An-Najah National University

Faculty of Graduate Studies

Best Approximation In General Normed Spaces

By:

Mu'tas Hasan Mahmoud Al-Sayed

Supervisor:

Dr. Abdallah A. Hakawati

Submitted In Partial Fulfillment Of The Requirements For The Degree Of Master Of Mathematics, Faculty of Graduate Studies, At An-Najah National University, At Nablus, Palestine

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This thesis was successfully defended on 8th of July of 2001 and approved by:

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Dedication

I present this work to whom I loved, my family for the support and encouragement, to my land, it's people who suffer more and more .

Acknowledgment

First of all, I thank my God for all the blessings, he bestowed on me and continues to bestow on me.

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Contents

Preface

Let $X=(X,\|\cdot\|)$ be a normed space and suppose that any given x in X is to be approximated by an element y in Y, where Y is a fixed subspace of X. We let d denoted the distance from x to Y. By definition,

$d = d(x,Y) = \inf_{y\in Y}\|x-y\|$.Clearly, d depends on both x and Y, which we keep, fixed, so that the simple notation d is in order. If there exists a $y_0 \in Y$ such that $\|x-y_0\| = d$. then y_0 is called a best approximation of Y to x or a best approximant of x in Y. We see that a best approximation y_0 is an element of minimum distance from the given x. Such a $y_0 \in Y$ may or may not exist; this raises the problem of existence. The problem of uniqueness is of practical interest, too, since for a given x and Y there may be more than one best approximation.

My thesis consists of three chapters. In chapter one we summarize some of the essential and basic concepts which shall be needed in the following chapters, this chapter consists of two sections; in the first one we present metric, normed, Banach spaces, and the last one we present inner product, and Hilbert spaces. This chapter is absolutely fundamental.

In chapter two, we define best approximations in section one. In section two we study some properties of the set of all best approximations $P(x,Y)$. In section three we study some properties of the proximinal set and show that compact subspace and finite-dimensional subspace are proximinal. In section four we consider the problem of uniqueness of best approximation. In section five we review the properties of Orlicz spaces in which we introduce some of the basic theory of proximinality

In chapter three, which is the main body of our thesis, we, in section one, study the main characterizations and properties of best approximations and some consequences of the characterization in arbitrary normed linear spaces. In sections two and three we gives some application in several spaces like $L^1(T,v)$, C(K) and $C_R(K)$.

Chapter One

Preliminaries

In this chapter we summarize some of the essential and basic definitions, theorems, and concepts in functional analysis which will be used in the sequent chapters.

Section (1.1) Metric Spaces, Normed Spaces, and Banach Spaces

<u>Definition 1.1.1:</u> (Metric space, metric).

A metric space is a pair (X,d), where X is a nonempty set and d is a metric on X, that's, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

i) d is real-valued, finite and nonnegative.

ii) d(x,y)=0 if and only if $x=y$.

iii) d(x,y)=d(y,x) (Symmetry)

iv) $d(x,y) \leq d(x,z)+d(z,y)$ (Triangle inequality). •

The diameter $D(A)$ of a nonempty set A in a metric space (X,d) is defined to be

$$D(A) = \sup_{x,y \in A} d(x,y)$$

A is said to be bounded if $D(A) < \infty$.

Let M be a subset of a metric space X, then a point x_0 of X is called an **accumulation point** of M if every neighborhood of x_0 contains at least one point $y \in M$ distinct from x_0. The set consisting of the points of M and the accumulation points of M is called the **closure** of M and is denoted by $\overline{M}$.

Definition 1.1.2: (Dense set).

A subset M of a metric space X is said to be dense in X if $\overline{M} = X$; where $\overline{M}$ is the closure of M in X. •

Definition 1.1.3: (Convergence of a sequence, limit).

A sequence (x_n) in a metric space (X,d) is said to converge in X if there is an $x \in X$ such that

$\lim_{n\to\infty} d(x_n, x) = 0$. The point x is called the limit of the sequence (x_n).

Definition 1.1.4: (Vector space).

A vector space (or linear space) over a field F is a nonempty set X of elements $x, y, \ldots$ (called vectors) together with two algebraic operations. These operations are called vector addition and multiplication of vectors by scalars, that is, by elements of F.

A **subspace** of a vector space X is a nonempty subset Y of X such that for all $y_1, y_2 \in Y$ and all scalars α, β we have $\alpha y_1 + \beta y_2 \in Y$.

A **linear combination** of vectors $x_1, x_2, \ldots, x_n$ of a vector space X is an expression of the form $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$ where the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ are any scalars.

A subset $Y=\{x_1, x_2, \ldots, x_n\}$ of X is said to be **linearly independent** if *whenever* we have $\sum_{i=1}^{i=n} k_i x_i = 0$ then $k_i=0$ for all $i=1,2,\ldots,n$.

<u>Definition 1.1.5:</u> (Finite and infinite dimensional Vector spaces)

A vector space X is said to be finite dimensional if there is a positive integer n such that X contains a linearly independent set of n vectors whereas any set of $n+1$ or more vectors of X is linearly dependent. n is called the dimension of X, written $n=dimX$. If X is not finite dimensional, it is said to be infinite dimensional. •

Definition 1.1.6: (Cauchy sequence, Completeness).

A sequence (x_n) in a metric space (X,d) is said to be **Cauchy** if for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for every $m,n>N$.

The space X is said to be **complete** if every Cauchy sequence in X converges to an element of X. .

Definition 1.1.7: (Normed space, Banach space).

A normed space X is a vector space with a norm defined on it. A Banach space is a complete normed space. A **norm** on a vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ and which has the properties

i) $\|x\| \geq 0$.

ii) $\|x\| = 0$ if and only if $x=0$.

iii) $\|\alpha x\| = |\alpha|\|x\|$.

iv) $\|x+y\| \leq \|x\| + \|y\|$.

Here x and y are arbitrary vectors in X and α any scalar.

A norm on X defines a metric d on X, which is given by $d(x,y) = \|x-y\|$ $(x, y \in X)$ and is called metric induced by the norm $\|.\|$. The normed space just defined is denoted by $(X, \|.\|)$.•

<u>Definition 1.1.8:</u> (Compactness).

A metric space X is said to be compact if every sequence in X has a convergent subsequence. A subset M of X is said to be *compact* if M is compact considered as a subspace of X, that is, if every sequence in M has a

[illegible]ment of M

We present the following result

Theorem 1.1.9: [7]

Every finite dimensional subspace Y of a normed space X is closed in X.

For a finite dimensional normed space we have

Theorem 1.1.10:[7]

In a finite dimensional normed space X, any subset of X is compact if and only if it is closed and bounded.

Theorem 1.1.11:[7]

A continuous mapping T of a compact subset M of a metric space X into $\mathbb{R}$ assumes a maximum and a minimum on M.

Definition 1.1.12: (Linear operator).

A linear operator T is an operator such that

i) the domain $D(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field.

ii) for all x,y in D(T) and scalar k,

$$T(x+y)=T(x)+T(y)$$

$$T(kx)=k\,T(x).$$

<u>Definition 1.1.13:</u> (Linear functional).

A linear functional f is a linear operator with domain in a vector space X and range in the scalar field F of X.

Section (1.2) Inner Product Spaces. Hilbert Spaces

<u>Definition 1.2.1:</u> (Inner product space, Hilbert space)

An inner product space is a vector space X with an inner product defined on X. A Hilbert space is a complete inner product space. An **inner product** on X is a mapping of $X \times X$ into the scalar field F of X; that is, with every pair of vectors x and y in X there is an associated scalar which is written $<x,y>$ and is called the inner product of x and y, such that for all vectors x,y,z and scalar k we have

i) $<x+y,z>=<x,z>+<y,z>$

ii) $<kx,y>=k<x,y>$

iii) $<x,y>=\overline{<y,x>}$

iv) $<x,x>\geq 0$
$<x,x>=0 \quad \textit{if and only if} \qquad x=0.$

In (iii), the bar denotes complex conjugate. Consequently, if X is a real vector space, we simply have $<x,y>=<y,x>$

Inner product spaces are normed spaces, with norm defined as $\|x\|^2=<x,x>$. Hilbert spaces are Banach spaces with norms induced by inner products. Not all normed spaces are inner product spaces.

