Stability Functions For Second Order Elastic Analysis

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Abstract

ان اجراء مقارنة بين اقترانات الاتزان الحقيقية واقترانات الاتزان المشتقة من نظرية العناصر المحدودة يظهر ان اقترانات الاتزان الحقيقية تتعرض الى مشاكل عددية عندما تكون القوى المحورية صغيرة في حين تكون هذه الاقترانات المشتقة من نظرية العناصر المحدودة غير دقيقة عندما تكون القوى المحورية كبيرة.

في هذه الورقة تم تطبيق مبدأ التوافق التربيعي الادنى لتحسين نتائج نظرية العناصر المحدودة تحت تأثير القوى الكبيرة ومعالجة المشاكل العددية تحت تأثير القوى المحورية الصغيرة .

A comparison between closed form stability functions and finite element stability functions shows that the former has numerical problems under small axial forces and the later is inaccurate under large axial forces. The least square fitting technique is applied to improve the finite element solution to get better accuracy under large axial forces and to solve numerical problems under small axial forces.

Introduction

It is customary in design practice to use a first-order analysis to determine the distribution of bending moments and internal forces throughout a structure. This type of analysis is based on the undeformed configuration of the structure, and so it disregards the additional stresses and deformations that occur due to changes in geometry. In a second-order analysis the equilibrium equations are formulated on the deformed structure, i.e. the second-order effects produced by the loads acting on the displaced structure are accounted for in the analysis. The second order analysis results in changes in the member stiffness matrix S_M. The new stiffness matrix is composed from the regular first-order elastic stiffness matrix with each term being multiplied by a function called a stability function.

The Closed Form Approach

The changes in the member stiffness matrix can be derived by elementary beam analysis and the new stiffness matrix is given by (Weaver and Gere 1980):

$$S_M = \frac{2EI}{L^3}\begin{bmatrix} 6s_1 & 3Ls_2 & -6s_1 & 3Ls_2 \\ 3Ls_2 & 2L^2s_3 & -3Ls_2 & L^2s_4 \\ -6s_1 & -3Ls_2 & 6s_1 & -3Ls_2 \\ 3Ls_2 & L^2s_4 & -3Ls_2 & 2L^2s_3 \end{bmatrix} \quad (1)$$

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Where E stands for elastic modulus, I for moment of inertia, L for length of member and P for axial force in member. Each term in Eq. 1 is expressed as the product of the first -order stiffness multiplied by the stability function s. The four stability functions appearing in Eq. 1 are defined in Table 1 for axial forces which are compression and tension . Notice as the axial force approaches zero the denominator approaches zero which causes numerical problems at small axial forces.

Function	Compression	Tension
s_1	$\frac{(KL)^3 \sin KL}{12\Phi_c}$	$\frac{(KL)^3 \sinh KL}{12\Phi_t}$
s_2	$\frac{(KL)^2(1-\cos KL)}{6\Phi_c}$	$\frac{(KL)^2(1-\cosh KL)}{6\Phi_t}$
s_3	$\frac{KL(\sin KL - KL\cos KL)}{4\Phi_c}$	$\frac{KL(KL\cosh KL - KL)}{4\Phi_t}$
s_4	$\frac{KL(KL - \sin KL)}{2\Phi_c}$	$\frac{KL(\sinh KL - KL)}{2\Phi_t}$
$K = \sqrt{\frac{P}{EI_z}}$	$\Phi_c = 2 - 2\cos KL - KL * \sin KL$	$\Phi_t = 2 - 2\cosh KL - KL * \sinh KL$

Table 1 Stability functions for a beam subjected to axial force.

The Finite Element Approach (First order)

We can derive the stiffness matrix using the finite element approach (first order) as is usually done in many classical textbooks. The final result will be (Weaver and Johnston 1984):

$$\underline{S}_{m} = \underline{S}_{mr} + \underline{S}_{mo} \qquad (2)$$

Where S_{mr} is the regular form of the member stiffness matrix for first-order analysis and is given by:

$$S_{mr} = \frac{2EI}{L^3}\begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \qquad (3)$$

S_{mg} is called the geometric stiffness matrix and is given by:

$$S_{mg} = \frac{P}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \qquad (4)$$

where the axial force P is positive for tensile loading.

