An Najah National University
Faculty of Graduate Studies

Numerical Methods for Solving Elliptic Boundary-Value Problems

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This Thesis was defended successfully on 02/08/2005 and approved by

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"قل هل يستوي الذين يعلمون والذين لا يعلمون"

صدق الله العظيم

TO MY PARENTS, SISTERS, BROTHERS, MY WIFE, MY SONS AND FRIENDS FOR THEIR MORAL SUPPORT

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Numerical Methods for solving Elliptic Boundary value problems

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Abstract

Elliptic Partial Differential Equations of second order have been studied using some numerical methods. This type of differential equations has specific applications in physical and engineering models. In most applications, first- order and second-order formulas are used for the derivatives. In this work higher order formulas such as: seven-points and nine-points formulas are used. Using these formulas will transform the partial differential equation into finite difference equations. To solve the resulting finite difference equations the following iterative methods have been used: Jacobi method, Gauss-Seidel method, Successive Over-Relaxation method (SOR) and Multigrid method.

In this thesis, we found that multigrid methods are the most efficient among all other methods. The execution time for multigrid methods is of order three while the other methods is of order five.

Chapter One

Introduction

Introduction

The majority of the problems of physics and engineering fall naturally into one of the following three physical categories: equilibrium problems, eigenvalue problems and propagation problems.

Partial Differential Equations (PDEs), which are considered at the heart of many mathematical models used in engineering and physics, has given rise to extensive computations. Often the problems that one would like to solve exceed the capacity of even the most powerful computers. On the other hand, the time required is too large to all inclusion of advanced mathematical models in the design process.

The solutions of PDEs are important in many fields of science and engineering notably in electromagnetism, astronomy, and fluid dynamics, because they describe the behavior of electric, gravitational and fluid potential. Most of the PDEs that arise in mathematical models of physical phenomena are difficult (if not impossible) to solve analytically, so we have used numerical methods to approximate the solution.

Linear Second-Order Partial Differential Equations

The general form of a linear second order PDE in two-dimensions is:

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu + g = 0 \qquad \textbf{(1.1)}$$

where $a, b, c, d, e, f,$ *and* g are coefficients which may be constants, or functions of the independent variables x and y.

This linear second order PDE in two independent variables can be classified as one of three standard or canonical form which we identify as hyperbolic, parabolic, or elliptic. The classifications of the PDEs are:

- Hyperbolic if $b^2 - 4ac > 0$
- Parabolic if $b^2 - 4ac = 0$
- Elliptic if $b^2 - 4ac < 0$

In this research, we compare different numerical methods for solving linear elliptic boundary-value problems(bvps). We use several different methods to solve the elliptic PDEs such as:

Successive Over- Relaxation (SOR), Jacobi method, Gauss-Seidel,and Multigrid method.

1.1 Study Approach

The aim of this work is to:

study, analyze and develop some numerical methods for solving elliptic partial differential equations with boundary conditions.

Also to investigate the most efficient method among the different methods used.we used MATLAB as a computational tool.

1.2 Model Problem

Elliptic PDEs arise usually from equilibrium or steady-state problems. A typical example of elliptic equation in two dimensions is the well-known Helmholtz equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = f(x, y) \qquad \textbf{(1.2)}$$

Important special cases:

-when λ=0, the above equation leads to Poisson's equation which is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad \textbf{(1.3)}$$

-when λ=0 and $f = 0$ the above equation is called Laplace's equation which is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \textbf{(1.4)}$$

If the solution $u(x, y)$ is satisfying, Poisson's equation in a square region

$$\mathrm{S} = \{ (x, y) \mid a < x < b, a < y < b \}$$

with boundary conditions $u(x, y) = g(x, y)$ on ∂S ,where ∂S is the boundary of S.

Poisson's equation summarizes the flow motion of incompressible viscous fluid, and the inverse square law theories of electricity, magnetism and gravity matter at points where the charged density pole strength or mass density respectively is non-zero.

The solution to an electrostatic problem is straight forward for the case in which the charge distribution is everywhere specified for them; the potential and electric field are given directly as integrals over this charge distribution.

$$u = \frac{1}{4\pi\varepsilon_0}\int \frac{dq^{\backslash}}{|r - r^{\backslash}|} \tag{1.5}$$

where $q^{\backslash}$ is the charge inside Gauss sureface.

$\in_0$ is the permitivity of the air.

$|r - r^{\backslash}|$ is the distance between the charge and the point.

$$E = \frac{1}{4\pi\varepsilon_0}\int \frac{(r - r^{\backslash})dq^{\backslash}}{|r - r^{\backslash}|} \tag{1.6}$$

First, we have the differential form of Gauss's law:

$$\text{Diverge } E = \frac{1}{\varepsilon_0}\rho \tag{1.7}$$

Furthermore, in a purely electrostatic field, E may be expressed as the negative gradient of the potential:

$$E = -grad\ u \tag{1.8}$$

Combining equation (1.7) and (1.8) we obtain:

$$div\ (grad\ u) = -\frac{1}{\varepsilon_0}\rho$$

It is convenient to think of $div\ \ grad$ as a single differential operator $\nabla.\nabla$ or ∇^2. Where ∇^2 is called Laplacian operator.

$$\nabla^2 u = -\frac{1}{\varepsilon_0}\rho \tag{1.9}$$

and this is a Poisson's equation.

The operator ∇^2 involves differentiation with respect to more than one independent variable which may be solved once we know the functional dependence of $\rho(x, y)$, and the appropriate boundary conditions.

Poisson's equation can be expressed in three dimensions as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{\rho}{\varepsilon_0} \quad \text{in rectangular coordinates or}$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial u}{\partial \theta}) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 u}{\partial \phi^2} = -\frac{\rho(r,\theta)}{\varepsilon_0}$$

(1.10)

in spherical coordinates, or

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}(r \frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{\rho(r,\theta,z)}{\varepsilon_0} \qquad \textbf{(1.11)}$$

in cylindrical coordinates.

Chapter Two

Finite Difference Discretization of Elliptic Equation

2.1 Introduction

Many engineering applications involve boundary value problems that require solving elliptic partial differential equations (PDEs). The discretization of such boundary-value problems leads to linear system $A\mathbf{u} = \mathbf{f}$, where $\mathbf{u}$ is the set of unknowns corresponding to the unknown variables in the PDE and $\mathbf{f}$ is the set of discrete values of the known function in the PDE.

Boundary conditions

For elliptic PDEs, there are given boundary conditions where in some property of u is specified ∂S. With either Laplace's or Poisson's equations, we can define three types of boundary conditions.

Types of boundary conditions

The solution in an interior points of the region S depends on all the data given on the boundary ∂S. The conditions on this boundary are of three types:

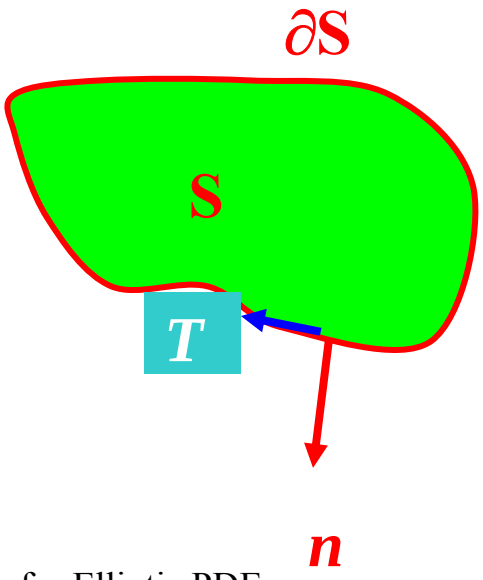


Fig. (2.1) Illustration of the region for Elliptic PDEs

$$\begin{cases} \text{(i) Dirichlet condition } : u = g \text{ on } \partial S \\ \text{(ii) Neumann condition } : \dfrac{\partial u}{\partial n} = V(x,y) \text{ on } \partial S \\ \text{(iii) Robbin (mixed) condition } : \dfrac{\partial u}{\partial n} + ku = g \text{ on } \partial S \end{cases}$$

It is often the case that an elliptic boundary-value problem is specified by boundary conditions that are of different parts of ∂S.