<u>Definition 1.2.2:</u> (Orthogonality).

An element x of an inner product space X is said to be orthogonal to an element y in X if $< x, y >= 0$. We also say that x and y are orthogonal, and we write $x \perp y$.

Similarly, for subsets $A, B \subset X$ we write

$x \perp A \;\; if \;\; x \perp a$ for all $a \in A$, and $A \perp B \;\; if \;\; a \perp b$ for all $a \in A$ and all $b \in B$.

<u>Definition 1.2.3:</u> (Distance of a point from a subset)

In a metric space *X*, the distance δ from an element $x \in X$ to a nonempty subset *Y* of *X* is defined to be

$\delta = \inf_{y \in Y} d(x, y)$. In a normed space this becomes

$\delta = \inf_{y \in Y} \|x - y\|$.

<u>Theorem 1.2.4:</u> [7]

In an inner product space,

$if \;\; x_n \to x \;\; and \;\; y_n \to y \;\; then < x_n, y_n > \to < x, y >$

Proof:

$$\begin{aligned}|<x_n,y_n> - <x,y>| &= |<x_n,y_n> - <x_n,y> + <x_n,y> - <x,y>| \\ &\le |<x_n,y_n - y>| + |<x_n - x,y>| \\ &\le \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| \to 0\end{aligned}$$

since $y_n - y \to 0$ & $x_n - x \to 0 \quad as \quad n \to \infty$.

Remark 1.2.5:

It is evident that, a Hilbert space X, with the induced norm, is a Banach space which satisfies the parallelogram law.

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Definition 1.2.6: (Direct sum).

A vector space X is said to be the direct sum of two subspaces Y and Z of X, written $X = Y \oplus Z$, if each $x \in X$ has a unique representationn

$x = y + z \qquad\qquad y \in Y,\ z \in Z$. Here Z and Y are called algebraic complements of each other in X, and (Y,Z) is called a complementary pair of subspaces in X.

Theorem 1.2.7:[7]

Let Y be any closed subspace of a Hilbert space H. then $H = Y \oplus Y^{\perp}$ where $Y^{\perp} = \{z \in H \mid z \perp Y\}$.

Chapter Two

Approximation in General Normed Spaces

Section (2.1) Introduction

The problem of best approximation amounts to the problem of finding, for a given point x and a given set Y in a metric space X, a point $y_0 \in Y$ which should be nearest to x among all points of the set Y, i.e. such that $d(x, y_0) = \inf_{y \in Y} d(x, y)$ where d denotes the distance in the metric space X. We shall take as X not an arbitrary metric space, but a normed linear space. Naturally, the distance in X is that induced by the norm, i.e.

$d(x, y) = \|x - y\| \quad \textit{for all } x, y \in X$.

Thus, the problem of best approximation consists of finding, for a given element x and a given set Y in a normed linear space X, an element $y_0 \in Y$ such that

$$\|x - y_0\| = \inf_{y \in Y} \|x - y\|$$.

Every $y_0 \in Y$ with this property is called an element of best approximation of x in y.

Section (2.2) The Set of Best approximations $P(x, Y)$

The set of all elements of best approximation of x in Y is denoted by $P(x,Y) = \{y \in Y : \|x - y\| = d(x,Y)\}$

In this section we study some properties of the set of best approximation of x in Y.

Theorem 2.2.1:

Let Y be a subspace of a normed space X, then

i) If $x \in Y$, then $P(x,Y) = \{x\}$.

ii) If $x \in \overline{Y} / Y$, then $P(x,Y) = \phi$.

Proof:

For (i) Let $x \in Y$ then $d(x,x)=0$

$P(x,Y) = \{y \in Y : \|x - y\| = 0\} = \{x\}$ sice d is a metric.

For (ii) let $x \in \bar{Y} \setminus Y$ then for each natural number n, there is an element x_n of Y such that $\|x_n - x\| < \frac{1}{n}$ i.e. in the limit case $d(x,Y) = 0$ then $P(x,Y) = \phi$.

A subspace A of a vector space X is said to be convex if $x, y \in A$ implies that the set $M = \{z \in X \mid z = \alpha x + (1-\alpha)y,\ 0 \le \alpha \le 1\} \subset A$. M is called a closed segment with boundary points x and y; any other $z \in M$ is called an interior point of M.

Theorem 2.2.2:

Let Y be a subspace of a normed space X, and $x \in X$, then $P(x,Y)$ is a convex set.

Proof:

Let $z, y \in P(x,Y)$, and let $r \ge 0$ so that $\|x - y\| = \|x - z\| = r$. For $0 \le \alpha \le 1$, we have

$$\|x-[\alpha y+(1-\alpha)z]\|=$$

$$\|x-\alpha y-z+\alpha z\|=\|x-\alpha y-z+\alpha z-\alpha x+\alpha x\|$$

$$=\|\alpha(x-y)+(1-\alpha)(x-z)\|$$

$$\leq\alpha\|x-y\|+(1-\alpha)\|x-z\|=\alpha r+r-\alpha r=r$$

thus; $\alpha y+(1-\alpha)z\in P(x,Y)$.

Hence $P(x,Y)$ is convex.

Corollary 2.2.3:

If $P(x,Y)$ is not empty; then it either contains exactly one point or an infinite number of points.

Proof:

If $x\in Y$ then by (2.2.1)(i) $P(x,Y)$ contains exactly one point, namely x.

If $P(x,Y)$ contains more than one points, say, y,z then by (2.2.2) for each scalar α with $|\alpha|\leq 1, P(x,Y)$ a convex set so contains all points of the form $\alpha y+(1-\alpha)z$, hence it contains an infinite number of points.

Theorem 2.2.4:

Let Y be a subspace of a normed space X, $P(x,Y)$ is a bounded set.

Proof:

Let $z, y \in P(x,Y)$ and let $\|x - y\| = r$.

$$\|z-y\| = \|z-x+x-y\| \le \|z-x\| + \|x-y\| = 2r$$

Therefore $P(x,Y)$ is a bounded set.

An element x is said to be orthogonal to an element y in a normed linear space X, if we have $\|x\| \le \|x - \alpha y\|$ for every scalar α. We write $x \perp y$ to say that x is orthogonal to y. In a Hilbert space we have $x \perp y$ if and only if $\langle x, y \rangle = 0$.

An element x of a normed linear space X is said to be orthogonal to a set $Y \subset X$ and we write $x \perp Y$ if $x \perp y$ for each $y \in Y$.

Theorem 2.2.5:

Let Y be a subspace of a normed space X, $x \in X / \overline{Y}$ and $y_0 \in Y$, we have $y_0 \in P(x,Y)$ if and only if $(x - y_0) \perp Y$.

Proof:

Suppose, $y_0 \in P(x,Y)$. Since Y is a subspace $y_0 - \alpha y \in Y$ for each scalar α. So; for each scalar α, we have; $\|x - y_0\| \le \|x - y + \alpha y_0\|$ Which implies that $x - y_0 \perp Y$.

Conversely, assume that $x - y_0 \perp Y$ then $x - y_0 \perp y$ for all $y \in Y$. Thus; $\|x - y_0\| \le \|x - y_0 - \alpha y\|$. For $z \in Y$, let

$y=z-y_0$. Since Y is a subspace $y \in Y$. Now ,

$\|x - y_0\| \leq \|x - y_0 - y\| = \|x - y_0 - (z - y_0)\| = \|x - z\|$.

