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Numerical Techniques for Solving Integral Equations with Carleman Kernel

By

Wala' Mohammad AmeenDraidi

Supervisor

Prof. Naji Qatanani

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This thesis was defended successfully on 2/3/2016 and approved by:

Defense committee Members	**Signature**
– **Prof. Naji Qatanani /Supervisor**	…..……………..
– **Dr. Maher Qarawani /External Examiner**	…………………
– **Dr. Adnan Daraghmeh /Internal Examiner**	…………………

Dedication

I dedicate this thesis to my parents, to my husband Emad and my sisters Tasneem, Sanaa and Amal, to my brothers, without their patience, understanding, support, and most of all love, this work wouldn’t have been possible.

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الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان

Numerical Techniques for Solving Integral Equations with Carleman Kernel

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Declaration

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Student's Name: **اسم الطالب: ولاء محمد أمين دريدي**

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Table of contents

List of Figures

List of Tables

Numerical Techniques for Solving Integral Equation with a Carleman Kernel

By
Wala' Mohammad AmeenDraidi
Supervisor
Prof. NajiQatanani

Abstract

Integral equations with Carleman type kernel arise frequently in physics and engineering,theory of elasticity, mathematical problems of radiativeheat transformationsand radiative equilibrium.

In this work we focus our attention mainly on the numerical handling of the Fredholm and Volterra integral equations with Carleman kernel.

The numerical treatment of such equations can be achieved by using the following numerical techniques, namely; Toeplitz matrix method, Product Nystrom method, sinc-collocation method and Laplace Adomian decomposition method.

To test the efficiency of these methods, we consider some numerical test cases.

Numerical results have shown that product Nystrom method is one of the most powerful numerical techniques for solving Fredholm integral equation with a Carleman kernel in comparison with the other numerical techniques. On the other hand, we see clearly that the Laplace Adomian decomposition method is a very reliable and efficient method for solving Volterra integral equations with Carleman kernel.

Introduction

In recent years singular integral equations have attracted the attention of many scientists and researchers due to their wide range of applications in science and technology.

Many problems related to mathematical physics, engineering, theory of elasticity and the mixed problems of continuous media lead to integral equations of the first or second kind [7,21]. The solution of these problems can be obtained analytically using the theory developed by Muskhelishvili[34]. On the other hand, numerical methods play a very important rule in solving singular integral equations. Abdalkhani[35] obtained a numerical solution of the nonlinear Volterra integral equation with Carleman kernel. Carleman was the first scientist who realized the importance of the Fredholm integral equation and its applications[19]. The importance of Carleman kernel came from the work of Arytiunian[8] who has shown that the contact problem of nonlinear theory of plasticity in the first approximation reduces to a Fredholm integral equation of the first kind with Carleman kernel.

Krein's technique [5] is used to find the relationship between integral which has Carleman kernel and integral with logarithmic kernel . Abdou, in his work [1,4] used a series in the Legendre polynomials form to obtain the solution of Fredholm-Volterra integral equation of the second kind under certain conditions.Guoqiang[22] obtained a numerical solution of two dimensional Volterra integral equations by collocation and iterated collocation method. Brunner [14] has used the sinc-collocation method to solve Fredholm integral equation of the second kind. Hendi and Al-Hazmi[23,24] have obtained the solution of Volterra integral equation using Laplace Adomian decomposion method. Gragam[20] applied the Galerkin method to obtain the solution of singular integral equation. Mohamed and Ismail [3]implemented Toeplitz matrix method to obtain numerical solution. Furthermore, Orsi[35] has used product Nystrom as numerical method.

This thesis is organized as follows: chapter one presents some of the fundamental principles of integral equations and their classifications such as linearity, homogeneity and singularity. Carleman kernel is also included in this chapter. Some of numerical techniques used to solve integral equations with Carleman kernel are introduced in chapter two. These techniques are Laplace Adomian decomposition method, Product Nystrom method, Toeplitz matrix method and Sinc-collocation method. In chapter three, the simulation results for solving the Fredholm integral equation with a Carleman kernel using the previous techniques are presented. Finally, chapter four involves solving the Volterra integral equations of the second kind with a Carleman kernel using Laplace Adomian decomposition method, Toeplitz matrix method, product Nystrom method and sinc-collocation method.

Chapter One

Review of Integral Equations

1.1 Classifications of Integral Equations

Definition (1.1)[29]: An integral equation is the equation where the unknown function u appears under the integral sign.

An integral equation may be written into the form:

$$u(x) = f(x) + \lambda \int_{r(x)}^{h(x)} g(x,t)\, u(t) \qquad (1.1)$$

where $f(x)$ is a given function, λ is a constant parameter which may be complex or real, $r(x)$ and $h(x)$ are the limits of integration that can either be constants or variables or mixed, $g(x\,,t)$ is a known function of two variables x and t called the kernel of the integral equation and $u(x)$ is unknown function.

1.1.1 Types of integral equations:

1. Fredholm integral equation

The well-known form of the *Fredholm integral equation of the second kind* is expressed as:

$$u(x) = f(x) + \lambda \int_{D} g(x,t)\, u(t)dt \qquad (1.2)$$

where D is a closed bounded set in R^m, for some $m \geq 1$

If the equation takes the form:

$$f(x) = \lambda \int_D g(x,t)\, u(t)dt \qquad (1.3)$$

it is called *Fredholm integral equation of the first kind.* For more details see [40]

2. Volterra integral equation

The standard form of the *Volterra integral equation of the second kind* is:

$$u(x) = f(x) + \lambda \int_a^x g(x,t)\, u(t)dt \qquad (1.4)$$

where the upper limit of the integral is a variable.

If the Volterra integral equation takes the form:

$$u(x) = \lambda \int_a^x g(x,t)\, u(t)dt \qquad (1.5)$$

then it is called a *Volterra integral equation of the first kind.* see [40]

3. Volterra–Fredholm integral equation

Volterra-Fredholm integral equation is a mixed of two different types of Fredholm integral equation and Volterra integral equation. The mixed Volterra-Fredholm integral equation has two forms[1,37]. For example;

$$u(x) = f(x) + \lambda_1 \int_a^x g_1(x,t)\, u(t)dt + \lambda_2 \int_a^b g_2(x,t)\, u(t)\, dt \qquad (1.6)$$

Generally speaking, Volterra integral part and its kernel $g_1(x,t)$ are measured for position while Fredholm integral part and its kernel $g_2(x,t)$are measured for the time. Moreover,Volterra-Fredholm integral equation may appears as :

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_a^b F(x,\tau)\, g(x,y)\, u(y,\tau)dyd\,\tau \qquad (1.7)$$

Definition (1.2)[10]: L^{p}-space: Let $p \in [1, \infty)$, L^{p} is the linear space of measurable functions $f: X \to R$, such that

$$\|f\|_p = \left(\int |f(x)|^p dt\right)^{1/p} < \infty.$$

When $p = 2$ we have L^2-space with

$$\|f\|_2 = \left(\int |f(x)|^2 dt\right)^{1/2}$$

1.1.2 Linearity of integral equations

Definition (1.3)[26]: if the exponent of unknown $u(x)$ function inside the integral is one the integral equation is called linear, otherwise it is called nonlinear, or if the equation contains nonlinear functions of $u(x)$, such as $e^u, sinhu, \ln(1+u)$.

Examples of linear integral equation:

$$u(x) = (1+x) + \int_0^1 (x-t)\, u(t)dt \tag{1.8}$$

$$u(x) = x + \int_0^x (x+t)\, u(t)dt \tag{1.9}$$

Examples of nonlinear integral equation:

$$u(x) = x^2 + \int_0^x (1-x+t)(u(t))^4 dt \tag{1.10}$$

$$u(x) = 1 + \int_0^1 xte^{u(t)} dt \tag{1.11}$$

1.1.3 Homogeneity of integral equation

Integral equations of second kind are classified as homogenous and inhomogeneous, if the function $f(x)$in the second kind ofFredholm or Volterra integral equation are identically zero then it is called homogenous, otherwise is called inhomogeneous[39].

Examples of homogenous integral equations:

$$u(x) = \int_0^x xt\, u(t)\; dt \tag{1.12}$$

$$u(x) = \int_0^1 (x-t)^2\, u(t)\, dt \tag{1.13}$$

Examples of inhomogeneous integral equations:

$$u(x) = \cos(x) + \int_0^x xt\, u(t) dt \tag{1.14}$$

$$u(x) = (1+x) + \int_0^1 (x-t)^4\, u(t) dt \tag{1.15}$$

1.3 Kinds of kernels

Integral equation can be classified according to its kernel [37,7]

1. Continuous kernel on interval $[\mathrm{a}, \mathrm{b}]$

$|g(x,t)| \leq \mu$where μ is constant

2. Carleman kernel

$$g(x,t) = \frac{A(x,t)}{|x-t|^{\alpha}} \quad 0 \leq \alpha < 1 \tag{1.16}$$

where $A(x,t)$ is continuous and have continuous derivative.

3. Logarithmic kernel

$$g(x,t) = A(x,t) ln|x-t| \tag{1.17}$$

4. Cauchy kernel

$$g(x,t) = \frac{A(x,t)}{|x-t|} \tag{1.18}$$

5. Abel's kernel

$$g(x,t) = \frac{A(x,t)}{(x-t)^{\alpha}} \quad 0 \leq \alpha < 1 \tag{1.19}$$

6. Degenerate kernel

The degenerate kernel $g(x,t)$ has the form

$$g(x,t) = \sum_{i=1}^{n} a_i(x)\, b_i(t) \tag{1.20}$$

7. Hilbert-Schmidt kernel

A Hilbert-Schmidt kernel $g(x,t)$: $a \le x \le b$ and $a \le t \le b$ satisfies

$$\int_a^b \int_a^b |g(x,t)|^2 dx\, dt < \infty$$

For each x in $a \le x \le b$

$$\int_a^b |g(x,t)|^2\, dt < \infty$$

For each value of t in $a \le t \le b$

$$\int_a^b |g(x,t)|^2\, dx < \infty$$

has a finite value ,then we call the kernel is *regular kernel* .

8. Hilbert kernel

$$g(x,t) = cot\left(\frac{x-t}{2}\right) \tag{1.21}$$

9. Skew –symmetric

$$g(x,t) = -g(t,x) \tag{1.22}$$

9. Difference kernel

$$g(x,t) = g(x-t)$$

and often called convolution type kernel.

1.3 SingularIntegral Equations

Definition (1.4)[34]: An integral equation is said to be singular if one of the limits of integration $r(x), h(x)$ or both become infinite,orif the kernel becomes unbounded at one or more points in the range of integration.

For example:

8

$$u(x) = f(x) + \lambda \int_{-\infty}^{\infty} e^{|x-t|}\, u(t)dt \qquad (1.23)$$

There are two types of singularities

(i) Weakly singular integral [33]kernel has the forms:

$$g(x,t) = \frac{B(x,t)}{|x-t|^{2}_{\alpha}}$$

or $$g(x,t) = B(x,t) ln|x-t|$$

$B(x,t)$ is bounded function (i.e. several times continuous and differentiable) with $B(x,t) \neq 0$ and α is a constant such that $0 < \alpha < 1$.

For example:

$$f(x) = \int_0^x \frac{1}{(x-t)^{\alpha}}\, u(t)dt\,, 0 < \alpha < 1 \qquad (1.24)$$

is a weakly singular integral equation .If $\alpha = \frac{1}{2}$, the equation

$$f(x) = \int_0^x \frac{1}{\sqrt{(x-t)}} u(t)dt$$

is called Abel's integral equation.

(ii) Strongly singular integral, the kernel takes the form:

$$g(x,t) = \frac{B(x,t)}{(x-t)^2}$$

1.4 Integral Equation with Carleman Kernel and Carleman Operator

The standard form of the Fredholm integral equation of the second kind with

a

Carleman kernel may take the form[27,31,39]

$$u(x) = f(x) + \lambda \int_a^b |x-t|^{-v}\, u(t)dt\,, 0 \leq v < 1 \qquad (1.25)$$

The Volterra integral equation with a Carleman kernel may has the form

$$u(x) = f(x) + \lambda \int_a^x |x-t|^{-v}\, u(t)dt\,, 0 \le v < 1 \qquad (1.26)$$

Definition (1.5) [**10**]The Operator: A mapping between two function spaces $A: X \to Y$ which assigns to every function $f \in X$ a function $Af \in Y$ is called operator.

Definition (1.6) [**10**]Measurable function:A function $f: X \to R$ is measurable if, for every real number$a$, the set:

$$\{x \in X: f(x) > a\}$$

is measurable.

When X=R with Lebesgue measure, or more generally any Borel measure, then all continuous functions are measurable. In fact, practically any function that can be described is measurable. Measurable functions are closed under addition and multiplication, but no composition.

Definition (1.7)[**10**]**:** inner product space

Let V be a linear space over $K = R$ or C. An inner product $\langle .,. \rangle$ is a function from $V \times V$ to $K$with the following properirtes:

1. For any $u \in V$,$\langle u,u \rangle \ge 0$ and $\langle u,u \rangle = 0$ if and only if $u = 0$.
2. For any $u, v \in V$, $\langle u,v \rangle = \overline{\langle v,u \rangle}$.
3. For any $u, v, w \in V$ and any $\alpha, \beta \in K$,$\langle \alpha u + \beta v, w \rangle =$

$$\alpha\langle u,w \rangle + \beta\langle v,w \rangle.$$

The space V together with the inner product$\langle .,. \rangle$ is called inner product space.

Definition (1. 8)[10]: Hilbert space (H)**:** It is a complete inner product space.

Definition (1.9)[39]: Carleman operator is defined as a densely operator G in the Hilbert spaceL^2, which allows the representation

$$Gu(x) = \int g(x,t)u(t)dt \qquad\qquad almost\ everywhere\ in\ X$$

This domain of G consists of all elements $u(x) \in L^2(a,b)$ such that $\int g(x,t)u(t)$ represents the element of $L^2(a,b)$.

Also, an operator G on the space $L^2(a,b)$ is called a Carleman operator if and only if :

$|Gu(x)| \leq k(x)$ for all f in the domain G and $\|u\| \leq 1$ wher k is measurable.

Types of Carleman kernels[38, 27]:

1. semi-Carleman kernel:

If one of these two conditions is satisfied the kernel is called semi-Carleman

$$\int_a^b |g(x,t)|^2\, dt < \infty\ almost\ every\ where\ in\ x \tag{1.27}$$

$$\int_a^b |g(x,t)|^2\, dx < \infty\ almost\ every\ wehere\ in\ t \tag{1.28}$$

2. bi-Carleman kernel:

if both conditions (1.27) and (1.28) are satisfied then the kernel it is called bi- Carleman.

3 Strong Carleman kernel:

An operator in a Hilbert space is a Hilbert-Schmidt operator ($\text{i.e.} \sum \|X\emptyset_n\|^2 < \infty$ for $\{\emptyset_n\}$ is complete orthonormal), and every Hilbert-Schmidt operator is a strong Carleman operator.

Remark (1.1)[39]: An operator in $L^2(a,b)$ is a strong Carleman operator if and only if it is a Hilbert-Schmidt operator.

Theorem (1.1)[38]: If A is a strong Carleman operator and B is bounded, then AB and BA are strong Carleman operators.

1.5 The Existence of a Unique Solution

Theorem(1.2)[17]: (the existence of unique solution)

To test the existence and the uniqueness of a solution to the Fredholm integral equation with Carleman kernel we consider the following theorem:

(i). The kernel satisfies the Fredholm condition

$$\left(\iint_a^b g^2(|x-t|)dxdt\right)^{1/2} = \mu \qquad \mu \text{ is constant} \qquad (1.29)$$

(ii). The given function $f(x)$, with it is first derivatives, it is continuous in $L^2[a,b]$ and its norm is defined as

$$|f| = \left(\int_a^b f^2(x)dx\right)^{1/2} = N \qquad N \text{ is constant} \qquad (1.30)$$

(iii). The unknown function $u(x)$ behaved as the known function $f(x)$ in $L^2[a,b]$.

To prove the existence of the solution we use the Picard method, for this we construct a sequence function $u_n(x)$ defined by

$$u_n(x) = f(x) + \lambda \int_a^b g(x-t)u_{n-1}(t)dt\, u_0(x) = f(x) \quad (1.31)$$

For ease the manipulation, it is convenient to introduce

$$\phi_n(x) = u_n(x) - u_{n-1}(x) \quad (1.32)$$

$$\phi_n = \lambda \int_a^b g(|x-t|)[u_{n-1}(t) - u_{n-2}(t)dt] \qquad n = 1,2,\dots$$

then, we have

$$\phi_n = \lambda \int_a^b g(|x-t|)\phi_{n-1}(t)dt \quad (1.33)$$

and we can deduce that

$$u_n(x) = \sum_{i=1}^{n} \phi_i\,(x) \quad (1.34)$$

using the property of the norm

$$\|\phi_n(x)\| = |\lambda| \left\| \int_a^b g(x-t)\phi_{n-1}(t)dt \right\| \quad (1.35)$$

we get by induction

$$\|\phi_n(x)\| \leq N(\mu\lambda)^n$$

This bounds makes the sequence$\phi_n(x)$ converges: so that when $n \to \infty$

$$u(x) = \lim_{n\to\infty} u_n\,(x) = \sum_{i=1}^{\infty} \phi_i\,(x) \leq \frac{N}{1-\mu\lambda}.$$

Theorem $(\mathbf{1.3})$: $[\mathbf{11, 25}]$ (Fredholm alternative theorem)

If the homogenous Fredholm integral equation has only trivial solution $u(x) = 0$, then the corresponding inhomogeneous integral equation has only a unique solution .

Theorem (1.4)[24]: To test the existence and the uniqueness of a solution to the Volterra integral equation with Carleman kernel we consider the following theorem:

(i). The kernel of Volterra integral equation belong to

$$c[0,T], 0 \leq t \leq T \leq \infty$$

and satisfy the condition

$$|g(x,t)| \leq A \qquad \text{where} A\ is\ constant \tag{1.36}$$

(ii). The given function $f(x)$ with its partial derivatives with respect to x are continuous in $L^2(\Omega)$.

(iii). The unknown function $u(x)$ in the space $L^2(\Omega)$, satisfy the Lipchitz condition with respect to position

$$|u(x_1) - u(x_2)| \leq A(t)|x_1 - x_2|, A(t) \leq c, \qquad c\ is\ constant.$$

and Hōlder condition with respect to time

$$|u(t_1) - u(t_2)| \leq B(t)(|t_1 - t_2|)^{-v}, B(t) \leq c, c\ is\ constant.$$

Chapter Two

Numerical Techniques for Solving Integral Equation with a Carleman Kernel of the Second Kind

In this chapter we introduce some important numerical techniques for solving integral equation of the second kind with a Carleman kernel.

2.1 Theoretical Frameworks

2.1.1 Lagrange Interpolation polynomial[9, 10].

Let u be a continuous function defined on a finite closed interval$[a, b]$. Furthermore, let

$$\mu: a \leq x_0 < x_1 < \cdots < x_n \leq b$$

be a partition of the interval$[a, b]$,choose $V = C[a, b]$be the space of the continuous function u is $u: [a, b] \to U$(which is complex or real), and choose V_{n+1} to be P_n, the space of polynomials of degree less than or equal to n. Then the Lagrange interpolate of degree n of u is defined by the conditions:

$$p_n(x_i) = u(x_i) \qquad 0 \leq i \leq n \qquad p_n \in P_n \tag{2.1}$$

here, the interpolation linear functional are [22]:

$$L_i\, u = u(x_i) \qquad 0 \leq i \leq n \tag{2.2}$$

If we choose the monomials $v_j(x) = x_j, 0 \leq j \leq n$ as the basis of P_n then it is shown that

$$\det(L_i v_j)_{(n+1)\times(n+1)} = \prod_{j>i}(x_j - x_i) \neq 0$$

Thus there exists a unique Lagrange interpolation polynomial.

Furthermore, we have the representation

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\varphi_i\,(x), \varphi_i(x) = \prod_{j\neq i} \frac{x - x_j}{x_i - x_j} \tag{2.3}$$

called Lagrange formula for interpolation polynomial.

where $\varphi_i(x)$ satisfies the condition:

$$\varphi_i(x_i) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{2.4}$$

The functions $\{\varphi_i\}_{i=0}^{n}$ form a basis for P_n , and they are often called Lagrange basis functions.

Theorem (2.1)[15]: If the function $u(x)$ has $n+1$ continuous derivatives on the interval $[a, b]$, and $p(x)$ is the Lagrange interpolation polynomial, then for all $x_n \in [a, b]$ there exists $z(x) \in (a, b)$ such that:

$$u(x) = p(x) + \frac{u^{(n+1)}\big(z(x)\big)}{(n+1)!}(x - x_1)(x - x_2) \dots (x - x_n) \tag{2.5}$$

Theorem (2.2): The following statements are equivalent:

1. The interpolating problem has a unique solution.
2. The functions $L_1, L_2, \dots, L_n$ are linearly independent.
3. The only element $u_n \in X_n$ satisfying the condition

$$u_n L_i = 0 \qquad 0 \leq i \leq n \tag{2.6}$$

is $u_n = 0$.

For the proof see [15].

2.1.2 Projection method

With all projection methods, we consider solving integral equation within the complete function space, usually $C(D)$ or $L^2(D)$ space, we choose a sequence of finite dimensional approximation subspace $X_n \subseteq X$,$n \geq 1$ where X_n has a dimension K_n[33]. Suppose that X_n has a basis $\{w_1, \dots, w_k\}$ with $k = K_n$, we want to find a function $u_n \in X_n$ which can be written as[26]:

$$u_n(x) = \sum_{j=1}^{K_n} b_j\, w_j(x) \qquad , \qquad x \in D \tag{2.7}$$

To find the coefficients $\{b_1, \dots, b_k\}$, we use a forcing equation which has an exact solution in some sense, the substitution in equation (1.25) gives us:

$$r_n = u_n(x) - \lambda \int |x-t|^{-v} u_n(t)dt - f(x) \tag{2.8}$$

$$r_n = \sum_{j=1}^{K_n} b_j \left\{ w_j(x) - \lambda \int |x-t|^{-v} u_n(t)dt \right\} - f(x) \tag{2.9}$$

This is called the residual in the approximation of the equation when used $u \approx u_n$. Equation (1.25) can be written in the operator form as:

$$(I - \lambda G)u = f \tag{2.10}$$

The residues can be written as:

$$r_n = (I - \lambda G)u_n - f \tag{2.11}$$

We assume $r_n(x)$ to be approximately zero in some sense to find the coefficients$\{b_1, \dots, b_k\}$.

