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THE FARTHEST POINT PROBLEM IN NORMED SPACES

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Dedication

إلى من غرسوا في نفسي روح التحدي وحب العلم منذ نعومة أظافري....... أمي وأبي

إلى من قدمت كل المساعدة والدعم والعطاء......... والدة زوجي الغالية

إلى الداعم الأول والسند...... شريك حياتي زوجي الحبيب

إلى فلذات كبدي و مصدر السعادة والإلهام....... إلى ابنتيً الغاليتين

إلى أخوتي وعائلتي الغاليه

إلى كل من دعمني وساندني أهدي ثمرة جهدي المتواضع

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وأتقدم بجزيل الشكر و الإمتنان لكل من ساندني وقدم يد العون والمساعدة لي وأخص بالشكر الدكتورالفاضل معاذ الكركي لما قدمه لي من نصائح و توجيهات ولدعمه المستمر ولمساعدته لي لاتمام هذه الاطروحه بهذا الشكل, كما وأتقدم بالشكر لكل الهيئة التدريسيه في قسم الرياضيات في جامعة النجاح الوطنية.

وأشكر أساتذتي أعضاء لجنة المناقشة الدكتور خالد عداربه والدكتور عبد القادر مصطفى الذين لم يتوانوا لحظة عن تقديم كل ما هو مفيد، وأشكرهم أيضاً على تفضلهم بقبول مناقشة هذه الد ارسة، واسداء النصح لي في استكمال ما فاتني من ضعف أو قصور.

والشكر موصول لعائلتي وصديقاتي في العمل على كلماتهم وتشجيعهم ودعمهم المتواصل لي.

Declaration

I, the undersigned, declare that I submitted the thesis entitled:

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I declare that the work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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Date: 13/3/2023

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Notations

- $B_r(x) = \{y \in X : d(x, y) = r\}$: Sphere with center x and radius r.
- $B_r\overline{(x)} = \{y \in X : d(x, y) \leq r\}$: Ball with center x and radius r
- U(x)=$\{x \in X : ||x|| \leq 1\}$ Unit ball,
- S(x)=$\{x \in X : ||x|| = 1\}$ Unit sphere.
- $D(x, E) : X \to$ R Farthest distance function.
- $P(x, E) : X \to P(E)$ Farthest function.
- $r(E) = \inf D(x, E)$ Chebyshev radius.
- $[x, y] = \{z \in X : d(x, z) + d(z, y) = d(x, y)\}$ Line segment between x and y.
- $[x, y, -[= \{z \in X : d(x, y) + d(y, z) = d(x, z)\}$Half ray from x passing through y.
- $K(E)_{m,n}$ Cardinality of a set $E \cap [m, n]$.
- $d(E)$ natural density of a set E.
- UR: is an abbreviation of uniquely remotal.

THE FARTHEST POINT PROBLEM IN NORMED SPACES

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Abstract

In this study, the researcher discusses the longstanding problem in Approximation Theory, which is called the Farthest Point Problem (FPP).

The FPP is partially an unsolved problem asking whether every uniquely remotal set in a Banach/Metric space is a singleton.

The researcher considers the convex metric space and demonstrates that every bounded subset is singleton if and only if *SF* condition is satisfied. Also, the researcher focuses on the Banach spaces. Firstly, in normed space the researcher proved that the singletoness occurs if partially continuous condition of farthest point map P satisfies. Then, the researcher takes a specific space in Banach space, the sequence space $\ell 1$(R) and shows the positive answer that every uniquely remotal subset of $\ell 1$(R) is singleton.

Finally, the researcher presents partially ideal statistically continuous notion of far- thest point map P and presents the main result, provided that E is a bounded, uniquely remotal set in a Banach space X over R with a Chebyshev center c and the farthest point map P defined on $[c, P(c)]$ is partially ideal statistically continuous at c, then E consists of one element only.

Keyword: A Banach/Metric Space; Approximation Theory; Convex Metric Space; Partially Ideal Statistically Continuous.

Chapter One
Introduction

The main objective of this thesis is to study the Farthest Point Problem (FPP), which is an open problem in Approximation Theory: for a normed space X, if E is uniquely remotal subset of X, then E must consist of one element? (Precise definitions below)

Many researchers and mathematicians study this problem and want to solve it, but in general, no positive solution in infinite dimensional space is found right now. We only have a positive answer in specific cases with additional conditions.

In 1961, Klee [11] studied this problem. He was the first one who found the direct connection between the farthest point problem and the nearest point problem which questions if Chebyshev sets are convex in Hilbert spaces (The later question is related to best approximation). Those two problems are equivalent, that is why farthest point problem gained its importance.

In 1983, Astaneh [3] obtained a partial positive answer; he proved that if a subset E that every element in normed space X has unique farthest point with respect to E (uniquely remotal) admitting a center O and satisfies the continuous condition (this will be described in details in Chapter 2), then E must be singleton. After that, Baronti [4] proved that a Banach Space X with a center O. If E be a remotal set such that and there exists a center O in the collection of Chebyshev centers of E such that $P(O, d)$ is true with $d > 0$ then E is a singleton. In 2010, Sababheh and Khalil [10] demonstrated that if E is a uniquely remotal set of a normed space, having a center O and if P defined on $[O, P(O)]$ is continuous at O then E consists of only one element. In 2017, Sababheh et al. [20] showed that if a compact subset E of a normed space X, with a center is uniquely remotal with respect to its center, then E is a singleton.

In this thesis, we firstly introduce the background that contains definitions, theorems and examples to reach the complete description of our open problem. In the first section, we define the Banach space with examples and also a special spaces of a normed space as dual spaces and inner product space. In addition, we present the Hahn-Banach theorem.

In the next section, we move to a closely related subject to our problem called approximation theory in the nearest point problem, and we consider this subject because the strong relation between the two problems is such a positive solution that one of them leads directly to the solution of the other as Klee shows in [11], so in the normed space X we define the best approximation to an element w out of a subspace L by $y_\circ$(if exists) such that:

$$d(w,L) = \|w - y_0\| = \inf\{\|w - \beta\|, \beta \in L\}$$

and we present some basic facts about the best approximation to an element w by introducing some theorems and Haar condition definition. After that, we return to our main subject and define the basics we need as farthest distance function $D(x,E)$, remotal set and UR set, also we explain different condition for a set E to be remotal set and define the convex metric space. In the last section of Preliminaries, we present Chebyshev sets, Chebyshev radius and Chebyshev centers with examples. We also introduce the meaning by nearly compact set and uniformly convex space to clarify when the set has a unique Chebyshev center.

The next chapter, we reproduce all results of papers that study the open problem specifically in convex metric space. Firstly, we consider the paper of Sangeeta and Narang, [15] On the farthest points in convex metric spaces and linear metric spaces which concentrate about the farthest point problem in a convex metric space and we study the relation between farthest point problem and nearest point problem. The following are some discussed topics in this section:

1. The existence of a farthest point $z \in E$ from an element x leads to the existence of nearest point also, such that z will be a nearest point to a specific point on line between x and z on the opposite side of x.

2. The definitions of convex metric space, M-space and we discussed the relation between them which is as follows: every strictly convex metric space is an M-space but the converse is not.

3. Explaining the condition on a subset E to be singleton, we have, a bounded subset E in a strictly convex metric space (X, d) has property SF if and only if E is singleton.

Secondly, in this section we consider the paper in 2011 of Sangeeta and Narang, On singletonness of remotal and UR sets which discussed the property of remotal sets and clarify theorems which aims at getting a positive answer to farthest point problem with minimum conditions. The most important result and theorems we show is (the specified space is the externally convex metric space):

1. When E be UR subset of the space (X,d). Then E is singleton if and only if

$$d(\lambda, P(\lambda)) = d(\eta, P(\eta)) \Longrightarrow P(\lambda) = P(\eta).$$

2. The existence of center $x_\circ$ in a bounded subset E by state the propositions and condition for that also we showed the proofs.

3. Explaining the result that if E is UR subset of convex metric space (X,d) and O is a center of E, then property $P(O, \Lambda)0$ is not satisfied.

In the next chapter, we dealt with our topic in more specifically. So we consider the studies and papers in normed linear space and in $l^1(\mathbb{R})$. In the first section, Panda and Kapoor in {On farthest points of sets} study our problem in normed linear space and clarify the condition must be satisfied to get a positive result. In the second section, we reformulate the paper of M.Sababheh et al. ,[20] Uniquely remotal sets in Banach spaces, and proved all important result .So we consider the following topics in this section:

1. The singletoness of a UR subset M of a normed space depending on partially continuous condition.
2. The relation between singleton of a remotal set and isolation result, we show that if M is a non singleton set, then surely $P(O)$ can be isolated which means that $P(\lambda) - P(O) > \delta$ for some $\epsilon > 0$ in the neighborhood of the centre O.
3. The main result in this section is proving that in a normed space, if a compact subset M with centre O is UR then M must be singleton.

The last section considers the newest studies about farthest point problem. In 2019 Yousef, Abdelrahman and Khalil, Roshdi and Mutabagani, B. in the paper" On the farthest point problem in Banach spaces" discussed our problem in a specific space (the sequence space $l^1(\mathbb{R})$). They showed that in $l^1(\mathbb{R})$ every UR subset must be singleton. For this, we define a relation $\leq$ on the set of closed balls

$$(J = \{\overline{B_{\|y-e_\circ\|)}(y)} : P(y) = e_\circ\})$$

by

$$B_1 \leq B_2 \; if \; B_2 \subseteq B_1$$

and they proved that J has a maximal element. Finally, we use this fact to present the principle theorem.

The last chapter, we define the statistical continuity (weaker than the standard continuity) and statistically convergent sequences. In this case we have the following generalization, if E is uniquely remotal with a chebyshev center $e_\circ$ and the FPM $P : X \to E$ limited to $[e_\circ, P(e_\circ)]$ is partially ideal statistically continuous at $e_\circ$ then E is singleton. This result is due to [24].

