An-Najah National University
Faculty of Graduate studies

On Fractional Differential Equations and Linear Fractional systems: An ECG Implementation

By

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3. Dr. Samir Matar	/ Internal Examiner	....

Dedication

I dedicate this thesis to my parents who give me a great source of inspiration and support, also I dedicate this work to my dear brothers, sisters and my friends, and I thank my doctors and everyone who contributed to stand by my side for the completion of this work.

Acknowledgments

Firstly, I want to thank Allah for giving me the strength, courage and success. I am grateful to my supervisor Dr. Mohammad Ass'ad for his continuous help, encouragement and guidance. Furthermore, I would like to thank all my teachers in Mathematics Department at An-Najah National University for their contribution during my studies.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

On Fractional Differential Equations and Linear Fractional systems: An ECG Implementation

أقر بأن ما اشتملت عليه هذه الرسالة إنما هو من نتاج جهدي الخاص باستثناء ما تمت الإشارة إليه حيثما ورد، وأن هذه الرسالة ككل، او أي جزء منها لم يقدم لنيل أي درجه أو لقب علمي أو بحثي لدى أيه مؤسسه تعليميه او بحثيه أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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Content

On Fractional Differential Equations and Linear Fractional systems: An ECG Implementation

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Abstract

Fractional calculus is a current research topic in applied sciences such as applied mathematics, physics, biophysics, aerodynamics, control theory, capacitor theory, electrical circuit, description of memory and hereditary properties etc. used the fractional models instead of classical models.

In this work, we develop analytical and numerical method to find the solutions of fractional differential equation. also characterize ECG graph and compare normal ECG with (LVH,RVH) ECG by finding P.T. values at the non-differentiable point.

Chapter 1

Introduction

1.1 Basic definition of fractional derivative

1.1.1 Riemann-Liouvellie (R-L) fractional derivative

- Riemann- Liouvellie definition [4][5]

  The left R-L definition of fractional derivative is

$$_{a}D_{t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}(t-\tau)^{n-1-\alpha}f(\tau)d\tau$$

  The Right R-L definition of fractional derivative is

$$_{t}D_{b}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^{n}\int_{t}^{b}(\tau-t)^{n-1-\alpha}f(\tau)d\tau$$

  Where $n-1\leq\alpha<n$ n:is positive integer.

- Example of Riemann-Liouvellie fractional derivative

1. The constant function

   Let $a=0$, $n-1<\alpha<n$

   Applying the Left R-L definition for $f(t)=C$, $C\in\mathbb{R}$

$$\begin{aligned}
{}_{0}D_{t}^{\alpha}[C] &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}(t-\tau)^{n-1-\alpha}Cd\tau \\
&= \frac{C}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}(t-\tau)^{n-1-\alpha}d\tau \\
&= \frac{C}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\frac{t^{n-\alpha}}{n-\alpha}
\end{aligned}$$

Using this formula $\left(\frac{d}{dt}\right)^n t^k = \frac{\Gamma(k+1)}{\Gamma(k-n+1)} t^{k-n}$

$$_0D_t^\alpha[C] = \frac{C}{\Gamma(1-\alpha)} t^{-\alpha}$$

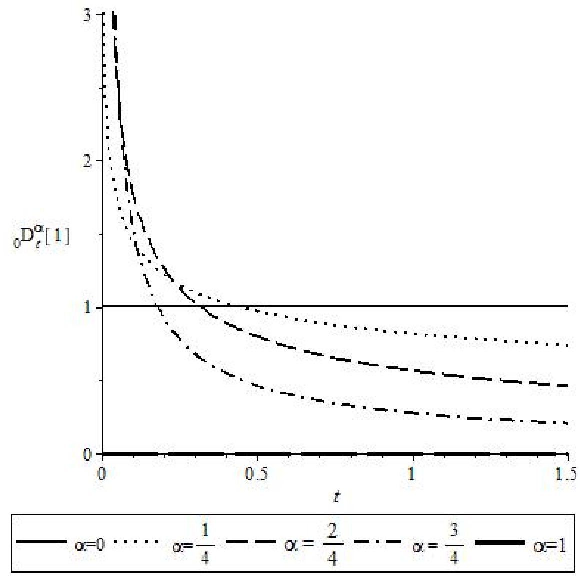


Figure 1.1: R-L fractional derivative of $f(t) = 1$

This shows that the fractional derivative of a constant C is non-zero but in classical calculus derivative of a constant is zero .

2. The Power function

   Let $a = 0$, $n - 1 < \alpha < n$

   Applying the Left R-L definition for $f(t) = t^k$

$$_0D_t^\alpha[t^k] = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-1-\alpha}(\tau^k)d\tau$$

Using the substitution $\tau = \varepsilon t$ then we have for

$\tau = 0, \varepsilon = 0$ and for $\tau = t, \varepsilon = 1$,$t - \tau = t(1-\varepsilon)$,$d\tau = td\varepsilon$

$$_0D_t^\alpha[t^k] = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^1 t^{n-\alpha-1}(1-\varepsilon)^{n-\alpha-1}\varepsilon^k t^k t d\varepsilon$$

$$= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n t^{n-\alpha+k}\int_0^1 \varepsilon^k(1-\varepsilon)^{n-\alpha-1}d\varepsilon$$

Using Beta-function

$\beta(\alpha,\gamma) = \int_0^1 \varepsilon^{\gamma-1}(1-\varepsilon)^{\alpha-1}d\varepsilon$;$Re(\alpha) > 0, Re(\gamma) > 0$

$$\beta(\alpha,\gamma) = \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}$$

$$_0D_t^\alpha[t^k] = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n t^{n-\alpha+k}\,\frac{\Gamma(k+1)\Gamma(n-\alpha)}{\Gamma(n-\alpha+k+1)}$$

$$_0D_t^\alpha[t^k] = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\,t^{k-\alpha}, (k > -1)$$

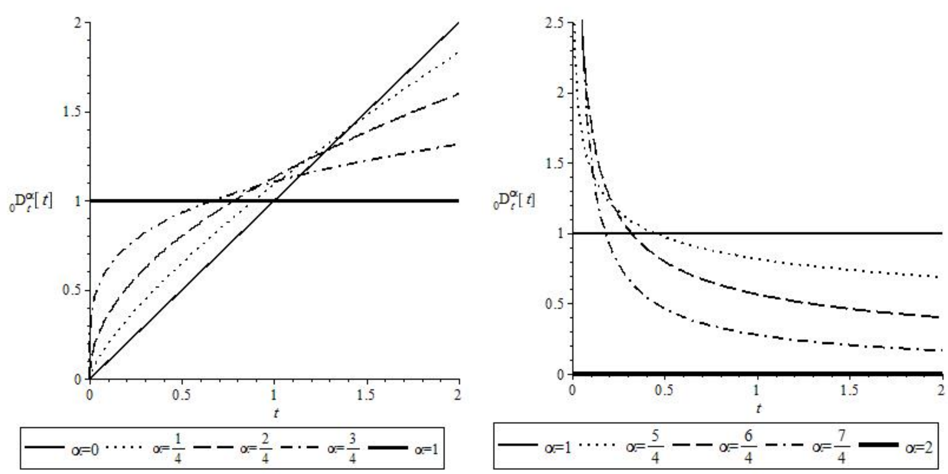


Figure 1.2: R-L fractional derivative of $f(t) = t$

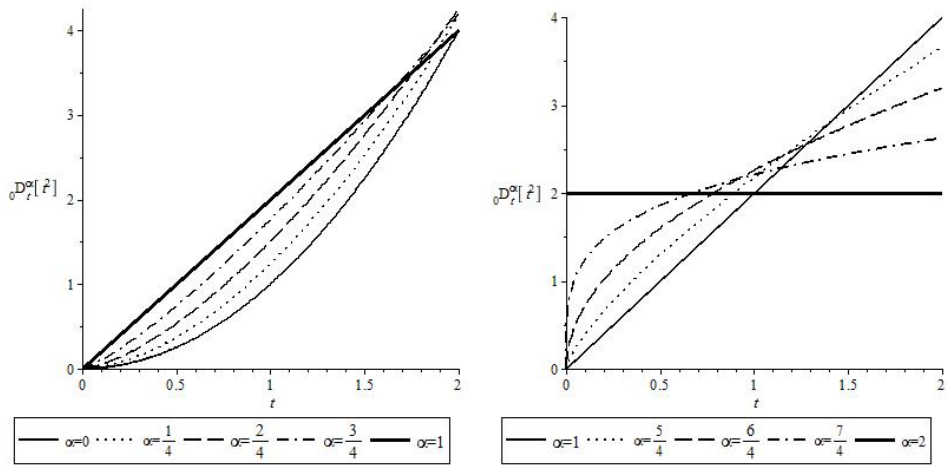


Figure 1.3: R-L fractional derivative of $f(t) = t^2$

3. The exponential function

   Let $a = 0$, $n - 1 < \alpha < n$

   Applying the Left R-L definition for $f(t) = e^{ct}, c \in \mathbb{C}$

$$e^{ct} = \sum_{k=0}^{\infty} \frac{(ct)^k}{\Gamma(k+1)} = 1 + \frac{ct}{\Gamma(2)} + \frac{c^2t^2}{\Gamma(3)} + \frac{c^3t^3}{\Gamma(4)+} + ...$$

$$\begin{aligned}
D^\alpha(e^{ct}) &= D^\alpha[1 + \frac{ct}{\Gamma(2)} + \frac{c^2t^2}{\Gamma(3)} + ...] \\
&= \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{c}{\Gamma(2)}\frac{\Gamma(2)t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{c^2}{\Gamma(3)}\frac{\Gamma(3)t^{2-\alpha}}{\Gamma(3-\alpha)} + ... \\
&= \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{c^2t^{2-\alpha}}{\Gamma(3-\alpha)} + ... \\
&= t^{-\alpha} \sum_{k=0}^{\infty} \frac{(ct)^k}{\Gamma((1-\alpha)+k)} \\
&= t^{-\alpha} E_{1,1-\alpha}(ct)
\end{aligned}$$

$${}_0D_t^\alpha[e^{ct}] = t^{-\alpha} E_{1,1-\alpha}(ct)$$

where $E_{1,1-\alpha}(ct)$ is two parameter Mittag-Leffler Function given in(1.8)

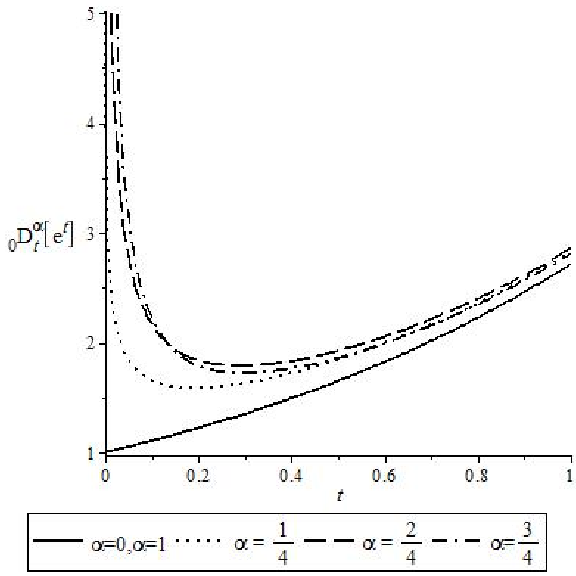


Figure 1.4: R-L fractional derivative of $f(t) = e^t$

4. The cos(ct),sin(ct)function,

   Let $a = 0$, $n - 1 < \alpha < n$

   Applying the Left R-L definition for $f(t) = cos(ct)$

$$cos(ct) = \sum_{k=0}^{\infty}(-1)^k \frac{(ct)^{2k}}{\Gamma(2k+1)} = 1 - \frac{(ct)^2}{\Gamma(3)} + \frac{(ct)^4}{\Gamma(5)} - ...$$

$$\begin{aligned} {}_0D_t^{\alpha}[cos(ct)] &= {}_0 D_t^{\alpha}[1 - \frac{(ct)^2}{\Gamma(3)} + \frac{(ct)^4}{\Gamma(5)} - ...] \\ &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)} - \frac{c^2}{\Gamma(3)}\frac{\Gamma(3)t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{c^4}{\Gamma(5)}\frac{\Gamma(5)t^{4-\alpha}}{\Gamma(5-\alpha)} - ... \end{aligned}$$

$$= \frac{t^{-\alpha}}{\Gamma(1-\alpha)} - \frac{c^2 t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{c^4 t^{4-\alpha}}{\Gamma(5-\alpha)} - ..$$

$$= t^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \frac{(ct)^{2k}}{\Gamma((1-\alpha)+2k)}$$

$$= t^{-\alpha} cos_{1,1-\alpha}(ct)$$

$$_0D_t^{\alpha}[cos(ct)] = t^{-\alpha} cos_{1,1-\alpha}(ct)$$

Similarly we can get

$$_0D_t^{\alpha}[sin(ct)] = t^{-\alpha} sin_{1,1-\alpha}(ct)$$

where $cos_{1,1-\alpha}(ct)$ *and* $sin_{1,1-\alpha}(ct)$ is two parameter fractional sine and cosine function given in(1.9,1.10).

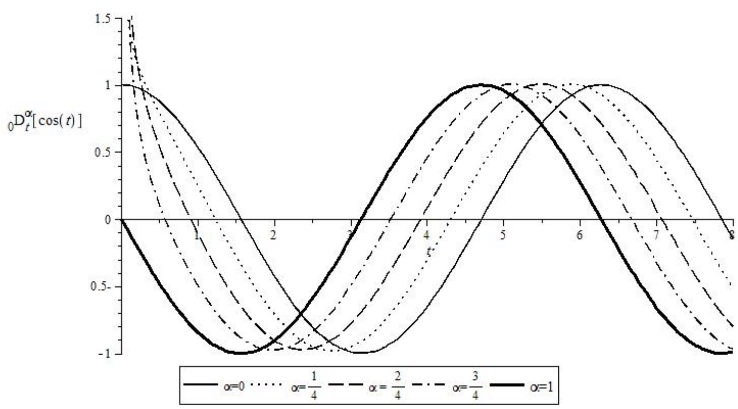


Figure 1.5: R-L fractional derivative of $f(t) = cos(at)$

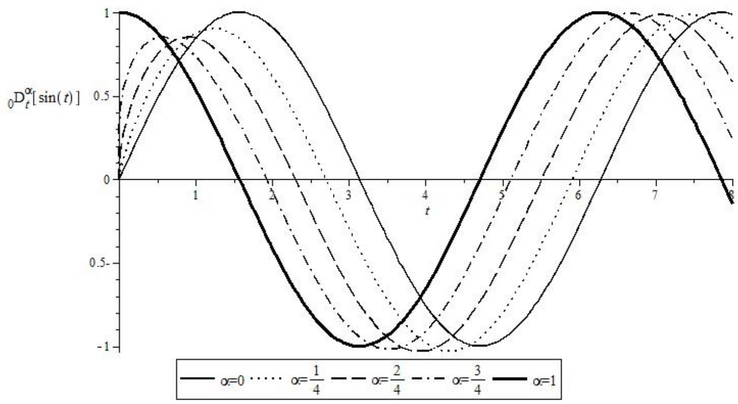


Figure 1.6: R-L fractional derivative of $f(t) = sin(t)$

1.1.2 Caputo fractional derivative

- Caputo definition[4]

$$ {}_a^cD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-1-\alpha} f^{(n)}(\tau) d\tau $$

Where $n-1 \leq \alpha < n$ n:is positive integer.

$f(t)$, must be differentiable n-times, if the function is non-differentiable then this definition is not applicable

- Example of Caputo fractional derivative

1. The constant function

Let $a = 0$, $n-1 < \alpha < n$

Applying the Caputo definition for $f(t) = C$,$C \in \mathbb{R}$

$$ {}_a^cD_t^{\alpha}[C] = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-1-\alpha} 0 d\tau = 0 $$

$$ {}_a^cD_t^{\alpha}[C] = 0 $$

Caputo derivative of a constant is zero

2. The Power function

Let $a=0$, $n-1<\alpha<n$

Applying the Caputo definition for $f(t)=t^k$

$$ {}_0^cD_t^\alpha[t^k]=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-1-\alpha}(\tau^k)^{(n)}d\tau $$

$$ =\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-1-\alpha}\frac{\Gamma(k+1)}{\Gamma(k-n+1)}\tau^{k-n}d\tau $$

$$ =\frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(k+1)}{\Gamma(k-n+1)}\int_0^t (t-\tau)^{n-1-\alpha}\tau^{k-n}d\tau $$

Using the substitution $\tau=\varepsilon t$ then we have for

$\tau=0,\varepsilon=0$ and for $\tau=t,\varepsilon=1$,$t-\tau=t(1-\varepsilon)$,$d\tau=td\varepsilon$

$$ {}_0^cD_t^\alpha[t^k]=\frac{\Gamma(k+1)}{\Gamma(n-\alpha)\Gamma(k-n+1)}\,t^{k-\alpha}\int_0^1 (1-\varepsilon)^{n-\alpha-1}\varepsilon^{k-n}d\varepsilon $$

$$ =\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha} $$

$$ {}_0^cD_t^\alpha f(t)=\begin{cases}\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}t^{k-\alpha} & k>n-1\\ 0 & k\le n-1\end{cases} $$

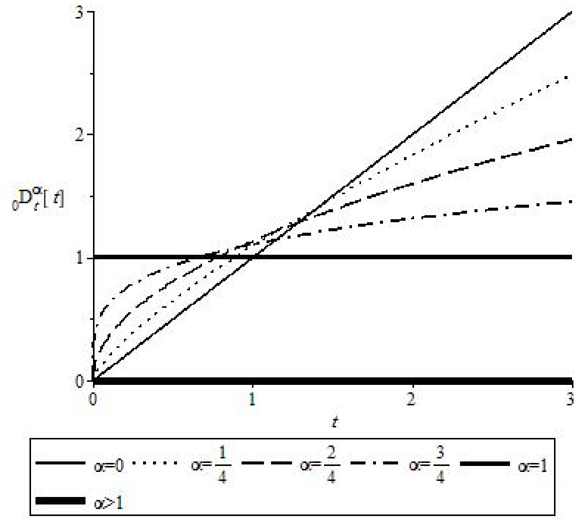


Figure 1.7: Caputo derivative of$\boldsymbol{f(t) = t}$

3. The exponential function

The relation between Caputo fractional derivative and Riemann-Liouvellie fractional derivative [7]

$$ {}_0^cD_t^{\alpha} f(t) ={}_0 D_t^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(t)^{k-\alpha}}{\Gamma(k+1-\alpha)} f^k(0) $$

Let $a = 0$, $n-1 < \alpha < n$

Applying this Property for $f(t) = e^{ct}, c \in \mathbb{C}$

$$ \begin{aligned} {}_0^cD_t^{\alpha}[e^{ct}] &={}_0 D_t^{\alpha}[e^{ct}] - \sum_{k=0}^{n-1} \frac{(t)^{k-\alpha} c^k}{\Gamma(k+1-\alpha)} \\ &= t^{-\alpha} E_{1,1-\alpha}(ct) - \sum_{k=0}^{n-1} \frac{(t)^{k-\alpha} c^k}{\Gamma(k+1-\alpha)} \\ &= t^{-\alpha} \sum_{k=0}^{\infty} \frac{(ct)^k}{\Gamma(k+1-\alpha)} - \sum_{k=0}^{n-1} \frac{(t)^{k-\alpha} c^k}{\Gamma(k+1-\alpha)} \end{aligned} $$

$$= \sum_{k=n}^{\infty} \frac{c^k t^{k-\alpha}}{\Gamma(k+1-\alpha)}$$
$$= \sum_{k=0}^{\infty} \frac{c^{k+n} t^{k+n-\alpha}}{\Gamma(k+n+1-\alpha)}$$
$$= c^n t^{n-\alpha} \sum_{k=0}^{\infty} \frac{c^k t^k}{\Gamma(k+n+1-\alpha)}$$
$$= c^n t^{n-\alpha} E_{1,n+1-\alpha}(ct)$$

$${}_0^cD_t^{\alpha}[e^{ct}] = c^n t^{n-\alpha} E_{1,n+1-\alpha}(ct)$$

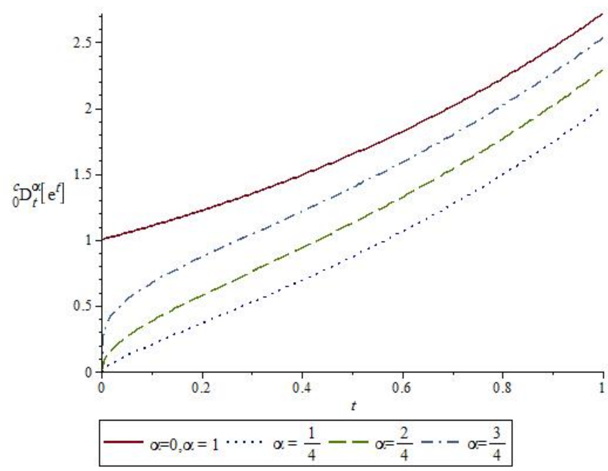


Figure 1.8: Caputo derivative of $f(t) = e^t$

4. The cos(ct),sin(ct) function

   Let $a = 0$, $n - 1 \leq \alpha < n$

   using the following interpretation of cos(ct),sin(ct)

$$cos(ct) = \frac{e^{ict} + e^{-ict}}{2}$$

$$sin(ct) = \frac{e^{ict} - e^{-ict}}{2i}$$

$${}_0^c D_t^\alpha [cos(ct)] = \tfrac{1}{2} {}_0^c D_t^\alpha (e^{ict} + e^{-ict})$$

$$= \frac{1}{2}\big((ic)^n t^{n-\alpha} E_{1,n+1-\alpha}(ict) + (-ic)^n t^{n-\alpha} E_{1,n+1-\alpha}(-ict)\big)$$

$${}_0^c D_t^\alpha [cos(ct)] = \frac{(ic)^n t^{n-\alpha}}{2} (E_{1,n+1-\alpha}(ict) + (-1)^n E_{1,n+1-\alpha}(-ict))$$

similarly we can get

$${}_0^c D_t^\alpha [sin(ct)] = \frac{(ic)^n t^{n-\alpha}}{2i} (E_{1,n+1-\alpha}(ict) - (-1)^n E_{1,n+1-\alpha}(-ict))$$

1.1.3 Jumarie modified fractional derivative

- Jumarie modified definition ,[4][5]

  The left Jumarie modified definition is

$$ {}_a^J D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_a^t (t-\tau)^{-\alpha-1} f(\tau) d\tau & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} [f(\tau) - f(a)] d\tau & 0 < \alpha < 1 \\ (f^{\alpha-n}(t))^n & n \le \alpha < n+1 \end{cases} \tag{1.1}$$

The right Jumarie modified definition is

$$
{}_{t}^{J}D_{b}^{\alpha}f(t)=\begin{cases}\frac{-1}{\Gamma(-\alpha)}\int_t^b(\tau-t)^{-\alpha-1}f(\tau)d\tau & \alpha<0\\ \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b(\tau-t)^{-\alpha}[f(b)-f(\tau)]d\tau & 0<\alpha<1\\ (f^{\alpha-n}(t))^{n} & n\leq\alpha<n+1\end{cases}\tag{1.2}
$$

Where $n-1\leq\alpha<n$ n:is positive integer.