Projection operator

Definition (2.1)[15]: Let A be linear space, A_1 and A_2 subspaces of A , we say A is direct sum of A_1 and A_2 and write $\mathrm{A} = \mathrm{A}_1 \oplus \mathrm{A}_2$ if any element of $a \in A$ can be decomposed as:

$$a = a_1 + a_2 \ , a_1 \in \mathrm{A}_1 \text{ and } , a_2 \in \mathrm{A}_2.$$

Proposition (2.1): Let A be linear space, then $\mathrm{A} = \mathrm{A}_1 \oplus \mathrm{A}_2$ if and only if there is a linear operator$P: A \to A$ with $P^2 = P$, such that $a_1 = Pa$ and $a_2 = (1 - \mathrm{P})a$. Also,$\mathrm{A}_1 = P(\mathrm{A})$ and $\mathrm{A}_2 = (1 - P)(\mathrm{A})$.

Definition (2.2): Let A be a Banach space, an operator $P \in \mathcal{L}(A)$ with the property $P^2 = P$ is called projection operator. The subspace $P(A)$is called the corresponding projection space.

Properties and classification:

Let Y be a finite dimensional vector space and P is a projection on W. Suppose the subspaces$Q$ and L are the range and kernel of P respectively. Then P has the following main properties[10]:

1. $P$is idempotent *(i.e.*$P^2 = P$*).*
2. P is the identity operator Ion Q.
3. If we assume that $Y = Q \oplus L$ this called the direct sum of subspace, also every vector $x \in Y$may decomposed uniquely as:

$x = q + l$ with$q = Px$ and $l = x - Px$, and where $q \in Q, l \in L$.

2.1.3 Composite Trapezoidal Rule

Trapezoidal rule is an average of the left and right hand sum of Riemann integration. Moreover, this technique is used to approximate the area beneath a curve.

Let $a < b \in R$, and we divide the interval into subintervals with [15]

$$h = \frac{b - a}{N}$$

so,

$$\int_a^b f(x)dx = h\left[\frac{f(a)+f(b)}{2} + \sum_{i=2}^{N-1} f(x_i)\right] \tag{2.12}$$

This formula is called the trapezoidal rule since if f is a function of positive values, $\int_a^b f(x)dx$ is approximated by the area in the trapezoid.

The error is given by:

$$E = -\frac{h^2}{12} f^{(2)}\big(z(x)\big) \tag{2.13}$$

2.1.4 Composite Simpson's Rule

Simpsons' rule results from integration over $[a,b]$of the second Lagrange polynomial with nodes $x_0 = a, x_1 = b, x_2 = a + h$

where,

$$h = \frac{b-a}{n} \text{and } x_j = a + jh, j = 0,1,2,\dots,n \tag{2.14}$$

The Simpson's rule is given as[15]:

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(b)\right] \tag{2.15}$$

the error term:

$$E = \frac{-(b-a)h^4}{180} f^{(4)}\big(z(x)\big) \quad where\ z \in (a,b) \tag{2.16}$$

Since the error term involves the fourth derivative of f, Simpson's rule give exact result when apply to any polynomial of degree three or less.

2.1.5 Laplace transform

Definition (2.3)[40]: let $u(x)$ be a function defined for $x \geq 0$, then the Laplace transform of $u(x)$, denoted by $L\big(u(x)\big)$ is:

$$U(s) = L\big(u(x)\big) = \int_0^\infty e^{-sx}\, u(x) dx \qquad (2.17)$$

where *s* is real.

Existence of the Laplace Transform

The conditions for the existence of the Laplace transform $U(s)$ of a function $u(x)$ are:

1. $u(x)$ is piecewise continuous on the interval of integration

   $0 \leq x < B$ for any positive B.

2. $u(x)$ is of exponential order e^{rx} as $x \to \infty$, i.e. $|u(x)| \leq Ae^{rx}, x \geq W$

where *r* is real constant, and A and W are positive constants. Accordingly, the Laplace transform $U(s)$ exist and must satisfy

$$\lim_{s\to\infty} U(s) = 0.$$

Moreover, all polynomials, simple exponentials, sine and cosine functions, and products of these functions are of exponential order. The function without an exponential order such as e^{x^2} grow too rapidly. The integral

$$\int_0^\infty e^{-xs}\, e^{x^2} dx \qquad (2.19)$$

does not converge for any value of s[40,41].

2.2 Numerical Techniques for Solving Integral Equation with a Carleman Kernel

Several numerical techniques for solving integral equations will be presented. These are:

2.2.1 Product Nystrom Method

To use the product Nystrom method as a numerical technique[2,9,16], we consider the integral equation

$$u(x) = f(x) + \lambda \int g(|x-t|)\, u(t)dt \tag{2.20}$$

so the kernel has the form:

$$g(x,t) = |x-t|^{-v} where \quad 0 \le v < 1 \tag{2.21}$$

we often factor out the singularity in g by writing

$$g(x,t) = p(x,t)\tilde{g}(x,t) \tag{2.22}$$

where $p(x,t)$ is "badly behaved" function and $\tilde{g}(x,t)$ is "well behaved "function. By a suitable Lagrange interpolation polynomial. Equation (2.20) can be written as:

$$u(x_i) = f(x_i) + \lambda \int \sum_{j=1}^{N} w_{ij} g\left(x_i, t_j\right) u\left(t_j\right) \tag{2.23}$$

where:

$x_i = t_i = a + ih$ and $i = 0,1,2,\dots,N$ with $h = (b-a)/N$ and N is even and $w_{ij}$are the weights which can be determined directly.

Moreover, we approximate the integral terms by a product integration using Simpson's rule, where $x = x_i$

$$\int p(x_i,t)\,\tilde{g}(x_i,t)u(t)dt = \sum_{j=0}^{\frac{N-2}{2}} \int_{t_{2j}}^{t_{2j+2}} p\left(x_{i,}t\right)\tilde{g}\,(x_i,t_i)u(t)dt \quad (2.24)$$

Hence we get

$$\sum_{j=1}^{N} w_{ij}g\left(x_i,t_j\right)u\left(t_j\right) = \sum_{j=0}^{\frac{N-2}{2}} \int_{t_{2j}}^{t_{2j+2}} p\left(x_{i,}t\right)\tilde{g}\,(x_i,t_i)u(t)dt \qquad (2.25)$$

if we approximate the nonsingular part $\tilde{g}(x,t)u(t)$ by integration over the interval $\left[t_{2j},t_{2j+2}\right]$ using the second degree of Lagrange interpolation polynomial with the pointes t_{2j},t_{2j+1}, t_{2j+2} we get

$$\begin{aligned}
&\int_a^b p(t_i,t)\tilde{g}\,(t_i,t)u(t)dt \\
&\approx \sum_{j=0}^{(N-2)/2} \int_{t_{2j}}^{t_{2j+2}} p(x_{i,}t)\left\{\frac{\left(t_{2j+1}-t\right)\left(t_{2j+2}-t\right)}{2\,h^2}\tilde{g}\left(t_i,x_{2j}\right)u\left(t_{2j}\right)\right. \\
&\qquad + \frac{\left(t-t_{2j}\right)\left(t_{2j+2}-t\right)}{h^2}\tilde{g}\left(t_i,t_{2j+1}\right)u\left(t_{2j+1}\right) \\
&\qquad \left. + \frac{\left(t-t_{2j}\right)(t-t)}{2\,h^2}\tilde{g}\left(t_i,t_{2j+2}\right)u\left(t_{2j+2}\right)dt\right\} \\
&\qquad = \sum_{j=0}^{N} w_{ij}\,\tilde{g}\left(t_i,t_j\right)u\left(t_j\right)(2.26)
\end{aligned}$$

where

$$w_{i,0} = \frac{1}{h^2}\int_{t_0}^{t_2} p(t_i,t)(t_1-t)(t_2-t)dt \qquad (2.27)$$

$$w_{i,2j+1} = \frac{1}{h^2}\int_{t_{2j}}^{t_{2j+1}} p(t_i,t)\left(t_{2j}-t\right)\left(t_{2j+2}-t\right)dt \qquad (2.28)$$

$$\begin{aligned}
w_{i,2j} &= \int_{t_{2j}}^{t_{2j+1}} p(t_i,t)\left(t_{2j+1}-t\right)\left(t_{2j+2}-t\right)dt \\
&\quad + \frac{1}{2h^2}\int_{t_{2j}-2}^{t_{2j}} p(t_i,t)\left(t_{2j-2}-t\right)\left(t_{2j-1}\right. \\
&\quad \left.-\,t\right)dt \qquad (2.29)
\end{aligned}$$

$$w_{i,N} = \frac{1}{2h^2}\int_{N-2}^{t_N} p(t_i,t)(t_{N-2}-t)(t_{N-1}-t)dt \qquad (2.30)$$

or

$$w_{i,0} = \beta_j(t_i),\ w_{i,2j+1} = 2\gamma_{j+1}(t_i)\ \text{,and}\ w_{i,2j} = \alpha_j(t_i + \beta_{j+1}(t_i))$$
$$,w_{i,N} = \alpha_{\frac{N}{2}}$$

Therefore:

$$\alpha_j(t_i) = \frac{1}{2h^2}\int_{N-2}^{t_N} |t_i - t|^{-v}(t - t_{2j-2})\big(t - t_{2j-1}\big)dt \qquad (2.31)$$

$$\beta_j(t_i) = \frac{1}{2h^2}\int_{t_{2j-2}}^{t_{2j}} |t_i - t|^{-v}\,(t_{2j-2} - t)\big(t_{2j} - t\big)dt \qquad (2.32)$$

$$\gamma_j(t_i) = \frac{1}{2h^2}\int_{t_{2j-2}}^{t_{2j}} |t_i - t|^{-v}\,(t - t_{2j-2})\big(t_{2j} - t\big)dt \qquad (2.33)$$

We introduce the change of variable

let $t = t_{2j-2} + \mu h$ where $0 \le \mu \le 2$, equations (2.31),(2.32),(2.33)

becomes

$$\alpha_j(t_i) = \frac{h^{1-v}}{2}\int_0^2 \mu(\mu-1)|i-2j+2-\mu|^{-v}\,d\mu \qquad (2.34)$$

$$\beta_j(t_i) = \frac{h^{1-v}}{2}\int_0^2 (1-\mu)(2-\mu)|i-2j+2-\mu|^{-v}\,d\mu \qquad (2.35)$$

$$\gamma_j(t_i) = \frac{h^{1-v}}{2}\int_0^2 \mu(2-\mu)|i-2j+2-\mu|^{-v}\,d\mu \qquad (2.36)$$

Now, we define:

$$\varphi_i(z) = \int_0^2 \mu^i|z-\mu|^{-v}\,d\mu \quad i = 0,1,2 \qquad (2.37)$$

where $z = i - 2j + 2$

This implies that:

$$\alpha_j(t_i) = \frac{h^{1-v}}{2}[\varphi_2(z) - \varphi_1(z)] \tag{2.38}$$

$$\beta_j(t_i) = \frac{h^{1-v}}{2}[\varphi_0(z) - 3\varphi_1(z) + \varphi_2(z)] \tag{2.39}$$

$$\gamma_j(t_i = \frac{h^{1-v}}{2}[\varphi_1(z) - \varphi_2(z)] \tag{2.40}$$

so we get:

$$\omega_{i,0} = \frac{h}{2}[2\varphi_0(z) - 3\varphi_1(z) + \varphi_2(z)], \qquad z = i \tag{2.41}$$

$$\omega_{i,2j+1} = h[2\varphi_1(z) - \varphi_2(z)], \qquad z = i - 2j \tag{2.42}$$

$$\omega_{i,2j} = \frac{h}{2}[\varphi_2(z) - \varphi_1(z) + 2\varphi_0(z-2) - 3\varphi_1(z-2) + \varphi_2(z-2)], \qquad z = i - 2j + 2 \tag{2.43}$$

$$\omega_{i,N} = \frac{h}{2}[\varphi_2(z) - \varphi_1(z)], \qquad z = i - N + 2 \tag{2.44}$$

Hence, equation (2.20) yields a system of linear algebraic equations

$$(I - \lambda W)U = F \tag{2.45}$$

solution of this system can be obtained directly as

$$U = [I - \lambda W]^{-1}F \qquad |I - \lambda W| \neq 0. \tag{2.46}$$

where I is a unit matrix.

Theorem (2.3)[2]: The product Nystrom method is said to be convergent of order r in $[a, b]$ if and only if, for N sufficiently large there exists $C > 0$, independent of N such that:

$$\|u(x) - u_n(x)\|_\infty \leq CN^{-r}. \tag{2.47}$$

2.2.2 Toeplitz Matrix Method

In this section, we present Toeplitz matrix method [6] to obtain numerical solution for Fredholm integral equation of the second kind with Carleman kernel.

The idea of this method is to obtain a system $2N+1$ linear algebraic equations, where $2N+1$ is the number of discrimination points. The coefficient matrix is expressed as sum of two matrices, on of them is the Toeplitz matrix and the other is a matrix with zero elements except the first and last column, see[2,30,32].

Consider the following Fredholm integral equation with discontinuous kernel

$$u(x) = f(x) + \lambda \int_{-a}^{a} g(|x-t|)\, u(t)dt \quad where \quad 0 \leq v < 1 \qquad (2.48)$$

The integral term in (2.48) can be written as

$$\int_{-a}^{a} |x-t|^{-v}\, u(t)dt = \sum_{n=-N}^{N-1} \int_{nh}^{nh+h} |x-t|^{-v}\, u(t)dt \quad ,h = \frac{a}{N} \qquad (2.49)$$

and the kernel is:

$$g(x,t) = |x-t|^{-v} \qquad 0 \leq v < 1 \qquad (2.50)$$

we approximate the integral in the right hand side of (5.49):

$$\int_{nh}^{nh+h} |x-t|^{-v} u(t)dt = A_n(x)\phi(nh) + B_n(x)\phi(nh+h) + R_n \qquad (2.51)$$

Where $A_n$and B_n are two arbitrary functions to be determined andR_n is the error. Putting $u(t) = 1, t$ in (2.51) yields a set of two equations in terms of the two functions A_n and B_n . If R_n is assumed negligible we can clearly solve this set of equations A_n and B_n, then we obtain

$$A_n(x) = \frac{1}{h}\big((nh+h)I(x) - J(x)\big) \qquad (2.52)$$

and

$$B_n(x) = \frac{1}{h}\big(J(x) - nh\, I(x)\big) \qquad (2.53)$$

$I(x)$ and $J(x)$ have the following forms:

$$I(x) = \int_a^{a+h} |x-t|^{-v}\, dt = A_n(x) + B_n(x) \qquad (2.54)$$

and

$$J(x) = \int_a^{a+h} t\, |x-t|^{-v}\, dt = aA_n(x) + (a+h)B_n(x) \qquad (2.55)$$

then the system of (2.49) becomes:

$$\int_a^a |x-t|^{-v}\, u(t)dt = \sum_{n=-N}^{N-1} [A_n(x)u(nh) + B_n(x)u(nh+h)] \qquad (2.56)$$

$$= \sum_{n=-N}^{N-1} A_n(x)u(nh) + \sum_{n=-N+1}^{N} B_{n-1}(x)u(nh)$$

$$= \sum_{n=-N}^{N} D_n(x)u(nh) \qquad (2.57)$$

where:

$$D_n(x) = \begin{cases} A_N(x) & n = N \\ A_n(x) + B_{n-1}(x) & -N < n < N \\ B_{-N-1}(x) & n = -N \end{cases}$$

thus, the integral equation (2.48) becomes

$$u(x)-\lambda\sum_{n=-N}^{N} D_n\,(x)u(nh)$$

$$= f(x) \qquad (2.58)$$

If we put $x = mh$ we get the following system algebraic equation:

$$u(mh)-\lambda\sum_{n=-N}^{N} D_{m,n}\,u(nh)$$

$$= f(mh) \qquad (2.59)$$

the matrix $D_{m,n}$may be written as$D_{m,n} = H_{m,n} - E_{m,n}$, where

$$H_{m,n}$$

$$= A_n(mh) + B_{n-1}(mh) \qquad (2.60)$$

is a Toeplitz matrix of order $2N + 1$ and

$$E_{m,n} = \begin{cases} B_{-N-1}\,(mh) & n = -N \\ 0 & -N < n < N \\ A_N(mh) & n = N \end{cases}$$

represents a matrix of order $2N + 1$ whose elements are zero except the first and last columns.

However, the solution of the system of equation (2.59) can be obtained in the form

$$u(mh) = [I - \lambda(H_{mn} - E_{mn})]^{-1} f(mh)$$

Where I is identity matrix and $|I - \lambda(H_{mn} - E_{mn})| \neq 0$.

Theorem (2.4)[**2**] : The Toeplitz matrix method is said to be convergent of order r in $[-a, a]$ if, for N sufficiently large, there exists a constant $D > 0$ independent of N such that:

$$\|u(x) - u_N(x)\| \leq DN^{-r} \quad . \qquad (2.61)$$

the error term R is determined form (2.51) by letting $u(t) = t^2$ to get

$$R_n = \left| \int_{nh}^{nh+n} t^2 |x-t|^{-v} dt - A_n(x)(nh)^2 - B_n(x)(nh+h)^2 \right| = \quad O(h^3)$$

2.2.3 Laplace Adomian Decomposition Method

The Adomian decomposition method consists of decomposing the unknown function $u(x)$ into sum of an infinite number of components defined by the decomposition series[23,40]

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots$$

where the components $u_n(x), n \geq 0$ are to be determined in recursive manner.

To establish the recurrence relation we substitute into Volterra integral equation(1.4) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \int_0^x g(x,t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt \qquad (2.62)$$

Usually the component $u_0(x)$ is defined by all terms that are not included under the integral side. Also, other components of the unknown function are completely determined by the recurrence relation

$$u_0(x) = f(x)$$
$$u_{n+1}(x) = \lambda \int_0^x g(x,t) u_n(t) dt \qquad n \geq 0$$

The traditional Laplace transforms cannot be used to solve Volterra integral equation with Carleman kernel , so we use combining between the Laplace transforms and Adomian decomposition method.

Consider Volterra integral equation with Carleman kernel

$$u(x) = f(x) + \lambda \int_0^x |x-t|^{-v} u(t)dt \qquad (2.63)$$

applying Laplace transformation to both sides of (2.63)

$$U(s) = F(s) + \lambda\, L\{|x-t|^{-v}\} * L\{u(t)\} \qquad (2.64)$$

Adomian decomposition method can be used in (2.64) as

$$U(s) = \sum_{n=0}^{\infty} U_n(s) \qquad (2.65)$$

Then,we get

$$\sum_{n=0}^{\infty} U_n(s) = G(s) + \lambda\, L\{|x-t|^{-v}\} \sum_{n=0}^{\infty} U_n(s) \qquad (2.66)$$

the Adomian decomposition method introduces the recursive relation:

$$U_0(s) = G(s) \qquad (2.67)$$

$$U_{n+1}(s) = \lambda[s^{v-1}\Gamma(1-v)]\{U_n(s)\} \qquad n \geq 1 \qquad (2.68)$$

take the inverse Laplace transformation of both side of (2.68) to obtain $u_0(x), u_1(x), u_2(x), \ldots$

2.2.4 Sinc-Collocation Method

The sinc approximation with single exponential (SE) and double exponential (DE) are considered as one of the efficient methods to solve this type of integral equations [6,12,13,18].

Sinc approximation for a function y is expressed as:

$$y(x) = \sum_{j=-N}^{N} y(jh)S(jh)(x) \qquad x \in R \qquad (2.69)$$

where the sinc-function is given by

$$S(jh)(x) = \frac{\sin \pi(\frac{x}{h} - j)}{\pi(\frac{x}{h} - j)} \qquad (2.70)$$

To construct approximation on the interval $L = (a, b)$ we consider $tanh$ transformation and its inverse

$$t = \varphi^{SE}(x) = \frac{b-a}{2}\tanh\left(\frac{x}{2}\right) + \frac{b+a}{2} \tag{2.71}$$

$$x = \{\varphi^{SE}(t)\}^{-1} = \log\left(\frac{t-a}{b-t}\right) \tag{2.72}$$

interpolation formula for $\mathrm{y(t)}$ over $(\mathrm{a,b})$ is

$$y(t) = \sum_{j=-N}^{N} y(\varphi^{SE}(jh))S(jh)\{\varphi^{SE}\}^{-1}(t) \tag{2.73}$$

Theorem (2.5)[13]: Let N be a positive integer and $0 < d < \pi$ then h is given as

$$h = \sqrt{\frac{\pi d}{\alpha\, N}} \tag{2.74}$$

where $\alpha = 1 - v$.

Also, double exponential transformation can be use

$$t = \varphi^{DE}(x) = \frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh(x)\right) + \frac{b+a}{2} \tag{2.75}$$

$$x = \{\varphi^{DE}(t)\}^{-1} = \log\left(\frac{1}{\pi}\log\left(\frac{t-a}{b-t}\right) + \sqrt{1 + \left(\frac{1}{\pi}\log\left(\frac{t-a}{b-t}\right)\right)^2}\right) \tag{2.76}$$

Theorem (2.6)[13]: For double exponential transformation,let N be a positive integer and $0 < d < \frac{\pi}{2}$ then h is given as

$$h = \frac{\log\left(\frac{2dN}{(1-v)}\right)}{N} \quad . \tag{2.77}$$

the sinc-function is required to be zero at the end points, $t = a$ and $t = b$, which seems to be an impractical assumption. In order to handle more general cases, we introduce the translation function[30]:

$$\Gamma[y](t) = y(t) - \left[\left(\frac{b-t}{t-a}\right)y(a) + \left(\frac{t-a}{b-t}\right)y(b)\right] \tag{2.78}$$

Single Exponential – Sinc Schemes

the function can be accurately approximated as

$$\Gamma[u](t) \approx \sum_{j=-N}^{N} \Gamma[u]\left(\varphi^{SE}(jh)\right) S(jh)\{\varphi^{SE}(t)\}^{-1} \tag{2.79}$$

so, the approximate solution $u(x)$ is considered as

$$P_N^{SE}[u](t) = u(a)w_a(t) + \sum_{j=-N}^{N} \Gamma[u]\left(\varphi^{SE}(jh)\right) S(jh)\{\varphi^{SE}(t)\}^{-1} + u(b)w_b(t) \tag{2.80}$$

where

$$w_a(t) = \frac{b-t}{b-a} \tag{2.81}$$

and

$$w_b(t) = \frac{t-a}{b-a} \tag{2.82}$$

where h is given in (2.74).

The unknown coefficient should be determined. For this purpose, let us approximate the solution as

$$u_N^{SE} = c_{-N-1}w_a(t) + \sum_{j=-N}^{N} \mathrm{c_j}\left(\varphi^{SE}(jh)\right) S(jh)\{\varphi^{SE}(t)\}^{-1} + c_{N+1}w_b(t) \tag{2.83}$$

There are $2N + 3$ unknown coefficients on the right hand side of (2.83) that should be determined.