Theerfore $y_0 \in P(x, Y)$. •

Section (2.3) Proximinal set:

In this section we study some properties of the proximinal set, and show that compact subspaces and finite-dimensional subspaces are proximinal.

A subspace Y is called proximinal in X if,

$\forall x \in X,\ \exists y \in Y$ s.t. $d(x, Y) = \|x - y\|$.

We begin this section by the following theorem

Theorem 2.3.1:

Every compact subspace Y of X is proximinal in X.

Proof:

Let $x \in X$, be arbitrary

Let $\delta = \inf\{d(x,y) : y \in Y\}$

Define a sequence of points $y_1, y_2, \ldots$ in Y such that $d(x, y_n) \to \delta$ as $n \to \infty$. By compactness of Y, we may assume that the sequence converges to a point $\tilde{y}$ of Y. We will show that $\tilde{y}$ is a point of Y of minimum distance from x.

$d(x,\tilde{y}) \le d(x, y_n) + d(\tilde{y}, y_n)$

The left hand side is independent of n and the right hand side converges δ to as n converges to ∞.

thus $d(x,\tilde{y}) \le \delta$

Since $\tilde{y} \in Y$, $d(x,\tilde{y}) \ge \delta$ Hence $d(x,\tilde{y}) = \delta$.

If Y is not compact then the next example shows that a best approximation may not exist.

Example 2.3.2:

Let X be the Euclidean Space $\mathbb{R}^2$ and let Y be the set of points that are strictly inside the unit circle. There is no best approximation in Y to any point of X that is out side or in the unit circle.

Theorem 2.3.3:

A finite-dimensional linear subspace Y of a normed linear space X is proximinal in X.

Proof:

Let Y be such a subspace and x in X. Let y_0 be an arbitrary point of Y. Then the point we seek lies in the set $\{y : y \in Y, \|y - x\| \le \|y_0 - x\|\}$ which, by theorem (1.1.10), is compact. By theorem (2.3.1), it contains a point of minimal distance from x. Hence Y is proximinal.

A natural question is suggested by theorem (2.3.1): Is the finite-dimensionality necessary? We will answer this question by an example. The example will show that the finite dimensionality can't be omitted from the last theorem and we consider the space c_0 of infinite sequence $f = (x_1, x_2, \ldots)$ such that $x_n \to 0$ with the norm $\|f\| = \max_n \|x_n\|$.

Example 2.3.4:

In the space c_0 the subspace Y of points $f = (x_1, x_2, \ldots)$ for which $\sum_{k=1}^{\infty} 2^{-k} x_k = 0$ is not proximinal in c_0.

Proof:

Let $g = (g_1, g_2, ..)$ be any point of (c_0) not in Y. Then the number $\lambda = \sum_{k-1}^{\infty} 2^{-k} g_k \neq 0$

Obviously $\lambda \neq \infty$

Let $f_1 = \frac{-2}{1}(\lambda,0,0,\ldots)+g$

$$f_2 = \frac{-4}{3}(\lambda,\lambda,0,0,\ldots)+g$$

$$f_3 = \frac{-8}{7}(\lambda,\lambda,\lambda,0,\ldots)+g \text{ etc}$$

Our claim is that $f_n \in Y$ for all n

Proof of claim:

By induction

For n=1;

$$f_1 = \frac{1}{2}(-2\lambda + g_1) + \frac{1}{2^2}g_2 + \frac{1}{2^3}g_3 + \ldots = -\lambda + \frac{1}{2}g_1 + \frac{1}{2^2}g_2 + \frac{1}{2^3}g_3 + \ldots = 0$$

hence $f_1 \in Y$.

Now, assume it true for n=k-1

$$f_k = \frac{2^k}{2^k - 1}(\lambda,\lambda,\lambda,\ldots,\underset{k^{th}-place}{\lambda},0,0,\ldots)+(g_1,g_2,\ldots)$$

$$f_k = (\frac{2^k}{2^k - 1}\lambda + g_1,\ldots,\underset{k^{th}-place}{\frac{2^k}{2^k-1}\lambda+g_k},g_{k+1},\ldots)$$

Now,

$$\frac{1}{2}\left(\frac{2^k}{2^k-1}\lambda+g_1\right)+\frac{1}{2^4}\left(\frac{2^k}{2^k-1}\lambda+g_2\right)+\ldots+\underbrace{\frac{1}{2^k}\left(\frac{2^k}{2^k-1}\lambda+g_k\right)}_{k\ place}+g_{k+1},\ldots=0$$

Then $f_k \in Y$.

Hence $f_n \in Y$ for all n

$$\|f_n-g\|=\frac{2^n}{2^n-1}|\lambda| \to |\lambda|$$

Then the distance from g to Y is not more than $|\lambda|$.

Now, we want to show that no point of Y is of distance $|\lambda|$ or less from g.

If $f=(x_1,x_2,\ldots)$ is an element of Y with $\|g-f\| \le |\lambda|$,

choose n s.t. $|x_k-g_k| \le \frac{1}{2}|\lambda|$ whenever $k \ge n$ [Possible since the elements of c_0 are sequance converging to 0].

Then

$$\left|\sum 2^{-k} g_k\right| = \left|\sum 2^{-k} g_k - \sum_k 2^{-k} x_k\right| = \left|\sum 2^{-k}(g_k - x_k)\right| \le \sum 2^{-k}\left|g_k - x_k\right| \le$$

$$\le |\lambda| \sum_{k<n} 2^{-k} + \frac{1}{2}|\lambda| \sum_{k \ge n} 2^{-k} < |\lambda|$$

which is a contradiction.

Section (2.4) Uniqueness of Best Approximation

In this section we consider the problem of uniquness of best approximant.

A subspace Y in a normed linear space X is called a semi-Chebyshev subspace if $\forall x \in X$, $P(x,Y)$ contains at most one point and if it contains exactly one point then Y is called Chebyshev.

<u>Definition 2.4.1</u>:(Strictly convex)

The set E of a linear space X is strictly convex if,

$\forall s_1, s_2 \in E,\ s_1 \ne s_2$, the points

$\{\alpha s_1 + (1-\alpha) s_2 : 0 < \alpha < 1\}$ are interior points of E.

Theorem2.4.2:

Let X be a normed linear space. Then, $\forall x \in X$ and for any $r \geq 0$, the closed ball $N(x,r) = \{y : \|y - x\| \leq r, y \in X\}$ is convex.

Proof:

Let y_0 and y_1 be in N(x,r).

$$\|\alpha\, y_0 + (1-\alpha)\, y_1 - x\| \leq \|\alpha\, y_0 - \alpha x\| + \|(1-\alpha)\, y_1 - (1-\alpha)x\| =$$

$$= |\alpha|\|y_0 - x\| + |1-\alpha|\|y_1 - x\|$$

$$\leq r\{|\alpha| + |1-\alpha|\} = r \qquad 0 \leq \alpha \leq 1$$

Theorem 2.4.3:

Let Y be a compact and strictly convex set in a normed linear space X. Then Y is Chebyshev.

Proof:

Let Y be a compact and strictly convex set in a normed linear space X, $x \in X$, and let $\delta = d(x,Y)$

Since Y is compact, theorem (2.3.1) shows that there is a best approximant for x in Y.

Suppose s_1 and s_2 are different best approximants of Y to x. Now; we have

$\left\|\frac{1}{2}(s_1+s_2)-x\right\| \le \frac{1}{2}\|s_1-x\| + \frac{1}{2}\|s_2-x\| = \frac{1}{2}\delta + \frac{1}{2}\delta = \delta$. And because Y is convex, it follows that $\frac{1}{2}(s_1+s_2)$ is also a best approximant of x in Y and therefore it satisfies the equation $\left\|\frac{1}{2}(s_1+s_2)-x\right\| = \delta$.................(1)

Let λ be the largest number in the interval $0 \le \lambda \le 1$ such that the point

$s = \frac{1}{2}(s_1+s_2) + \lambda(x - \frac{1}{2}(s_1+s_2))$.........(2) is in Y.

