forall g_1 = \frac{1}{\alpha}g \in G$ iff $\frac{1}{\alpha}g_0 \in R_{cG}(x)$ this implies that $g_0 \in \alpha R_{cG}(x)$.

Therefore, $R_{c\alpha G}(\alpha x) = \alpha R_{cG}(x)$ for any scalar α.

Corollary 4.2.4:

Let G be a subspace of a cone - normed linear space X. Then:

1. G is a co - proximinal iff G + y is a co - proximinal for every $y \in X$.
2. G is a co - proximinal iff αG is a co - proximinal for any scalar α.

Proof:

For (1):

G is a co - proximinal iff $R_{cG}(x) \neq \emptyset$ iff $R_{cG}(x) + y \neq \emptyset$ iff $R_{cG+y}(x + y) \neq \emptyset$.
(by proposition 4.2.3)

Hence, $G + y$ is a co - proximinal.

For (2):

G is a co - proximinal iff $R_{cG}(x) \neq \emptyset$ iff $\alpha R_{cG}(x) \neq \emptyset$ iff $R_{c\alpha G}(\alpha x) \neq \emptyset$

(by proposition 4.2.3)

Hence, αG is a co - proximinal.

Example 4.2.5:

Suppose $X = R^2$ with norm $\| (x , y) \|_c = | x | + | y |$.

And $G = \{ (x , y) : x \geq 0 , y \geq 0 \}$, G is subset of X then

1. $(0,1) \in R_{cG} (-1,1)$ but $(0 ,1) \notin P_{cG} (-1,1)$.
2. $(0,0) \notin R_{cG} (-1,1)$ but $(0,0) \in P_{cG} (-1,1)$.

Number (1) showed in example 2.2.3 and example 3.1.4

Now, we will show number (2)

Proof:

For any $(g_1 ,g_2) \in G$.

We have $\| (0 , 0) - (g_1 ,g_2) \|_c = \| (- g_1 , -g_2) \|_c = | -g_1| + | -g_2|$

$$= g_1 + g_2 \leq 2+ g_1 +g_2 \geq 2 + g_1-g_2 .$$

$$=1+ g_1 +1 - g_2.$$

$$= |-(1 +g_1) | + | 1-g_2| .$$

$$= \|-(1+ g_1) ,(1-g_2)\|_c .$$

$$= \| (-1 ,1) -(g_1 ,g_2)\|_c.$$

Then $\| (0 , 0) - (g_1 ,g_2) \|_c \geq \| (-1 ,1) -(g_1 ,g_2)\|_c$.

This implies that $(0,0) \notin R_{cG} (-1,1)$.

We see that in our observed occasions, best co- approximation in cone-normed space is a counter copy of best approximation in cone-normed space.

Conclusion

In this thesis, we were able to apply the theorems of Best Approximation and Best co-approximation of normed-space on cone normed space.

We also found that a good number of major results on Best Approximation and Best Co-Approximation in normed spaces are transformed, word for word, to the theory of Best Approximation and Best Co-Approximation on cone normed spaces.

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جامعة النجاح الوطنية
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التقريب الأفضل في فضاءات القياس المخروطي

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قدمت هذه الاطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية، نابلس فلسطين.
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الملخص

قمنا بإجراء دراسة لنوعين من التقريب الأمثل في فضاءات القياس العادية و فضاءات القياس المخروطية.

كما قمنا بمحاولة زرع لنظريات التقريب الموجودة في فضاءات القياس العادية في فضاءات القياس المخروطية .

كما تطرقنا لفكرة الـ

Proximinal additivity and ϕ_c– summand in cone –Banach space.

وتوصلت هذه الدراسة إلى بعض النتائج الجديدة ومنها أننا يمكننا تطبيق نظريات التقريب الامثل الموجودة في فضاءات القياس العادية إلى فضاءات القياس المخروطية .