In order to determine the unknowns, we apply the collocation method, by setting $t = t_i^{SE}$

$$u_N^{SE}\left(t_i^{SE}\right) = f\left(t_i^{SE}\right) + \int \left|t_i^{SE} - x\right|^{-v} u_N^{SE}(x)dx \qquad (2.84)$$

where $t = t_i^{SE}$, $i = -N-1, -N, \dots, N, N+1$

$$\text{Also, } t_i^{SE} = \begin{cases} a & (i = -N-1) \\ \varphi^{SE}(ih) & (i = -N, \dots, N) \\ b & (i = N+1) \end{cases}$$

since the sinc-collocation method does not allow any singularity in the interval, we split the integral in (2.84) into two integrals to remove the singularity using $x = t_i^{SE}$

$$\int \left|t_i^{SE} - x\right|^{-v} u_N^{SE}(x)dx$$

$$= \int^{t_i^{SE}} (t_i^{SE} - x)^{-v} u_N^{SE}(x)d + \int_{t_i^{SE}} \left(x - t_i^{SE}\right)^{-v} u_N^{SE}(x)dx \qquad (2.85)$$

The first integral can accurately be approximated by $A_N^{SE}[u_N^{SE}]\left(t_i^{SE}\right)$, the operator A_N^{SE} is:

$$A_N^{SE}[y](t) = (t-a)^{1-v} h \sum_{m=-M}^{N} \frac{y\left(\varphi_{a,t}^{SE}(mh)\right)}{(1+e^{-mh})(1+e^{mh})^{1-v}} \qquad (2.86)$$

where N is a positive integer, h as expressed in (2.78) and $M\lceil(1-v)N\rceil$.

The second integral is given by :

$$B_N^{SE}[y](t) = (b-t)^{1-v} h \sum_{m=-M}^{N} \frac{y\left(\varphi_{t,b}^{SE}(mh)\right)}{(1+e^{mh})(1+e^{-mh})^{1-v}} \qquad (2.87)$$

if we introduce $K_N^{SE} = A_N^{SE} + B_N^{SE}$, we obtain the linear system

$$(E_n^{SE} - K_n^{SE})c_n = f_n^{SE} \qquad (2.88)$$

Where

$$c_n = [c_{-N-1}, c_{-N}, \dots, c_N, c_{N+1}]^T \qquad (2.89)$$

$$f_n^{SE} = [f(a), f(t_{-N}^{SE}), \dots, f(t_N^{SE}), f(b)]^T \qquad (2.90)$$

$$E_n^{SE}=\begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ w_a(t_{-N}^{SE}) & 1 & & 0 & w_b(t_{-N}^{SE}) \\ \vdots & & \ddots & & \vdots \\ w_a(t_N^{SE}) & 0 & & 1 & w_b(t_{-N}^{SE}) \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

$$K_n^{SE}=\begin{bmatrix} B_N^{SE}[w_a](a) & \cdots & B_N^{SE}[S(jh)\{\varphi^{SE}(.)\}^{-1}](a) & \dots & B_N^{SE}[w_b](a) \\ K_N^{SE}[w_a](t_{-N}^{SE}) & \cdots & K_N^{SE}[S(jh)\{\varphi^{SE}(.)\}^{-1}](t_{-N}^{SE}) & \dots & K_N^{SE}[w_b](t_{-N}^{SE}) \\ \vdots & & \vdots & & \vdots \\ K_N^{SE}[w_a](t_N^{SE}) & \dots & K_N^{SE}[S(jh)\{\varphi^{SE}(.)\}^{-1}](t_N^{SE}) & \dots & K_N^{SE}[w_b](t_N^{SE}) \\ A_N^{SE}[w_a](b) & \dots & A_N^{SE}[S(jh)\{\varphi^{SE}(.)\}^{-1}](b) & \dots & A_N^{SE}[w_b](b) \end{bmatrix}$$

where n has a dimension $2N+3$. By finding c_n we can approximate the solution u_N^{SE} using (2.84).

Double Exponential-Sinc Schemes

if we use double exponential we get the following[33]:

$$t_i^{DE}=\begin{cases} a & (i=-N-1) \\ \varphi^{DE}(ih) & (i=-N,\dots,N) \\ b & (i=N+1) \end{cases}$$

The operator A_N^{DE} is given as

$$A_N^{DE}[y](t)=(t-a)^{1-v}h\sum_{m=-M}^{N}\frac{y\left(\varphi_{a,t}^{DE}(mh)\right)\pi\cosh(mh)}{(1+e^{-\pi\sinh(mh)})(1+e^{\pi\sinh(mh)})^{1-v}}.$$

where N is a positive integer, h as expressed in (2.77) and ,

$$M=N+\lceil\log((1-v)/h\rceil.$$

The second operator $B_N^{DE}[y](t)$is

$$B_N^{DE}[y](t)=(b-t)^{1-v}h\sum_{m=-M}^{N}\frac{y\left(\varphi_{t,b}^{DE}(mh)\right)\pi\cosh(mh)}{(1+e^{\pi\sinh(mh)})(1+e^{-\pi\sinh(mh)})^{1-v}}\ .$$

Chapter Three

Numerical examples for Fredholm integral equations with a Carleman kernel

In this chapter, three numerical techniques are considered to solve Fredholm integral equation of the second kind with a Carleman kernel. These techniques are Toeplitz matrix, Product Nystrom and Sinc-collocation methods. The algorithm of each technique will be implemented and Matlab software is used to solve some numerical examples.

Example (3.1):

Consider the Fredholm integral equation of second kind with a Carleman kernel:

$$u(x) = (1-x^2)^{\frac{3}{4}} - \frac{\pi}{2\sqrt{2}}(2-x^2) + \frac{2}{3}\int_{-1}^{1} |x-t|^{-\frac{1}{2}} u(t)dt \qquad (3.1)$$

The exact solution of (3.1) [16] is:

$$u(x) = (1-x^2)^{\frac{3}{4}}$$

The following algorithm is applied to obtain the solution of equation (3.1) using the Toeplitz matrix method.

Algorithm (3.1): ToeplitzMatrix Method

1- Input $N, a, v, N1, h, Lamda$
where $h = \frac{a}{N},\ N1 = 2 * N + 1.$

2- Calculate $I(x)$ and $J(x)$

$$I(x) = \int_{a}^{a+h} |x-t|^{-v}\, dt = A_n(x) + B_n(x)$$

$$J(x) = \int_a^{a+h} t\,|x-t|^{-v}\,dt = aA_n(x) + (a+h)B_n(x)$$

3- Calculate $A_n(x)$ and $B_n(x)$

$$A_n(x) = \frac{1}{h}\big((nh+h)I(x) - J(x)\big)$$

$$B_n(x) = \frac{1}{h}\big(J(x) - nh\,I(x)\big)$$

4- Put $x = mh$ and $a = nh$

5- Calculate $D_{m.n} = H_{m,n} - E_{m,n}$

where $H_{m,n} = A_n(mh) + B_{n-1}(mh) - N \leq m, n \leq N$

$$E_{m,n} = \begin{cases} B_{-N-1}\,(mh) & n = -N \\ 0 & -N < n < N \\ A_N(mh) & n = N \end{cases}$$

6- Solve the following linear system

$$u(x) = \left[I - \lambda(H_{m,n} - E_{m,n}\right]^{-1} f(x)$$

The Carleman kernel has two cases whether x$> t$ or x$< t$. So, we get the following results:

Case (1): if $\boldsymbol{x} > t$

$$I(x) = \int_a^{a+h} |x-t|^{-v}dt = \frac{\big(x-(a+h)\big)^{1-v}}{v-1} - \frac{(x-a)^{1-v}}{v-1}$$

$$J(x) = \int_a^{a+h} t|x-t|^{-v}dt = -\frac{\big(x+(1-v)(a+h)\big)\big(x-(a+h)\big)^{1-v}}{(v-1)(v-2)} + \frac{(x+(1-v)a)(x-a)^{1-v}}{(v-1)(v-2)}$$

Then,

$$A_n(x) = \frac{1}{h}\left[(nh+h)\left(\frac{(x-(a+h))^{1-v}}{v-1} - \frac{(x-a)^{1-v}}{v-1}\right) + \frac{\big(x+(1-v)(a+h)\big)\big(x-(a+h)\big)^{1-v}}{(v-1)(v-2)} - \frac{(x+(1-v)a)(x-a)^{1-v}}{(v-1)(v-2)}\right]$$

$$B_n(x) = \frac{1}{h}\left[\frac{-\big(x+(1-v)(a+h)\big)\big(x-(a+h)\big)^{1-v}}{(v-1)(v-2)} + \frac{(x+(1-v)a)(x-a)^{1-v}}{(v-1)(v-2)} - \frac{nh(x-(a+h))^{1-v}}{v-1} + \frac{nh(x-a)^{1-v}}{v-1}\right]$$

by substituting $a = nh,$ and $x = mh, -N \leq n, m \leq N$ we get

$$A_n(mh) = (n+1)\left(\frac{(h(m-(n+1))^{1-v}}{v-1} - \frac{(h(m-n))^{1-v}}{v-1}\right) + \frac{\big(m+(1-v)(n+1)\big)(h(m-(n+1))^{1-v}}{(v-1)(v-2)} - \frac{(m+(1-v)n)(h(m-n))^{1-v}}{(v-1)(v-2)}$$

$$B_n(mh) = \frac{-\big(m+(1-v)(n+1)\big)(h(m-(n+1))^{1-v}}{(v-1)(v-2)} + \frac{(m+(1-v)n)(h(m-n))^{1-v}}{(v-1)(v-2)} - \frac{n(m-(n+1))^{1-v}}{v-1} + \frac{n(h(m-n))^{1-v}}{v-1}$$

Then,

$$H_{m,n} = (n+1)\left(\frac{(h(m-(n+1))^{1-v}}{v-1} - \frac{(h(m-n))^{1-v}}{v-1}\right) + \frac{\big(m+(1-v)(n+1)\big)(h(m-(n+1))^{1-v}}{(v-1)(v-2)} - \frac{(m+(1-v)n)(h(m-n))^{1-v}}{(v-1)(v-2)} - \frac{(m+(1-v)n)(h(m-n)^{1-v}}{(v-1)(v-2)} + \frac{(m+(1-v)(n-1))(h(m-n+1))^{1-v}}{(v-1)(v-2)} - \frac{(n-1)(m-n)^{1-v}}{v-1} + \frac{(n-1)(h(m-n+1))^{1-v}}{v-1}$$

Case (2): if $x < t$

$$I(x) = \int_a^{a+h} |x-t|^{-v} dt = -\frac{((a+h)-x)^{1-v}}{v-1} + \frac{(a-x)^{1-v}}{v-1}$$

$$J(x) = \int_a^{a+h} t|x-t|^{-v} dt = \frac{\big(x+(1-v)(a+h)\big)\big((a+h)-x\big)^{1-v}}{(v-1)(v-2)} + \frac{(x+(1-v)a)(x-a)^{1-v}}{(v-1)(v-2)}$$

Then,

$$A_n(x) = \frac{1}{h}\left[(nh+h)\left(-\frac{((a+h)-x)^{1-v}}{v-1} + \frac{(a-x)^{1-v}}{v-1}\right) - \frac{\big(x+(1-v)(a+h)\big)\big((a+h)-x\big)^{1-v}}{(v-1)(v-2)} + \frac{(x+(1-v)a)(a-x)^{1-v}}{(v-1)(v-2)}\right]$$

$$B_n(x) = \frac{1}{h}\left[\frac{\big(x+(1-v)(a+h)\big)\big((a+h)-x\big)^{1-v}}{(v-1)(v-2)} - \frac{(x+(1-v)a)(a-x)^{1-v}}{(v-1)(v-2)} + \frac{nh((a+h)-x)^{1-v}}{v-1} - \frac{nh(a-x)^{1-v}}{v-1}\right]$$

by substituting $a = nh,$ and $x = mh, -N \leq n, m \leq N$ we get

$$A_n(mh) = (n+1)\left(\frac{-(h((n+1)-m)^{1-v}}{v-1} + \frac{(h(n-m))^{1-v}}{v-1}\right) - \frac{\big(m+(1-v)(n+1)\big)(h((n+1)-m)^{1-v}}{(v-1)(v-2)} + \frac{(m+(1-v)n)(h(n-m))^{1-v}}{(v-1)(v-2)}$$

$$B_n(mh) = \frac{-\big(m+(1-v)(n+1)\big)(h((n+1)-m)^{1-v}}{(v-1)(v-2)} - \frac{(m+(1-v)n)(h(n-m))^{1-v}}{(v-1)(v-2)} + \frac{n(h((n+1)-m)^{1-v}}{v-1} - \frac{n(h(n-m))^{1-v}}{v-1}$$

Then,

$$H_{m,n} = (n+1)\left(-\frac{(h((n+1)-m)^{1-v}}{v-1} + \frac{(h(n-m))^{1-v}}{v-1}\right) - \frac{\big(m+(1-v)(n+1)\big)(h((n+1)-m)^{1-v}}{(v-1)(v-2)} + \frac{(m+(1-v)n)(h(n-m))^{1-v}}{(v-1)(v-2)} + \frac{(m+(1-v)n)(h(n-m)^{1-v}}{(v-1)(v-2)} - \frac{(m+(1-v)(n-1))(h(n-m-1))^{1-v}}{(v-1)(v-2)} + \frac{(n-1)(h(n-m)^{1-v}}{v-1} - \frac{(n-1)(h(n-m-1))^{1-v}}{v-1}$$

The elements of the first column of the matrix $E_{mn}$are given by

$$E_{m,-N} = B_{-N-1}(mh)$$

Hence,

$$E_{m,-N} = \frac{-\big(m + (1-v)(-N)\big)(h(m+N))^{1-v}}{(v-1)(v-2)} + \frac{(m + (1-v)(-N-1))(h(m+N+1))^{1-v}}{(v-1)(v-2} - \frac{(-N-1)(h(m+N)^{1-v}}{v-1} + \frac{(-N-1)(h(m+N+1))^{1-v}}{v-1}$$

While the element of the last column of the matrix E_{mn} is given by:

$$E_{m,N} = A_N(mh)$$

Then,

$$E_{m,N} = (\mathrm{N}+1)\left(\frac{-(h((N+1)-m)^{1-v}}{v-1} + \frac{(h(N-m))^{1-v}}{v-1}\right) - \frac{\big(m + (1-v)(N+1)\big)(h((N+1)-m)^{1-v}}{(v-1)(v-2)} + \frac{(m + (1-v)N)(h(N-m))^{1-v}}{(v-1)(v-2)}$$

From above equations we obtain the following linear system of algebraic equations, namely;

$$u(x) = [I - \lambda(H_{mn} - E_{mn})]^{-1} f(x)$$

This linear system has a dimension $2\mathrm{N}+1$. Example(3.1) yields a linear system of dimension 201×201.

Table (3.1) contains both the exact and the numerical results using the Toeplitz matrix method for example (3.1).

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
-1	0	-0.566942971	56.6942971e-2
-0.9	0.287783043	0.359953584	7.2170541e-2
-0.8	0.464758002	0.536405370	7.1647369e-2
-0.7	0.603500497	0.666735339	6.3234842e-2
-0.6	0.715541753	0.766685810	5.1144057e-2
-0.5	0.805927449	0.843508893	3.7581444e-2
-0.4	0.877423909	0.901618906	2.4194996e-2
-0.3	0.931710617	0.944036529	1.2325913e-2
-0.2	0.969847442	0.972895774	0.3048332e-2
-0.1	0.992490586	0.989646207	0.2844378e-2
0	1	0.995136344	0.4863656e-2
0.1	0.992490586	0.989646207	0.2844378e-2
0.2	0.969847442	0.972895774	0.3048332e-2
0.3	0.931710617	0.944036529	1.2325913e-2
0.4	0.877423909	0.901618906	2.4194996e-2
0.5	0.805927449	0.843508893	3.7581444e-2
0.6	0.715541753	0.766685810	5.1144057e-2
0.7	0.603500497	0.666735339	6.3234842e-2
0.8	0.464758002	0.536405370	7.1647369e-2
0.9	0.287783043	0.359953584	7.2170541e-2
1	0	-0.566942971	56.6942971e-2

Table (3.1):The exact and numerical solutions using Toeplitz matrix algorithm where N=100

It can be observed that the maximum error is 56.6942971e-2.

The exact and approximate results of $u(x)$ are shown in Fig. 3.1 (a) and the resulted error is shown in Fig. 3.1 (b).

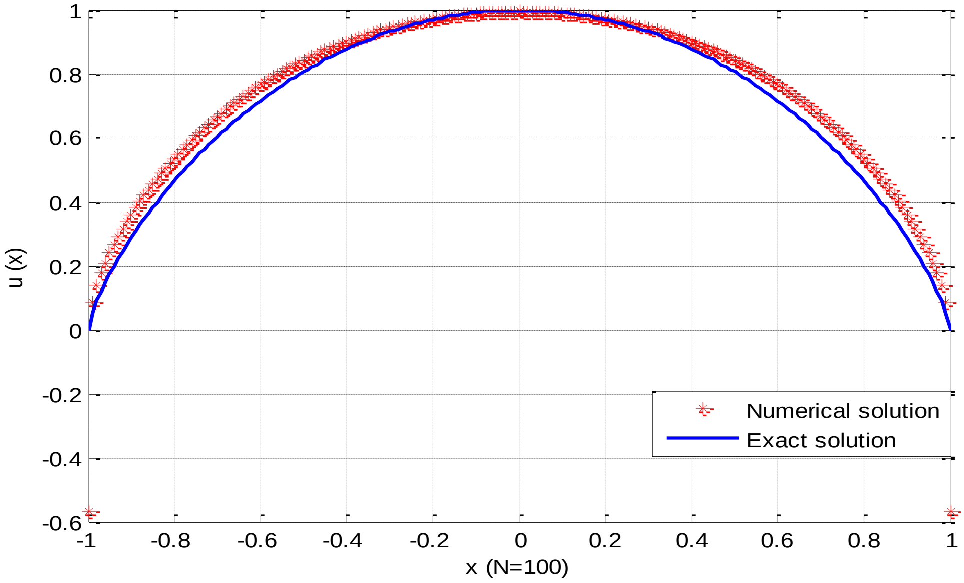

Fig. 3.1(a) A comparison between the exact and approximate solutionin example 3.1

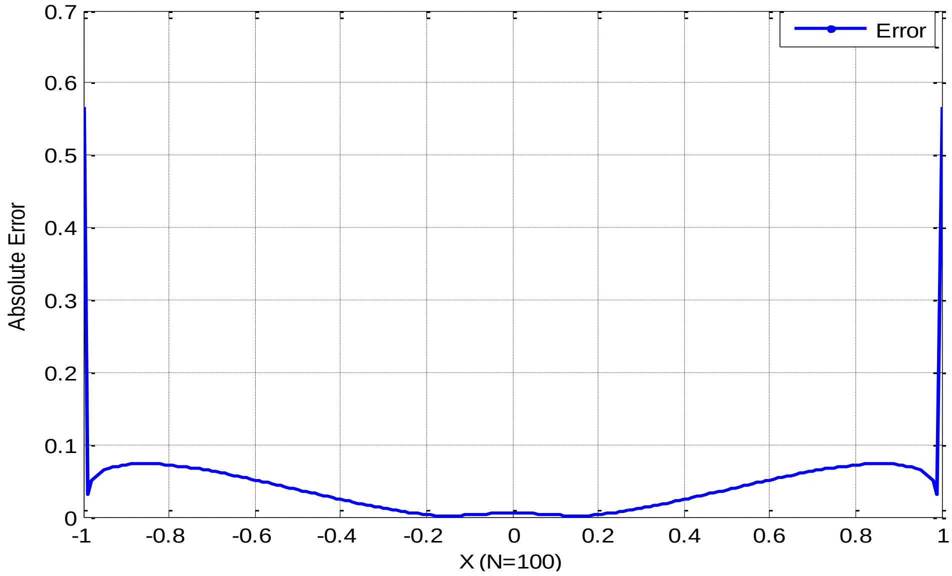

Fig. 3.1 (b) Absolute error between exact and numerical solution in example 3.1

Algorithm (3.2): Product Nystrom Method

This algorithm can be illustrated as follows:

1- Input $N, a, b, v, N1, h, Lamda$

where$h = \frac{(b-a)}{N}$, $N1 = N + 1$

2- Calculate φ_0, φ_1, and φ_2

$$\varphi_0 = \int_0^2 |x-t|^{-v} dt$$
$$\varphi_1 = \int_0^2 t|x-t|^{-v} dt$$
$$\varphi_2 = \int_0^2 t^2|x-t|^{-v} dt$$

3- Calculate $w_{i,0}$, $w_{i,2j+1}$, $w_{i,2j}$ and $w_{i,N}$

$$w_{i,0} = \frac{h^{1-v}}{2}(2\varphi_0(x) - 3\varphi_1(x) + \varphi_2(x)) \qquad x = i$$
$$w_{i,2j+1} = h^{1-v}(2\varphi_1(x) - \varphi_2(x)) \qquad x = i - 2j$$
$$w_{i,2j} = \frac{h^{1-v}}{2}\big(\varphi_2(x) - \varphi_1(x) + 2\varphi_0(x-2) - 3\varphi_1(x-2) + \varphi_2(x-2)\big)$$
$$x = i - 2j + 2$$
$$w_{i,N} = \frac{h^{1-v}}{2}(\varphi_2(x) - \varphi_1(x)) \qquad x = i - N + 2$$

where $0 \le i \le N$ and $0 \le j \le \frac{N-2}{2}$

4- Construct the matrix W

5- Solve the following linear system

$$U = [I - \lambda W]^{-1} F \qquad |I - \lambda W| \neq 0$$

We will usethe algorithm (3.2) to solve the numerical example (3.1). It worth mentioning that the Carleman kernel has two cases either$0 < t < x < 2$ in case 1 or $0 < x < t < 2$ in case 2. According to that,we get the following results:

<u>φFunctions</u>

- $\varphi_0 = \int_0^2 |x-t|^{-v} dt$

Case(1)

$$\varphi_0 = \frac{(x-t)^{1-v}}{v-1}\Big|_0^2 = \frac{(x-2)^{1-v}}{v-1} - \frac{x^{1-v}}{v-1}$$

Case(2)

$$\varphi_0 = -\frac{(2-x)^{1-v}}{v-1} + \frac{(-x)^{1-v}}{v-1}$$

- $\varphi_1 = \int_0^2 t|x-t|^{-v}dt$

Case(1)

$$\varphi_1 = \frac{-(x+t-vt)(x-t)^{1-v}}{(v-1)(v-2)} \Big|_0^2$$
$$= \frac{-(x+2(1-v))(x-2)^{1-v}}{(v-1)(v-2)} + \frac{x^{2-v}}{(v-1)(v-2)}$$