- Example of Jumarie modified fractional derivative

1. The constant function

   Let $a=0$

   Applying The left Jumarie modified definition for

   $f(t)=c$ $c\in\mathbb{R}$, when $\alpha<0$

$$
\begin{aligned}
{}_{0}^{J}D_{t}^{\alpha}[c] &= \frac{1}{\Gamma(-\alpha)}\int_0^t(t-\tau)^{-\alpha-1}cd\tau\\
&= \frac{c}{\Gamma(-\alpha)}\int_0^t(t-\tau)^{-\alpha-1}d\tau\\
&= \frac{ct^{-\alpha}}{\Gamma(1-\alpha)}
\end{aligned}
$$

   when $0<\alpha<1$

$$
{}_{0}^{J}D_{t}^{\alpha}[c]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\tau)^{-\alpha}[c-c]d\tau=0
$$

when $n \leq \alpha < n+1, n \geq 1$

$$ {}_0^J D_t^\alpha[c] = \left(f^{\alpha-n}(c)\right)^n = 0 $$

$$ {}_0^J D_t^\alpha[c] = \begin{cases} \frac{ct^{-\alpha}}{\Gamma(1-\alpha)} & \alpha < 0 \\ 0 & \alpha > 0 \end{cases} $$

2. The power function

   Let $a = 0$

   Applying The left Jumarie modified definition for

   $f(t) = t^k$ when $0 > \alpha$

$$ {}_0^J D_t^\alpha[t^k] = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\tau)^{-\alpha-1} \tau^k d\tau $$

   Using the substitution $\tau = \varepsilon t$ then we have for

   $\tau = 0, \varepsilon = 0$ and for $\tau = t, \varepsilon = 1$;$t - \tau = t(1-\varepsilon)$;$d\tau = td\varepsilon$

$$ \begin{aligned} {}_0^J D_t^\alpha[t^k] &= \frac{1}{\Gamma(-\alpha)} \int_0^1 t^{-\alpha-1}(1-\varepsilon)^{-\alpha-1} \varepsilon^k t^k t d\varepsilon \\ &= \frac{t^{-\alpha+k}}{\Gamma(-\alpha)} \int_0^1 \varepsilon^k (1-\varepsilon)^{-\alpha-1} d\varepsilon \\ &= \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha} \end{aligned} $$

$$ {}_0^J D_t^\alpha[t^k] = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha}, (k > -1), (\alpha < 0) $$

when $0 < \alpha < 1$

$$ {}_0^JD_t^{\alpha}[t^k] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}\tau^k d\tau $$

Using the substitution $\tau = \varepsilon t$ then we have for

$\tau = 0, \varepsilon = 0$ and for $\tau = t, \varepsilon = 1$;$t - \tau = t(1-\varepsilon)$;$d\tau = td\varepsilon$

$$ \begin{aligned} {}_0^JD_t^{\alpha}[t^k] &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^1 t^{-\alpha}(1-\varepsilon)^{-\alpha}\varepsilon^k t^k t d\varepsilon \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}t^{k-\alpha+1}\int_0^1 \varepsilon^k(1-\varepsilon)^{-\alpha}d\varepsilon \\ &= \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha} \end{aligned} $$

$$ {}_0^JD_t^{\alpha}[t^k] = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha}, (k>0), (0<\alpha<1) $$

when $n < \alpha < n+1$

$$ \begin{aligned} {}_0^JD_t^{\alpha}[t^k] = \left(f^{\alpha-n}[t^k]\right)\right)^n &= \left(\frac{\Gamma(k+1)}{\Gamma(k-\alpha+n+1)}t^{k-\alpha+n}\right)^{(n)} \\ &= \frac{\Gamma(k+1)}{\Gamma(k-\alpha+n+1)}\frac{\Gamma(k-\alpha+n+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha} \\ &= \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha} \end{aligned} $$

$$ {}_0^JD_t^{\alpha}[t^k] = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha} \quad , (k>0) \tag{1.3} $$

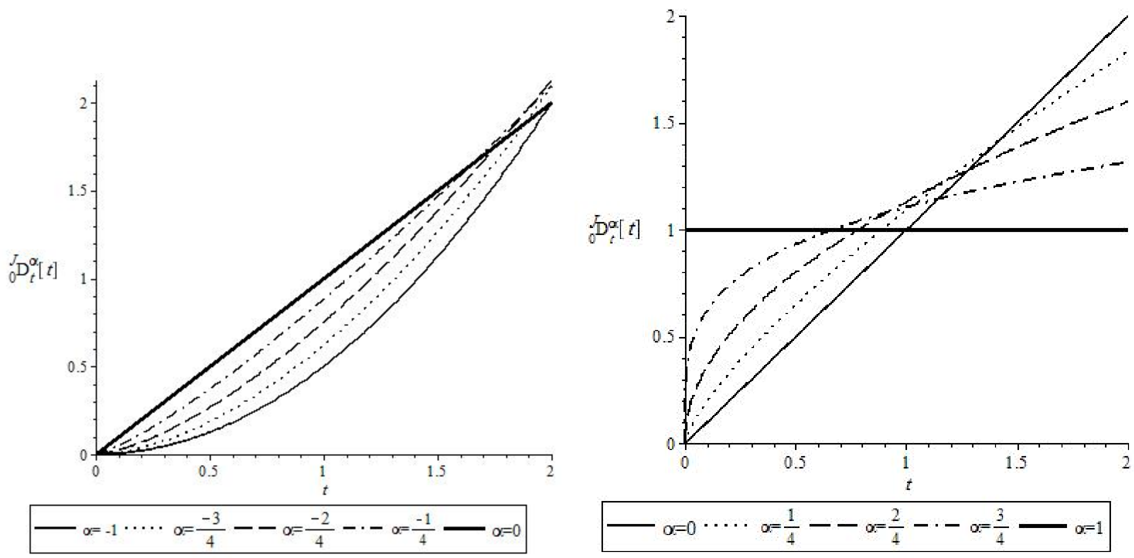


Figure 1.9: Jumarie modified derivative of $f(t) = t$

3. The exponential function

   Let $a = 0$, Applying the left Jumarie modified definition for $f(t) = e^{ct}, c \in \mathbb{C}$

$$e^{ct} = \sum_{k=0}^{\infty} \frac{(ct)^k}{\Gamma(k+1)} = 1 + \frac{ct}{\Gamma(2)} + \frac{c^2t^2}{\Gamma(3)} + \frac{c^3t^3}{\Gamma(4)} + ...$$

$$\begin{aligned}
{}_0^J D_t^\alpha(e^{ct}) &= D^\alpha[1 + \frac{ct}{\Gamma(2)} + \frac{c^2t^2}{\Gamma(3)} + \frac{c^3t^3}{\Gamma(4)} + ...] \\
&= 0 + \frac{c}{\Gamma(2)} \frac{\Gamma(2)t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{c^2}{\Gamma(3)} \frac{\Gamma(3)t^{2-\alpha}}{\Gamma(3-\alpha)} + ... \\
&= \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{c^2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{c^3t^{3-\alpha}}{\Gamma(4-\alpha)} + ... \\
&= ct^{1-\alpha} \sum_{k=0}^{\infty} \frac{(ct)^k}{\Gamma((2-\alpha)+k)} \\
&= ct^{1-\alpha} E_{1,2-\alpha}(ct)
\end{aligned}$$

$$
{}_0^J D_t^{\alpha}[e^{ct}] = ct^{1-\alpha} E_{1,2-\alpha}(ct)
$$

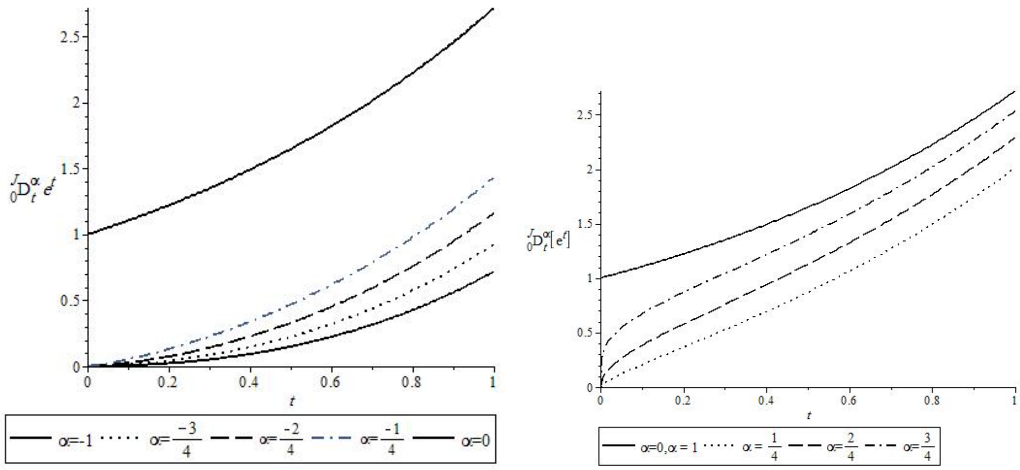


Figure 1.10: Jumarie modified derivative of $f(t) = e^t$

4. cos(ct),sin(ct) function Let $a = 0$ Applying The left Jumarie modified definition for $f(t) = cos(ct)$

$$
cos(ct) = \sum_{k=0}^{\infty} (-1)^k \frac{(ct)^{2k}}{\Gamma(2k+1)}
$$

$$
\begin{aligned}
{}_0^J D_t^{\alpha} cos(ct) &= D^{\alpha}[1 - \frac{(ct)^2}{\Gamma(3)} + \frac{(ct)^4}{\Gamma(5)} - \frac{(ct)^6}{\Gamma(7)} + ...] \\
&= 0 - \frac{c^2}{\Gamma(3)} \frac{\Gamma(3) t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{c^4}{\Gamma(5)} \frac{\Gamma(5) t^{4\alpha}}{\Gamma(5-\alpha)} ... \\
&= -\frac{c^2 t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{c^4 t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{c^6 t^{6-\alpha}}{\Gamma(7-\alpha)} + ... \\
&= -ct^{1-\alpha} \sum_{k=0}^{\infty} (-1)^k \frac{(ct)^{2k+1}}{\Gamma((2-\alpha) + 2k + 1)}
\end{aligned}
$$

$$= -ct^{1-\alpha} sin_{1,2-\alpha}(t)$$

$${}_0^J D_t^\alpha cos(ct) = -ct^{1-\alpha} sin_{1,2-\alpha}(t)$$

similarly we can get

$${}_0^J D_t^\alpha sin(ct) = ct^{1-\alpha} cos_{1,2-\alpha}(ct)$$

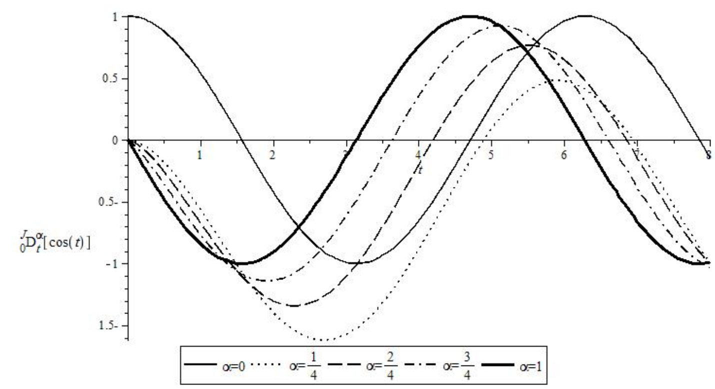


Figure 1.11: Jumarie modified derivative of $f(t) = cos(t)$

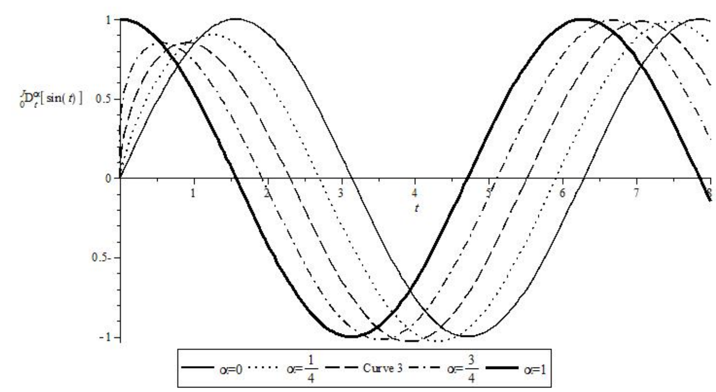


Figure 1.12: Jumarie modified derivative of $f(t) = sin(t)$

1.1.4 Properties of fractional derivatives

- Riemann-Liouvellie fractional derivative

  Let $n-1<\alpha<n,\ n,m\in\mathbb{N},\ \lambda,\mu\in\mathbb{C}$ and the functions $f(t)$ and $g(t)$ be such that both ${}_aD_t^{\alpha}f(t)$and ${}_aD_t^{\alpha}g(t)$ exist. Then [14],[20]

  1. Linearity

  $$ {}_aD_t^{\alpha}\left(\lambda\, f(t)+\mu\, g(t)\right)=\lambda\, {}_aD_t^{\alpha}f(t)+\mu\, {}_aD_t^{\alpha}g(t) $$

  2. Non-commutation

  $$ D_a^m D_t^{\alpha} f(t) =_a D_t^{\alpha+m} f(t) \neq_a D_t^{\alpha} D^m f(t) \tag{1.4} $$

  The inequalities in equation 1.4 become equalities under the the following additional condition

  $$ D_a^m D_t^{\alpha} f(t) =_a D_t^{\alpha+m} f(t) = D_t^{\alpha} D^m f(t) $$
  $$ f^{(s)}(0)=0\ ,\, s=0,1,...,m $$

  3. Leibniz rule

  If f(t) and g(t) and all it's derivatives are continuous in [0,t],then the following holds

  $$ {}_0D_t^{\alpha}(f(t)g(t))=\sum_{k=0}^{\infty}\binom{\alpha}{k}D_t^{\alpha-k}f(t)\, g^k(t) $$

4. Let $0 < \alpha_i < 1$ $,i = 1, 2, .., n,$ then the following equality holds,

$$ {}_aD_t^{\alpha} f(t) = {}_aD_t^{\alpha_1} \, {}_aD_t^{\alpha_2} \, ... \, {}_aD_t^{\alpha_n} $$
$$ \alpha = \alpha_1 + \alpha_2 \, ... \, \alpha_n $$

- Caputo fractional derivative

  Let $n - 1 < \alpha < n, \quad n, m \in \mathbb{N}, \; \lambda, \mu \in \mathbb{C}$ and the functions $f(t)$ and $g(t)$ be such that both ${}_a^cD_t^{\alpha} f(t)$and ${}_a^cD_t^{\alpha} g(t)$ exist. Then[7],[20]

  1. Linearity

$$ {}_a^cD_t^{\alpha} \left(\lambda \, f(t) + \mu \, g(t)\right) = \lambda \, {}_a^cD_t^{\alpha} f(t) + \mu \, {}_a^cD_t^{\alpha} g(t) $$

  2. Non-commutation

$$ {}_a^cD_t^{\alpha} \, D^m f(t) = {}_a^cD_t^{\alpha+m} f(t) \neq D^m \, {}_a^cD_t^{\alpha} f(t) \tag{1.5} $$

  The inequalities in equation 1.5 become equalities under the the following additional condition

$$ {}_a^cD_t^{\alpha} \, D^m f(t) = {}_a^cD_t^{\alpha+m} f(t) = D^m \, {}_a^cD_t^{\alpha} f(t) $$
$$ f^{(s)}(0) = 0 \, , s = n, n + 1 ... m $$

  3. Leibniz rule

  If f(t) and g(t) and all its derivatives are continuous in [0,t],then the following holds

$$ {}_{0}^{c}D_t^{\alpha}(f(t)g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} D_t^{\alpha-k} f(t) g^{(k)}(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \left((f(t)g(t))^k(0)\right) $$

4. Let $0 < \alpha_i < 1$,$i = 1, 2, .., n$, then the following equality holds,

$$ {}_{a}^{c}D_t^{\alpha} f(t) = {}_{a}^{c}D_t^{\alpha_1}\, {}_{a}^{c}D_t^{\alpha_2} \ldots {}_{a}^{c}D_t^{\alpha_n} $$

$$ \alpha = \alpha_1 + \alpha_2 \ldots \alpha_n $$

- Jumarie modified fractional derivative

  Let $n-1 < \alpha < n$, $n, m \in \mathbb{N}$, $\lambda, \mu \in \mathbb{C}$ and the functions $f(t)$ and $g(t)$ be such that both ${}_{a}^{J}D_t^{\alpha} f(t)$and ${}_{a}^{J}D_t^{\alpha} g(t)$ exist. Then[10],[11],[8]

  1. Linearity

  $$ {}_{a}^{J}D_t^{\alpha} \left(\lambda\, f(t) + \mu\, g(t)\right) = \lambda\, {}_{a}^{J}D_t^{\alpha} f(t) + \mu\, {}_{a}^{J}D_t^{\alpha}\, g(t) $$

  2. Non-commutation

  $$ {}_{a}^{J}D_t^{\alpha}\, D^m f(t) \neq D^m\, {}_{a}^{J}D_t^{\alpha} f(t) $$

  3. Leibniz rule

  Let $0 < \alpha < 1$, and the functions $f(t)$ and $g(t)$ are two non-differentiable functions in [0,t],then the following holds

$$ {}_a^J D_t^{\alpha}(f(t)g(t)) = f(t)\, {}_a^J D_t^{\alpha} g(t) + g(t)\, {}_a^J D_t^{\alpha} f(t) $$

4. For any positive integer n and ,$0 < \alpha_i < 1$,$i = 1, 2, .., n$, then the following equality holds,

$$ {}_a^J D_t^{\alpha} f(t) = {}_a^J D_t^{\alpha_1}\, {}_a^J D_t^{\alpha_2} \dots {}_a^J D_t^{\alpha_n} $$

$$ \alpha = \alpha_1 + \alpha_2 \dots \alpha_n $$

1.2 Mittag-Leffler Function

The applications of the Mittag-Leffler function and its extensions are discussed recently in a rapidly increasing number of papers, related to Fractional Calculus and fractional order differential equations.

1.2.1 One Parameter Mittag-Leffler Function

The one parameter Mittag-Leffler function was defined by

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \alpha k)} $$

This function play important role in fractional calculus like exponential function in classical calculus [3, 4]

$$E_1(z) = 1 + \frac{z}{\Gamma(1+1)} + \frac{z^2}{\Gamma(1+2)} + ... = e^z$$

$$E_2(z) = 1 + \frac{z}{\Gamma(1+2)} + \frac{z^2}{\Gamma(1+4)} + ... = \cosh\sqrt{z}$$

We now rewrite the Mittag-Leffler function in the following form by an infinite

$$E_\alpha(at^\alpha) = \sum_{k=0}^{\infty} \frac{(at^\alpha)^k}{\Gamma(1+\alpha k)} = 1 + \frac{at^\alpha}{\Gamma(1+\alpha)} + \frac{a^2t^2\alpha}{\Gamma(1+2\alpha)} + ...$$

Then we find Jumarie fractional derivative of order α for Mittag-Leffler function $E_\alpha(at^\alpha)$

$$\begin{aligned}
{}_0^JD_t^\alpha(E_\alpha(at^\alpha)) &= D^\alpha[1 + \frac{at^\alpha}{\Gamma(1+\alpha)} + \frac{a^2t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{a^3t^{3\alpha}}{\Gamma(1+3\alpha)} + ...] \\
&= 0 + \frac{a}{\Gamma(1+\alpha)}\frac{\Gamma(1+\alpha)}{\Gamma(1)} + \frac{a^2t^\alpha}{\Gamma(1+2\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} + ... \\
&= [a + \frac{a^2t^\alpha}{\Gamma(1+\alpha)} + \frac{a^3t^{2\alpha}}{\Gamma(1+2\alpha)} + ...] \\
&= a[1 + \frac{at^\alpha}{\Gamma(1+\alpha)} + \frac{a^2t^{2\alpha}}{\Gamma(1+2\alpha)} + ...] \\
&= aE(at^\alpha)
\end{aligned}$$

$$ {}_0^J D_t^\alpha (E_\alpha(at^\alpha)) = aE(at^\alpha) \tag{1.6} $$

This shows that $AE_\alpha(at^\alpha)$ is a solution of the fractional differential equation

$$ {}_0^J D_t^\alpha y = ay \tag{1.7} $$

Where A is arbitrary constant.

Therefore 1.7 with using the initial condition y(0)=1 has solution

$$ y = E_\alpha(at^\alpha) $$

1.2.2 Two parameter Mittag-Leffler function

The two parameter Mittag-Leffler function [3] was defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + k\alpha)} \tag{1.8} $$

$$ E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k)} = e^z $$

$$ E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^k + 1}{\Gamma(k+2)} = \frac{1}{z}(e^z - 1) $$

$$ E_{2,1}\left(z^2\right) = \sum_{k=0}^{\infty} \frac{z^2 k}{\Gamma(2k+1)} = \cosh z $$

now we find Jumarie fractional derivative of order β for Mittag-Leffler function $E_\alpha(at^\alpha)$

$$\begin{aligned} D^\beta(E_\alpha(at^\alpha)) &= D^\beta[1+\frac{at^\alpha}{\Gamma(1+\alpha)}+\frac{a^2t^{2\alpha}}{\Gamma(1+2\alpha)}+...] \\ &= 0+\frac{a}{\Gamma(1+\alpha)}\frac{t^{\alpha-\beta}\Gamma(1+\alpha)}{\Gamma(1+\alpha-\beta)}+... \\ &= \frac{a}{\Gamma(1+\alpha-\beta)}t^{\alpha-\beta}+\frac{a^2}{\Gamma(1+2\alpha-\beta)}t^{2\alpha-\beta}+... \\ &= at^{\alpha-\beta}[\frac{1}{\Gamma(1+\alpha-\beta)}+\frac{at^\alpha}{\Gamma(1+2\alpha-\beta)}+...] \\ &= at^{\alpha-\beta}\sum_{k=0}^{\infty}\frac{(at^\alpha)^k}{\Gamma((1+\alpha-\beta)+k\alpha)} \\ &= at^{\alpha-\beta}E_{\alpha,1+\alpha-\beta}(at^\alpha) \end{aligned}$$

$${}_0^JD_t^\beta(E_\alpha(at^\alpha)) = at^{\alpha-\beta}E_{\alpha,1+\alpha-\beta}(at^\alpha)$$

1.2.3 Complex Mittag-Leffler Function and Its Properties

Jumarie [9] defined the one parameter fractional sine and cosine function in the following form,

$$cos_\alpha(t^\alpha) = \frac{E_\alpha(it^\alpha)+E_\alpha(-it^\alpha)}{2} = \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k\alpha}}{\Gamma(1+2k\alpha)}$$

$$sin_\alpha(t^\alpha) = \frac{E_\alpha(it^\alpha)-E_\alpha(-it^\alpha)}{2} = \sum_{k=0}^{\infty}(-1)^k\frac{t^{(2k+1)\alpha}}{\Gamma(1+2k\alpha+\alpha)}$$

with this definition and with definition of one parameter Mittag-Leffler function we get the following identity

$$E_\alpha(it^\alpha) = 1 + \frac{it^\alpha}{\Gamma(1+\alpha)} + \frac{i^2t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{i^3t^{3\alpha}}{\Gamma(1+3\alpha)}...$$

$$= [1 - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + ...] + i[\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ...]$$
$$= \sum_{k=0}^{\infty}(-1)^k \frac{t^{2k\alpha}}{\Gamma(1+2k\alpha)} + i\sum_{k=0}^{\infty}(-1)^k \frac{t^{(2k+1)\alpha}}{\Gamma(1+2k\alpha+\alpha)}$$

Thus

$$E_\alpha(it^\alpha) = cos_\alpha(t^\alpha) + isin_\alpha(t^\alpha)$$

From figure 1.13 and figure 1.14 can be observed both the fractional trigonometric functions $cos_\alpha(t^\alpha), sin_\alpha(t^\alpha)$:

- for $\alpha < 1$ is fade oscillatory motion
- for $\alpha = 1$ harmonic motion with sustained oscillations
- for $\alpha > 1$ is increase oscillatory motion

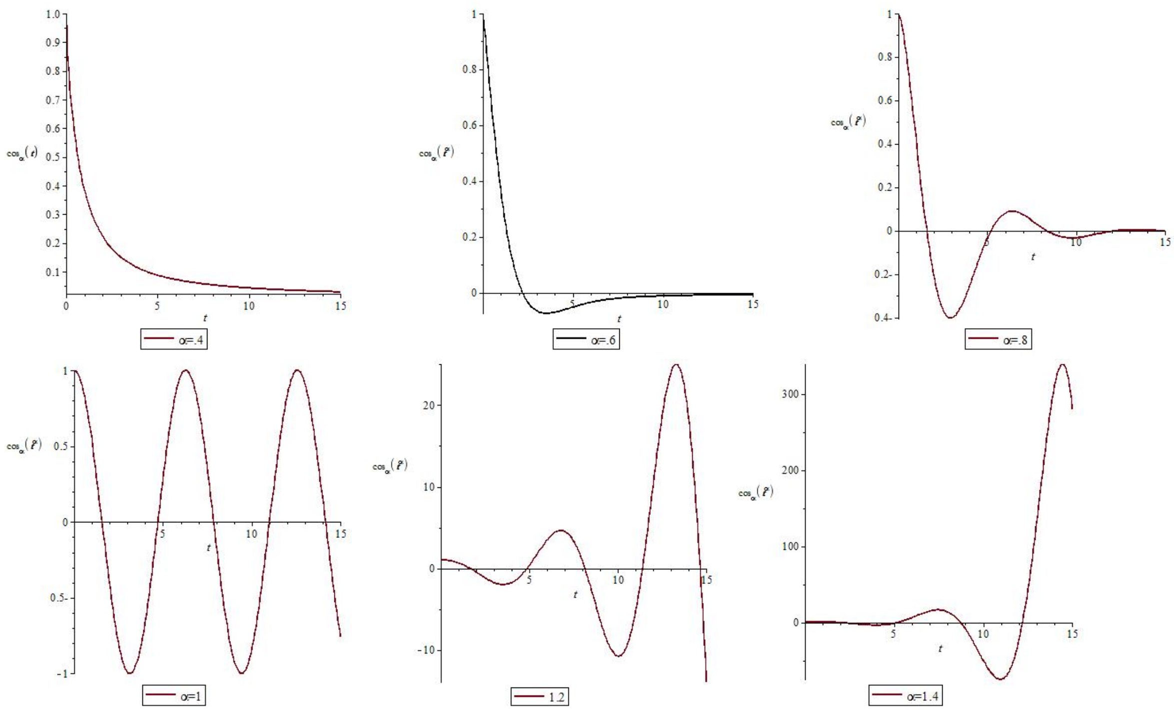


Figure 1.13: $cos_\alpha(t^\alpha)$

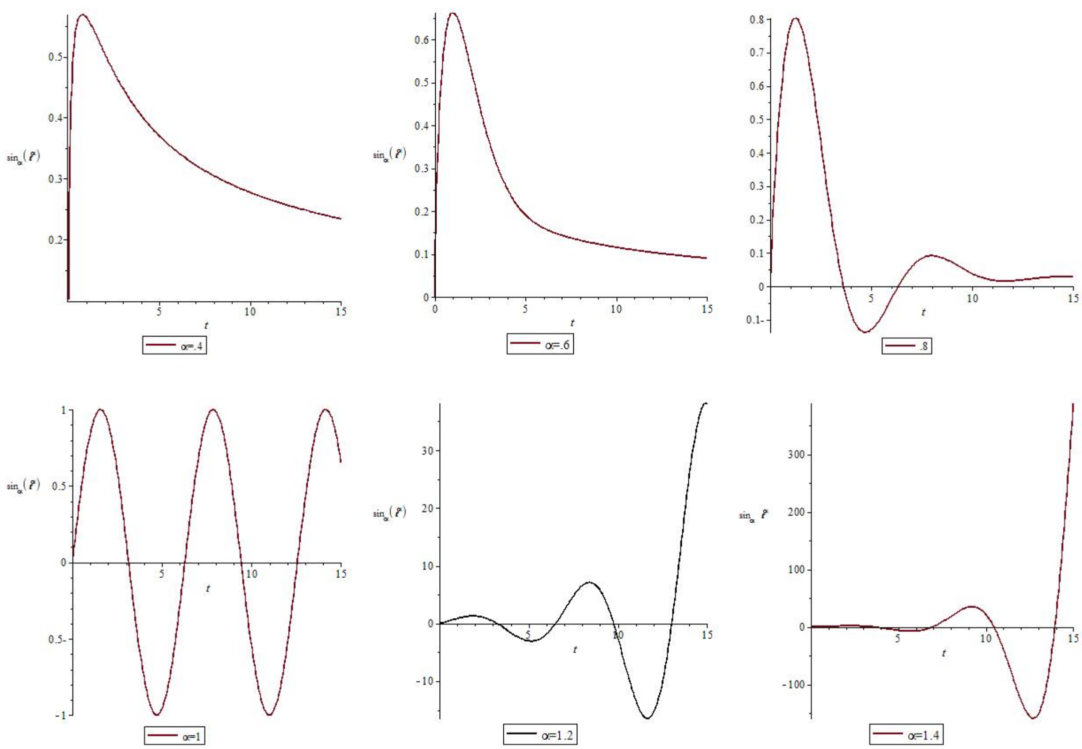


Figure 1.14: $sin_\alpha(t^\alpha)$

The series presentation of $cos_\alpha(t^\alpha)$ is

$$cos_\alpha(t^\alpha) = 1 - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} - ...$$

Taking term by term Jumarie fractional derivative of order α we get

$$\begin{aligned}
{}_0^J D_t^\alpha(cos_\alpha(t^\alpha)) &= D^\alpha[1 - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} - ...] \\
&= 0 - \frac{1}{\Gamma(1+2\alpha)} \frac{t^\alpha \Gamma(1+2\alpha)}{\Gamma(1+\alpha)} + ... \\
&= -\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} - ... \\
&= -[\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ...] \\
&= -\sum_{k=0}^{\infty} (-1)^k \frac{t^{(2k+1)\alpha}}{\Gamma(2k\alpha+\alpha+1)} \\
&= -sin_\alpha(t^\alpha)
\end{aligned}$$

$${}^J D^\alpha(cos_\alpha(t^\alpha)) = -sin_\alpha(t^\alpha)$$

The series presentation of $sin_\alpha(t^\alpha)$ is

$$sin_\alpha(t^\alpha) = \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{5\alpha}}{\Gamma(1+5\alpha)} - ...$$

Taking term by term Jumarie derivative of order α we get

$$\begin{aligned}
{}_0^J D_t^\alpha sin_\alpha(t^\alpha) &= {}_0^J D_t^\alpha [\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{5\alpha}}{\Gamma(1+5\alpha)} - ...] \\
&= 1 - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} - ... \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k\alpha}}{\Gamma(2k\alpha+1)} \\
&= cos_\alpha(t^\alpha)
\end{aligned}$$

$${}_0^J D_t^\alpha (sin_\alpha(t^\alpha)) = cos_\alpha(t^\alpha)$$

Also Jumarie [9] defined the two parameter fractional sine and cosine function in the following form,

$$cos_{\alpha,\beta}(t^\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k\alpha}}{\Gamma(\beta+2k\alpha)} \tag{1.9}$$

$$sin_{\alpha,\beta}(t^\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{(2k+1)\alpha}}{\Gamma(\beta+2k\alpha+\alpha)} \tag{1.10}$$

with this definition and with definition of two parameter Mittag-Leffler function we get the following identity

$$E_{\alpha,\beta}(it^\alpha) = cos_{\alpha,\beta}(t^\alpha) + isin_{\alpha,\beta}(t^\alpha)$$

Chapter 2

Linear Fractional Differential Equation with Jumarie Derivative

A linear $n\alpha$-order fractional differential equation of the form [14]

$$a_n {}^J D^{n\alpha} y(t) + a_{n-1} {}^J D^{(n-1)\alpha} y(t) + ... + a_0 y(t) = g(t^\alpha) \qquad (2.1)$$

where the coefficients $a_i, i = 0..n$ are real constants and

$a_n \neq 0, 0 < \alpha < 1$

From now we indicate Jumarie fractional derivative with the starting point of differentiation as 0 as ${}^J D^{n\alpha} y(t)$ instead${}_0^J D_t^{n\alpha} y(t)$

2.1 linear homogeneous fractional differential equation

$$a_n {}^J D^{n\alpha} y(t) + a_{n-1} {}^J D^{(n-1)\alpha} y(t) + ... + a_0 y(t) = 0 \qquad (2.2)$$

The above differential equation is said to be linear homogeneous fractional differential equation when $g(t^\alpha)$ identically zero in(2.1), otherwise it is non-homogeneous.