The origins of the farthest point problem stem from the best approximation, which is an important topic and has many applications in mathematics, physics and engineering.

Chapter Two

Preliminaries

This chapter is devoted for the basic backgrounds. The farthest point problem stemmed from approximation theory. So, we will introduce Banach spaces, best approximation, remotal sets and Chebyshev sets.

2.1 Banach Spaces

In this section we recall the basic theory of Banach spaces [22, 19, 6]. Let W be a real or complex vector space. A norm on W is a function that takes each vector w in W into a non negative number $\|w\|$ such that

1. $\|w\| = 0$ if and only if $w = 0$
2. $\|\gamma w\| = |\gamma|\|w\|$, for every w in W and every scalar γ.
3. $\|w + v\| \leq \|w\| + \|v\|$ for every w, v in W.

A normed space is said to be complete if every Cauchy sequence (x_n) in W converges. A complete normed space W said to be a Banach Space.

The basic examples of Banach spaces include $\mathbb{F}^n$, with the norm

$$\|x\| = \sqrt{\sum_{j=1}^{n} |\xi_j|^2},$$

$x = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{F}^n$.

Another example of Banach spaces are l^p spaces, it is a Banach Space with norm defined by

$$\|x\| = \left(\sum_{j=1}^{n} |\xi_j|^p \right)^{1/p}$$

Also l^∞ is a Banach space with the norm

$$\|x\| = \sup_j |\xi_j|.$$

An essential space is $C[a, b]$, the space of all continuous real valued function defined on $[c, d]$ with norm defined by

$$\|z\| = \max_{\alpha \in [c,d]} |z(\alpha)|.$$

Definition 2.1.1. [12] Let X be a normed space. Then the dual space of X is the set

$$X^* = \{\phi : \phi \text{ is a bounded linear functional on } X \text{ to } \mathbb{R} \}$$

with the norm.

$$\|f\| = \sup_{x \in X} \frac{|f(x)|}{\|x\|} = \sup_{x \in X, \|x\|=1} |f(x)|$$

Theorem 2.1.2. [12] X^* is a Banach space.

Theorem 2.1.3. [12] Consider the real vector space X and let q is a sublinear functional on X. Also, suppose f is a linear functional defined on Y the subspace of X and let f satisfies $f(x) \leq q(x)$ $\forall x \in Y$. Then f has a linear extension $f^\sim$ from Y to X satisfying:

$$f^\sim(x) \leq q(x) \ \forall x \in X$$

We have $f^\sim$ is a linear functional on X and $f^\sim(x) = f(x)$ for every $x \in Y$.

As a special case of a normed space we have an inner product space and a Hilbert space, so we will define it.

Definition 2.1.4. [12] A space W is called inner product space if it is a vector space with inner product $< w, z >$, which is attached to a complex number for every pair

$(u,v) \in X^2$ and satisfies these conditions, for all vectors z, u and w, scalars γ we have:

1. $< z+u, w > = < z, w > + < u, w >$
2. $< \gamma z, v > = \gamma < z, v >$
3. $< u, z > = \overline{< z, u >}$
4. $< w, w > \geq 0$, and $< w, w > = 0 \iff w = 0$

Also this inner product defines a norm on W defined by:

$$\|w\| = \sqrt{< w, w >}$$

Hence, the inner product space classified as normed space. Also, Hilbert spaces are Banach spaces.

As a matter of fact, we can not say that all normed spaces are inner product spaces and the norm on inner product space must obey the parallelogram equality

$$\|\phi + \psi\|^2 + \|\phi - \psi\|^2 = 2\|\phi\|^2 + 2\|\psi\|^2.$$

Otherwise we conclude that the space cannot be inner product space.

2.2 Best Approximation

Farthest point problem is strongly related to Best approximation. So, in this section we will introduce the Best Approximation theory in normed spaces. We follow the terminology of [12].

Let $(X, \|.\|)$ be a normed space, and Y be a subspace of X.For $\phi \in X$ define

$$d(\phi, Y) = \inf_{\psi \in Y} \|\phi - \psi\|.$$

Definition 2.2.1. [12] Let Y be a subspace of a normed space X. If there exists ψ_0 in Y such that

$$\|\phi - \psi_0\| = d(\phi, Y)$$

then ψ_0 is called the best approximation to ϕ out of Y.

The existence of such ψ_0 is a main and historical question in mathematics and attracted many mathematicians for many decades. It is clear that if Y is finite dimensional subspace of X, then for any x in X there exists a best approximation to x out of Y, [12]. In general existence and uniqueness of best approximation are not guaranteed, we present some cases where existence and uniqueness are ensured. Strictly convex normed spaces are of special interest.

Definition 2.2.2. [12] Let X be a normed space, if for all ψ, ϕ of norm 1 we have,

$$\|\psi + \phi\| < 2 \qquad (\psi \neq \phi),$$

then we say that X is a strictly convex normed space.

Strictly convex normed spaces can have at most one best approximation.

Theorem 2.2.3 (Uniqueness Theorem). [12] Consider Y be a subspace of a normed space X. If X is strictly convex then there is at most one best approximation to an x in X out of Y.

Applying of uniqueness theorem depends on what space we use. For example, In [12] Hilbert spaces are strictly convex but the space $C[a, b]$ is not.

As a result, Hilbert spaces clearly satisfy the uniqueness of best approximation condition. However to deal with other spaces we need to introduce the approximation in $C[a, b]$ with the supremum norm, this approximation is called the uniform approximation.

The following condition will be essential and sufficient for the uniqueness of approximations in the space $C[a,b]$.

Definition 2.2.4. [12] Let Y be a subspace of $C[a,b]$ and of finite dimension. Y is said to satisfy the Haar condition if every non zero function y in Y has at most $(n-1)$ zeros in $[a,b]$ such that $n = \dim Y$.

Theorem 2.2.5. [12] Let Y be a subspace of $C[a,b]$ and of finite dimension. Then a necessary and sufficient condition for each x to have unique best approximation is Y satisfies the Haar condition.

We present results that help to determine whether certain spaces satisfy the uniqueness of best approximation by checking Haar Condition. Let $\mathcal{P}_n$ denote the polynomial of degree less than or equal n, for some $n \in \mathbb{Z}$.

Theorem 2.2.6. [12] The best approximation to an x in $C[a,b]$ out of $\mathcal{P}_n$ is unique.

At the end of this section we confirm the strong relation between existence of nearest point and farthest point by the following theorem:

Theorem 2.2.7. [18]Suppose X is a normed linear space and A be a closed bounded convex subset such that $0 \in intA$, Then a necessary and sufficient condition for A to be remotal (antiremotal) is that $cS(0;d)$ is proximinal(antiproximinal) for any $d > 0$ with respect to P_A

where P_A is the Minkowski functional associated to the set A. Given that proximinal set means

Definition 2.2.8 (Proximinal Set). Consider a metric space (X,d) and a subset S, S is called proximinal set if for each point $x \in X$ there exists a point $s \in S$ such that s is nearest to x.

If there is a unique such point s for all $x \in X$, then S is said to be uniquely proximinal (Chebyshev).

2.3 Farthest Function P (x), Remotal Set E

In this section we present our problem. We present remotal sets, farthest distance function $D(.,E)$ and their basic properties.

Definition 2.3.1. [11] Suppose X is a normed space and $E \subset X$ be closed and bounded .

1. We define the farthest distance function $D(.,E) : X \to \mathbb{R}$ by

$$D(x,E) = \sup\{\|x-e\| : e \in E\}$$

2. E is called Remotal set if for all x in X, there exists e in E such that

$$D(x,E) = \|x-e\|.$$

3. We define the farthest function

$$\begin{aligned} &P(\cdot,E) : X \to \mathcal{P}(E), \\ &\qquad x \mapsto P(x,E) = \{e \in E : D(x,E) = \|x-e\|\}. \end{aligned}$$

In general $P(x,E)$ need not be a singleton. But if it is a singleton then E is called uniquely remotal, more precisely:

Definition 2.3.2. [11] In normed space X, closed and bounded subset E. If $P(x,E)$ has a unique point then E called a uniquely remotal set.

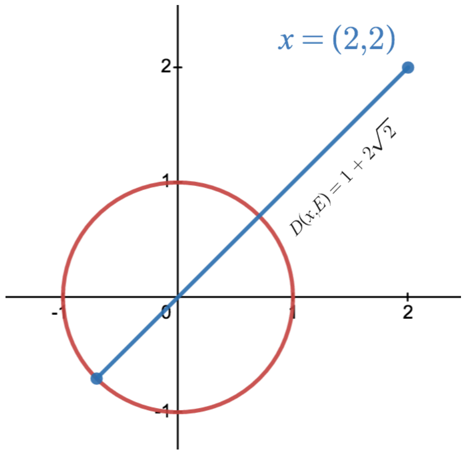
Consider the following example.

Example 2.3.3. Let $X = \mathbb{R}$ with the standard norm. Let

$$\{E := x_1^2 + x_2^2 = 1\},$$

be the unit circle. Then it is not difficult to see that $D((2,2),E) = 1+2\sqrt{2}$, and the supremum (max in this case) is attained at one point namely at $(-1/\sqrt{2}, -1/\sqrt{2})$

and $D((0,0),E)=1$ and the supremum (max in this case) is attained at all points of E.



Remotal sets have special significance in this thesis , so we discuss it in more details. And the condition on X give that every closed and bounded subset E be remotal when the space$X$ is a Banach space having a monotone basis. Hence We remind that a monotone basis is a basis $\{e_1, e_2, \ldots\}$ such that:

$$\mathrm{span}\{\mathrm{e_1}, \mathrm{e_2}, \ldots, \mathrm{e_n}\} \perp \mathrm{span}\{\mathrm{e_{n+1}}, \mathrm{e_{n+2}}, \ldots, \mathrm{e_{n+m}}\},$$

for each natural numbers n and m, where $w \perp v$ means

$$\|w\| \leq \|w+\alpha v\|$$

for each real number α. [4].