Since Y is compact the value λ is well-defined.

(1) and (2) imply the equation

$\|s-x\| = (1-\lambda)\delta$............(3).

However δ is positive, otherwise $s_1 = x = s_2$, and λ is positive because the strict convexity of Y implies that

$\frac{1}{2}(s_1 + s_2)$ is an interior point of Y. It therefor follows

from 3 that $\|s - x\|$ is less than δ. This contradiction

proves the theorem.

Definition 2.4.4: (strictly convex norm)

A strictly convex norm is a norm $\|.\|$ such that. for

all x, y of norm 1, $\|x + y\| < 2$. A normed space with

such a norm is called a strictly convex normed space.

Linear subspaces of strictly convex spaces are,

clearly, strictly convex.

Theorem 2.4.5:

Let Y be a convex set in a normed linear space X, whose

norm is strictly convex. Then, Y is Semi-Chebyshev.

Proof:

Suppose that s_1 and s_2 are different best approximations from Y to x and that $\delta = d(x, Y)$. Since the strict convexity of the norm implies that the set $N(x,\delta)$ is strictly convex, the point $\frac{1}{2}(s_1 + s_2)$ is an interior point of $N(x,\delta)$, which is the condition $\left\|\frac{1}{2}(s_1 + s_2) - x\right\| < \delta$.

This is a contradiction, however, because $\frac{1}{2}(s_1 + s_2) \in Y$.

Theorem 2.4.6:

Let Y be a finite-dimensional subspace in a strictly convex normed linear space X. Then Y is Chebyshev.

Proof:

Theorem (2.3.3) shows that there is a best approximation from Y to x.

Suppose that s_1 and s_2 are different best approximations from Y to x

$\|s_1 - x\| = \|s_2 - x\| = \delta$.

$\left\|\frac{1}{2}(s_1 + s_2) - x\right\| \leq \frac{1}{2}\|s_1 - x\| + \frac{1}{2}\|s_2 - x\| = \delta$. Since Y is a linear subspace, $\frac{1}{2}(s_1 + s_2) \in Y$; $\frac{1}{2}(s_1 + s_2) \geq \delta$. Now if $\delta = 0$, it is clear that $s_1 = x = s_2$

If $\delta \neq 0$, then the vectors $\frac{s_1 - x}{\delta}, \frac{s_2 - x}{\delta}$, and there midpoint are all of norm 1, and by the strict convexity, $s_1 = s_2$.

Theorem 2.4.7:

Hilbert spaces are strictly convex.

Proof:

For all x and $y \neq x$ of norm one we have, say, $\|x - y\| = \alpha$, where $\alpha > 0$, and the parallelogram equality gives

$\|x+y\|^2 = -\|x-y\|^2 + 2(\|x\|^2 + \|y\|^2) = -\alpha^2 + 2(1+1) < 4$,

hence $\|x+y\| < 2$.

Corollary 2.4.8:[7]

Any subspace of Hilbert space is Chebyshev.

Theorem 2.4.9:

Let X be an inner product space and $Y \neq \phi$ a convex subset which is complete. Then for every given $x \in X$ there exists a unique $y \in Y$ such that

$$d = \inf_{\tilde{y} \in Y} \|x - \tilde{y}\| = \|x - y\|$$

Proof:

By definition of an infimum there is a sequence (y_n) in Y such that

$d_n \to d \qquad where \qquad d_n = \|x - y_n\| \qquad \ldots*$

We show that (y_n) is Cauchy. Writing $y_n - x = v_n$, we have

$\|v_n\| = d_n$ and

$\|v_n + v_m\| = \|y_n + y_m - 2x\| = 2\left\|\frac{1}{2}(y_n + y_m) - x\right\| \geq 2d$ because

Y is convex, so that $\frac{1}{2}(y_n + y_m) \in Y$. Furthermore, we

have $y_n - y_m = v_n - v_m$. Hence by the parallelogram

equality,

$\|y_n - y_m\|^2 = \|v_n - v_m\|^2 = -\|v_n + v_m\|^2 + 2(\|v_n\|^2 + \|v_m\|^2) \leq -(2d)^2 + 2(d_n^2 + d_m^2)$
And (*) implies that (y_n) is Cauchy. Since Y is complete,

(y_n) converges, say, $y_n \to y \in Y$. Since $y \in Y$, we have

$\|x - y\| \geq d$. Also, by (*),

$\|x - y\| \leq \|x - y_n\| + \|y_n - y\| = d_n + \|y_n - y\| \to d$. This shows

that $\|x - y\| = d$.

Now we assume that $y, y_0 \in Y$ both satisfy $\|x - y\| = d$

and $\|x - y_0\| = d$ and then show that $y_0 = y$. By the

parallelogram equality,

$$\|y-y_0\|^2 = \|(y-x)-(y_0-x)\|^2 = 2\|y-x\|^2 + 2\|y_0-x\|^2 -$$
$$-\|(y-x)+(y_0-x)\|^2$$
$$= 2d^2 + 2d^2 - 2^2\left\|\frac{1}{2}(y+y_0)-x\right\|^2$$

On the right, $\frac{1}{2}(y+y_0) \in Y$, so that $\left\|\frac{1}{2}(y+y_0)-x\right\| \geq d$.

This implies that the right hand side is less than or equal to $2d^2 + 2d^2 - 4d^2 = 0$. Hence we have the inequlity $\|y-y_0\| \leq 0$. Clearly, $\|y-y_0\| \geq 0$, so that we must have equality.

Section (2.5) Best Approximation in Orlicz Space

A function ϕ is said to be subadditive if for all x and y we have $\phi(x+y) \leq \phi(x)+\phi(y)$.

Let ϕ be a strictly increasing continuous subadditive function defined on $[0,\infty]$ with $\phi(0)=0$. Such a function is called a modulus function. Some examples of modulus functions are

$\phi(x) = x^p$ $,0 < p < \infty$, and $\phi(x) = \ln(1+x)$. In fact if ϕ is

a modulus function then so is $\frac{\phi}{1+\phi}$. Further, the

composition of two modulus functions is a modulus

function. Let (μ, X) be a finite measure space. The

Orlicz space $L^\phi(\mu, X)$ is the set of all complex-valued

measurable function, f which are defined on X and

satisfy $|f|_\phi = \int_X \phi(|f|)d\mu < \infty$.

With the metric $\|.\|_\phi$, the space $L^\phi(\mu, X)$ becomes a

complete linear topological space [3].

For a modulus function ϕ and a measure space

(T,μ) we set,

$$\ell^\phi(X) = \{x_n \in X : \sum \phi(\|x_n\|) < \infty\}$$

$$L^{\phi}(\mu, X) = \left\{ g : T \to X : \int_T \phi(\|g(t)\| d\mu(t) < \infty \right\}$$

For $x = (x_n) \in \ell^{\phi}$ and $f \in L^{\phi}(\mu,)$ we have,

$\|x\|_{\phi} = \sum_{n=1}^{\infty} \phi|x_n|$ and $\|f\|_{\phi} = \int \phi|f| d\mu$

<u>Theorem 2.5.1:</u>

if Y is a closed subspace of X, then $L^{\phi}(\mu, Y)$ is a closed subspace of $L^{\phi}(\mu, X)$.

Proof:

$$L^{\phi}(\mu, Y) = \{ y : T \to Y \|y\|_{\phi} < \infty \}$$

First we show that $L^{\phi}(\mu, Y)$ is a subspace of $L^{\phi}(\mu, X)$

Let $y_1, y_2 \in L^{\phi}(\mu, Y)$.

$(y_1 + y_2)(t) = y_1(t) + y_2(t) \in Y$. Since $y_1(t), y_2(t) \in Y$ and Y is a subspace.

$\|y_1 + y_2\|_\phi \le \|y_1\|_\phi + \|y_2\|_\phi < \infty$.