Case(2)

$$\varphi_1 = \frac{(x+t-vt)(t-x)^{1-v}}{(v-1)(v-2)} \Big|_0^2$$
$$= \frac{(x+2(1-v))(2-x)^{1-v}}{(v-1)(v-2)} + \frac{(-x)^{2-v}}{(v-1)(v-2)}$$

- $\varphi_2 = \int_0^2 t^2|x-t|^{-v}dt$

Case(1)

$$\varphi_2 = \frac{(x-t)^{1-v}(t^2v^2 - 2xtv - 3y^2v + 2xt + 2x^2 + 2t^2)}{(v-1)(v-2)(v-3)} \Big|_0^2$$
$$= \frac{(x-2)^{1-v}(4v^2 - 4xv - 12v + 4x + 2x^2 + 8) - 2x^{3-v}}{(v-1)(v-2)(v-3)}$$

Case(2)

$$\varphi_2 = \frac{-(t-x)^{1-v}(t^2v^2 - 2xtv - 3y^2v + 2xt + 2x^2 + 2t^2)}{(v-1)(v-2)(v-3)} \Big|_0^2$$
$$= \frac{-(2-x)^{1-v}(4v^2 - 4xv - 12v + 4x + 2x^2 + 8) + 2(-x)^{3-v}}{(v-1)(v-2)(v-3)}$$

<u>*W* matrix entries</u>

- $w_{i,0} = \frac{h^{1-v}}{2}(2\varphi_0(x) - 3\varphi_1(x) + \varphi_2(x)) \qquad x = i$

$$w_{i,0} = \frac{h^{1-v}}{2}\left[\frac{2(x-2)^{1-v}}{v-1} - \frac{2x^{1-v}}{v-1} + \frac{3\big(x+2(1-v)\big)(x-2)^{1-v}}{(v-1)(v-2)} - \frac{3x^{2-v}}{(v-1)(v-2)} + \frac{(x-2)^{1-v}(4v^2-4xv-12v+4x+2x^2+8)-2x^{3-v}}{(v-1)(v-2)(v-3)}\right]$$

- $w_{i,2j+1} = h^{1-v}(2\varphi_1(x) - \varphi_2(x)) \qquad x = i-2j$

Case(1)

$$w_{i,2j+1} = h^{1-v}\left(\frac{-2\big(x+2(1-v)\big)(x-2)^{1-v}}{(v-1)(v-2)} + \frac{2x^{2-v}}{(v-1)(v-2)} + \frac{-(x-2)^{1-v}(4v^2-4xv-12v+4x+2x^2+8)+2x^{3-v}}{(v-1)(v-2)(v-3)}\right)$$

Case(2)

$$w_{i,2j+1} = h^{1-v}\left[\frac{2\big(x+2(1-v)\big)(2-x)^{1-v}}{(v-1)(v-2)} + \frac{2(-x)^{2-v}}{(v-1)(v-2)} + \frac{(2-x)^{1-v}(4v^2-4xv-12v+4x+2x^2+8)-2(-x)^{3-v}}{(v-1)(v-2)(v-3)}\right]$$

$$w_{i,2j} = \frac{h^{1-v}}{2}\big(\varphi_2(x) - \varphi_1(x) + 2\varphi_0(x-2) - 3\varphi_1(x-2) + \varphi_2(x-2)\big) \qquad x = i-2j+2$$

Case(1)

$$w_{i,2j} = \frac{h^{1-v}}{2}\left[\frac{(x-2)^{1-v}(4v^2-4xv-12v+4x+2x^2+8)-2x^{3-v}}{(v-1)(v-2)(v-3)}\right.$$
$$+\frac{\left(x+2(1-v)\right)(x-2)^{1-v}}{(v-1)(v-2)}-\frac{x^{2-v}}{(v-1)(v-2)}+\frac{2\left((x-2)-2\right)^{1-v}}{v-1}-\frac{2(x-2)^{1-v}}{v-1}$$
$$+\frac{3\left((x-2)+2(1-v)\right)\left((x-2)-2\right)^{1-v}}{(v-1)(v-2)}-\frac{3(x-2)^{2-v}}{(v-1)(v-2)}$$
$$\left.+\frac{\left((x-2)-2\right)^{1-v}(4v^2-4(x-2)v-12v+4(x-2)+2(x-2)^2+8)-2(x-2)^{3-v}}{(v-1)(v-2)(v-3)}\right]$$

Case(2)

$$w_{i,2j} = \frac{h^{1-v}}{2}\left[\frac{-(2-x)^{1-v}(4v^2-4xv-12v+4x+2x^2+8)+2(-x)^{3-v}}{(v-1)(v-2)(v-3)}\right.$$
$$-\frac{(x+2(1-v))(2-x)^{1-v}}{(v-1)(v-2)}-\frac{(-x)^{2-v}}{(v-1)(v-2)}-\frac{2(2-(x-2))^{1-v}}{v-1}$$
$$+\frac{2(-(x-2))^{1-v}}{v-1}+\frac{-3((x-2)+2(1-v))(2-(x-2))^{1-v}}{(v-1)(v-2)}-\frac{3(-(x-2)^{2-v}}{(v-1)(v-2)}$$
$$\left.+\frac{-\left(2-(x-2)\right)^{1-v}(4v^2-4(x-2)v-12v+4(x-2)+2(x-2)^2+8)+2(-(x-2))^{3-v}}{(v-1)(v-2)(v-3)}\right]$$

- $w_{i,N} = \frac{h^{1-v}}{2}(\varphi_2(x)-\varphi_1(x))$ $\qquad x = i - N + 2$

$$w_{i,N} = \frac{h^{1-v}}{2}\left[\frac{-(2-x)^{1-v}(4v^2-4xv-12v+4x+2x^2+8)+2(-x)^{3-v}}{(v-1)(v-2)(v-3)} - \frac{\left(x+2(1-v)\right)(2-x)^{1-v}}{(v-1)(v-2)} - \frac{(-x)^{2-v}}{(v-1)(v-2)}\right]$$

From above equations we obtain the linear system of algebraic equations:

$$U = [I - \lambda W]^{-1} F$$

This linear system has a dimension $\mathrm{N} + 1$. For example (3.1) the dimension of the system is 101×101.

Table (3.2) contains both the exact and the numerical results using the Product Nystrom method for example (3.1).

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
-1	0	0.100734	10.0734e-2
-0.9	0.287783	0.398842	11.1059 e-2
-0.8	0.464758	0.563723	9.8965 e-2
-0.7	0.6035	0.684173	8.0672 e-2
-0.6	0.715542	0.772481	5.6939 e-2
-0.5	0.805927	0.839905	3.3978 e-2
-0.4	0.877424	0.887441	1.0017 e-2
-0.3	0.931711	0.92295	0.876 e-2
-0.2	0.969847	0.944529	2.532 e-2
-0.1	0.992491	0.958654	3.384 e-2
0	1	0.961642	3.836 e-2
0.1	0.992491	0.958654	3.384 e-2
0.2	0.969847	0.944529	2.532 e-2
0.3	0.931711	0.92295	0.876 e-2
0.4	0.877424	0.887441	1.0017 e-2
0.5	0.805927	0.839905	3.3978 e-2
0.6	0.715542	0.772481	5.6939 e-2
0.7	0.6035	0.684173	8.0672 e-2
0.8	0.464758	0.563723	9.8965 e-2
0.9	0.287783	0.398842	11.1059 e-2
1	0	0.100734	10.0734 e-2

Table (3.2): The exact and numerical solutions using Product Nystrom algorithm where N=100

It can be observed that the maximum error is 11.1059 e-2.

The exact and approximate results of $u(x)$ are shown in Fig. 3.2 (a) and the resulted error is shown in Fig. 3.2 (b).

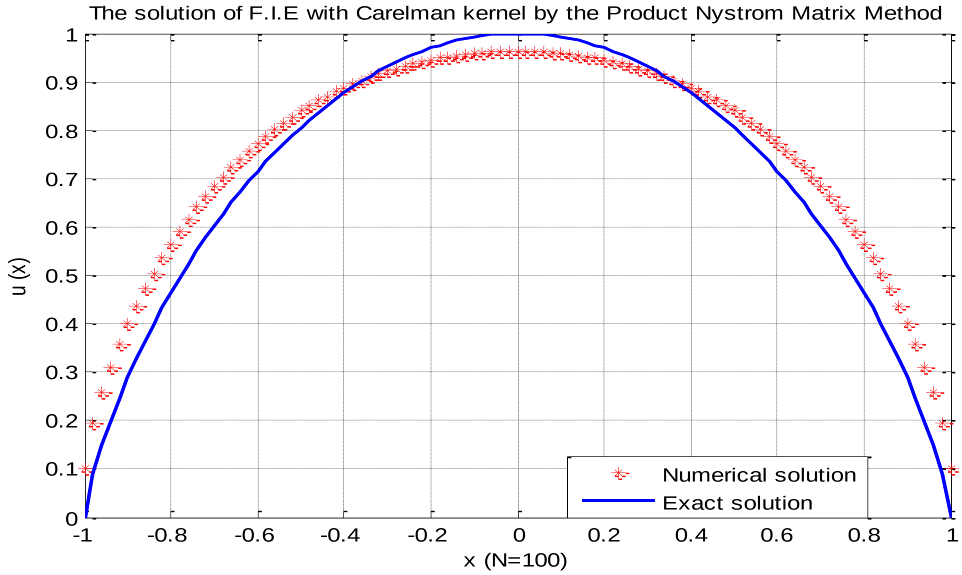

Fig. 3.2(a) A comparison between the exact and approximate solution in example 3.1

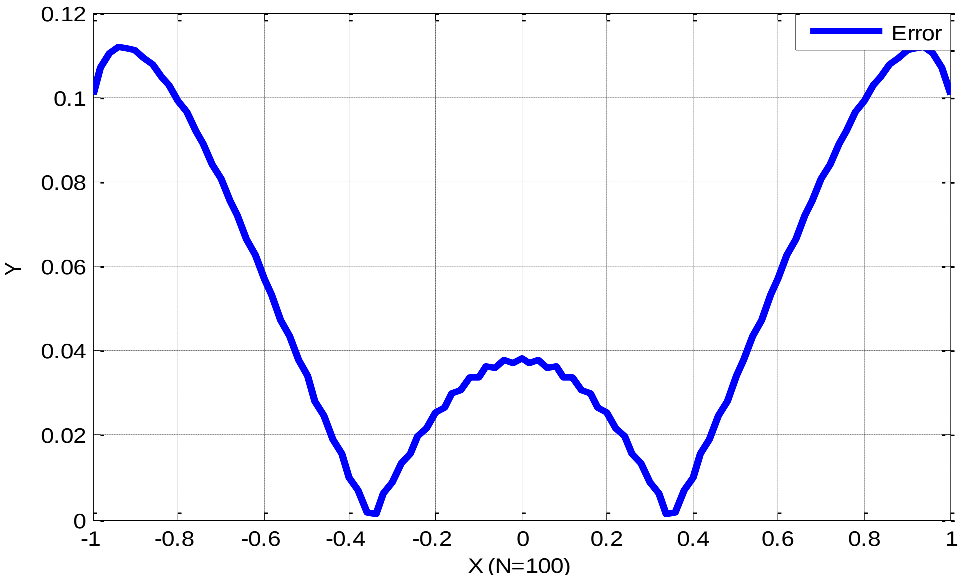

Fig. 3.2(b) Absolute error between exact and numerical solution in example 3.1

Example (3.2):

Consider the Fredholm integral equation of second kind with a Carleman kernel:

$$u(x) = x^2(1-x)^2 - \frac{27}{30800}\left(x^{\frac{8}{3}}(54x^2 - 126x + 77) + (1-x)^{\frac{8}{3}}\right.$$
$$\times\ (54x^2 + 18x + 5)$$
$$+ \frac{1}{10}\int_0^1 |x-t|^{-\frac{1}{3}} u(t)dt \qquad (3.2)$$

The exact solution of (3.2) [20] is:

$$u(x) = x^2(1-x)^2.$$

Applying algorithm (3.1) for example (3.2).Table (3.3) contains both the exact and the numerical results usingthe Toeplitz matrix method for example (3.2).

Table (3.3):The exact and numerical solutionsusing Toeplitz matrix algorithm where N=50

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
0	0	-0.00000887	0.88718e-05
0.1	0.0081	0.008075	2.45906e-05
0.2	0.0256	0.025542	5.82241e-05
0.3	0.0441	0.044006	9.35426e-05
0.4	0.0576	0.057481	11.9251e-5
0.5	0.0625	0.062371	12.8574e-5
0.6	0.0576	0.057481	11.925e-5
0.7	0.0441	0.044006	9.35391e-05
0.8	0.0256	0.025542	5.82182e-05
0.9	0.0081	0.008075	2.45806e-05
1	0	-0.00000887	0.883702e-05

It can be observed that the maximum error is0.000129.The exact and approximate results of $u(x)$ are shown in Fig. 3.3 (a) and the resulted error is shown in Fig. 3.3 (b).

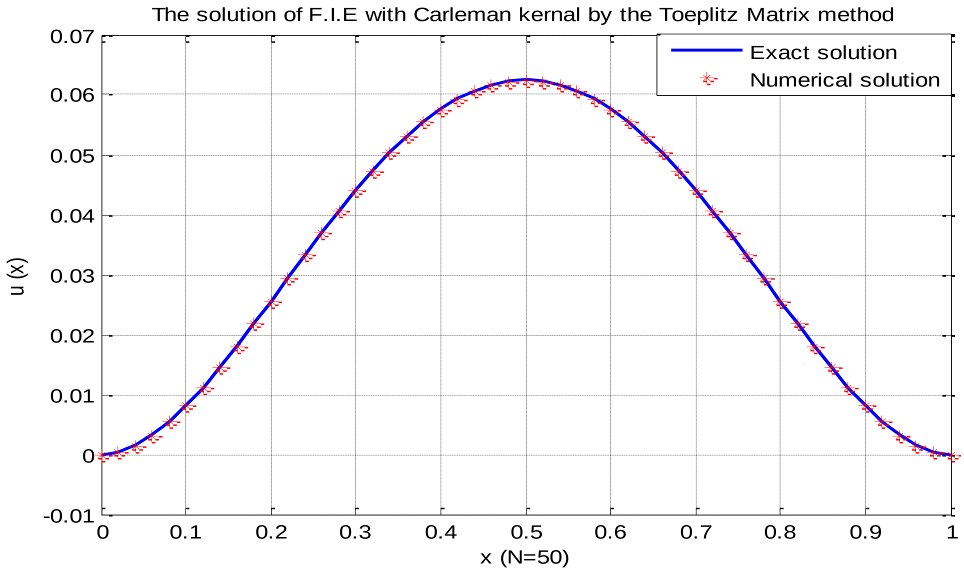

Fig. 3.3(a) A comparison between the exact and approximate solution in example 3.2

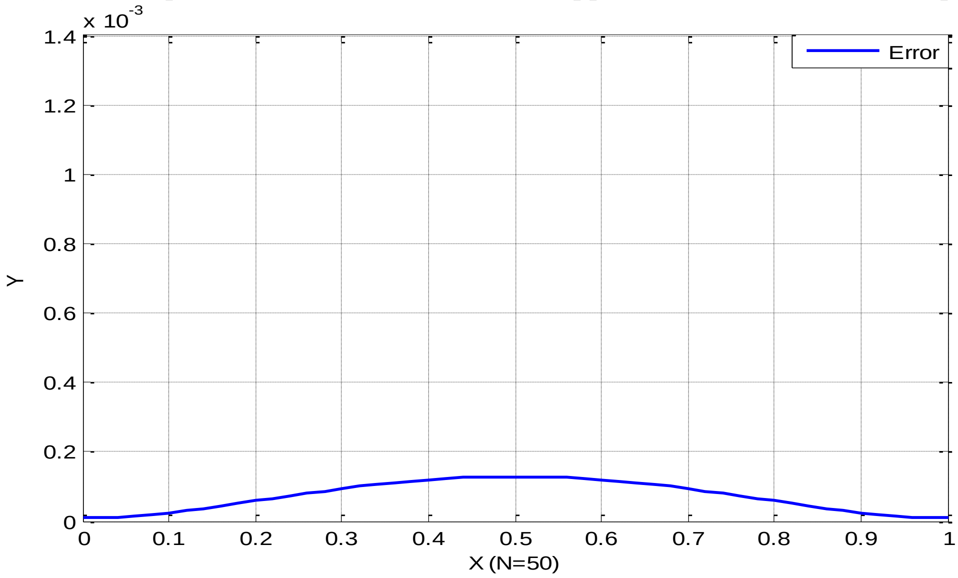

Fig. 3.3 (b) Absolute error between exact and numerical solution in example 3.2

Applying algorithm (3.2) for example (3.2).Table (3.4) contains both the exact and the numerical results using theProduct Nystrommethod for example (3.2).

Table (3.4):The exact and numerical solutionsusingProduct Nystromalgorithm where N=50

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
0	0	-0.00000729	0.728912e-05
0.1	0.0081	0.008081344	1.86565e-05
0.2	0.0256	0.025554337	4.56627e-05
0.3	0.0441	0.044031788	6.82118e-05
0.4	0.0576	0.057507077	9.29228e-05
0.5	0.0625	0.06240657	9.34303e-05
0.6	0.0576	0.057507077	9.29228e-05
0.7	0.0441	0.044031788	6.82118e-05
0.8	0.0256	0.025554337	4.56627e-05
0.9	0.0081	0.008081344	1.86565e-05
1	0	-0.00000729	7.28912e-05

It can be observed that the maximum error is 9.98614e-05.The exact and approximate results of $u(x)$ are shown in Fig. 3.4 (a) and the resulted error is shown in Fig. 3.4 (b).

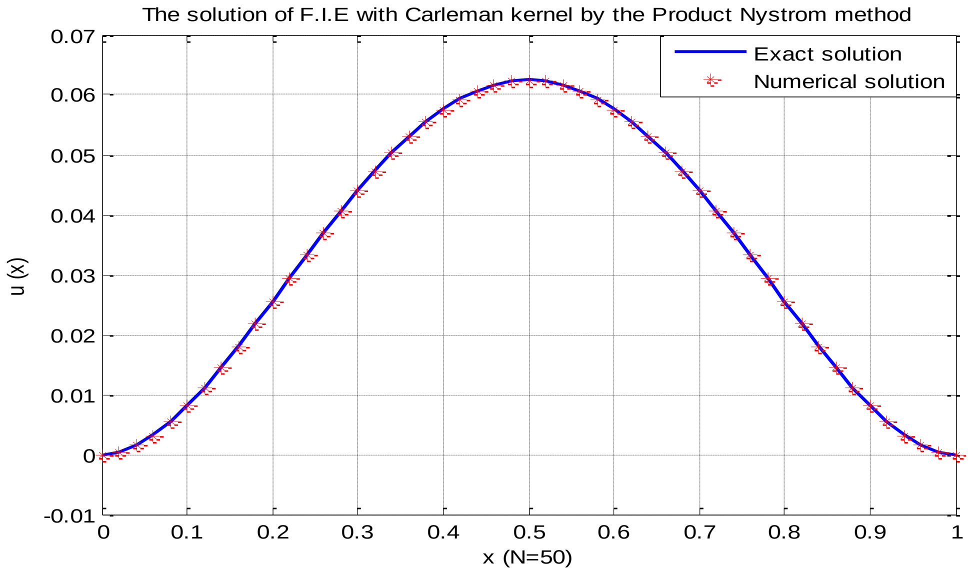

Fig. 3.4 (a) A comparison between the exact and approximate solution in example 3.2

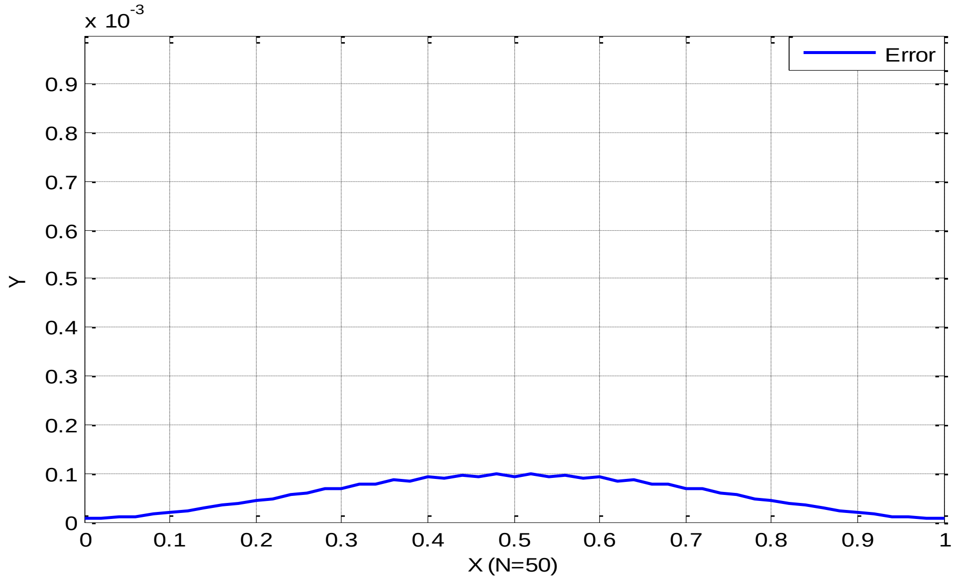

Fig. 3.4(b) Absolute error between exact and numerical solution in example 3.2

The following algorithm is applied for example (3.2).

Algorithm (3.3): Sinc-Collocation Algorithm

1- Input $d, Alpha, N, a, b, \lambda, v$

2- Find $N1, H, M$

where $h = \sqrt{\frac{\pi d}{alpha * N}}$, $N1 = 2N + 3,\ \ M = \lceil (1 - v) * N \rceil$.

