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THE S-PROPERTY AND BEST APPROXIMATION

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الإهداء

إلى الذين فضلهم بلغ السماء

وسيظل رضاهم موضع الرجاء

إليكم والدي ووالدتي واخوتي

حباً وعهداً مني بالوفاء

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Preface

The problem of best approximation is the problem of finding , for a given point x and a given set G in a normed space $(X,\|.\|)$, a point g_0 in G which should be nearest to x a mong all points of the set G .

However , in our study , we shall mainly take as X not an arbitrary normed space but Orlicz space , we shall denote by P(x,G) , the set of all elements of best approximants of x in G .

i.e $P(x,G) = \{ g_0 \in G : \|x - g_0\| = \inf \{ \|x - g\| : g \in G\} \}$

The problem of best approximation began , in 1853 , with P. L. Chebyshev who considered the problem in the space of all real valued continuous function defined on [a,b] , a closed real interval in R .

My thesis consist of four chapters . Each chapter is divided into sections . A number like 2.1.3 indicates item (definition , theorem , corollary or lemma) number 3 in section 1 of chapter 2 . Each chapter begins with a clear statement of the pertinent definitions and theorems together with illustrative and descriptive material . At the end of this thesis we present a collection of references .

In chapter (1) we introduce the basic results and definitions which shall be needed in the following chapters . The topics include projection , normed space , compactness , Hilbert space and measure theory . This chapter is absolutely fundamental . The results have been stated without proofs , for theory may be looked up in any standard text book in Functional Analysis . A reader who is familiar with these topics may skip this chapter and refer to it only when necessary .

Chapter (2) will be devoted to give an introduction to fundamental ideas of Best Approximation in Normed Space . We will start by introducing the definition of best approximants of $x \in X$ in a closed subspace G of X . We denote the set of all best approximation of x in G by P(x,G) . In section (2) we study the properties of P(x,G) . In section (3) we define proximinal set and Chebyshev subspace , and we mention some conditions that can assure that G is proximinal in X . Finally , we define L^p-summand and give a simpler proof for the fact that "every a closed subspace of a Hilbert space is proximinal " .

Chapter (3) has two purposes . First , we review the properties of Orlicz spaces . Second , we introduce some of the basic theory of proximinality in Orlicz space . This material was designed to meet the needs of chapter (4) .

W. Deeb and R. Khalil proved the following results .

(1) If G is 1-complemented in X , then G is proximinal in X . [1 , p.529] .

(2) If $L^\phi(\mu,G)$ is proximinal in $L^\phi(\mu,X)$, then G is proximinal in X . [3 , p.8] , [2 , p.297] , [4 , p.37] .

(3) If $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$, then $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$. [1 , p.528] .

Some questions a bout proximinality in $L^{\phi}(\mu,X)$ now suggest themselves .

(1) Let X be a Banach space and let G be proximinal in X . Under what conditions can it be asserted that G is 1-complemented in X ?

(2)If G is proximinal in X ,Under what conditions can it be concluded that $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$? In particular , is the proximinatily of G in X a sufficient condition ?

(3) If $L^{\infty}(\mu,G)$ is proximinal in $L^{\infty}(\mu,X)$. Under what condition can be asserted that $L^{1}(\mu,G)$ is proximinal in $L^{1}(\mu,X)$.

These questions are addressed in the section (1) of chapter (4) .

The answer depends on the S-property .

Some interesting results have been a chieved . A mong of which it is shown that if G has the S-property then $L^{\phi}(\mu,G)$ has the S-property . It is also proved that if G has the S-property then

$$L^{\phi}(\mu, P_G^{-1}(0)) = P^{-1}_{L^{\phi}(\mu,G)}(0) .$$

I ask our God to be our assistant to continue our efforts so as to a chieve the hopes and desires of all scholars in mathematics .

Chapter 1

Preliminaries

This chapter contains some definitions and basic result about normed space , Hilbert space , compactness , Banach space , projection and measure theory which will be used in the subsequent chapters .

1. Normed Linear Spaces :

Definition 1.1.1 : [9 , p.35] . Let X be a linear space over K . A norm on X is a function $||.|| : X \to \mathbf{R}$ such that for $x , y \in X$ and $k \in K$, we have :

(i) $||x|| \geq 0$ and $||x|| = 0$ if and only if $x = 0$.

(ii) $||x + y|| \leq ||x|| + ||y||$.

(iii) $||kx|| = |k| \, ||x||$.

A normed linear space X is a linear space X with a norm $||.||$ on it .

Theorem 1.1.2 : [9 , p.35] .

(1) Every normed space is a metric space with respect to the metric

$$d(x , y) = ||x - y|| .$$

(2) For any two elements x and y of a normed space we have ,

$$|\, ||x|| - ||y|| \,| \leq ||x - y|| .$$

(3) A norm is a real-valued continuous function .

Definition 1.1.3 : [9 , p.18] . Let $(X , ||.||)$ be a normed linear space and $G \subset X$. For $x \in X$, we define the distance of a point x from the subset G as :

$$d(x,G) = \inf \{ \|x - y\| \ : \ y \in G \}$$

Theorem 1.1.4 : [12 , p.147] . Let X be a normed linear space and G be a subspace of X , then

(1) $d(x + g,G) = d(x,G)$ $\qquad (x \in X , g \in G)$

(2) $d(x + y,G) \leq d(x,G) + d(y,G)$ $\qquad (x , y \in X)$

(3) $d(\alpha x,G) = |\alpha| \ d(x,G)$ $\qquad (x \in X , \alpha = \text{scalar})$

(4) $| \ d(x,G) - d(y,G) \ | \leq \|x - y\|$ $\qquad (x , y \in X)$

Proof : For (1) . Let $x \in X$, $g \in G$ and $\in > 0$ be arbitrary . By the definition of $d(x,G) = \inf \{ \|x - g\| \ : \ g \in G \}$ there exist $g_0 \in G$ s.t

$$\|x - g_0\| \leq d(x , G) + \in$$

consequently , we have

$$d(x + g,G) \leq \| \ x + g - (g_0 + g)\| = \|x - g_0\| \leq d(x,G) + \in$$

But $x \in X$, $g \in G$ and $\in > 0$ were arbitrary , hence

$$d(x + g,G) \leq d(x,G) \qquad (x \in X , g \in G) \ ....(1)$$

Applying these relation for $x + g \in X$ instead of x and $-g \in G$ instead of $g \in G$, we obtain

$$d(x,G) \leq d(x + g,G) \qquad (x \in X , g \in G) \ ....(2)$$

From (1) , (2) we get $d(x + g,G) = d(x,G)$ $\quad (x \in X , g \in G)$.

For (2) of the theorem ; Let $x , y \in X$ and $\in > 0$ be arbitrary . By the definition of $d(x,G)$ and $d(y,G)$ there exist $g_1 , g_2 \in G$ s.t

$$\|x - g_1\| \le d(x,G) + \in / 2 \ , \qquad \|y - g_2\| \le d(y,G) + \in / 2 \ .$$

Consequently , we have

$$d(x+y,G) \le \|x + y - (g_1 + g_2)\| \le \|x - g_1\| + \|y - g_2\| \le d(x,G) + d(y,G) + \in$$

But x , y $\in$ X , and $\in > 0$ were arbitrary , hence

$$d(x + y,G) \le d(x,G) + d(y,G) \qquad (x , y \in X) .$$

For (3) : Let $x \in X$, $\alpha \neq 0$ be a scalar and $\in > 0$ be arbitrary and take

$g_0 \in G$ satisfying

$$\|x - g_0\| \le d(x,G) + \in / |\alpha|$$

We have

$$d(\alpha x,G) \le \|\alpha x - \alpha g_0\| = |\alpha| \ \|x - g_0\| \le |\alpha| \ d(x,G) + \in$$

But x , $\alpha \neq 0$ and $\in$ were arbitrary , it follows that

$$d(\alpha x,G) \le |\alpha| \ d(x,G) \ ....(1)$$

Applying this relation for αx instead of x and $1/\alpha$ instead of α we obtain

$$d(x,G) = d(1/\alpha \ . \ \alpha x,G) \le 1/|\alpha| \ d(\alpha x,G)$$

Hence $|\alpha| \ d(x,G) \le d(\alpha x,G)$ ....(2)

From (1) and (2) and since $d(0,G) = 0$ we get

$$d(\alpha x,G) = |\alpha| \ d(x,G)$$

For (4) : Let x , y $\in$ X and $\in > 0$ be arbitrary and take $g_0 \in G$ satisfying

$$\|y - g_0\| \le d(y,G) + \in$$

We have

$$d(x,G) \le \|x - g_0\| \le \|x - y\| + \|y - g_0\| \le \|x - y\| + d(y,G) + \epsilon$$

But x , y and ϵ were arbitrary , there follows

$$d(x,G) - d(y,G) \le \|x - y\| \qquad (x , y \in X)$$

In these relations , interchange x and y yields ;

$$d(y,G) - d(x,G) \le \|x - y\|$$

Hence $| d(x,G) - d(y,G) | \le \|x - y\|$. ∎

Definition 1.1.5 : [11 , p.153] .

1) A subset H of a vector space X is called a hyperplane if there exists a linear functional $f \neq 0$ defined on X such that

$$H = \{ x \in X : f(x) = 0 \}$$

2) A subset H of a vector space X is called an affine hyperplane if there exists a linear functional $f \neq 0$ defined on X and a real number α such that

$$H = \{ x \in X : f(x) = \alpha \} .$$

Theorem 1.1.6 : [12 , p.24] . Let X be a normed linear space , and $H = \{y \in X , f(y) = \alpha\}$ be a hyperplane of X , f being a continuous linear functional on X , α a scalar and let $x \in X$. Then the distance of the point x to the hyperplane H is

$$d(x,H) = |f(x) - \alpha| / \|f\|$$

Theorem 1.1.7 : [7 , p.74] . Every finite dimensional subspace G of a normed space X is closed in X .

Now for linear maps , we have the following theorem .

Theorem 1.1.8 : [11 , p.26] . Let X and Y be normed spaces .

(a) A linear map $T : X \to Y$ is continuous if and only if T is bounded .

(b) The null space N(T) of a non zero continuous linear map is a closed subspace of X .

2. Compactness :-

Definition 1.2.1 : [7 , p.77] . A metric space X is said to be compact if every sequence in X has a convergent subsequence . A subset M of X is said to be compact if every sequence in M has a convergent subsequence whose limit is an element of M .

A general property of compact sets is expressed in :

Lemma 1.2.2 : [7 , p.77] . A compact subset of a metric space is closed and bounded .

However , for a finite dimensional normed space we have :

Theorem 1.2.3 : [7 , p.77] . In a finite dimensional normed space X , a subset of X is compact if and only if it is closed and bounded .

In connection with continuous mapping a fundamental property is that compact sets have compact images , as follow we have :

Theorem 1.2.4 : [7 , p.81] . Let X and Y be metric spaces and $T : X \to Y$ be a continuous mapping . Then the image of a compact subset M of X is compact .

From this theorem we conclude that the following property , weii-known from Calculus for continuous functions , carries over to metric space .

Theorem 1.2.5 : [7 , p.81] . A continuous mapping T of a compact subset M of a metric space X into R assumes its maximum and minimum on M .

3. Banach Space and Hilbert Space :-

Definition 1.3.1 : [7 , p.58] . A complete normed space is called a Banach space .

Remark 1.3.2 : [9 , p.47] .