Hence $y_1 + y_2 \in L^\phi(\mu, Y)$ ------(1)

Now, let α be a real number and $y \in L^\phi(\mu, Y)$. We want

$\alpha y \in L^\phi(\mu, Y)$

$\alpha(y(t)) \in Y \Rightarrow \|\alpha y\|_\phi \le |\alpha| \|y\|_\phi < \infty$

Hence $\alpha y \in L^\phi(\mu, Y)$ --------(2).

Then (1) and (2) imply that $L^\phi(\mu, Y)$ is a subspace of

$L^\phi(\mu, X)$.

To show that $L^\phi(\mu, Y)$ is closed let (y_n) be a sequnce in

$L^\phi(\mu, Y)$ such that $y_n \to y$.

$y_n \to y \Rightarrow y_n(t) \to (y(t))\, for\, all\, t \in T$. But $(y_n(t))$ is a

sequence in Y *for all* $t \in T$ and Y is closed subspace of

X, then $y(t) \in Y \quad \forall t \in T$

$\|y\|_\phi = \|y - y_n + y_n\|_\phi \le \|y - y_n\|_\phi + \|y_n\|_\phi$

As $n \to \infty$, $\|y\|_\varphi \le \lim\|y_n\|_\phi < \infty \Rightarrow L^\phi(\mu, Y)$ is a closed

subspace of $L^\phi(\mu, X)$.

Theorem 2.5.2:

If Y is a proximinal subspace of X, then $\ell^\phi(Y)$ is

a proximinal subspace of $\ell^\phi(X)$

Proof:

let $x_n \in \ell^\phi(Y)$ and Y proximinal in X.

For all $n, \exists\, y_n \in Y$ such that. $d(x_n, Y) = \|x_n - y_n\|$ and

$\|y_n\| \le 2\|x_n\|$. $[\forall y \in Y; \|x_n - y_n\| \le \|x_n - y\|$, in particular

take $y=0$ $\|x_n - y_n\| \le \|x_n\|$, $\|y_n\| \le \|y_n - x_n\| + \|x_n\| \le 2\|x_n\|]$.

$y = (y_n) \in \ell^\phi(Y)$

Our claim is that y is a best approximation for x in $\ell^\phi(Y)$

Let $z \in \ell^{\phi}(Y)$ then

$\|x-z\|_{\phi} = \sum \phi\|x_n - z_n\| \geq \sum \phi\|x_n - y_n\| = \|x-y\|_{\phi}$ hence

$d(x, \ell^{\phi}(Y)) = \|x-y\|_{\phi}$ *and* $y \in p(x, \ell^{\phi}(Y))$.

<u>Theorem 2.5.3:</u>[8]

let Y be a closed subspace of a Banach space X, if g is a best approximation of f in $L^{\phi}(\mu, Y)$, then *y(t)* is a best approximation of *f(t)* in Y for almost all $t \in T$.

<u>Theorem 2.5.4:</u>[6]

let Y be a reflexive subspace of X, then $L^{1}(\mu, Y)$ is proximinal in $L^{1}(\mu, X)$.

<u>Theorem 2.5.5:</u>[5]

Let $1 < p < \infty$. The following are equivalent:

1) $L^{p}(\mu, Y)$ is proximinal in $L^{p}(\mu, X)$

2) $L^{1}(\mu, Y)$ is proximinal in $L^{1}(\mu, X)$

Proof:

(1) implies (2)

Let $f \in L^p(\mu, X)$. Since the measure space (T,μ) is finite, $f \in L^1(\mu, X)$, by assumption, there exists $g \in L^1(\mu, Y)$ such that $\|f - g\|_1 \leq \|f - h\|_1$ for all $h \in L^1(\mu, Y)$. then $\|f(t) - g(t)\| \leq \|f(t) - y\|$ *for all* $y \in Y$

hence

$$\|f(t) - g(t)\| \leq \|f(t) - w(t)\| \qquad \textit{for all } w \in L^p(\mu, Y).$$

Since $0 \in Y$, it follows that $\|g(t)\| \leq 2\|f(t)\|$. Hence

$$g \in L^p(\mu, Y) \textit{ and } \|f - g\|_p \leq \|f - w\|_p \quad \forall w \in L^p(\mu, Y)$$

(2) implies (1)

Consider the map $J : L^1(\mu, X) \to L^p(\mu, X)$

By

$$\begin{cases} J(f)(t) = \|f(t)\|^{\frac{1}{p}-1} f(t) & \textit{if } f(t) \neq 0 \\ J(f)(t) = 0 & \textit{otherwise} \end{cases}$$

Then $\|J(f)(t)\| = \|f(t)\|^{\frac{1}{p}}$ hence $\|J(f)\|_p^p = \|f\|$ since J is one to one.

Further if $g \in L^p(\mu, X)$, then

$f(t) = \|g(t)\|^{p-1} g(t) \in X$ & $\|f(t)\| = \|g(t)\|^p$ thus

$f \in L^1(\mu, X)$.

Further

$$J(f)(t) = [\|f(t)\|]^{\frac{1}{p}-1} \cdot \|g(t)\|^{p-1} g(t) = \|g(t)\|^{1-p} \cdot \|g(t)\|^{p-1} g(t) = g(t)$$

then J is onto. Also $J(L^1(\mu, Y)) = L^p(\mu, Y)$.

Now, let $f \in L^1(\mu, X)$. With no loss of generality we can assume that $f(t) \neq 0$, otherwise we can restrict our measure to the support of f.

Since $J(f) \in L^p(\mu, X)$ then by assumption, there exists some $g \in L^1(\mu, X)$ such that

$\|J(f) - J(g)\|_p \leq \|J(f) - J(h)\|_p \qquad \forall h \in L^p(\mu, Y)$ then by

(2.5.3), we get

$$\|J(f)(t)-J(g)(t)\| \leq \|J(f)(t)-y\| \quad \forall y \in Y \text{ hence}$$

$$\|J(f)(t)-J(g)(t)\| \leq \left\|J(f)(t)-\|f(t)\|^{\frac{1}{p}-1} y\right\| \quad \forall y \in Y$$

multiply both side by $\|f(t)\|^{1-\frac{1}{p}}$ to get

$$\left\|f(t)-\|f(t)\|^{1-\frac{1}{p}}.\|g(t)\|g(t)\right\| \leq \|f(t)-y\| \quad \forall y \in Y.$$

Let $w(t)=\|f(t)\|^{1-\frac{1}{p}}\|g(t)\|^{\frac{1}{p}-1} g(t)$. Since *g(t)* is a best approximation of *f(t)* in Y and $0 \in Y$, it follows that $\|g(t)\| \leq 2\|f(t)\|$. hence $w \in L^1(\mu,Y)$ and

$\|f(t)-w(t)\| \leq \|f(t)-\theta(t)\| \quad \forall \theta \in L^1(\mu,Y)$ and so g is a best approximation of $f \in L^1(\mu,Y)$.

In a similar way we can prove the following theorem.

<u>Theorem 2.5.6:</u>[5]

let Y be a closed subspace of X, then the following are equivalent:

i) $L^{\phi}(\mu, Y)$ is proximinal in $L^{\phi}(\mu, X)$.

ii) $L^{1}(\mu, Y)$ is proximinal in $L^{1}(\mu, X)$ •

Theorem 2.5.7:

If Y is a reflexive subspace in X, then

(i) $L^{\phi}(\mu, Y)$ is proximinal in $L^{\phi}(\mu, X)$.

(ii) $L^{p}(\mu, Y)$ is proximinal in $L^{p}(\mu, X)$.

Proof:

(i) By theorem (2.5.4) Y is reflexive in X, $L^{1}(\mu, Y)$ is proximinal in $L^{1}(\mu, X)$ and hence by (2.5.6) $L^{\phi}(\mu, Y)$ is proximinal in $L^{\phi}(\mu, X)$.