3- Find E_n matrix

where

$$E_n^{SE} = \lambda \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ w_a(t_{-N}^{SE}) & 1 & & 0 & w_b(t_{-N}^{SE}) \\ \vdots & & \ddots & & \vdots \\ w_a(t_{N}^{SE}) & 0 & & 1 & w_b(t_{-N}^{SE}) \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

4- Use the following equations to fill the matrix in (3)

$$t = \varphi_{ab}{}^{SE}(x) = \frac{b-a}{2}\tanh\left(\frac{x}{2}\right) + \frac{b+a}{2}$$

$$w_a(t) = \frac{b-t}{b-a}$$

$$w_b(t) = \frac{t-a}{b-a}$$

5- Find K_n matrix

$$K_n^{SE} = \begin{bmatrix} B_N^{SE}[w_a](a) & \cdots & B_N^{SE}[S(jh)\{\varphi^{SE}(.)\}^{-1}](a) & \ldots & B_N^{SE}[w_b](a) \\ K_N^{SE}[w_a](t_{-N}^{SE}) & \cdots & K_N^{SE}[S(jh)\{\varphi^{SE}(.)\}^{-1}](t_{-N}^{SE}) & \ldots & K_N^{SE}[w_b](t_{-N}^{SE}) \\ \vdots & & \vdots & & \vdots \\ K_N^{SE}[w_a](t_N^{SE}) & \ldots & K_N^{SE}[S(jh)\{\varphi^{SE}(.)\}^{-1}](t_N^{SE}) & \ldots & K_N^{SE}[w_b](t_N^{SE}) \\ A_N^{SE}[w_a](b) & \ldots & A_N^{SE}[S(jh)\{\varphi^{SE}(.)\}^{-1}](b) & \ldots & A_N^{SE}[w_b](b) \end{bmatrix}$$

6- To fill the matrix in (5), consider the equations in (4) and use the following:

$$A_N^{SE}[y](t) = (t-a)^{1-v}h\sum_{m=-M}^{N}\frac{y\left(\varphi_{a,t}^{SE}(mh)\right)}{(1+e^{-mh})(1+e^{mh})^{1-v}}$$

$$B_N^{SE}[y](t) = (b-t)^{1-v}h\sum_{m=-M}^{N}\frac{y\left(\varphi_{t,b}^{SE}(mh)\right)}{(1+e^{mh})(1+e^{-mh})^{1-v}}$$

$$K_N^{SE} = A_N^{SE} + B_N^{SE}$$

$$x = \{\varphi^{SE}(t)\}^{-1} = \log\left(\frac{t-a}{b-t}\right)$$

$$t_i^{SE} = \begin{cases} a & (i=-N-1) \\ \varphi^{SE}(ih) & (i=-N,\ldots,N) \\ b & (i=N+1) \end{cases}$$

7- Find f_n matrix

Where $f_n^{SE} = [f(a), f(t_{-N}^{SE}), \ldots, f(t_N^{SE}), f(b)]^T$

8- Solve the following linear system to get C_n matrix

$(E_n^{SE} - K_n^{SE})c_n = f_n^{SE}$

where

$c_n = [c_{-N-1}, c_{-N}, \ldots, c_N, c_{N+1}]^T$

9- Substitute the coefficients of matrix c_n in the following equation to get the approximate solution u_N^{SE}

$$u_N^{SE} = c_{-N-1}w_a(t) + \sum_{j=-N}^{N} \mathrm{c_j}\left(\varphi^{SE}(jh)\right)S(jh)\{\varphi^{SE}(t)\}^{-1} + c_{N+1}w_b(t)$$

Table (3.5) contains both the exact and the numerical results using theSinc-collocation method for example (3.2).

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
0	0	3.14 e-03	3.142512e-3
0.05	2.26e-03	5.95 e-03	3.698471 e-3
0.1	8.10 e-03	1.21 e-02	3.950354 e-3
0.15	1.63 e-02	2.03 e-02	4.013146 e-3
0.2	2.56 e-02	2.95 e-02	3.94698 e-3
0.25	3.52 e-02	3.90 e-02	3.804759 e-3
0.3	4.41 e-02	4.77 e-02	3.626509 e-3
0.35	5.18 e-02	5.52 e-02	3.450183 e-3
0.4	5.76 e-02	6.09 e-02	3.304274 e-3
0.45	6.13 e-02	6.45 e-02	3.208296 e-3
0.5	6.25 e-02	6.57 e-02	3.174789 e-3
0.55	6.13 e-02	6.45 e-02	3.208296 e-3
0.6	5.76 e-02	6.09 e-02	3.304274 e-3
0.65	5.18 e-02	5.52 e-02	3.450183 e-3
0.7	4.41 e-02	4.77 e-02	3.626509 e-3
0.75	3.52 e-02	3.90 e-02	3.804759 e-3
0.8	2.56 e-02	2.95 e-02	3.94698 e-3
0.85	1.63 e-02	2.03 e-02	4.013146 e-3
0.9	8.10 e-03	1.21 e-02	3.950354 e-3
0.95	2.26 e-03	5.95 e-03	3.698471 e-3
1	0.00	3.16 e-03	3.160432 e-3

Table (3.5):The exact and numerical solutions usingsinc-collocation method algorithm where N=50

It can be observed that the maximum error is 4.013146 e-3.The exact and approximate results of $u(x)$ are shown in Fig. 3.5 (a) and theresulted error is shown in Fig. 3.5 (b).

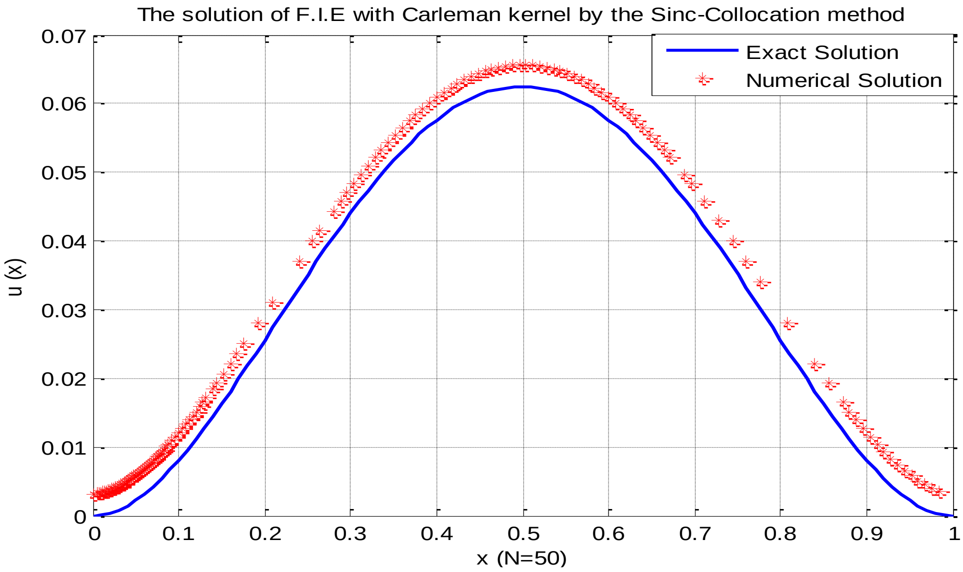

Fig. 3.5 (a) A comparison between the exact and approximate solution in example 3.2

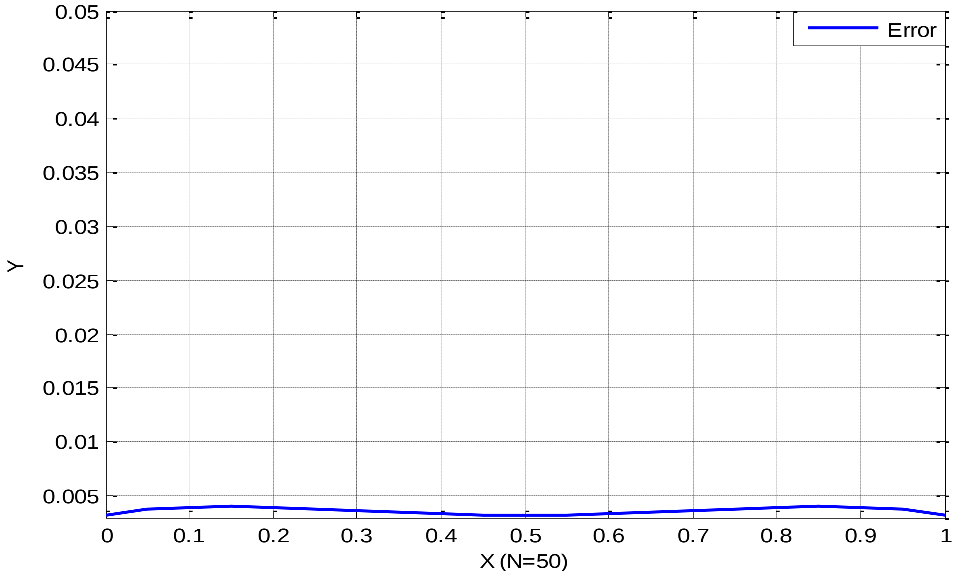

Fig. 3.5(b) Absolute error between exact and numerical solution in example 3.2

Chapter Four

Numerical examples for Volterra integral equation with Carleman kernel

In this chapter, four different numerical techniques are usedto solve Volterra integral equation of the second kind with a Carleman kernel. These techniques are Laplace Adomian Decomposition method, Toeplitz Matrix method, Product Nystrom method and Sinc-Collocation method. The algorithms introduced in chapter three are used and Matlab software is implemented to solve some numerical examples.

Example (4.1):

Consider the Volterra integral equation of second kind with a Carleman kernel:

$$u(x) = x^2 - 0.00675x^{8/3} + 0.01\int_0^x |x-t|^{-\frac{1}{3}}\, u(t)dt \qquad (4.1)$$

The exact solution of (4.1) [27] is:

$$u(x) = x^2$$

The following algorithm is applied to obtain the solution of equation (4.1) using the Laplace Adomian decomposition method.

Algorithm (4.1): Laplace Adomian Decomposition Method

1- suppose that $f(x) = x^2 - 0.00675x^{8/3}$

where $f_0(x) = x^2 \qquad and \qquad f_1(x) = -0.00675x^{8/3}$

2- By using Laplace Adomian decomposition method we get a recursive relation:

$$u_0 = x^2$$

$$u_1 = -0.00675x^{8/3} + L^{-1}[0.01 * \frac{\Gamma\left[\frac{2}{3}\right]}{s^{\frac{2}{3}}} L(u_0)]$$

$$u_1 = -0.00675x^{8/3} + [0.01 * x^{8/3} \frac{\Gamma\left[\frac{2}{3}\right]}{\Gamma\left[\frac{11}{3}\right]}] = 0$$

so, $u_{n+1}(x) = 0, n \geq 1$

The solution becomes:

$$u(x) = \sum_{n=0}^{\infty} u_n\,(x) = x^2$$

Table (4.1) contains both exact and numerical results using the Laplace Adomian decomposition transformation method for example (4.1).

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
0	0.0000	0.0000	0
0.09	0.0081	0.0081	0
0.18	0.0324	0.0324	0
0.27	0.0729	0.0729	0
0.36	0.1296	0.1296	0
0.45	0.2025	0.2025	0
0.54	0.2916	0.2916	0
0.63	0.3969	0.3969	0
0.72	0.5184	0.5184	0
0.81	0.6561	0.6561	0
0.9	0.8100	0.8100	0

Table (4.1):The exact and numerical solutions using Laplace Adomian decomposition transformation algorithm

It can be observed that the maximum error is 0. The exact and approximate results of $u(x)$ are shown in Fig. 4.1 (a) and the resulted error is shown in Fig. 4.1 (b).

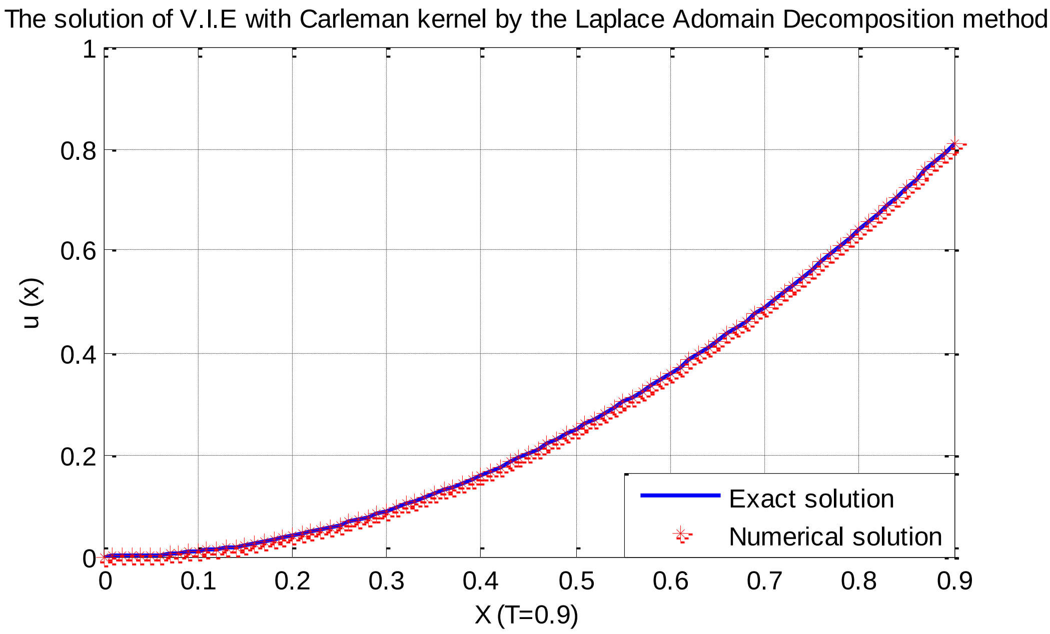

Fig. 4.1(a) A comparison between the exact and approximate solution in example 4.1

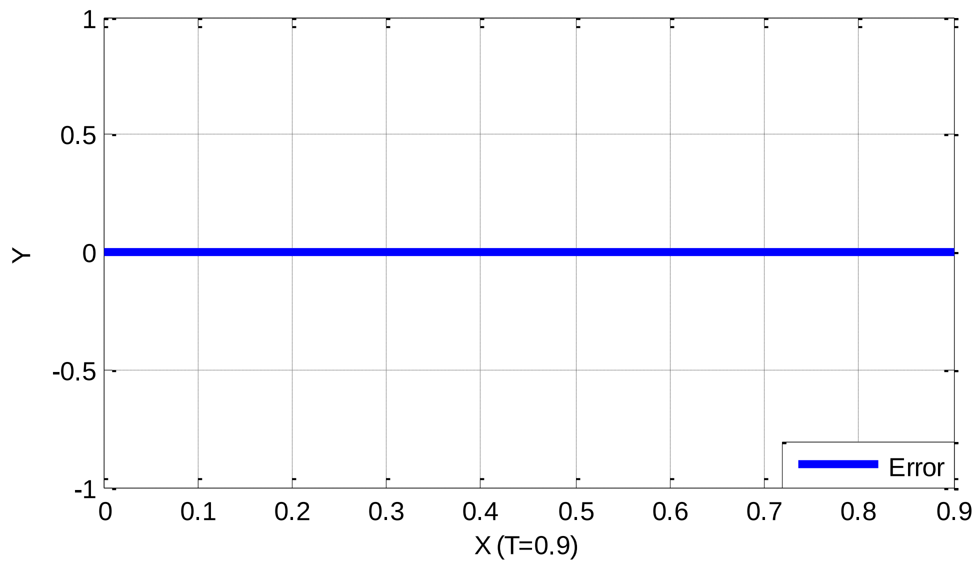

Fig. 4.1 (b) Absolute error between exact and numerical solution in example 4.1

Toeplitz Matrix Method

The upper bound of the Volterra integral equation is less than 1. Equation (4.1) is solved when the upper bound is 0.1, 0.5 and 0.9, respectively. Applying algorithm (3.1) for example (4.1).

Table (4.2) contains both the exact and the numerical results using the Toeplitz matrix method for example (4.1) where the upper bound is 0.1.

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
0	0.0000	0.0000083	8.3261E-06
0.01	0.0001	0.0001088	8.8252E-06
0.02	0.0004	0.0004094	9.3889E-06
0.03	0.0009	0.0009099	9.9411E-06
0.04	0.0016	0.0016104	1.0395E-05
0.05	0.0025	0.0025106	1.0649E-05
0.06	0.0036	0.0036106	1.0579E-05
0.07	0.0049	0.0049100	1.0026E-05
0.08	0.0064	0.0064088	8.7544E-06
0.09	0.0081	0.0081063	6.3356E-06
0.1	0.0100	0.0099937	6.3256E-06

Table (4.2):The exact and numerical solutions using Toeplitz matrix algorithm where N=50.

It can be observed that the maximum error is 1.0649E-05.

The exact and approximate results of $u(x)$ are shown in Fig. 4.2 (a) and the resulted error is shown in Fig. 4.2 (b).

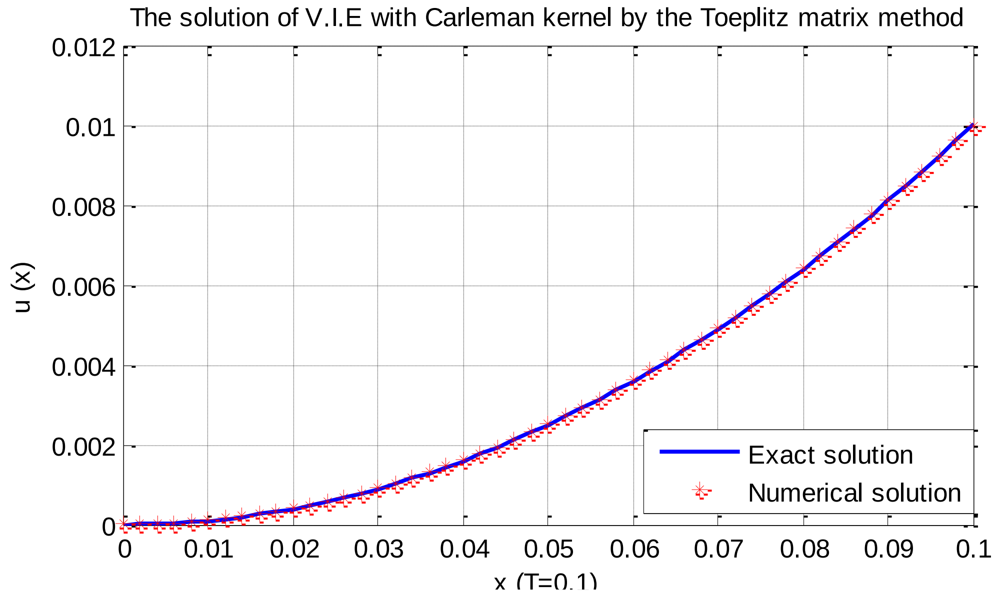

Fig. 4.2(a) A comparison between the exact and approximate solution in example 4.1

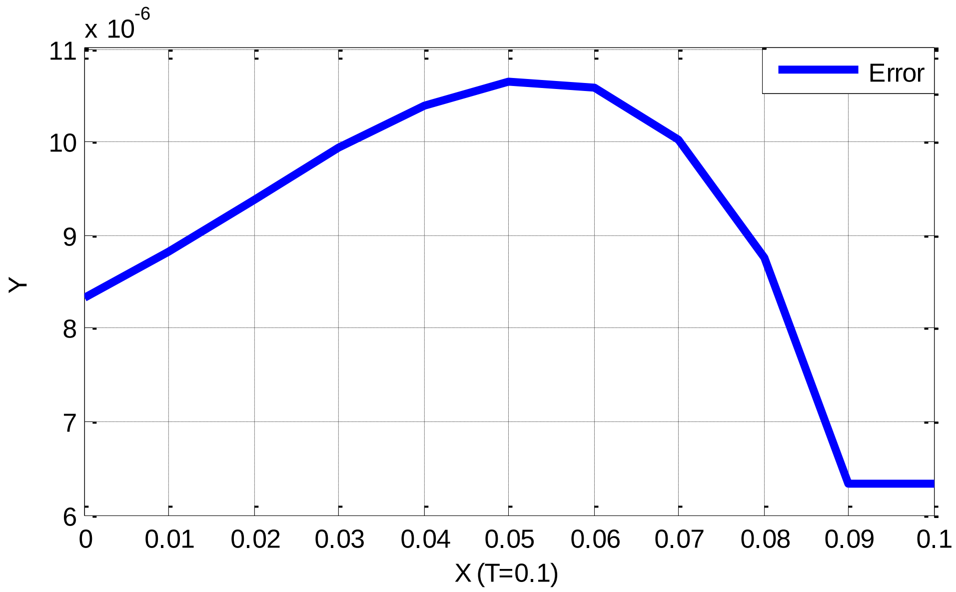

Fig. 4.2(b) Absolute error between exact and numerical solution in example 4.1

Table (4.3) contains both the exact and the numerical results using the Toeplitz matrix method for example (4.1) where the upper bound is 0.5.

x_i	u_e	u_i	$\|u_e - u_i\|$
0	0.0000	0.0006129	0.0006129
0.05	0.0025	0.0031499	0.0006499
0.1	0.0100	0.0106914	0.0006914
0.15	0.0225	0.0232320	0.0007320
0.2	0.0400	0.0407652	0.0007652
0.25	0.0625	0.0632838	0.0007838
0.3	0.0900	0.0907787	0.0007787
0.35	0.1225	0.1232380	0.0007380
0.4	0.1600	0.1606448	0.0006448
0.45	0.2025	0.2029675	0.0004675
0.5	0.2500	0.2495394	0.0004606

Table (4.3):The exact and numerical solutions using Toeplitz matrix algorithm where N=50.

It can be observed that the maximum error is 0.0007838. The exact and approximate results of $u(x)$ are shown in Fig. 4.3 (a) and the resulted error is shown in Fig. 4.3 (b).