Numerical Solution for Boundary- Value Problem

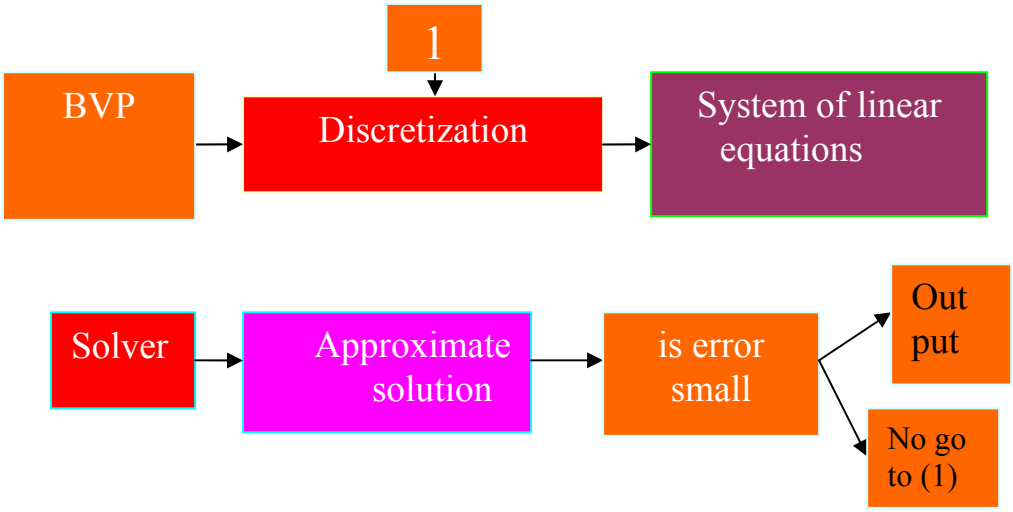
The methods that have been used are based on finite difference methods for solving linear boundary value problems.

1) Define discrete mesh points within the domain S of boundary-value problems.

2) Replace derivatives in the given PDE by some finite differences.

3) Solve the system of linear equations at all mesh points.

Diagramatically, the above steps are shown:



Approximate solution is determined at all mesh points simultaneously by solving single system of linear equations.

Consider the elliptic boundary problem Eqs (1.4) with a square domain:

$$S= \{(x,y) : 0 \le x \le a , 0 \le y \le a\} \quad \textbf{(2.1)}$$

Define the mesh point as the points of intersection of the straight lines

$x_i = ih$, $i = 1,2,...,n$ and $y_j = jk$, $j = 1,2,...,m.$

$h = \frac{a}{n}$ and $k = \frac{c}{m}$.

Using Taylor's theorem, the second order central –difference approximations to first and second derivatives at the interior mesh point (i, j) are:

$$(\frac{\partial u}{\partial x})_{i,j} = \frac{u(i+1, j) - u(i-1, j)}{2h} + O(h^2) \quad \textbf{(2.2.a)}$$

$$(\frac{\partial u}{\partial y})_{i,j} = \frac{u(i, j+1) - u(i, j-1)}{2k} + O(k^2) \quad \textbf{(2.2.b)}$$

$$(\frac{\partial^2 u}{\partial x^2})_{i,j} = \frac{u(i+1,j) - 2u(i,j) + u(i-1,j)}{h^2} + O(h^2) \quad \textbf{(2.2.c)}$$

$$(\frac{\partial^2 u}{\partial y^2})_{i,j} = \frac{u(i,j+1) - 2u(i,j) + u(i,j-1)}{k^2} + O(k^2) \quad \textbf{(2.5.d)}$$

(Lapidus, 1982)

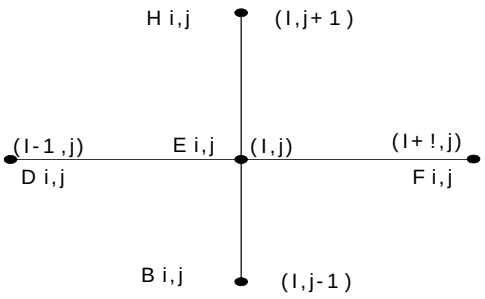


Fig.(2.2):Five-point scheme for Laplace equation.

Taking $x_i = ih$ and $y_j = jk$ the five points finite difference approximation to the elliptic PDE at the interior mesh point $(x_i.y_j)$ Fig. (2.2) is

$$B_{i,j}\,u_{i,j-1} + D_{i,j}\,u_{i-1,.j} + E_{i,j}\,u_{i,j} + F_{i,j}u_{i+1,j} + H_{i,j}u_{i,j+1} = g_{i,j}.$$
$$i = 1,2...n$$
$$j = 1,2,...m \qquad \textbf{(2.3)}$$

where $B_{i,j}$, $D_{i,j}$, $E_{i,j}$, $F_{i,j}$, $H_{i,j}$ *and* $g_{i.j}$ are known values. In the case of Laplace's equation B, D, E, F *and* H are 1, 1, –4, 1 and 1 respectively.

Linear Systems:

Defining the vector u to be

$$[u_{1,1}...u_{M-1,1}; u_{1,2}...u_{M-1,2}; ...; u_{1,M-1}...u_{M-1,M-1}]^T \qquad \textbf{(2.4)}$$

Where $[\ ]^T$ denotes the transpose and imposes an ordering on the $(M-1)^2$ unknown grid values. With this ordering, the totality of equations at the $(M-1)^2$ internal nodes of the unit square leads to the matrix equation

$$A\,\mathbf{u} = \mathbf{b} \qquad \textbf{(2.5)}$$

Where A is a matrix of order $(M-1)^2$ given by:

$$A = \begin{bmatrix} B & -J & & & 0 \\ -J & B & -J & & \\ . & . & . & & \\ & . & . & & \\ & & -J & B & -J \\ 0 & & & -J & B \end{bmatrix} \qquad \textbf{(2.6)}$$

M : the number of interior points in each direction

with J the identity matrix of order $(M-1)$ and B the following matrix of order $(M-1)$ given by

$$B = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ . & . & . & & \\ & . & . & & \\ & & -1 & 4 & -1 \\ 0 & & & -1 & 4 \end{bmatrix} \qquad \textbf{(2.7)}$$

The components of the vector **b** depend on the boundary values of $g(x,y)$ at the grid points on the perimeter of the square region and the equation .(Mitchell, 1980)

2.2 Difference Equations for Boundary Nodes

(a) Dirichlet conditions:

When $u(x,y) = g(x,y)$ on the boundary $\partial S.$, the nodal values are $u_{i,j} = g(x_i, y_j)$ on $\partial S.$, also when the node (i,j) is adjacent to the boundary then this node is either adjacent to one boundary node Fig (2.3.a) or

adjacent to two boundary nodes (Fig.2.3.b).

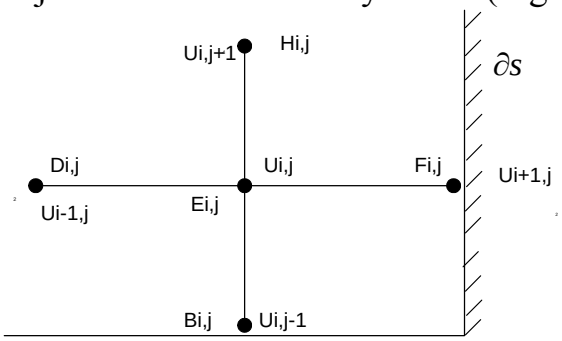


Fig.(2.3.a)

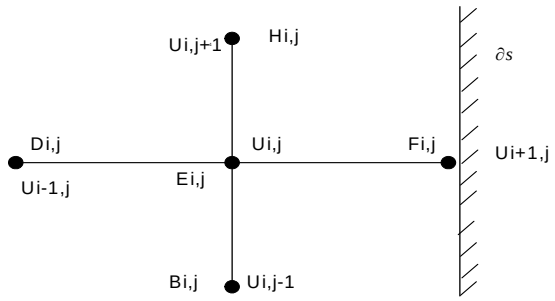


Fig.(2.3.b)

In the first case let the boundary node be $(i+1, j)$ then the difference equation at the point (x_i, y_j) can be written as:

$$B_{i,j}\, u_{i,j-1} + D_{i.j}\, u_{i-1,j} + E_{i,j}\, u_{i,j} + H_{i,j} u_{i,j+1} = g_{i,j} - F_{i,j} u_{i+1,j}$$

(2.8)

where $u_{i+1,j}$ is given (known value) by the boundary codition.