Theorem 2.3.4. [4] Let X be a Banach space admitting a monotone basis. Let $E \neq \emptyset$ be closed and bounded subset. Then a sufficient and necessary condition of E to be remotal is that X is finite dimensional.

Proof. The first part is clearly true, that if X is a finite dimensional space then obviously E will be a remotal set.

Secondly, we want show that if E remotal set then X must be finite dimensional.

For sake of contradiction, let X be an infinite dimensional Banach space, and $\{e_n\}$ be a monotone normalized basis. Now, consider the set E to be:

$$E = \left\{ \left(1 - \frac{1}{n}\right) e_n : n \in \mathbb{N} \right\}$$

Here E is remotal, since it is closed and bounded. But we get:

$$\sup \left\{ \left\| (1 - \frac{1}{n}) e_n \right\| : n \in \mathbb{N} \right\} = 1 > \left\| (1 - \frac{1}{n}) e_n \right\|,$$

for any $n \in \mathbb{N}$ So, E is not remotal and this make a contradiction.

□

In Hilbert spaces the condition for a set to be remotal is easier, the next theorem shows that.

Theorem 2.3.5. [4] Consider a Hilbert space X. If E is bounded and closed subset then it is remotal.

Theorem 2.3.6. [18] A nonempty bounded set E in a linear normed space X is remotal if and only if the following associated set

$$K_d = A + cS(0; d)$$

is closed for every $d > 0$. (cS(0;d) defined as the closure of circle with center (0,0) and radius=d)

Definition 2.3.7 (Convex Metric Space). A convex metric space (X, d) is a metric space such that for any two distinct points x, y in X, we have at least one point z in X with $x < z < y, 0 \leq t \leq 1$

$$d(x, z) = (1 - t)d(x, y),$$

and

$$d(z, y) = td(x, y)$$

Also, there is another definition of convex metric space depends on convex structure W (the reader can found in [16])

In metric space (X,d), a subset E is convex when any two points $x, y \in E$, any point lie between x and y is also in E.

For remotal sets in convex metric space we have:

Proposition 2.3.8. [15]A bounded subset E of (X, ρ) is remotal if and only if $Conv(E)$ is remotal.

This proposition satisfy also in normed linear space([5])

2.4 Chebyshev Sets

Definition 2.4.1 (Chebyshev Set). : E is Chebyshev set if for all x in X, there exists unique e in E which is the best approximation of x in E.

In metric space (X, d) and a subset E, we consider the problem of choosing an element x of X which represent the set E. Then the error incurred when choosing x will be sup$\{d(x, y) : y \in E\}$ = D(x,E), so to have the minmum value of the error we make this assumption:

An element $\alpha \in X$ will best represent the set E when

$$D(\alpha, E) = \inf_{x \in X} D(x, E)$$

which lead to the Chebyshev center definition below:

Definition 2.4.2 (Chebyshev Center). Let X be a normed(metric) space, a chebyshev center of a subset E is $\alpha \in X$ such that:

$$D(\alpha, E) = \inf_{x \in X} D(x, E)$$

.

A Chebyshev Center of E is also defined as the center of minimal closed ball containing E. This concept is an important motif in the optimization theory. Hence a several studies talk about the existence and uniqueness of chebyshev center which we present some of these studies(for more details see [13]).

For example,

Definition 2.4.3 (Chebyshev Radius). The number

$$r(E) = \inf_{x \in X} D(x, E)$$

called chebyshev radius, also can describe as the smallest radius of balls in X that contains E.

Centers of all such balls are just the centers of E .This means

$$Z(E) = \{x_E \in X : D(x_E, E) = r(E)\}$$

is the set of centers of balls have minimal radius covering E.

Example 2.4.4. The chebyshev radius of the line segment [a,b] in real line $\mathbb{R}$ is 1/2(b-a) and it is chebyshev center 1/2(a+b).

Example 2.4.5. In $(\mathbb{R}^2, ||.||_\infty)$ the centers of the set$\{(x, 0) : |x| \leq 1\}$ is the set $\{(0, y) : |y| \leq 1\}$

Definition 2.4.6 (Nearly Compact). E is nearly compact set if any maximizing sequence $\{y_n\}$ in E for an $x \in X$ has a convergent sub-sequence in. (A sequence $\{y_n\}$ in a subset E of X is maximizing sequence for $x \in X$ if

$$\lim_{n \to \infty} = D(x, E))$$

Let E be a subset of metric space (X,d) and $z \in E$, we define

$$E_z = \{x \in X : d(x, z) = d(x, E)\}$$

which is the set of all points of X having z as nearest point in E. [16]

Theorem 2.4.7 (Unicity Theorem). [16] Consider a strictly convex metric space X. If a convex set E is proximinal then E will be uniquely proximinal.

The converse of this theorem also true with condition (for details return to [16]

Definition 2.4.8 (Uniformly convex). [13] A convex metric space (X,d,W) is said to be uniformly convex if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $r > 0$, given x,y,p $\in X$ with

$$d(x,p) \leq r, d(y,p) \leq r$$

and

$$d(W(x,y,1/2),p) > r - \epsilon,$$

we have

$$d(x,y) \leq \epsilon.$$

As an example of uniformly convex spaces: every uniformly convex Banach space is uniformly convex metric space. Goebel, Sekowski and Stachura in [7] showed that a hyperbolic metric is uniformly convex in some sense.

Considering the existence and uniqueness of chebyshev centers, we have the following results of Smith[8]:
1) If X is strictly convex Banach space, then every compact set in X has at most one center in X.

2) If X is an E-space, then each compact convex set in X has a unique center.

3)It is known that every bounded subset of uniformly convex Banach space has a unique center.

4)In ([23]) we have a generalization of previous result, if X is a reflexive strictly convex dual Banach space, then every convex remotal set E in X has unique center.

After that in 2019 Narang reach to new result. He study convex metric space and show that in uniformly convex metric space every bounded subset has a unique center.

In [13] Narang state and proved many theorems and lemma to reach to the result above. Here we will state some of them:

Theorem 2.4.9. Let E be a remotal subset of a strictly convex metric space , then Z(E) is at most a singleton.

Theorem 2.4.10. Let E be a remotal subset of a strictly convex metric space (X, d, W) such that

$$P : X \longrightarrow \mathbb{R}$$

attains it is infimum on X, then $Z(E)$ is exactly a singleton.

By Lemma 2.2 in([13]) a continuous mapping defined on a compact space always attains it is infimum. so we get this corollary:

Corollary 2.4.11. Let E be a remotal subset of a strictly convex compact metric space then Z(E) is exactly a singleton.

Theorem 2.4.12. Let E be a bounded subset of a uniformly convex metric space (X,d,W), then Z(E) is at most a singleton.

Theorem 2.4.13. Let E be a bounded subset of a uniformly convex metric space (X,d,W) such that$P : X \longrightarrow \mathbb{R}$ attains it is infimum on X, then Z(E) is exactly a singleton.

Chapter Three
Farthest Point Problem in Metric Space

3.1 Farthest Point Problem in Convex Metric Space

In this section we will study and represent the most important result related to the Farthest point problem in convex metric spaces.

In 2014, Sangeeta and Narang studied the Farthest point problem and dedicate their studies on the convex metric space and relation between farthest point and nearest point problems.(for details we point out the reader to [15])
Consider (X, d) is an externally convex M-space and let E be a bounded closed subset of X. for y in$E$ be a farthest point with respect to $x \in X$ then we have also y as a nearest point in E from any $t \in [x, y >$.

As a result, if there exists no nearest point in E then there exists no farthest point in E.[15]

Definition 3.1.1 (Convex Metric Space). A mapping $W : X \times X \times [0,1] \longrightarrow X$ is called a convex structure on X if for any $x, y \in X$ and $\alpha \in [0,1]$

$$d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1-\alpha) d(u, y), \quad for\ all\ u \in X$$

A metric space (X, d) with the convex structure becomes a convex metric space.

Now we consider a specefic types of convex metric spaces namely M-space

Definition 3.1.2 (M-space). [15] (X, d) called an M-space if it is convex metric space such that for any distinct points x, y in X with $d(x, y) = \rho$ and all $r \in [0, \rho]$, we have a unique point z_r in X satisfies:

$$\overline{B_r(x)} \cap \overline{B_{\rho-r}(y)} = \{z_r\}$$

where, $\overline{B_r(x)} = \{y \in X : d(x, y) \leq r\}$.

Definition 3.1.3. [9]There is another equivalent definition of strictly convex space. A convex space (X, d) is named strictly convex if for $x, y \in \overline{B_r(z)}$ such that $d(x, y) = \lambda$, then

$$\overline{B_{(1-t)\lambda}(x)} \cap \overline{B_{t\lambda}(y)} \subseteq \overline{B_r(z)} \ \forall t \in (0,1)$$

It is not difficult to show that every strictly convex metric space is an M-space. Indeed:

Proof. let $x, y \in X$ and $d(x, y) = \lambda$
since (X, d) convex metric space then the set $\overline{B_{(1-t)\lambda}(x)} \cap \overline{B_{t\lambda}(y)}$ not empty for all $t \in (0,1)$.
Using contradiction method,
assume $z_1, z_2 \in \overline{B_{(1-t)\lambda}(x)} \cap \overline{B_{t\lambda}(y)}$, by strict convexity we have

$$\overline{B_{(1-s)\xi}(z_1)} \cap \overline{B_{s\xi}(z_2)} \subseteq \overline{B_{(1-t)\lambda}(x)} \cap \overline{B_{t\lambda}(y)}$$

for all $s \in (0,1)$ where $\xi = d(z_1, z_2)$ and this true only if $z_1 = z_2$
this mean the intersection contain unique point.
Hence, (x, d) is an M-space. ☐

but converse not true and this example show that:

Example 3.1.4. [9] Define

$$S_r = \{(x, y, z) : x^2 + y^2 + z^2 = r^2\},$$

be the two dimensional spherical space of radius r in $\mathbb{R}^3$. Let $X = \{w \in S_r : d(w, (0,0,r)) \leq \rho\}$ be a closed ball of S_r with $\pi r/4 < \rho < \pi r/2$. Clearly X is convex space and contains no dimetral points pairs of $S_{2,r}$ (this means that any intersection of two balls will contain a unique point), hence (X, d) is an M-space.