(1) The set of all bounded linear maps on a normed space X into a normed space Y is denoted by B[X,Y] . If X = Y , B[X] denotes B[X,X] .

(2) The set of all bounded linear functionals on a normed space X is denoted by X* .

For $f \in X^*$, any element $x \in X - \{0\}$ with the property that .

$$f(x) = \|f\| \, \|x\| \text{ is called a maximal element of } f .$$

Definition 1.3.3 : [12 , p.110] . A normed space X is strictly convex if and only if every functional $f \neq 0 \in X^*$ has at most one maximal element of norm 1. An equivalent form : a normed space X is strictly convex if

$$\|x + y\| = \|x\| + \|y\| \text{ and } \|x\| = \|y\| \text{ implies that } x = y .$$

Remark 1.3.4 : [11 , p.32] . An arbitrary $f \in c^*$ can be expressed as

$$f(x) = y_0 \lim_{n\to\infty} x_n + \sum_{n=1}^{\infty} y_n x_n \quad \text{where}$$

$x = (x_1, x_2, x_3, \ldots..) \in c$ and $y = (y_0, y_1, y_2, \ldots.)$ such that $\sum_{i=1}^{\infty} |y_i| < \infty$ and

$$||f|| = |y_0| + \sum_{i=1}^{\infty} |y_i|$$

Definition 1.3.5 : [9 , p.176] . Let X be a vector space over the field K . An inner product on X is a function

$< , > : X \times X \to K$ such that for all $x, x', y \in X$ and $\alpha \in K$.

(1) $<x + x', y) = <x , y> + <x' , y>$

(2) $<\alpha x , y> = \alpha<x , y>$

(3) $\overline{<x , y>} = <y , x>$

(4) $<x , x> \geq 0 \ \forall x \in X$ and $<x , x> = 0$ iff $x = 0$.

An inner product space is a linear space with an inner product on it .

Definition 1.3.6 : [9 , p.182] . A Hilbert space X is a Banach space in which the norm satisfies the parallelogram law .

$$||x + y||^2 + ||x - y||^2 = 2\,||x||^2 + 2\,||y||^2 , \quad x , y \in X$$

Definition 1.3.7 : [7 , p.131] . Two vectors x and y in an inner-product space are called orthogonal , (written $x \perp y$) , if $<x , y> = 0$.

Theorem 1.3.8 : [7 , p.135] . Let X be an inner-product space and $x , y \in X$. Then for $x \perp y$ we have $||x + y||^2 = ||x||^2 + ||y||^2$.

In the case of a general Hilbert space X , we obtain an interesting representations of X as a direct sum of a closed subspace M and its orthogonal complement

$$M^{\perp} = \{ x \in X \ , \ x \perp M \}$$

which is the set of all vectors orthogonal to each member of M . The next theorem is sometimes called the projection theorem .

Theorem 1.3.9 : [11 , p.96] . If M is a closed subspace of a Hilbert space X , then

$$X = M \oplus M^{\perp}$$

4. Projection :-

Definition 1.4.1 : [9 , p.85] . If X is a normed space and $P \in B[X]$ satisfies $P^2 = P$, then P is called a projection .

Theorem 1.4.2 : [9 , p.85] . If P is a projection on a Banach space X , and if M and N are its range and null space , respectively , then M and N are closed subspaces and

$$X = M \oplus N$$

Theorem 1.4.3 : [9 , p.85] . Let X be a Banach space , and let M and N be closed subspace of X such that

$$X = M \oplus N$$

The mapping defined on each $z = x + y$, $x \in M$, $y \in N$ by $P(z) = x$ is a projection on X whose range is M and whose null space is N .

5. Lebesgue Measure and Integrable Function :-

In this section we shall review the theory of the Lebesgue measure and the p-integrable function .

Definition 1.5.1 : [10 , p.9] .

(a) A collection $\mathcal{M}$ of subsets of a set X is said to be a σ-algebra of X if it has the following properties .

(i) $X \in \mathcal{M}$

(ii) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$ where A^c is the complement of A relative to X .

(iii) If $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{M}$ for n = 1,2,3,.... , then $A \in \mathcal{M}$.

(b) If $\mathcal{M}$ is a σ-algebra in X , then X is called a measurable space , and the members of $\mathcal{M}$ are called the measurable sets in X .

(c) If X is a measurable space , Y is a topological space , and f is a mapping of X into Y , then f is said to be measurable provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y .

For $E \subset X$, let χ_E denote the characteristic function of E ; i.e ,

$$\chi_E(t) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{if } t \notin E \end{cases}$$

It is measurable iff E is measurable .

Definition 1.5.2 : [8 , p.113] . A function $f : T \to X$ is said to be simple if its range contains only finitely many points $x_1, x_2, \ldots\ldots, x_n$ and if $f^{-1}(x_i)$ is measurable for $i = 1, 2, \ldots.., n$. Such a function then can be written as

$f = \sum_{i=1}^{n} x_i \, \chi_{E_i}$ where for each i , $E_i = f^{-1}(x_i)$. Define

$$\int_E f \, d\mu = \sum_{i=1}^{n} x_i \, \mu(E_i \cap E)$$

If f is a non-negative measurable function on E , define

$$\int_E f \, d\mu = \sup \{ \int_E s \, d\mu : 0 \le s \le f \text{ , s is simple and measurable on E} \}$$

Definition 1.5.3 : [10 , p.17] .

(a) A measure is a function μ , defined on a σ-algebra $\mathcal{M}$, whose range is in $[0,\infty]$ and which is countably additive . This means that if $\{A_n\}$ is a disjoint countable collection of members of $\mathcal{M}$, then

$$\mu \left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n)$$

(b) A measure space is a measurable space which has a measure defined on the σ-algebra of its measurable sets .

A property which is true except for a set of measure zero is said to hold almost everywhere .

Remark 1.5.4 : [10 , p.21] . The following propositions are immediate consequences of the definitions . Functions and sets are assumed to be measurable on a measure space E .

(a) If $0 \leq f \leq g$ then $\int_E f \leq \int_E g$.

(b) If $A \subset B$ and $f > 0$, then $\int_A f\, d\mu \leq \int_B f\, d\mu$.

(c) If c is constant , then $\int_E c\, d\mu = c\mu(E)$.

(d) If $E = E_1 \cup E_2$ where E_1 and E_2 are disjoint , then

$$\int_E f\, d\mu = \int_{E_1} f\, d\mu + \int_{E_2} f\, d\mu$$

(e) If $f \geq 0$ and $\int_E f\, d\mu = 0$, then $f = 0$ almost everywhere on E .

Definition 1.5.5 : [1 , p.527] . Let X be a real Banach space , and (T,μ) be a finite measure space . The space of Bochner p-integrable functions defined on (T,μ) with values in a Banach space X is denoted $L^p(\mu,X)$.

For $f \in L^p(\mu,X)$, we write

$$\|f\|_p = \begin{cases} \left(\int_T \|f(t)\|^p \, d\mu(t)\right)^{1/p} & 1 \leq p < \infty \\ \int_T \|f(t)\|^p \, d\mu(t) & 0 < p < 1 \\ \text{ess.} \sup\limits_{t \in T} \|f(t)\| & p = \infty \end{cases}$$

where $\text{ess.} \sup\limits_{t \in T} \|f(t)\| = \inf \{ M : \mu \{ t : \|f(t)\| > M \} = 0 \}$.

Chapter 2

Approximation in Normed Spaces

1. Introduction : Let $X = (X, \|.\|)$ be a normed space, G be a subspace of X and $x \in X$, an element $g_0 \in G$ is called **a best approximant** of x in G if

$$\|x - g_0\| = d(x,G) = \inf \{\|x - g\| : g \in G\} .$$

We see that for $x \in X$ a best approximant $g_0 \in G$ is an element of minimal distance from the given x. Such a $g_0 \in G$ may or may not exist.

We shall denote the set of all elements of best approximants of x in G by P(x,G) i.e $P(x,G) = \{ g \in G : \|x - g\| = d(x,G) \}$.

2. The Set of Best Approximants :-

In this section, we introduce some basic properties of P(x,G), we use the book of Erwin Kreyszig for this purpose, so we start with the following.

Example 2.2.1 : Let $X = (X, \|.\|_1)$ be the normed space of ordered pairs $x = (x_1,x_2)$ of real numbers with norm defined by $\|x\|_1 = |x_1| + |x_2|$.

Let us take $x = (1,-1)$ and the subspace $G = \{ g = (g_1,g_1) : g_1 \in \mathbf{R} \}$ then for all $g \in G$, we clearly have

$$\|x - g\|_1 = |1 - g_1| + |-1 - g_1| \geq |1 - g_1 + 1 + g_1| = 2.$$

Hence $d(x,G) \geq 2$ ....(1)

Also $(1,1) \in G$, and $d(x,G) \leq \|x - (1,1)\| = 2$ ....(2)

From (1) and (2) we get $d(x,G) = 2$ and

$P(x,G) = \{ g \in G : g = (g_1,g_1) \text{ and } |g_1| \leq 1 \}$.

From this example we conclude that P(x,G) need not be a subspace .

Theorem 2.2.2 : Let G be a subspace of a normed space X .

(i) if $x \in G$, then $P(x,G) = \{x\}$.

(ii) if G is not closed and $x \in \overline{G} \backslash G$, then $P(x,G) = \phi$.

Proof :- For (i) , let $x \in G$, then $\inf \{ \|x - g\| : g \in G \} = 0$. Thus , if $y \in P(x,G)$ then $\|x - y\| = 0$ hence $x = y$ since X is a normed space .

For (ii) , let $x \in \overline{G} \backslash G$. This means that $\forall\, n \in \mathbf{N}$, $\exists\, x_n \in G$ s.t $\|x_n - x\| \leq 1/n$. i.e $d(x,G) = 0$. Hence $P(x,G) = \phi$. ∎

Now provided $P(x,G) \neq \phi$, we shall prove that either P(x,G) contains exactly one element or else an infinite number of elements .

Theorem 2.2.3 : Let X be a normed space , $x \in X$, and let G be a subspace of X , then P(x,G) is a convex set .

Proof : Let δ be the distance from x to G . The statement holds if P(x,G) is empty or has just one point . Now suppose that $y , z \in P(x,G)$ such that $y \neq z$. So

$$\|x - y\| = \|x - z\| = \delta .$$

We will show that if $0 \leq \alpha \leq 1$, and if

$$w = \alpha y + (1 - \alpha)z , \text{ then } w \in P(x,G)$$

to show this :

$$\|x - w\| = \|x - (\alpha y + (1 - \alpha)z)\|$$

$$= \|x - \alpha y - (1 - \alpha)z + \alpha x - \alpha x\|$$

$$= \|\alpha(x - y) + (1 - \alpha)(x - z)\|$$

$$\leq \alpha \|x - y\| + (1 - \alpha) \|x - z\|$$

$$= \alpha \delta + (1 - \alpha)\delta$$

$$= \delta$$

Therefore $\|x - w\| \leq \delta$ ....(1)

Also $w \in G$, since G is a subspace , so ; $\delta \leq \|x - w\|$ ....(2)

From (1) and (2) we get that $\|x - w\| = \delta$, so $w \in P(x,G)$. Since $y , z \in P(x,G)$ were arbitrary ; $P(x,G)$ is convex . ∎

Theorem 2.2.4 : Let G be a subspace of a normed space X , then for $x \in X$.