Comparison Between the Closed Form Approach and the Finite Element Approach

If we write Eq. 2 in the same form of Eq. 1, the stability functions for the finite element approach will be as follows:

$$S_{1f} = 1 + \frac{(KL)^2}{10} \qquad (5a)$$

$$S_{2f} = 1 - \frac{(KL)^2}{30} \qquad (5b)$$

$$S_{3f} = 1 + \frac{(KL)^2}{30} \qquad (5c)$$

$$S_{4f} = 1 - \frac{(KL)^2}{30} \tag{5d}$$

The subscript f is used in the stiffness functions to indicate that these are found from the finite element approach.

Figs. 2 and 3 show the comparison between the stability functions from the closed form approach (e) and the stability functions from the finite element approach (f). From these figures it is apparent that the finite element approach is inaccurate under large axial forces. Therefore, there is a need to improve this approach.

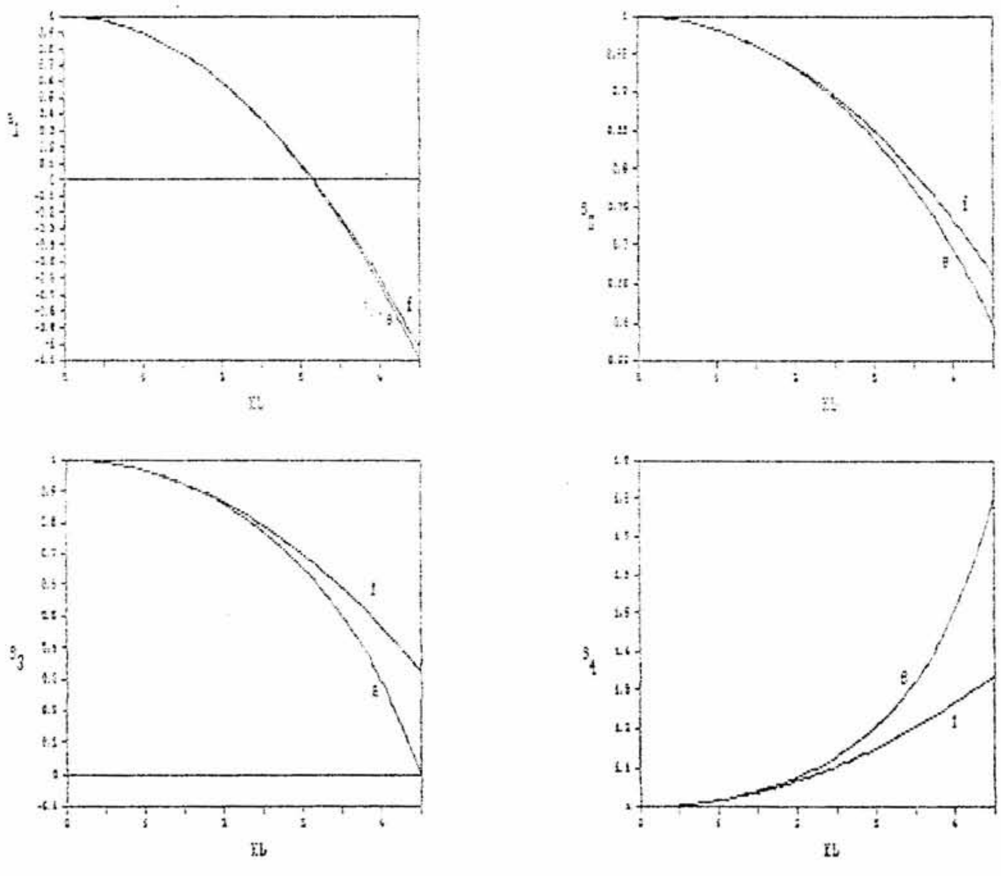


Fig. 1 Comparison between closed form stability functions (e) and finite element stability functions (f) for compression loading.