However, if we take points a, b and c in X which satisfy $d(a, b) = d(a, c) = \frac{\pi r}{2}$, then any point $u \in X$ which lies between b and c has property $d(a,u) = \frac{\pi r}{2}$ and this contradicts the strict convexity definition.

Proposition 3.1.5. [15]) Let E be bounded subset of M-space and $e_\circ \in P(x_\circ)$ then $e_\circ \in P(x)$ for all $x \in [e_\circ, x_\circ > \backslash [e_\circ, x_\circ]$

Definition 3.1.6. (Property SF)
[14] In convex metric space (X,d), a bounded subset E satisfy SF property if $x_\circ \in X$ and $e_\circ \in P(x_\circ)$ imply $e_\circ$ is a farthest point from y, forall $y \in [e_\circ, x_\circ]$

Based on the SF property we have the following results.

Proposition 3.1.7. In convex metric space (X,d) a bounded subset E has the previous property if and only if E has a single point.

Now we define the $P(x,\lambda)$ property in convex metric space.

Definition 3.1.8. $P(x,\lambda)$
Let (X,d) be a convex metric space and $E \subseteq X$,we say $P(x,\lambda)$ is true for some $\lambda \in (0,1)$ if $z \in P(x), z' \in [x,y]$ such that $d(z',z) = (1-t)d(x,z)$,
and $d(z',x) = td(x,z)$ for $0 < t \leq \lambda$, imply that $z \in P(z')$

this mean if z is the farthest point from x in E then z is also a farthest point from $W(x,z,t)$.
we introduce some example explain the previous property:

Example 3.1.9. [14] Consider $X = \mathbb{R} \backslash 0$ with usual metric and $T = [-1,1] \backslash 0$ In this space the property $P(x,\lambda)$ is true for $x = 1,-1$ and $\lambda = 1/2$ $P_T(-1) = 1, P_T(x) = 1$ forall $x \in [-1,0)$
then

$$P_T(\alpha y + (1-\alpha)x) = 1, 0 < \alpha \leq \lambda$$

Example 3.1.10. Let $E = \{(x,y) : x = -\sqrt{1-y^2}, -1 \leq y \leq 1\} \subseteq \mathbb{R}^2$ with usual metric, then clearly $P(x,\lambda)$ true for $z = (-1,0)$ and $\lambda = 1/2$ but $P(x,\lambda)$ not true for $z = (0,0)$.

It is not difficult to show that, if E is a remotal subset of convex metric space (X, d), then $P(x, \lambda)$ cann't be true if c is the center of the set.([15] p.233)

3.2 Farthest Point Problem in Strictly Convex Metric Space

In 2011 [14] also Sangeeta and T.D.Narang discuss properties of remotal and uniquely remotal set and restrictions under which remotal sets are singelton in convex metric space. At first, they introduce phrases and propositons, we will present some of them to reach final and main result.

Theorem 3.2.1. [14] Let X be an externally convex M-space and suppose E is uniquely remotal subset. Then E is singelton if and only if

$$d(a, P(a)) = d(b, P(b)) \Longrightarrow P(a) = P(b)$$

Proof. Firstly, suppose E has a single point then $P(a) = P(b)$ for all $a, b \in X$ and so the condition absolutely is satisfied.

Conversely, if the condition is true we want to show E is a singleton.(we show that for all $a, b \in X, P(a) = P(b)$). Consider $a_\circ, b_\circ \in X$ $(a_\circ \neq b_\circ)$, without loss of generality assume that:

$$0 < d(a_\circ, P(a_\circ)) < d(b_\circ, P(b_\circ)).$$

Take

$$c_\circ \in [P(a_\circ), a_\circ, -[\ \backslash [P(a_\circ), a_\circ]$$

be such that: $d(c_\circ, P(a_\circ)) = d(b_\circ, P(b_\circ))$, but we know that:

$$\begin{aligned} d(b_\circ, P(b_\circ)) &= d(c_\circ, P(a_\circ)) \\ &\leq d(c_\circ, P(c_\circ)) \\ &= d(c_\circ, a_\circ) + d(a_\circ, P(c_\circ)) \\ &\leq d(c_\circ, a_\circ) + d(a_\circ, P(a_\circ)) \\ &= d(c_\circ, P(a_\circ)). \end{aligned}$$

Hence $d(b_\circ, P(b_\circ)) = d(c_\circ, P(a_\circ)) = d(c_\circ, P(c_\circ))$. The hypothesis implies

$$P(b_\circ) = P(c_\circ) = P(a_\circ)$$

□

Definition 3.2.2. [14] Let E be a uniquely remotal subset of a convex M-space (X, d), then a point $x \in X$ is said have property ϵ_x if

$$w \in [x, P(x)] \Longrightarrow w \in A_x,$$

Given,

$$A_x = \{z \in X : d(z, P(z)) \geq d(x, P(x))\}$$

.

Proposition 3.2.3. [14] Let X be an externally convex M-space and suppose E is uniquely remotal subset. Suppose $x_\circ$ is the unique Chebyshev centre with respect to $A_{x_\circ}$. Then

$$X = A_{x_0} \cup [x_\circ, P(x_\circ)]$$

in X.

Proof. Let

$$A'_{x_\circ} = \{x \in X : d(x, P(x)) < d(x_\circ, P(x_\circ))\}$$

be the complement of $A_{x_\circ}$. So, we have to show that

$$A'_{x_\circ} \subset [x_\circ, P(x_\circ)].$$

Consider $x_1 \in A'_{x_\circ}$, that is

$$d(x_1, P(x_1)) < d(x_\circ, P(x_\circ)).$$

If $x_1 = P(x_1)$, then absolutely E is singleton and therefore $x_1 = P(x_\circ) \in [x_\circ, P(x_\circ)]$. Let $x_\circ \neq P(x_\circ)$ and $x_1 \neq P(x_1)$, choose $y_\circ \in [P(x_1), x_1, -[$ such that

$$d(y_\circ, P(x_1)) = d(x_\circ, P(x_\circ))$$

then,

$$d(x_\circ, P(x_\circ)) = d(y_\circ, P(x_1)) \leq d(y_\circ, P(y_\circ)) \tag{3.1}$$

and

$$d(y_\circ, x_1) = d(y_\circ, P(x_1)) - d(x_1, P(x_1)) \tag{3.2}$$

so,

$$d(y_\circ, x_1) = d(x_\circ, P(x_\circ)) - d(x_1, P(x_1)) \tag{3.3}$$

hence,

$$d(x_\circ, P(x_\circ)) = d(y_\circ, x_1) + d(x_1, P(x_1)). \tag{3.4}$$

Since

$$d(y_\circ, P(y_\circ)) \leq d(y_\circ, x_1) + d(x_1, P(y_\circ)),$$

(3.1) implies

$$\begin{aligned} d(x_\circ, P(x_\circ)) &= d(y_\circ, P(x_1)) \\ &\leq d(y_\circ, P(y_\circ)) \\ &\leq d(y_\circ, x_1) + d(x_1, P(y_\circ)). \end{aligned}$$

Also,

$$d(x_1, P(y_\circ)) \leq d(x_1, P(x_1)),$$

by the definition of farthest point map P with respect to x_1. (3.4) implies

$$\begin{aligned} d(x_\circ, P(x_\circ)) &\leq d(y_\circ, x_1) + d(x_1, P(x_1)) \\ &= d(x_\circ, P(x_\circ)) \end{aligned}$$

Hence all previous inequalities will be equal that is

$$d(x_\circ, P(x_\circ)) = d(y_\circ, P(x_1)).$$

That means $x_\circ = y_\circ$ and $P(x_1) = P(x_\circ)$. So replace $y_\circ$ by $x_\circ$ in (3.3) to get the following

$$d(x_\circ, x_1) = d(x_\circ, P(x_\circ)) - d(x_1, P(x_1)),$$

in other words

$$d(x_\circ, P(x_\circ)) = d(x_\circ, x_1) + d(x_1, P(x_1))$$

. Since $P(x_1) = P(x_\circ)$ we conclude that

$$d(x_\circ, P(x_\circ)) = d(x_\circ, x_1) + d(x_1, P(x_\circ)),$$

which implies

$$x_1 \in [x_\circ, P(x_\circ)],$$

hence

$$A'_{x_\circ} \subset [x_\circ, P(x_\circ)]$$

□

Proposition 3.2.4. [14] Consider the externally convex M-space and E be uniquely remotal subset, then e_0 is a centre of E if and only if $[e_\circ, P(e_\circ)] \subset A_{e_\circ}$ i.e. e_0 satisfies property $\epsilon_{e_\circ}$.

Proof. The first direction, if $e_\circ$ is a centre of E then by definition of the centre

$$d(e_\circ, P(e_\circ)) = \inf_{x \in X} d(x, P(x))$$

i.e. $d(x, P(x)) \leq d(e_\circ, P(e_\circ))$ for all $x \in X$. So, by definition of $A_{e_\circ}$ this implies $X = A_{e_\circ}$. Hence, $[e_\circ, P(e_\circ)] \subset A_{e_\circ}$

Conversely, suppose $[e_\circ, P(e_\circ)] \subset A_{e_0}$, then by previous proposition

$$X = A_{e_0} \cup [e_\circ, P(e_\circ)] = A_{e_\circ}$$

So this gives

$$d(x, P(x)) \leq d(e_\circ, P(e_\circ))$$

for all $x \in X$. Hence

$$d(e_\circ, P(e_\circ)) = \inf_{x \in X} d(x, P(x)).$$

We get that, $e_\circ$ is a centre of E.

□

The next proposition represent that in general for remotal sets the property $P(e_\circ, \lambda)$ cannot be true if $e_\circ$ is the center of the set:

Proposition 3.2.5. [14] In convex metric space (X, d), let E be UR subset. If we have $e_\circ$ is a center of set E then the property $P(e_\circ, \lambda)$ is not satisfied.