In the second case when the node (i, j) is near the south-east corner of the boundary of a square region. Let the two known boundary nodes be $(i+1, j)$ and $(i, j-1)$ as in Fig. (2.3.b), then the finite difference equation for this point is

$$D_{i,j} u_{i-1,j} + E_{i,j} u_{i,j} + H_{i,j} u_{i,j+1} = g_{i,j} - B_{i,j} u_{i,j-1} - F_{i,j} u_{i+1,j}$$

(2.9)

where $u_{i,j-1}$ and $u_{i+1,j}$ are given by the boundary conditions.

(b) Neuman condition:

When $\frac{\partial u}{\partial n} = V(x, y)$ is on the boundary, then in the five- point difference scheme, three of the node lie on the boundary line, one inside the boundary at $(i-1, j)$ and the fifth at $(i+1, j)$ as in Fig. (2.4).

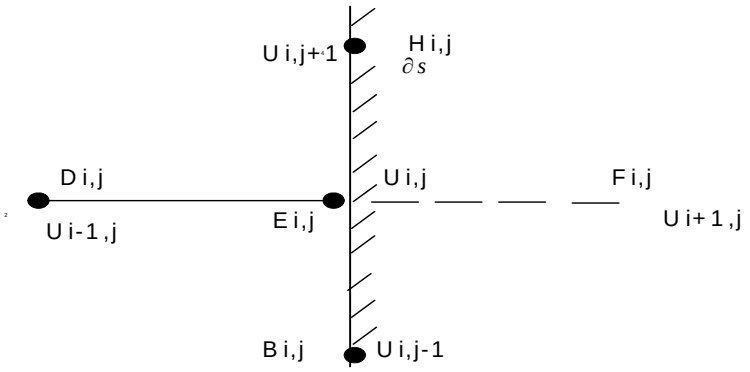


Fig . (2.4)

To find the value of $u(i+1, j)$, we use either the following forward difference approximation namely:

$$V(x_i, y_j) = \frac{u_{i+1,j} - u_{i.j}}{h} \qquad \textbf{(2.10)}$$

which gives:

$$u_{i+1,j} = u_{i,j} + hV_{i,j} \quad ,$$

or the central difference approximation (2.5.a)

Using Eq. (2.13) the finite difference equation (2.6) at (i, j) can be written as:

$$B_{i,j}u_{i,j-1} + D_{i,j}u_{i-1.j} + (E_{i,j} + F_{i,j})u_{i.j} + H_{i,j}u_{i,j+1} = g_{i,j} - F_{i,j}V_{i,j}\ h.$$

(see figure (2.2)). **(2.11)**

Taking the central difference approximation $\frac{\partial u}{\partial n}$, the five- point difference equation at (i, j) , Fig (2.4), can be written as:

$$B_{i,j}u_{i,j-1} + (D_{i,j} + F_{i,j})u_{i-1,j} + E_{i,j}u_{i,j} + H_{i,j}u_{i,j+1} = g_{i,j} - 2h\ F_{i,j}V_{i,j} \quad \textbf{(2.12)}$$

The totality of equations at the $(M+1)^2$ grid points of the square leads to the matrix equation

$$A\,\mathbf{u} = 2h\mathbf{G}, \quad \textbf{(2.13)}$$

Where A is a matrix of order $(M+1)^2$ given by:

$$A = \begin{bmatrix} B & -2I & & & & 0 \\ -I & B & -I & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & . & . \\ & & & -I & B & -I \\ 0 & & & & -2I & B \end{bmatrix} \quad \textbf{(2.14)}$$

With I the identity matrix of order $(M+1)$, and B matrix of order $(M+1)$ given by:

$$B = \begin{bmatrix} 4 & -2 & & & & 0 \\ -1 & 4 & -1 & & & \\ & . & . & . & & \\ & & . & . & . & . \\ & & & . & . & . \\ & & & -1 & 4 & -1 \\ 0 & & & & -2 & 4 \end{bmatrix} \quad \textbf{(2.15)}$$

The vectors u and G of eq's (2.12) are respectively given by:

$$[u_{0,0} \ldots u_{M,0}; u_{0,1} \ldots u_{M,1}; \ldots; u_{0,M} \ldots u_{M,M}]^T$$

and

$$[2V_{0,0}, V_{1,0}, \ldots V_{M-1,0}, 2V_{M,0}; V_{0,1}, 0 \ldots 0, V_{M,1}; \ldots; V_{0,m-1}, 0 \ldots o, V_{M.m-1}; 2V_{0,M}, V_{1,M} \ldots V_{M-1,M}, 2V_{M,M}]^T$$

(Smith, 1978; Mitchel, 1969).

(c) Robbin's conditions

The boundary conditions of the form (2.3) can be incorporated into the difference equations for the boundary nodal points by an extension of the methods outlined in the Dirichlet and Neuman problems (a) and (b).

(Smith, 1978; Mitchel, 1969).

2.3 Error Analysis

The following is an error analysis of five points difference approximation to Laplace's equation over the region . S.

Consider the Dirichlet problem:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , (x,y) \quad on \ S \qquad \textbf{(2.16.a)}$$

$$u(x,y) = g(x,y), \qquad (x,y) \ on \ \partial S \qquad \textbf{(2.16.b)}$$

where S is defined by equation (2.4) and g is a known function.

Define the set of all interior mesh points in S by S_h . At the points of S_h we replace $\nabla^2 u(x,y)$ by the five-point difference approximation of Laplace's equation, namely,

$$\nabla_h{}^2 u_h(x,y) = \frac{u_{i,j-1} + u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2} \qquad \textbf{(2.17)}$$

where $u_h(x,y)$ is the computed solution . To bound the truncation error $|\nabla_h{}^2 u_h(x,y) - \nabla^2 u(x,y)|$ for all $(x_i, y_j) \in S_h$,

We use the Taylor's series expansions for the points $u_{i+1,j}$, $u_{i-1,j}$, $u_{i,j+1}$, and $u_{i,j-1}$:

$$u_{i+1,j} = u + h\frac{\partial u}{\partial x} + \frac{1}{2}h^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{6}h^3\frac{\partial^3 u}{\partial x^3} + \frac{1}{24}h^4\frac{\partial^4 u(x+\beta h, y)}{\partial x^4} \qquad \textbf{(2.18)}$$

$$u_{i-1,j} = u - h\frac{\partial u}{\partial x} + \frac{1}{2}h^2\frac{\partial^2 u}{\partial x^2} - \frac{1}{6}h^3\frac{\partial^3 u}{\partial x^3} + \frac{1}{24}h^4\frac{\partial^4 u(x-\beta h, y)}{\partial x^4} \qquad \textbf{(2.19)}$$

Where $0 < \beta < 1$.

Adding these two equations leads to

$$u_{i+1,j} - 2u_{i,j} + u_{i-1,j} = (h^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{12}h^4\frac{\partial^4 u(x+\beta h, y)}{\partial x^4})_{i,j} . \qquad \textbf{(2.20)}$$

Similarly, for $u_{i,j+1}$ and $u_{i,j-1}$ we get that

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = (h^2\frac{\partial^2 u}{\partial y^2} + \frac{1}{12}h^4\frac{\partial^4 u(x, y+\gamma h)}{\partial y^4})_{i,j} \qquad \textbf{(2.21)}$$

where $0 < \gamma < 1$.