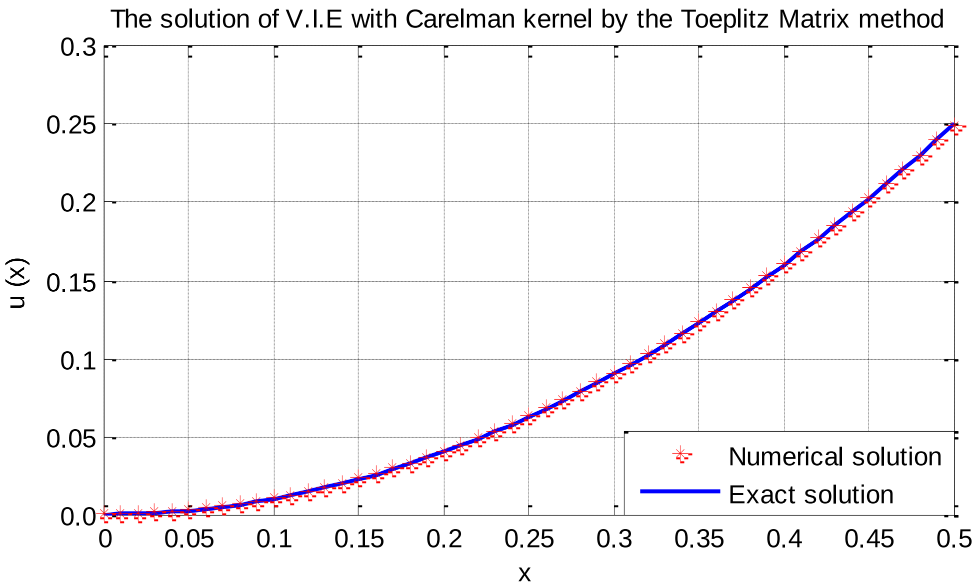

Fig. 4.3 (a) A comparison between the exact and approximate solution in example 4.1

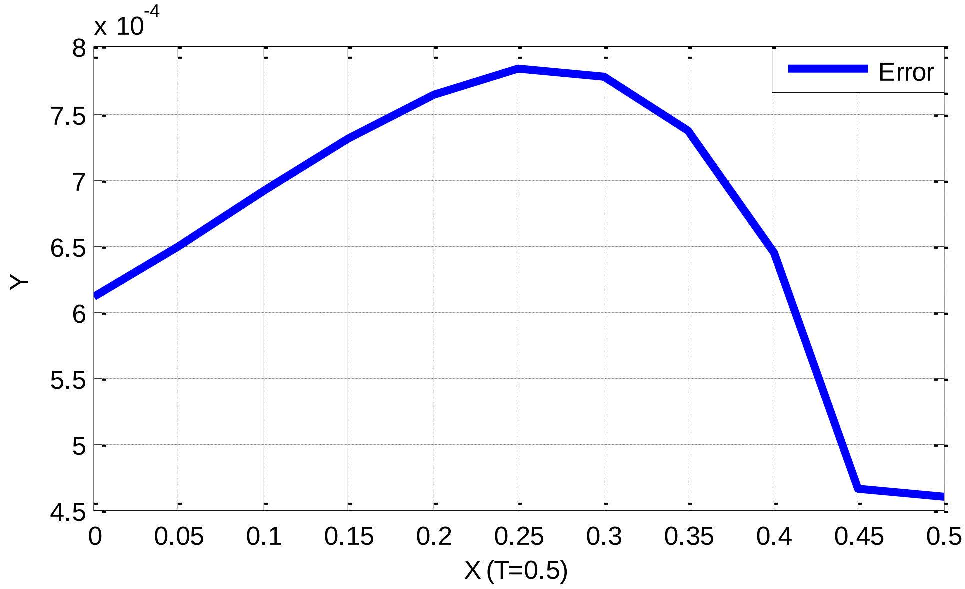

Fig. 4.3(b) Absolute error between exact and numerical solution in example 4.1

Table (4.4) contains both the exact and the numerical results using the Toeplitz matrix method for example (4.1) where the upper bound is 0.9.

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
0	0.0000	0.0029535	0.0029535
0.09	0.0081	0.0112327	0.0031327
0.18	0.0324	0.0357326	0.0033326
0.27	0.0729	0.0764278	0.0035278
0.36	0.1296	0.1332878	0.0036878
0.45	0.2025	0.2062770	0.0037770
0.54	0.2916	0.2953520	0.0037520
0.63	0.3969	0.4004566	0.0035566
0.72	0.5184	0.5215083	0.0031083
0.81	0.6561	0.6583566	0.0022566
0.9	0.8100	0.8077980	0.0022020

Table (4.4):The exact and numerical solutions using Toeplitz Matrix algorithm where N=50

It can be observed that the maximum error is 0.0037770. The exact and approximate results of $u(x)$ are shown in Fig. 3.4 (a) and the resulted error is shown in Fig. 4.4 (b).

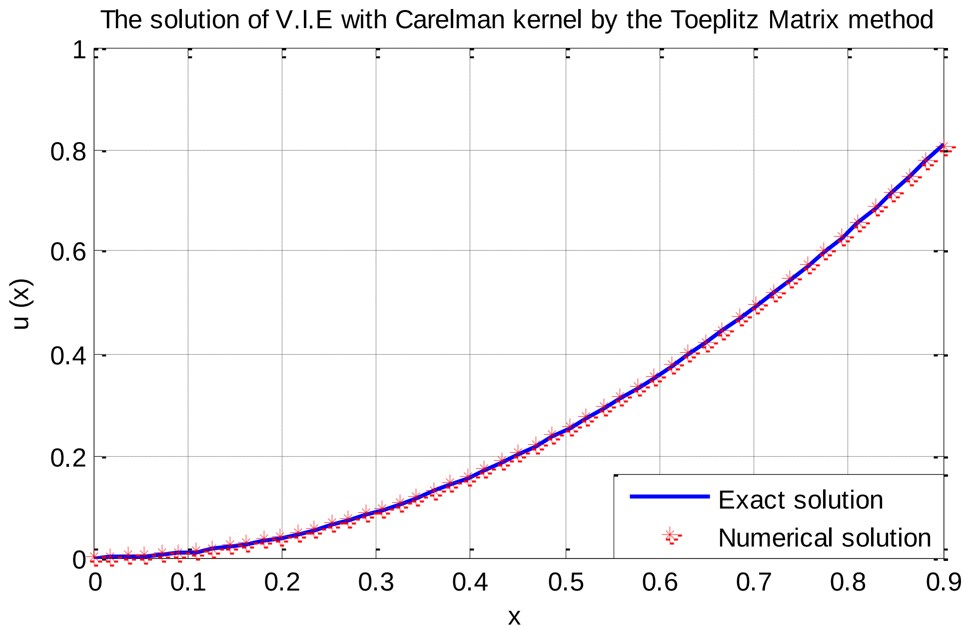

Fig. 4.4 (a) A comparison between the exact and approximate solution in example 4.1

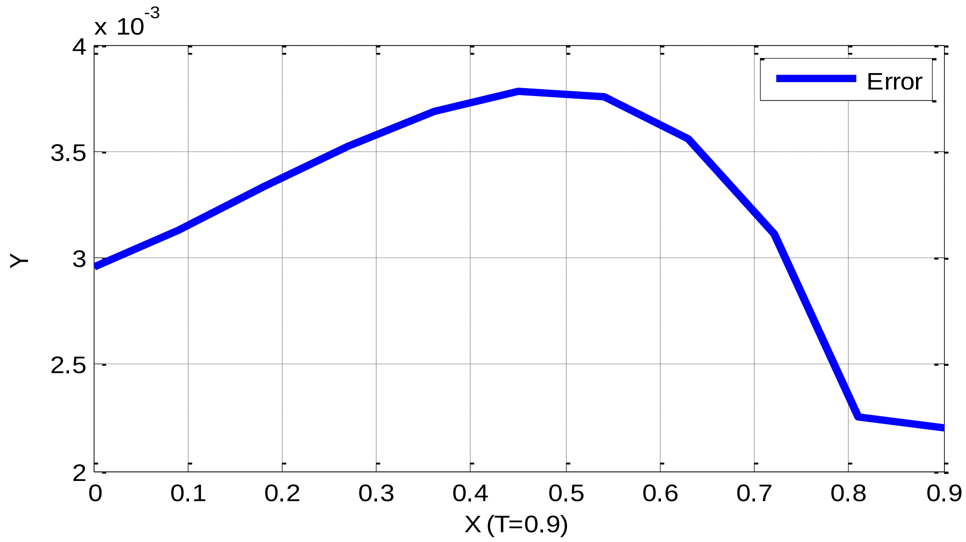

Fig. 4.4 (b) Absolute error between exact and numerical solution in example 4.1

Product Nystrom Method

Equation (4.1) is solved when the upper bound is 0.1, 0.5 and 0.9, respectively. Applying algorithm (3.2) for example (4.1).

Table (4.5) contains both the exact and the numerical results using the Product Nystrom method for example (4.1) where the upper bound is 0.1.

x_i	u_e	u_i	$\|u_e - u_i\|$
0	0.0000	0.0000081	0.0000081
0.01	0.0001	0.0001086	0.0000086
0.02	0.0004	0.0004092	0.0000092
0.03	0.0009	0.0009097	0.0000097
0.04	0.0016	0.0016101	0.0000101
0.05	0.0025	0.0025104	0.0000104
0.06	0.0036	0.0036103	0.0000103
0.07	0.0049	0.0049098	0.0000098
0.08	0.0064	0.0064084	0.0000084
0.09	0.0081	0.0081060	0.0000060
0.1	0.0100	0.0100000	0.0000000

Table (4.5):The exact and numerical solutions using Product Nystrom algorithm where N=50

It can be observed that the maximum error is 0.0000104.

The exact and approximate results of $u(x)$ are shown in Fig. 4.5 (a) and the resulted error is shown in Fig. 4.5 (b).

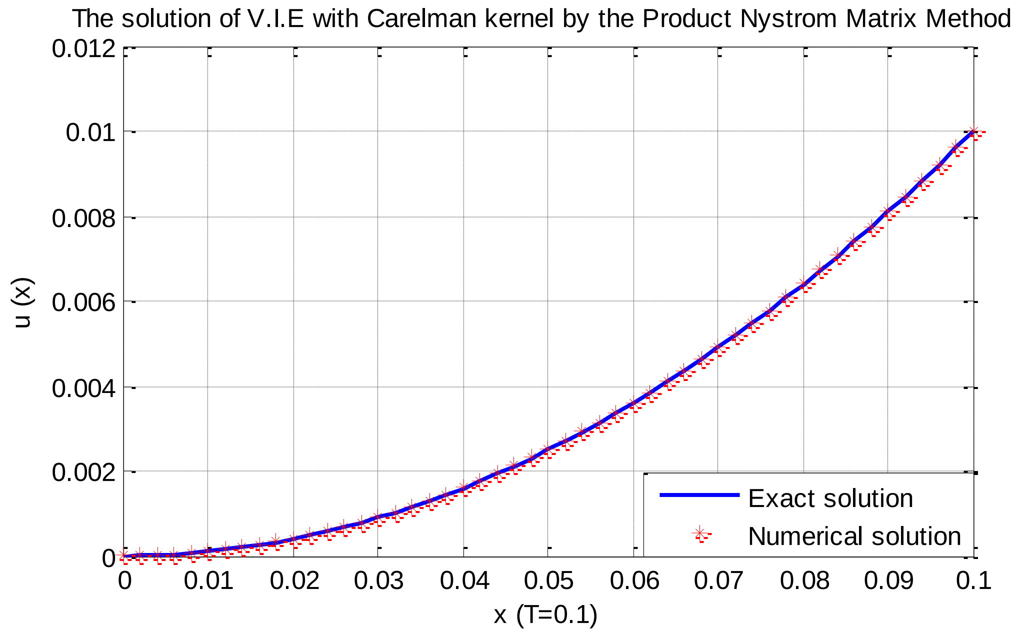

Fig. 4.5 (a) A comparison between the exact and approximate solution in example 4.1

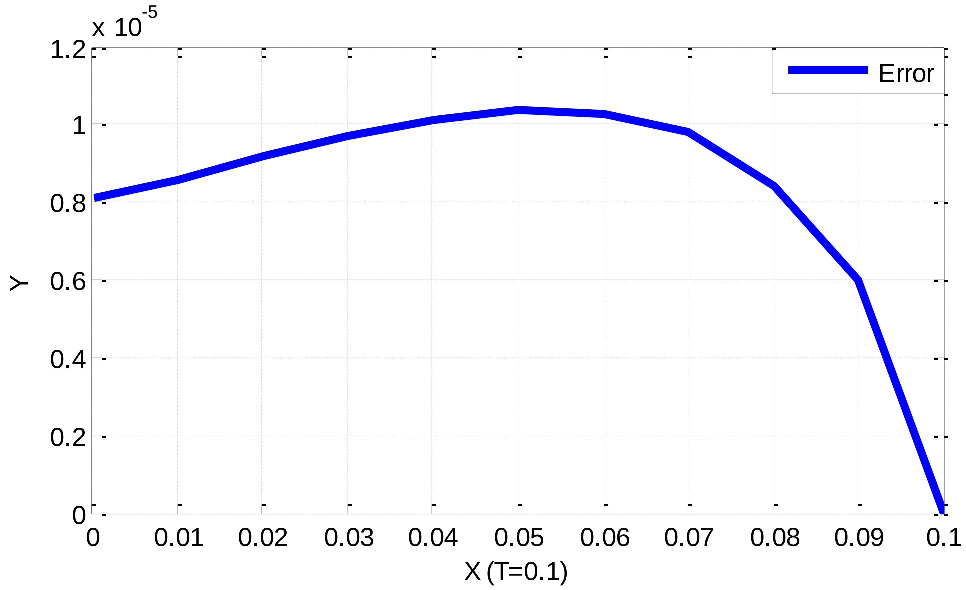

Fig. 4.5 (b) Absolute error between exact and numerical solution in example 4.1

Table (4.6) contains both the exact and the numerical results usingthe Product Nystrom method for example (4.1) where the upper bound is 0.5.

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
0	0.0000	0.0005967	0.0005967
0.05	0.0025	0.0031333	0.0006333
0.1	0.0100	0.0106743	0.0006743
0.15	0.0225	0.0232146	0.0007146
0.2	0.0400	0.0407472	0.0007472
0.25	0.0625	0.0632656	0.0007656
0.3	0.0900	0.0907591	0.0007591
0.35	0.1225	0.1232180	0.0007180
0.4	0.1600	0.1606215	0.0006215
0.45	0.2025	0.2029400	0.0004400
0.5	0.2500	0.2500055	0.0000055

Table (4.6):The exact and numerical solutions using Product Nystrom algorithm where N=50

It can be observed that the maximum error is 0.0007656. The exact and approximate results of $u(x)$ are shown in Fig. 4.6 (a) and the resulted error is shown in Fig. 4.6 (b).

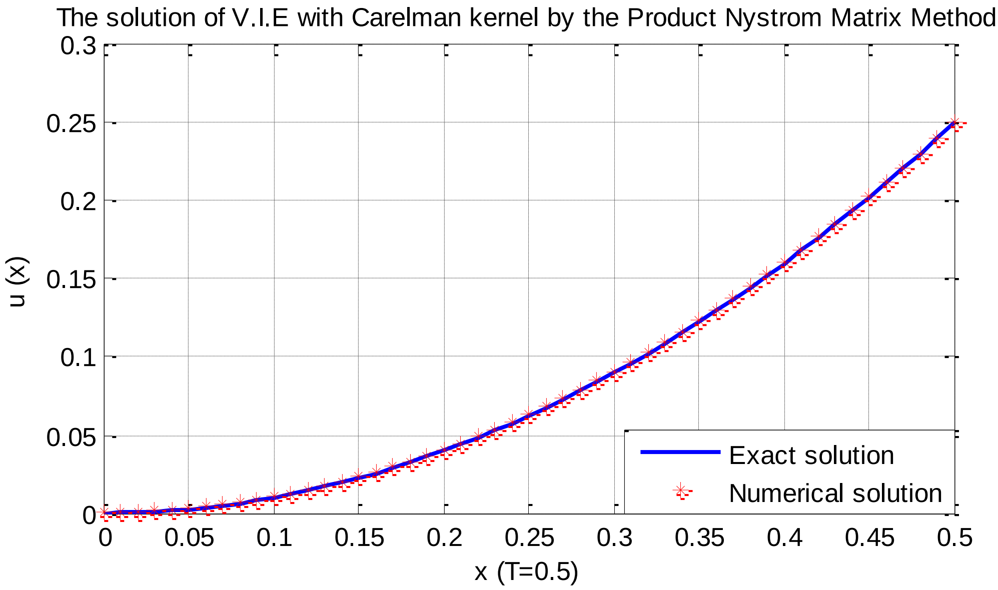

Fig. 4.6 (a) A comparison between the exact and approximate solution in example 4.1

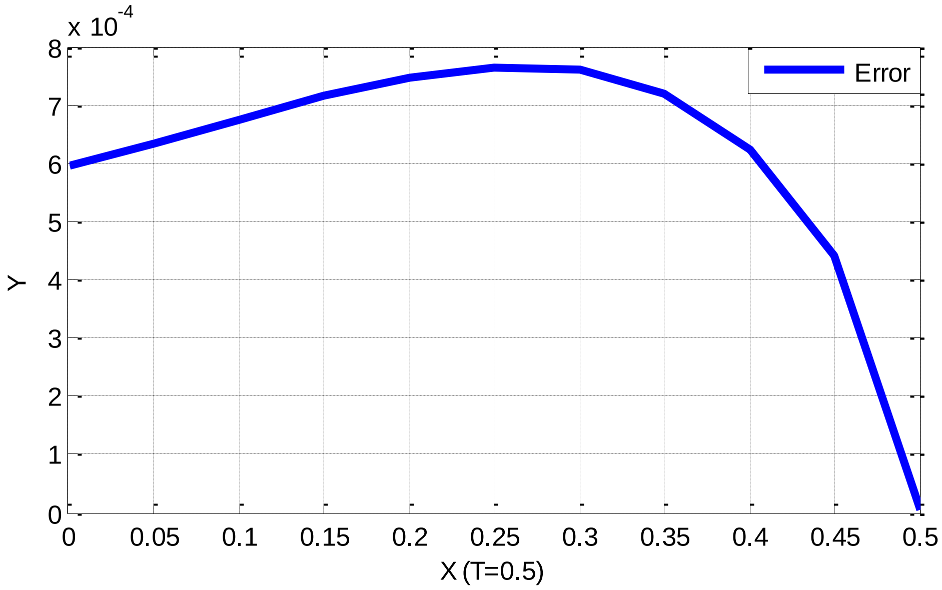

Fig. 4.6 (b) Absolute error between exact and numerical solution in example 4.1

Table (4.7) contains both the exact and the numerical results using the Product Nystrom method for example (4.1) where the upper bound is 0.9.

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
0	0.0000	0.0028752	0.0028752
0.09	0.0081	0.0111528	0.0030528
0.18	0.0324	0.0356503	0.0032503
0.27	0.0729	0.0763440	0.0034440
0.36	0.1296	0.1332007	0.0036007
0.45	0.2025	0.2061890	0.0036890
0.54	0.2916	0.2952579	0.0036579
0.63	0.3969	0.4003602	0.0034602
0.72	0.5184	0.5213963	0.0029963
0.81	0.6561	0.6582247	0.0021247
0.9	0.8100	0.8100392	0.0000392

Table (4.7):The exact and numerical solutions using Product Nystrom algorithm where N=50

It can be observed that the maximum error is 0.0036890.

The exact and approximate results of $u(x)$ are shown in Fig. 4.7 (a) and the resulted error is shown in Fig. 4.7 (b).

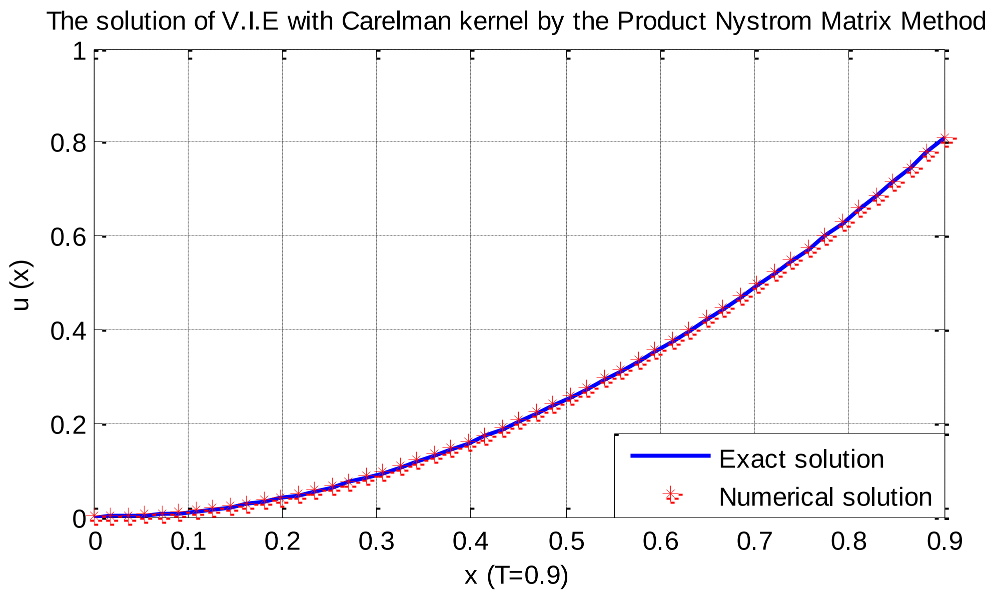

Fig. 4.7 (a) A comparison between the exact and approximate solution in example 4.1

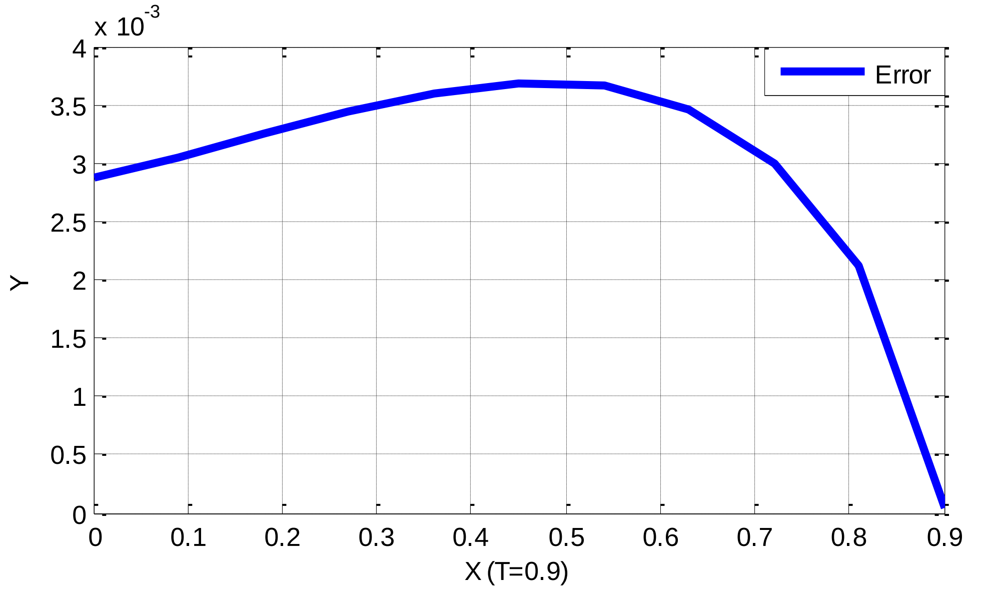

Fig. 4.7 (b) Absolute error between exact and numerical solution in example 4.1

Sinc - Collocation Method

Equation (4.1) is solved when the upper bound is 0.1, 0.5 and 0.9, respectively. Applying algorithm (3.3) for example (4.1).

Table (4.8) contains both the exact and the numerical results using the Sinc-Collocation method for example (4.1) where the upper bound is 0.1.

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
0	0.0000	0.0000130	0.0000130
0.01	0.0001	0.0001133	0.0000133
0.02	0.0004	0.0004132	0.0000132
0.03	0.0009	0.0009081	0.0000081
0.04	0.0016	0.0015915	0.0000085
0.05	0.0025	0.0024821	0.0000179
0.06	0.0036	0.0035917	0.0000083
0.07	0.0049	0.0049082	0.0000082
0.08	0.0064	0.0064126	0.0000126
0.09	0.0081	0.0081109	0.0000109
0.1	0.0100	0.0100049	0.0000049

Table (4.8):The exact and numerical solutions using Sinc-Collocation algorithm where N=50.