Theorem 2.1 let $y_1, y_2, ..., y_k$ be solutions of the homogeneous $n\alpha$-order fractional differential equation (2.2) . Then the linear combination

$$A_1 y_1(t) + A_2 y_2(t) + ... A_k y_k(t)$$

where the $A_i, i = 1...k$ are arbitrary constants, is also a solution

Proof. let $L(y(t)) = a_n {}^J D^{n\alpha} y(t) + a_{n-1} {}^J D^{(n-1)\alpha} y(t) + ... + a_0 y(t)$ and let $y_1(t), y_2(t), ..., y_k(t)$ be solutions of the homogeneous equation $L(y_1(t)) = 0, L(y_2(t)) = 0, ..., L(y_k(t)) = 0$ If we define

$y(t) = A_1y_1(t) + A_2y_2(t) + ... + A_ky_k(t)$, then by linearity of L we have
$L[A_1y_1(t) + A_2y_2(t) + ... + A_ky_k(t)]$
$= A_1L(y_1(t)) + A_2L(y_2(t)) + ... + A_kL(y_k(t))$
$= A_1.0 + A_2.0 + ... + A_k.0 = 0$
$A_1y_1(t) + A_2y_2(t) + ... + A_ky_k(t)$ is also solution $\square$

We begin by considering the special case of the second order fractional differential equation

$$a_2{}^JD^{2\alpha}y(t) + a_1{}^JD^{\alpha}y(t) + a_0y(t) = 0 \tag{2.3}$$

where a_0, a_1, a_2 are real constant

$$({}^JD^{\alpha} - m_1)({}^JD^{\alpha} - m_2)y(t) = 0$$

where $m_1 = \frac{-a_1+\sqrt{a_1^2-4a_2a_0}}{2a_2}$,$m_2 = \frac{-a_1-\sqrt{a_1^2-4a_2a_0}}{2a_2}$
there will be three forms of the general solution of (2.3) corresponding to the three cases [5]:

- **Case I** : m_1, m_2 are real and distinct

  The fractional differential equation

$${}^JD^{2\alpha}y(t) - (m_1 + m_2){}^JD^{\alpha}y(t) + (m_1m_2)y(t) \tag{2.4}$$

has solution of the form

$$y(t) = A_1 E_\alpha(m_1 t^\alpha) + A_2 E_\alpha(m_2 t^\alpha)$$

where A_1 and A_2 are constants

Proof. Let $y(t) = A_1 E_\alpha(m_1 t^\alpha) + A_2 E_\alpha(m_2 t^\alpha)$.

Differentiating α and 2α times with Jumarie derivative we get

${}^J D^\alpha y(t) = A_1 m_1 E_\alpha(m_1 t^\alpha) + A_2 m_2 E_\alpha(m_2 t^\alpha)$ and

${}^J D^{2\alpha} y(t) = A_1 {m_1}^2 E_\alpha(m_1 t^\alpha) + A_2 {m_2}^2 E_\alpha(m_2 t^\alpha)$ Substituting ${}^J D^\alpha y(t)$ and ${}^J D^{2\alpha} y(t)$ into the fractional differential equation(2.4) we get

$[A_1 {m_1}^2 - A_1 m_1 (m_1 + m_2) + A_1 (m_1 m_2)] E_\alpha(m_1 t^\alpha) + [A_2 {m_2}^2 - A_2 m_2 (m_1 + m_2) + A_2 (m_1 m_2)] E_\alpha(m_2 t^\alpha) = 0$

This shows that the fractional differential equation

$${}^J D^{2\alpha} y(t) - (m_1 + m_2) {}^J D^\alpha y(t) + (m_1 m_2) y(t) = 0$$

has solution in the form

$$y(t) = A_1 E_\alpha(m_1 t^\alpha) + A_2 E_\alpha(m_2 t^\alpha)$$

On the other hand consider the fractional differential equation

$${}^J D^{2\alpha} y(t) - (m_1 + m_2) {}^J D^\alpha y(t) + (m_1 m_2) y(t) = 0$$

it can be expressed in the following form

$$({}^{J}D^{\alpha} - m_1)({}^{J}D^{\alpha} - m_2)y(t) = 0 \tag{2.5}$$

Let, $x(t) = ({}^{J}D^{\alpha} - m_2)y(t)$ then equation (2.5) reduce to the form

$$({}^{J}D^{\alpha} - m_1)x(t) = 0 \qquad or \qquad {}^{J}D^{\alpha}x(t) = m_1 x(t)$$

Solution of the above equation is same as the solution of the equation (1.2) which is

$$x(t) = AE_{\alpha}(m_1 t^{\alpha})$$

$$({}^{J}D^{\alpha} - m_2)y(t) = AE_{\alpha}(m_1 t^{\alpha})$$

multiply both sides by $E_{\alpha}(-m_2 t^{\alpha})$

$$E_{\alpha}(-m_2 t^{\alpha})\,{}^{J}D^{\alpha}y(t) - E_{\alpha}(-m_2 t^{\alpha})m_2 y(t) = AE_{\alpha}(m_1 t^{\alpha})E_{\alpha}(-m_2 t^{\alpha})$$

$${}^{J}D^{\alpha}[y(t)E_{\alpha}(-m_2 t^{\alpha})] = \frac{A}{m_1 - m_2}\,{}^{J}D^{\alpha}[E_{\alpha}(m_1 t^{\alpha})E_{\alpha}(-m_2 t^{\alpha})]$$

On integrating both sides and applying ${}^{J}D^{-\alpha}$ on both sides of above, we get

$$y(t)E_{\alpha}(-m_2 t^{\alpha}) = \frac{A}{m_1 - m_2}[E_{\alpha}(m_1 t^{\alpha})E_{\alpha}(-m_2 t^{\alpha}) + A_2$$

$$y(t) = \frac{A}{m_1 - m_2}[E_\alpha(m_1 t^\alpha) + A_2 E_\alpha(m_2 t^\alpha)$$

Let $A_1 = \frac{A}{m_1 - m_2}$ Therefore $y(t) = A_1 E_\alpha(m_1 t^\alpha) + A_2 E_\alpha(m_2 t^\alpha)$ is a solution of the fractional differential equation (2.5) □

- **Case II**: $m_1 = m_2$ are real and equal

The fractional differential equation

$$^J D^{2\alpha} y(t) - 2m^J D^\alpha y(t) + m^2 y(t)$$

has solution of the form

$$y = A_1 E_\alpha(mt^\alpha) + A_2 t^\alpha E_\alpha(mt^\alpha)$$

where A_1 and A_2 are constants

Proof. Let $y(t) = A_1 E_\alpha(mt^\alpha) + A_2 t^\alpha E_\alpha(mt^\alpha)$. Differentiating α and 2α times with Jumarie derivative we get

$^J D^\alpha y(t) = A_1 m E_\alpha(mt^\alpha) + A_2 m t^\alpha E_\alpha(mt^\alpha) + A_2 \Gamma(\alpha+1) E_\alpha(mt^\alpha)$ and

$^J D^{2\alpha} y(t) = A_1 m^2 E_\alpha(mt^\alpha) + 2A_2 m \Gamma(\alpha+1) E_\alpha(mt^\alpha) + A_2 m^2 t^\alpha E_\alpha(mt^\alpha)$

Substituting $^J D^\alpha y(t)$ and $^J D^{2\alpha} y(t)$ into the fractional differential equation we get

$[A_1 m^2 + 2A_2 m \Gamma(\alpha + 1) + A_2 m^2 t^\alpha] E_\alpha(mt^\alpha) - 2m E_\alpha(mt^\alpha)[A_1 m + A_2 m t^\alpha + A_2 \Gamma(\alpha + 1)] + m^2 E_\alpha(mt^\alpha)[A_1 + A_2 t^\alpha] = 0$

This shows that the fractional differential equation

$$ {}^{J}D^{2\alpha}y(t) - 2m\,{}^{J}D^{\alpha}y(t) + m^2 y(t) = 0 $$

has solution in the form

$$ y(t) = A_1 E_\alpha(mt^\alpha) + A_2 t^\alpha E_\alpha(mt^\alpha) $$

On the other hand consider the fractional differential equation

$$ {}^{J}D^{2\alpha}y(t) - 2m\,{}^{J}D^{\alpha}y(t) + m^2 y(t) $$

it can be expressed in the following form

$$ ({}^{J}D^{\alpha} - m)({}^{J}D^{\alpha} - m)y(t) = 0 \tag{2.6} $$

Let, $x(t) = ({}^{J}D^{\alpha} - m)y(t)$ then equation (2.6) reduce to the form

$$ ({}^{J}D^{\alpha} - m)x(t) = 0 \qquad or \qquad {}^{J}D^{\alpha}x(t) = mx(t) $$

Solution of the above equation is same as the solution of the equation (1.2) which is

$$ x(t) = AE_\alpha(mt^\alpha) $$

$$ ({}^{J}D^{\alpha} - m)y(t) = AE_\alpha(mt^\alpha) $$

multiply both side by $E_\alpha(-mt^\alpha)$

$$E_\alpha(-mt^\alpha)\,{}^JD^\alpha y(t) - E_\alpha(-mt^\alpha)\, m\, y(t) = AE_\alpha(mt^\alpha)E_\alpha(-mt^\alpha)$$

$${}^JD^\alpha[y(t)E_\alpha(-mt^\alpha)] = \frac{A}{\Gamma(\alpha+1)}\,{}^JD^\alpha[t^\alpha]$$

On integrating both sides and applying ${}^JD^{-\alpha}$ on both sides of above, we get

$$y(t)E_\alpha(-mt^\alpha) = \frac{A}{\Gamma(\alpha+1)}t^\alpha + A_2$$

Let $A_1 = \frac{A}{\Gamma(\alpha+1)}$ Therefore $y = A_1 t^\alpha E_\alpha(mt^\alpha) + A_2 E_\alpha(mt^\alpha)$ is a solution of the fractional differential equation (2.6) □

- **Case III**: $m_1 = p + iq, m_2 = p - iq$ are complex

$${}^JD^{2\alpha}y(t) - 2p\,{}^JD^\alpha y(t) + (p^2+q^2)y(t) = 0$$

has solution of the form

$$y(t) = E_\alpha(pt^\alpha)[A_1 cos_\alpha(qt^\alpha) + A_2 sin_\alpha(qt^\alpha)]$$

Proof. The given fractional differential equation can be written in the following form

$$[{}^JD^\alpha - (p+iq)][{}^JD^\alpha - (p-iq)]y(t) = 0 \tag{2.7}$$

Using Case I we get the solution of the fractional differential (2.7) can be written in the following form

$$y(t) = a_1 E_\alpha((p - iq)t^\alpha) + a_2 E_\alpha((p + iq)t^\alpha)$$

$$y(t) = a_1 E_\alpha(pt^\alpha) E_\alpha(-iqt^\alpha) + a_2 E_\alpha(pt^\alpha) E_\alpha(iqt^\alpha)$$

$$y(t) = a_1 E_\alpha(pt^\alpha)[cos_\alpha(-qt^\alpha) + isin_\alpha(-qt^\alpha)]$$

$$+a_2 E_\alpha(pt^\alpha)[cos_\alpha(qt^\alpha) + isin_\alpha(qt^\alpha)]$$

Since $cos_\alpha(-qt^\alpha) = cos_\alpha(qt^\alpha)$ and $sin_\alpha(-qt^\alpha) = -sin_\alpha(qt^\alpha)$
Therefore

$$y(t) = E_\alpha(pt^\alpha)[(a_1 + a_2)cos_\alpha(qt^\alpha) + (-a_1 + a_2)isin_\alpha(qt^\alpha)]$$

let $A_1 = a_1 + a_2$ and $A_2 = (-a_1 + a_2)i$

$$y(t) = E_\alpha(pt^\alpha)[A_1 cos_\alpha(qt^\alpha) + A_2 sin_\alpha(qt^\alpha)]$$

□

Example 2.1. *2α-order FDEs*

Solve the following fractional differential equations.

1. $2\,{}^J D^{2\alpha} y(t) - 5\,{}^J D^\alpha y(t) - 3y(t) = 0$

2. ${}^J D^{2\alpha} y(t) - 4\,{}^J D^\alpha y(t) + 4y(t) = 0$

3. ${}^{J}D^{2\alpha}y(t) + 4y(t) = 0$

Solution

1. $({}^{J}D^{\alpha} + \frac{1}{2})({}^{J}D^{\alpha} - 3)y(t) = 0$

   From(Case I) $y(t) = A_1E_{\alpha}(\frac{-1}{2}) + A_2E_{\alpha}(3t^{\alpha})$

2. $({}^{J}D^{\alpha} - 2)({}^{J}D^{\alpha} - 2)y(t) = 0$

   From(Case II) $y(t) = A_1E_{\alpha}(2t^{\alpha}) + A_2t^{\alpha}E_{\alpha}(2t^{\alpha})$

3. $({}^{J}D^{\alpha} - 2i)({}^{J}D^{\alpha} + 2i)y(t) = 0$

   From(Case III) $y(t) = A_1cos_{\alpha}(2t^{\alpha}) + A_2sin_{\alpha}(2t^{\alpha})$

Example 2.2. *An Initial-Value Problem*

Solve ${}^{J}D^{2\alpha}y(t) - 3{}^{J}D^{\alpha}y(t) + 2y(t) = 0, y(0) = -1,\ {}^{J}D^{\alpha}y(0) = 2$

Solution

$({}^{J}D^{\alpha} - 2)({}^{J}D^{\alpha} - 1)y(t) = 0$

From(Case I) the general solution is

$y(t) = A_1E_{\alpha}(2t^{\alpha}) + A_2E_{\alpha}(t^{\alpha})$

Putting the initial condition.$y(0) = -1,\ {}^{J}D^{\alpha}y(0) = 2$

$y(0) = A_1 + A_2 = -1$

${}^{J}D^{\alpha}y(0) = 2A_1 + A_2 = 2$ *and solving we get* $A_1 = 3$,$A_2 = -4$

Hence the solution is,

$$y(t) = 3E_{\alpha}(2t^{\alpha}) - 4E_{\alpha}(t^{\alpha})$$

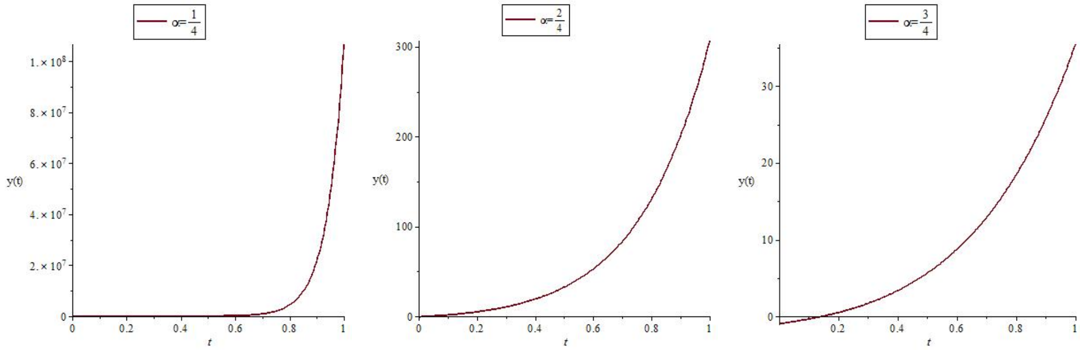


Figure 2.1: Solution curve of IVP in Example 2.2 for different values of $\boldsymbol{\alpha = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}}$

In ref [5] had solved Linear Second Order Fractional Differential Equation , Now we will see that the foregoing procedure can produce solutions for homogeneous linear higher Order Fractional Differential Equation

HIGHER-ORDER EQUATIONS

In general, to solve an nth-order differential equation

$$a_n {}^J D^{n\alpha} y(t) + a_{n-1} {}^J D^{(n-1)\alpha} y(t) + ... + a_0 y(t) = 0 \tag{2.8}$$

where $a_i, i = 0..n$ are real constants. we must solve an nth -degree polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + ... a_2 m^2 + a_1 m + a_0 = 0 \tag{2.9}$$

If all the roots of (2.9) are real and distinct, then the general solution of (2.8) is

$$y(t) = A_1 E_\alpha(m_1 t^\alpha) + A_2 E_\alpha(m_2 t^\alpha) + ... + A_n E_\alpha(m_n t^\alpha)$$

It is somewhat harder to summarize the analogues of Cases II and III because the roots of (2.9) of degree greater than two can occur in many combinations.

When m_1 is a root of multiplicity k of (2.9) equation (that is, k roots are equal to m_1), it can be shown that the general solution of (2.8) must contain

$$A_1 E_\alpha(m_1 t^\alpha) + A_2 E_\alpha t^\alpha (m_1 t^\alpha) + ... + A_n t^{(k-1)\alpha} E_\alpha(m_1 t^\alpha)$$

Finally, when the coefficients are real, complex roots of (2.9) always appear in conjugate pairs.

Example 2.3. *Third-Order FDE*

Solve ${}^J D^{3\alpha} y(t) + 3{}^J D^{2\alpha} y(t) - 4y(t) = 0$

Solution

to solve a third degree polynomial equation

$m^3 + 3m^2 - 4 = (m-1)(m+2)^2 = 0$

so the roots are $m_1 = 1, m_2 = m_3 = -2$

Thus the general solution of the FDE is

$y(t) = A_1 E_\alpha(t^\alpha) + A_2 E_\alpha(-2t^\alpha) + A_3 t^\alpha E_\alpha(-2t^\alpha)$

Example 2.4. *Fourth-Order FDE*

Solve ${}^J D^{4\alpha} y(t) + 2{}^J D^{2\alpha} y(t) + y(t) = 0$

Solution

solve a Fourth degree polynomial equation

$m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0$

so the roots are $m_1 = i, m_2 = -i, m_3 = i, m_3 = -i$

Thus the general solution of the FDE is

$y(t) = A_1 cos_\alpha(t^\alpha) + A_2 sin_\alpha(t^\alpha) + t^\alpha[A_3 cos_\alpha(t^\alpha) + A_4 sin_\alpha(t^\alpha)]$

2.2 linear non-homogeneous fractional differential equation

Any function y_p, free of arbitrary parameters, that satisfies (2.1) is said to be a particular solution or particular integral of the equation.

Theorem 2.2 Let y_p be any particular solution of the non-homogeneous linear $n\alpha$-order fractional differential equation (2.1) and let $y_1, y_2, ..., y_n$ be a set of solutions of the associated homogeneous fractional differential equation (2.2) . Then the general solution of the equation(2.1) is

$$A_1 y_1(t) + A_2 y_2(t) + ...A_n y_n(t) + y_p(t)$$

where the $A_i, i = 1...n$ are arbitrary constants, is also a solution

Proof. let $L(y(t)) = a_n {}^J D^{n\alpha} y(t) + a_{n-1} {}^J D^{(n-1)\alpha} y(t) + ... + a_0 y(t)$
and let $Y(t)$ and $y_p(t)$be particular solutions of the non-homogeneous fractional differential equation,$L(Y(t)) = g(t^\alpha)$ and $L(y_p(t)) = g(t^\alpha)$. If we define
$u(t) = Y(t) - y_p(t)$ then by linearity of L we have
$L(u(t)) = L(Y(t) - y_p(t)) = L(Y(t)) - L(y_p(t)) = g(t^\alpha) - g(t^\alpha) = 0$
This shows that u(t) is a solution of the homogeneous equation

$L(u(t)) = 0$. Hence by Theorem 2.1

$u(t) = A_1 y_1(t) + A_2 y_2(t) + ...A_n y_n(t)$ and so

$Y(t) - y_p(t) = A_1 y_1(t) + A_2 y_2(t) + ...A_n y_n(t)$

or $Y(t) = A_1 y_1(t) + A_2 y_2(t) + ...A_n y_n(t) + y_p(t)$ □

We see in Theorem 2.6 that the general solution of a non-homogeneous linear fractional differential equation consists of the sum of two functions:

$$Y(t) = A_1 y_1(t) + A_2 y_2(t) + ...A_n y_n(t) + y_p(t) = y_c(t) + y_p(t)$$

The linear combination $y_c(t) = A_1 y_1(t) + A_2 y_2(t) + ...A_n y_n(t)$ which is the general solution of (2.2), is called the complementary function for equation (2.1). Thus the general solution will be

$$y = y_p + y_c$$

Method of Undetermined Coefficients can be used to find a particular solution to a non-homogeneous differential equation.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y^{(1)} + a_0 = g(t) \tag{2.10}$$

The method is quite simple. All that we need to do is look at $g(t)$ and make a guess as to the form of y_p leaving the coefficients undetermined . Plug the guess into the differential equation and see if we can determine values of the coefficients. If we can determine values for the coefficients then our guess is correct, if we cant find values for the coefficients then our guess is incorrect. The general method is limited to linear differential equation such as (2.10) where $g(t)$ is a polynomial function, an exponential function e^{ct}, a sine or cosine function $sin(ct), cos(ct)$, or finite sums and products of these functions [12].

Now we will use the same idea of undetermined coefficients to find a particular solution to a non-homogeneous fractional differential equation(2.1). where $g(t^{\alpha})$ is fractional polynomial function, a Mittag - Leffer function $E_{\alpha}(ct^{\alpha})$, a fractional sine or cosine function $sin_{\alpha}(ct), cos_{\alpha}(ct)$,or finite sums and products of these functions.