(i) $P(x,G)$ is a bounded set .

(ii) If G is a closed subspace , then $P(x,G)$ is a closed set .

Proof : For (i) , let $g_0 \in P(x,G)$, we have by definition $\|x - g_0\| = \delta$ where

$$\delta = \inf \{ \|x - g\| : g \in G \} .$$

Now $\|g_0\| = \|g_0 - x + x\|$

$$\leq \|g_0 - x\| + \|x\|$$

$$\leq \|x\| + \|x\| , \qquad \text{since } 0 \in G$$

$$= 2 \|x\|$$

Thus $P(x,G)$ is bounded .

For (ii) , we show that if a sequence $(g_n) \in P(x,G)$ such that $g_n \to g$ then $g \in P(x,G)$. Now $g_n \in P(x,G)\ \forall\ n \in \mathbf{N}$, so $||x - g_n|| = d(x,G) = \delta$, $\forall\ n \in N$. Also $g_n \in G$. Since G is a closed subspace , then $g \in G$.

But the function $F_x : G \to R$ defined by $F_x(g) = ||x - g||\ \forall\ g \in G$ is continuous by part (3) of Theorem (1.1.2) . So $F_x(g_n) \to F_x(g)$ implies that

$$||x - g_n|| \to ||x - g||$$

But $||x - g_n|| = \delta\ \forall\ n \in N$, so $||x - g|| = \delta$.

Therefore $g \in P(x,G)$. ∎

Theorem 2.2.5 : Let G be a subspace of a normed space X . For $x \in X$:

(i) if $z \in P(x,G)$ then $\alpha z \in P(\alpha x,G)$ for all scalars α .

(ii) if $z \in P(x,G)$ then $z + g \in P(x + g,G)$ for all $g \in G$.

Proof : For (i) ; if $g \in G$ and α a scalar $\neq 0$ we have

$$||\alpha x - g|| = |\alpha|\ ||x - (1/\alpha).g|| \geq |\alpha|\ ||x - z|| = ||\alpha x - \alpha z||$$

Thus $\alpha z \in P(\alpha x,G)$.

For (ii) ; if $g' \in G$ we have

$$||x + g - g'|| \geq ||x - z|| = ||x + g - (z + g)||$$

whence $z + g \in P(x + g,G)$. ∎

An element x of a normed linear space X is said to be **orthogonal** to an element $y \in X$, and we write $x \perp y$, if we have $||x + \alpha y|| \geq ||x||$ for every scalar α. In a Hilbert space X we have $x \perp y$ if and only if $\langle x,y \rangle = 0$.

An element x of a normed linear space X is said to be orthogonal to a set $G \subset X$ and we write $x \perp G$, if we have $x \perp g$ $(g \in G)$.

The relationship between orthogonality and best approximation is given by the following theorem.

Theorem 2.2.6 : [12 , p.92] . Let X be a normed space , G a subspace of X, $x \in X\backslash\overline{G}$ and $g_0 \in G$. We have $g_0 \in P(x,G)$ if and only if $x - g_0 \perp G$.

Proof : By the definition , orthogonality means that

$$||x - g_0 + \alpha g|| \geq ||x - g_0|| \quad (g \in G,\ \alpha = \text{scalar})$$

and this is obviously equivalent to saying that $g_0 \in P(x,G)$. ∎

3. Proximinal Set :-

Theorem 2.2.2 shows that if X is a normed space and G a subspace of X , then for every $x \in G$ the set P(x,G) is non empty , and if the subspace G is not closed , then for every $x \in \overline{G}\backslash G$ the set P(x,G) is empty . Furthermore , for the elements x of $X\backslash\overline{G}$ the set P(x,G) may or may not be empty . The subspaces $G \subset X$ which have the property that $P(x,G) \neq \phi$ for every $x \in X$ are called **proximinal sets in X** . Some authors use the term distance set , or existence set for proximinal sets .

Remark 2.3.1 : In a normed spaces X, the condition that G is a closed subspace of X is not sufficient for G to be proximinal in X as shown by the following example .

Example : Let $X = c_0$ = the space of all sequences of complex numbers that converge to zero , with norm $\|x\| = \sup_n |x_n|$, and let

$$G = \{ x = (x_n) \in c_0 : \sum_{n=1}^{\infty} 2^{-n} x_n = 0 \} .$$

G is closed but not proximinal . To show that define a linear functional

$$f : c_0 \to \mathbf{C} \text{ by } f(x) = \sum_{n=1}^{\infty} 2^{-n} x_n \quad \text{for all } x = (x_n) \in c_0 .$$

Then $f \in c_0^*$ and $\|f\| = 1$ by Remark (1.3.4) and hence G is a closed subspace of c_0 by part (2) of Theorem (1.1.8) .

Now let $x = e^{(1)} = (1,0,0,......) \in c_0$.

Then $d(e^{(1)},G) = 1/2$ by Theorem (1.1.6) .

We claim that there does not exist any $g \in G$ s.t $\|e^{(1)} - g\| = 1/2$.

Assume on the contrary that $\exists\, g = (g_i) \in G$ s.t $\|e^{(1)} - g\| = 1/2$, then

$|1 - g_1| \leq 1/2$ and

$|g_k| \leq 1/2$ for all $k \geq 2$. Since $\sum 2^{-n} g_n = 0$, we get that

$$1/4 \leq 1/2\,|g_1| = |\sum_{n\geq 2}^{\infty} 2^{-n} g_n| \leq \sum_{n\geq 2}^{\infty} 2^{-n} |g_n| \leq 1/2 \sum_{n=2}^{\infty} 2^{-n} = 1/4 \text{ .... } *$$

So we must have equalities in (*) , and that can happen only if $|g_n| = 1/2$ for all n . But this contradicts our assumption that $g \in c_0$. Thus G is not proximinal in c_0 . ∎

We call a linear subspace G of a normed space X a **semi-Chebyshev** subspace if for every $x \in X$ the set P(x,G) contains at most one element.

An example of such subspaces is that of the subspaces G with the property that the set P(x,G) is empty for all $x \in X \setminus G$. We will see such a space in Remark (4.1.2). G is called a **Chebyshev subspace** if it is simultaneously proximinal and semi-Chebyshev. i.e if for every $x \in X$ the set P(x,G) contains exactly one element.

Theorem 2.3.2 : For a subspace G of a normed space X, the following are equivalent :

(i) G is proximinal in X.

(ii) $X = G + P_G^{-1}(0)$ where $P_G^{-1}(0) = \{ x \in X : 0 \in P(x,G) \}$.

Proof : If G is proximinal and $x \in X$, $g_0 \in P(x,G)$, then

$$x = g_0 + (x - g_0) \in G + P_G^{-1}(0).$$

Conversely if we have (ii) and $x \in X$, $x = g_0 + y$ where $g_0 \in G$, $y \in P_G^{-1}(0)$ then $0 \in P(y,G) = P(x - g_0,G)$, implies $d(x - g_0,G) = ||x - g_0||$.

$$\Rightarrow d(x,G) = ||x - g_0||. \text{ Hence } g_0 \in P(x,G).$$ ∎

Theorem 2.3.3 : Let G be a compact subspace of a normed space X, then G is proximinal in X.

Proof : Fix $x \in X$, since the mapping $T_x : G \to \mathbf{R}$ defined by $T_x(g) = ||x - g||$ is continuous by part (3) of Theorem (1.1.2), then $T(G) \subset \mathbf{R}$ is compact by

Theorem (1.2.4). So T(G) is closed and bounded in **R** by Theorem (1.2.3). Theorem (1.2.5) implies $\inf T(G) \in T(G) = \{ \|x - g\| : g \in G \}$. Hence there exists $g_0 \in G$ such that $\|x - g_0\| = \inf\{ \|x - g\| : g \in G \}$. This proves that $g_0 \in P(x,G)$. Since x is an arbitrary, $P(x,G) \neq \phi$ for all $x \in X$. ∎

While compactness of G is a sufficient condition for a best approximant to exist, it is clearly not necessary. For, consider the noncompact subspace $G = \{ (x,0) : x \in \mathbf{R} \}$ of $\mathbf{R}^2$. Clearly G is proximinal in $\mathbf{R}^2$.

Corollary 2.3.4 : Every closed and bounded subspace of a finite dimensional normed space X is proximinal in X.

Proof : Every closed and bounded subspace of a finite dimensional space is compact by Theorem (1.2.3). ∎

Theorem 2.3.5 : Let G be a finite dimensional subspace of a normed space X, then G is proximinal in X.

Proof : Let $x \in X$ be given, consider the closed Ball.

$$\overline{B} = \{ g \in G : \|g\| \leq 2\|x\| \}$$

Then $0 \in \overline{B}$, so that for the distance from x to $\overline{B}$ we obtain

$$d(x,\overline{B}) = \inf\{ \|x - g'\| : g' \in \overline{B} \} \leq \|x - 0\| = \|x\|$$

Now if $g \in G$ and $g \notin \overline{B}$, then $\|g\| > 2\|x\|$ and

$$\|x - g\| \geq \|g\| - \|x\| > \|x\| \geq d(x,\overline{B}) \;..........(1)$$

If $g \in G$ and $g \in \overline{B}$, then $d(x, \overline{B}) \leq \|x - g\|$ ....(2)

(1) and (2) imply that $d(x, \overline{B})$ is a lower bound of the set

$$\{ \|x - g\| : g \in G \} .$$

Hence $d(x, \overline{B}) \leq d(x,G)$.

Also $\overline{B} \subset G \Rightarrow d(x,G) \leq d(x, \overline{B})$.

This shows that $d(x,\overline{B}) = d(x,G)$. Since $\overline{B}$ is closed and bounded and since G is finite dimensional ; Theorem (1.2.3) implies $\overline{B}$ is compact . Theorem (2.3.3) implies that $\overline{B}$ is proximinal in X . Hence , if a best approximant to x exists , it must lie in $\overline{B}$. Thus G is proximinal in X . ∎

It is not possible to drop the finite dimensional requirement of this theorem . For , let X be the space of continuous functions defined on [0,1/2] with $L\infty$ norm . i.e $\|f\|_\infty = \max_{0 \leq x \leq 1/2} |f(x)|$.

Let G be the subspace of polynomials ,and let $g(x) = 1/(1-x)$. For any $\epsilon > 0$, there exists a polynomial p_ϵ such that $|g(x)-p_\epsilon(x)| < \epsilon$ for all $x \in [0,1/2]$ [Weierstrass Approximation Theorem].

Hence $d(g,G) = 0$. However , since g is not a polynomial , we see that there is no $p \in G$ satisfying $d(g,G) = \|g - p\| = 0$.

We present the following example to show that proximinal subspaces need not be finite dimensional ; thus proving that the converse of the foregoing theorem is not true .

Example 2.3.6 : The infinite dimensional subspace c_0 of c is proximinal in c .

Proof : On c , define the linear functional f by $f(x) = \lim x_n = \mathbf{x}$.

Then $c_0 = \{ x = (x_n) \in c : f(x) = 0 \}$ is the hyperplane of c and $d(x,c) = |\mathbf{x}|$ by Theorem (1.1.6) .

Let $g = (g_n)$ be defined as , $g_n = x_n - \mathbf{x}$.

Now $g \in c_0$ and $\|x - g\| = \sup \{ |x_n - g_n| : n \in \mathbf{N} \}$

$$= \sup \{ |x_n - (x_n - x)| : n \in \mathbf{N} \}$$

$$= |\mathbf{x}|$$

Hence $d(x,c) = \|x - g\|$ and so ; $g \in P(x,c)$.