(ii) By theorems (2.5.4) and (2.5.5).

Definition 2.5.8: **(L^{p}-summand)**

A closed subspace Y of a Banach space X is said to be an L^p-summand, $1 < p < \infty$ if there is a bounded projection $P : X \to Y$ which is onto, and

$\|x\|^p = \|P(x)\|^p + \|x - P(x)\|^p$.

Theorem 2.5.9:

If Y is an L^1-summand of X, then Y is proximinal in X.

Proof:

Let

$$
\begin{aligned}
& x \in X, y \in Y, \quad y = P(z) \quad , say \\
& \|x - y\| = \|x - P(z)\| = \|P(x - P(z))\| + \|x - y - P(x - y\| \\
& \qquad = \|P(x) - P(P(x))\| + \|x - P(z) - P(x) + P(P(z))\| \\
& \qquad = \|P(x) - P(z)\| + \|x - P(x)\| \\
& \therefore \|x - y\| \geq \|x - P(x)\|
\end{aligned}
$$

Then Y is proximinal.

Theorem 2.5.10:

Let Y be a closed subspace of X. If $L^1(T, Y)$ is proximinal in $L^1(T,X)$ then $L^\infty(T,Y)$ is proximinal in $L^\infty(T,X)$.

Proof:

Let $f \in L^\infty(T,X)$ *so* $f \in L^1(T,X)$ *and* $\|f\|_1 < \|f\|_\infty$.

But $L^1(T,Y)$ is proximinal in $L^1(T,X)$, then there exists

$f_1 \in L^1(T,Y)$ such that $\|f - f_1\| = d(f, L^1(T,Y))$ and by

(2.5.3), it follows that

$$\begin{aligned} \|f(t) - f_1(t)\| &= d(f(t), Y) \qquad && a.e.t \\ \|f(t) - f_1(t)\| &\le \|f(t) - y\| \qquad && a.e.t \end{aligned}$$

Hence *for all* $y \in Y$. In particular

$\|f(t) - f_1(t)\| \le \|f(t) - g(t)\| \qquad a.e.t$ *for all* $g \in L^1(T,Y)$.

But $L^\infty(T,Y) \subseteq L^1(T,Y)$, and hence, for every $h \in L^\infty(T,Y)$

we have $\|f(t) - f_1(t)\| \le \|f(t) - h(t)\| \qquad a.e.t$ *

Now, since $0 \in Y$, we get $\|f_1(t)\| \le 2\|f(t)\| \qquad a.e.t$.

Hence $f_1 \in L^{\infty}(T,Y)$. Thus it follows from (*) that

$\|f - f_1\|_{\infty} \leq \|f - h\|_{\infty}$ For every $h \in L^{\infty}(T,Y)$. And hence

$L^{\infty}(T,Y)$ is proximinal in $L^{\infty}(T,X)$.

Theorem 2.5.11:[6]

If Y is an L^1-summand of X, then $L^1(T,Y)$ is L^1-summand in $L^1(T,X)$.

Chapter Three

Characterization of Best Approximation

Section (3.1) Introduction

In the present chapter we shall give characterizations of elements of best approximation and some consequences of these characterizations in arbitrary normed linear spaces, and we shall see how they apply to various concrete spaces. Since we have

$$P(x,Y)=\begin{cases} x & for \quad x\in Y \\ \phi & for \quad x\in \bar{Y}\setminus Y \end{cases}$$

for any linear subspace Y of a normed linear space X, it will be sufficient to characterize the elements of best approximation of the elements $x\in X\setminus \bar{Y}$. In order to exclude the trivial case when such elements x doesn't exist, throughout the sequel by "linear subspace" $Y\subset X$

we shall understand "proper linear subspace Y which is not dense in X", that is, we shall assume, without special mention, that $\overline{Y} \neq X$.

Section (3.2) Characterization in General Normed Spaces

We recall by X^* the conjugate space of X, i.e. the space of all continuous linear functionals on X,with the norm $\|f\| = \sup\limits_{\substack{x\in X \\ \|x\|\leq 1}} |f(x)|$.

Theorem 3.2.1:

Let Y be a subspace of a normed space X, $x \in X / \overline{Y}$ and $y_0 \in Y$. We have $y_0 \in P(x,Y)$ if and only if there exists an $f \in X^*$ with the following properties:

$$\|f\| = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.1)$$
$$f(y) = 0 \quad y \in Y \ldots\ldots\ldots (2.2)$$
$$f(x - y_0 = \|x - y_0\| \ldots. (2.3)$$

Proof:

Assume that $y_0 \in P(x,Y)$, since $x \in X/\overline{Y}$ we have $d(x,Y) = \|x - y_0\| > 0$. Then by Hahn-Banach theorem, there exists $f_0 \in X^*$ such that

$$f_0(x) = 1, \|f_0\| = \frac{1}{\|x - y_0\|}, \quad f_0(y) = 0 \quad y \in Y \quad .$$

Let $f = \|x - y_0\| f_0$. $f \in X^*$, and satisfies (2.1),(2.2) and (2.3).

Conversely, assume that there exists an $f \in X^*$ satisfying (2.1),(2.2) and (2.3). Then for any $y \in Y$ we have

$$\|x - y_0\| = |f(x - y_0)| = |f(x - y)| \le \|f\| \|x - y\| = \|x - y\| \quad .$$

Therefore $y_0 \in P(x,Y)$.

__**Lemma 3.2.2:**__

Let Y be a subspace of a normed space X, $x \in X/\overline{Y}, y_0 \in Y and f \in X^*$, then

i) f satisfies (2.1) and (2.3) if and only if it satisfies (2.1)

and $\quad \operatorname{Re} f(x-y_0)=\|x-y_0\|$.....(2.4)

ii) f satisfies (2.2) if and only if

$\operatorname{Re} f(y)=0 \quad y\in Y$.....(2.5)

iii) f satisfies (2.1),(2.2) and $|f(x-y_0)|=\|x-y_0\|$.....(2.6)

if and only if the function $f_1=[sign\, f(x-y_0)]f$

satisfies (2.1),(2.2) and (2.3) where for a complex

number α,

$$sign\alpha=\begin{cases} e^{-i\arg\alpha}=\dfrac{\overline{\alpha}}{|\alpha|} & if \quad \alpha\neq 0 \\ 0 & if \quad \alpha=0 \end{cases}$$

iv) f satisfies (2.1),(2.2)

and $|\operatorname{Re} f(x-y_0)|=\|x-y_0\|$.....(2.7) if and only if either

$f_1=f$, or the function $f_2=-f$ satisfies (2.1),(2.2), and (2.3)

Proof:

i) If $f(x-y_0)=\|x-y_0\| \, then \operatorname{Re} f(x-y_0)=\|x-y_0\|$.

Conversely, if f satisfies (2.1) and (2.4), we have

$\|x-y_0\|=\operatorname{Re} f(x-y_0)\leq |f(x-y_0)|\leq \|x-y_0\|$ then

$\operatorname{Re} f(x-y_0)=|f(x-y_0)|$.

$f(x-y_0)$ is real and positive and hence $f(x-y_0)=\|x-y_0\|$.

ii) (2.2) implies (2.5) is obvious.

Conversely, if f satisfies (2.5), then for

$iy\in Y \; we \; have \operatorname{Im} f(y)=-\operatorname{Re} f(iy)$,

$f(y)=\operatorname{Re} f(y)-i\operatorname{Re} f(iy)=0 \quad y\in Y$

iii) if f satisfies (2.1),(2.2) and (2.6), f_1 satisfies (2.1),

(2.2) and $f_1(x-y_0)=|f(x-y_0)|=\|x-y_0\|$.