Example 2.5. *$g(t^{\alpha})$ is a fractional polynomial function*

Solve

$$^{J}D^{2\alpha}y(t) - 2^{J}D^{\alpha}y(t) - 3y(t) = 3t^{2\alpha} + 4t^{\alpha} - 5 \tag{2.11}$$

Solution

We first solve the associated homogeneous equation $^{J}D^{2\alpha}y(t) - 2^{J}D^{\alpha}y(t) - 3y(t) = 0$. *From the quadratic formula we find that the roots of the equation $m^2 - 2m - 3 = 0$ are $m_1 = 3$ and $m_2 = -1$. Hence the complementary function is*

$$y_c = A_1E_{\alpha}(3t^{\alpha}) + A_2E_{\alpha}(-t^{\alpha})$$

Now, because the function $g(t^{\alpha})$ is fractional polynomial of degree 2α, *let us assume a particular solution that is also in the form of fractional polynomial of degree* 2α

$$y_p = At^{2\alpha} + Bt^{\alpha} + C$$

We seek to determine specific coefficients A*,* B*, and* C *for which* y_p *is a solution of (2.11).Substituting* y_p *and the Jumarie Fractional Derivative* ${}^{J}D^{\alpha}y_p = A\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}t^{\alpha} + B\Gamma(\alpha+1)$ *and* ${}^{J}D^{2\alpha}y_p = A\Gamma(2\alpha+1)$
into the given fractional differential equation (2.11), we get
$A\Gamma(2\alpha+1) - 2(A\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1)}t^{\alpha} + B\Gamma(\alpha+1)) - 3(At^{2\alpha} + Bt^{\alpha} + C) = 3t^{2\alpha} + 4t^{\alpha} - 5$
The coefficients of like powers of t *must be equal, That is*
$-3A = 3, \quad -2A\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} - 3B = 4, \quad A\Gamma(2\alpha+1) - 2B\Gamma(\alpha+1) - 3c = -5$
Solving this system of equations leads to the values
$A = -1, \qquad B = \frac{-4}{3} + \frac{2\Gamma(2\alpha+1)}{3\Gamma(\alpha+1)}, \qquad C = \frac{15+8\Gamma(\alpha+1)-7\Gamma(2\alpha+1)}{9}$

$$y_p = -t^{2\alpha} + \left(\frac{-4}{3} + \frac{2\Gamma(2\alpha+1)}{3\Gamma(\alpha+1)}\right)t^{\alpha} + \frac{15 + 8\Gamma(\alpha+1) - 7\Gamma(2\alpha+1)}{9}$$

The general solution of the given equation(2.11) is
$y = A_1E_{\alpha}(3t^{\alpha}) + A_2E_{\alpha}(-t^{\alpha}) - t^{2\alpha} + \left(\frac{-4}{3} + \frac{2\Gamma(2\alpha+1)}{3\Gamma(\alpha+1)}\right)t^{\alpha}$
$+\frac{15+8\Gamma(\alpha+1)-7\Gamma(2\alpha+1)}{9}$

Example 2.6. $g(t^{\alpha})$ *fractional sine function*

Find a particular solution of

$${}^{J}D^{2\alpha}y(t) - {}^{J}D^{\alpha}y(t) + y(t) = 2sin_{\alpha}(3t^{\alpha}) \tag{2.12}$$

Solution

because ${}^{J}D^{2\alpha} and {}^{J}D^{\alpha}$ *of* $sin_{\alpha}(3t^{\alpha})$ *produce* $sin_{\alpha}(3t^{\alpha})$ *and* $cos_{\alpha}(3t^{\alpha})$, *we are assume a particular solution that includes both of these terms:*

$$y_p = Acos_{\alpha}(3t^{\alpha}) + Bsin(3t^{\alpha})$$

Substituting y_p *and the Jumarie fractional derivative*
${}^{J}D^{\alpha}y_p = -3Asin_{\alpha}(3t^{\alpha}) + 3Bcos_{\alpha}(3t^{\alpha})$ *and*
${}^{J}D^{2\alpha}y_p = -9Acos_{\alpha}(3t^{\alpha}) - 9Bsin_{\alpha}(3t^{\alpha})$
into the given fractional differential equation (2.12), we get
$(-8A - 3B)cos_{\alpha}(3t^{\alpha}) + (3A - 8B)sin_{\alpha}(3t^{\alpha}) = 2sin_{\alpha}(3t^{\alpha})$
From the resulting system of equations,

$$-8A - 3B = 0 \qquad and \qquad 3A - 8B = 2$$

we get $A = \frac{6}{73}$ *and* $B = \frac{-16}{73}$. *A particular solution of the equation is*

$$y_p = \frac{6}{73}cos_{\alpha}(3t^{\alpha}) - \frac{16}{73}sin(3t^{\alpha})$$

As we mentioned, the form that we assume for the particular solution y_p is an educated guess. This educated guess must take into consideration not only the types of functions that make up $g(t^{\alpha})$ but also, as we shall see in Example (2.7), the functions that make up the complementary function y_c

Example 2.7. *$g(t^\alpha)$ is a Mittag - Leffer*

solve

$$^JD^{2\alpha}y(t) - (a+b)^JD^\alpha y(t) + (ab)y(t) = E_\alpha(ct^\alpha) \tag{2.13}$$

Solution

- $a \neq b \neq c$

  We first solve the associated homogeneous equation

  $^JD^{2\alpha}y(t) - (a+b)^JD^\alpha y(t) + (ab)y(t) = 0$

  the complementary function is

$$y_c = A_1E(at^\alpha) + A_2E(bt^\alpha)$$

  Now, because the function $g(t^\alpha)$ is a Mittag - Leffer, let us assume a particular solution that is also in the form of a Mittag - Leffer

$$y_p = AE_\alpha(ct^\alpha)$$

  We seek to determine specific coefficients A which $y_p$is a solution of (2.13).Substituting y_p and the Jumarie Fractional Derivative

  $^JD^\alpha y_p = AcE_\alpha(ct^\alpha)$ *and* $^JD^{2\alpha}y_p = Ac^2E_\alpha(ct^\alpha)$

  into the given fractional differential equation (2.13), we get

$$Ac^2E_\alpha(ct^\alpha) - (a+b)AcE_\alpha(ct^\alpha) + (ab)AE_\alpha(ct^\alpha) = E_\alpha(ct^\alpha)$$

That is, $Ac^2 - (a+b)Ac + (ab) = 1$ *we get* $A = \frac{1}{(c-a)(c-b)}$ *Thus a particular solution is*

$$y_p = \frac{1}{(c-a)(c-b)} E_\alpha(ct^\alpha)$$

The general solution of the given equation is

$$y = A_1 E(at^\alpha) + A_2 E(bt^\alpha) + \frac{1}{(c-a)(c-b)} E_\alpha(ct^\alpha)$$

- $c = a \neq b$

  the complementary function is

$$y_c = A_1 E(at^\alpha) + A_2 E(bt^\alpha)$$

Observe that our assumption $AE(at^\alpha)$ *is already present in* y_c. *This means that* $E(at^\alpha)$ *is a solution of the associated homogeneous fractional differential equation, and* $AE(at^\alpha)$ *when substituted into the fractional differential equation necessarily produces zero. Inspired by Case II of Section 2.1, lets see whether we can find a particular solution of the form*

$$y_p = At^\alpha E_\alpha(at^\alpha)$$

Substituting ${}^J D^\alpha y_p = Aat^\alpha E(at^\alpha) + A\Gamma(\alpha+1)E(at^\alpha)$ *and*
${}^J D^{2\alpha} y_p = Aa^2 t^\alpha E(at^\alpha) + 2Aa\Gamma(\alpha+1)E(at^\alpha)$

into the fractional differential equation and simplifying gives

$A(a-b)\Gamma(\alpha+1)E_\alpha(at^\alpha) = E_\alpha(at^\alpha)$

From the last equality we see that the value of A is now determined as $A = \frac{1}{(a-b)\Gamma(\alpha+1)}$ *Therefore a particular solution is*

$$y_p = \frac{t^\alpha}{(a-b)\Gamma(\alpha+1)} E_\alpha(at^\alpha)$$

The general solution of the given equation is

$$y = A_1 E(at^\alpha) + A_2 E(bt^\alpha) + \frac{t^\alpha}{(a-b)\Gamma(\alpha+1)} E_\alpha(at^\alpha)$$

- $c = a = b$

  the complementary function is

$$y_c = A_1 E(at^\alpha) + A_2 t^\alpha E(at^\alpha)$$

we can find a particular solution of the form

$$y_p = At^{2\alpha} E_\alpha(at^\alpha)$$

Substituting ${}^J D^\alpha y_p = Aat^{2\alpha}E(at^\alpha) + A\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t^\alpha E(at^\alpha)$ *and* ${}^J D^{2\alpha} y_p = Aa^2 t^{2\alpha} E(at^\alpha) + 2Aa\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t^\alpha E(at^\alpha) + A\Gamma(2\alpha+1)E(at^\alpha)$ *into the fractional differential equation and simplifying gives*

$A\Gamma(2\alpha+1)E_\alpha(at^\alpha) = E_\alpha(at^\alpha)$

From the last equality we see that the value of A is now determined as $A = \frac{1}{\Gamma(2\alpha+1)}$ Therefore a particular solution is

$$y_p = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} E_\alpha(at^\alpha)$$

The general solution of the given equation is

$$y = A_1 E(at^\alpha) + A_2 t^\alpha E(at^\alpha) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} E_\alpha(at^\alpha)$$

When $g(t^\alpha)$ is a sum of several terms

When $g(t^\alpha) = g_1(t^\alpha) + g_2(t^\alpha) + ... + g_n(t^\alpha)$, we can break the equation into n parts and solve them separately.

$$^J D^{2\alpha} y(t) + a_1 {}^J D^\alpha y(t) + a_0 y(t) = g_1(t^\alpha) + g_2(t^\alpha) + ... + g_n(t^\alpha) \quad (2.14)$$

If y_1 is a solution of the equation

$$^J D^{2\alpha} y(t) + a_1 {}^J D^\alpha y(t) + a_0 y(t) = g_1(t^\alpha)$$

and y_2 is a solution of the equation

$$^J D^{2\alpha} y(t) + a_1 {}^J D^\alpha y(t) + a_0 y(t) = g_2(t^\alpha)$$

and so on y_n is a solution of the equation

$$^J D^{2\alpha} y(t) + a_1 {}^J D^{\alpha} y(t) + a_0 y(t) = g_n(t^\alpha)$$

Then, $\quad y_p = y_1 + y_2 + ... + y_n \quad$ is a solution of the equation (2.14)

Example 2.8.

$$^J D^{2\alpha} y(t) - 2^J D^{\alpha} y(t) - 3y(t) = 3t^{2\alpha} + 4t^\alpha - 5 + E_\alpha(2t^\alpha) + cos_\alpha(t^\alpha)$$

Solution

Solve each of the sub-parts:

- $^J D^{2\alpha} y(t) - 2^J D^{\alpha} y(t) - 3y(t) = 3t^{2\alpha} + 4t^\alpha - 5$

  $y_1(t) = -t^{2\alpha} + \left(\frac{-4}{3} + \frac{2\Gamma(2\alpha+1)}{3\Gamma(\alpha+1)}\right) t^\alpha + \frac{15+8\Gamma(\alpha+1)-7\Gamma(2\alpha+1)}{9}$

- $^J D^{2\alpha} y(t) - 2^J D^{\alpha} y(t) - 3y(t) = E_\alpha(2t^\alpha)$

  $y_2(t) = \frac{-1}{3} E_\alpha(2t^\alpha)$

- $^J D^{2\alpha} y(t) - 2^J D^{\alpha} y(t) - 3y(t) = cos_\alpha(t^\alpha)$

  $y_3(t) = -cos_\alpha(t^\alpha) - \frac{1}{10} sin_\alpha(t^\alpha)$

Then $\quad y_p(t) = -t^{2\alpha} + \left(\frac{-4}{3} + \frac{2\Gamma(2\alpha+1)}{3\Gamma(\alpha+1)}\right) t^\alpha$
$+\frac{15+8\Gamma(\alpha+1)-7\Gamma(2\alpha+1)}{9} - \frac{1}{3} E_\alpha(2t^\alpha) - cos_\alpha(t^\alpha) - \frac{1}{10} sin_\alpha(t^\alpha)$

The general solution is

$y(t) = A_1 E_\alpha(3t^\alpha) + A_2 E_\alpha(-t^\alpha) - t^{2\alpha} + \left(\frac{-4}{3} + \frac{2\Gamma(2\alpha+1)}{3\Gamma(\alpha+1)}\right) t^\alpha$
$+\frac{15+8\Gamma(\alpha+1)-7\Gamma(2\alpha+1)}{9} - \frac{1}{3} E_\alpha(2t^\alpha) - cos_\alpha(t^\alpha) - \frac{1}{10} sin_\alpha(t^\alpha)$

When $g(t^\alpha)$ is a product of several terms :

when $g(t^\alpha)$ is a product of basic functions, $y_p(t)$ is chosen based on:

1. $y_p(t)$ is a product of the corresponding choices of all the parts of $g(t^\alpha)$
2. There are as many coefficients as the number of distinct terms in $y_p(t)$
3. Each distinct term must have its own coefficient, not shared with any other term.

Example 2.9.

$$^J D^{2\alpha} y(t) - 2^J D^{\alpha} y(t) - 3y(t) = E_\alpha(2t^\alpha) cos_\alpha(3t^\alpha) \tag{2.15}$$

Solution

The corresponding homogeneous equation $^J D^{2\alpha} y(t) - 2^J D^{\alpha} y(t) - 3y(t) = 0$ *Therefore, the complementary solution is*

$$y_c = A_1 E_\alpha(3t^\alpha) + A_2 E_\alpha(-t^\alpha)$$

Now,start with the basic forms of the corresponding functions $g(t^\alpha)$ *that are to appear in the product, without assigning any coefficient. In the above example, they are* $E_\alpha(2t^\alpha)$ *and* $cos_\alpha(3t^\alpha) + sin_\alpha(3t^\alpha)$

Multiply them together to get all the distinct terms in the product: $cos_\alpha(3t^\alpha)E_\alpha(2t^\alpha) + sin_\alpha(3t^\alpha)E_\alpha(2t^\alpha)$ *Then we insert the undetermined coefficients into the expression, one for each term:*

$$y_p(t) = Acos_\alpha(3t^\alpha)E_\alpha(2t^\alpha) + Bsin_\alpha(3t^\alpha)E_\alpha(2t^\alpha)$$

We seek to determine specific coefficients A, B *for which* y_p *is a solution of (2.15). Substituting* y_p *and the Jumarie fractional derivatives,*
${}^J D^\alpha y_p(t) = [(2A + 3B)cos_\alpha(3t^\alpha) + (-3A + 2B)sin_\alpha(3t^\alpha)]E_\alpha(2t^\alpha)$ *and*
${}^J D^{2\alpha} y_p(t) = [(-5A + 12B)cos_\alpha(3t^\alpha) + (-12A - 5B)sin_\alpha(3t^\alpha)]E_\alpha(2t^\alpha)$
into the given differential equation (2.15), we get
$[(-12A + 6B)cos_\alpha(3t^\alpha) + (-6A - 12B)sin_\alpha(3t^\alpha)]E_\alpha(2t^\alpha)$
$= E_\alpha(2t^\alpha)cos_\alpha(3t^\alpha)$ *From the resulting system of equations,*

$$-12A + 6B = 1 \quad and \quad -6A - 12b = 0$$

we get $A = \frac{-1}{15}$ *and* $B = \frac{1}{30}$. *A particular solution of the equation is*

$$y_p(t) = \frac{-1}{15}cos_\alpha(3t^\alpha)E_\alpha(2t^\alpha) + \frac{1}{30}sin_\alpha(3t^\alpha)E_\alpha(2t^\alpha)$$

The general solution of the given equation is

$$y(t) = A_1E_\alpha(3t^\alpha)+A_2E_\alpha(-t^\alpha)\frac{-1}{15}cos_\alpha(3t^\alpha)E_\alpha(2t^\alpha)+\frac{1}{30}sin_\alpha(3t^\alpha)E_\alpha(2t^\alpha)$$

The Rules-of the Method of Undetermined Coefficients

1. If $g(t^\alpha) = AE_\alpha(at^\alpha)$ is a Mittag-Leffler function , the starting choice for $y_p(t)$ is a Mittag-Leffler function.$y_p(t) = A_1E_\alpha(at^\alpha)$

2. If a fractional polynomial function appears in $g(t^\alpha)$, the starting choice for $y_p(t)$ is a generic fractional polynomial of the same degre

3. If either fractional cosine $Acos_\alpha(at^\alpha)$ or fractional sine $Bsin_\alpha(at^\alpha)$ appears in $g(t^\alpha)$, the starting choice for $y_p(t)$ needs to contain both fractional cosine and fractional sine $y_p = A_1cos_\alpha(at^\alpha)+ A_2sin_\alpha(at^\alpha)$

4. If $g(t^\alpha)$ is a sum of several functions, $g(t^\alpha) = g_1(t^\alpha) + g_2(t^\alpha) + ... + g_n(t^\alpha)$,separate it into n parts and solve them individually.

5. If $g(t^\alpha)$ is a product of basic functions, the starting choice for $y_p(t)$ is chosen based on:

   (a) $y_p(t)$ is a product of the corresponding choices of all the parts of $g(t^\alpha)$

   (b) There are as many coefficients as the number of distinct terms in $y_p(t)$

   (c) Each distinct term must have its own coefficient, not shared with any other term.

6. Before finalizing the choice of $y_p(t)$, compare it against $y_c(t)$. If there is any shared term between the two, the present choice of $y_p(t)$ needs to be multiplied by t^α. Repeat until there is no shared term.

2.3 Systems of linear α-order fractional differential equations

2.3.1 Homogeneous linear systems

Ref [4] had solved systems of two linear first α-order fractional differential equations in two unknowns,but now we illustrate how to solve of n linear first α-order fractional differential equations in n unknowns of the form

$$
\begin{aligned}
{}^{J}D^{\alpha}\, x_1 &= f_1(t, x_1, x_2, ..., x_n) \\
{}^{J}D^{\alpha}\, x_2 &= f_2(t, x_1, x_2, ..., x_n) \\
&. \\
&. \\
{}^{J}D^{\alpha}\, x_n &= f_n(t, x_1, x_2, ..., x_n)
\end{aligned}
\tag{2.16}
$$

LINEAR SYSTEMS When each of the functions f_i , $i = 1..n$ is a linear function with constant coefficients,in the dependent variables $x_1, x_2, ..., x_n$, then the system of equations has the general form

$$
\begin{aligned}
{}^{J}D^{\alpha}\, x_1 &= a_{11}x_1(t) + a_{12}x_2(t) + ... + a_{1n}x_n(t) + g_1(t^{\alpha}) \\
{}^{J}D^{\alpha}\, x_2 &= a_{21}x_1(t) + a_{22}x_2(t) + ... + a_{2n}x_n(t) + g_2(t^{\alpha}) \\
&. \\
&. \\
{}^{J}D^{\alpha}\, x_n &= a_{n1}x_1(t) + a_{n2}x_2(t) + ... + a_{nn}x_n(t) + g_n(t^{\alpha})
\end{aligned}
\tag{2.17}
$$

When $g_i(t^{\alpha}) = 0$, $i = 1, 2, ..., n$, the linear system (2.17) said to be homogeneous; otherwise, it is nonhomogeneous.

MATRIX FORM OF A LINEAR SYSTEM

If X , A , G denote the respective matrices

$$
X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{pmatrix}, \qquad G = \begin{pmatrix} g_1(t^{\alpha}) \\ g_2(t^{\alpha}) \\ . \\ . \\ . \\ g_n(t^{\alpha}) \end{pmatrix}
$$

then the system of linear first-α-order fractional differential equations (2.17) can be written as

$$
{}^{J}D^{\alpha}\begin{pmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{pmatrix} + \begin{pmatrix} g_1(t^{\alpha}) \\ g_2(t^{\alpha}) \\ . \\ . \\ . \\ g_n(t^{\alpha}) \end{pmatrix}
$$

or simply

$$
{}^{J}D^{\alpha}X = AX + G
$$

If the system is homogeneous, its matrix form is

$$
{}^{J}D^{\alpha}X = AX \tag{2.18}
$$

then to solve the system of fractional differential equation (2.18) we use the method similar to as used in classical differential equations.

To construct a general solution to (2.18), assume a solution of the form $X = KE_{\alpha}(\lambda t^{\alpha})$, where the λ and the constant K vector are to be determined. Substituting $X = KE_{\alpha}(\lambda t^{\alpha})$ into ${}^{J}D^{\alpha}X = AX$, we obtain $\lambda KE_{\alpha}(\lambda t^{\alpha}) = AKE_{\alpha}(\lambda t^{\alpha})$.

After dividing out $E_\alpha(\lambda t^\alpha)$ and rearranging, we obtain

$$(A - \lambda I)K = 0 \tag{2.19}$$

Thus to solve the homogeneous system of fractional differential equations ${}^J D^\alpha X = AX$, we must find the eigenvalues and eigenvectors of A. Therefore $KE_\alpha(\lambda t^\alpha)$ is a solution of ${}^J D^\alpha X = AX$ provided that λ is an eigenvalue and K is an eigenvector of the coefficient matrix A. In the discussion that follows we examine three cases: real and distinct eigenvalues , repeated eigenvalues, and, finally, complex eigenvalues.

- Case I: **Distinct real eigenvalues**

  Let $\lambda_1, \lambda_2, ..., \lambda_n$ be n distinct real eigenvalues of the coefficient matrix A of the homogeneous system (2.18) and let $K_1, K_2, ..., K_n$ be the corresponding eigenvectors. Then the general solution of (2.18) is given by

  $$X = A_1 K_1 E_\alpha(\lambda_1 t^\alpha) + A_2 K_2 E_\alpha(\lambda_2 t^\alpha) + ... + A_n K_n E_\alpha(\lambda_n t^\alpha)$$

- Case II: **Repeated eigenvalues**

  we have following cases:

  1. For some $n \times n$ matrices A it may be possible to find m linearly independent $K_1, K_2, ..., K_m$ corresponding to an eigenvalue λ_1 of multiplicity $m \leq n$ eigenvectors.In this case the general

solution of the system is

$$X = [A_1\, K_1 + A_2\, K_2 + ... + A_m\, k_m]E_\alpha(\lambda_1 t^\alpha)$$

2. If there is only one eigenvector corresponding to the eigenvalue λ_1 of multiplicity m, then m linearly independent solutions of the form

$$\begin{aligned}
x_1 &= k_1 E_\alpha(\lambda_1 t^\alpha) \\
x_2 &= [k_1 \tfrac{t^\alpha}{\Gamma(\alpha+1)} + k_2]E_\alpha(\lambda_1 t^\alpha) \\
&\;. \\
&\;. \\
x_m &= [k_1 \tfrac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} + k_2 \tfrac{t^{(n-2)\alpha}}{\Gamma((n-2)\alpha+1)} + ... + k_m]E_\alpha(\lambda_1 t^\alpha)
\end{aligned} \tag{2.20}$$

where $k_i\; i = 1, 2, ..., m$ are column vectors, can always be found.we must have

$$(A - \lambda_1 I)k_1 = 0 \tag{2.21}$$

$$(A - \lambda_1 I)k_2 = k_1 \tag{2.22}$$

$$.$$

$$(A - \lambda_1 I)k_m = k_{m-1}$$

k_1 must be an eigenvector of A associated with λ_1. By solving (2.21), we find one solution $x_1 = k_1 E_\alpha(\lambda_1 t^\alpha)$. To find the second solution x_2, we need only solve the additional system (2.22) for the vector k_2 and so on. In this case the general solution of the system is

$$X = A_1\, x_1 + A_2\, x_2 + ... + A_m\, x_m$$

- Case III: **Complex eigenvalue**

  Let A be the coefficient matrix having real entries of the homogeneous system (2.18), and let k_1 be an eigenvector corresponding to the complex eigenvalue $\lambda_1 = p + iq$, p and q real, and let $B1 = Re(k_1)$ and $B_2 = Im(k_1))$. Then

$$\begin{aligned} x_1 &= [B_1 cos_\alpha(qt^\alpha) - B_2 sin_\alpha(qt^\alpha)]E_\alpha(pt^\alpha) \\ x_2 &= [B_2 cos_\alpha(qt^\alpha) + B_1 sin_\alpha(qt^\alpha)]E_\alpha(pt^\alpha) \end{aligned} \tag{2.23}$$

  the general solution of the system

$$X = A_1 x_1 + A_2 x_2$$

Example 2.10. *Distinct Eigenvalues*

Solve the initial-value problem

$$^{J}D^{\alpha}x = 2x + y$$

$$^{J}D^{\alpha}y = x + 2y \qquad 0 < \alpha \leq 1 \; with \; x(0) = 2, y(0) = 0$$

Solution

We first find the eigenvalues and eigenvectors of the matrix of coefficients.

From the characteristic equation

$$det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$$

we see that the eigenvalues are $\lambda_1 = 1$ *and* $\lambda_2 = 3$

Now for $\lambda_1 = 1$ *, (2.19)is equivalent to*

$$k_1 + k_2 = 0$$

$$k_1 + k_2 = 0$$

Thus $k_1 = -k_2$. *When* $k_1 = 1$, *the related eigenvector is*

$$k_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 3$ *we have*

$$-k_1 + k_2 = 0$$
$$k_1 - k_2 = 0$$

so $k_1 = k_2$ *;therefore with* $k_1 = 1$ *the corresponding eigenvector is*

$$k_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The general solution of the system is

$$X = A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} E_\alpha(t^\alpha) + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} E_\alpha(3t^\alpha)$$

Putting the initial condition.$x(0) = 2, y(0) = 0$

$$A_1 + A_2 = 2$$
$$-A_1 + A_2 = 0$$

and solving we get $A_1 = A_2 = 1$. *Hence the solution is,*

$$x(t) = E_\alpha(3t^\alpha) + E_\alpha(t^\alpha)$$
$$y(t) = E_\alpha(3t^\alpha) - E_\alpha(t^\alpha)$$

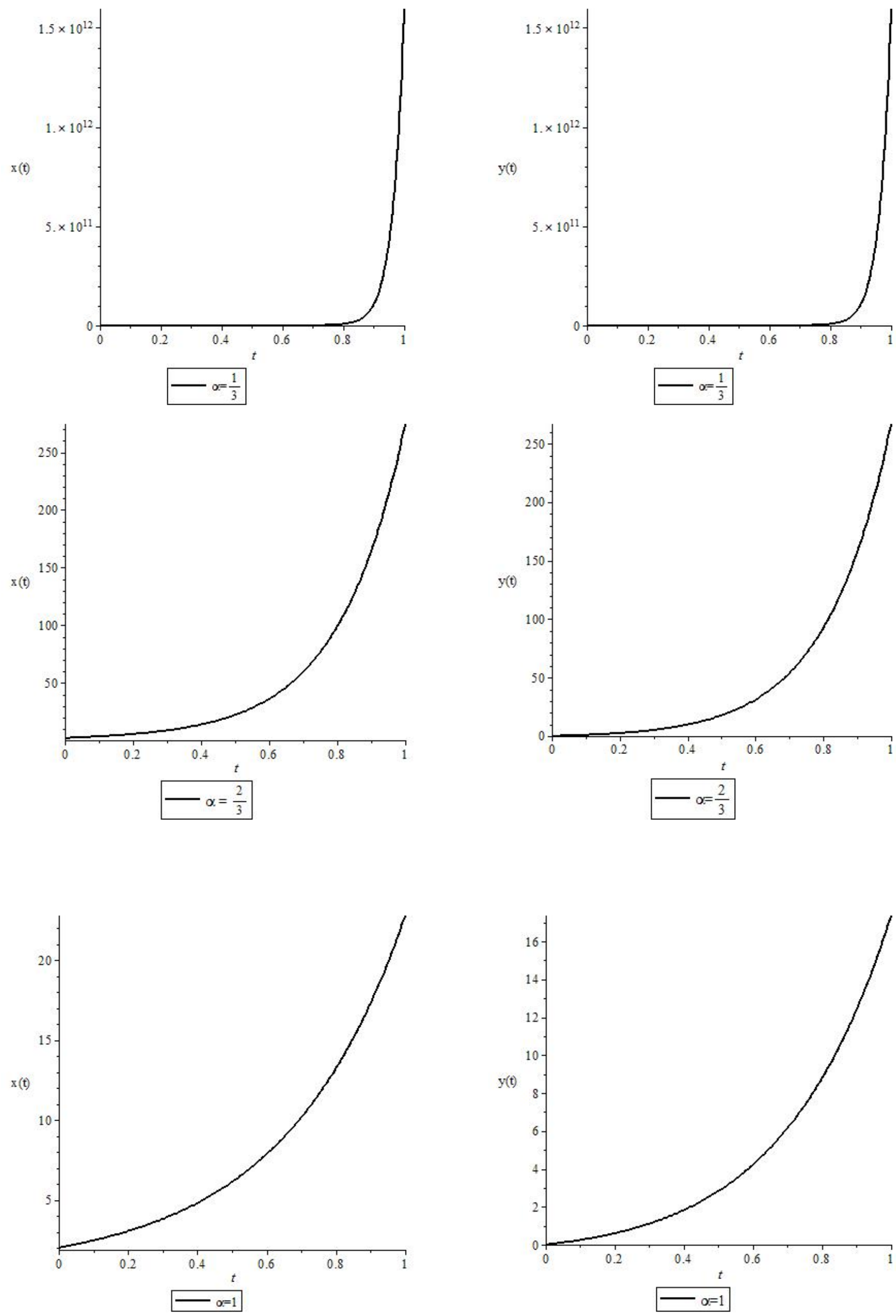


Figure 2.2: solutions of fractional differential equation in Example 2.10 for x(t) and y(t) for different values of $\alpha = \frac{1}{3}, \frac{2}{3}, 1$