Proof. Suppose the contrary, let $P(e_\circ, \lambda)$ be true and $e_\circ$ is a centre of E. this means $D(e_\circ, E) = \inf_{x \in X} D(x, E)$, So, $D(e_\circ, E) \leq D(x, E)$.

For all $x \in X$. Suppose $y \in P(e_\circ)$ be a farthest point of $e_\circ$ and since $P(e_\circ, \lambda)$ is true, then for all $y' \in [e_\circ, y]$ such that

$$d(y', y) = (1 - t)d(e_\circ, y)$$

and $d(y', e_\circ) = td(e_\circ, y)$ for $0 < t \leq \lambda$ we have $y \in P(y')$. This implies

$$D(y', E) = d(y', y) < d(e_\circ, y)$$

but $d(e_\circ, y) = D(e_\circ, E)$ so, $D(y', E) < D(e_\circ, E)$ which make a contradiction with the definition of a centre .

□

About the singletonness of only a remotal set, Sangeeta and T.D.Narang conclude this result by using the previous proposition.

Theorem 3.2.6. [14] In convex metric space (X, d), let E be UR subset demonstrate centers. If there is a center $e_\circ$ such that $P(e_\circ, \lambda)$ is true with $\lambda > 0$ then E is a Singleton.

For Banach space the previous result was also proved in [4].

Considering singeltonness of remotal sets in M-space, we have this result:

Theorem 3.2.7. Consider the M-space (X, d) and E be a remotal set such that for some $\epsilon > 0$, $\lambda > 0$ for which $P(y, \lambda)$ is true for every y in X satisfying the condition

$$r(E, y) < r(E) + \epsilon,$$

then E is a singleton.

Chapter Four
Farthest Point Problem in Banach Space

4.1 Farthest Point Problem in Normed Linear Space

More previously in 1978, Panda and Kapoor [17] studied the farthest point problem by define the known farthest point map and study the condition must happened to get the positive result of the open problem. To summarize their result we firstly introduce some definitions:

Definition 4.1.1. [17] In a normed linear space X a subset E is called an M-compact set when every maximizing sequence in E is compact.

And a sequence $\{g_n\}$ in E is called maximizing sequence if for t in X, $||t - g_n||$ converges to farthest distance between t and E.(i.e. $||t - g_n|| \Longrightarrow P(t)$).

In [17] Panda and Kapoor proved that in a normed linear space which admits centers any nonempty uniquely remotal M-compact set must be singleton.

We will consider the continuity of farthest point map P by these theorems:

Theorem 4.1.2. [17] Let X be reflexive Banach space satisfy M-compact condition, and let $E \neq \phi$ be a closed and bounded subset of X. Then there exists a subset T dense in X such that:

a) For t in T, we have any maximizing sequence in E for t is compact.

b) The F.P.M P restricted to $T$is upper semicontinous.

Theorem 4.1.3. [17] Suppose E is a uniquely remotal subset of a locally uniformly convex Banach space X. Then the F.P.M supported by E is continuous on a dense subset of X.

Proof. Consider the set

$$G = \cup_{x \in X} \{\alpha x + (1 - \alpha) P(x)\}$$

Clearly G is dense in X, and for all $\alpha \geq 1$,

$$P(\alpha x + (1 - \alpha) P(x)) = P(x)$$

If $x_n \longrightarrow \alpha x + (1 - \alpha) P(x)$, then $P(x_n)$ is a maximizing sequence for $\alpha x + (1 - \alpha) P(x)$ by the local uniform convexity of the norm

$$P(x_n) \longrightarrow P(x)$$

□

As we mentioned before, this open problem has been considered by many researchers and scientists. But except when X is finite dimensional no solution know for general infinite dimensional normed linear space. An Positive Partial results has been provided by Asplund [1] and others.
In [17] they used the idea of chebyshev center. As an examples of space which admit center we have all conjugate Banach spaces, the space $L^1(\mu)$ of absolutely integrable function and the space $C_R(\Omega)$ of real valued, bounded continuous functions (where Ω is paracompact)

Theorem 4.1.4. [17] Let X be normed linear space with centers, and let $E \neq \phi$ be UR subset of X. Suppose that for all x in $E + r(E)U(x)$, the farthest point map $q : X \longrightarrow E$ restricted to the line segment $[x, q(x)]$ is continuous at x, then E is singleton.

Corollary 4.1.5. [17] Let X be as above and let K be nonempty M-compact uniquely remotal subset of X, then K must be a singleton.

4.2 Farthest Point Problem in Normed Space

In this section we will discuss the farthest point problem on normed space by present the main theorems and previous results to prove the singletoness of just remotal set without need to the continuity of farthest point map condition. Specifically, we show the singletonnes of a compact set which demonstrate a unique farthest point to it's center.

Our aim here is to clarify the positive answer of farthest point problem if FPM P limited to [c,P(c)] satisfy the partial continuity definition at c which says as follow:

Definition 4.2.1 (Partial Continuous). [20] Let $P : E \longrightarrow X$ be a function and let $e \in E.P$ satisfy partial continuous definition at e if $\exists$ a non constant sequence $e_n \subset E$, such that $e_n \longrightarrow e$ and $P(e_n) \longrightarrow P(e)$.

The next aim is to consider the case of a non-singleton sets, and discuss the result that $P(e)$ will be isolated i.e. $\|P(x) - P(e)\| > \delta$ for some $\delta > 0$ and all x in a neighborhood of e. And, we use the isolation result to prove the our open problem.

Since the partial continuous of P is already weaker than the continuous condition and the function can be easily partially continuous we prove the singletoness by using the partial continuous condition as explained in the following theorem:

Theorem 4.2.2. [20]Suppose X is normed space and E bounded and closed subset in X with a center c. When E is uniquely remotal and

$$P : [c, P(c)] \longrightarrow E$$

satisfy the partial continuity definition at c, then E will be singleton.

Proof. We can assume, without loss of generality, that $c = 0$. Suppose E is not a singleton, therefore $P(0) = e \neq 0$. Since P restricted to $[0, e]$ is partially continuous at 0, then we have a non constant sequence $a_n \subset [0,e]$ such that $a_n \to 0$ and

$$P(a_n) \to e,$$

because $e \neq 0$ then

$$a_n = \lambda_n e.$$

for some positive sequence λ_n such that $\lambda_n \to 0$. For all $n \in \mathbb{N}$, let $\Lambda_n \in X^*$ be defined as follows

$$\Lambda_n(P(a_n) - a_n) = \|P(a_n) - a_n\|,$$

and $\|\Lambda_n\| = 1$. Now,

$$\begin{aligned}
\Lambda_n(a_n) &= \Lambda_n(P(a_n)) - \Lambda_n(P(a_n) - a_n) \\
&\leq \|\Lambda_n\|\|P(a_n)\| - \|P(a_n) - a_n\| \\
&= \|P(a_n) - 0\| - \|P(a_n) - a_n\| \\
&\leq D(0,E) - D(x_n,E) \\
&\leq 0.
\end{aligned}$$

Note that $D(0,E) = \inf_{x \in X} D(x,E)$, since 0 is considered as a center of E. We proved that $\Lambda_n(a_n) \leq 0$. but $a_n = \lambda_n e$, $\lambda_n > 0$ which implies that

$$\Lambda_n(\lambda_n e) \leq 0 \to \lambda_n \Lambda_n(e) \leq 0.$$

Hence,

$$\Lambda_n(e) \leq 0,$$

for all $n \in \mathbb{N}$. Now,

$$\Lambda_n(P(a_n) - a_n) = \|P(a_n) - a_n\| \to \|e - 0\| = \|e\|,$$

since $\|P(a_n)\| \to \|e\|$, and $\|a_n\| \to 0$. Also,

$$\begin{aligned} \Lambda_n(P(a_n) - a_n) - \phi_n(e) &= \phi_n(P(a_n) - a_n - e) \\ &\leq \|\Lambda_n\| \|P(a_n) - a_n - e\| \\ &= \|P(a_n) - a_n - e\| \to 0 \end{aligned}$$

Notice that both $\Lambda_n(P(a_n) - a_n)$ and $\Lambda_n(P(a_n) - a_n)$ - $\Lambda_n(e)$ converge, then we deduce the convergence of $\Lambda_n(e)$. Moreover

$$\begin{aligned} \lim_{n\to\infty} \Lambda_n(e) &= \lim_{n\to\infty} \Lambda_n(P(a_n) - a_n) \\ &= \lim_{n\to\infty} \|P(a_n) - a_n\| \end{aligned}$$

So, $\lim_{n\to\infty} \Lambda_n(e) = \|e\|$. But since $\Lambda_n(e) \leq 0$ for all $n \in \mathbb{N}$ then this convergence can happen only if $e = P(0) = 0$ and this contradicts the assumption. Hence E must be singleton. □

In this proof we use the uniquely remotal condition just to confirm that the function P is well defined, so we get the following stronger version of theorem:

Theorem 4.2.3. [20] Suppose X is a normed space and E is bounded and closed set in X with a center $e_\circ$, consider $P : X \longrightarrow E$ be a function from $P(x, E)$ such that P is single valued . If P is partially continous at $e_\circ$, then E is a singelton.

As a result of those theorems we have:

Corollary 4.2.4. [20] In normed space X, when a bounded and closed subset E with center $e_\circ$ has more than one point, then no deduced function P from $P(x, E)$ will satisfy the partial continuity definition at $e_\circ$.

The second aim is to show that isolation result will help demonstrate the singletoness of remotal set and the above result will conclude immediately.

Theorem 4.2.5. [20] In normed space X, when a bounded and closed subset E with center $e_\circ$ has more than one point, then for all deduced function P from $P(x, E)$, $\exists$ $\delta > 0$ such that

$$\|(P(x) - P(e_\circ))\| > \delta$$

for all $x \in (e_\circ, P(e_\circ)]$.