Adding (2.23) and (2.24) we get:

$$|\nabla_h{}^2 u_h(x,y) - \nabla^2 u(x,y)| = \frac{h^2}{12}[\frac{\partial^4 u(x+\beta h, y)}{\partial x^4} + \frac{\partial^4 u(x, y+\gamma h)}{\partial y^4}] . \qquad \textbf{(2.22)}$$

Hence if $u(x,y)$ has bounded derivatives of all order up to and including the fourth in $\bar{S} = S \cup \partial S$,then for all $(x,y) \in S_h$,

$$|\nabla_h{}^2 u_h(x,y) - \nabla^2 u(x,y)| \le \frac{h^2}{6} M_4 \qquad \textbf{(2.23)}$$

Where $u(x,y)$ is the solution of problem (2.19) and

$$M_4 = \max\{\max_{(x,y)\in\bar{S}} |\frac{\partial^4 u(x,y)}{\partial x^4}|, \max_{(x,y)\in\bar{S}} |\frac{\partial^4 u(x,y)}{\partial y^4}|\}. \qquad \textbf{(2.24)}$$

To find abound for error in $u_h(x,y)$ which is called discretization error e_h we apply a Theorem (smith, 1978 ; p.223) which states that :

If V is any function defined on the set of mesh points $\overline{s_h}$ in the square region S (2.19) then

$$\max_{S_h} |V| \le \max_{\partial S_h} |V| + \frac{1}{4}(a^2 + a^2)\max_{S_h} |\nabla_h{}^2 - \nabla^2|$$

Applying this theorem to the discretization error e_h, we get:

$$\max_{S_h} |e_h| \le \max_{\partial S_h} |e_h| + \frac{1}{4}(a^2 + a^2)\max_{S_h} |\nabla_h{}^2 - \nabla^2|$$

but e_h=0 on ∂S_h because $u_h + u = g$ *on* ∂S_h by Eq. (2.19.b).Hence, by Eq.(2.26), we conclude that

$$\max_{S_h} |e_h| \le \frac{1}{24}(a^2 + a^2)h^2 M_4 .$$

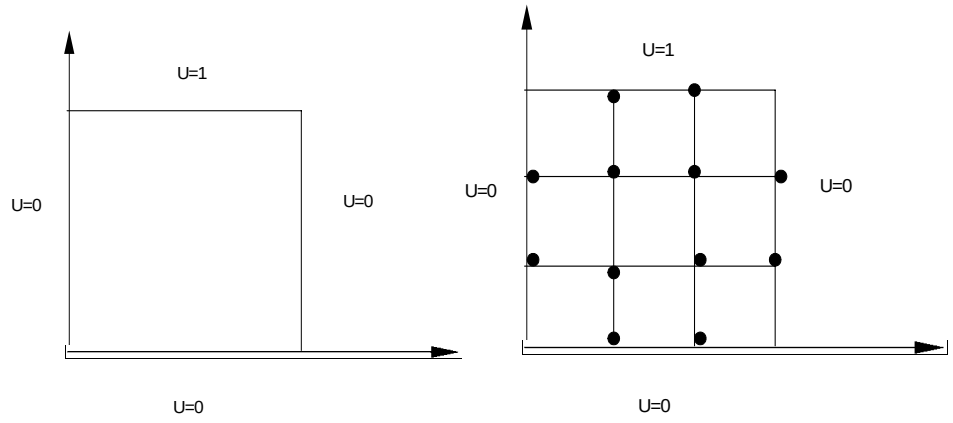
Therefore u_h converges to the exact solution u as h tends to zero and the discretization error is proportional to h^2 (smith, 1978; Whiteman, 1982).

Example:

Consider Laplace's equation

$$u_{xx} + u_{yy} = 0$$

On unit square with boundary conditions $u(x,y)=\begin{cases}0:x=0\\0:x=1\\0:y=0\\1:y=1\end{cases}$ as shown below



Define a discrete mesh in this domain, including boundaries with $n=2$, $h=\frac{1}{3},\ and\ k=\frac{1}{3}$, as shown on right above

Interior grid points where we will compute approximate solution are given by:

$$(x_i,y_j)=(ih,ky)\quad ,\qquad i,j=1,2,...,n$$

After replacing the derivatives by central difference approximation at each interior mesh point we get the finite difference equation

$$\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2}=0$$

where $u_{i,j}$ approximates the true solution $u(x_i,y_j)$ for $i,j=1,2$

The point (x_i,y_j) is a boundary point if $i,j=0$ or $i,j=3$

The result is the following linear system:

$$4u_{1,1} - u_{0,1} - u_{2,1} - u_{1,0} - u_{1,2} = 0$$
$$4u_{2,1} - u_{1,1} - u_{3,1} - u_{2,0} - u_{2,2} = 0$$
$$4u_{1,2} - u_{0,1} - u_{2,2} - u_{1,1} - u_{1,3} = 0$$
$$4u_{2,2} - u_{1,2} - u_{3,2} - u_{2,1} - u_{2,3} = 0$$

The above equations can be written in matrix form as:

$$A\,\mathbf{u} = \mathbf{b}$$

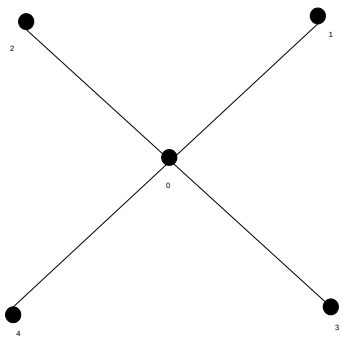
where $A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$, $u = \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Using Gaussian elimination with backward substitution or any direct method, gives the exact solution:

$$u = \begin{bmatrix} 0.125 & 0.125 & 0.375 & 0.375 \end{bmatrix}^T$$

2.4 Rotated Five-Points Formula

Another type of discretization, so-called rotated discretization , can be achieved by rotating the i - and j – plane axis clockwise by 45^0 .Thus , the discretized form of Laplace's equation at the mesh point (x_i, y_j) with the finite differences formula will lead to the rotated five points schemel as:



$$u_1 = u_0 + (s_1k)u_x + (s_1k)u_y + \frac{(s_1k)^2}{2}u_{x^2} + \frac{(s_1k)^2}{2}u_{y^2} + (s_1k)^2 u_{xy} + \frac{(s_1k)^3}{6}u_{x^3} + \frac{(s_1k)^3}{6}u_{y^3} + \frac{(s_1k)^3}{2}u_{x^2y} + \frac{(s_1k)^3}{2}u_{xy^2}$$

$$u_2 = u_0 - (s_2k)u_x + (s_2k)u_y + \frac{(s_2k)^2}{2}u_{x^2} + \frac{(s_2k)^2}{2}u_{y^2} - (s_2k)^2 u_{xy} - \frac{(s_2k)^3}{6}u_{x^3} + \frac{(s_2k)^3}{6}u_{y^3} + \frac{(s_2k)^3}{2}u_{x^2y} - \frac{(s_2k)^3}{2}u_{xy^2} + \ldots$$

$$u_3 = u_0 + (s_3k)u_x - (s_3k)u_y + \frac{(s_3k)^2}{2}u_{x^2} + \frac{(s_3k)^2}{2}u_{y^2} - (s_3k)^2 u_{xy} - \frac{(s_3k)^3}{6}u_{x^3} - \frac{(s_3k)^3}{6}u_{y^3} - \frac{(s_3k)^3}{2}u_{x^2y} + \frac{(s_3k)^3}{2}u_{xy^2} + \ldots$$

$$u_4 = u_0 - (s_4k)u_x - (s_4k)u_y + \frac{(s_4k)^2}{2}u_{x^2} + \frac{(s_4k)^2}{2}u_{y^2} + (s_4k)^2 u_{xy} - \frac{(s_4k)^3}{6}u_{x^3} - \frac{(s_4k)^3}{6}u_{y^3} - \frac{(s_4k)^3}{2}u_{x^2y} - \frac{(s_4k)^3}{2}u_{xy^2} + \ldots$$

Now Laplace's equation can be written as:

$$u_{xx} + u_{yy} = \alpha_1\ u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4 - \alpha_0 u_0 = 0$$

After we solve the above equations for $\alpha's$ we obtain

$$\alpha_0 = \frac{2}{k^2}, \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2k^2}.$$