Example 2.11. *Distinct Eigenvalues*

Solve

$$
\begin{aligned}
{}^{J}D^{\alpha}x &= x + 4z \\
{}^{J}D^{\alpha}y &= 2y \\
{}^{J}D^{\alpha}z &= 3x + y - 3z \qquad\qquad 0 < \alpha \leq 1
\end{aligned}
$$

Solution

Using the cofactors of the third row, we find

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ 3 & 1 & -3-\lambda \end{vmatrix} = (\lambda - 2)(\lambda - 3)(\lambda + 5) = 0$$

For $\lambda_1 = 2$ *we have the corresponding eigenvector is*

$$k_1 = \begin{pmatrix} 1 \\ \frac{-7}{4} \\ \frac{1}{4} \end{pmatrix}$$

For $\lambda_2 = 3$ *we have the corresponding eigenvector is*

$$k_2 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

And for $\lambda_3 = -5$ *we have the corresponding eigenvector is*

$$k_3 = \begin{pmatrix} 1 \\ 0 \\ \frac{-3}{2} \end{pmatrix}$$

The general solution is

$$Y = A_1 \begin{pmatrix} 1 \\ \frac{-7}{4} \\ \frac{1}{4} \end{pmatrix} E_\alpha(2t^\alpha) + A_2 \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} E_\alpha(3t^\alpha)$$

$$+ A_3 \begin{pmatrix} 1 \\ 0 \\ \frac{-3}{2} \end{pmatrix} E_\alpha(-5t^\alpha)$$

Example 2.12. *Repeated Eigenvalues*

Find solution of the system

$${}^J D^\alpha x = 4x - y$$

$${}^J D^\alpha y = x + 2y \qquad 0 < \alpha \leq 1 \; with \; x(0) = 2, y(0) = 1$$

Solution

We first find the eigenvalues and eigenvectors of the matrix of coefficients.

$$det(A-\lambda I)=\begin{vmatrix}4-\lambda & -1\\ 1 & 2-\lambda\end{vmatrix}=(\lambda-3)(\lambda-3)=0$$

For $\lambda_1=\lambda_2=3$ *we have the corresponding eigenvector is* $K_1=\begin{pmatrix}1\\1\end{pmatrix}$ *Thus from (2.20) we find*

$$x_1=\begin{pmatrix}1\\1\end{pmatrix}E_\alpha(3t^\alpha)$$

We find from (2.22) that we must

$$(A-3I)k_2=K_1$$

$$\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

The row of the last matrix means $a_1-a_2=1$ *or* $a_1=a_2+1$ *by choosing* $a_1=1,$ *we find* $a_2=0.$ *Hence* $k_2=\begin{pmatrix}1\\0\end{pmatrix}$ *Thus from (2.20) we find*

$$x_2=\begin{pmatrix}1\\1\end{pmatrix}\frac{t^\alpha}{\Gamma(\alpha+1)}E_\alpha(3t^\alpha)+\begin{pmatrix}1\\0\end{pmatrix}E_\alpha(3t^\alpha).$$

The general solution is

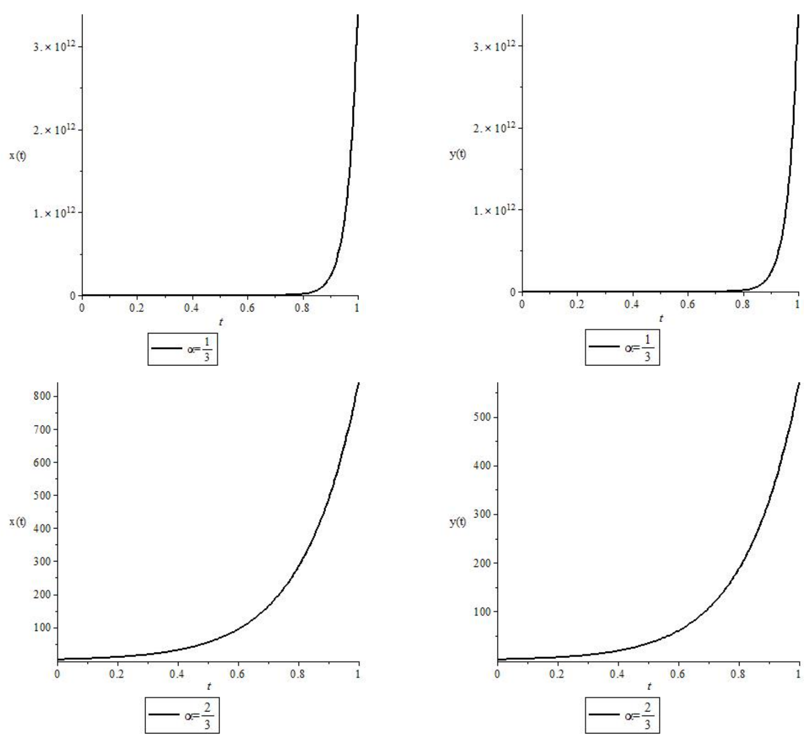
$$\begin{pmatrix}x\\y\end{pmatrix}=A_1\begin{pmatrix}1\\1\end{pmatrix}E_\alpha(3t^\alpha)+A_2\left[\begin{pmatrix}1\\1\end{pmatrix}\frac{t^\alpha}{\Gamma(\alpha+1)}+\begin{pmatrix}1\\0\end{pmatrix}\right]E_\alpha(3t^\alpha)$$

Putting the initial condition.$x(0) = 2, y(0) = 1$

$$A_1 + A_2 = 2$$
$$A_1 = 1$$

and solving we get $A_1 = A_2 = 1$. *Hence the solution is,*

$$x(t) = \left[\frac{t^\alpha}{\Gamma(1+\alpha)} + 2\right] E_\alpha(3t^\alpha)$$
$$y(t) = \left[\frac{t^\alpha}{\Gamma(1+\alpha)} + 1\right] E_\alpha(3t^\alpha)$$



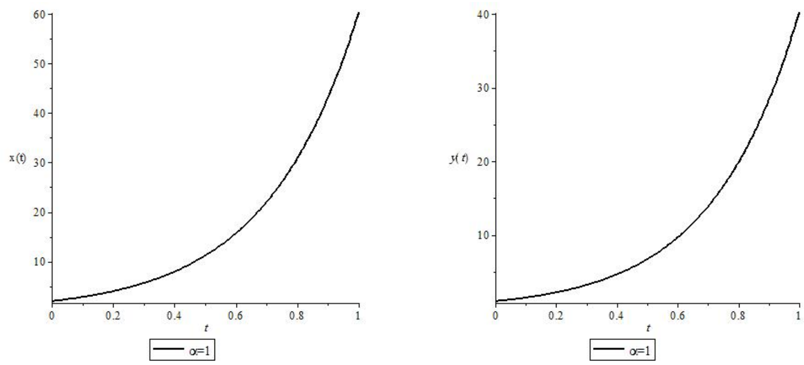


Figure 2.3: solutions of fractional differential equation in Example 2.12 for x(t) and y(t) for different values of $\boldsymbol{\alpha = \frac{1}{3}, \frac{2}{3}, 1}$

Example 2.13. *Repeated Eigenvalues*

Find the general solution of the system

$$
\begin{aligned}
{}^{J}D^{\alpha}x &= x - 2y + 2z \\
{}^{J}D^{\alpha}y &= -2x + y - 2z \\
{}^{J}D^{\alpha}z &= 2x + -2y + z \qquad\qquad 0 < \alpha \leq 1
\end{aligned}
$$

Solution

We first find the eigenvalues and eigenvectors of the matrix of coefficients.

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = (\lambda+1)^2(\lambda-5) = 0$$

We see that $\lambda_1 = \lambda_2 = -1$ *and* $\lambda_3 = 5$

$$2k_1 - 2k_2 + 2k_3 = 0$$

$$-2k_1 + 2k_2 - 2k_3 = 0$$

$$2k_1 - 2k_2 + 2k_3 = 0$$

Thus $k_1 - k_2 + k_3 = 0$ *or* $k_1 = k_2 - k_3$*.The choices* $k_2 = 1$, $k_3 = 0$ *and* $k_2 = 1$, $k_3 = 0$ *yield, in turn,*$k_1 = 1$ *and* $k_1 = 0$*.Thus two eigenvectors corresponding to* $\lambda_1 = -1$ *are*

$$K_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad and \qquad K_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

we have found two linearly independent solutions,

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} E_\alpha(-t^\alpha) \qquad and \qquad x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} E_\alpha(-t^\alpha)$$

corresponding to the same eigenvalue. Last, for$\lambda_3 = 5$

$$-4k_1 - 2k_2 + 2k_3 = 0$$

$$-2k_1 - 4k_2 - 2k_3 = 0$$

$$2k_1 - 2k_2 - 4k_3 = 0$$

Thus $k_2 = -k_3$ *and* $k_1 = k_3$. *Picking* $k_3 = 1$ *gives* $k_1 = 1,\ k_2 = -1$; *thus a third eigenvector is*

$$K_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

We conclude that the general solution of the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} E_\alpha(-t^\alpha) + A_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} E_\alpha(-t^\alpha) + A_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} E_\alpha(5t^\alpha)$$

Example 2.14. *Complex Eigenvalues*

Solve the initial-value problem

$$^{J}D^\alpha x = 3x + 2y$$

$$^{J}D^\alpha y = -5x + y \qquad 0 < \alpha \leq 1 \text{ with } x(0) = 2, y(0) = 1$$

Solution

First we obtain the eigenvalues from.

$$det(A-\lambda I)=\begin{vmatrix}3-\lambda & 2\\ -5 & 1-\lambda\end{vmatrix}=\lambda^2-4\lambda+13=0$$

The eigenvalues are $\lambda_1=2+3i$ *and* $\lambda_2=\overline{\lambda_1}=2-3i$. *For* λ_1 *the system*

$$(1-3i)k_1+2k_2=0$$

$$-5k_1-(1+3i)k_2=0$$

gives $k_1=\frac{-1-3i}{5}k_2$.*By choosing* $k_2=1$, *we get*

$$K_1=\begin{pmatrix}\frac{-1-3i}{5}\\ 1\end{pmatrix}$$

Now from (2.23) we form

$$B_1=Re(K_1)=\begin{pmatrix}\frac{-1}{5}\\ 1\end{pmatrix}\qquad and \qquad B_2=Im(K_1)=\begin{pmatrix}\frac{-3}{5}\\ 0\end{pmatrix}$$

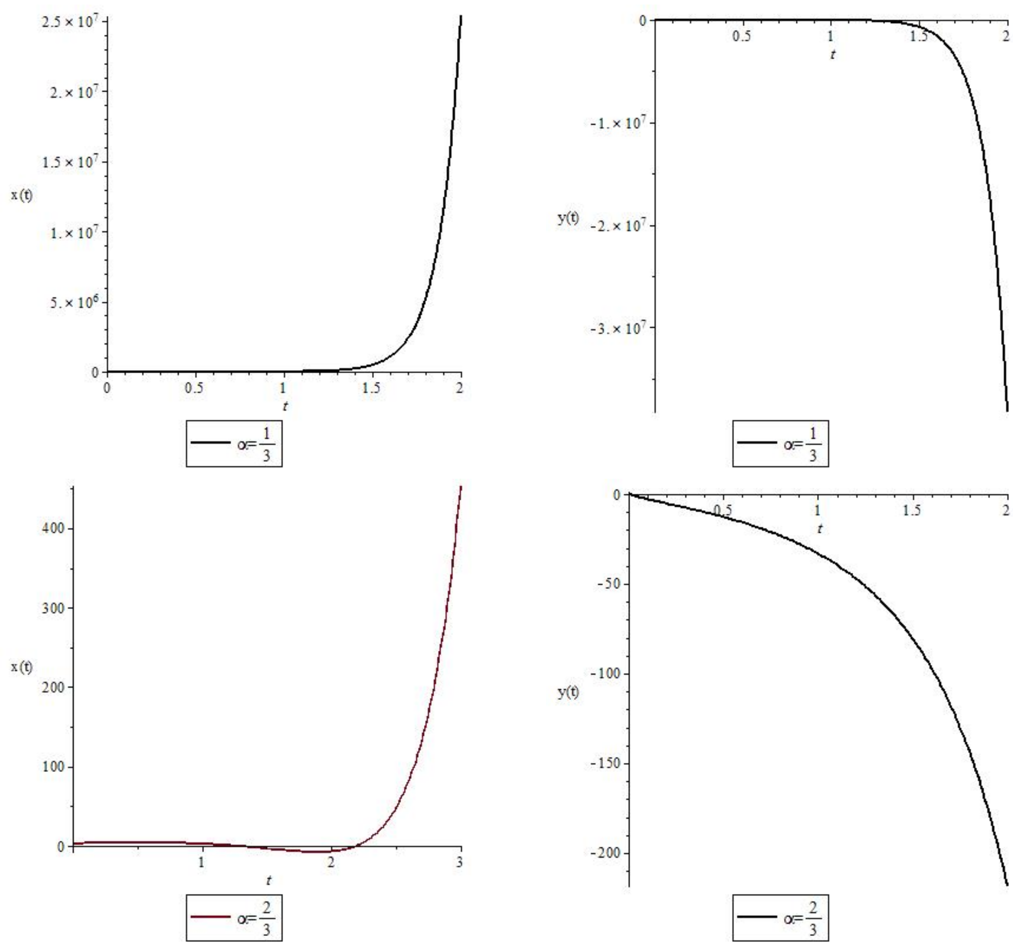
the general solution of the system is

$$\begin{pmatrix}x\\ y\end{pmatrix}=A_1\left[\begin{pmatrix}\frac{-1}{5}\\ 1\end{pmatrix}cos_\alpha(3t^\alpha)-\begin{pmatrix}\frac{-3}{5}\\ 0\end{pmatrix}sin_\alpha(3t^\alpha)\right]E_\alpha(2t^\alpha)$$

$$+A_2\left[\begin{pmatrix}\frac{-3}{5}\\ 0\end{pmatrix}cos_\alpha(3t^\alpha)+\begin{pmatrix}\frac{-1}{5}\\ 1\end{pmatrix}sin_\alpha(3t^\alpha)\right]E_\alpha(2t^\alpha)$$

Now the initial condition , $x(0) = 2, y(0) = 1$ *yields the algebraic system* $\frac{-1}{5}A_1 + \frac{-3}{5}A_2 = 2$ *and* $A_1 = 1$, *whose solution is* $A_1 = 1$ *and* $A_2 = \frac{-11}{3}$.
Thus the solution to the problem is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} cos_\alpha(3t^\alpha) + \begin{pmatrix} \frac{4}{3} \\ \frac{-11}{3} \end{pmatrix} sin_\alpha(3t^\alpha) \right] E_\alpha(2t^\alpha)$$



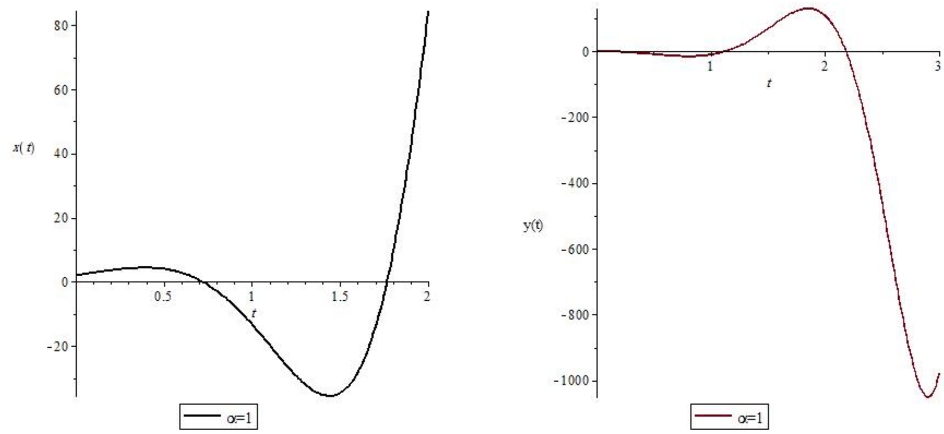


Figure 2.4: solutions of fractional differential equation in Example 2.14 for x(t) and y(t) for different values of $\alpha = \frac{1}{3}, \frac{2}{3}, 1$

2.3.2 Non-Homogeneous linear systems

The general solution of a non-homogeneous linear system $^{J}D^{\alpha}X = AX + G$ is $X = X_c + X_p$, X_c is the complementary function or general solution of the associated homogeneous linear system $^{J}D^{\alpha}X = AX$ and X_p is any particular solution of the non-homogeneous system. In Section 2.3.1 we show how to obtain X_c when the coefficient matrix A was an $n \times n$ matrix.In this section we used methods of Undetermined coefficients for obtainingX_p.

Method of Undetermined Coefficients

As in Section 2.2, the method of undetermined coefficients consists of making an educated guess about the form of a particular solution vector X_p, the guess is motivated by the types of functions that make up the entries of the column matrix G. Not surprisingly, the matrix version of undetermined coefficients is applicable to ${}^{J}D^{\alpha}X = AX+G$ only when the entries G are fractional polynomials, Mittag-Leffler function, fractional sines and cosines, or finite sums and products of these functions.

Example 2.15. *Solve the system*

$$ {}^{J}D^{\alpha}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 6t^{\alpha} \\ -10t^{\alpha}+4 \end{pmatrix} $$

Solution

The eigenvalues and corresponding eigenvectors of the associated homogeneous system ${}^{J}D^{\alpha}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *are found to be* $\lambda_1 = 2, \lambda_2 = 7$, $K_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}, K_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. *Hence the complementary function is*

$$ X_c = A_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} E_{\alpha}(2t^{\alpha}) + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} E_{\alpha}(7t^{\alpha}) $$

Now because G *can be written* $G = \begin{pmatrix} 6 \\ -10 \end{pmatrix} t^{\alpha} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$, *we shall try to find a particular solution of the system that possesses the same form:*

$$ X_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t^{\alpha} + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} $$

Substituting this last assumption into the given system yields

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \Gamma(1+\alpha) = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} \left[\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t^\alpha + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \right] + \begin{pmatrix} 6 \\ -10 \end{pmatrix} t^\alpha + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

From the last identity we obtain four algebraic equations in four unknowns

$$\begin{aligned} 6a_2 + b_2 + 6 &= 0 & \qquad 6a_1 + b_1 - a_2\Gamma(\alpha+1) &= 0 \\ 4a_2 + 3b_2 - 10 &= 0 & \qquad 4a_1 + 3b_1 - b_2\Gamma(\alpha+1) + 4 &= 0 \end{aligned}$$

Solving the first two equations simultaneously yields $a_2 = -2, b_2 = 6$. *We then substitute these values into the last two equations and solve for* a_1 *and* b_1. *The results are* $a_1 = \frac{-6}{7}\Gamma(\alpha+1) + \frac{2}{7}$, $b_1 = \frac{22}{7}\Gamma(\alpha+1) - \frac{12}{7}$. *It follows, therefore, that a particular solution vector is*

$$X_p = \begin{pmatrix} -2 \\ 6 \end{pmatrix} t^\alpha + \begin{pmatrix} \frac{-6}{7}\Gamma(\alpha+1) + \frac{2}{7} \\ \frac{22}{7}\Gamma(\alpha+1) - \frac{12}{7} \end{pmatrix}$$

The general solution of the system

$$X = A_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} E_\alpha(2t^\alpha) + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} E_\alpha(7t^\alpha) + \begin{pmatrix} -2 \\ 6 \end{pmatrix} t^\alpha + \begin{pmatrix} \frac{-6}{7}\Gamma(\alpha+1) + \frac{2}{7} \\ \frac{22}{7}\Gamma(\alpha+1) - \frac{12}{7} \end{pmatrix}$$

Example 2.16. *Determine the form of a particular solution vector* X_p

for the system

$$^{J}D^{\alpha}x = 5x + 3y - 2E_{\alpha}(t^{\alpha}) + 1$$

$$^{J}D^{\alpha}y = -x + y + E_{\alpha}(t^{\alpha}) - 5t^{\alpha} + 7$$

Solution

The eigenvalues and corresponding eigenvectors of the associated homogeneous system $^{J}D^{\alpha}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *are found to be* $\lambda_1 = 4$,$\lambda_2 = 2$, $K_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$,$K_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. *Hence the complementary function is*

$$X_c = A_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} E_{\alpha}(4t^{\alpha}) + A_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} E_{\alpha}(2t^{\alpha})$$

Because G can be written in matrix terms as

$$G = \begin{pmatrix} -2 \\ 1 \end{pmatrix} E_{\alpha}(t^{\alpha}) + \begin{pmatrix} 0 \\ -5 \end{pmatrix} t^{\alpha} + \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

an assumption for a particular solution would be

$$X_p = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} E_{\alpha}(t^{\alpha}) + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t^{\alpha} + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

The method of undetermined coefficients for linear systems is not as straight forward as the last examples would seem to indicate. In Example (2.16) if we replace $E_\alpha(t^\alpha)$ in G by $E_\alpha(2t^\alpha)$ (λ is an eigenvalue of A),then the correct form of the particular solution vector is

$$X_p = \begin{pmatrix} a_4 \\ b_4 \end{pmatrix} t^\alpha E_\alpha(2t^\alpha) + \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} E_\alpha(2t^\alpha) + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t^\alpha + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

Chapter 3

Fractional Shifted Legendre Polynomials

Legendre polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formula[19]

$$L_{j+1}\,(z) = \frac{2j+1}{j+1}\, z\, L_j(z) - \frac{j}{j+1}\, L_{j-1}(z) \qquad j = 1, 2, ...$$

where $L_0(z) = 1$ and $L_1(z) = z$. In order to use these polynomials on the interval $t \in [0, L]$ we define the so called shifted Legendre polynomials by introducing the change of variable $z = \frac{2t}{L} - 1$. Let the shifted Legendre polynomials $L_J(\frac{2t}{L} - 1)$ be denoted by $P_j(t)$. Then $P_j(t)$ can be obtained as follows:

$$P_{j+1}(t) = \frac{(2j+1)(\frac{2t}{L} - 1)}{j+1} P_j(t) - \frac{j}{j+1} P_{j-1}(t)$$

where $P_0(t) = 1$ and $P_1(t) = \frac{2t}{L} - 1$. The analytic form of the shifted Legendre polynomials $P_j(t)$ of degree j given by:

$$P_j(t) = \sum_{i=0}^{j} (-1)^{j+i} \frac{(j+i)!t^i}{L^i(j-i)!(i!)^2}$$

The first few shifted Legendre polynomials are:

$$P_0 = 1$$

$$P_1 = 2\,\frac{t}{L} - 1$$

$$P_2 = 6\left(\frac{t}{L}\right)^2 - 6\frac{t}{L} + 1$$
$$p_3 = 20\left(\frac{t}{L}\right)^3 - 30\left(\frac{t}{L}\right)^2 + 12\left(\frac{t}{L}\right) - 1$$

Note that $P_j(0) = (-1)^k$ and $P_j(L) = 1$. The orthogonality condition is [2]:

$$\int_0^L P_i(t)P_j(t)dt = \begin{cases} \frac{L}{2i+1} & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

The function x(t), which is a square integrable in [0, L], may be expressed in terms of shifted Legendre polynomials as:

$$X(t) = \sum_{i=0}^{\infty} C_i P_i(t)$$

where the coefficients C_i are given by:

$$C_i = (2i+1)\int_0^1 X(t)P_i(t)dti = 1, 2, ...$$

In practice, only the first (m+1)-terms shifted Legendre polynomials are considered. Then we have:

$$X_m(t) = \sum_{i=0}^{m} C_i P_i(t) \tag{3.1}$$

Now,we will approximate the Jumarie modified fractional derivative by shifted Legendre polynomial.

$$X_m(t) = \sum_{i=0}^{m} C_i P_i(t) \tag{3.2}$$

$$\begin{aligned} {}^{J}D^{\alpha}(X_m(t)) &= \sum_{i=0}^{m} C_i \, {}^{J}D^{\alpha} P_i(t) \\ &= \sum_{i=0}^{m} C_i \, \sum_{k=0}^{i} (-1)^{k+i} \, \frac{(k+i)!}{L^k (i-k)!(k!)^2} \, {}^{J}D^{\alpha} \, t^k \end{aligned}$$

$$\begin{aligned} &= \sum_{i=0}^{m} \sum_{k=0}^{i} C_i (-1)^{k+i} \, \frac{(k+i)!}{L^k (i-k)!(k!)^2} \, \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} t^{k-\alpha} \\ &= \sum_{i=0}^{m} \sum_{k=0}^{i} C_i (-1)^{k+i} \, \frac{(k+i)!}{L^k (i-k)!(k!)\Gamma(k+1-\alpha)} \, t^{k-\alpha} \end{aligned}$$

Thus

$${}^{J}D^{\alpha}(X_m(t)) = \sum_{i=0}^{m} \sum_{k=0}^{i} C_i \, w_{i,k}^{\alpha} \, t^{k-\alpha} \tag{3.3}$$

Where $w_{i,k}^{\alpha}$ is given by :

$$w_{i,k}^{\alpha} = (-1)^{k+i} \, \frac{(k+i)!}{L^k (i-k)!(k!)\Gamma(k+1-\alpha)} \, t^{k-\alpha}$$

Example 3.1. *Consider the case when* $x(t) = t^3$, $m = 3$ *and* $\alpha = 0.5$, *the series of* t^3 *is:*

$$t^3 = \sum_{i=0}^{3} C_i P_i(t) = \frac{1}{4}P_0(t) + \frac{9}{20}P_1(t) + \frac{1}{4}P_2(t) + \frac{1}{20}P_3(t)$$

$${}^{J}D^{\alpha}t^3 = \sum_{i=0}^{3}\sum_{k=0}^{i} C_i\, w_{i,k}^{\frac{1}{2}}\, t^{k-\frac{1}{2}}$$

$$w_{0,0}^{\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})},\quad w_{1,0}^{\frac{1}{2}} = \frac{-1}{\Gamma(\frac{1}{2})},\quad w_{1,1}^{\frac{1}{2}} = \frac{2}{\Gamma(\frac{3}{2})},\quad w_{2,0}^{\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})},\quad w_{2,1}^{\frac{1}{2}} = \frac{-6}{\Gamma(\frac{3}{2})}$$

$$w_{2,2}^{\frac{1}{2}} = \frac{12}{\Gamma(\frac{5}{2})},\quad w_{3,0}^{\frac{1}{2}} = \frac{-1}{\Gamma(\frac{1}{2})},\quad w_{3,1}^{\frac{1}{2}} = \frac{12}{\Gamma(\frac{3}{2})},\quad w_{3,2}^{\frac{1}{2}} = \frac{-60}{\Gamma(\frac{5}{2})},\quad w_{3,3}^{\frac{1}{2}} = \frac{120}{\Gamma(\frac{7}{2})}$$

Therefore :

$${}^{J}D^{\alpha}t^3 = \frac{6}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}$$

${}^{J}D^{\alpha}t^3$ *approximated by shifted Legendre polynomial equal the analytic fractional derivative* ${}^{J}D^{\alpha}t^3 = \frac{\Gamma(4)}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} = \frac{6}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}$ *by(1.3)*

3.1 Numerical solution of Nonlinear α-Order Fractional Differential Equation Using Shifted Legendre polynomial

In this section, we will use the shifted Legendre polynomials to approximate solution of nonlinear fractional α-order differential equation

Example 3.2. *Consider the following fractional differential equation:*

$${}^{J}D^{\alpha}u(t) = -u^2(t) + 1 \qquad u(0) = 0, \quad 0 < \alpha \leq 1 \tag{3.4}$$

The exact solution, when $\alpha = 1$, *is*

$$u(t) = \frac{e^{2t} - 1}{e^{2t} + 1}$$

In order to use the shifted Legendre polynomials method, we first approximate $u(t)$ *as*

$$u(t) = \sum_{i=0}^{m} C_i P_i(t_p) \tag{3.5}$$

From Eqs.(3.4) and (3.3), we have

$$\sum_{i=0}^{m} \sum_{k=0}^{i} C_i \, w_{i,k}^{\alpha} \, t^{k-\alpha} = -\left(\sum_{t=0}^{m} C_i P_i(t) \right)^2 + 1 \tag{3.6}$$

We now collocate Eqs.(3.6) at m *points as:*

$$\sum_{i=0}^{m} \sum_{k=0}^{i} C_i \, w_{i,k}^{\alpha} \, t_p^{k-\alpha} = -\left(\sum_{i=0}^{m} C_i P_i(t_p) \right)^2 + 1 \tag{3.7}$$

For suitable collocation points we use the roots of shifted Legendre polynomial $P_m(t)$. *Also, by substituting Eq.(3.5) in the initial conditions* $u(0) = 0$, *we can find*

$$\sum_{i=0}^{m} (-1)^i c_i = 0 \tag{3.8}$$

Equations (3.7) and (3.8), give (m + 1) of non-linear algebraic equations which can be solved using the Newton iteration method, for the unknowns $a_i \, , i = 0, 1, ..., m$ *This is a nonlinear system of algebraic equations.*
The numerical solution, for m = 8, is shown in Figure (3.1).