Proof. Suppose P be any extracted function, and since $x \in (e_\circ, P(e_\circ)]$ we can write x in the form:

$$x = \alpha e_\circ + (1 - \alpha)P(e_\circ),$$

such that $0 < \alpha < 1$. Then,

$$\begin{aligned}
\|x - P(x)\| &= \|\alpha e_\circ + (1 - \alpha)P(e_\circ) - P(x)\| \\
&= \|\alpha e_\circ - \alpha P(x) + \alpha P(x) + (1 - \alpha)P(e_\circ) - P(x)\| \\
&= \|\alpha(e_\circ - P(x) + (1 - \alpha)(P(e_\circ) - P(x))\| \\
&\leq \alpha\|(e_\circ - P(x)\| + (1 - \alpha)\|(P(e_\circ) - P(x))\| \\
&\leq \alpha\|(e_\circ - P(e_\circ)\| + (1 - \alpha)\|(P(e_\circ) - P(x))\| \\
&\leq \alpha\|(x - P(x)\| + (1 - \alpha)\|(P(e_\circ) - P(x))\|.
\end{aligned}$$

Since $e_\circ$ is the center of E,

$$\|x - P(x)\| \geq \|e_\circ - P(c)\|,$$

for all $x \in X$, now we have:

$$\|x - P(x)\| - \alpha\|(x - P(x)\| \leq (1 - \alpha)\|(P(e_\circ) - P(x))\|,$$

so:

$$\|(P(x) - P(e_\circ))\| \geq \|x - P(x)\| \geq \|e_\circ - P(ce_\circ\| = r,$$

Hence

$$\|(P(x) - P(e_\circ))\| \geq r > \delta$$

(by considering δ be such that $\delta \leq r$) $\square$

It is worth mentioning that in 1983 [2], it was demonstrated that specifically in an inner product space when a bounded and closed subset E with center $e_\circ$ not singleton and uniquely remotal then we have a $\delta > 0$ with above isolation property.

However, here the result generalizes the old result in a way, where the space can be any normed space and also E need not be uniquely remotal but only remotal.

Now having proved this isolation result, we follow steps to prove singletoness of certain uniquely remotal sets.

Theorem 4.2.6. [20] In normed space X, consider a bounded and closed subset E with center $e_\circ$ has more than one point and let δ be as in previous theorem. If the distance $D(c, E \setminus B(P(c), \delta))$ is accomplished, then E cannot be uniquely remotal.

Proof. The proof will be by contradiction. suppose E is uniquely remotal and let $e_\circ = 0$. Since E in not singleton then there exists $\alpha \in (0, P(0)]$ such that

$$\|(P(x) - P(0))\| \geq \delta,$$

for all $x \in (0, \alpha]$. Now, let $a_n \to 0, \{a_n\} \subset (0, \alpha]$. By isolation result

$$P(a_n) \in E \setminus B(P(0), \delta).$$

So,

$$\begin{aligned} D(0, E) &= \lim_{n \to \infty} D(a_n, E) \\ &= \lim_{n \to \infty} D(a_n, E \setminus B(P(0), \delta)) \\ &= D(0, E \setminus B(P(0), \delta)) \end{aligned}$$

And since $D(0, E \setminus B(P(0), \delta))$ is satisfied by assumption, there exists $e \in E \setminus$

$B(P(0),\delta)$ such that

$$D(0,E) = \|0 - e\|$$

but $e \neq P(0)$. So this make a contradiction with assumption that E is uniquely remotal. $\square$

This unique remotality of E is implied by the behavior with regard to $e_\circ$ which showed by the following corollary.

Corollary 4.2.7. [20] In normed space X, consider a bounded and closed subset E with center $e_\circ$, and let P be any deduced single valued mapping of $P(.,E)$. If a sequence $a_n \in (e_\circ, P(e_\circ)]$ exists with $a_n \longrightarrow e_\circ$ and $P(a_n)$ also converges, then E cannot be uniquely remotal.

The proof of this corollary is from the fact: if $a_n \longrightarrow e_\circ$ and
$P(a_n) \longrightarrow a$ then $a \in P(e_\circ, E)$

Corollary 4.2.8. [20] In normed space, when subset E is compact, uniquely remotal and admitting a center $e_\circ$, then E will be a singleton set.

Proof. By contradiction, suppose E is not a singleton and δ as in theorem 4.10. E is compact, so $E \setminus B(P(e_\circ),\delta)$ is also compact, hence it's remotal.
So $D(e_\circ, E \setminus B(P(e_\circ),\delta))$ is attained. By previous theorem(4.6) E is not uniquely remotal which contradicts the assumption. $\square$

Note that, in this proof we use the condition of uniquely remotal of E with respect to c only. So the singletoness of E need not the uniquely remotal condition, in other words:

Theorem 4.2.9. [20]In normed space X, let subset E is compact, and admitting a center, when E is uniquely remotal just on it's center, then E will be a singleton set.

At the end, by previous results and theorems we cut off the study of the continuity behaviour of Farthest Point mapping.

4.3 Farthest Point Problem in Banach Space($\mathbf{l^1}$(R))

In this section we will consider the most recent work about the farthest point problem which study on a specific space (sequence space $l^1(\mathbb{R})$). Before that in 2017, they proved that farthest point problem is absolutely true without continuity condition but only in a specific space. Here in this article [25] we continue the same criteria, it takes the open problem in more specific space the $l^1(\mathbb{R})$ space.

The main result of this article is to show that every uniquely remotal subset of sequence space $l^1(\mathbb{R})$ is singleton. Firstly we will proved the following propositions which played a key role to reach to the main result.

Proposition 4.3.1. Suppose E is an uniquely remotal set in Banach space X, and a_n is a sequence in X with $a_n \to a$ and also $a \in X$. Whenever $P(a_n) = b$ for all $n \in \mathbb{N}$, such that $b \in E$ then $P(a) = b$.

Proof. By contradiction method, suppose $P(a) \neq b$, however there must exists $c \in E$ such that $P(x) = c$ since E isuniquely remotal. $\|a - c\| > \|a - a'\|$ for all $a' \in X$, so there exists $\epsilon > 0$ such that:

$$\|a - c\| > \|a - b\| + \epsilon \tag{4.1}$$

Also, there exists $n_\circ \in N$ such that

$$\|a_n - a\| < \frac{\epsilon}{2}, \tag{4.2}$$

for all $n \geq n_0$. If $m \geq n_0$ then we have:

$$
\begin{aligned}
\|a_m - c\| &= \|a_m - a + a - c\| \\
&\geq \|a - c\| - \|a_m - a\| \\
&> \|a - b\| + \epsilon - \frac{\epsilon}{2} \text{ by (4.1) and (4.2)} \\
&> \|a_m - b\| - \|a_m - a\| + \frac{\epsilon}{2} \\
&> \|a_m - b\| + \frac{\epsilon}{2} - \frac{\epsilon}{2} = \|a_m - b\|.
\end{aligned}
$$

We have $\|a_m - c\| > \|a_m - b\|$ and this contradicts that $P(a_m) = b$. Hence $P(a) = b$ □

Proposition 4.3.2. [25] In a Banach space X, consider M be a compact subset and E be uniquely remotal subset in X. Then $\exists m \in M$ and $a \in E$ such that :

$$D(E, M) = \sup\{\|s - t\| : s \in M, t \in E\} = \|a - m\| = \|P(m) - m\|$$

(Here $a = P(m)$ since E is uniquely remotal)

Proof. Depending on definition of $D(E, M)$, there exists two sequences $(a_n) \in E$ and $(m_n) \in M$ with,

$$D(E, M) = \lim_{n\to\infty} \|a_n - m_n\|$$

(We want to prove that $\lim_{n\to\infty} \|a_n - m_n\| = \|P(m) - m\|$). Now, since M is compact, then there exists a sub sequence m_{n_k} of m_n such that $m_{n_k} \to m$ in M so,

$$D(E, M) = \lim_{n\to\infty} \|a_{n_k} - m_{n_k}\|$$

From the definition of $D(E, M)$ we conclude that:

$$D(E, M) \geq \|a' - m'\|,$$

for all $a' \in E$ and $m' \in M$. Therefore

$$\lim_{n\to\infty} \|a_{n_k} - m_{n_k}\| \geq \|P(m) - m\|. \tag{4.3}$$

But

$$\begin{aligned}\|a_{n_k} - m_{n_k}\| &\leq \|a_{n_k} - m\| + \|m - m_{n_k}\| \\ &\leq \|m - P(m)\| + \|m_{n_k} - m\|.\end{aligned}$$

Take the limit as $k \to \infty$ we get

$$\lim_{n\to\infty} \|a_{n_k} - m_{n_k}\| \leq \|m - P(m)\|. \tag{4.4}$$

Since $m \in M$, $P(m) \in E$ and from (4.3) and (4.4) we have:

$$\lim_{n\to\infty} \|a_{n_k} - m_{n_k}\| = \|m - P(m)\|$$

and hence

$$D(E, M) = \|m - P(m)\|.$$

□

For E be a uniquely remotal of Banach space X, we define the closed ball as follow: let $x_\circ \in X$ and $P(x_\circ) = e_\circ$ with $e_\circ \in E$ we have

$$\overline{B_{\|x_\circ - e_\circ\|}(x_\circ)} = \overline{B_{D(x_\circ, E)}(x_\circ)}$$

Consider $J = \{\overline{B_{\|y-e_\circ\|)}(y)} : P(y) = e_\circ\}$, and define the relation "$\leq$" on J as follow:

$$B_1 \leq B_2 \ if \ B_2 \subseteq B_1$$

Finally, we want to prove the main result

Theorem 4.3.3. [25] If E is a uniquely remotal set in the sequence space $l^1(\mathbb{R})$, then E will be singleton.

Proof. Suppose E is a uniquely remotal set in $l^1(\mathbb{R})$ and e_1 be unique farthest point in E from 0 i.e. $P(0) = e_1$. By [25, Theorem 3.1]

$$J = \{\overline{B_{\|y-e_1\|)}(y)} : P(y) = e_1\}$$

has a maximum element called $\overline{B_{\|u-e_1\|)}(u)}$.