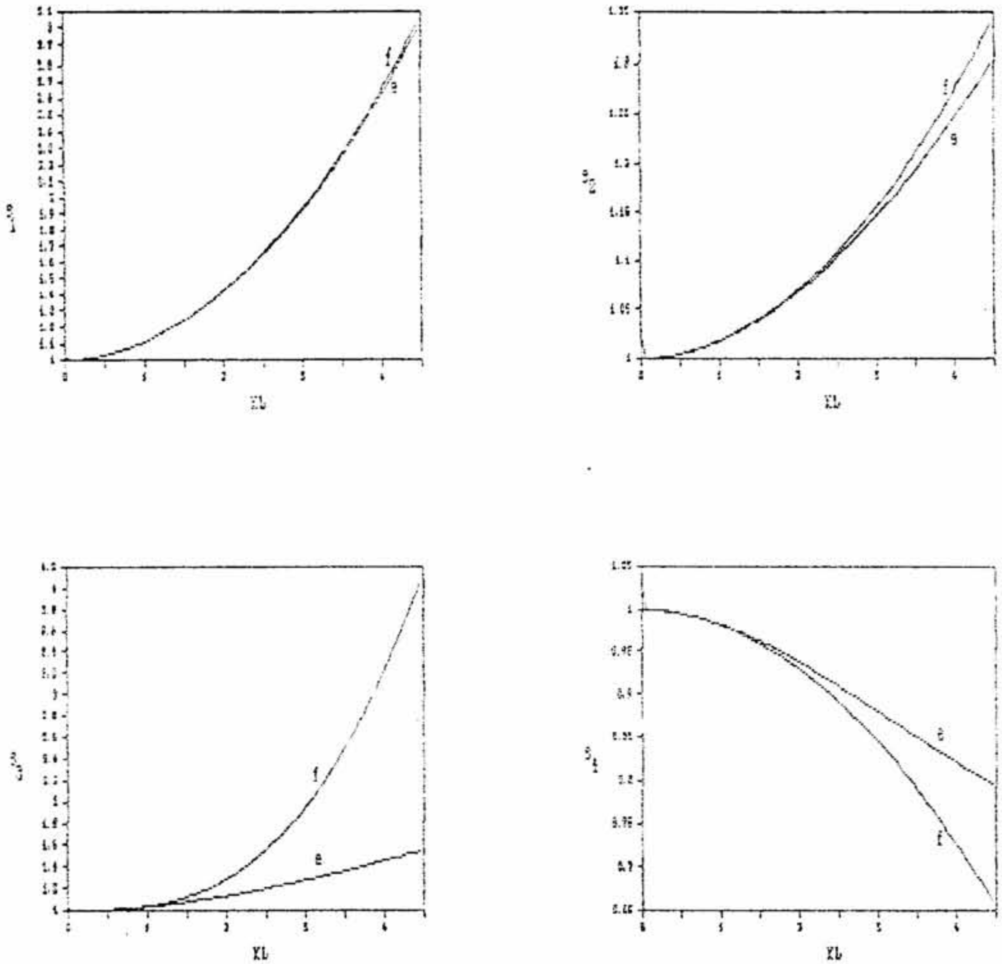


Fig. 2 Comparison between closed form stability functions (e) and finite element stability functions (f) for tensile loading.

Proposed Approach:

To improve the finite element approach, we suggested adding another term to the stability functions of equations 5a to 5d. As kl was raised to even powers (0 or 2) in the stability functions, we suggested that the additional term will be a function of $(kl)^4$. The new form of stability functions (S_n) is:

$$S_{in}(kl) = 1 + \alpha (kl)^2 + \beta (kl)^4 \qquad (6)$$

Where the values of α are taken from the finite element approach and the values of β are to be estimated.

Looking at this new form, we can notice that it could be considered a polynomial Taylor approximation around zero of the original stability functions. However, those functions are not defined at zero. Only their limits as the axial force approaches zero are known and equal to 1. Therefore, the Taylor approximation should be done around a point near zero. However, such a Taylor approximation requires evaluating the second and fourth order derivatives of the original functions (Burden et. al., 1978). Looking at those functions, we can notice that evaluating such derivatives is mathematically very tedious and difficult. Therefore, we decided to determine the values of β statistically using the least squares approach.

In the least squares approach, we define a set of points for which we are interested in estimating the best fit curve. For us, the best fit curve is a polynomial given by equation 6 above with β is unknown for each stability function. The set of points are all the possible values of *kL* that we have in nature versus their corresponding values of stability functions as given by the closed form approach. The possible values of *kL* range from zero to approximately 4.5 for unbraced frames. Zero is where we have no axial force applied and 4.5 is the maximum probable *kL* in unbraced frames (Toucan, 1989).