Since x was arbitrary ; c_0 is proximinal in c . ∎

Theorem 2.3.7: [12 , p.93] . Let X be a normed space and G a hyperplane in X , passing through 0 . G is proximinal in X if and only if there exists an element $z \in X - \{0\}$ such that $0 \in P(z,G)$. (i.e $z \perp G$) .

Proof : Assume that G is proximinal and take arbitrary $x \in X - G$, $y_0 \in P(x,G)$. Then for $z = x - y_0 \neq 0$ we have $0 \in P(z,G)$.

Conversely , assume that there exists $z \in X - \{0\}$ such that $0 \in P(z,G)$ and let $x \in X - G$ be arbitrary .

Take $f \in X^+$ such that

$$G = \{ y \in X \quad : \quad f(y) = 0 \},$$

and put

$$y_0 = x - (f(x) / f(z))\, z$$

(we have $f(z) \neq 0$ since , otherwise $z \in G$, so $0 \in P(z,G) = \{z\}$, and hence $z = 0$, a contradiction to the hypothesis) .

We have then

$$f(y_0) = 0$$

Whence $y_0 \in G$. Also , since $(f(z) / f(x))\,(y - y_0) \in G$ for every $y \in G$, we have

$$\|x - y_0\| = | f(x) / f(z)|\ \|z\| \leq | f(x) / f(z)|\ \|z - (f(z) / f(x))\,(y - y_0)\| = \|x - y\|$$

Whence $y_0 \in P(x,G)$. Since $x \in X - G$ has been arbitrary , it follows that G is proximinal in X . ∎

4. Approximation in Hilbert space

Before we prove that all closed subspaces of a Hilbert space are proximinal we need the following definition .

Definition 2.4.1 : [6 , p.279] . A closed subspace G of a Banach space X is called an L^p-summand , $1 \leq p < \infty$, if there is a bounded projection $P : X \rightarrow G$ which is onto , and $\|x\|^p = \|P(x)\|^p + \|x - P(x)\|^p$ for all $x \in X$.

Theorem 2.4.2 : If G is an L^p-summand of a Banach space X , then G is proximinal in X . $1 \leq p < \infty$.

Proof : Let $x \in X$, for every $g \in G$ we have

$$\|x - g\|^p = \|P(x - g)\|^p + \|x - g - P(x - g)\|^p$$
$$= \|P(x) - g\|^p + \|x - P(x)\|^p$$
$$\geq \|x - P(x)\|^p$$

Hence $\|x - g\| \geq \|x - P(x)\|$

i.e $P(x) \in P(x,G)$. Thus G is proximinal in X . ∎

Erwin kreyszig [7] proved that if G is a closed subspace of a Hilbert space X , then G is proximinal in X . He used Cauchy sequence and parallelogram law to prove this theorem . Here we give a simpler proof .

Theorem 2.4.3 : Let G be any closed subspace of a Hilbert space X , then G is a Chebyshev subspace .

Proof : Since G is a closed subspace of a Hilbert space X . Theorem (1.3.9) implies $X = G \oplus G^{\perp}$ where $G^{\perp} = \{ g \in X : g \perp G \}$. Hence every element $x \in X$ has a unique representation i.e $x = g + z$ where $g \in G$, $z \in G^{\perp}$. Now we define the projection $P : X \to G$ by $P(x) = g$. Clearly P is onto and bounded .

Also if $x = g + z$ and $z \perp g$ then $\|z + g\|^2 = \|z\|^2 + \|g\|^2$ by Theorem (1.3.8) , so $\|x\|^2 = \|x - P(x)\|^2 + \|P(x)\|^2$ i.e G is an L^2-summand of X , hence Theorem (2.4.2) implies that G is proximinal in X .

Now we show that P(x,G) contains exactly one element .

Let $x \in X$, assume $g_1, g_2 \in P(x,G)$ such that $g_1 \neq g_2$. Since $g_1, g_2 \in G$ and G is subspace , then $(g_1 + g_2)/2 \in G$. By the parallelogram law for :

$(x - g_1)/2$ and $(x - g_2)/2$ we have

$$\|(x - g_1)/2 + (x - g_2)/2\|^2 + \|-g_1/2 + g_2/2\|^2 = 2\,\|(x - g_1)/2\|^2 + 2\,\|(x - g_2)/2\|^2$$

or $\|x - (g_1 + g_2)/2\|^2 < 1/2\,\|x - g_1\|^2 + 1/2\,\|x - g_2\|^2 = [d(x,G)]^2$. This implies that $\|x - (g_1 + g_2)/2\,\| < d(x,G)$.

Which contradicts the definition of d(x,G) . Hence G is a Chebyshev subspace . ∎

Corollary 2.4.4 : Any closed subspace of R^n , C^n is a Chebyshev subspace.

Chapter 3

Best Approximation in Orlicz Space

1. Orlicz Spaces :

In order to study Orlicz spaces , it is necessary to introduce the definition of modulus functions .

Definition 3.1.1 : [5 , p.159] . A function ϕ : $[0,\infty)\to[0,\infty)$ is called a modulus function if the following are satisfied :

(i) ϕ is continuous at zero from the right and strictly increasing .

(ii) $\phi(0) = 0$.

(iii) ϕ is subadditive i.e $\phi(x + y) \leq \phi(x) + \phi(y)$. $\forall$ x,y $\in [0,\infty)$

Examples of such functions are $\phi(x) = x^p$, $0 < p \leq 1$, and $\phi(x) = \ln(1+x)$. In fact , if ϕ is a modulus function and $a \geq 0$ then $\Psi_1(x) = \phi(x) / (1+\phi(x))$ and $\Psi_2(x) = a\phi(x)$ are modulus functions . Further , the composition of two modulus functions is a modulus function .

Theorem 3.1.2 : Every modulus function is continuous on $[0,\infty)$.

Proof : Let $x_0 \in [0,\infty)$. We show that ϕ is continuous at x_0 , i.e $\lim_{x\to x_0} \phi(x) = \phi(x_0)$. At first we show that

$$|\phi(x) - \phi(y)| \leq \phi(|x - y|) \text{ for all } x,y \in [0,\infty) .$$

Now $|x| = |x - y + y| \leq |x - y| + |y|$, since ϕ is strictly increasing and subadditive we get $\phi(|x|) \leq \phi(|x - y|) + \phi(|y|)$.

So $\phi(|x|) - \phi(|y|) \leq \phi(|x - y|)$ ....(1)

Also

$|y| = |y - x + x| \leq |x - y| + |x|$, implies $\phi(|y|) - \phi(|x|) \leq \phi(|x-y|)$ ......(2)

From (1) and (2) we get :

$$|\phi(|x|) - \phi(|y|)| \leq \phi(|x - y|) \qquad \text{for all } x,y \in [0,\infty) .$$

Now given $\in > 0$, there exist $\delta(\in) > 0$ such that if $0 < x < \delta(\in)$ then $|\phi(x)| < \in$, because ϕ is continuous at 0 . But $|\phi(x) - \phi(x_0)| \leq \phi(|x - x_0|) < \in$ if $|x - x_0| < \delta(\in)$. Hence ϕ is continuous at x_0 . Since x_0 is arbitrary , ϕ is continuous on $[0,\infty)$. ∎

Definition 3.1.3 : [5 , p.159] . Let X be a real Banach space , and (T,μ) be finite measure space . For a given modulus function ϕ , we define the Orlicz space as :

$$L^{\phi}(\mu,X) = \{ f : T \to X : \int_T \phi(\|f(t)\|)\, d\mu(t) < \infty \} .$$

The function $d : L^{\phi}(\mu,X) \times L^{\phi}(\mu,X) \to [0,\infty)$, given by

$$d(f,g) = \int_T \phi(\|f(t) - g(t)\|)\, d\mu(t)$$

defines a metric on $L^{\phi}(\mu,X)$, under which it becomes a complete metric linear space [4 , p.70] .

For $f \in L^{\phi}(\mu,X)$, we write

$$\|f\|_{\phi} = \int_T \phi(\|f(t)\|)\, d\mu(t) .$$

The spaces of p-Bochner integrable functions $L^p(\mu,X)$, $0 < p \leq 1$, by Definition (1.5.5) are reduced to be a special case of the Orlicz space under the modulus function $\phi(x) = x^p \quad 0 < p \leq 1$.

The following theorem relates $L^1(\mu,X)$ and $L^\phi(\mu,X)$ for all modulus functions ϕ .

Theorem 3.1.4 : [5 , p.159] . If ϕ is a modulus function , then

$$L^1(\mu,X) \subset L^\phi(\mu,X) .$$

Proof : For each real number x , we have $[x] \leq x < [x] + 1$; where [ ] denotes the greatest integer function . But ϕ is strictly increasing and subadditive , then

$$\phi(x) \leq \phi([x] + 1) \leq \phi([x]) + \phi(1) \leq [x]\, \phi(1) + \phi(1)$$

$$\leq x\, \phi(1) + \phi(1) = (x + 1)\, \phi(1) .$$

If $x > 1$, then $\phi(x) \leq 2x\, \phi(1)$; and if $x \leq 1$, then $\phi(x) \leq \phi(1)$.

Now , let $f \in L^1(\mu,X)$, and let $A = \{ t \in T \ : \ \|f(t)\| \leq 1 \}$

$$B = \{ t \in T \ : \ \|f(t)\| > 1 \}$$

$$\|f\|_\phi = \int_T \phi(\|f(t)\|)\, d\mu(t)$$

$$= \int_A \phi(\|f(t)\|)\, d\mu(t) + \int_B \phi(\|f(t)\|)\, d\mu(t)$$

$$\leq \int_T \phi(1)\, d\mu(t) + \int_T 2\, \|f(t)\|\, \phi(1) d\mu(t)$$

$$\leq \phi(1)\, \mu(T) + 2\phi(1)\, \|f\|_1 < \infty \quad \text{since } f \in L^1(\mu,X) .$$

Hence $f \in L^{\phi}(\mu,X)$.

2. Best Approximation in $L^{\phi}(\mu,X)$

In this section we investigate when $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$ where G is a closed subspace of a Banach space X .

We present the following useful theorem .

Theorem 3.2.1 : [4 , p.73] . Let G be a closed subspace of a Banach space X , if g is a best approximant of f in $L^{\phi}(\mu,G)$. Then g(t) is a best approximant of f(t) in G for almost all $t \in T$.

The following theorem establishes the relation between proximinality in $L^{\phi}(\mu,X)$ and $L^{1}(\mu,X)$.

Theorem 3.2.2 : [4 , p.73] . Let G be a closed subspace of Banach space X , the following are equivalent :

(i) $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$.

(ii) $L^{1}(\mu,G)$ is proximinal in $L^{1}(\mu,X)$.

Proof : (i)→(ii) . Let $f \in L^{1}(\mu,X)$, since $L^{1}(\mu,X) \subset L^{\phi}(\mu,X)$ then $f \in L^{\phi}(\mu,X)$, but $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$ so there exist $g \in L^{\phi}(\mu,G)$ such that

$$\|f - g\|_{\phi} \leq \|f - h\|_{\phi} \quad \forall\, h \in L^{\phi}(\mu,G) .$$

Theorem (3.2.1) implies

$$\|f(t) - g(t)\| \leq \|f(t) - y\| \quad \forall\, y \in G \text{ for almost all } t \in T \ldots\ldots (1)$$

Since $0 \in G$ we get

$$\|f(t) - g(t)\| \leq \|f(t) - 0\| = \|f(t)\|$$

Hence

$$\|g(t)\| = \|g(t) - f(t) + f(t)\|$$

$$\leq \|g(t) - f(t)\| + \|f(t)\|$$

$$\leq 2\,\|f(t)\|$$

Therefore $g \in L^1(\mu,G)$.