Without loss of generality, let $u = 0$ and $\|e_1\| = 1$, so the maximum element will be in the unit ball of l^1. Assume

$$e_1 = (a_1, a_2, \ldots),$$

since $\|e_1\| = 1$ then we can assume $a_1 \neq 0$ and also let $a_1 > 0$. Hence $a_1 > \frac{1}{m_0}$ for some $m_0 \in \mathbb{N}$. Let

$$\lambda_1 = (1, 0, 0, ...)$$

be the unit sequence element and consider the sequence element $(\frac{\lambda_1}{n})$ such that $n > m_0$. We want to show that $P(\frac{\lambda_1}{n}) \neq e_1$ for all $n > m_\circ$. By contrary, let $P(\frac{\lambda_1}{n}) = e_1$ for some $n > m_\circ$, then for $w \in \overline{B_{\|\frac{\lambda_1}{n}-e_1\|)}(\frac{\lambda_1}{n})}$ we have:

$$\begin{aligned} \|w\| - \left\|\frac{\lambda_1}{n}\right\| &\leq \left\|w - \frac{\lambda_1}{n}\right\| \\ &\leq \left\|\frac{\lambda_1}{n} - e_1 \,\|\right. \end{aligned}$$

And

$$\left\|\frac{\lambda_1}{n} - e_1\right\| = \|e_1\| - \|\frac{\lambda_1}{n}\|,$$

thus $\|w\| < \|e_1\| = 1$

Hence $w \in \overline{B_1(0)}$, and this contradicts the fact of $\overline{B_1(0)}$ is the maximal element. So,

$$P\left(\frac{\lambda_1}{n}\right) \neq e_1,$$

for all $n > m_\circ$. Now, suppose that

$$P\left(\frac{\lambda_1}{n}\right) = z_n = (c_1^n, c_2^n, \ldots),$$

then we must have $c_1^n < \frac{1}{n}$, since if not we get

$$\begin{aligned}\left\|z_n - \frac{\lambda_1}{n}\right\| &= \left\|(c_1^n, c_2^n, \ldots) - (\frac{1}{n}, 0, 0, ...)\right\| \\ &= \|z_n\| - \left\|\frac{\lambda_1}{n}\right\|\end{aligned}$$

$\square$

Chapter Five

Statistical continuity and the Farthest Point Problem

In this chapter we present the farthest point problem in a different a pproach. We present the concept of partial ideal statistical continuity of a function. And we will show that this concept is weaker than the continuity condition of the function by example. Finally we prove the main aim of this chapter by show that in a real Banach space X, if E is a bounded uniquely remotal subset of X with a chebyshev center $e_\circ$ and the FPM

$$P : X \longrightarrow E$$

limited to $[e_\circ, P(e_\circ)]$ is partially ideal statistically continuous at $e_\circ$ then E is singleton.

5.1 l statistically convergent sequence

In the first section, we state the basic notations and concepts needed to reach to the main definition in this section. Firstly, we clarify the notion of cardinality of $E(E \subset \mathbb{N})$ and also the natural density $(d(E))$ which depends on the cardinality definition. After that, we state the definition of statistically convergent sequence, then we explain the conditions of set ℓ $(\ell \subset 2^N)$ to be an ideal. Finally, we state the main definition, the ℓ-statistically convergent sequence and explain this concept by example.

Definition 5.1.1 (Cardinality of E). [24] Let $\mathbb{N}$ be the natural numbers and E subset of $\mathbb{N}$, then $K(E)_{m,n}$ is the cardinality of a set $E \cap [m, n]$.

We can define also the notion of natural density of E like this [24]:

$$\overline{d}(E) = \lim_{n\to\infty} \sup \frac{K(E)_{1,n}}{n},$$
$$\underline{d}(E) = \lim_{n\to\infty} \inf \frac{K(E)_{1,n}}{n}$$

When $\overline{d}(E) = \underline{d}(E)$ then natural density of E exists and it is denoted by:

$$d(E) = \lim_{n\to\infty} \frac{K(E)_{1,n}}{n}$$

Definition 5.1.2 (Statistically Convergent). [24] Let a_n be a sequence of real numbers, it called statistically convergent to a if for all $\epsilon > 0$, we have set

$$E = \{n \in \mathbb{N} : |a_n - a| \geq \epsilon\}$$

such that $d(E) = 0$ and it is denoted by $a_n \xrightarrow{S} a$.

Definition 5.1.3 (Ideal Set). [21] Let ℓ be a family of subsets of $\mathbb{N}$ ($\ell \subset 2^N$), it is said to be an ideal if the following condition satisfied:

1. $\phi \in \ell$,
2. $A, B \in \ell \Longrightarrow A \cup B \in \ell$,
3. $A \in \ell, B \subseteq A \Longrightarrow B \in \ell$

The ideal ℓ defined above called a non-trivial ideal if $\ell \neq \{\phi\}$, and proper ideal if $\mathbb{N} \notin \ell$. Admissible ideal ℓ is a proper ideal such that forall $n \in \mathbb{N}$, we have $\{n\} \in \ell$.

Definition 5.1.4 (ℓ Statistically Convergent). [21] Suppose ℓ be a non-trivial, admissible ideal. A sequence $\{a_n\}$ of real numbers is called ℓ statistically convergent to a ($a \in \mathbb{R}$) if for each $\epsilon, \delta > 0$

$$\left\{n \in \mathbb{N} : \frac{1}{n}\Big|\{k \leq n : \|a_k - a\| \geq \epsilon\}\Big| \geq \delta\right\} \in \ell$$

and is denoted by $a_n \xrightarrow[S]{\ell} a$

We consider an example to clarify the notion of ℓ-statistically convergent:

Example 5.1.5. [24] Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in [-1,0] by:

$$x_n = \begin{cases} 0 & \text{if } n \neq m^2 \forall m \in \mathbb{N} \\ -\frac{1}{n} & \text{if } n = m^2 for\ some\ m \in \mathbb{N} \end{cases}$$

it is easy to show that the natural density of set $M = \{m^2 : m \in \mathbb{N}\}$ is zero. (d(M)=0):

$M = \{1, 4, 9, 16, ...\}$, clearly $K(M)_{1,n} \leq \sqrt{n}$. We have

$$\begin{aligned} d(M) &= \lim_{n\to\infty} \frac{K(E)_{1,n}}{n} \\ &\leq \lim_{n\to\infty} \frac{\sqrt{n}}{n} = 0 \end{aligned}$$

For $\epsilon > 0$,

$$\{n \in \mathbb{N} : \|x_n - 0\| \geq \epsilon\} \subset \{n \in \mathbb{N} : n = m^2 for\ some\ m \in \mathbb{N}\}$$

Since $d(M) = 0$, we get

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \|x_k - x\| \geq \epsilon\}| = 0$$

Hence $\{x_n\}_{n\in\mathbb{N}} \xrightarrow{S} 0$. Let $\delta > 0$ and since ℓ is admissible :

$$\left\{n \in \mathbb{N} : \frac{1}{n} \Big| \{k \leq n : \|x_k - x\| \geq \epsilon\} \Big| \geq \delta \right\} \in \ell$$

This proves that $\{x_n\}_{n\in\mathbb{N}} \xrightarrow[\ell]{S} 0$.

5.2 Partial Ideal Statistical Continuity and Singeltoness

Now, we reach to introduce the main definition we need to prove the final result. The notion of partial ideal statistically continuity of a function:

Definition 5.2.1. [24] Consider a non-trivial, admissible ideal ℓ and a real Banach space X. Let E be a nonempty set in X we called the function $P : E \longrightarrow X$

partially ideal statistically continuous at a, if we have a sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ such that $\{x_n\}_{n\in\mathbb{N}}$ is ℓ-statistically convergent to a and $\{P(x_n)\}_{n\in\mathbb{N}}$ is ℓ-statistically convergent to $P(a)$.

This new definition of contunuity is weaker than the partial continuity inserted by Sababheh et al.[20]. So we consider the last example given and show that the partial ideal statistical continuity is much easier to satisfy.

Example 5.2.2. Consider a non-trivial, admissible ideal ℓ and a function $f : [-1, 0] \to \mathbb{R}$ defined by:

$$f(x) = [x], \ x \in [-1, 0]$$

It is not difficult to show that this function is not partially continous at $x = 0$. Our mission is to show that f is partially ideal statistically continuous at $x = 0$?

By using the previous sequence $\{x_n\}_{n\in\mathbb{N}}$ we define in example 5.1.5, and define the function sequence $\{P(x_n)\}_{n\in\mathbb{N}}$ by:

$$f(x_n) = \begin{cases} 0 & \text{if } n \neq m^2 \ \forall m \in \mathbb{N} \\ -1 & \text{if } n = m^2 for\ some\ m \in \mathbb{N} \end{cases}$$

We easily show that $\{f(x_n)\}_{n\in\mathbb{N}} \xrightarrow[S]{\ell} 0$:
Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we have

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \Big| \{k \leq n : \|f(x_k) - f(0)\| \geq \epsilon\} \Big| \geq \delta \right\} \in \ell$$

So, we showed $\{f(x_n)\}_{n\in\mathbb{N}} \xrightarrow[\ell]{S} 0$ $(f(0) = 0)$, and previously we also proved that $\{x_n\}_{n\in\mathbb{N}} \xrightarrow[\ell]{S} 0$.
This complete the proof and show the above function $f(x)$ is partially ideal statistically continuous at $x = 0$. So the patial ideal statistical is much weaker than partial continuity.

Now we reach to the main and important result about the singletoness of a uniquely remotal set with the condition explained below:

Theorem 5.2.3. [24]Consider a non-trivial, admissible ideal ℓ and nonempty bounded subset E of real Banach space X. If E is uniquely remotal with a chebyshev center $e_\circ$ and the FPM $P : X \to E$ limited to $[e_\circ, P(e_\circ)]$ is partially ideal statistically continuous at $e_\circ$ then E is singleton.