Conversely, if $f_1=[sign\, f(x-y_0)]f$, satisfies

(2.1),(2.2), and (2.3) then $f=e^{i\arg f(x-y_0)}f_1$ satisfies

(2.1),(2.2) and (2.6)

iv) if $f_1=f$ or $f_2=-f$ satisfies (2.1),(2.2) and

(2.3), then, f satisfies (2.1),(2.2), and (2.7)

Conversely, if f satisfies (2.1),(2.2), and (2.7), we have $\|x-y_0\| = |\operatorname{Re} f(x-y_0)| \leq |f(x-y_0)| \leq \|x-y_0\|$ whence $|\operatorname{Re} f(x-y_0)| = |f(x-y_0)|$ then $f(x-y_0)$ is real, whence either $f_1=f$, or $f_2=-f$ satisfies (2.1),(2.2),and (2.4).

We shall now give a number of equivalent variants of the conditions of theorem (3.2.1).

<u>Corollary 3.2.3:</u>[11]

Let Y be a subspace of a normed space X if $x \in X/\overline{Y}$ and $y_0 \in Y$ the following statements are equivalent:

1) $y_0 \in P(x,Y)$

2) there exists an $f \in X^*$ satisfying (2.1),(2.2), and (2.4).

3) there exists an $f \in X^*$ satisfying (2.1),(2.2), and (2.6).

4) there exists an $f \in X^*$ satisfying (2.1),(2.2), and (1.7).

5) there exists an $f \in X^*$ satisfying (2.1),(2.5), and (2.6).

6) there exists an $f \in X^*$ satisfying (2.1),(2.5), and (2.7).

Corollary 3.2.4:[11]

Let Y be a subspace of a normed space X, $x \in X / \overline{Y}$ $and M \subset Y$. We have $M \subset P(x,Y)$ if and only if there exists an $f \in X^*$ satisfying (2.1),(2.2) and

$$f(x - y_0) = \|x - y_0\| \qquad y_0 \in M$$

Section (3.3) Applications in the Space $L^1(T,v)$:

Let (T,v) be a positive measure space and, for $1 \le p < \infty$ $(p = \infty)$, let $L^p(T,v)$ be the space of all equivalence classes of functions of p^{th} power v-integrable (v-measurable and v-essentially bounded on T), endowed with the usual vector operations and with the norm

$$\|x\| = \left[\int_T |x(t)|^p \, dv(t) \right]^{\frac{1}{p}} \; (res. \|x\| = ess \sup_{t \in T} |x(t)|).$$ for a

function x on T. where

$$ess\sup_{t\in T}\|f(t)\| = \inf\{M : \mu\{t : \|f(t)\| > M\} = 0\}$$

We shall use the notation $Z(x) = \{t \in T / x(t) = 0\}$.

Theorem 3.3.1:[11]

Let $X=L^1(T,v)$, where (T,v) is a positive measure space, let Y be a linear subspace of X, $x \in X / \overline{Y}$ and $y_0 \in Y$. The following statement are equivalent:

1) $y_0 \in P(x,Y)$

2) There exist countably additive and v-absolutely continuous set functions m defined on the sets and finite measure, such that

$$\sup_{0\pi u(t)\pi\infty} \frac{|m(A)|}{v(A)} = 1 \quad \ldots\ldots\ldots\ldots\ldots(3.1)$$

$$\int y(t)dm(t) = 0 \qquad (y \in Y)\ldots.(3.2)$$

$$\int_T [x(t) - y_0(t)]dm(t) = \int_T |x(t) - y_0(t)|dv(t)\ldots\ldots(3.3)$$

3) we have

$$\int_{T\setminus Z(x-y_0)} y(t)\, sign[x(t)-y_0(t)]dv(t) \le \int_{Z(x-y_0)} |y(t)|dv(t) \quad y\in Y \text{....(3.4)}$$

4) We have

$$\int_{T\setminus p_0} y(t)\, sign[x(t)-y_0(t)]dv(t) \le \int_{p_0} |y(t)|dv(t) \quad y\in Y \text{...(3.5)}$$

$$where \;\; p_0 = Z(x-y_0)\setminus \bigcap_{y\in Y} Z(y) \text{...........(3.6)}$$

5) there exists a

$$\beta \in L^{\infty}(T,v) \;\; s.t \;\; ess\sup_{t\in T}|\beta(t)| = 1 \text{.....(3.7)}$$

$$\int_T y(t)\beta(t)dv(t) = 0 \qquad y\in Y \text{.................(3.8)}$$

$$\int_T [x(t)-y_0(t)]\beta(t)dv(t) = \int_T |x(t)-y_0(t)|dv(t) \text{...(3.9)}$$

6) There exists a $\beta \in L^{\infty}(T, v)$ satisfying (3.7,3.8)

and $\beta(t)[x(t)-y_0(t)] = |x(t)-y_0(t)| \;\; v-a.e\, on T$ ...(3.10)

7) There exists a v-measurable function α on the set $Z(x-y_0)$ such that

$|\alpha(t)| \leq 1 \ v - a.e \ on \ Z(x - y_0)$......(3.11)

$$\int_{Z(x-y_0)} y(t)\alpha(t)dv(t) + \int_{T \setminus Z(x-y_0)} y(t) sign[x(t) - y_0(t)]dv(t) = 0 \ y \in Y \text{...(3.12)}$$

Corollary 3.3.2:

Let $X = L^1(T,v)$, where *(T, v)* is a positive measure space with the property that the dual $L^1(T,v)^*$ is canonically equivalent to $L^\infty(T,v)$, and let *Y* be a linear subspace of *X*, $x \in X / \overline{Y}$ & $y_0 \in P(x,Y)$. Then there exists a v-measurable set $U_{g_0} \subset T$ with

$v(U_{g_0}) > 0$ *and a member* $\beta \in L^\infty(T,v)$ so that we have

(3.7),(3.8) and

$|\beta(t)| = 1 \qquad v - a.e \ on \ U_{g_0},$ 3.13

$g_0(t) = x(t) \qquad v - a.e \ on \ T \setminus U_{g_0}$ 3.14

Proof:

By virtue of the implication (1 implies 6) of theorem (3.3.1) there exists a $\beta \in L^{\infty}(T,\nu)$ satisfying

(3.7),(3.8) and $U_{g_0} = T \setminus Z(x - g_0)$, then by

$x \in X / \overline{Y}$ we have $\nu(U_{g_0}) > 0$, and by the definition of

U_{g_0} we have (3.14 by 3.15 and 3.10) we have 3.13.

We recall that in a linear spaces L any set of the form $\{\lambda x + (1-\lambda)y /\ 0 \le \lambda \le 1\}$, where $x, y \in L$, is called a segment; the points $\lambda x + (1-\lambda)y$ with $0 < \lambda < 1$ are called interior points of the segment.

A set $A \subset L$ is called convex if together with any two points x, y it contains the whole segment generated by them, that is, if the

relation

$x, y \in A$ *and* $0 \le \lambda \le 1$ *implies that* $\lambda x + (1-\lambda)y \in A$.

A set M in a topological linear space L is called an extermal subset of a closed convex set A if M is a closed convex subset of A and together with every interior point of a segment in A it contains the whole segment, i.e. the relations $x, y \in A \, and \, 0 < \lambda < 1, \ \lambda x + (1-\lambda)y \in M$. An extermal subset of A consisting of a single point is called an extermal point of A, we shall denote by $\vartheta(A)$ the set of all extermal points of A.