For the above case, we need only to define a new $k^{\backslash}$ as:

$$k^{\backslash} = \frac{k}{(2)^{1/2}}$$

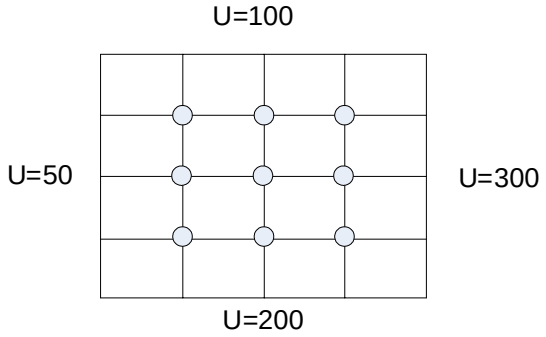
so Laplace's equation can be written as:

$$u_{xx} + u_{yy} = (u_1 + u_2 + u_3 + u_4 - 4u_0) = 0$$

$$u_0 = u(i, j) = \frac{1}{4}[u(i+1, j+1) + u(i-1, j+1) + u(i-1, j-1) + u(i+1, j-1)]$$

Now let's write the above equation in a vector matrix form:

$$A\,\mathbf{u} = \mathbf{b}$$



Let's take ($n = m = 4$) in each direction

For node (1,1): $u_{1,1} = \frac{1}{4}[u_{2,2} + u_{0,2} + u_{2,0} + u_{0,0}]$

For node (2, 1): $u_{2,1} = \frac{1}{4}[u_{3,2} + u_{1,2} + u_{3,0} + u_{1,0}]$

For node (3, 1): $u_{3,1} = \frac{1}{4}[u_{4,2} + u_{2,2} + u_{4,0} + u_{2,0}]$

For node (1, 2): $u_{1,2} = \frac{1}{4}[u_{2,3} + u_{0,3} + u_{2,1} + u_{0,1}]$

For node (2, 2): $u_{2,2} = \frac{1}{4}[u_{3,3} + u_{1,3} + u_{3,1} + u_{1,1}]$

For node (3, 2): $u_{3,2} = \frac{1}{4}[u_{4,3} + u_{2,3} + u_{4,1} + u_{2,1}]$

For node (1, 3): $u_{1,3} = \frac{1}{4}[u_{2,4} + u_{0,4} + u_{2,2} + u_{0,2}]$

For node (2, 3): $u_{2,3} = \frac{1}{4}[u_{3,4} + u_{1,4} + u_{3,2} + u_{1,2}]$

1.3 2.3 3.3

1.2 2.2 3.2

1.1 2.1 3.1

For node (3,3): $u_{3,3} = \frac{1}{4}[u_{4,4} + u_{2,4} + u_{2,2} + u_{4,2}]$

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 4 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 4 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 \end{bmatrix},$$

$$\mathbf{u} = \begin{bmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{1,2} & u_{2,2} & u_{2,3} & u_{1,3} & u_{2,3} & u_{3,3} \end{bmatrix}^T$$

$$\mathbf{b} = \begin{bmatrix} 400 & 600 & 700 & 100 & 0 & 0 & 200 & 200 & 500 \end{bmatrix}^T$$

the above system can be solved and the approximated solution will be:

$$u = \begin{bmatrix} 137.5 & 187.9808 & 212.5 & 88.4615 & 150.00 & 63.4615 & 87.500 & 65.8654 & 162.5 \end{bmatrix}^T$$

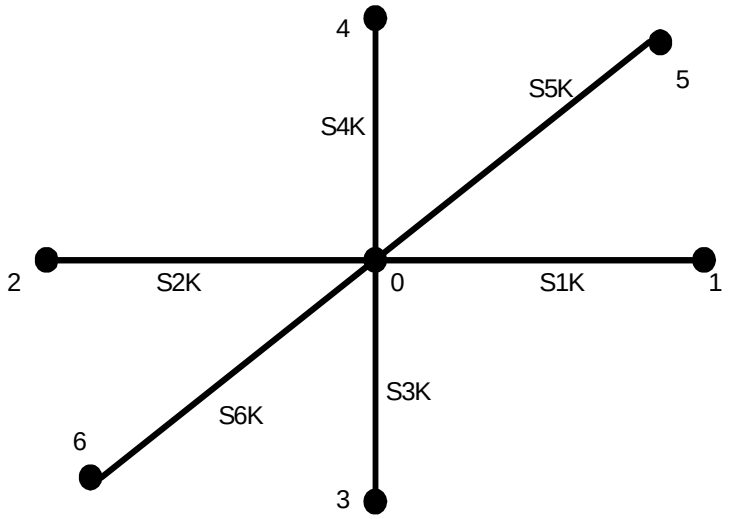
2.5 A Seven-Point Formula

We begin this discussion by developing a series of seven point finite difference approximations for the model elliptic PDE.

We consider the interior of S but ignore in the present, the boundary conditions. The subscripts on u will be i and j respectively.

In a more general sense, we may desire a seven –point finite formula that either uses u_0 and any sixth other points or has variable spacing throughout or in some specific region of S .This can be done using Laplace's equation as an illustration, by setting

$$u_{xx} + u_{yy} = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4 + \alpha_5 u_5 + \alpha_6 u_6 - \alpha_0 u_0$$



Where the $u_1,.....,u_6$ are the values of U selected mesh points around U_0 and $\alpha_0,.....,\alpha_6$ are parameters to be determined by Taylor's

$$u_1 = [u_0 + (s_1k)u_{x^1} + \frac{(s_2k)^2}{2}u_{x^2} + \frac{(s_1k)^3}{6}u_{x^3} + \frac{(s_1k)^4}{24}u_{x^4} + ...]_0$$

$$u_3 = [u_0 - (s_3k)u_{y^1} + \frac{(s_3k)^2}{2}u_{y^2} - \frac{(s_3k)^3}{6}u_{y^3} + \frac{(s_3k)^4}{24}u_{y^4} - .....]_0$$

$$u_2 = [u_0 - (s_2k)u_{x^1} + \frac{(s_2k)^2}{2}u_{x^2} - \frac{(s_2k)^3}{6}u_{x^3} + \frac{(s_2k)^4}{24}u_{x^4} + ...]_0$$

$$u_4 = [u_0 = (s_4k)u_y + \frac{(s_4k)^2}{2}u_{y^2} + \frac{(s_4k)^3}{6}u_{y^3} + \frac{(s_4k)^4}{24}u_{y^4} + ...]$$

$$u_5 = [u_0 + (s_5k)u_{x^1} + (s_5k)u_{y^1} + \frac{(s_5k)^2}{2}u_{x^2} + \frac{(s_5k)^2}{2}u_{y^2} + (s_5k)^2u_{x^1y^1} +$$
$$\frac{(s_5k)^3}{6}u_{x^3} + \frac{(s_5k)^3}{2}u_{x^2y^1} + \frac{(s_5k)^3}{2}u_{x^1y^2} + \frac{(s_5k)^3}{6}u_{y^3} + ...$$

$$u_6 = [u_0 - (s_6k)u_{x^1} - (s_6k)u_{y^1} + \frac{(sk)^2}{2}u_{y^2} + \frac{(s_6k)^2}{2}u_{x^2} +$$

$$(s_6k)^2 u_{x^1y^1} - \frac{(s_6k)^3}{6}u_{x^3} - \frac{(s_6k)^3}{2}u_{x^2y^1} - \frac{(s_6k)^3}{6}u_{xy^2} - \frac{(s_6k)^3}{6}u_{y^3} + ...]$$

$$u_{xx} + u_{yy} = u_0[-\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6] +$$

$$u_{x^1}[\alpha_1(s_1k) - \alpha_2(s_2k) + \alpha_5(s_5k) - \alpha_6(s_6k)] +$$

$$u_{y^1}[-\alpha_3(s_3k) + \alpha_4(s_4k) + \alpha_5(s_5k) - \alpha_6(s_6k)] +$$

$$\frac{u_{x^2}}{2}[\alpha_1(s_1k)^2 + \alpha_2(s_2k)^2 + \alpha_5(s_5k)^2 + \alpha_6(s_6k)^2] +$$

$$\frac{u_y}{2}[\alpha_3(s_3k)^2 + \alpha_4(s_4k)^2 + \alpha_5(s_5k)^2 + \alpha_6(s_6k)^2] +$$

$$\frac{u_{x^3}}{6}[\alpha_2(s_1k)^3 - \alpha_2(s_2k)^3 + \alpha_3(s_3k)^3 + \alpha_5(s_5k)^3 - \alpha_6(s_6k)^3] +$$

$$\frac{u_{y^3}}{6}[-\alpha_3(s_3k)^3 + \alpha_4(s_4k)^3 + \alpha_5(s_5k)^3 - \alpha_6(s_6k)^3]$$