From (1) we get

$$\|f(t) - g(t)\| \leq \|f(t) - h(t)\| \quad \forall\, h \in L^{\phi}(\mu,G) \text{ a.e.t}$$

Integrating both sides we get

$$\|f - g\|_1 \leq \|f - h\|_1 \ \forall\, h \in L^{\phi}(\mu,G) .$$

Therefore $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$.

Conversely , (ii)→(i) . Define the map $J : L^{\phi}(\mu,X) \to L^1(\mu,X)$ by $J(f) = \hat{f}$

where

$$\hat{f}(t) = \begin{cases} \dfrac{\phi(\|f(t)\|)}{\|f(t)\|}\, f(t) & f(t) \neq 0 \\ 0 & f(t) = 0 \end{cases}$$

At first we show that $\hat{f} \in L^1(\mu,X)$

$$\|\hat{f}\|_1 = \int_T \|\hat{f}(t)\|\, d\mu(t)$$

$$= \int_T \frac{\phi(\|f(t)\|)}{\|f(t)\|} \|f(t)\| \, d\mu(t)$$

$$= \int_T \phi(\|f(t)\|) \, d\mu(t)$$

$$= \|f\|_\phi < \infty .$$

Second , we claim that J is onto .

Let $g \in L^1(\mu,X)$ and let $f(t) = \begin{cases} \frac{\phi^{-1}(\|g(t)\|)\, g(t)}{\|g(t)\|} & g(t) \neq 0 \\ 0 & g(t) = 0 \end{cases}$

Then $\|f\|_\phi = \int_T \phi(\|f(t)\|) \, d\mu(t)$

$$= \int_T \phi\left[\frac{\phi^{-1}(\|g(t)\|)}{\|g(t)\|} \|g(t)\| \right] d\mu(t)$$

$$= \int_T \|g(t)\| \, d\mu(t)$$

$$= \|g\|_1$$

Hence $f \in L^\phi(\mu,X)$ and $J(f) = g$.

Finally since ϕ is one-to-one it follows that J is one-to-one . It is now clear that

$$J(L^\phi(\mu,G)) = L^1(\mu,G) .$$

Now , let $f \in L^\phi(\mu,X)$. Then $J(f) = \hat{f} \in L^1(\mu,X)$ and there exists $\hat{g} \in L^1(\mu,G)$ such that $\|\hat{f} - \hat{g}\|_1 \leq \|\hat{f} - \hat{h}\|_1$ for all $\hat{h} \in L^1(\mu,G)$. By Theorem (3.2.1) ; we have

$$\|\hat{f}(t) - \hat{g}(t)\| \le \|\hat{f}(t) - y\| \text{ for all } y \in G \quad \text{a.e.t}.$$

Since $\hat{g} \in L^1(\mu,G)$ and J is onto , there exists $g \in L^\phi(\mu,G)$ s.t . $J(g) = \hat{g}$.

Hence $\left\| f(t) - \dfrac{\phi(\|g(t)\|)\,\|f(t)\|\,g(t)}{\phi(\|f(t)\|)\,\|g(t)\|} \right\| \le \left\| f(t) - \dfrac{\|f(t)\|}{\phi(\|f(t)\|)}\, y \right\|$ a.e.t

and for all $y \in G$.

Now take $h \in L^\phi(\mu,G)$. Then $\dfrac{\phi(\|f(t)\|)}{\|f(t)\|}\, h(t) \in G$ a.e.t

Hence $\|f(t) - w(t)\| \le \|f(t) - h(t)\|$ a.e.t and for all $h \in L^\phi(\mu,G)$,

where $w(t) = \dfrac{\phi(\|g(t)\|)\,\|f(t)\|}{\phi(\|f(t)\|)\,\|g(t)\|}\; g(t)$

using the fact that $\|g(t)\| \le 2\,\|f(t)\|$ we will show that $w \in L^\phi(\mu,G)$ as follows

$$\|w(t)\| = \frac{\phi(\|g(t)\|)\,\|f(t)\|}{\phi(\|f(t)\|)\,\|g(t)\|} \cdot \|g(t)\|$$

$$\le \frac{2\phi(\|f(t)\|)\,.\,\|f(t)\|}{\phi(\|f(t)\|)}$$

$$= 2\,\|f(t)\|$$

Hence $w \in L^\phi(\mu,G)$. Thus $L^\phi(\mu,G)$ is proximinal in $L^\phi(\mu,X)$. ∎

In a similar way we can prove the following theorem .

Theorem 3.2.3 : [2 , p.297] . Let G be a closed subspace of Banach space X . If $1 < p < \infty$ the following are equivalent :

(i) $L^p(\mu,G)$ is proximinal in $L^p(\mu,X)$.

(ii) $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$.

For $p = \infty$, we have :

Theorem 3.2.4 : [1 , p.528] . Let G be a closed subspace of Banach space X . If $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$, then $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$.

Proof : Let $f \in L^\infty(\mu,X)$. Since $L^\infty(\mu,X) \subseteq L^1(\mu,X)$, we have $f \in L^1(\mu,X)$. By assumption , there exists $f_1 \in L^1(\mu,G)$ such that

$$||f - f_1||_1 = d(f,L^1(\mu,G)) .$$

By Theorem (3.2.1) , it follows that

$$||f(t) - f_1(t)|| = d(f(t),G) \quad \text{a.e.t.}$$

Hence $||f(t) - f_1(t)|| \leq ||f(t) - y||$ a.e.t . , and for all $y \in G$.

In particular

$$||f(t) - f_1(t)|| \leq ||f(t) - g(t)|| \text{ a.e.t. , and for all } g \in L^1(\mu,G) .$$

But $L^\infty(\mu,G) \subseteq L^1(\mu,G)$, and hence , for every $h \in L^\infty(\mu,G)$ we have

$$||f(t) - f_1(t)|| \leq ||f(t) - h(t)|| \text{ a.e.t. } \ldots\ldots\ldots\ldots\ldots\ldots(*)$$

Now , since $0 \in G$, we get $||f_1(t)|| \leq 2||f(t)||$ a.e.t . Hence $f_1 \in L^\infty(\mu,G)$. Thus it follows from (*) that

$$||f - f_1||_\infty \leq ||f - h||_\infty \text{ for every } h \in L^\infty(\mu,G) .$$

Consequently $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$. ∎

Theorem 3.2.5 : Let G be a closed subspace of a Banach space X . If

$L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$, then G is proximinal in X .

Proof : Let $x \in X$, we define $f(t) = x$ for all $t \in T$. Then $f \in L^{\phi}(\mu,X)$. Since $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$, there exists $g \in L^{\phi}(\mu,G)$ such that $\|f - g\|_{\phi} = d(f,L^{\phi}(\mu,G))$. Theorem (3.2.1) implies $\|f(t) - g(t)\| \leq \|f(t) - y\|$ a.e.t and for all $y \in G$. Hence $\|x - g(t)\| \leq \|x - y\|$ for all $y \in G$. Consequently G is proximinal in X . ∎

For $p = \infty$, we have :

Theorem 3.2.6 : Let G be a closed subspace of a Banach space X . If $L^{\infty}(\mu,G)$ is proximinal in $L^{\infty}(\mu,X)$, then G is proximinal in X .

Proof : Let $x \in X$. Consider the function $f(t) = x$ for all $t \in T$. Then $f \in L^{\infty}(\mu,X)$. Hence there exists $g \in L^{\infty}(\mu,G)$ s.t $\|f - g\|_{\infty} = d(f,L^{\infty}(\mu,G))$. Theorem [11 , p.36] ; $\|f - g\|_{\infty} = \sup\limits_{t} d(f(t),G)$.

Hence $\|f - g\|_{\infty} = \sup\limits_{t} d(x,G)$. Since $f(t) = x$ for all $t \in T$.

$\|f - g\|_{\infty} = d(x,G)$. But $d(x,G) = \sup \{ \|x - g(t)\| \ : \ t \in T \}$.

$\Rightarrow \|x - g(t)\| \leq d(x,G)$, for all $t \in T$.

Therefore G is proximinal in X . ∎

For $1 < p < \infty$, we have :

Theorem 3.2.7 : [3 , p.8] . Let G be a closed subspace of a Banach space X . If $L^{p}(\mu,G)$ is proximinal in $L^{p}(\mu,X)$, then G is proximinal in X for $1 < p < \infty$.

Proof : If $L^p(\mu,G)$ is proximinal in $L^p(\mu,X)$. Theorem (3.2.3) implies $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$. Theorem (3.2.4) implies $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$. Theorem (3.2.6) implies G is proximinal in X . ∎

Definition 3.2.8 : [1 , p.529] . A subspace G of a Banach space X is called 1-complemented in X if there is a closed subspace W in X such that $X = G \oplus W$ and the projection $P : X \to W$ is a contractive projection .

Lemma 3.2.9 : [1 , p.529] . If G is 1-complemented in X , then G is proximinal in X .

Proof : Let $X = G \oplus W$ and $x \in X$. Then $x = g + w$, where $g \in G$, $w \in W$ and $\|w\| \leq \|x\|$. We show that $\|x - g\| \leq \|x - y\|$ for all $y \in G$. If possible assume that there exist $g_1 \neq g \in G$ s.t $\|x-g_1\| < \|x-g\|$.

Set $w_1 = x - g_1$. By the uniqueness of the representation of x , we have

$w_1 \notin W$.

Hence $w_1 = g_2 + w_2$, where $g_2 \in G$, $w_2 \in W$ and $\|w_2\| \leq \|w_1\|$. Therefore

$$x = w_1 + g_1 = (g_2 + w_2) + g_1 = (g_1 + g_2) + w_2 ,$$

and consequently $g = g_1 + g_2$ and $w = w_2$. Thus

$$\|w\| = \|w_2\| \leq \|w_1\|.$$

But by assumption , $\|w_1\| = \|x - g_1\| < \|x - g\| = \|w\|$.

This contradicts the assumption . Consequently $\|x - g\| \leq \|x - y\|$ for all $y \in G$. Hence G is proximinal in X . ∎

Remark 3.2.10 : If G is 1-complemented in X , then G may not be a Chebyshev subspace .

Proof : Let $X = \mathbf{R}^2$ and $G = \{ (g,g) : g \in \mathbf{R} \}$ with $\|(x,y)\| = |x| + |y|$.

Then G is not Chebyshev by Example (2.2.1) .

Now , let $W = \{ (0,w) : w \in \mathbf{R} \}$. Then

$$(x,y) = (x,x) + (0,y - x) .$$

Hence $\mathbf{R}^2 = G \oplus W$.

We define $P : X \to W$ as

$$P(x,y) = P[\, (x,x) + (0,y - x) \,] = (0,y - x)$$

Now $\|w\| = \|(0,y - x)\| = |y - x| \leq |y| + |x| = \|(x,y)\|$.

Hence P is a contractive projection .

Therefore G is 1-complemented in $\mathbf{R}^2$.