In the method of least squares, we estimate the coefficient of a suggested polynomial which minimizes the sum of squared errors which are given by:

$$J = \sum_{i=1}^{n} \left(1 + \alpha (kL)_i^2 + \beta (kL)_i^4 - S_i\right)^2$$

Where J is the sum of squared errors, i is a point under consideration and n is the number of points. As *kL* is a continuous random variable from 0 to 4.5, the above sum is replaced by an integral which is:

$$J = \int_0^{4.5} \left(1 + \alpha (kL)^2 + \beta (kL)^4 - S_{ie}(kL)\right)^2 d(kL)$$

As our objective is to find the value of β which minimizes J, the necessary condition for that is:

$$\frac{\partial J}{\partial \beta} = 0$$

Therefore:

$$\beta = \frac{\int_0^{4.5} [S_{1e}(kL)(kL)^4 - (kL)^4 - \alpha(kL)^6]\, d(kL)}{\int_0^{4.5} (kL)^8\, d\,kL}$$

The integral of $S_{1e}{*}(kL)^4$ cannot be evaluated analytically for the stability functions. Therefore, it was evaluated numerically. The interval from 0 to 4.5 was divided into 90 trapezoids with constant width of 0.05. Then trapezoidal numerical integration was applied to evaluate the integral. The results were used in the above equation to estimate the value of β for each stability function. The final results are shown in table 2.

Function	Compression	Tension
S_1	$1 - \frac{(KL)^2}{10} - \frac{(KL)^4}{5800}$	$1 + \frac{(KL)^2}{10} - \frac{(KL)^4}{10000}$
S_2	$1 - \frac{(KL)^2}{60} - \frac{(KL)^4}{5800}$	$1 - \frac{(KL)^2}{60} + \frac{(KL)^4}{10000}$
S_3	$1 - \frac{(KL)^2}{30} - \frac{(KL)^4}{1400}$	$1 - \frac{(KL)^3}{30} - \frac{(KL)^4}{3000}$
S_4	$1 - \frac{(KL)^2}{60} + \frac{(KL)^4}{1000}$	$1 - \frac{(KL)^2}{60} - \frac{(KL)^4}{3000}$

Table 2 Proposed stability functions.

The proposed stability functions shown in table 2 were plotted versus kL and compared with those obtained using the closed form approach. Figs. 3 and 4 show a comparison between these two approaches. From those figures, we can notice that the proposed stability functions fit accurately the functions used in closed form approach. Also the proposed approach does not have any numerical inaccuracy at low values of kL. The following example demonstrates the power of the technique.