It can be observed that the maximum error is 0.0000179.

The exact and approximate results of $u(x)$ are shown in Fig. 4.8 (a) and the resulted error is shown in Fig. 4.8 (b).

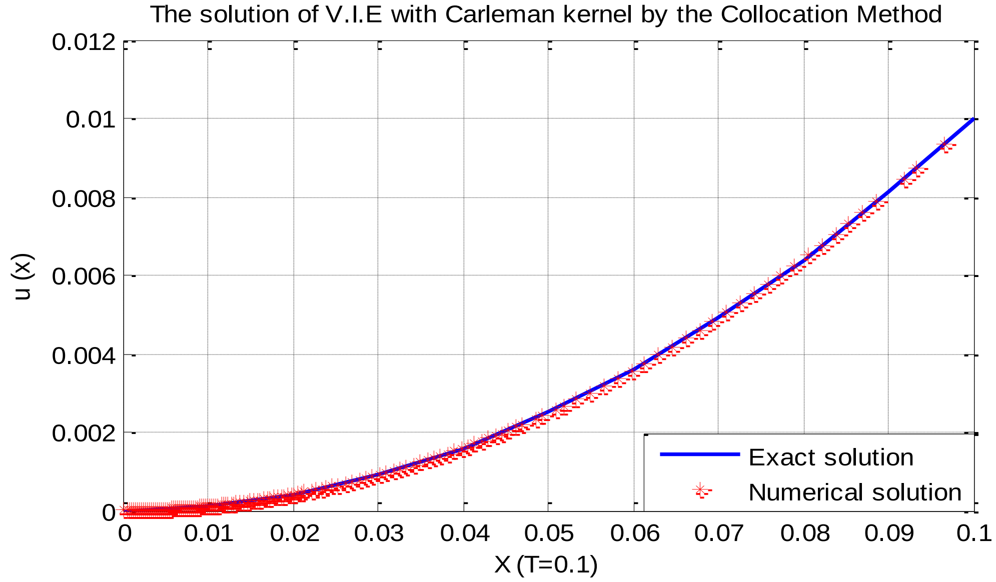

Fig. 4.8 (a) A comparison between the exact and approximate solution in example 4.1

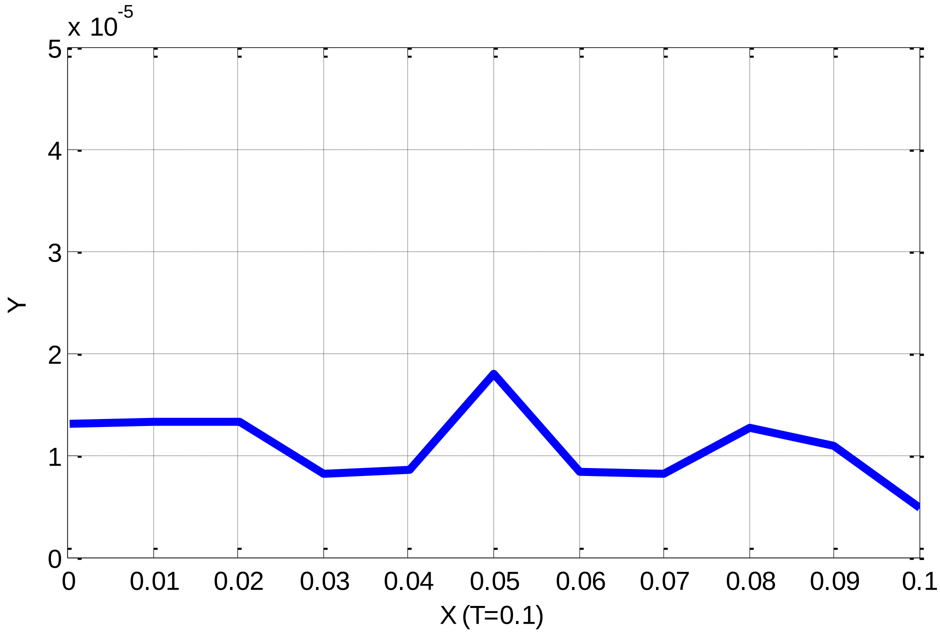

Fig. 4.8(b) Absolute error between exact and numerical solution in example 4.1

Table (4.9) contains both the exact and the numerical results using the Sinc-Collocation method for example (4.1) where the upper bound is 0.5.

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
0	0.0000	0.0009513	0.0009513
0.05	0.0025	0.0034804	0.0009804
0.1	0.0100	0.0109753	0.0009753
0.15	0.0225	0.0231266	0.0006266
0.2	0.0400	0.0394792	0.0005208
0.25	0.0625	0.0613296	0.0011704
0.3	0.0900	0.0894952	0.0005048
0.35	0.1225	0.1231375	0.0006375
0.4	0.1600	0.1609348	0.0009348
0.45	0.2025	0.2033026	0.0008026
0.5	0.2500	0.2503595	0.0003595

Table (4.9):The exact and numerical solutions using Sinc-Collocation algorithm where N=50.

It can be observed that the maximum error is 0.0011704.

The exact and approximate results of $u(x)$ are shown in Fig. 4.9 (a) and the resulted error is shown in Fig. 4.9 (b).

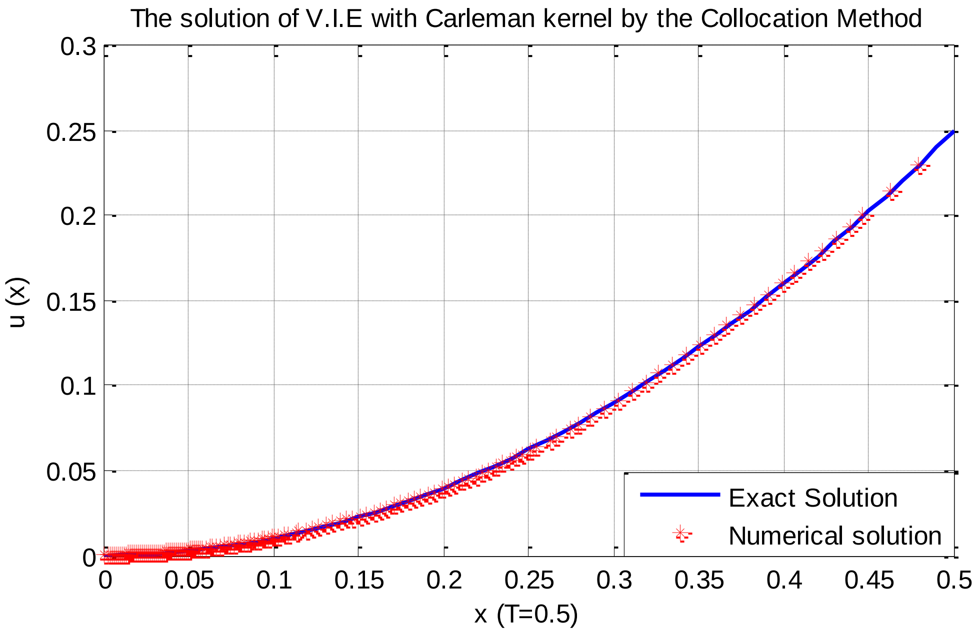

Fig. 4.9 (a) A comparison between the exact and approximate solution in example 4.1

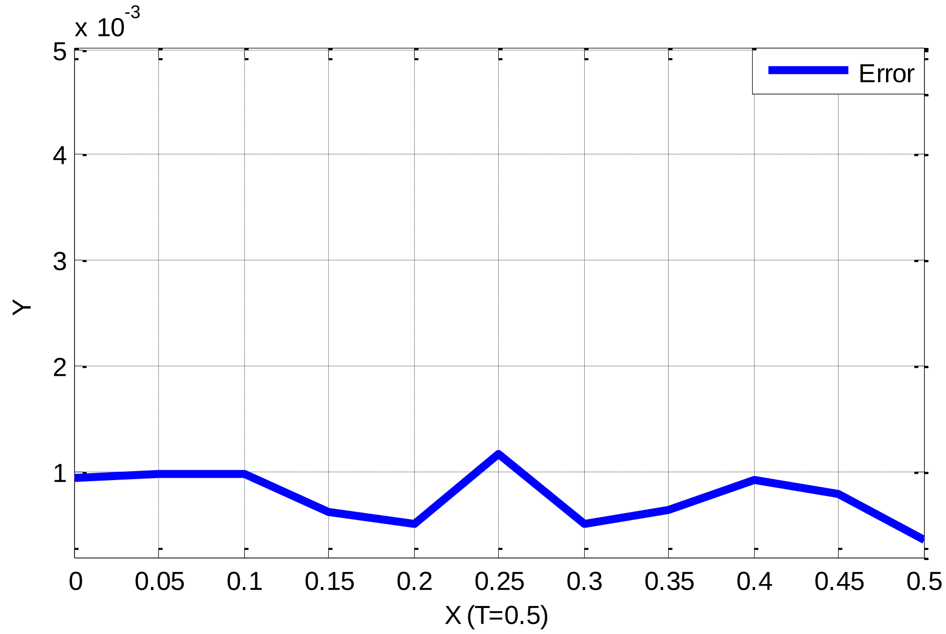

Fig. 4.9(b) Absolute error between exact and numerical solution in example 4.1

Table (4.10) contains both the exact and the numerical results using the Sinc-Collocation method for example (4.1) where the upper bound is 0.9.

x_i	u_e	u_i	$\lvert u_e - u_i \rvert$
0	0.0000	0.0045622	0.0045622
0.09	0.0081	0.0128197	0.0047197
0.18	0.0324	0.0371140	0.0047140
0.27	0.0729	0.0760201	0.0031201
0.36	0.1296	0.1274352	0.0021648
0.45	0.2025	0.1973416	0.0051584
0.54	0.2916	0.2895107	0.0020893
0.63	0.3969	0.4000702	0.0031702
0.72	0.5184	0.5229171	0.0045171
0.81	0.6561	0.6599638	0.0038638
0.9	0.8100	0.8096076	0.0003924

Table (4.10):The exact and numerical solutions using Sinc-Collocation algorithm where N=50.

It can be observed that the maximum error is 0.0051584.

The exact and approximate results of $u(x)$ are shown in Fig. 4.10 (a) and the resulted error is shown in Fig. 4.10 (b).

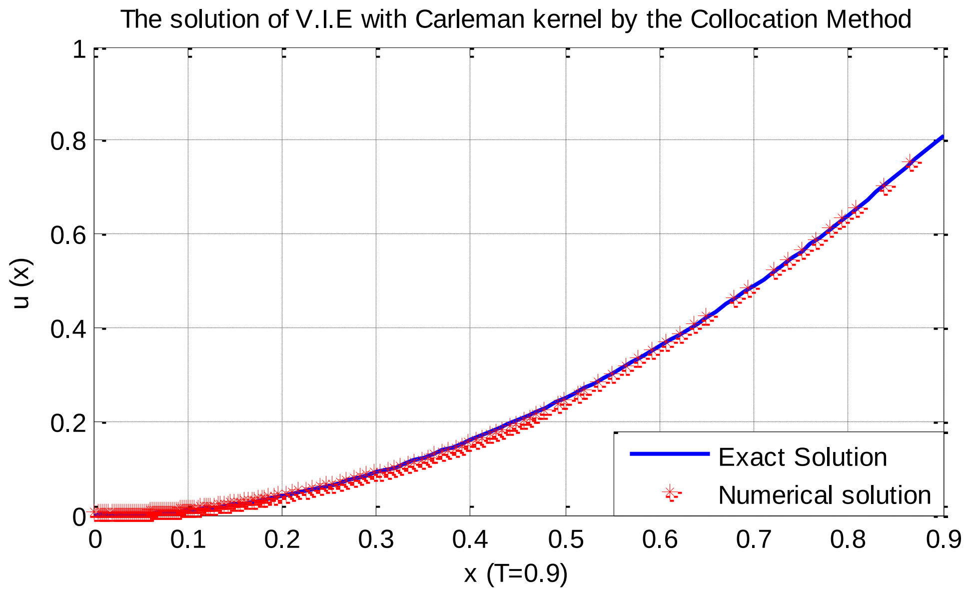

Fig. 4.10 (a) A comparison between the exact and approximate solution in example 4.1

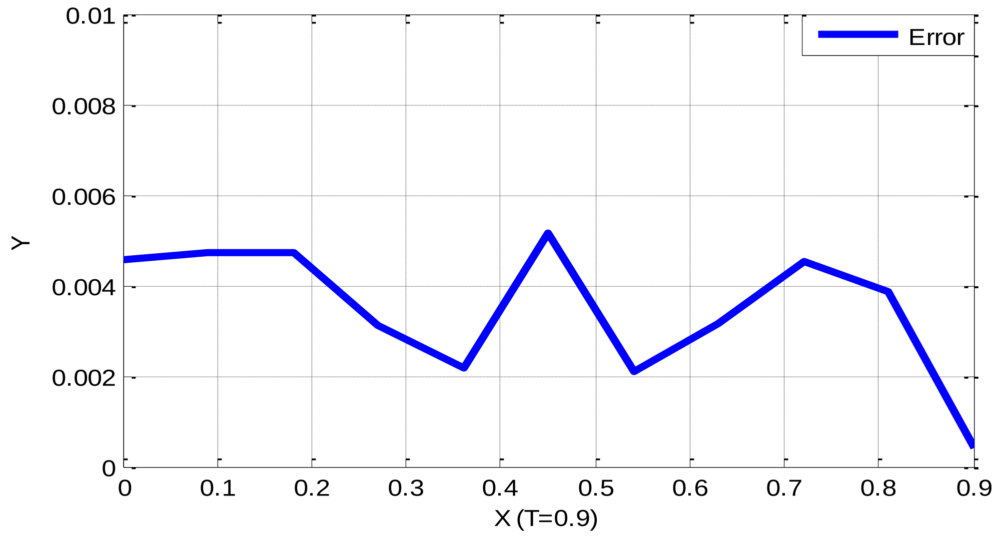

Fig. 4.10 (b) Absolute error between exact and numerical solution in example 4.1

Conclusion

Integral equations with a Carleman kernel are used frequently in various fields of science and technology.

We have attempted to solve such equations using various numerical techniques, namely; Toeplitz matrix method, product Nystrom method, Sinc-collocation method and Laplace Adomian decomposition method.

The numerical methods were performed in a form of algorithms to solve some numerical test cases using Matlab software.

It has been observed that the Product Nystrom method is one of the most powerful numerical technique for solving Fredholm integral equation of the second kind with a Carleman kernel in comparison with Toeplitz matrix method and sinc-collocation method. On the other hand, the Laplace Adomian decomposition method has shown to be one of the most efficient method for solving Volterra integral equation with Carleman kernel. In fact, it is more accurate than Toeplitz matrix method, product Nystrom method and sinc-collocation method.

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Appendix

MATLAB code for Toeplitz matrix method for Fredholm integral

```
%The solution of F.S.I.E with Carelman kernel by the Toeplitz matrix
method

clear all
tic
N=10; N1=2*N+1; a=1; v=0.5;
Be=zeros(N1,N1); De=zeros(N1,N1); Ee=zeros(N1,N1); Phi=zeros(N1);
h=a/N;
lamda=2/3;

for i=1:N1
for j=1:N1
ii=(-N-1)+i;
jj=(-N-1)+j;
Be(i,j)=elementB(h,v,ii,jj);
end
end

for i=1:N1
for j=1:N1
ii=(-N-1)+i;
jj=(-N-1)+j;
```

```
Ee(i,j)=elementC(h,v,N,ii,jj);
end
end

EeT=Ee';
total=Be-EeT;
abs(total)

for i=1:N1
for j=1:N1
ii=(-N-1)+i;
jj=(-N-1)+j;
Total(i,j)=Be(i,j)+EeT(i,j);   % a_n,m
end
end

for i=1:N1
for j=1:N1
if(i==j)
De(i,j)=1-(2/3)*(abs(total(i,j)));
else
De(i,j)=-(2/3)*(abs(total(i,j)));
end
end
```

```
end

x=-1:h:1;
f=elementf(x);
phi=De\f'
Exact=(1-x.^2).^(3/4);
%Er=abs(sin(x)'-phi);
%plot(x,phi,'r',x,f,'b^')
plot(x,phi,'r',x,Exact,'b')
toc
functionBmn=elementB(h,v,m,n)
if m>=n

Bmn=(n+1)*(((h*(m-(n+1)))^(1-v)/(v-1))-((h*(m-n))^(1-v)/(v-
1)))+(((m+(1-v)*(n+1))*(h*(m-(n+1)))^(1-v))/((v-1)*(v-2)))-2*(((m+(1-
v)*n)*(h*(m-n))^(1-v))/((v-1)*(v-2)))+(((m+(1-v)*(n-1))*(h*(m-n+1))^(1-
v))/((v-1)*(v-2)))-((n-1)*(h*(m-n))^(1-v))/(v-1)+(((n-1)*(h*(m-n+1))^(1-
v))/(v-1));

else

Bmn=(n+1)*((-(h*(n+1-m))^(1-v)/(v-1))+((h*(n-m))^(1-v)/(v-1)))-
(((m+(1-v)*(n+1))*(h*(n+1-m))^(1-v))/((v-1)*(v-2)))+2*(((m+(1-
v)*n)*(h*(n-m))^(1-v))/((v-1)*(v-2)))-(((m+(1-v)*(n-1))*(h*(n-1-m))^(1-
```

```
v))/((v-1)*(v-2)))+(((n-1)*(h*(n-m))^(1-v))/(v-1))-(((n-1)*(h*(n-m-1))^(1-v))/(v-1));

end
function E=elementC(h,v,N,m,n)

if(n==-N)
% B(-N-1)
E=-(((m+(1-v)*(-N))*(h*(m+N))^(1-v))/((v-1)*(v-2)))+(((m+(1-v)*(-N-1))*(h*(m+N+1))^(1-v))/((v-1)*(v-2)))-(((-N-1)*(h*(m+N))^(1-v))/(v-1))+(((-N-1)*(h*(m+N+1))^(1-v))/(v-1));
else
if(n==N)
E=(N+1)*((-(h*(N+1-m))^(1-v)/(v-1))+((h*(N-m))^(1-v)/(v-1)))-(((m+(1-v)*(N+1))*(h*(N+1-m))^(1-v))/((v-1)*(v-2)))+(((m+(1-v)*N)*(h*(N-m))^(1-v))/((v-1)*(v-2)));
else
E=0;
end
end
function y=elementf(x)
y=(1-(x.^2)).^(3/4)-(2-(x.^2))*(pi/(2*sqrt(2)));
```

MATLAB code for Toeplitz matrix method for Volterra integral

```
%The solution of V.I.E with Carelman kernel by the Toeplitz matrix method

clear all
clc
format longE
tic
N=10;
N1=N+1;
T=1;a=T;h=a/N;
v=1/3;
Be=zeros(N1,N1); Phi=zeros(N1);
Lamda=0.01;

% Gmn matrix
for i=1:N1
for j=1:N1
ii=i-1;
jj=j-1;
Be(i,j)=elementB(h,v,ii,jj);
end
end
```

```
% Emn matrix
for i=1:N1
for j=1:N1
ii=i-1;
jj=j-1;
Ee(i,j)=elementC(h,v,N,ii,jj);
end
end

% Dmn matrix=Gmn-Emn
EeT=Ee';
Total=Be-EeT;
abs(Total);

% Soliution of the system of equations
for i=1:N1
for j=1:N1
if(i==j)
De(i,j)=1-Lamda*abs(Total(i,j));
else
De(i,j)=-Lamda*abs(Total(i,j));
end
end
end
```

```
x=0:h:1;
f=elementf(x);
phi=De\f';
Exact=x.^2;

% Soliution plot
plot(x,phi,'r*',x,Exact,'b')
toc
function E=elementC(h,v,N,m,n)

if(n==-N)
% B(-N-1)
E=-(((m+(1-v)*(-N))*(h*(m+N))^(1-v))/((v-1)*(v-2)))+(((m+(1-v)*(-N-1))*(h*(m+N+1))^(1-v))/((v-1)*(v-2)))-(((-N-1)*(h*(m+N))^(1-v))/(v-1))+(((-N-1)*(h*(m+N+1))^(1-v))/(v-1));
else
if(n==N)

E=(N+1)*((-(h*(N+1-m))^(1-v)/(v-1))+((h*(N-m))^(1-v)/(v-1)))-(((m+(1-v)*(N+1))*(h*(N+1-m))^(1-v))/((v-1)*(v-2)))+(((m+(1-v)*N)*(h*(N-m))^(1-v))/((v-1)*(v-2)));
else
E=0;
end
```

```
end

%function y=elementf(x)
y=x.^2;

%functionBmn=elementB(h,v,m,n)
if m>=n

Bmn=(n+1)*(((h*(m-(n+1)))^(1-v)/(v-1))-((h*(m-n))^(1-v)/(v-
1)))+(((m+(1-v)*(n+1))*(h*(m-(n+1)))^(1-v))/((v-1)*(v-2)))-2*(((m+(1-
v)*n)*(h*(m-n))^(1-v))/((v-1)*(v-2)))+(((m+(1-v)*(n-1))*(h*(m-n+1))^(1-
v))/((v-1)*(v-2)))-((n-1)*(h*(m-n))^(1-v))/(v-1)+(((n-1)*(h*(m-n+1))^(1-
v))/(v-1));

else

Bmn=(n+1)*((-(h*(n+1-m))^(1-v)/(v-1))+((h*(n-m))^(1-v)/(v-1)))-
(((m+(1-v)*(n+1))*(h*(n+1-m))^(1-v))/((v-1)*(v-2)))+2*(((m+(1-
v)*n)*(h*(n-m))^(1-v))/((v-1)*(v-2)))-(((m+(1-v)*(n-1))*(h*(n-1-m))^(1-
v))/((v-1)*(v-2)))+(((n-1)*(h*(n-m))^(1-v))/(v-1))-(((n-1)*(h*(n-m-1))^(1-
v))/(v-1));


end
```