Theorem 3.2.11 : If G is 1-complemented in X , then $L^1(\mu,G)$ is 1-complemented in $L^1(\mu,X)$.

Proof : Let $X = G \oplus W$ and let $P : X \to W$ be contractive projection . Hence $x = (I - P)(x) + P(x)$ and $\|P(x)\| \leq \|x\|$. For $f \in L^1(\mu,X)$, set $f_1 = (I - P)of$,

$f_2 = Pof$ a.e.t . Then

$$\|f_2\|_1 = \int_T \|f_2(t)\| \, d\mu(t) = \int_T \|P(f(t))\| \, d\mu(t) \leq \int_T \|f(t)\| \, d\mu(t) = \|f\|_1 < \infty .$$

Hence $f_2 \in L^1(\mu,W)$. Also

$$\|f_1\|_1 = \int_T \|f_1(t)\|\, d\mu(t) = \int_T \|(I-P)(f(t))\|\, d\mu(t) = \int_T \|f(t) - P(f(t))\|\, d\mu(t)$$

$$\leq \int_T \|f(t)\|\, d\mu(t) + \int_T \|P(f(t))\|\, d\mu(t) \leq \int_T \|f(t)\|\, d\mu(t) + \int_T \|f(t)\|\, d\mu(t)$$

$$= 2\,\|f\|_1 < \infty .$$

Hence $f_1 \in L^1(\mu,G)$. Clearly $f = f_1 + f_2$.

Since W is a closed subspace of X , then $L^1(\mu,W)$ is a closed subspace of $L^1(\mu,X)$. Also if $f \in L^1(\mu,W) \cap L^1(\mu,G)$ then $f \in L^1(\mu,W)$ and $f \in L^1(\mu,G)$

$\Rightarrow f(t) \in W$ and $f(t) \in G \;\; \forall\, t \in T$, but $G \cap W = \{0\}$.

$\Rightarrow f(t) = 0 \;\; \forall\, t \in T \Rightarrow f = \hat{0}$ (zero function) .

Hence $L^1(\mu,X) = L^1(\mu,G) \oplus L^1(\mu,W)$. Define the map $\hat{p}$: $L^1(\mu,X) \rightarrow L^1(\mu,W)$ by $\hat{p}(f) = Pof = f_2$ for all $f \in L^1(\mu,X)$, $\hat{p}$ is a contractive projection . So $L^1(\mu,G)$ is 1-complemented in $L^1(\mu,X)$. ∎

Corollary 3.2.12 : If G is 1-complemented in X , then $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$.

Proof : The corollary follows from the above Theorem and Lemma (3.2.9) . ∎

Definition 3.2.13 : [4 , p.72] . A closed subspace G of a Banach space X is called a ϕ-summand of X if there is a bounded projection $P : X \rightarrow G$ such that

$$\phi(\|x\|) = \phi(\|P(x)\|) + \phi(\|x - P(x)\|) \text{ for all } x \in X .$$

where ϕ is a modulus function .

Theorem 3.2.14 : [4 , p.72] . If G is a ϕ-summand of Banach space X , then G is proximinal in X .

Proof : Let $x \in X$, for every $g \in G$ we have

$$\phi(\|x-g\|) = \phi(\|P(x-g)\|) + \phi(\|(x-g) - P(x-g)\|)$$

$$= \phi(\|P(x) - g\|) + \phi(\|x - P(x)\|)$$

$$\geq \phi(\|x - P(x)\|)$$

Since ϕ^{-1} exists and strictly increasing . Hence $\|x - g\| \geq \|x - P(x)\|$.

i.e $P(x) \in P(x,G)$. Thus G is proximinal in X . ∎

Remark 3.2.15 : If G is a ϕ-summand of Banach space X , then G is a Chebyshev subspace .

Proof : Assume that G is a ϕ-summand of X . Theorem (3.2.14) implies $P(x) \in P(x,G)$.

Now suppose g^* is another closest element to x .

i.e $\|x - g^*\| = \|x - P(x)\|$ ....*

But $x - g^* \in X$

$$\phi(\|x - g^*\|) = \phi(\|P(x - g^*)\|) + \phi(\|x - g^* - P(x - g^*)\|)$$

$$= \phi(\|P(x) - g^*\|) + \phi(\|x - P(x)\|)$$

by (*) $\phi(\|P(x) - g^*\|) = 0$. So , $P(x) = g^*$.

Therefore P(x) is the unique closest element .

Thus G is a Chebyshev subspace . ∎

Theorem 3.2.16 : [4 , p.73] . Let G be a proximinal subspace of Banach space X . Then for every simple function $f \in L^{\phi}(\mu,X)$, $P(f,L^{\phi}(\mu,G))$ is not empty .

Proof : Let $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ where E_i are disjoint measurable sets in T .

Set $g = \sum_{i=1}^{n} b_i \chi_{E_i}$ where $b_i \in P(a_i,G)$. If h is an element in $L^{\phi}(\mu,G)$, then

$$\|f - h\|_{\phi} = \int_T \phi(\|f(t) - h(t)\|)\, d\mu(t)$$

$$= \sum_{i=1}^{n} \int_{E_i} \phi(\|f(t) - h(t)\|)\, d\mu(t)$$

$$= \sum_{i=1}^{n} \int_{E_i} \phi(\|a_i - h(t)\|)\, d\mu(t)$$

$$\geq \sum_{i=1}^{n} \int_{E_i} \phi(\|a_i - b_i\|)\, d\mu(t)$$

$$= \int_T \phi(\|f(t) - g(t)\|)\, d\mu(t)$$

$$= \|f - g\|_{\phi}$$

Therefore $\|f - g\|_{\phi} \leq \|f - h\|_{\phi}$ $\forall$ $h \in L^{\phi}(\mu,G)$. Thus $g \in P(f,L^{\phi}(\mu,G))$. ∎

Chapter 4

S-property

Introduction : Let X be a Banach space and G a closed subspace of X. The space G is said to have the S-property if $z_1 \in P(x_1,G)$ and $z_2 \in P(x_2,G)$ then $z_1 + z_2 \in P(x_1 + x_2,G)$ for all $x_1, x_2 \in X$. In this chapter we prove that if G has the S-property, then $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$ if and only if G is proximinal in X. As an application of this result we prove that $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$ if and only if $L^{\infty}(\mu,G)$ is proximinal in $L^{\infty}(\mu,X)$, in case G has the S-property.

An example of a subspace $G \subset X$ which has the S-property is the following.

Example 4.1.1 : Let $X = \mathbf{R}^2$. Set $G = \{ (x,0) : x \in \mathbf{R} \}$ with the Euclidean norm.

Now if $x_1 = (m,n)$, then $P(x_1,G) = \{(m,0)\}$ i.e $z_1 = (m,0)$.

and if $x_2 = (r,s)$, then $P(x_2,G) = \{(r,0)\}$ i.e $z_2 = (r,0)$.

But $x_1 + x_2 = (m+r,n+s)$; so $P(x_1 + x_2,G) = \{(m+r,0)\}$ i.e

$$(m+r,0) = z_1 + z_2 .$$

Therefore G has the S-property.

Remark 4.1.2 : In a Banach space X, if G has the S-property, it does not necessarily follow that G is proximinal in X. For example, if $X = c_0$ with

$||x|| = \sup_n |x_n|$ and $G = \{ x = (x_n) \in c_0 : \sum_{n=1}^{\infty} 2^{-n} x_n = 0 \}$. Remark (2.3.1)

implies G is not proximinal in c_0 .

Now let $x \in c_0 - G$ and suppose that $P(x,G) \neq \phi$ i.e $\exists\, z \in G$ s.t $z \in P(x,G)$; so $0 \in P(x - z,G)$ by part (2) of Theorem (2.2.5) . This means there exists $x - z \in c_0 - \{0\}$ such that $0 \in P(x - z,G)$. Theorem (2.3.7) implies G is proximinal in c_0 which is a contradiction . Therefore $P(x,G) = \phi$ for every

$x \in c_0 - G$.

Hence G has the S-property .

We shall now give various closed subspaces of a Banach space which have the S-property .

Theorem 4.1.3 : If G is a ϕ-summand of X , then G has the S-property .

Proof : Let $z_1 \in P(x_1,G)$, $z_2 \in P(x_2,G)$. Since G is a ϕ-summand of X , then there exist a projection $E : X \to G$ s.t $E(x)$ is a unique best approximant of x in G for all $x \in X$ by Theorem (3.2.14) and Remark (3.2.15) . Hence

$z_1 = E(x_1)$ and $z_2 = E(x_2)$.

But $z_1 + z_2 = E(x_1) + E(x_2) = E(x_1 + x_2)$ since E is linear .

This implies that $z_1 + z_2 \in P(x_1 + x_2,G)$. Thus G has the S-property . ∎

Theorem 4.1.4 : If G is 1-complemented and Chebyshev in X , then G has the S-property .

Proof : Let $z_1 \in P(x_1,G)$, $z_2 \in P(x_2,G)$, since G is 1-complemented in X , there is a closed subspace $W \subset X$ s.t $X = G \oplus W$. This implies x_1 and x_2 can be written , uniquely , in the form

$$x_1 = g_1 + w_1 , \qquad x_2 = g_2 + w_2$$

where $g_1 , g_2 \in G$, and $w_1 , w_2 \in W$.

From the proof of Lemma (3.2.9) and the assumption that G is Chebyshev , we get that $z_1 = g_1$, $z_2 = g_2$.

Now $x_1 + x_2 = (g_1 + g_2) + (w_1 + w_2)$.

Since G is a subspace , so $g_1 + g_2 \in G$.

Also W is a subspace , so $w_1 + w_2 \in W$.

It now follows that $z_1 + z_2 = g_1 + g_2 \in P(x_1 + x_2,G)$.

Thus G has the S-property . ∎

Theorem 4.1.5 : Let G be a closed subspace of a Hilbert space X , then G has the S-property .

Proof : Let $z_1 \in P(x_1,G)$ and $z_2 \in P(x_2,G)$. We show $z_1 + z_2 \in P(x_1 + x_2,G)$.

Theorem (2.2.6) implies $x_1 - z_1 \perp G$, and $x_2 - z_2 \perp G$. Hence

$$\langle x_1 - z_1,g\rangle = 0 \text{ and } \langle x_2 - z_2,g\rangle = 0 \text{ for all } g \in G .$$

Now $\langle x_1 + x_2 - (z_1 + z_2),g\rangle = \langle x_1 - z_1,g\rangle + \langle x_2 - z_2,g\rangle = 0$ for all $g \in G$.

Hence $x_1 + x_2 - (z_1 + z_2) \perp G$. Theorem (2.2.6) implies

$$z_1 + z_2 \in P(x_1 + x_2,G)$$

Thus G has the S-property . ∎

Theorem 4.1.6 : If G is a semi-Chebyshev hyperplane in a Banach space X passing through zero , then G has the S-property .

Proof : Case (1) : If G is proximinal in X . Let $f \in X^*$ so that $G = \{ y \in X , f(y) = 0 \}$. Fix an arbitrary $z \in X \backslash G$ so ; $f(z) \neq 0$, and let

$$y_0 = x - f(x)/f(z) . z \quad \text{where } x \in X$$

So $f(y_0) = 0$, whence $y_0 \in G$ .Consequently

$$X = G \oplus W \text{ where } W = \{ w = \alpha z \ : \ \alpha \text{ scalar} \} \text{ ....*}$$

Now let $z_1 \in P(x_1,G)$, $z_2 \in P(x_2,G)$. It will be shown that

$$z_1 + z_2 \in P(x_1 + x_2,G) .$$

By (*) every $x_1 \in X$, $x_2 \in X$ can be written , uniquely , in the form

$$x_1 = g_1 + \alpha_1 z , \qquad x_2 = g_2 + \alpha_2 z \text{ .... **}$$

where g_1 , $g_2 \in G$ and α_1 , α_2 are scalars .