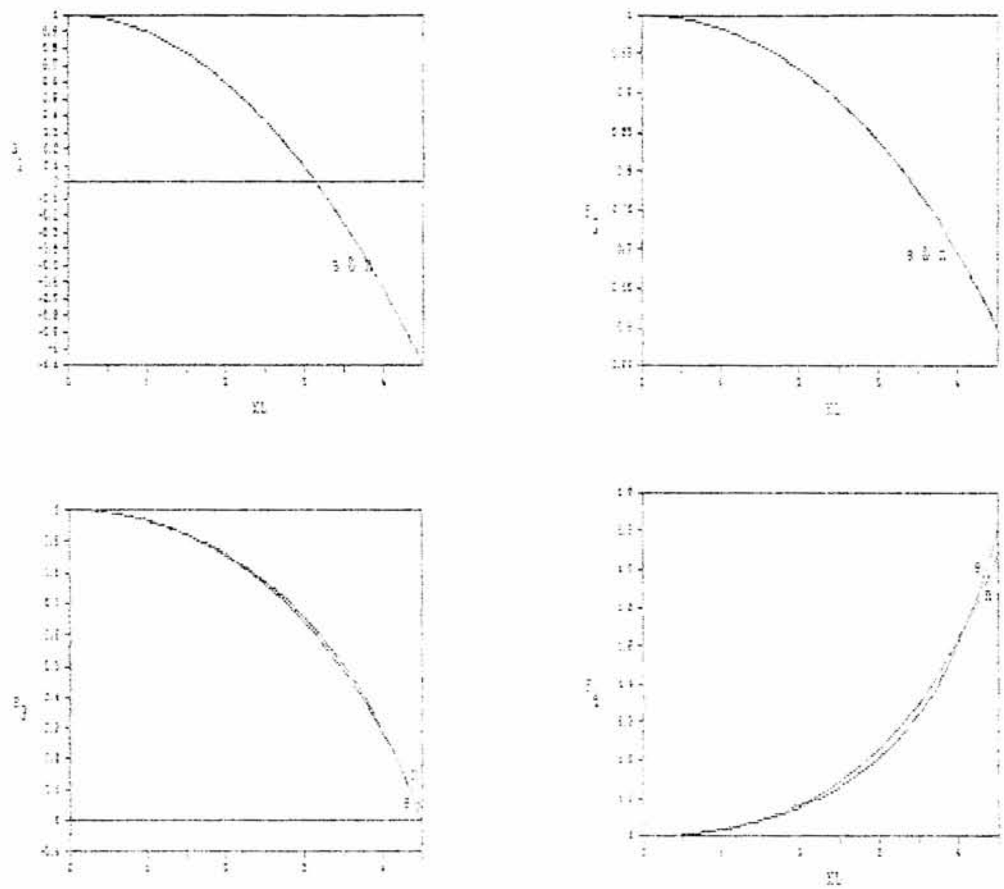


Fig. 3 Comparison between closed form stability functions (e) and proposed approach stability functions (n) for compression loading.

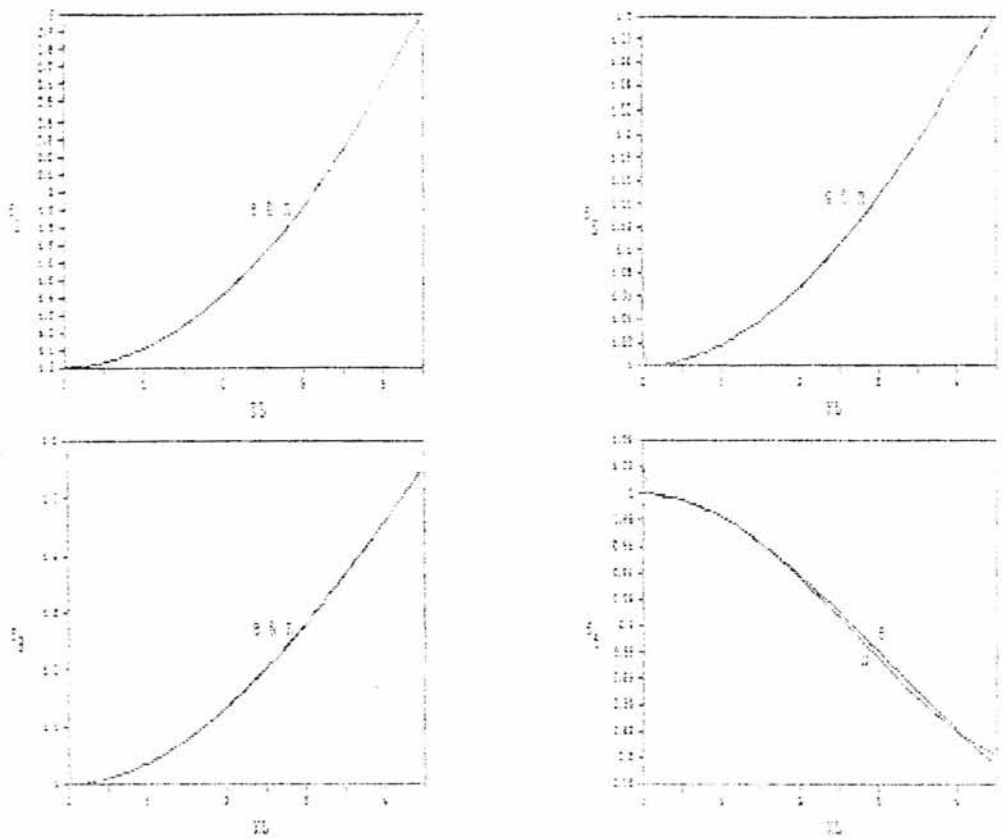
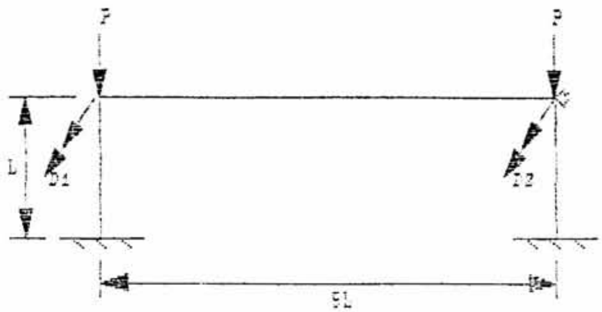


Fig. 4 Comparison between closed form stability functions (e) and proposed approach stability functions (n) for tensile loading.