MATLAB code for product Nystrom method for Fredholm integral

```
% The solution of F.I.E with Carelman kernel by the Product Nystrom
Matrix Method

clear all
clc
format longE
tic
N=200;   % N should be even.
N1=N+1; a=-1; b=1; h=(b-a)/N ;v=0.5;
Be=zeros(N1,N1); De=zeros(N1,N1); Phi=zeros(N1);
Lamda=2/3;

% First coloumn
for i=0:N
j=0;
jj=j;
ii=i;
Be(i+1,j+1)=elementB(h,v,ii,jj);
end

%Cases 1 & 2, odd and even
for j=1:N-1
for i=0:N
```

```
jj=j;
ii=i;

if i>j

if (rem(jj,2)==0)
Be(i+1,j+1)=elementCE1(h,v,ii,jj);
end

if (rem(jj,2)~=0)
Be(i+1,j+1)=elementCO1(h,v,ii,jj);
end

else%%%%%%%%%%%%%%%%%%

if (rem(jj,2)==0)
Be(i+1,j+1)=elementCE2(h,v,ii,jj);
end

if (rem(jj,2)~=0)
Be(i+1,j+1)=elementCO2(h,v,ii,jj);
end

end
```

```
end
end

% Last coloumn
for i=0:N
j=N;
jj=j;
ii=i;
Be(i+1,j+1)=elementD(h,v,N,ii);
end

Total=Be;
abs(Total);

% Soliution matrix
for i=1:N1
for j=1:N1
if(i==j)
De(i,j)=1-Lamda*abs(Total(i,j));
else
De(i,j)=-Lamda*abs(Total(i,j));
end
end
end
```

```
x=-1:h:1;

f=elementf(x);

phi=De\f';

Exact=(1-x.^2).^(3/4);


% plotting the solution

plot(x,phi,'r*',x,Exact,'b')

toc

function y=elementf(x)

y=(1-(x.^2)).^(3/4)-(2-(x.^2))*(pi/(2*sqrt(2)));

function W_L=elementD(h,v,N,ii)


% Last coloumn


x=ii-N+2;


E=(h^(1-v))/2;

A_b=[4*v^2-4*x*v-12*v+4*x+2*x^2+8]*(2-x)^(1-v);

D_h=(-x)^(3-v);

C_c=(v-1)*(v-2)*(v-3);

B_a=(x+2*(1-v))*(2-x)^(1-v);

C_b=(v-1)*(v-2);

D_f=(-x)^(2-v);
```

```
W_L=E*[((-A_b+2*D_h)/C_c)-(B_a/C_b)-(D_f/C_b)];
function W=elementCO2(h,v,ii,jj)

% case 2, odd coloumns
jj=(jj-1)/2;

x=ii-2*jj;

%%%
E=(h^(1-v))/2;
B_a=(x+2*(1-v))*(2-x)^(1-v);
D_f=(-x)^(2-v);
C_b=(v-1)*(v-2);
A_b=[4*v^2-4*x*v-12*v+4*x+2*x^2+8]*(2-x)^(1-v);
C_c=(v-1)*(v-2)*(v-3);
D_h=(-x)^(3-v);
%%%

W=2*E*[((2*B_a+2*D_f)/C_b)+(A_b/C_c)-(2*D_h/C_c)];
end
function W=elementCO1(h,v,ii,jj)

% case 1, odd coloumns
jj=(jj-1)/2;
```

```
x=ii-2*jj;


%%%
E=(h^(1-v))/2;
B_b=(x+2*(1-v))*(x-2)^(1-v);
D_g=(x)^(2-v);
C_b=(v-1)*(v-2);
A_a=[4*v^2-4*x*v-12*v+4*x+2*x^2+8]*(x-2)^(1-v);
C_c=(v-1)*(v-2)*(v-3);
D_e=(x)^(3-v);
%%%


W=2*E*[((-2*B_b+2*D_g)/C_b)-(A_a/C_c)+(2*D_e/C_c)];
end



function W=elementCE2(h,v,ii,jj)


% case 2, even coloumns
jj=(jj)/2;


x=ii-2*jj+2;


%%%
```

```
E=(h^(1-v))/2;

A_b=[4*v^2-4*x*v-12*v+4*x+2*x^2+8]*(2-x)^(1-v);

D_h=(-x)^(3-v);

C_c=(v-1)*(v-2)*(v-3);

B_a=(x+2*(1-v))*(2-x)^(1-v);

D_f=(-x)^(2-v);

C_b=(v-1)*(v-2);

D_m=(2-(x-2))^(1-v);

C_a=(v-1);

D_l=(-(x-2))^(1-v);

B_d=((x-2)+2*(1-v))*(2-(x-2))^(1-v);

D_o=(-(x-2))^(2-v);

A_d=[4*v^2-4*(x-2)*v-12*v+4*(x-2)+2*(x-2)^2+8]*(2-(x-2))^(1-v);

D_k=(-(x-2))^(3-v);

C_c=(v-1)*(v-2)*(v-3);

%%%


W=E*[((-A_b+2*D_h)/C_c)-((B_a+D_f)/C_b)-

(2*D_m/C_a)+2*(D_l/C_a)-(3*B_d/C_b)-3*(D_o/C_b)+((-

A_d+2*D_k)/C_c)];

end

function W=elementCE1(h,v,ii,jj)


% case 1, even coloumns
```

```
jj=(jj)/2;

  x=ii-2*jj+2;

%%%
E=(h^(1-v))/2;
A_a=[4*v^2-4*x*v-12*v+4*x+2*x^2+8]*(x-2)^(1-v);
D_e=(x)^(3-v);
C_c=(v-1)*(v-2)*(v-3);
B_b=(x+2*(1-v))*(x-2)^(1-v);
D_g=(x)^(2-v);
C_b=(v-1)*(v-2);
D_n=((x-2)-2)^(1-v);
C_a=(v-1);
D_a=(x-2)^(1-v);
B_c=((x-2)+2*(1-v))*((x-2)-2)^(1-v);
C_b=(v-1)*(v-2);
D_i=(x-2)^(2-v);
A_c=[4*v^2-4*(x-2)*v-12*v+4*(x-2)+2*(x-2)^2+8]*((x-2)-2)^(1-v);
D_j=(x-2)^(3-v);
C_c=(v-1)*(v-2)*(v-3);
%%%
```

```
W=E*[((A_a-2*D_e)/C_c)+((B_b-D_g)/C_b)+(2*D_n/C_a)-
2*(D_a/C_a)+(3*B_c/C_b)-3*(D_i/C_b)+((A_c-2*D_j)/C_c)];
end

function W_F=elementB(h,v,ii,jj)

% First coloumn

x=ii;

E=(h^(1-v))/2;
D_a=(x-2)^(1-v);
C_a=(v-1);
D_c=(x)^(1-v);
B_b=(x+2*(1-v))*(x-2)^(1-v);
C_b=(v-1)*(v-2);
D_g=(x)^(2-v);
A_a=[4*v^2-4*x*v-12*v+4*x+2*x^2+8]*(x-2)^(1-v);
D_e=(x)^(3-v);
C_c=(v-1)*(v-2)*(v-3);

W_F=E*[(2*D_a/C_a)-(2*D_c/C_a)+(3*B_b/C_b)-(3*D_g/C_b)+((A_a-
2*D_e)/C_c)];
```

MATLAB code for product Nystrom method for Volterra integral

```
clear all
clc
format longE
tic
N=50;   % N should be even.

T=0.9;a=T;h=a/N;
v=1/3;

N1=N+1;
Be=zeros(N1,N1); De=zeros(N1,N1); Phi=zeros(N1);
Lamda=0.01;

% First coloumn
for i=0:N
j=0;
jj=j;
ii=i;
Be(i+1,j+1)=elementB(h,v,ii,jj);
end

%Cases 1 & 2, odd and even
```

```
for j=1:N-1
for i=0:N
jj=j;
ii=i;

if i>j

if (rem(jj,2)==0)
Be(i+1,j+1)=elementCE1(h,v,ii,jj);
end

if (rem(jj,2)~=0)
Be(i+1,j+1)=elementCO1(h,v,ii,jj);
end

else%%%%%%%%%%%%%%%%%%

if (rem(jj,2)==0)
Be(i+1,j+1)=elementCE2(h,v,ii,jj);
end

if (rem(jj,2)~=0)
Be(i+1,j+1)=elementCO2(h,v,ii,jj);
end
```

```
end
end
end

% Last coloumn
for i=0:N
j=N;
jj=j;
ii=i;
Be(i+1,j+1)=elementD(h,v,N,ii);
end

Total=Be;
abs(Total);

% Soliution matrix
for i=1:N1
for j=1:N1
if(i==j)
De(i,j)=1-Lamda*abs(Total(i,j));
else
De(i,j)=-Lamda*abs(Total(i,j));
end
end
```

```
end

x=0:h:T;
f=elementf(x);
phi=De\f';
Exact=x.^2;

% plotting the solution
plot(x,phi,'r*',x,Exact,'b')
toc
function y=elementf(x)
y=x.^2;
function W_L=elementD(h,v,N,ii)

% Last coloumn

x=ii-N+2;

E=(h^(1-v))/2;
A_b=[4*v^2-4*x*v-12*v+4*x+2*x^2+8]*(2-x)^(1-v);
D_h=(-x)^(3-v);
C_c=(v-1)*(v-2)*(v-3);
B_a=(x+2*(1-v))*(2-x)^(1-v);
C_b=(v-1)*(v-2);
```

```
D_f=(-x)^(2-v);

W_L=E*[((-A_b+2*D_h)/C_c)-(B_a/C_b)-(D_f/C_b)];
function W=elementCO2(h,v,ii,jj)

% case 2, odd coloumns
jj=(jj-1)/2;

x=ii-2*jj;

%%%
E=(h^(1-v))/2;
B_a=(x+2*(1-v))*(2-x)^(1-v);
D_f=(-x)^(2-v);
C_b=(v-1)*(v-2);
A_b=[4*v^2-4*x*v-12*v+4*x+2*x^2+8]*(2-x)^(1-v);
C_c=(v-1)*(v-2)*(v-3);
D_h=(-x)^(3-v);
%%%

W=2*E*[((2*B_a+2*D_f)/C_b)+(A_b/C_c)-(2*D_h/C_c)];
end

function W=elementCO1(h,v,ii,jj)
```

```
% case 1, odd coloumns
jj=(jj-1)/2;

x=ii-2*jj;

%%%
E=(h^(1-v))/2;
B_b=(x+2*(1-v))*(x-2)^(1-v);
D_g=(x)^(2-v);
C_b=(v-1)*(v-2);
A_a=[4*v^2-4*x*v-12*v+4*x+2*x^2+8]*(x-2)^(1-v);
C_c=(v-1)*(v-2)*(v-3);
D_e=(x)^(3-v);
%%%

W=2*E*[((-2*B_b+2*D_g)/C_b)-(A_a/C_c)+(2*D_e/C_c)];
end
function W=elementCE2(h,v,ii,jj)

% case 2, even coloumns
jj=(jj)/2;

x=ii-2*jj+2;
```

```
%%%
E=(h^(1-v))/2;
A_b=[4*v^2-4*x*v-12*v+4*x+2*x^2+8]*(2-x)^(1-v);
D_h=(-x)^(3-v);
C_c=(v-1)*(v-2)*(v-3);
B_a=(x+2*(1-v))*(2-x)^(1-v);
D_f=(-x)^(2-v);
C_b=(v-1)*(v-2);
D_m=(2-(x-2))^(1-v);
C_a=(v-1);
D_l=(-(x-2))^(1-v);
B_d=((x-2)+2*(1-v))*(2-(x-2))^(1-v);
D_o=(-(x-2))^(2-v);
A_d=[4*v^2-4*(x-2)*v-12*v+4*(x-2)+2*(x-2)^2+8]*(2-(x-2))^(1-v);
D_k=(-(x-2))^(3-v);
C_c=(v-1)*(v-2)*(v-3);
%%%

W=E*[((-A_b+2*D_h)/C_c)-((B_a+D_f)/C_b)-
(2*D_m/C_a)+2*(D_l/C_a)-(3*B_d/C_b)-3*(D_o/C_b)+((-
A_d+2*D_k)/C_c)];
end
function W_F=elementB(h,v,ii,jj)
```

```
% First coloumn

x=ii;

E=(h^(1-v))/2;
D_a=(x-2)^(1-v);
C_a=(v-1);
D_c=(x)^(1-v);
B_b=(x+2*(1-v))*(x-2)^(1-v);
C_b=(v-1)*(v-2);
D_g=(x)^(2-v);
A_a=[4*v^2-4*x*v-12*v+4*x+2*x^2+8]*(x-2)^(1-v);
D_e=(x)^(3-v);
C_c=(v-1)*(v-2)*(v-3);

W_F=E*[(2*D_a/C_a)-(2*D_c/C_a)+(3*B_b/C_b)-(3*D_g/C_b)+((A_a-
2*D_e)/C_c)];
function W=elementCE1(h,v,ii,jj)

% case 1, even coloumns
jj=(jj)/2;

  x=ii-2*jj+2;
```

```
%%%
E=(h^(1-v))/2;
A_a=[4*v^2-4*x*v-12*v+4*x+2*x^2+8]*(x-2)^(1-v);
D_e=(x)^(3-v);
C_c=(v-1)*(v-2)*(v-3);
B_b=(x+2*(1-v))*(x-2)^(1-v);
D_g=(x)^(2-v);
C_b=(v-1)*(v-2);
D_n=((x-2)-2)^(1-v);
C_a=(v-1);
D_a=(x-2)^(1-v);
B_c=((x-2)+2*(1-v))*((x-2)-2)^(1-v);
C_b=(v-1)*(v-2);
D_i=(x-2)^(2-v);
A_c=[4*v^2-4*(x-2)*v-12*v+4*(x-2)+2*(x-2)^2+8]*((x-2)-2)^(1-v);
D_j=(x-2)^(3-v);
C_c=(v-1)*(v-2)*(v-3);
%%%
W=E*[((A_a-2*D_e)/C_c)+((B_b-D_g)/C_b)+(2*D_n/C_a)-
2*(D_a/C_a)+(3*B_c/C_b)-3*(D_i/C_b)+((A_c-2*D_j)/C_c)];
End
```

MATALB sowfwear to find sinc-collocation method

```
% The solution of F.I.E with Carelman kernel by the Sin-Collocation method

clear all
clc
format longE
tic

d=3.14;
Alpha=2/3;
N=50;
N1=2*N+3;
h=sqrt(pi*d/(Alpha*N));
h=0.8*h;
a=0;
b=1;

Lamda=10

P=2/3;    M=ceil(P*N);

%En matrix
```

```
for i=1:N1
   R=-N;
for j=1:N1
if i==j
      E(j,i)=1;

elseif (i==1)&&((j~=1)&&(j~=N1))
   x=R*h;
   t=((b-a)/2)*tanh(x/2)+((b+a)/2);
Wa=((b-t)/(b-a));
   E(j,i)=Wa;
   R=R+1;

elseif(i==N1)&&((j~=1)&&(j~=N1))
   x=R*h;
   t=((b-a)/2)*tanh(x/2)+((b+a)/2);
Wb=((t-a)/(b-a));
   E(j,i)=Wb;
   R=R+1;

else
      E(j,i)=0;
end
```

```
end

end

E1=Lamda*E;


%gn matrix

R=-N;

for j=1:N1


if j==1;

 g(1)=elementf(a);

elseif j==N1

 g(N1)=elementf(b);

else

    x=R*h;

    t=((b-a)/2)*tanh(x/2)+((b+a)/2);

    F=elementf(t);     % Initially, it will be changed later on.

    g(j)=F;

    R=R+1;

end


end
```

```
%kn matrix
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
                    XY=-N-2; %
for i=1:N1
  R=-N;               XY=XY+1; %
%XY=-N-2;  %Here
for j=1:N1
%XY=XY+1;  %Here

if (i==1)&&(j==1)
    z=0;
for m=-N:M
    t=((b-a)/2)*tanh((m*h)/2)+((b+a)/2);
Wa=((b-t)/(b-a));
    z=z+[Wa]/[(1+exp(m*h))*(1+exp(-m*h))^P];
end
    K(j,i)=[(b-a)^P]*h*z;
end

if (i==N1)&&(j==1)
    z=0;
for m=-N:M
    t=((b-a)/2)*tanh((m*h)/2)+((b+a)/2);
```

```
Wb=((t-a)/(b-a));
        z=z+[Wb]/[(1+exp(m*h))*(1+exp(-m*h))^P];
end
        K(j,i)=[(b-a)^P]*h*z;
end

if (i==1)&&(j==N1)
        z=0;
for m=-M:N
        t=((b-a)/2)*tanh((m*h)/2)+((b+a)/2);
Wa=((b-t)/(b-a));
        z=z+[Wa]/[(1+exp(-m*h))*(1+exp(m*h))^P];
end
        K(j,i)=[(b-a)^P]*h*z;
end

if (i==N1)&&(j==N1)
        z=0;
for m=-M:N
        t=((b-a)/2)*tanh((m*h)/2)+((b+a)/2);
Wb=((t-a)/(b-a));
        z=z+[Wb]/[(1+exp(-m*h))*(1+exp(m*h))^P];
end
        K(j,i)=[(b-a)^P]*h*z;
```

```
end


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 1st coloumn
if (i==1)&&((j~=1)&&(j~=N1))
    x=R*h;
    Q=((b-a)/2)*tanh(x/2)+((b+a)/2);      %t-N


   z=0;  %(BN)
for m=-N:M
        t=((b-Q)/2)*tanh((m*h)/2)+((b+Q)/2);
Wa=((b-t)/(b-a));
        z=z+[Wa]/[(1+exp(m*h))*(1+exp(-m*h))^P];
end
        S1=[(b-Q)^P]*h*z;


  z=0;  %(AN)
for m=-M:N
        t=((Q-a)/2)*tanh((m*h)/2)+((Q+a)/2);
Wa=((b-t)/(b-a));
        z=z+[Wa]/[(1+exp(-m*h))*(1+exp(m*h))^P];
end
        S2=[(Q-a)^P]*h*z;
```

```
        K(j,i)=S1+S2;


    R=R+1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Last column
if (i==N1)&&((j~=1)&&(j~=N1))
    x=R*h;
    Q=((b-a)/2)*tanh(x/2)+((b+a)/2);       %t-N


   z=0;  %(BN)
for m=-N:M
        t=((b-Q)/2)*tanh((m*h)/2)+((b+Q)/2);
Wb=((t-a)/(b-a));
       z=z+[Wb]/[(1+exp(m*h))*(1+exp(-m*h))^P];
end
        S1=[(b-Q)^P]*h*z;


  z=0;  %(AN)
for m=-M:N
        t=((Q-a)/2)*tanh((m*h)/2)+((Q+a)/2);
Wb=((t-a)/(b-a));
        z=z+[Wb]/[(1+exp(-m*h))*(1+exp(m*h))^P];
end
```

```
        S2=[(Q-a)^P]*h*z;


        K(j,i)=S1+S2;


    R=R+1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Middle


if ((i>1 && i<N1)&&(j==1))      % 1st row in the middle
    z=0;
for m=-N:M
        t=((b-a)/2)*tanh((m*h)/2)+((b+a)/2);
        x=log10(((t-a)/(b-t)));
%XY;


if x==0   %
            S=1;  %
        z=z+[S]/[(1+exp(m*h))*(1+exp(-m*h))^P];%


else%


        S=sin(pi*(x/h-XY))/(pi*(x/h-XY));
```

```
        z=z+[S]/[(1+exp(m*h))*(1+exp(-m*h))^P];
end%
end
        K(j,i)=[(b-a)^P]*h*z;


end
if ((i>1 && i<N1)&&(j==N1))  % Last row in the middle
    z=0;
for m=-M:N
        t=((b-a)/2)*tanh((m*h)/2)+((b+a)/2);
        x=log10(((t-a)/(b-t)));


if x==0   %
            S=1;  %
        z=z+[S]/[(1+exp(m*h))*(1+exp(-m*h))^P];%


else%


        S=sin(pi*(x/h-XY))/(pi*(x/h-XY));
        z=z+[S]/[(1+exp(-m*h))*(1+exp(m*h))^P];
end%
end
        K(j,i)=[(b-a)^P]*h*z;
end
```

```
if ((i>1 && i<N1)&&(j>1 && j<N1))  % The middle
    x=R*h;
    Q=((b-a)/2)*tanh(x/2)+((b+a)/2);       %t-N

   z=0;  %(BN)
for m=-N:M
        t=((b-Q)/2)*tanh((m*h)/2)+((b+Q)/2);
        x=log10(((t-a)/(b-t)));
        S=sin(pi*(x/h-XY))/(pi*(x/h-XY));
        z=z+[S]/[(1+exp(m*h))*(1+exp(-m*h))^P];
end
        S1=[(b-Q)^P]*h*z;

  z=0;  %(AN)
for m=-M:N
        t=((Q-a)/2)*tanh((m*h)/2)+((Q+a)/2);
        x=log10(((t-a)/(b-t)));
        S=sin(pi*(x/h-XY))/(pi*(x/h-XY));
        z=z+[S]/[(1+exp(-m*h))*(1+exp(m*h))^P];
end
        S2=[(Q-a)^P]*h*z;

        K(j,i)=S1+S2;
```

```
    R=R+1;
end

end
end

phi=(E1-K)\g';

% Creating the solution function
symstSolutionab
a=0;  b=1;
F1=0;
jj=-N;
for j=2:(2*N+2)

F1=F1+phi(j)*[sin(pi*((log((t-a)/(b-t)))/h-jj))/(pi*((log((t-a)/(b-t)))/h-jj))];
% x=F

jj=jj+1;

end

toc
```

```
function y=elementf(t)

y=[10*(t.^2).*(1-t).^2]-(270/30800)*[(t.^(8/3)).*(54*t.^2-126*t+77)+((1-
t).^(8/3)).*((54*(t.^2))+18*t+5)];
```

جامعة النجاح الوطنية
كلية الدراسات العليا

التقنيات العددية لحل المعادلات التكاملية ذات نواة كارلمن

اعداد
ولاء محمد أمين دريدي

اشراف
أ.د.ناجي قطناني

قدمت هذه الاطروحة استكمالا لمتطلبات الحصول على درجه الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية,نابلس-فلسطين .
2017

التقنيات العددية لحل المعادلات التكاملية ذات نواة كارلمن

اعداد

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أ.د.ناجي قطناني

الملخص

المعادلات التكاملية التي تحوي على نواةكارلمن لها العديد من التطبيقات في الفيزياء والهندسة وبالأخص في التوازن الاشعاعي و الإنتقال الحراري والمشاكل الرياضية.

في هذه الرسالة قمنا بالتركيز على الطرق العددية التي تتعامل مع معادله فريدهولم التكاملية ومعادلةفولتيرا التكاملية والتي تحتوي على النواه الشاذةكارلمن، وقد استخدمت هذه الطرق العددية لحل المعادلات التكاملية ذات نواةكارلمن،وتشمل: طريقة المصفوفات المتراصة، طريقة ضرب نيسروم، معادلة لبلاس التحويلية، وطريقة التجميع.

وللتحقق من كفاءة هذه الطرق العددية قمنا بحل بعض الأمثلة العددية، والتي أظهرت قربها من الحلول الدقيقة.وقد توصلنا من خلال الحلول العددية أن طريقة ضرب نيستروم هي الأفضل لحل معادلة فريدهولم التي تحتوي على النواة الشاذة كارلمن مقارنة بالطرق الأخرى.بالإضافة إلى أن طريقة لبلاس التحويلية هي الطريقة الأفضل لحل معادلة فولتيرا.