$$+\frac{u_{x^4}}{24}[\alpha_1(s_1k)^4 + \alpha_2(s_2k)^4 + \alpha_5(s_5k)^4 + \alpha_6(s_6k)^6]$$

$$+\frac{u_{y^4}}{24}[\alpha_3(s_3k)^4 + \alpha_4(s_4k)^4 + \alpha_5(s_5k)^4 + \alpha_6(s_6k)^4]$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 - \alpha_0 = 0$$

$$\alpha_1(s_1k) - \alpha_2(s_2k) + \alpha_5(s_5k) - \alpha_6(s_6k) = 0$$

$$-\alpha_3(s_3k) + \alpha_4(s_4k) + \alpha_5(s_5k) - \alpha_6(s_6k) = 0$$

$$\alpha_1(s_1k)^2 + \alpha_2(s_2k)^2 + \alpha_5(s_5k)^2 + \alpha_6(s_6k)^2 = 2$$

$$\alpha_3(s_3k)^2 + \alpha_4(s_4k)^2 + \alpha_5(s_5k)^2 + \alpha_6(s_6k)^2 = 2$$

$$\alpha_1(s_1k)^3 - \alpha_2(s_2k)^3 + \alpha_5(s_5k)^3 - \alpha_6(s_6k)^3 = 0$$

$$-\alpha_3(s_3k)^3+\alpha_4(s_4k)^3+\alpha_5(s_5k)^3-\alpha_6(s_6k)^3=0$$

$$\begin{array}{ccccccc|cc}
-1 & 1 & 1 & 1 & 1 & 1 & 1 & \alpha_0 & 0 \\
0 & (s_1k) & -(s_2k) & 0 & 0 & (s_5k) & -(s_6k) & \alpha_1 & 0 \\
0 & 0 & 0 & -(s_3k) & (s_4k) & (s_5k) & -(s_6k) & \alpha_2 & 0 \\
0 & (s_1k)^2 & (s_2k)^2 & 0 & 0 & (s_5k)^2 & (s_6k)^2 & \alpha_3 & 2 \\
0 & 0 & 0 & (s_3k)^2 & (s_4k)^2 & (s_5k)^2 & (s_6k)^2 & \alpha_4 & 2 \\
0 & (s_1k)^3 & -(s_2k)^3 & 0 & 0 & (s_5k)^3 & -(s_6k)^3 & \alpha_5 & 0 \\
0 & 0 & 0 & -(s_3k)^3 & (s_4k)^3 & (s_5k)^3 & -(s_6k)^3 & \alpha_6 & 0
\end{array}$$

For regular region:

$$s_1=s_2=s_3=s_4=s_5=s_5=s_6=1$$

Let $\alpha_6=r=1$

$\alpha_5=t=1$

$\alpha_0+\alpha_5+\alpha_6=400$

$\alpha_1+\alpha_5=100$

$\alpha_2+\alpha_6=100$

$\alpha_3+\alpha_6=100$

$\alpha_4+\alpha_5=100$

When we solve the above equations, we obtain:

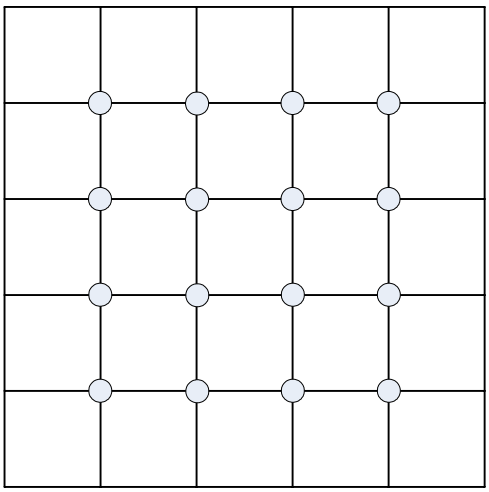
$$\alpha_0=398\ ,\alpha_1=99\ ,\alpha_2=99\ ,\alpha_3=99\ ,\alpha_4=99\ ,\alpha_6=\alpha_5=1$$

$$u_{xx}+u_{yy}=99u_1+99u_2+99u_3+99u_4+u_5+u_6-398u_0=0$$

$$u_0=\frac{1}{398}(99\,u_1+99\,u_2+99\,u_3+99\,u_4+u_5+u_6)$$

Let us write the above equation in a matrix form as:

$$A\,\mathbf{u} = \mathbf{b}$$



U=100

For node (1, 1): $u_{1,1} = \frac{1}{398}[99(u_{2,1} + u_0 + u_0 + u_{1,2}) + u_{2,2} + u_0]$

For node (2, 1): $u_{2,1} = \frac{1}{398}[99(u_{3,1} + u_{1,1} + u_0 + u_{2,2}) + u_{3,2} + u_0]$

For node (3, 1): $u_{3,1} = \frac{1}{398}[99(u_{4,1} + u_{2,1} + u_0 + u_{3,2}) + u_{4,2} + u_0]$

1.4 2.4 3.4

For node (4, 1): $u_{4,1} = \frac{1}{398}[99(u_0 + u_{3,1} + u_0 + u_{4,2}) + u_0 + u_0]$

For node (1,2): $u_{1,2} = \frac{1}{398}[99(u_{2,2} + u_{0,2} + u_{1,1} + u_{1,3}) + u_{2,3} + u_{0,1}]$

1.3 2.3 3.3

U=50

For node (2, 2): $u_{2,2} = \frac{1}{398}[99(u_{3,2} + u_{1,2} + u_{2,1} + u_{2,3}) + u_{3,3} + u_{1,1}]$

For node (3, 2): $u_{3,2} = \frac{1}{398}[99(u_{4,2} + u_{2,2} + u_{3,1} + u_{3,3}) + u_{4,3} + u_{2,1}]$

1.2 2.2 3.2

1.1 2.1 3.1

For node (4, 2): $u_{4,2} = \frac{1}{398}[99(u_0 + u_{3,2} + u_{4,1} + u_{4,3}) + u_0 + u_{3,1}]$

For node (1, 3): $u_{1,3} = \frac{1}{398}[99(u_{2,3} + u_0 + u_{1,2} + u_{1,4}) + u_{2,4} + u_0]$

For node (2, 3): $u_{2,3} = \frac{1}{398}[99(u_{3,3} + u_{1,3} + u_{2,2} + u_{2,4}) + u_{3,4} + u_{1,2}]$

For node (3, 3): $u_{3,3} = \frac{1}{398}[99(u_{4,3} + u_{2,3} + u_{3,2} + u_{3,4}) + u_{4,4} + u_{2,2}]$

For node (4, 3): $u_{4,3} = \frac{1}{398}[99(u_{3,3} + u_0 + u_{4,2} + u_{4,4}) + u_0 + u_{3,2}]$

For node (1, 4): $u_{1,4} = \frac{1}{398}[99(u_{2,4} + u_0 + u_{1,3} + u_0) + u_0 + u_0]$

For node (2, 4): $u_{2,4} = \frac{1}{398}[99(u_{3,4} + u_{2,3} + u_{1,4} + u_0) + u_0 + u_{1,3}]$

For node (3, 4): $u_{3,4} = \frac{1}{398}[99(u_{4,4} + u_{2,4} + u_{3,3} + u_0) + u_0 + u_{2,3}]$

For node (4, 4): $u_{4,4} = \frac{1}{398}[99(u_0 + u_{3,4} + u_{4,3} + u_0) + u_0 + u_{3,3}]$

$$A = \begin{bmatrix}
398 & -99 & 0 & 0 & -99 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-99 & 398 & -99 & 0 & 0 & -99 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -99 & 398 & -99 & 0 & 0 & -99 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -99 & 398 & 0 & 0 & 0 & -99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-99 & 0 & 0 & 0 & 398 & -99 & 0 & 0 & -99 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -99 & 0 & 0 & -99 & 398 & -99 & 0 & 0 & -99 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -99 & 0 & 0 & -99 & 398 & -99 & 0 & 0 & -99 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -99 & 0 & 0 & -99 & 398 & 0 & 0 & 0 & -99 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -99 & 0 & 0 & 0 & 398 & -99 & 0 & 0 & -99 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -99 & 0 & 0 & -99 & 398 & -99 & 0 & 0 & -99 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -99 & 0 & 0 & -99 & 398 & -99 & 0 & 0 & -99 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -99 & 0 & 0 & -99 & 398 & 0 & 0 & 0 & -99 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -99 & 0 & 0 & 0 & 398 & -99 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -99 & 0 & 0 & -99 & 398 & -99 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -99 & 0 & 0 & -99 & 398 & -99 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -99 & 0 & 0 & -99 & 398
\end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{1,3} & u_{2,3} & u_{3,3} & u_{4,3} & u_{1,4} & u_{2,4} & u_{3,4} & u_{4,4} \end{bmatrix}^T$$

$$\mathbf{b} = \begin{bmatrix} 400 & 600 & 600 & 800 & 100 & 0 & 0 & 400 & 100 & 0 & 0 & 400 & 300 & 200 & 200 & 500 \end{bmatrix}^T$$

The above matrix can be written as:

$$\begin{bmatrix} A_1 & B_1 & 0 \\ D & A_2 & B_2 \\ 0 & B^T{}_2 & A_3 \end{bmatrix}$$ This is Tridiagonal Block Matrix.