Example

We are interested in finding the buckling load of the braced frame structure shown in the figure below. All members have constant *EI*. To simplify analysis we'll neglect the effect of axial deformations in members.



The stiffness matrix for free displacements is given by

$$\underline{S} = \frac{EI}{L}\begin{bmatrix} 4\left(s_3+\frac{1}{5}\right) & 1 \\ 1 & 4\left(s_3+\frac{1}{5}\right) \end{bmatrix} \qquad a$$

Buckling occurs when

$$|\underline{S}| = 0 \qquad b$$

From Eq. b we get

$$16\left(s_3+\frac{1}{5}\right)^2-1=0 \Rightarrow s_3=\frac{1}{20} \qquad c$$

From Eq. c we can determine the value of KL from Table 1 for closed form, Eq. 5c for finite element and Table 2 for proposed approach. The value of the critical load is determined from the value of KL as follows

$$P_{cr} = EI\,(KL)^2 \qquad d$$

Following the above procedure we get P_{cr} equal to $19.5EI/L^2$ for closed form approach, $28.5EI/L^2$ for finite element approach and $19.9EI/L^2$ for proposed approach.

The previous result shows the inaccuracy of finite element approach at large values of axial forces where the error was of the order 46%, while the error in the proposed approach was of the order 2%.

To demonstrate the numerical problems of the closed form approach at small values of KL, we draw Fig. 5 which relates the stiffness functions for closed form approach against values of KL. The stiffness functions must be equal to zero, however Fig. 5 shows fluctuation in their values. Such problems do not occur in the finite element or proposed approach.

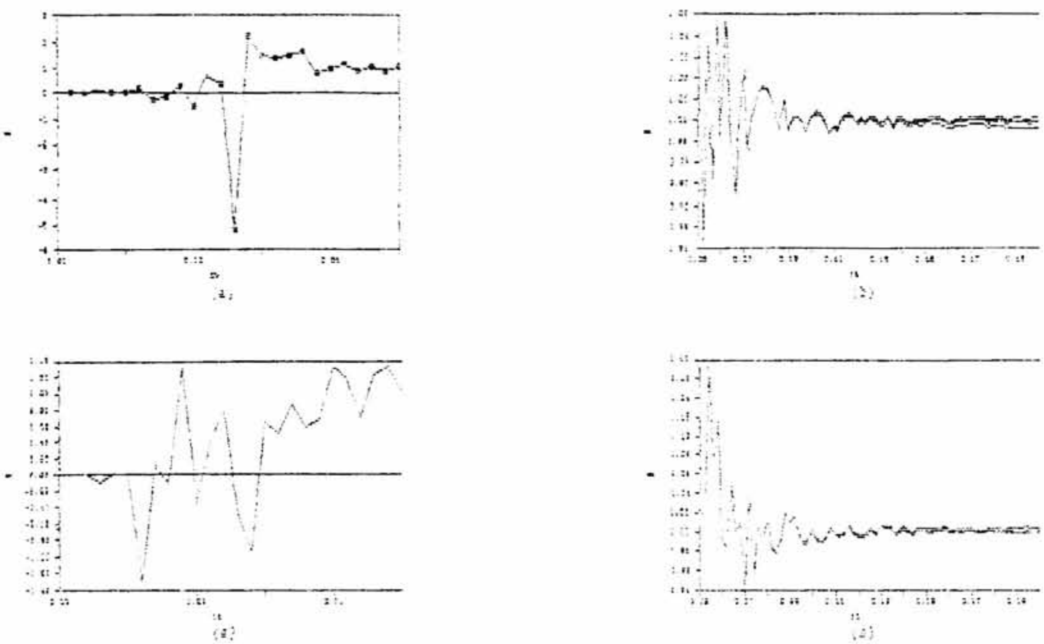


Fig. 5 Fluctuations of the values of stiffness functions for closed form approach at small values of *KL*.

Conclusion

Closed form stability functions has numerical problems under small axial forces, thus their use especially in computers is restricted. The widely used finite element stability functions are inaccurate under large axial forces as was shown in the example above. The finite element solution was improved by deriving new stability functions using least squares technique. The resulting functions are accurate even under large axial force. The figures and example shown in this paper illustrated that accuracy. At the same time, the suggested functions are easy to use without any numerical difficulty. This makes using them advantageous to other methods.

References

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