Now assume that $g' \in P(x_1 + x_2,G)$, then by **

$g' \in P(g_1 + g_2 + (\alpha_1 + \alpha_2)z,G)$. Theorem (2.2.5) implies

$$g' = (g_1 + g_2) + (\alpha_1 + \alpha_2) w \quad \text{where} \quad w \in P(z,G)$$

$$= g_1 + \alpha_1 w + g_2 + \alpha_2 w$$

Since $w \in P(z,G)$, Theorem (2.2.5) implies

$$g_1 + \alpha_1 w \in P(g_1 + \alpha_1 z,G) = P(x_1,G) .$$

And $g_2 + \alpha_2 w \in P(g_2 + \alpha_2 z, G) = P(x_2, G)$.

Hence

$$g_1 + \alpha_1 w = z_1$$

$$g_2 + \alpha_2 w = z_2$$

Consequently $g' = z_1 + z_2$.

Therefore $z_1 + z_2 \in P(x_1 + x_2, G)$. Thus G has the S-property .

Case (2) : If G is not proximinal in X . Theorem (2.3.7) implies $P(x,G) = \phi$ for every $x \in X - G$. Thus G has the S-property .

We now state and prove an important result for G with S-property .

Theorem 4.1.7 : Let X be any Banach space , and G a closed subspace of X which has the S-property , then $P_G^{-1}(0)$ is a closed subspace of X and

$$P_G^{-1}(0) \cap G = \{0\} .$$

Proof : Let x_1 , $x_2 \in P_G^{-1}(0)$, so ; $0 \in P(x_1,G)$ and $0 \in P(x_2,G)$. Since G has the S-property we get $0 \in P(x_1 + x_2, G)$. Hence $x_1 + x_2 \in P_G^{-1}(0)$. *

Let $x \in P_G^{-1}(0)$ and α be any scalar . Then

$d(\alpha x, G) = |\alpha|\, d(x,G) = |\alpha|\, \|x\| = \|\alpha x\| \Rightarrow 0 \in P(\alpha x, G) \Rightarrow \alpha x \in P_G^{-1}(0)$. **

By (*) , (**) $P_G^{-1}(0)$ is a subspace of X .

Now let (x_n) be a sequence in $P_G^{-1}(0)$ and $x \in X$ such that $\lim x_n = x$. Since G is a subspace of X , $0 \in G$, $d(x,G) \le \|x\|$ ......(1) .

Given $\in > 0$ there exist a natural number $N(\in)$ such that $\|x_n - x\| < \in$ for all $n > N(\in)$.

Fix $n \geq N(\in)$ to have :

$$\begin{aligned}
\|x\| = \|x - x_n + x_n\| &\leq \|x - x_n\| + \|x_n\| \\
&< \in + |\, \|x_n\| - d(x,G) + d(x,G)\, | \\
&\leq \in + |\, \|x_n\| - d(x,G)\, | + d(x,G) \\
&= \in + |\, d(x_n,G) - d(x,G)\, | + d(x,G) \quad \text{since } x_n \in P_G^{-1}(0) \\
&\leq \in + \|x_n - x\| + d(x,G) \\
&\leq 2\in + d(x,G)
\end{aligned}$$

Hence $\|x\| \leq d(x,G)$ ......(2)

From (1) , (2) we get $\|x\| = d(x,G)$. So , $0 \in P(x,G)$ hence $x \in P_G^{-1}(0)$.

Thus $P_G^{-1}(0)$ is closed .

$$\begin{aligned}
\text{Let } g \in P_G^{-1}(0) \cap G &\Rightarrow g \in P_G^{-1}(0) \text{ and } g \in G . \\
&\Rightarrow 0 \in P(g,G) \text{ and } g \in G . \\
&\Rightarrow \|g\| = d(g,G) \text{ and } g \in G . \\
&\Rightarrow \|g\| = 0 . \\
&\Rightarrow g = 0
\end{aligned}$$

Therefore $P_G^{-1}(0) \cap G = \{0\}$. ∎

The next theorem shows that if we add the condition that G has the S-property then the converse of Lemma (3.2.9) will be true .

Theorem 4.1.8 : Let X be any Banach space , and G a closed subspace of X which has the S-property . G is proximinal in X if and only if G is 1-complemented in X .

Proof : If G is 1-complemented in X , then by Lemma (3.2.9) it is proximinal in X .

Suppose now that G is proximinal in X . Theorem (2.3.2) implies

$X = G + P_G^{-1}(0)$. Theorem (4.1.7) shows that $P_G^{-1}(0)$ is a closed subspace of X and $P_G^{-1}(0) \cap G = \{0\}$. Hence $X = G \oplus P_G^{-1}(0)$.

Now define $P : X \to P_G^{-1}(0)$ by

$$P(x) = P(g + z) = z \text{ where } x = g + z \,,\; g \in G \,,\; z \in P_G^{-1}(0) .$$

$$\begin{aligned} \|x\| \geq d(x,G) &= d(g + z,G) \\ &= d(z,G) \\ &= \|z\| \end{aligned}$$

Therefore $\|x\| \geq \|z\|$.

Hence P is a contractive projection . Thus G is 1-complemented in X . ∎

An important application of the previous theorem is the following .

Theorem 4.1.9 : Let X be any Banach space , and G be a closed subspace of X which has the S-property . The following are equivalent :

(i) G is proximinal in X .

(ii) $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$.

(iii) $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$.

Proof : (i)→(ii) , assume G is proximinal in X . Theorem (4.1.8) implies G is 1-complemented in X . Corollary (3.2.12) implies $L^{1}(\mu,G)$ is proximinal in $L^{1}(\mu,X)$.

(ii)→(iii) , assume $L^{1}(\mu,G)$ is proximinal in $L^{1}(\mu,X)$. Theorem (3.2.2) implies $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$.

(iii)→(i) , assume that $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$. Theorem (3.2.5) implies G is proximinal in X . ∎

We shall now give various corollaries of this theorem .

Corollary 4.1.10 : Let X be a Banach space , and G is a ϕ-summand of X , then $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$.

Proof : The corollary follows from the above Theorem and Theorems (4.1.3) , (3.2.14) . ∎

Corollary 4.1.11 : Let G be a closed subspace of a Hilbert space X , then $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$.

Proof : The corollary follows from the above Theorem and Theorems (2.4.3) , (4.1.5) . ∎

Corollary 4.1.12 : If G is a Chebyshev hyperplane in a Banach space X passing through zero , then $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$.

Proof : The corollary follows from the above Theorem and Theorem (4.1.6) . ∎

In particular , it follows that every Orlicz space $L^{\phi}(\mu,X)$ where X is a Banach space and strictly convex has at least one proximinal linear subspace .

Now we state and prove our main result .

Theorem 4.1.13 : Let X be any Banach space , and G be a closed subspace of X with the S-property . Then $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$ if and only if $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$.

Proof : If $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$, then by Theorem (3.2.4) $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$.

Conversely if $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$, then G is proximinal in X by Theorem (3.2.7) .

Theorem (4.1.9) , implies $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$. ∎

2. Further Results :-

Theorem 4.2.1 : Let X be a Banach space and G be a closed subspace of X , if G has the S-property in X , then $L^{\phi}(\mu,G)$ has the S-property in $L^{\phi}(\mu,X)$.

Proof : Let $g_1 \in P(f_1,L^{\phi}(\mu,G))$ and $g_2 \in P(f_2,L^{\phi}(\mu,G))$, we will show that :

$$g_1 + g_2 \in P(f_1 + f_2 , L^{\phi}(\mu,G))$$

Now

$g_1 \in P(f_1,L^{\phi}(\mu,G)$. Theorem (3.2.1) implies

$g_1(t) \in P(f_1(t),G)$ for all almost $t \in T$ ....(1)

Also

$g_2 \in P(f_2,L^{\phi}(\mu,G))$. Theorem (3.2.1) implies

$g_2(t) \in P(f_2(t),G)$ for all almost $t \in T$ ....(2)

Since G has the S-property , from (1) and (2) we get

$(g_1 + g_2)(t) \in P((f_1 + f_2)(t),G)$ for all almost $t \in T$.

Hence

$$d((f_1 + f_2)(t),G) = \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\|$$

Hence $\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \leq \|(f_1 + f_2)(t) - y\|$ a.e.t , and $\forall\ y \in G$

In particular

$\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \leq \|(f_1 + f_2)(t) - h(t)\|$ a.e.t , and $\forall\ h \in L^{\phi}(\mu,G)$

Since ϕ is strictly increasing , then

$\phi\,(\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\|) \leq \phi\,(\|(f_1 + f_2)(t) - h(t)\|)$ a.e.t and

$\forall\ h \in L^{\phi}(\mu,G)$

Integrating both sides we get

$$\|(f_1 + f_2) - (g_1 + g_2)\|_{\phi} \leq \|(f_1 + f_2) - h\|_{\phi} \quad \forall\ h \in L^{\phi}(\mu,G)$$

Hence $d(f_1 + f_2,L^{\phi}(\mu,G)) = \|(f_1 + f_2) - (g_1 + g_2)\|_{\phi}$

Therefore $g_1 + g_2 \in P(f_1 + f_2,L^{\phi}(\mu,G))$.

Thus $L^{\phi}(\mu,G)$ has the S-property .

The following theorem describes the relationship between $P^{-1}_{L^{\phi}(\mu,G)}(0)$ and

$L^{\phi}(\mu, P_G^{-1}(0))$.

Theorem 4.2.2 : Let X be a Banach space , and G be a closed subspace of X . If G has the S-property , then $P^{-1}_{L^{\phi}(\mu,G)}(0) = L^{\phi}(\mu, P_G^{-1}(0))$.

Proof : Let $f \in L^{\phi}(\mu, P_G^{-1}(0))$. This means $f(t) \in P_G^{-1}(0)$, and that $\|f\|_{\phi} < \infty$.

Now

$f(t) \in P_G^{-1}(0)$; so , $0 \in P(f(t),G)$; hence $d(f(t),G) = \|f(t)\|$.

i.e $\|f(t)\| \le \|f(t) - g\| \ \ \forall g \in G$

In particular

$$\|f(t)\| \le \|f(t) - h(t)\| \ \ \forall h \in L^{\phi}(\mu,G)$$

since ϕ is strictly increasing , then

$$\phi(\|f(t)\|) \le \phi(\|f(t) - h(t)\|) \ \ \forall h \in L^{\phi}(\mu,G)$$

Integrating both sides we get

$$\|f\|_{\phi} \le \|f - h\|_{\phi} \ \ \forall \ h \in L^{\phi}(\mu,G)$$

Hence $d(f,L^{\phi}(\mu,G)) = \|f\|_{\phi}$

Therefore $0 \in P(f,L^{\phi}(\mu,G)) \Rightarrow f \in P^{-1}_{L^{\phi}(\mu,G)}(0)$

Thus

$$L^{\phi}(\mu, P_G^{-1}(0)) \subset P^{-1}_{L^{\phi}(\mu,G)}(0) \quad ....(1)$$

Let $f \in P^{-1}_{L^{\phi}(\mu,G)}(0)$. We claim that $f \in L^{\phi}(\mu, P_G^{-1}(0))$ i.e. $\|f\|_{\phi} < \infty$ and

$f(t) \in P_G^{-1}(0)$.