2.6 A Nine-points formula and Truncations Error

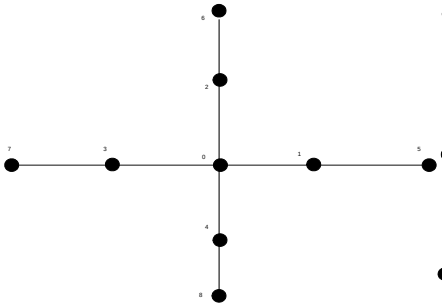


Figure 2.4.a

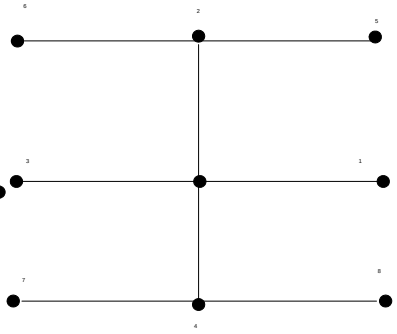


Figure 2.4.b

In this section, we develop higher-order nine-point approximations .

One approach is to use higher order approximation for U_{xx} and U_{yy} .

We turn to the set of points shown in the figure above to drive this approximation. We may follow our previous procedure and estimate $\alpha_0, \alpha_1, ...\alpha_4$ *and* $\alpha_5, ...\alpha_8$ in the form

$$u_{xx} + u_{yy} = -\alpha_0 u_0 + \sum_{i=1}^{4} \alpha_i u_i + \sum_{i=5}^{8} \alpha_i u_i$$

For Laplace's equation. Substituting Taylor's series

$$u_1 = [u_0 + (sk_1)u_{x^1} + \frac{(sk_1)^2}{2} u_{x^2} + \frac{(sk_1)^3}{6} u_{x^3} + \frac{(sk_1)^4}{24} u_{x^4} + ...]_0$$

$$u_2 = [u_0 + (sk_1)u_y + \frac{(sk_1)^2}{2} u_{y^2} + \frac{(sk_1)^3}{6} u_{y^3} + \frac{(sk_1)^4}{24} u_{y^4} + ...]$$

$$u_3 = [u_0 - (s\ k_1)u_x + \frac{(sk_1)^2}{2} u_{x^2} - \frac{(sk_1)^3}{6} u_{x^3} + \frac{(sk_1)^4}{24} u_{x^4} + ...]$$

$$u_4 = [u_0 - (sk_1)u_y + \frac{(sk_1)^2}{2} u_{y^2} - \frac{(sk_1)^3}{6} u_{y^3} + \frac{(sk)^4}{24} u_{y^4} + ...]$$

$$u_5 = u_0 + (sk_2)u_x + (sk_2)u_y + \frac{(sk_2)^2}{2} u_{x^2} + \frac{(sk_2)^2}{2} u_{y^2} + (sk_2)^2 u_{xy} + \frac{(sk_2)^3}{6} u_{x^3} + \frac{(sk_2)^3}{6} u_{y^3} + \frac{(sk_2)^3}{2} u_{x^2y} + \frac{(sk_2)^3}{2} u_{xy^2}$$

$$u_6 = u_0 - (sk_2)u_x + (sk_2)u_y + \frac{(sk_2)^2}{2} u_{x^2} + \frac{(sk_2)^2}{2} u_{y^2} - (sk_2)^2 u_{xy} - \frac{(sk_2)^3}{6} u_{x^3} + \frac{(sk_2)^3}{6} u_{y^3} + \frac{(sk_2)^3}{2} u_{x^2y} - \frac{(sk_2)^3}{2} u_{xy^2} + ...$$

$$u_7 = u_0 + (sk_2)u_x - (sk_2)u_y + \frac{(sk_2)^2}{2} u_{x^2} + \frac{(sk_2)^2}{2} u_{y^2} - (sk_2)^2 u_{xy} - \frac{(sk_2)^3}{6} u_{x^3} - \frac{(sk_2)^3}{6} u_{y^3} - \frac{(sk_2)^3}{2} u_{x^2y} + \frac{(sk_2)^3}{2} u_{xy^2} + ...$$

$$u_8 = u_0 - (sk_2)u_x - (sk_2)u_y + \frac{(sk_2)^2}{2} u_{x^2} + \frac{(sk_2)^2}{2} u_{y^2} + (sk_2)^2 u_{xy} - \frac{(sk_2)^3}{6} u_{x^3} - \frac{(sk_2)^3}{6} u_{y^3} - \frac{(sk_2)^3}{2} u_{x^2y} - \frac{(sk_2)^3}{2} u_{xy^2} + ...$$

Collecting terms and solving for the α_i. We obtain for $k_1 = k_2 = k$,

$$\alpha_0 = \frac{10}{3k^2}, \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{2}{3k^2},$$

$$\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \frac{1}{6k^2}.$$

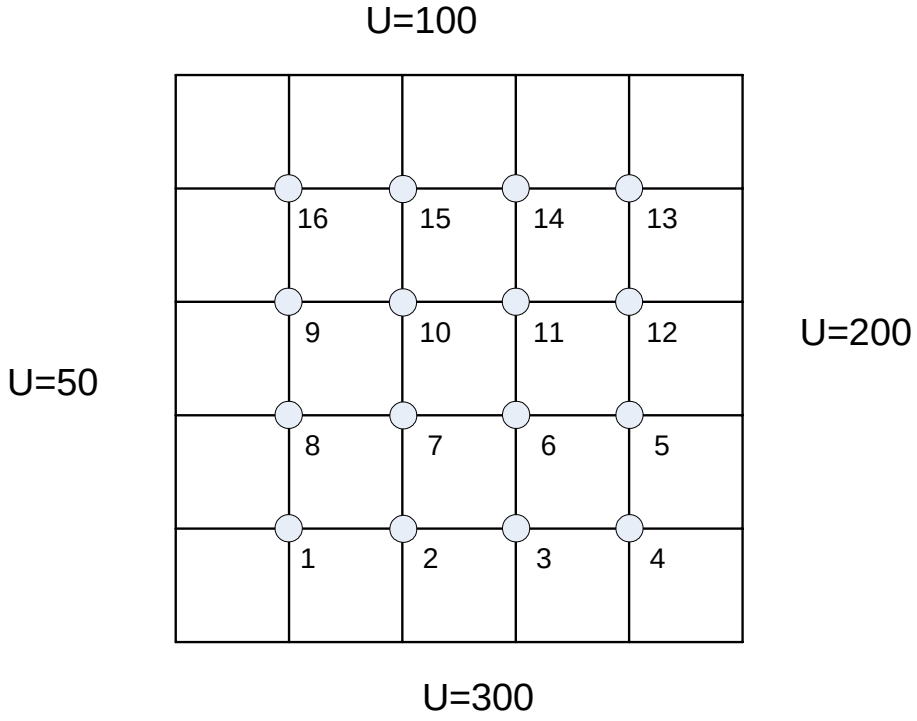
(Lapidus, 1982; Pinder, 1982)

Thus a forth-order approximation for Laplace's equation is given by:

$$u_0 = \frac{1}{20}[4(u_1 + u_2 + u_3 + u_4) + (u_5 + u_6 - u_7 + u_8)]$$

Let's write the above equation in a matrix form,

First, assume that $n = 5$ in each direction and the function in each side equals zero.