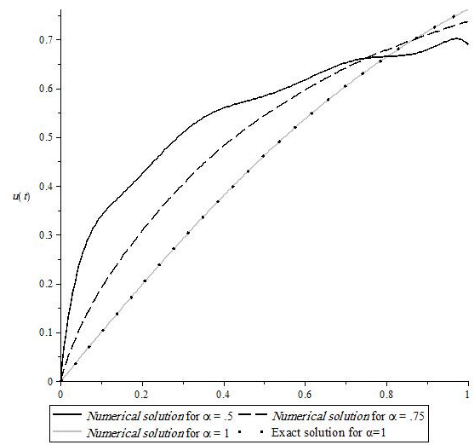


Figure 3.1: The behavior of the exact and approximate solution of example with m = 8

t	$\alpha = .5$	$\alpha = .75$	$\alpha = 1$	*Exact*	$\lvert u_A - u_N \rvert_{error}$
.1	0.3381606335	0.1899465355	0.09966790365	0.09966799462	$9.097\ 10^{-8}$
.2	0.4241413039	0.3071555379	0.1973754023	0.1973753202	$8.21\ 10^{-8}$
.3	0.5088055209	0.4033784164	0.2913126467	0.2913126125	$3.42\ 10^{-8}$
.4	0.5591782979	0.481450290	0.3799488411	0.3799489623	$1.212\ 10^{-7}$
.5	0.5840484299	0.543860843	0.4621171644	0.4621171573	$7.1\ 10^{-9}$
.6	0.6169471519	0.596465471	0.5370496876	0.5370495670	$1.206\ 10^{-7}$
.7	0.6521833639	0.641698329	0.6043677308	0.6043677771	$4.63\ 10^{-8}$
.8	0.6646549159	0.678526912	0.6640366921	0.6640367703	$7.82\ 10^{-8}$
.9	0.6791947179	0.709265802	0.7162979656	0.7162978702	$9.54\ 10^{-8}$
1	0.6902414599	0.736244942	0.7615941560	0.7615941560	0

We can see the numerical solution is in very good agreement with the exact solution when $\alpha = 1$(max error$= 1.212\ 10^{-7}$). Therefore, we hold that the solution for $\alpha = .5$ and $\alpha = .75$ is also credible.

Example 3.3. *Consider the following fractional differential equation:*

$$^{J}D^{\alpha}u(t) = 2u(t) - u^2(t) + 1 \qquad u(0) = 0, \quad 0 < \alpha \leq 1 \tag{3.9}$$

The exact solution, when $\alpha = 1$, is

$$u(t) = \frac{-1}{2}\left(-\sqrt{2} + 2\,tanh(\frac{1}{2}\,(\sqrt{2}\,arctanh(\frac{1}{2}\,\sqrt{2}) - 2t)\,\sqrt{2})\right)\,\sqrt{2}$$

The numerical solution, for m = 8, is shown in Figure (3.2).

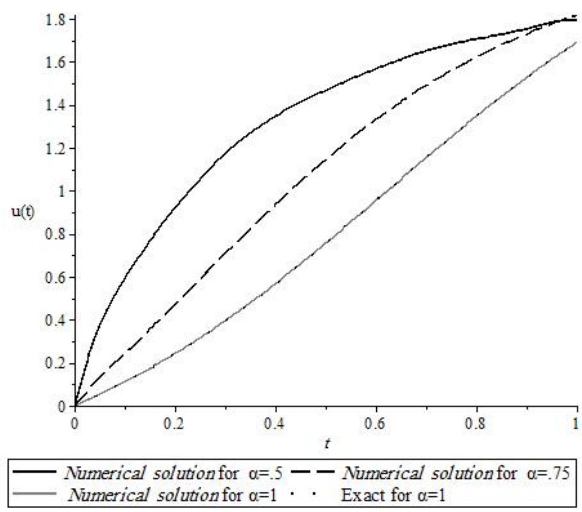


Figure 3.2: The behavior of the exact and approximate solution of example with m = 8

We can see the numerical solution is in very good agreement with the exact solution when $\alpha = 1$(max error= $1.248\ 10^{-6}$). Therefore, we hold that the solution for $\alpha = .5$ and $\alpha = .75$ is also credible.

All numerical results are obtained by using Maple 2017

t	$\alpha = .5$	$\alpha = .75$	$\alpha = 1$	$Exact$	$\|u_A - u_N\|_{error}$
.1	0.5930405854	0.2435496386	0.1102958617	0.1102951980	$6.637\ 10^{-7}$
.2	0.919158542	0.4706262447	0.2419759117	0.2419768004	$8.887\ 10^{-7}$
.3	1.172138698	0.7062798847	0.3951044304	0.3951048494	$4.190\ 10^{-7}$
.4	1.347573710	0.935667539	0.5678141860	0.5678121670	$2.0190\ 10^{-6}$
.5	1.467139243	1.145767479	0.7560142262	0.7560143945	$1.683\ 10^{-7}$
.6	1.566199492	1.331260760	.9535634551	0.9535662170	$0.27619\ 10^{-5}$
.7	1.649424880	1.489969269	1.152950216	1.152948968	$1.248\ 10^{-6}$
.8	1.704887949	1.621252974	1.346365887	1.346363656	$2.231\ 10^{-6}$
.9	1.752887187	1.728783296	1.526908275	1.526911314	$3.039\ 10^{-6}$
1	1.791528286	1.817121395	1.689498390	1.689498392	$2\ 10^{-9}$

3.2 Optimization problem using fractional Shifted Legendre Polynomials

In this section, we apply fractional shifted Legendre polynomial method to approximate the optimal policy for a non-linear optimization problem and its corresponding system of the fractional differential equations. Consider the non-linear programming problem with equality constraints defined by

$$\begin{gathered} min f(x) \\ s.t \quad g_i(x) = 0\,,\, i = 1..n \end{gathered} \tag{3.10}$$

To obtain solution of 3.10, One of the most effective methods for solving is the quadratic penalty function method.Which turns a constrained optimization problem to an unconstrained [1]. The quadratic penalty function for this problem is given by [19]

$$minF(x,\beta) = f(x) + \frac{\beta}{2}\sum_{i=1}^{n}(g_i(x))^2 \qquad (3.11)$$

Consider the unconstrained optimization problem 3.11, an approach based on fractional dynamic system can be described by the following FDEs

$$^{J}D^{\alpha}x(t) = -\nabla_x F(x,\beta) \qquad 0 < \alpha \leq 1 \qquad (3.12)$$

With initial conditions $x(t_0) = c_i$, $i = 1, 2, ..., n$

Algorithm to choose β (an auxiliary penalty variable)

1. Given $\beta_0 > 0$, and a tolerance (tol)

2. For k=0,1,2,...

3. If $|\nabla_\beta F(x,\beta)| \leq tol$ stop

4. Else, $\beta_{k+1} > \beta_k$ and find a new x_{k+1}

Example 3.4. *optimization problem*

Consider the following non-linear programming problem

$$minimize\ f(x) = 2u + 5v + 3u^2 + 3uv + 2v^2 \qquad (3.13)$$

$$subject\ to\ g(x) = u - v - 2 \qquad (3.14)$$

The optimal solution is $(u = 0.4375, v = -1.5625)$.

For solving the above problem, we convert it to an unconstrained optimization problem with quadratic penalty function (3.11), then we have

$$F(x,\beta) = 2u + 5v + 3u^2 + 3uv + 2v^2 + \frac{\beta}{2}(u - v - 2)^2$$

The system of FDEs from (3.12) is defined as

$$\begin{aligned} {}^{J}D^{\alpha}\, u &= -6u - 3v - 2 - \beta(u - v - 2) \\ {}^{J}D^{\alpha}\, v &= -3u - 4v - 5 + \beta(u - v - 2) \end{aligned} \tag{3.15}$$

$u(t)$ *and* $v(t)$ *be approximated by shifted Legendre polynomial as (3.2)*

$$u(t) = \sum_{i=0}^{m} a_i P_i(t) \qquad v(t) = \sum_{i=0}^{m} b_i P_i(t) \tag{3.16}$$

From Eqs.(3.15) and (3.3), we have

$$\sum_{i=0}^{m}\sum_{k=0}^{i} a_i\, w_{i,k}^{\alpha}\, t^{k-\alpha} = -6\sum_{t=0}^{m} a_i P_i(t) - 3\sum_{t=0}^{m} b_i P_i(t) - 2$$

$$-\,\beta\Big(\sum_{t=0}^{m} a_i P_i(t) - \sum_{t=0}^{m} b_i P_i(t) - 2\Big) \tag{3.17}$$

$$\sum_{i=0}^{m}\sum_{k=0}^{i} b_i\, w_{i,k}^{\alpha}\, t^{k-\alpha} = -3\sum_{t=0}^{m} a_i P_i(t) - 4\sum_{t=0}^{m} b_i P_i(t) - 5$$

$$+\,\beta\Big(\sum_{t=0}^{m} a_i P_i(t) - \sum_{t=0}^{m} b_i P_i(t) - 2\Big) \tag{3.18}$$

We now collocate Eqs.(3.17)and (3.18) at m *points* $t_p\,(p = 0, 1, ..., m)$ *as:*

$$\sum_{i=0}^{m}\sum_{k=0}^{i} a_i\, w_{i,k}^{\alpha}\, t_p^{k-\alpha} = -6\sum_{t=0}^{m} a_i P_i(t_p) - 3\sum_{t=0}^{m} b_i P_i(t_p) - 2$$

$$-\beta(\sum_{t=0}^{m} a_i P_i(t_p) - \sum_{t=0}^{m} b_i P_i(t_p) - 2) \qquad (3.19)$$

$$\sum_{i=0}^{m}\sum_{k=0}^{i} b_i\, w_{i,k}^{\alpha}\, t_p^{k-\alpha} = -3\sum_{t=0}^{m} a_i P_i(t_p) - 4\sum_{t=0}^{m} b_i P_i(t_p) - 5$$

$$+\beta(\sum_{t=0}^{m} a_i P_i(t_p) - \sum_{t=0}^{m} b_i P_i(t_p) - 2) \qquad (3.20)$$

For suitable collocation points we use the roots of shifted Legendre polynomial $P_m(t)$. *Also, by substituting Eq.(3.5) in the initial conditions* $u(0) = 0\; v(0) = 0$, *we can find*

$$\sum_{i=0}^{m}(-1)^i a_i = 0 \qquad \sum_{i=0}^{m}(-1)^i b_i = 0 \qquad (3.21)$$

Equations (3.19) ,(3.20)and (3.21), give (2m + 2) of non-linear algebraic equations which can be solved using the Newton iteration method, for the unknowns a_i *and* b_i $\;,i = 0, 1, ..., m$.

Now we use the algorithm to choose β*,and we use the shifted Legendre polynomial with m = 12 at* $\alpha = 1$ *,* $t = 1$

β	u	v	$\|\nabla_\beta F(x,\beta)\|$
2	0.4445140271	-1.568818815	$8.888193790\ 10^{-5}$
4	0.4432596091	-1.569007361	$7.52392765\ 10^{-5}$
6	0.4416799708	-1.567233662	$3.972642663\ 10^{-5}$
8	0.4407959401	-1.56617236	$2.427860244\ 10^{-5}$
10	0.4404486199	-1.56548312	$1.759276972\ 10^{-5}$
12	0.43978238	-1.56506230	$1.173546215\ 10^{-5}$
14	0.43924356	-1.564678701	$7.692065675\ 10^{-6}$
16	0.4401927101	-1.564876371	$1.284779109\ 10^{-5}$
18	0.43909649	-1.564509911	$6.503064085\ 10^{-6}$
20	0.4397366899	-1.562921089	$3.531894606\ 10^{-6}$
21	0.4390099799	-1.562884609	$1.794733740\ 10^{-6}$
22	0.4371246199	-1.56437798	$1.128903380\ 10^{-6}$
23	0.4373339701	-1.56052865	$2.284196632\ 10^{-6}$
24	0.43659889	-1.559938081	$5.996284925\ 10^{-6}$
25	0.43630631	-1.559749569	$7.77804523\ 10^{-6}$

Best $\beta = 22$ *when* $|\nabla_\beta F(x,\beta) = 1.128903380\ 10^{-6}|$

Now we use the Algorithm to choose β*,and we use the shifted Legendre polynomial with* m = *12 at* $\alpha = .8$, $t = 1$

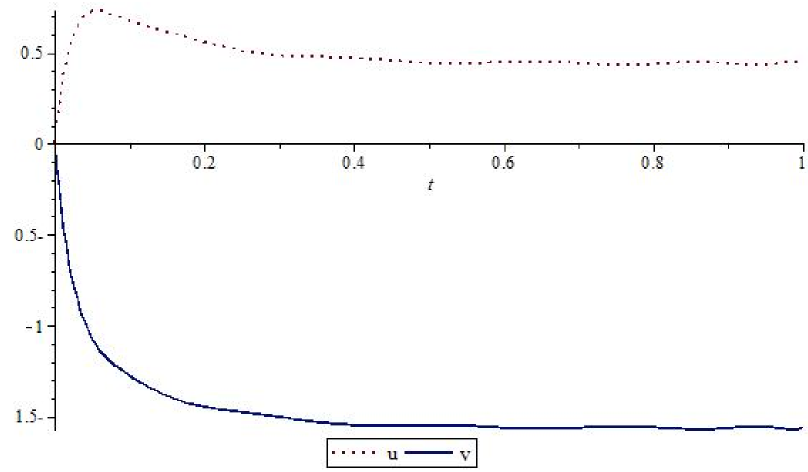


Figure 3.3: The behavior of approximate solution of example using shifted Legendre polynomial with m = 12 at $\alpha = 1$ at $\beta = 22$

β	u	v	$\nabla_\beta F(x,\beta)\|$
1	0.3941391146	1.460271379	0.01059815213
2	0.420906271	−1.496528909	0.003408474751
3	0.431636597	−1.51182914	0.001598061446
4	0.4362763401	−1.519029559	0.0009987813320
5	0.4402324501	−1.521178839	0.0007445443085
6	0.44135513	−1.523260809	0.0006260158865
7	0.4415458699	−1.52597994	0.0005272865080
8	0.4402697699	−1.52325607	0.0006651821740
9	0.43404604	−1.52347114	0.0009023949975
10	0.43487325	−1.521203091	0.0009646439100

Best $\beta = 7$ *when* $|\nabla_\beta F(x,\beta)| = 0.0005272865080$

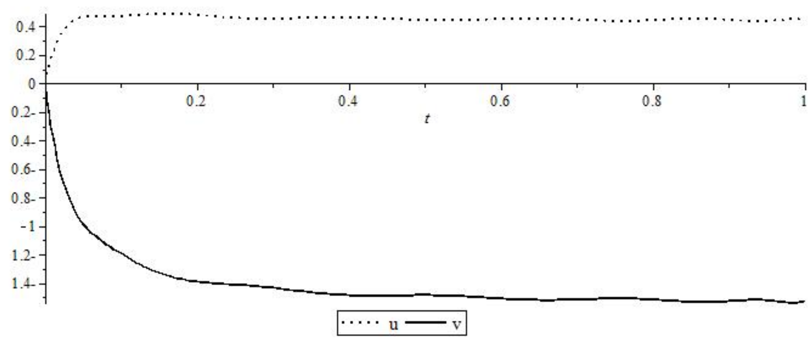


Figure 3.4: The behavior of approximate solution of example using shifted Legendre polynomial with m = 12 at $\boldsymbol{\alpha = .8}$, $\boldsymbol{\beta = 7}$

The numerical comparison among ($\alpha = .8$) with ($\alpha = 1$) this shows that ($\alpha = .8$) perform rapid convergency to the optimal solutions of the optimization problems.

Example 3.5. *optimization problem*

Consider the following non-linear programming problem

$$minimize f(x) = 100\left(u^2 - v\right)^2 + (u-1)^2 \tag{3.22}$$

$$subject\ to\ g(x) = u(u-4) - 2v + 12 = 0 \tag{3.23}$$

The optimal solution is ($u = 1.99937524420685, v = 4.00000019515963$). *For solving the aboveproblem, we convert it to an unconstrained optimization problem with quadratic penalty function (3.11), then we have*

$$F(x,\beta) = 100(u^2 - v)^2 + (u-1)^2 + \frac{\beta}{2}(u(u-4) - 2v + 12)^2$$

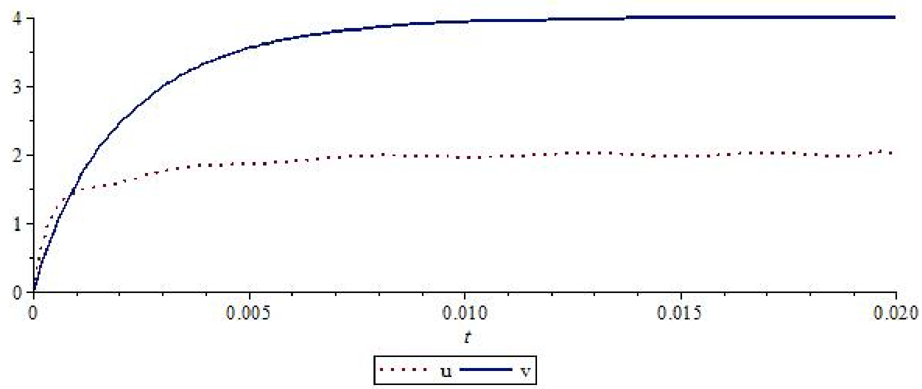


Figure 3.5: The behavior of approximate solution of example using shifted Legendre polynomial with m = 12 at $\alpha = 1$,$\beta = 99$

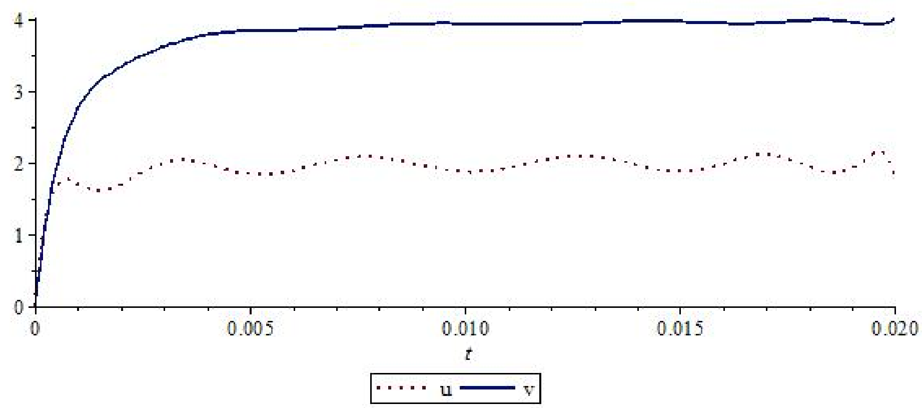


Figure 3.6: The behavior of approximate solution of example using shifted Legendre polynomial with m = 12 at $\alpha = .9$, $\beta = 130$

The numerical comparison among ($\alpha = .8$) with ($\alpha = 1$) this shows that ($\alpha = .8$) perform rapid convergency to the optimal solutions of the optimization problems.

In this section, we implemented numerical methods for solving the system of fractional differential equations which generated from the NLP problem. Our proposed methods using Jumarie modified , in Ref [19] proposed methods using Caputo . The properties of the Legendre polynomials are used to reduce the system of fractional differential equations to the solution of system of algebraic equations. From illustrative examples, it can be seen that the proposed numerical approaches can obtain accurate and satisfactory results. All numerical results are obtained using Maple 2017.

Chapter 4

Non-Differentiable Points of a Function

4.1 Characterization of Non-Differentiable Points of a Function by Jumarie modified Fractional Derivative

For differentiable functions the Jumarie modified definition (both left and right) of the fractional derivative gives the same value at any particular point but for functions having non differentiability at some point gives different value for the left and right Jumarie modified. The difference in values of the fractional derivative at the non differentiable points indicates the Phase transition at the non-differentiable points [13].

Example 4.1.

$$f(t) = t + 3 \qquad a \le t \le b$$

By using the Jumarie modified definition we obtain for The left Jumarie modified definition(1.1)

$$\begin{aligned}
{}_a^J D_t^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha}[\tau + 3 - (a+3)]d\tau \quad 0 < \alpha < 1 \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha}[\tau - a]d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha}[-(t-\tau) + (t-a)]d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_a^t -(t-\tau)^{-\alpha+1} d\tau + \int_a^t (t-\tau)^{-\alpha}(t-a) d\tau \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\frac{-(t-a)^{2-\alpha}}{2-\alpha} + \frac{(t-a)^{2-\alpha}}{1-\alpha} \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\frac{(t-a)^{2-\alpha}}{(1-\alpha)(2-\alpha)} \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\frac{(t-a)^{2-\alpha}}{(1-\alpha)} \right] \\
&= \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}
$$

Therefore

$$
{}_a^J D_t^\alpha f(\frac{a+b}{2}) = \frac{((\frac{a+b}{2}) - a)^{1-\alpha}}{\Gamma(2-\alpha)} = \frac{(\frac{b-a}{2})^{1-\alpha}}{\Gamma(2-\alpha)}
$$

Again using our right Jumarie modified definition 1.2 we obtain

$$
\begin{aligned}
{}_t^J D_b^\alpha f(t) &= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau - t)^{-\alpha} [(b+3) - (\tau+3)] d\tau \quad 0 < \alpha < 1 \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau - t)^{-\alpha} [b - \tau] d\tau \\
&= \frac{-1}{\Gamma(1-\alpha)} ddt \int_t^b (\tau - t)^{-\alpha} [(b-t) - (\tau - t)] d\tau \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_t^b -(\tau - t)^{-\alpha+1} d\tau + \int_t^b (\tau - t)^{-\alpha} (b-t) d\tau \right] \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\frac{-(b-t)^{2-\alpha}}{2-\alpha} + \frac{(b-t)^{2-\alpha}}{1-\alpha} \right] \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\frac{(b-t)^{2-\alpha}}{(2-\alpha)(1-\alpha} \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\frac{(b-t)^{1-\alpha}}{(1-\alpha)} \right] \\
&= \frac{(b-t)^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}
$$

Therefore

$$ {}_t^J D_b^{\alpha} f(\frac{a+b}{2}) = \frac{(b-(\frac{a+b}{2}))^{1-\alpha}}{\Gamma(2-\alpha)} = \frac{(\frac{b-a}{2})^{1-\alpha}}{\Gamma(2-\alpha)} $$

Thus in both the cases (for left and right Jumarie modified fractional derivative) value of $f(\frac{a+b}{2})$ *is equal. Thus for continuous and differentiable functions both the values are equal, and is equal to* $\frac{(\frac{b-a}{2})^{1-\alpha}}{\Gamma(2-\alpha)}$

Example 4.2.

$$ f(t) = \begin{cases} 8t+2 & 0 \le t \le .5 \\ 12t & .5 \le t \le 1 \end{cases} \tag{4.1} $$

The fractional order derivative using left Jumarie modified definition is

- *When* $0 \le t \le .5$

$$
\begin{aligned}
{}_0^J D_t^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} [(8\tau+2)-2] d\tau \quad 0<\alpha<1 \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} [8\tau] d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} [-8(t-\tau)+8t] d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_0^t -8(t-\tau)^{-\alpha+1} d\tau + \int_0^t (t-\tau)^{-\alpha}(8t) d\tau \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\frac{-8t^{2-\alpha}}{2-\alpha} + \frac{8t^{2-\alpha}}{1-\alpha} \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\frac{8t^{2-\alpha}}{(1-\alpha)(2-\alpha)} \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\frac{8t^{2-\alpha}}{(1-\alpha)} \right] \\
&= \frac{8t^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}
$$

- *When* $.5 \le t \le 1$

$$
\begin{aligned}
{}_{0}^{J}D_t^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_0^{.5} (t-\tau)^{-\alpha}[8\tau]d\tau + \int_{.5}^{t} (t-\tau)^{-\alpha}[12\tau-2]d\tau \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \begin{bmatrix} \int_0^{.5}(t-\tau)^{-\alpha}[-8(t-\tau)+8t]d\tau \\ + \int_{.5}^{t}(t-\tau)^{-\alpha}[12t-2-12(t-\tau)]d\tau \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \begin{bmatrix} \frac{8(t-.5)^{2-\alpha}-8t^{2-\alpha}}{2-\alpha} + \frac{-8t(t-.5)^{1-\alpha}+8t^{2-\alpha}}{1-\alpha} \\ + \frac{(12t-2)(t-.5)^{1-\alpha}}{1-\alpha} - \frac{12(t-.5)^{2-\alpha}}{2-\alpha} \end{bmatrix} \\
&= \frac{8t^{1-\alpha} + 4(t-.5)^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}
$$

Therefore

$$
{}_{0}^{J}D_t^{\alpha} f(t) = \begin{cases} \frac{8t^{1-\alpha}}{\Gamma(2-\alpha)} & 0 \le t \le .5 \\ \frac{8t^{1-\alpha}+4(t-.5)^{1-\alpha}}{\Gamma(2-\alpha)} & .5 \le t \le 1 \end{cases} \tag{4.2}
$$

The function f(t) is not differentiable at $t = \frac{1}{2}$ *but* $\alpha-$*order left Jumarie modified derivative at* $t = \frac{1}{2}$ *exists and equal to* $\frac{8(\frac{1}{2})^{1-\alpha}}{\Gamma(2-\alpha)}$ *The graphical presentation of* ${}_{0}^{J}D_t^{\alpha} f(t)$ *for different values of alpha is shown in the figure-[4.1], from the figure it clear that* ${}_{0}^{J}D_t^{\alpha} f(t)$ *exists at the non-differentiable point* $t = \frac{1}{2}$

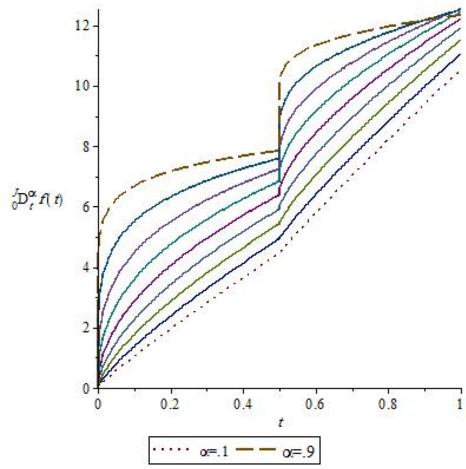


Figure 4.1: Graph of the function ${}_0^J D_t^\alpha f(t)$ for different values of alpha.