Now

$f \in P^{-1}_{L^{\phi}(\mu,G)}(0) \subset L^{\phi}(\mu,X)$. Hence $\|f\|_{\phi} < \infty$ ....(2)

Also $0 \in P(f,L^{\phi}(\mu,G))$. Theorem (3.2.1) implies

$0 \in P(f(t),G)$ a.e.t .

Thus $f(t) \in P_G^{-1}(0)$ ....(3)

From (2) and (3) we get

$f \in L^{\phi}(\mu, P_G^{-1}(0))$

Therefore

$P^{-1}_{L^{\phi}(\mu,G)}(0) \subset L^{\phi}(\mu, P_G^{-1}(0))$ ....(4)

From (1) and (4) we get $P^{-1}_{L^{\phi}(\mu,G)}(0) = L^{\phi}(\mu, P_G^{-1}(0))$. ∎

Lemma 4.2.3 : Let X be a Banach space , and G_1 , G_2 are closed subspaces of X . If $G_1 \subset G_2$ then $P^{-1}_{G_2}(0) \subset P^{-1}_{G_1}(0)$

Proof : Let $x \in P^{-1}_{G_2}(0) \Rightarrow 0 \in P(x,G_2) \Rightarrow d(x,G_2) = \|x\|$, but $G_1 \subset G_2$

$\Rightarrow \|x\| \geq d(x,G_1) \geq d(x,G_2) = \|x\|$.

Therefore $d(x,G_1) = \|x\|$. Hence $x \in P^{-1}_{G_1}(0)$. ∎

Theorem 4.2.4 : Let X be a Banach space , and G a closed subspace of X which has the S-property . If G is proximinal in X then $P_G^{-1}(0)$ is proximinal in X and has the S-property .

Proof : Let $x \in X$. The proof of Theorem (4.1.8) implies x can be written, uniquely, in the form

$$x = g + z \quad \text{where} \quad g \in G \text{ and } z \in P_G^{-1}(0) \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Now $g \in G \Rightarrow g \perp w \quad \forall w \in P_G^{-1}(0) \Rightarrow g \perp P_G^{-1}(0) \Rightarrow 0 \in P(g, P_G^{-1}(0))$

$\Rightarrow d(g, P_G^{-1}(0)) = \|g\|$ ....(2)

From (1) and (2) we get

$$d(x - z, P_G^{-1}(0)) = \|x - z\| \Rightarrow d(x, P_G^{-1}(0)) = \|x - z\|$$

Therefore $z \in P(x, P_G^{-1}(0))$ i.e $z = x - g$ where $g \in P(x,G)$ ....(3)

Thus $P_G^{-1}(0)$ is proximinal in X .

To show $P_G^{-1}(0)$ has the S-property .

Let $z_1 \in P(x_1, P_G^{-1}(0))$ and $z_2 \in P(x_2, P_G^{-1}(0))$

From (3) we get $x_1 - z_1 \in P(x_1,G)$ and $x_2 - z_2 \in P(x_2,G)$

Since G has the S-property , then

$$x_1 + x_2 - (z_1 + z_2) \in P(x_1 + x_2, G)$$

$$\Rightarrow z_1 + z_2 \in P(x_1 + x_2, P_G^{-1}(0))$$

Thus $P_G^{-1}(0)$ has the S-property . ∎

Theorem 4.2.5 : Let X be a Banach space , and G is proximinal in X . If G has the S-property then $P^{-1}_{P_G^{-1}(0)}(0) = G$.

Proof : Let $g \in G \Rightarrow z \perp g \quad \forall\ z \in P_G^{-1}(0) \Rightarrow g \perp P_G^{-1}(0)$

$\Rightarrow 0 \in P(g, P_G^{-1}(0)) \Rightarrow g \in P^{-1}_{P_G^{-1}(0)}(0)$.

Therefore $G \subset P^{-1}_{P_G^{-1}(0)}(0)$ ....(1)

Now , let $x \in P^{-1}_{P_G^{-1}(0)}(0)$. Then by the proof of Theorem (4.1.8) we have

$$x = x_1 + x_2 \quad \text{where} \quad x_1 \in G \quad \text{and} \quad x_2 \in P_G^{-1}(0) .$$

Since $G \subset P^{-1}_{P_G^{-1}(0)}(0)$, $x_1 \in P^{-1}_{P_G^{-1}(0)}(0)$. Then $x_2 = x - x_1 \in P^{-1}_{P_G^{-1}(0)}(0)$.

But $x_2 \in P_G^{-1}(0)$. Theorem (4.1.8) implies $x_2 = x - x_1 = 0$.

$$\Rightarrow x = x_1 \Rightarrow x \in G$$

Therefore $P^{-1}_{P_G^{-1}(0)}(0) \subset G$ ....(2)

Thus we proved that $G = P^{-1}_{P_G^{-1}(0)}(0)$. ∎

Theorem 4.2.6 : Let X be a Banach space , and G is proximinal in X . If G has the S-property then $L^{\phi}(\mu,G) = L^1(\mu,G)$.

Proof : Theorem (3.1.4) implies $L^1(\mu,G) \subset L^{\phi}(\mu,G)$ ....(1)

also $L^1(\mu, P_G^{-1}(0)) \subset L^{\phi}(\mu, P_G^{-1}(0))$.

Lemma (4.2.3) implies $P^{-1}_{L^{\phi}(\mu,P_G^{-1}(0))}(0) \subset P^{-1}_{L^1(\mu,P_G^{-1}(0))}(0)$.

Theorem (4.2.2) implies $L^{\phi}(\mu, P^{-1}_{P_G^{-1}(0)}(0)) \subset L^1(\mu, P^{-1}_{P_G^{-1}(0)}(0))$.

Theorem (4.2.5) implies $L^{\phi}(\mu,G) \subset L^1(\mu,G)$ ....(2)

From (1) and (2) we get $L^{\phi}(\mu,G) = L^1(\mu,G)$. ∎

Let X be a Banach space , and G be proximinal in X , then any map which associates with each element of X one of its best approximant in G is called a proximity map . This mapping is , in general , non linear .

Theorem 4.2.7 : Let X be a Banach space , and G be a Chebyshev subspace of X . There exists a linear proximity map if and only if G has the S-property .

Proof : Let E be a linear proximity map . we claim that G has the S-property .

Let $z_1 \in P(x_1,G)$, and $z_2 \in P(x_2,G)$. We show that

$$z_1 + z_2 \in P(x_1 + x_2,G)$$

Now

$$z_1 + z_2 = E(x_1) + E(x_2) = E(x_1 + x_2) \in P(x_1 + x_2,G) .$$

Therefore G has the S-property .

Conversely , assume that G has the S-property .

Define $E : X \rightarrow G$ such that $E(x) \in P(x,G)$.

Now , we claim that E is linear .

Let $x_1 , x_2 \in X$, we show that $E(x_1 + x_2) = E(x_1) + E(x_2)$.

Now , $E(x_1) \in P(x_1,G)$ and $E(x_2) \in P(x_2,G)$.

Since G has the S-property , then $E(x_1) + E(x_2) \in P(x_1 + x_2,G)$.

Also $E(x_1 + x_2) \in P(x_1 + x_2,G)$. Since G is a Chebyshev subspace then

$$E(x_1 + x_2) = E(x_1) + E(x_2) \text{ ....(1)}$$

Let $x \in X$, α scalar then $E(x) \in P(x,G)$. Theorem (2.2.5) implies

$$\alpha \,.\, E(x) \in P(\alpha x,G) \text{ , also } E(\alpha x) \in P(\alpha x,G) .$$

Since G is a Chebyshev subspace of X then

$$E(\alpha x) = \alpha \,.\, E(x) \text{ ....(2)}$$

By (1) and (2) E is a linear . ∎

If S is a compact Hausdroff space and X is a Banach space , then C(S,X) will denotes the Banach space of all continuous maps f from S into X with norm defined as $||f|| = \sup_{s} ||f(s)||$.

Theorem 4.2.8 : Let S be a compact Hausdroff space , and G be a Chebyshev subspace of a Banach space X which has the S-property . If C(S,G) be a Chebyshev subspace of C(S,X) then C(S,G) has the S-property in C(S,X) .

Proof : Since G is a Chebyshev and has the S-property . Theorem (4.2.7) implies there exists a linear proximity map $E : X \rightarrow G$. Then define the map.

$$F : C(S,X) \rightarrow C(S,G) \text{ by}$$

$$F(f) = E \circ f \text{ .}$$

Our claim is that F is a linear proximity map .

If $g \in C(S,G)$ then

$$||f(s) - E(f(s))|| \leq ||f(s) - g(s)||$$

For all $s \in S$, $f \in C(S,X)$. Hence

$$||f - E \circ f|| \leq ||f - g|| \text{ for all } g \in C(S,G)$$

Therefore

$$F(f) \in P(f,C(S,G))$$

And consequently

F is a proximity map .

Let $f,g \in C(S,X)$, we claim that $F(f+g) = F(f) + F(g)$

$$\begin{aligned} F(f+g)(s) &= E(f(s) + g(s)) \\ &= E(f(s)) + E(g(s)) \qquad \text{since E is linear} \\ &= E \circ f(s) + E \circ g(s) \\ &= (E \circ f + E \circ g)(s) \end{aligned}$$

Hence $F(f + g) = E \circ f + E \circ g$

$$= F(f) + F(g) \text{ ....(1)}$$

Let $f \in C(S,X)$, and α (scalar)

$$\begin{aligned} F(\alpha f)(t) &= E(\alpha f(t)) \\ &= \alpha E(f(t)) \qquad \text{since E is linear} \\ &= \alpha [(E \circ f)(t)] \\ &= \alpha (F(f))(t) \end{aligned}$$

$$F(\alpha f) = \alpha F(f) \text{ ....}(2)$$

By (1) and (2) F is linear . ∎

Theorem (4.2.7) implies C(S,G) has the S-property .

In [7 , p.333] , it is shown that if X is a Hilbert space , then X is strictly convex . In the following , we give another proof .

Theorem 4.2.9 : If X is a Hilbert space , then X is strictly convex .

Proof : Assume that X is not strictly convex , whence there exists an $f \in X^*$ with $\|f\| = 1$ which has two distinct maximal elements x,y of norm $\|x\| = \|y\| = 1$ by Definition (1.3.3) .

Put $G = [x - y] = \{ \alpha(x - y) : \alpha \text{ scalar} \}$.

$$\|x\| = |f(x)| = |f(x - g)| \leq \|f\| \, \|x - g\| = \|x - g\| \quad \text{for all } g \in G .$$

Hence $0 \in P(x,G)$.

Also $\|y\| = |f(y)| = |f(y - g)| \leq \|f\| \, \|y - g\| = \|y - g\|$ for all $g \in G$.

Thus $0 \in P(y,G) \Rightarrow 0 \in P(-y,G)$.

But G is a closed subspace of a Hilbert space . Theorem (4.1.5) implies G has the S-property . Then $0 \in P(x - y,G)$ which contradicts the fact $x - y \in G - \{0\}$. Therefore X is strictly convex . ∎

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