Proof. Since E is uniquely remotal set we have,
For all $x \in X$, $\exists! a \in E$ with $\|x - a\| = D(x, E)$ and the FPM is well defined.
Suppose the Chebyshev center of E occurred at $a = 0$.
Now we use the contradiction method, so assume E not singleton we have $P(0) \neq 0$
Since the FPM $P : X \to E$ limited to $[0, P(0)]$ is partially ideal statistically continuous at 0, then we have a sequence $\{a_n\}_{n\in\mathbb{N}} \in [0, P(0)]$ with

$$\{a_n\}_{n\in\mathbb{N}} \xrightarrow[S]{\ell} 0 \; and \; \{P(a_n)\}_{n\in\mathbb{N}} \xrightarrow[S]{\ell} P(0)$$

So, $a_n = \mu_n P(0)$ such that $\mu_n > 0$, forall $n \in \mathbb{N}$ and $\mu_n \xrightarrow[S]{\ell} 0$

Now, for all $n \in \mathbb{N}$, we define $\Psi_n \in X^*$ by: $\Psi_n(P(a_n) - a_n) = \|P(a_n) - a_n\|$.

Also $\|\Psi_n\| = 1$, hence

$$\begin{aligned}
\Psi_n(a_n) &= \Psi_n(P(a_n) - P(a_n) + a_n) \\
&= \Psi_n(P(a_n)) - \Psi_n(P(a_n) - a_n) \\
&\leq \|\Psi_n\|\|P(a_n)\| - \|P(a_n) - a_n\| \\
&= \|P(a_n)\| - \|P(a_n) - a_n\| \\
&= \|P(a_n) - 0\| - \|P(a_n) - a_n\| \\
&\leq D(0, E) - D(a_n, E) \; (since \; D(x, E) \geq \|P(a_n) - 0\|) \\
&\leq 0.
\end{aligned}$$

The last inequality follows from the definition of Chybeshev center: $D(0, E) \leq D(x, E)\ \forall x \in X$. So,

$$\Psi_n(a_n) \leq 0\ \forall n \in \mathbb{N},$$

hence

$$\Psi_n(\mu_n P(0)) \leq 0\ \forall n \in \mathbb{N}$$

so

$$\mu_n \Psi_n(P(0)) \leq 0 \forall n \in \mathbb{N},$$

since $\mu_n > 0$ we have

$$\Psi_n(P(0)) \leq 0$$

$\forall n \in \mathbb{N}$. We get,

$$\{P(a_n) - a_n\}_{n\in\mathbb{N}} \xrightarrow[S]{\ell} P(0).$$

So by the definition of ℓ-statistically convergent sequence, for each $\epsilon, \delta > 0$ we have:

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|P(a_k) - a_k - P(0)\| \geq \epsilon\}| \geq \delta\} \in \ell \tag{5.1}$$

But we get by triangle inequality that,

$$\Big|\|P(a_k) - a_k\| - \|P(0)\|\Big| \leq \|P(a_k) - a_k - P(0)\|$$

Suppose $\epsilon > 0$, we get:

$$\{k \leq n : \Big|\|P(a_k) - a_k\| - \|P(0)\|\Big| \geq \epsilon\} \subset \{k \leq n : \|P(a_k) - a_k - P(0)\| \geq \epsilon\}$$

And for $\delta > 0$,

$$\begin{aligned}&\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \Big|\|P(a_k) - a_k\| - \|P(0)\|\Big| \geq \epsilon\}| \geq \delta\} \\ &\qquad \subset \{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|P(a_k) - a_k - P(0)\| \geq \epsilon\}| \geq \delta\}\end{aligned}$$

By previous relation and by 5.1 we have:

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|P(a_k) - a_k\| - \|P(0)\| \geq \epsilon\}| \geq \delta\} \in \ell$$

Hence, the sequence $\{\|P(a_n) - a_n\|\}_{n\in\mathbb{N}} \xrightarrow[S]{\ell} \|P(0)\|$

Now we get:

$$\begin{aligned}\Psi_n(P(a_n) - a_n) - \Psi_n(P(0)) &= \Psi_n(P(a_n) - a_n - P(0)) \ (since\ \Psi_n\ is\ linear functional)\\ &\leq \|\Psi_n\|\|P(a_n) - a_n - P(0)\|\\ &= \|P(a_n) - a_n - P(0)\|\end{aligned}$$

Clearly the above sequence

$$\{\|P(a_n) - a_n - P(0)\|\}_{n\in\mathbb{N}}$$

is ℓ-statistically convergent to 0, so the two sequences in the left ℓ-statistically convergent to the same value. But we know that

$$\{\Psi_n(P(a_n) - a_n)\}_{n\in\mathbb{N}} \xrightarrow[S]{\ell} \|P(0)\|.$$

Hence,

$$\{\Psi_n(P(0)\}_{n\in\mathbb{N}} \xrightarrow[S]{\ell} \|P(0)\|,$$

but we have that $\Psi_n(P(0)) \leq 0$ $\forall n \in \mathbb{N}$ so the above is true only if $P(0) = 0$ which make a contradiction.

We proved that any set E is uniquely remotal will be a singleton. □

Theorem 5.2.4. [24] Consider a real Banach space X, the admissible ideal ℓ ($\ell \neq \{\phi\}$) and also bounded subset E such that $E \neq \phi$. When E is UR with a Chebyshev center c and E has more than one element, then the FPM P limited to $(e_\circ, P(e_\circ)]$ is not partially ideal statistically continuous at $e_\circ$.

Proof. we have $e_\circ$ as a center of E, so let $a \in (e_\circ, P(e_\circ)]$. Then $a = \alpha e_\circ + (1-\alpha)P(e_\circ)$

for $\alpha \in [0, 1)$.

Now,

$$\begin{aligned}
\|a - P(a)\| &= \|\alpha e_\circ + (1-\alpha)P(e_\circ) - P(a)\| \\
&= \|\alpha e_\circ - \alpha P(a) + (1-\alpha)P(e_\circ) - P(a) + \alpha P(a)\| \\
&= \|\alpha(e_\circ - P(a)) + (1-\alpha)(P(e_\circ) - P(a))\| \\
&\leq \alpha\|(e_\circ - P(a))\| + (1-\alpha)\|(P(e_\circ) - P(a))\| \ (\textit{by triangle inequality}) \\
&\leq \alpha\|(e_\circ - P(e_\circ))\| + (1-\alpha)\|(P(e_\circ) - P(a))\| \\
&\leq \alpha\|(a - P(a))\| + (1-\alpha)\|(P(e_\circ) - P(a))\| \ (\textit{since } e_\circ \textit{ is chebyshev center})
\end{aligned}$$

$$\implies \|(e_\circ - P(e_\circ))\| \leq \|(a - P(a))\| \leq \|(P(e_\circ) - P(a))\|$$

(last inequality is true by substitute $\alpha = 0$)

Consider a radius $r = \|(e_\circ - P(e_\circ))\|$ and a sequence

$$\{a_n\}_{n\in\mathbb{N}} \xrightarrow[S]{\ell} e_\circ$$

(here,$\{a_n\}_{n\in\mathbb{N}} \in (e_\circ, P(e_\circ)]$), but when consider the function sequence $\{P(a_n)\}_{n\in\mathbb{N}}$ we have:

$$\frac{1}{n}\Big|\{k \leq n : \|P(a_k) - P(e_\circ)\| \geq r\}\Big| = 1 \ \forall n \in \mathbb{N}$$

Let $0 < \delta < 1$, we get:

$$\left\{n \in \mathbb{N} : \frac{1}{n}\Big|\{k \leq n : \|P(a_k) - P(e_\circ)\| \geq r\}\Big| \geq \delta\right\} = \mathbb{N} \notin \ell$$

Hence the farthest point map P is not partially ideal statistically continuous at c. □

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جامعة النجاح الوطنية

كلية الدراسات العليا

مسألة المسافات الأبعد في الفضاءات المعيارية

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أنغام خبيصة

إشراف

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قدمت هذه الرسالة استكمالا لمتطلبات الحصول على درجة الماجستير في رياضيات، من كلية الدراسات العليا، في جامعة النجاح الوطنية، نابلس– فلسطين.

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الملخص

في هذه الرسالة، نتناول المسألة طويلة الأمد في نظرية التقريب، والتي تدعى مسألة أبعد نقطة (FPP). على اعتبار أن هذه المسالة غير محلولة جزئياً، متسائلين فيما إذا اعتبرت كل مجموعة بعيدة بشكل فريد في فضاءات بناخ المترية تحتوي على عنصر واحد فقط.

ونفترض أيضاً أن الفضاءات المترية المحدبة تشير إلى أن كل مجموعة جزئية محدودة تحتوي على عنصر واحد فحسب، إذا وفقط إذا تحقق شرط SF. وبتركيزنا على فضاءات بناخ في الفضاء المعياري أولاً ، فقد أثبتنا أن المجموعة المنفردة تحدث عند تحقق شرط الاتصال الجزئي لاقتران أبعد مسافة. ثم نأخذ بعد ذلك فضاءًا محدداً من فضاءات بناخ هو الفضاء المتسلسل (R) $\ell 1$، فيظهر الحل الموجب للمسأله أنه لكل مجموعة جزئية بعيدة ومنفردة من الفضاء المتسلسل تحتوي على عنصر واحد فقط.

أخيراً، فإننا نعرض مفهوم الاتصال الإحصائي الجزئي لأقتران أبعد مسافة، ثم نعرض النتيجة الأساسية، مفترضين أن لكل مجموعة محدودة، بعيدة وفريدة من نوعها في فضاء بناخ تحقق مركز تشيبيشيف ويكون اقتران أبعد مسافة المعرًف على المركز يحقق شرط الاتصال الإحصائي الجزئي، فإن هذه المجموعه تتكون من عنصر واحد فقط.

الكلمات المفتاحية: نظرية التقريب، فضاءات بناخ المترية، الفضاءات المترية المحدبة، الاتصال الإحصائي الجزئي.