The fractional order derivative using right Jumarie modified Fractional Derivative definition is

- *When* $0 \le t \le .5$

$$
\begin{aligned}
{}_t^J D_1^\alpha f(t) &= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \begin{bmatrix} \int_t^{.5} (\tau - t)^{-\alpha} [10 - 8\tau] d\tau \\ + \int_{.5}^{1} (\tau - t)^{-\alpha} [12 - 12\tau] d\tau \end{bmatrix} \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \begin{bmatrix} \int_t^{.5} (\tau - t)^{-\alpha} [-8(\tau - t) - 8t + 10] d\tau \\ + \int_{.5}^{1} (\tau - t)^{-\alpha} [-12(\tau - t) - 12t + 12] d\tau \end{bmatrix} \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \begin{bmatrix} \frac{-8(.5-t)^{2-\alpha}}{2-\alpha} + \frac{-8t(.5-t)^{1-\alpha}}{1-\alpha} + \frac{10(.5-t)^{1-\alpha}}{1-\alpha} \\ + \frac{-12(1-t)^{2-\alpha} + 12(.5-t)^{2-\alpha}}{2-\alpha} + \frac{-12t(1-t)^{1-\alpha} + 12t(.5-t)^{1-\alpha}}{1-\alpha} \\ + \frac{12(1-t)^{1-\alpha} - 12(.5-t)^{1-\alpha}}{1-\alpha} \end{bmatrix} \\
&= \frac{-4(.5-t)^{1-\alpha} + 12(1-t)^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}
$$

- *When* $.5 \leq t \leq 1$

$$
\begin{aligned}
{}_{t}^{J}D_{1}^{\alpha}f(t) &= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t}^{1}(\tau-t)^{-\alpha}[12-12\tau]d\tau \quad 0<\alpha<1\\
&= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t}^{1}(\tau-t)^{-\alpha}[-12(\tau-t)-12t+12]d\tau\\
&= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\begin{bmatrix} \int_{t}^{1}-12(\tau-t)^{-\alpha+1}d\tau \\ +\int_{t}^{1}(\tau-t)^{-\alpha}(-12t+12)d\tau \end{bmatrix}\\
&= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\left[\frac{-12(1-t)^{2-\alpha}}{(2-\alpha)}+\frac{(-12t+12)(1-t)^{1-\alpha}}{(1-\alpha)}\right]\\
&= \frac{12(1-t)^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}
$$

Therefore

$$
{}_{t}^{J}D_{1}^{\alpha}f(t) = \begin{cases} \frac{-4(.5-t)^{1-\alpha}+12(1-t)^{1-\alpha}}{\Gamma(2-\alpha)} & 0 \leq t \leq .5 \\ \frac{12(1-t)^{1-\alpha}}{\Gamma(2-\alpha)} & .5 \leq t \leq 1 \end{cases} \tag{4.3}
$$

From figure [4.2] it is clear that the right Jumarie modified derivative exist at $t=\frac{1}{2}$ *and equal to* $\frac{12(\frac{1}{2})^{1-\alpha}}{\Gamma(2-\alpha)}$ *Thus; though the considered function is not differentiable at* $t=\frac{1}{2}$ *but its right modified fractional derivative exists and its value is* ${}_{t}^{J}D_{1}^{\alpha}f(\frac{1}{2}) = \frac{12(\frac{1}{2})^{1-\alpha}}{\Gamma(2-\alpha)}$ *which differ from the value* ${}_{0}^{J}D_{t}^{\alpha}f(\frac{1}{2}) = \frac{8(\frac{1}{2})^{1-\alpha}}{\Gamma(2-\alpha)}$ *of the derivative at* $t=\frac{1}{2}$ *obtained by left Jumarie modified derivative . Here the difference indicates there is a phase transition from the left hand to the right hand side about the point* $t=\frac{1}{2}$ *and the degree of phase transition is* $\frac{4(\frac{1}{2})^{1-\alpha}}{\Gamma(2-\alpha)}$

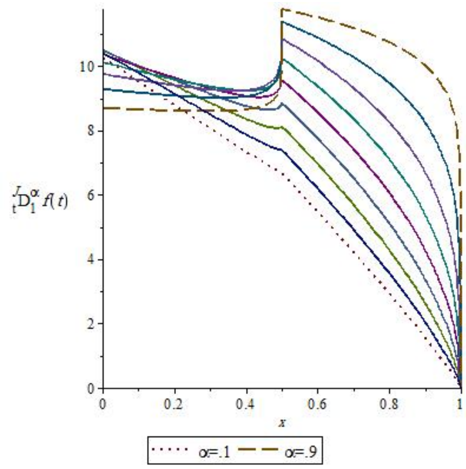


Figure 4.2: Graph of the function ${}_t^J D_1^\alpha f(t)$ for different values of alpha.

Example 4.3.

$$f(t) = \begin{cases} at{+}b & p \leq t \leq q \\ ct + d & q \leq t \leq r \end{cases} \tag{4.4}$$

The function $f(t)$ is continuous at $t{=}q$ such that $aq{+}b = cq{+}d$ but not differentiable at that point [16].

- *Left Jumarie modified* $\quad 0 < \alpha < 1$

$$\begin{aligned}
{}_p^J D_t^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_p^t (t-\tau)^{-\alpha} [(a\tau + b - (ap - b)] d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_p^t (t-\tau)^{-\alpha} [a(\tau - p)] d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_p^t (t-\tau)^{-\alpha} [-a(t-\tau) + a(t-p)] d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_p^t -a(t-\tau)^{-\alpha+1} d\tau + \int_p^t (t-\tau)^{-\alpha} [a(t-p)] d\tau \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt}\left[\frac{-a(t-p)^{2-\alpha}}{2-\alpha} + \frac{a(t-p)^{2-\alpha}}{1-\alpha}\right] \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt}\left[\frac{a(t-p)^{2-\alpha}}{(1-\alpha)(2-\alpha)}\right] \\
&= \frac{a(t-p)^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}
$$

- *Right Jumarie modified*

$$
\begin{aligned}
{}_t^J D_r^\alpha f(t) &= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^r (\tau-t)^{-\alpha}[(cr+d-(c\tau+d)]d\tau \quad 0<\alpha<1 \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^r (\tau-t)^{-\alpha}[c(r-\tau)]d\tau \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^r (\tau-t)^{-\alpha}[-c(\tau-t)+c(r-t)]d\tau \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt}\left[\int_t^r -c(\tau-t)^{-\alpha+1}d\tau + \int_t^r (\tau-t)^{-\alpha}[c(r-t)]d\tau\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt}\left[\frac{-c(r-t)^{2-\alpha}}{2-\alpha} + \frac{c(r-t)^{2-\alpha}}{1-\alpha}\right] \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt}\left[\frac{c(r-t)^{2-\alpha}}{(1-\alpha)(2-\alpha)}\right] \\
&= \frac{c(r-t)^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}
$$

- *phase transition at point* $t = q$

$$
P.T = \frac{a(q-p)^{1-\alpha} - c(r-q)^{1-\alpha}}{\Gamma(2-\alpha)}
$$

Example 4.4.

$$f(t)=\begin{cases} at^2+bt+c & p\le t\le q \\ gt^2+ht+m & q\le t\le r\end{cases} \tag{4.5}$$

The function $f(t)$ is continuous at t=q such that $aq^2+bq+c=gq^2+hq+m$ but not differentiable at that point [16] .

- *Left Jumarie modified* $0<\alpha<1$

$$\begin{aligned}
{}_{p}^{J}D_t^{\alpha}f(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_p^t (t-\tau)^{-\alpha}[(a\tau^2+b\tau+c-(ap^2+bp+c)]d\tau \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_p^t (t-\tau)^{-\alpha}[a\tau^2+b\tau-ap^2-bp]d\tau \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_p^t (t-\tau)^{-\alpha}\begin{bmatrix} a(t-\tau)^2-(b+2at)(t-\tau) \\ +bt-ap^2-bp+at^2\end{bmatrix}d\tau
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left[\tfrac{a(t-p)^{3-\alpha}}{3-\alpha}-\tfrac{(b+2at)(t-p)^{2-\alpha}}{2-\alpha}+\tfrac{(bt-ap^2-bp+at^2)(t-p)^{1-\alpha}}{1-\alpha}\right] \\
&= \frac{1}{\Gamma(1-\alpha)}\left[\tfrac{-a\alpha(t-p)^{2-\alpha}}{2-\alpha}+\tfrac{\alpha(b+2at)(t-p)^{1-\alpha}}{1-\alpha}+(at+ap+b)(t-p)^{1-\alpha}\right]
\end{aligned}$$

- *Right Jumarie modified* $0<\alpha<1$

$$\begin{aligned}
{}_{t}^{J}D_r^{\alpha}f(t) &= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^r (\tau-t)^{-\alpha}[gr^2+hr+m-(g\tau^2+h\tau+m)]d\tau \\
&= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^r (\tau-t)^{-\alpha}[gr^2+hr-g\tau^2-h\tau]d\tau
\end{aligned}$$

$$= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^r (\tau - t)^{-\alpha} \begin{bmatrix} -g(\tau-t)^2 - (h+2gt)(\tau-t) \\ +hr + gr^2 - ht - gt^2 \end{bmatrix} d\tau$$
$$= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\tfrac{-g(r-t)^{3-\alpha}}{3-\alpha} - \tfrac{(h+2gt)(r-t)^{2-\alpha}}{2-\alpha} + \tfrac{(hr+gr^2-ht-gt^2)(r-t)^{1-\alpha}}{1-\alpha} \right]$$
$$= \frac{1}{\Gamma(1-\alpha)} \left[\tfrac{g\alpha(r-t)^{2-\alpha}}{2-\alpha} + \tfrac{\alpha(2gt+h)(r-t)^{1-\alpha}}{1-\alpha} + (gt+gr+h)(r-t)^{1-\alpha} \right]$$

- *phase transition at point* $t = q$

$$P.T = \frac{1}{\Gamma(1-\alpha)} \begin{bmatrix} \tfrac{-a\alpha(q-p)^{2-\alpha}}{2-\alpha} + \tfrac{\alpha(b+2aq)(q-p)^{1-\alpha}}{1-\alpha} + (aq+ap+b)(q-p)^{1-\alpha} \\ -\tfrac{g\alpha(r-q)^{2-\alpha}}{2-\alpha} - \tfrac{\alpha(2gq+h)(r-q)^{1-\alpha}}{1-\alpha} - (gq+gr+h)(r-q)^{1-\alpha} \end{bmatrix}$$

Example 4.5.

$$f(t) = \begin{cases} at + b & p \le t \le q \\ gt^2 + ht + m & q \le t \le r \end{cases} \tag{4.6}$$

The function $f(t)$ *is continuous at t=q such that* $ap + b = gq^2 + hq + m$ *but not differentiable at that point [16].*

- *Left Jumarie modified*

$${}_p^J D_t^\alpha f(t) = \frac{a(t-p)^{1-\alpha}}{\Gamma(2-\alpha)}$$

- *Right Jumarie modified*

$${}_t^J D_r^\alpha f(t) = \left[\tfrac{g\alpha(r-t)^{2-\alpha}}{2-\alpha} + \tfrac{\alpha(2gt+h)(r-t)^{1-\alpha}}{1-\alpha} + (gt+gr+h)(r-t)^{1-\alpha} \right]$$

- *phase transition at point* $t = q$

$$P.T = \frac{1}{\Gamma(1-\alpha)}\begin{bmatrix} \frac{a(q-p)^{1-\alpha}}{1-\alpha} - \frac{g\alpha(r-q)^{2-\alpha}}{2-\alpha} - \frac{\alpha(2gq+h)(r-q)^{1-\alpha}}{1-\alpha} \\ \text{-}(gq+gr+h)(r\text{-}q)^{1-\alpha} \end{bmatrix}$$

Example 4.6.

$$f(t) = \begin{cases} at^2+bt+c & p \le t \le q \\ ht+m & q \le t \le r \end{cases} \tag{4.7}$$

The function $f(t)$ is continuous at t=q such that $ap+b = gq^2+hq+m$ but not differentiable at that point

- *Left Jumarie modified*

$$ {}_p^J D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\begin{bmatrix} \frac{-a\alpha(t-p)^{2-\alpha}}{2-\alpha} + \frac{\alpha(b+2at)(t-p)^{1-\alpha}}{1-\alpha} \\ +(at+ap+b)(t-p)^{1-\alpha} \end{bmatrix}$$

- *Right Jumarie modified*

$$ {}_t^J D_r^\alpha f(t) = \frac{h(r-t)^{1-\alpha}}{\Gamma(2-\alpha)}$$

- *phase transition at point $t = q$*

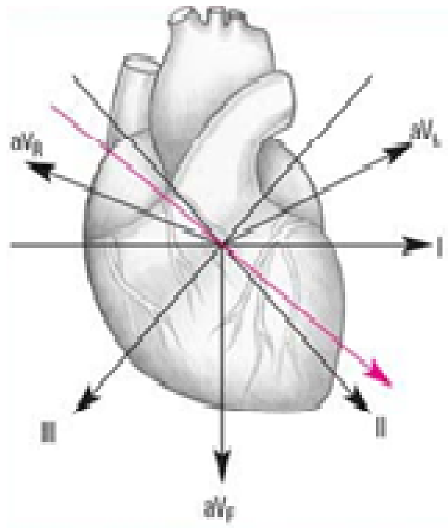
$$P.T = \frac{1}{\Gamma(1-\alpha)}\begin{bmatrix} \frac{a(q-p)^{1-\alpha}}{1-\alpha} - \frac{g\alpha(r-q)^{2-\alpha}}{2-\alpha} - \frac{\alpha(2gq+h)(r-q)^{1-\alpha}}{1-\alpha} \\ -(gq+gr+h)(r-q)^{1-\alpha} \end{bmatrix}$$

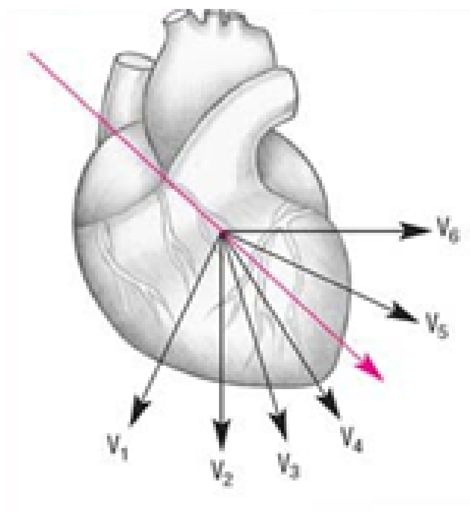
4.2 Application of Jumarie modified Fractional Derivatives in Characterization of Electrocardiogram (ECG) graphs

In this section we shall characterize non-differentiable points of ECG graphs using left and right Jumarie modified definitions of half order fractional derivatives and the Phase Transition (P.T.) at that point. In this section we also find the mean and standard deviation of all non-differentiable points of ECG to get a better solution to interpret this type of ECGs.ECG is the pictographic representation of electrical charge depolarization and repolarization of the heart muscle.There are several types of heart diseases such as right ventricular hypertrophy, left ventricular hypertrophy, right bundle branch block etc.Which can be detected by finding level of phase transition at some particular leads of patients ECGs. Our main objective find some measures which will help the medical experts to diagnose right ventricular hypertrophy (RVH) and left ventricular hypertrophy (LVH) from patients ECG [16],[17]

An ECG lead is a graphical description of the electrical activity of the heart and it is created by analysing several electrodes. The standard ECG which is referred to as a 12-lead ECG . These 12 leads consists of two sets of ECG leads: I, II, III, AVR, AVL, AVF are obtained from the limb leads and V1,V2, V3,V4, V5, V6 are obtained from the chest leads.

Leads I, II and VL look at the left lateral surface of the heart, leads III and VF at the inferior surface, and lead VR looks at the right atrium [6].





The six V leads (V1V6) look at the heart in a horizontal plane, from the front and the left side. Thus, leads V1 and V2 look at the right ventricle, V3 and V4 look at the septum between the ventricles and the anterior wall of the left ventricle, and V5 and V6 look at the anterior and lateral walls of the left ventricle [6].

Right ventricular hypertrophy (RVH)

Right ventricular hypertrophy is a heart disorder characterized by thickening of the walls of the right ventricle. It can be caused by excessive stress on the right ventricle [16].

The limb lead criteria of RVH in ECG are as follows [18]:

1. R wave in V1 lead + S wave in V5 and V6 lead is>10.5mm
2. R wave in V1 lead > 7 mm
3. R/S ratio in V1 lead > 1 mm
4. S wave in V5 or V6 lead > 7 mm
5. R/S ratio in V5 or V6 lead <1 mm

Left ventricular hypertrophy (LVH)

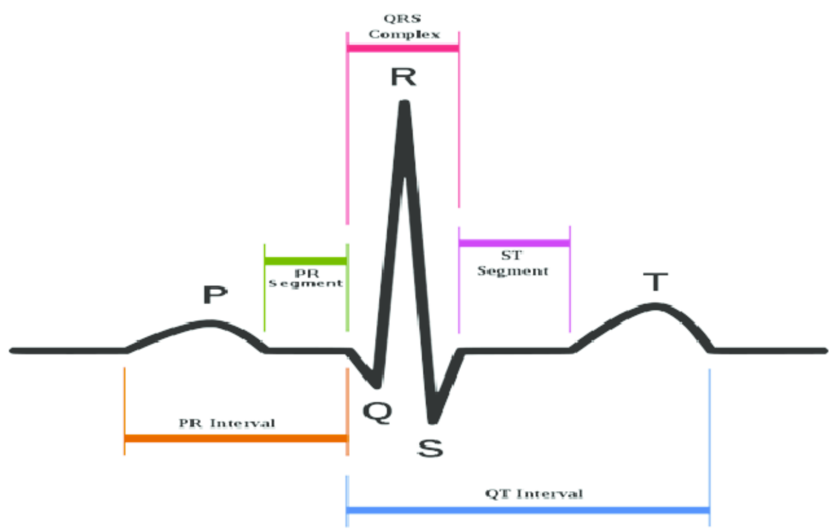
Left ventricular hypertrophy is a heart disorder characterized by thickening of the walls of the left ventricle.

The limb lead criteria of LVH in ECG are as follows[18]:

1. R in V5 or V6 + S in V1 or V2 >35 mm
2. R in V5 or V6 >25 mm
3. S in V1 or V2 >25 mm

Application of fractional derivative in ECG Graph:

Now we have to study the non-differentiable points of the QRS complex in ECG leads with the help of fractional derivatives to compare normal ECGs with abnormal ECGs (RVH,LVH).Here we have to find out the half-order fractional derivative (both left and right) and calculate the corresponding Phase Transition values(P.T) by using example(4.3,4.4,4.5,4.6).



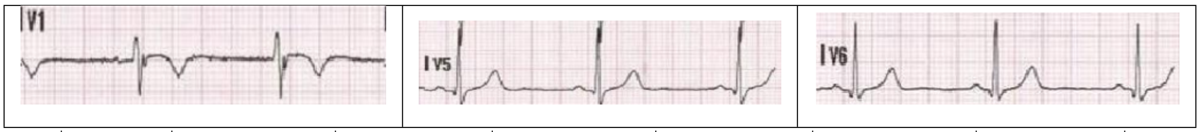
If Q or S point smooth at QRS complex of any lead of the ECGs under consideration then we cannot find the Left and Right Fractional Derivative at that point. We have denoted those cases by 'NA' i.e. 'Not Arise' [16].

- compare normal ECGs with abnormal ECGs (RVH).

Since RVH is characterized by R and S wave in V1, V5 and V6 leads. Thus we compute P.T. values at non-differentiable points only at those leads.So our concern is to find any distinguishing measurements of P.T values at non-differentiable points on those leads ,to compare the problematic ECG (in our case RVH) with normal ECG.

Examples (4.7,4.8,4.10,4.11) Taken from [16],But these examples(4.9,4.12) of my work.

Example 4.7. *Normal ECG*



QRS	R in V1	S in V1	R in V5	S in V5	R in V6	S in v6
1	4	9.5	18	21	14.5	15
2	4.5	10.5	18	21	14.5	15
3			18	20.5	13.5	14

Here, we see that in R wave in V1 <7mm but S wave in V5 and V6 $>$ 7 mm whereas $\frac{R}{S}$ in V1 <1, and in V5 and V6 <1. So from Doctors point of view this graph is normal ECG .

In the following tables we have presented the the Left and Right Fractional Derivative and phase transition values of the non-differentiable points of different leads of ECG.

Table 4.1: Phase transition at the non-differentiable points Q,R,S of V1,V5 and V6 leads

		P.T of V1	P.T of V5	P.T of V6
1	Q	6.383076	30.244018	25.489442
	R	21.542883	62.234996	40.064340
	S	25.532306	36.896290	20.117226
2	Q	5.077706	30.319613	21.243420
	R	21.833282	52.419806	42.829167
	S	24.734422	28.483269	25.239477
3	Q	–	31.915382	25.239477
	R	–	61.437111	23.823144
	S		35.534215	37.340191

Now we construct a table with mean and standard deviation of phase transition values at the non-differentiable points of considerable ECG graph.

Table 4.2: Mean and Standard Deviation of Phase transition at the non-differentiable points Q,R,S of V1,V5and V6 leads

Non-differentiable Points	Leads	Mean	SD
Q point	V1 lead	5.730391	0.923035979
	V5 lead	30.82633767	0.943897145
	V6 lead	23.51866867	2.139323404

R point	V1 lead	21.6880825	0.205343102
	V5 lead	58.69730433	5.451091094
	V6 lead	40.07789933	2.744513121
S point	V1 lead	25.133364	0.564189187
	V5 lead	33.63792467	4.515713494
	V6 lead	21.182555	3.643018712

Example 4.8. *Normal ECG*

QRS	R in V1	S in V1	R in V5	S in V5	R in V6	S in v6
1	2.8	19.7	17.9	17.6	19.2	18
2	3.1	18.8	19.4	19.6	19.5	18.2
3	3.2	19.5	18.6	18.3	19.9	18.6

Here, we see that in R wave in V1 <7mm but S wave in V5 and V6 > 7 mm whereas $\frac{R}{S}$ in V1 $<$ 1, and in V5 and V6 $>$ 1. So from Doctors point of view this graph is normal ECG .

In the following tables we have presented the the Left and Right Fractional Derivative and phase transition values of the non-differentiable points of different leads of ECG.

Table 4.3: Phase transition at the non-differentiable points Q,R,S of V1,V5 and V6 leads

		P.T of V1	P.T of V5	P.T of V6
1	Q	NA	23.4741	27.7903
	R	33.7053	53.9826	62.0691
	S	38.4275	32.0749	NA
2	Q	NA	30.0893	27.4551
	R	34.3529	60.7762	63.7091
	S	36.2674	34.6120	NA
3	Q	NA	24.4952	30.2656
	R	31.8500	50.1233	51.7140
	S	38.4311	26.6582	NA

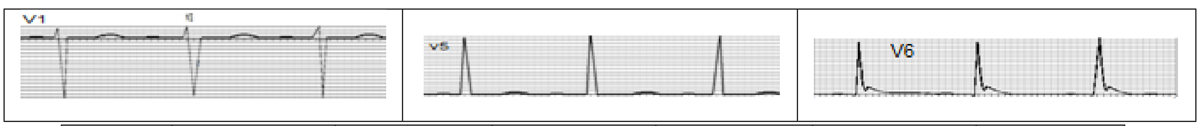
Now we construct a table with mean and standard deviation of phase transition values at the non-differentiable points of considerable ECG graph.