Lemma 3.3.3:

Let $X = L^1(T, v)$, where *(T, v)* is a positive measure space with the property that the dual $L^1(T,v)^*$ is canonically equivalent to $L^\infty(T,v)$, and let $f \in X^*$. We have $f \in \vartheta(S_{X^*})$ if and only if there exists a $\beta \in L^\infty(T,v)$ such that

$$|\beta(t)| \equiv 1 \qquad v-a.e \quad on \quad T \qquad 3.16$$
$$f(x) = \int_T x(t)\beta(t)\,dv(t) \qquad x \in X \qquad 3.17$$

Proof:

Let $f \in \vartheta(S_{X^*})$. Then there exists, by virtue of the canonical equivalence

$L^1(T,v)^* \equiv L^\infty(T,v),\ a\ \ \beta \in L^\infty(T,v)$ such that we have (3.17) and $|\beta(t)| \leq 1 \quad v-a.e \quad on \quad T$. Assume that β doesn't satisfy (3.16). Then there exists a measurable subset $A \subset T$ with $v(A)>0$ such that

$$|\beta| < 1 \qquad v-a.e \qquad on \qquad A \qquad 3.18$$

$$Put \quad \beta_1(t) = e^{i\arg\beta(t)} \qquad t \in T$$
$$\beta_2(t) = (2|\beta(t)|-1)e^{i\arg\beta(t)} \qquad t \in T$$

Then for $f_j(x) = \int_T x(t)\beta_j(t)\,dv(t) \qquad x \in X,\ j = 1,2$

We will have $f_1, f_2 \in S_{X^*}$, $f = \frac{1}{2}(f_1 + f_2)$ and by (3.18), $f_1 \neq f_2$ then $f \notin \vartheta(S_{X^*})$

Conversely, assume that $f \in S_{X^*} \setminus \vartheta(S_{X^*})$ then there exists $f = \frac{1}{2}(f_1 + f_2) \qquad 3.19$.

Whence also $\beta, \beta_1, \beta_2 \in L^{\infty}(T, v)$ uniquely determined by

f,f₁, and *f₂* respectively, and a measurable subset $A \subset T$

with *v(A)>0* s.t

$$\begin{aligned}
&|\beta(t)|, |\beta_j(t)| \leq 1 \quad v-a.e \;\; on \;\; T; \\
&\beta_1(t) \neq \beta_2(t) \quad v-a.e \;\; on \;\; A \qquad\qquad 3.20 \\
&f(x) = \int_T x(t)\beta(t)\,dv(t); \\
&f_j(x) = \int_T x(t)\beta_j(t)\,dv(t) \quad x \in X \quad j = 1,2 \qquad\qquad 3.21
\end{aligned}$$

Then from (3.20 and 3.21) it follows that we have

$\beta(t) = \frac{1}{2}[\beta_1(t) + \beta_2(t)] \;\; v-a.e \;\; on \; T$, whence by (3.20)

$|\beta(t) < 1 \;\; v-a.e. \;\; on \; T|$ and thus condition (3.16) is not

satisfied.

We will now deduce several characterizations of

elements of best approximation in the space $L^1(T, v)$;

collected in the following theorem.

Theorem 3.3.4:[11]

Let $X = L^1(T,v)$, where *(T, v)* is a positive measure space let Y be a linear subspace of $X, x \in X / \bar{Y}\ and y_0 \in Y$. The following are equivalent:

1) $y_0 \in P(x,Y)$

2) For every $y \in Y$ there exists a $\beta = \beta^y \in L^\infty(T,v)$, s.t

$$|\beta(t)| = 1 \qquad v - a.e.\ on\ T \qquad 3.22$$
$$\operatorname{Re}\int_T [y_0(t) - y(t)]\, dv(t) \geq 0 \qquad 3.23$$
$$\int_T [x(t) - y_0(t)]\beta(t)\, dv(t) = \int_T |x(t) - y_0(t)|\, dv(t) \qquad 3.24$$

3) For every $y \in Y$ there exists on the set Z(x-y_0) a v-measurable function $\alpha = \alpha^y$ s.t

$$|\alpha(t)| \equiv 1 \qquad v - a.e\ on\ Z(x - y_0) \qquad 3.25$$
$$\int_T |x(t) - y_0(t)|\, dv(t) \leq \operatorname{Re}\int_{Z(x-y_0)} [x(t) - y(t)]\alpha(t)\, dv(t) + \operatorname{Re}\int_{T \setminus Z(x-y_0)} [x(t) - y(t)]\, sign[x(t) - y_0(t)\, dv(t)$$

4) For every $y \in Y$ there exists on the set Z(x-y_0) a v-measurable function $\alpha = \alpha^y$ satisfying (3.24) and

$$\operatorname{Re} \int_{T \setminus Z(x-y_0)} [y_0(t) - y(t)]\, sign[x(t) - y_0(t)]\, dv(t)$$

$$\geq -\operatorname{Re} \int_{Z(x-y_0)} [y_0(t) - y(t)]\alpha(t)dv(t) \quad \ldots 3.27$$

5) We have

$$\operatorname{Re} \int_{T - Z(x-y_0)} [y_0(t) - y(t)] sign[x(t) - y_0(t)]\, dv(t))$$

$$\geq - \int_{Z(x-y_0)} |y_0(t) - y(t)| dv(t) \quad (y \in Y) \ldots 3.28$$

6) We have

$$\operatorname{Re} \int_{T \setminus Z(x-y_0)} y(t)\, sign[x(t) - y_0(t)]\, dv(t) \geq - \int_{Z(x-y_0)} |y(t)|\, dv(t)$$

Section (3.4) Application in the spaces *C(K)* & *C_R(K)*

For a compact space *K*, we shall denote by *C(K)*, the space of all continuous functions *x* on *K*, with the norm $\|x\| = \max_{k \in K} |x(k)|$.

For a compact space K, we shall denote by $C_R(K)$, the space of all numerical continuous real-valued function on K, with the norm $\|x\| = \max_{k \in K} |x(k)|$.

Theorem 3.4.1:[11]

Let $X=C(K)$, Y a linear subspace of X, $x \in X / \overline{Y}$ *and* $y_0 \in Y$ we have $y_0 \in P(x, Y)$ if and only if there exists a Radon measure μ on K, with the following properties

$$|\mu|(K) = 1 \qquad 4.1$$

$$\int_Q g(k)\, d\mu(k) = 0 \qquad g \in Y \qquad 4.2$$

$$\frac{d\mu}{d|\mu|} \in C(K) \qquad 4.3$$

$$x(k) - g_0(k) = [sign \frac{d\mu}{d|\mu|}(k)] \max_{t \in K} |x(t) - g_0(t)| \qquad k \in S(\mu) \qquad 4.4$$

Let us now consider the problem of simultaneous characterization of a set $M \subset Y$ of elements of best approximation.

Theorem 3.4.2:

Let $X=C(K)$, Y a linear subspace of X, $x \in X / \overline{Y}$ $and M \subset Y$. We have $M \subset P(x,Y)$ if and only if there exists a Radon measure μ on K satisfying (4.1,4.2 and 3.3) and

$$x(k) - g_0(k) = [sign\frac{d\mu}{d|\mu|}(k)] \max_{t \in K}|x(t) - g_0(t)| \qquad (k \in S(\mu); g_0 \in M \quad 4.5$$

Proof:

Assume that we have $M \subset P(x,Y)$. Then there exists, by virtue of corollary (3.2.4), a Radon measure μ on K satisfying (4.1,4.2) and

$$\int_K [x(k) - g_0(k)] d\mu(k) = \max_{k \in K}|x(k) - g_0(k)| \qquad (g_0 \in M) \quad 4.6$$

It follows from (4.1,4.6) and $x \in X \setminus \overline{Y}$ that for every $g_0 \in M$ there exists a set of $|\mu|$-measure zero $N_{g_0} \subset K$ such that

$$\frac{d\mu}{d|\mu|}(k) = \frac{\overline{x(k) - g_0(k)}}{\max\limits_{t \in Q}|x(t) - g_0(t)|} \qquad k \in K \setminus N_{g_0}$$

Consequently, for any pair $g_0, g_0' \in M$ we have

$$g_0(k) = g_0'(k) \qquad (k \in K \setminus (N_{g_0} \cup N_{g_0'})),$$
hence $\quad (g_0 - g_0')|\mu| = 0,$
whence $\quad g_0(k) = g_0'(k) \qquad k \in S(\mu)$

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