Table 4.4: Mean and Standard Deviation of Phase transition at the non-differentiable points Q,R,S of V1,V5and V6 leads

Non-differentiable Points	Leads	Mean	SD
Q point	V1 lead	NA	NA
	V5 lead	26.01953	3.561308
	V6 lead	28.50367	1.535056

R point	V1 lead	33.30273	1.299104
	V5 lead	54.9607	5.393383
	V6 lead	59.16407	6.503847
S point	V1 lead	37.70867	1.248175
	V5 lead	31.11503	4.062849
	V6 lead	NA	NA

Example 4.9. *Normal ECG*



QRS	R in V1	S in V1	R in V5	S in V5	R in V6	S in v6
1	2.8	19.7	17.9	18.2	19.2	18
2	3.1	18.8	19.4	19.6	19.5	18.2
3	3.2	19.5	18.6	18.9	19.9	18.6

Here, we see that in R wave in V1 <7mm but S wave in V5 and V6 > 7 mm whereas $\frac{R}{S}$ in V1 < 1, and in V5 and V6 > 1. So from Doctors point of view this graph is normal ECG .

In the following tables we have presented the the Left and Right Fractional Derivative and phase transition values of the non-differentiable points of different leads of ECG.

Table 4.5: Phase transition at the non-differentiable points Q,R,S of V1,V5 and V6 leads

		P.T of V1	P.T of V5	P.T of V6
1	Q	4.468153541	28.56426728	32.23453626
	R	26.69722313	49.10076812	58.51775171
	S	57.04525099	21.15453956	NA
2	Q	4.946884276	30.95792096	32.71326699
	R	26.16041262	53.07415263	59.29738355
	S	38.92908126	22.34190750	NA
3	Q	3.610813335	20.98785251	23.81681957
	R	34.72831120	51.14788891	51.00273835
	S	57.12853455	30.63876714	NA

Now we construct a table with mean and standard deviation of phase transition values at the non-differentiable points of considerable ECG graph.

Table 4.6: Mean and Standard Deviation of Phase transition at the non-differentiable points Q,R,S of V1,V5and V6 leads

Non-differentiable Points	Leads	Mean	SD
Q point	V1 lead	4.341950383	0.676917142251348
	V5 lead	26.83668025	5.20470786251155
	V6 lead	29.58820761	5.00389704933024

R point	V1 lead	29.19531565	4.79922610408714
	V5 lead	51.10760323	1.98699857037973
	V6 lead	56.27262453	5.00389704933024
S point	V1 lead	51.03428893	10.4835000667114
	V5 lead	24.71173807	5.16717678073229
	V6 lead	NA	NA

Example 4.10. *RVH ECG*

QRS	R in V1	S in V1	R in V5	S in V5	R in V6	S in v6
1	10.7	8.1	7.8	15.2	6.5	10.6
2	10	8	8.8	17.8	7.1	11.1

Here, we see that in R wave in V1 >7mm , S wave in V5,V6 > 7 mm, $\frac{R}{S}$ in V1 > 1, and in V5 and V6 < 1mm. So from Doctors point of view this patient has cardiac problem which called Right Ventricular Hypertrophy. In the following tables we have presented the the Left and Right Fractional Derivative and phase transition values of the non-differentiable points of different leads of ECG

Table 4.7: Phase transition at the non-differentiable points Q,R,S of V1,V5 and V6 leads

		P.T of V1	P.T of V5	P.T of V6
1	Q	26.0803932	NA	NA
	R	36.4947865	33.2203698	25.1153
	S	14.7037186	25.7151588	23.6948
2	Q	17.4327977	NA	NA
	R	26.8886555	39.3007419	20.8981
	S	16.333122	35.4445989	20.3390

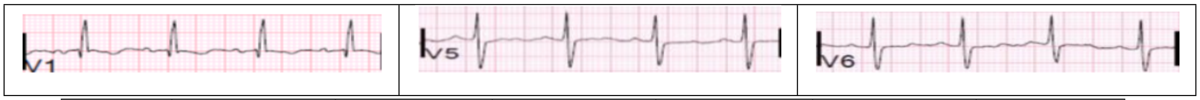
Now we construct a table with mean and standard deviation of phase transition values at the non-differentiable points of considerable ECG graph.

Table 4.8: Mean and Standard Deviation of Phase transition at the non-differentiable points Q,R,S of V1,V5and V6 leads

Non-differentiable Points	Leads	Mean	SD
Q point	V1 lead	21.7566	6.114773
	V5 lead	NA	NA
	V6 lead	NA	NA
R point	V1 lead	31.69172	6.79256
	V5 lead	36.26056	4.299472
	V6 lead	23.0067	2.982054

S point	V1 lead	15.51842	1.152162
	V5 lead	30.57988	6.879753
	V6 lead	22.01692	2.372916

Example 4.11. *RVH ECG*



QRS	R in V1	S in V1	R in V5	S in V5	R in V6	S in v6
1	10.7	8.8	9.1	19.2	8.6	15.6
2	10.1	9.3	10.3	19.4	8.8	15.3
3	11	9	9.1	17.1	8.1	14
4	10.9	8.9	10	19.8	8.8	15.2

Here, we see that in R wave in V1 >7mm , S wave in V5,V6 > 7 mm, $\frac{R}{S}$ in V1 > 1, and in V5 and V6 < 1. So from Doctors point of view this patient has cardiac problem which called Right Ventricular Hypertrophy.

In the following tables we have presented the the Left and Right Fractional Derivative and phase transition values of the non-differentiable points of different leads of ECG.

Table 4.9: Phase transition at the non-differentiable points Q,R,S of V1,V5 and V6 leads

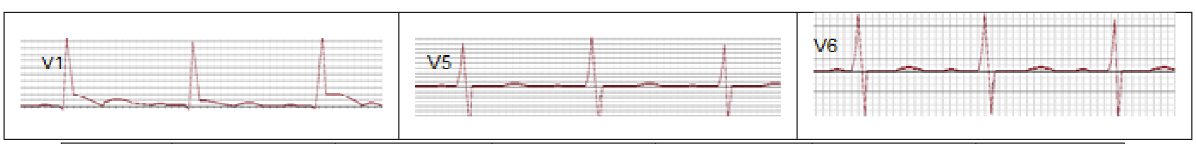
		P.T of V1	P.T of V5	P.T of V6
1	Q	21.210452	NA	NA
	R	30.1310726	37.1478021	31.2891687
	S	15.7002922	34.5669156	25.1595267
2	Q	16.7788313	NA	NA
	R	24.4743855	44.5219585	38.9823975
	S	13.7510722	36.6391728	21.4870607
3	Q	19.5922919	NA	NA
	R	32.0811607	34.5425282	24.6282722
	S	16.05711705	25.9435303	22.9338048
4	Q	19.4358294	NA	NA
	R	20.4990607	47.6872047	39.3608674
	S	8.199727814	30.6882091	21.7925225

Now we construct a table with mean and standard deviation of phase transition values at the non-differentiable points of considerable ECG graph.

Table 4.10: Mean and Standard Deviation of Phase transition at the non-differentiable points Q,R,S of V1,V5and V6 leads

Non-differentiable Points	Leads	Mean	SD
Q point	V1 lead	19.25435	1.834999
	V5 lead	NA	NA
	V6 lead	NA	NA
R point	V1 lead	26.79642	5.294556
	V5 lead	40.97487	6.155181
	V6 lead	33.56518	7.023408
S point	V1 lead	13.42705	3.629269
	V5 lead	31.95946	4.708362
	V6 lead	22.84323	1.664994

Example 4.12. *RVH ECG*



QRS	R in V1	S in V1	R in V5	S in V5	R in V6	S in v6
1	10.7	8.8	9.1	19.2	8.6	15.6
2	10.1	9.3	10.3	19.4	8.8	15.3
3	11	9	9.1	17.1	8.1	14

Here, we see that in R wave in V1 >7mm , S wave in V5,V6 > 7 mm, $\frac{R}{S}$ in V1 > 1, and in V5 and V6 < 1. So from Doctors point of view this patient has cardiac problem which called Right Ventricular Hypertrophy. In the following tables we have presented the the Left and Right Fractional Derivative and phase transition values of the non-differentiable points of different leads of ECG.

Table 4.11: Phase transition at the non-differentiable points Q,R,S of V1,V5 and V6 leads

		P.T of V1	P.T of V5	P.T of V6
1	Q	18.67049872	NA	NA
	R	27.80235083	35.35588057	30.54146279
	S	9.929736670	42.47218369	32.02361855
2	Q	17.71303725	NA	NA
	R	27.40907894	37.38696307	30.50385016
	S	10.49392625	36.41205485	27.63670055
3	Q	13.77424498	NA	NA
	R	27.33828251	48.20745346	39.47414698
	S	14.36192209	34.33893603	41.64957407

Now we construct a table with mean and standard deviation of phase transition values at the non-differentiable points of considerable ECG graph.

Table 4.12: Mean and Standard Deviation of Phase transition at the non-differentiable points Q,R,S of V1,V5and V6 leads

Non-differentiable Points	Leads	Mean	SD
Q point	V1 lead	16.71926032	2.59499896272331
	V5 lead	NA	NA
	V6 lead	NA	NA
R point	V1 lead	27.51657076	0.250011439602452
	V5 lead	40.31676570	6.9085844443570
	V6 lead	33.50648663	5.16817966965665
S point	V1 lead	11.595195	2.41260472303617
	V5 lead	37.74105820	4.22635975551328
	V6 lead	33.76996440	7.16780638445059

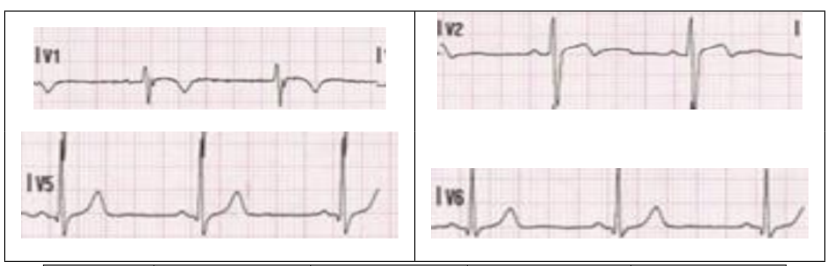
It is clear from tables values P.T values at S is higher than the P.T. values at R of V1 lead of normal ECGs whenever the opposite case holds for RVH ECGs. Also we see that for V5 and V6 leads the difference between P.T. values at R and S point is small (i.e. less than 20) for RVH ECG whereas these difference is large for normal ECGs. Also From tables it is cleared that maximum mean P.T. values are higher for normal ECG where as they are small for RVH ECG.

- compare normal ECGs with abnormal ECGs (LVH).

Since LVH is characterized by deep S wave in V1 and V2 leads and long R wave in V5 and V6 leads. Thus we compute P.T. values at non-differentiable points only at those leads.So our concern is to find any distinguishing measurements of P.T values at non-differentiable points on those leads ,to compare the problematic ECG (in our case LVH) with normal ECG.

Examples (4.13,4.15) Taken from [17],But these examples (4.14,4.16) of my work.

Example 4.13. *Normal ECG*



QRS	S in V1	S in V2	R in V5	R in v6
1	9.5	21.5	18	14.5
2	10.5	23	18	14.5
3			18	13.5

Here, we see that in table R in V5 and V6 <25 mm, S in V1 and V2 <25 mm and R in V5 or V6 + S in V1 or V2 are not all greater than 35 mm . So from Doctors point of view this graph is normal ECG .

In the following tables we have presented the the Left and Right Fractional Derivative and phase transition values of the non-differentiable points of different leads of ECG.

Table 4.13: Phase transition at the non-differentiable points Q,R,S of V1,V2,V5and V6 leads

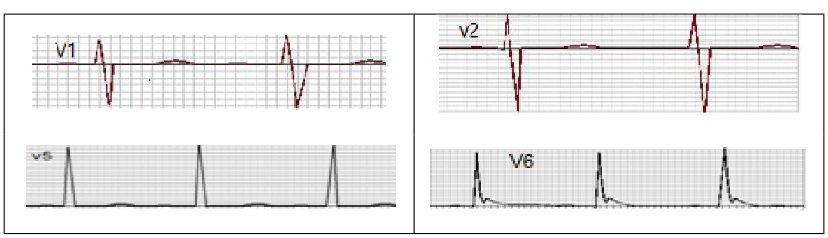
		P.T of V1	P.T of V2	P.T of V5	P.T of V6
1	Q	6.3831	NA	30.2440	25.4894
	R	21.5429	34.4156	62.2350	40.0643
	S	25.5323	44.2073	36.8963	20.1172
2	Q	5.0777	10.1554	30.3196	21.2434
	R	21.8333	46.8581	52.4198	42.8292
	S	24.7344	57.4477	28.4833	25.2395
3	Q	—	—	31.9154	23.8231
	R	—	—	61.437	37.3402
	S			35.5342	18.1910

Now we construct a table with mean and standard deviation of phase transition values at the non-differentiable points of considerable ECG graph.

Table 4.14: Mean and Standard Deviation of Phase transition at the non-differentiable points Q,R,S of V1,V2,V5and V6 leads

Non-differentiable Points	Leads	Mean	SD
Q point	V1 lead	5.7304	0.9230
	V2 lead	NA	NA
	V5 lead	30.8263	0.9439
	V6 lead	23.5187	2.1393
R point	V1 lead	21.6881	0.2053
	V2 lead	40.6368	8.7982
	V5 lead	58.6973	5.4511
	V6 lead	40.0779	2.7445
S point	V1 lead	25.1334	0.5642
	V2 lead	50.8275	9.3624
	V5 lead	33.6379	4.5157
	V6 lead	21.1826	3.6430

Example 4.14. *Normal ECG*



QRS	S in V1	S in V2	R in V5	R in v6
1	9.5	20.7	17.9	19.2
2	11	22.1	19.4	19.5
3			18.6	19.9

Here, we see that in table R in V5 and V6 <25 mm, S in V1 and V2 <25 mm and R in V5 or V6 + S in V1 or V2 are not all greater than 35 mm . So from Doctors point of view this graph is normal ECG .

In the following tables we have presented the the Left and Right Fractional Derivative and phase transition values of the non-differentiable points of different leads of ECG.

Table 4.15: Phase transition at the non-differentiable points Q,R,S of V1,V2,V5and V6 leads

		P.T of V1	P.T of V2	P.T of V5	P.T of V6
1	Q	5.5851919	13.244883	28.564267287	32.23453626
	R	16.304794	36.123601	49.10076812	58.51775171
	S	23.080376	50.139126	21.1545395639	NA
2	Q	7.1809610	12.446999	30.9579209	32.71326699
	R	19.5931318	37.224601	53.0741526	59.29738355
	S	19.746635	48.0758318	22.3419075	NA

3	Q	—	—	20.98785251	23.8168195
	R	—	—	51.14788891	51.0027383
	S			30.63876714	NA

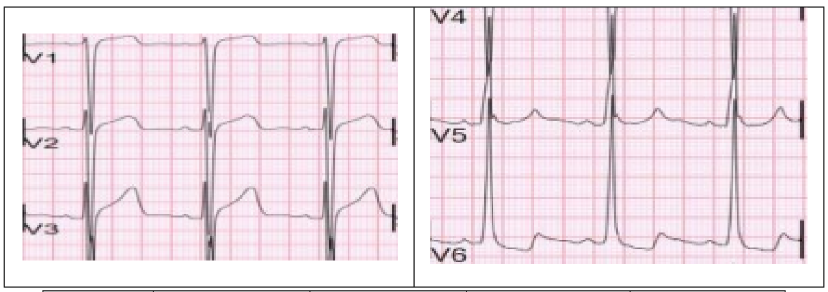
Now we construct a table with mean and standard deviation of phase transition values at the non-differentiable points of considerable ECG graph.

Table 4.16: Mean and Standard Deviation of Phase transition at the non-differentiable points Q,R,S of V1,V2,V5and V6 leads

Non-differentiable Points	Leads	Mean	SD
Q point	V1 lead	6.383076485	1.12837916737430
	V2 lead	12.84594143	0.564189582980045
	V5 lead	26.83668025	5.20470786251155
	V6 lead	29.58820761	5.00389704933024
R point	V1 lead	17.94896296	2.32520600670953
	V2 lead	36.67410178	0.778524636797067
	V5 lead	51.10760323	1.98699857037973
	V6 lead	56.27262453	5.00389704933024

S point	V1 lead	21.41350599	2.35731095267244
	V2 lead	49.1074793	1.45896982951975
	V5 lead	24.71173807	5.16717678073229
	V6 lead	NA	NA

Example 4.15. *LVH ECG*



QRS	S in V1	S in V2	R in V5	R in v6
1	34.2	46.7	27.5	36.8
2	35.2	49	27.7	38
3	36.1	50.2	28.8	37

Here, we see that in table R in V5 and V6 >25 mm, S in V1 and V2 >25 mm and R in V5 or V6 + S in V1 or V2 are all greater than 35 mm . So from Doctors point of view this patient with problematic ECG has cardiac problem which called Left Ventricular Hypertrophy.

In the following tables we have presented the the Left and Right Fractional Derivative and phase transition values of the non-differentiable points of different leads of ECG.

Table 4.17: Phase transition at the non-differentiable points Q,R,S of V1,V2,V5and V6 leads

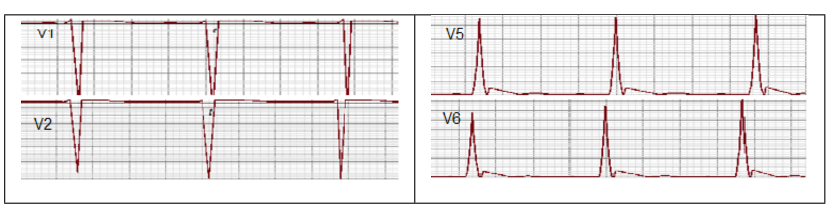
		P.T of V1	P.T of V2	P.T of V5	P.T of V6
1	Q	NA	NA	NA	NA
	R	45.7081	64.5818	116.4704	100.9253
	S	89.5825	123.3424	NA	NA
2	Q	NA	NA	NA	NA
	R	42.6324	65.4877	55.5799	74.1328
	S	108.3481	146.1234	NA	NA
3	Q	NA	NA	NA	NA
	R	44.8548	71.4775	58.2417	113.1625
	S	81.5230	106.5454	NA	NA

Now we construct a table with mean and standard deviation of phase transition values at the non-differentiable points of considerable ECG graph.

Table 4.18: Mean and Standard Deviation of Phase transition at the non-differentiable points Q,R,S of V1,V2,V5and V6 leads

Non-differentiable Points	Leads	Mean	SD
Q point	V1 lead	NA	NA
	V2 lead	NA	NA
	V5 lead	NA	NA
	V6 lead	NA	NA
R point	V1 lead	44.3984	1.5878
	V2 lead	67.1823	3.7472
	V5 lead	76.7640	34.4125
	V6 lead	96.0735	19.9621
S point	V1 lead	93.1512	13.7640
	V2 lead	125.3371	19.8642
	V5 lead	NA	NA
	V6 lead	NA	NA

Example 4.16. *LVH ECG*



QRS	S in V1	S in V2	R in V5	R in v6
1	34.2	46.7	27.4	34
2	35.6	49	27.8	38
3	36	50.2	28.6	42

Here, we see that in table R in V5 and V6 >25 mm, S in V1 and V2 >25 mm and R in V5 or V6 + S in V1 or V2 are all greater than 35 mm So from Doctors point of view this patient with problematic ECG has cardiac problem which called Left Ventricular Hypertrophy.

In the following tables we have presented the the Left and Right Fractional Derivative and phase transition values of the non-differentiable points of different leads of ECG.

Table 4.19: Phase transition at the non-differentiable points Q,R,S of V1,V2,V5and V6 leads

		P.T of V1	P.T of V2	P.T of V5	P.T of V6
1	Q	NA	NA	NA	NA
	R	40.9842211	57.7485759	77.6601805	96.36664735
	S	91.8892561	121.1538024	NA	NA
2	Q	NA	NA	NA	NA
	R	41.6330867	60.3438480	78.7939057	107.703900
	S	84.95590061	112.3133742	NA	NA

3	Q	NA	NA	NA	NA
	R	59.8366445	79.0437638	81.0613563	119.0411526
	S	108.114128	147.408593	NA	NA

Now we construct a table with mean and standard deviation of phase transition values at the non-differentiable points of considerable ECG graph.

Table 4.20: Mean and Standard Deviation of Phase transition at the non-differentiable points Q,R,S of V1,V2,V5and V6 leads

Non-differentiable Points	Leads	Mean	SD
Q point	V1 lead	NA	NA
	V2 lead	NA	NA
	V5 lead	NA	NA
	V6 lead	NA	NA
R point	V1 lead	47.48465080	10.7020590779060
	V2 lead	65.71206260	11.6182852416726
	V5 lead	79.17181417	1.73179395786860
	V6 lead	107.7039	11.337252625

S point	V1 lead	94.98642827	11.8857153835199
	V2 lead	126.9585899	18.2535001546237
	V5 lead	NA	NA
	V6 lead	NA	NA

From table it is clearly show that the P.T at the non differentiable points is higher for LVH ECG than normal ECG . The P.T. values at the non-differentiable points for normal ECG is less than 65 . The patient having LVH problem that P.T. value can be exceed 100. Also From tables it is cleared that maximum mean P.T. value and maximum standard deviation of the P.T. values are higher for LVH ECG.

Conclusions

Here we develop an analytical method to find the solutions of linear fractional differential equation and system of fractional differential equations, composed by Jumarie modified fractional derivative in terms of one parameter Mittag-Leffler function. The solutions obtained are similar as the solutions obtained usual calculus, in terms the exponential function. we find the approximate solution of fractional derivative using Legendre polynomials and implementing it to solve the nonlinear fractional differential equations. Illustrative example is included to demonstrate the validity and applicability of the presented technique.

In this thesis we have to characterize graph ECG and compare normal ECG with (LVH,RVH) ECG by finding P.T. values at the non-differentiable points and mean, standard deviation of the P.T. values of the non-differentiable points of considerable ECG samples.

References

[1] D. p. Bertsekas. **On penalty and multiplier methods for constrained minimization.** SIAM Journal on Control and Optimization 14(2):216-235,1976.

[2] A. Bhrawy, E. Doha, s. Ezz-Eldien, and M. Abdelkawy. **A numerical technique based on the shifted legendre polynomials for solving the time-fractional coupled Kdv equations.** Calcolo,53(1):1-17,2016.

[3] U. Ghosh, s. Sarkar, and S. Das. **Fractional weierstrass function by application of jumarie fractional trigonometric functions and its analysis**. arXiv preprint arXiv:1508.06862, 2015**.**

[4] U. Ghosh, s. Sarkar, and S. Das. **Solution of system of linear fractional differential equations with modified derivative of jumarie type**. arXiv preprint arXiv:1508.06862, 2015**.**

[5] U. Ghosh, s. Sengupta, S. Sarkar, and S. Das**. Analytic Solution of of linear fractional differential equations with jumarie derivative in term of mittag-leffler function.** arXiv preprint arXiv:1508.06862, 2015**.**

[6] J. Hampton. ***The ECG made easy***. Elsevier Health Sciences, 2013.

[7] M. Ishteva, L. Boyadjiev, and R.scherer. **On the caputo operator of fractional calculus and c-laguerre functions.** Mathematical Sciences Research Journal, 9(6):161, 2005.

[8] G. Jumarie. **Modified Riemann-liouville derivative and fractional taylor series of nondifferentiable functions further results**. Computers & Mathematics with Applications, 51(9-10):1367-1376, 2006.

[9] G. Jumarie. **Fourier's transform of fractional order via mittag-leffler function and modified Riemann-liouville derivative**. Journal of applied mathematics & informatics, 26(5-6):1101-1121, 2018.

[10] G. Jumarie. **An approach to differential geometry of fractional order via modified Riemann-liouville derivative**. Acta Mathematica sinica, English series, 28(9):1741-1768, 2012.

[11] G. Jumarie. **On the derivative chain-rules in fractional calculus via fractional difference and their application to systems modelling**. Open Physics, 11(6):617–633, 2013.

[12] D. Logan. **A first course in differential equations**. Springer, 2017.

[13] S. Ogrekci. **Generalized taylor series method for solving nonlinear fractional differential equations with modified riemann-liouville derivative**. Advances in Mathematical Physics, 2015, 2015.

[14] I. Podlubny. **Fractional differential equations, volume 198: An introduction to fractional derivatives, fractional differential equations, to methods of their...(mathematics in science and engineering)**. 1998.

[15] J. Sabatier, O. P. Agrawal, and J. T. Machado. Advances in fractional calculus, volume 4. Springer, 2007.

[16] S. Sengupta, U. Ghosh, S. Sarkar, and S. Das. **Application of fractional derivatives in characterization of ecg graphs of right ventricular hypertrophy patients.** arXiv preprint arXiv:1711.02332, 2017.

[17] S. Sengupta, U. Ghosh, S. Sarkar, and S. Das. **Application of fractional calculus to distinguish left ventricular hyper trophy with normal ecg**. In 2018 4th International Conference on Recent Advances in Information Technology (RAIT), pages 1–6. IEEE, 2018.

[18] B. Surawicz and T. Knilans. **Chou's electrocardiography in clinical practice: adult and pediatric**. Elsevier Health Sciences, 2008.

[19] N. Sweilam, M. Khader, and A. Mahdy. **Computional methods for fractional differential equations generated by optimization problem**. Journal of Fractional Calculus and Applications, 3:1–12, 2012.

[20] M. H. Tavassoli, A. Tavassoli, and M. O. Rahimi. **The geometric and physical interpretation of fractional order derivatives of polynomial functions**. Differential geometry Dynamical systems, 15:93–104, 2013.

المعادلات التفاضليه الكسريه والنظم الخطيه التفاضليه الكسريه :وتطبيق التفاضل الكسري في تخطيط القلب

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الملخص

حساب التفاضل والتكامل الكسري هو موضوع بحث حالي في العلوم التطبيقية مثل الرياضيات التطبيقية والفيزياء ، والفيزياء الحيوية ، والديناميكا الهوائية ، ونظرية التحكم ، ونظرية المكثف ، والدوائر الكهربائية ، ووصف الذاكرة والخصائص الوراثية ، وما إلى ذلك ، استخدمت النماذج الكسرية بدلاً من النماذج الكلاسيكية.

في هذا العمل ، نطور طريقة تحليلية وعددية لإيجاد حلول المعادلة التفاضلية الكسرية. قمنا أيضًا بتمييز رسم بياني تخطيط القلب وقارنا تختيط القلب الطبيعي بـ (تضخم البطين الأيمن ،وتضخم البطين الأيسر)من خلال إيجاد قيم المراحل الانتقاليه افي نقطة غير قابلة للإشتقاق.

جامعة النجاح الوطنية
كلية الدراسات العليا

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قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجه الماجستير في الرياضيات المحوسبة بكليه الدراسات العليا في جامعه النجاح الوطنية في نابلس - فلسطين

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