For node (1): $u_1 = \frac{1}{20}[4(u_2 + u_8 + 0 + 0) + (u_7 + 0 - 0 + 0)]$

For node (2): $u_2 = \frac{1}{20}[4(u_3 + u_7 + u_1 + 0) + (u_6 + u_8 - 0 + 0)]$

For node (3): $u_3 = \frac{1}{20}[4(u_4 + u_6 + u_2 + 0) + (u_5 + u_7 - 0 + 0)]$

For node (4): $u_4 = \frac{1}{20}[4(u_5 + u_3 + 0 + 0) + (u_6 + 0 - 0 + 0)]$

For node (5): $u_5 = \frac{1}{20}[4(u_{12} + u_6 + u_4 + 0) + (u_{11} + 0 - u_3 + 0)]$

For node (6): $u_6 = \frac{1}{20}[4(u_{11}+u_5+u_7+u_3)+(u_{12}+u_{10}-u_2+u_4)]$

For node (7): $u_7 = \frac{1}{20}[4(u_{10}+u_6+u_8+u_{\,2})+(u_{11}+u_9-u_1+u_3)]$

For node (8): $u_8 = \frac{1}{20}[4(u_7+u_9+u_1+0)+(u_{10}+0-0+u_2)]$

For node (9): $u_9 = \frac{1}{20}[4(u_{10}+u_{16}+u_8+0)+(u_{15}+0-0+u_7)]$

For node (10): $u_{10} = \frac{1}{20}[4(u_{11}+u_{15}+u_9+u_7)+(u_{14}+u_{16}-u_8+u_6)]$

For node (11): $u_{11} = \frac{1}{20}[4(u_{12}+u_{14}+u_{10}+u_6)+(u_{13}+u_{15}-u_7+u_5)]$

For node (12): $u_{12} = \frac{1}{20}[4(u_{13}+u_{11}+0+u_5)+(u_{14}+0-u_6+0)]$

For node (13): $u_{13} = \frac{1}{20}[4(u_{12}+u_{14}+0+0)+(0+0-u_{11}+0)]$

For node (14): $u_{14} = \frac{1}{20}[4(u_{13}+0+u_{15}+u_{11})+(0+0-u_{10}+u_{12})]$

For node (15): $u_{15} = \frac{1}{20}[4(u_{14}+0+u_{16}+u_{10})+(0+0-u_9+u_{11})]$

For node (16): $u_{16} = \frac{1}{20}[4(u_{15}+0+0+u_9)+(0+0-0+u_{10})]$

The above linear equations can be written in a matrix form as:

$$A\ \mathbf{u}\ =\ \mathbf{b}\quad ,$$

$\mathbf{u} = [u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \quad u_7 \quad u_8 \quad u_9 \quad u_{10} \quad u_{11} \quad u_{12} \quad u_{13} \quad u_{14} \quad u_{15} \quad u_{16}]^T$

$\mathbf{b} = [1700\ 1200\ 1200\ 2200\ 1200\ 0\ \ 0\ \ 200\ \ 200\ \ 0\ \ 0\ \ 1200\ 1700\ 600\ \ 600\ \ 600]^T$

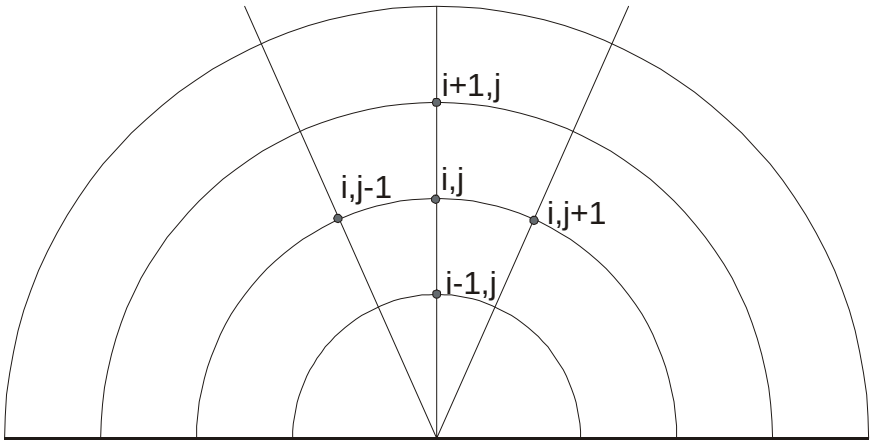
the above system can be solved and the approximated solution will be:

$\mathbf{u} = [1343\ \ 14561\ \ 15711\ \ 17471\ \ 13949\ \ 10778\ \ 9015\ \ 7835\ \ 623\ \ 7428\ \ 9364\ \ 13083\ \ 12492\ \ 922\ \ 782\ \ 668]^T$

2. 7 The Laplace equation in polar coordinate

The Laplace equation in polar coordinate is written as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \textbf{(2.25)}$$



Index notation in polar coordinate.

Let i and j be the index in the r and θ directions respectively, the finite difference form of the Laplace equation is written as:

$$\nabla^2 u = \frac{u(i-1,j) - 2u(i,j) + u(i+1,j)}{\Delta r^2} + \frac{1}{r_i}\frac{u(i+1,j) - u(i-1,j)}{2\Delta r} + \frac{1}{r_i^2}\frac{u(i,j-1) - 2u(i,j) + u(i,j+1)}{\Delta \theta^2} = 0$$

The value at the node (i, j) can be computed from

$$2\left[\frac{1}{\Delta r^2} + \frac{1}{r_i^2 \Delta \theta}\right] u(\mathrm{i,j}) = \frac{u(i-1,j) + u(i+1,j)}{\Delta r^2} + \frac{1}{r_i}\frac{u(i+1,j) - u(i-1,j)}{2\Delta r} + \frac{1}{r_i^2}\frac{u(i,j-1) + u(i,j+1)}{\Delta \theta^2} \quad .$$

Chapter Three

Iterative Methods for Solving Elliptic Equations

3.1 Introduction

In chapter two we discussed alternative methods for approximating the solution of linear elliptic equations, the finite difference method, using the differential equation. Each of these methods gives rise to a system of linear algebraic equations, which may be very large depending on the accuracy needed. A two-dimensional problem may lead to a system of several thousand unknowns, are common in real engineering situations. The solution of each a system is a major problem in itself and has been the subject of much detailed study. The most obvious property of the system is that it is extremely sparse. Even when there are many thousand unknowns each equation will involve one unknown and the unknowns at its immediate neighbours. In this chapter, we deals with some iterative methods.

3.2 Jacobi Iterative method:

Consider a linear system**:** $A\,u = F$

Take the matrix A and consider a split into: $A = D - L - U$, where D is diagonal and($-L$) and($-U$) are the strictly lower and upper triangular parts of A.

The linear system $Au = F$ can be written as:

$$(D - L - U)x = F$$
$$Dx = (L + U)x + F$$
$$x = D^{-1}(L + U)x + D^{-1}F$$

Let $R_J = D^{-1}(L + U)$, $R_J = D^{-1}(D - A) = I - D^{-1}A$

R_J is called the error propagation or iteration matrix for Jacobi method.

Then, the iteration is:

$$u^{(new)} = R_J u^{(old)} + D^{-1}F$$

$$\text{or} \quad u_{i,j}^{(kr+1)} = \frac{(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})^{(ik)} - h^2 f_{i,j}}{4} \qquad \textbf{(3.1)}$$

Error propagation matrix :

From the derivation:

$$x = D^{-1}(L+U)x + D^{-1}F$$
$$x = R_J + D^{-1}F \qquad \textbf{(3.2)}$$

The iteration is

$$u^{(new)} = Ru^{(old)} + D^{-1}F \qquad \textbf{(3.3)}$$

Subtracting eq's (3.3),(3.4)

$$x - u^{(new)} = R_J x + D^{-1}F - (R_J u^{(old)} + D^{-1}F)$$

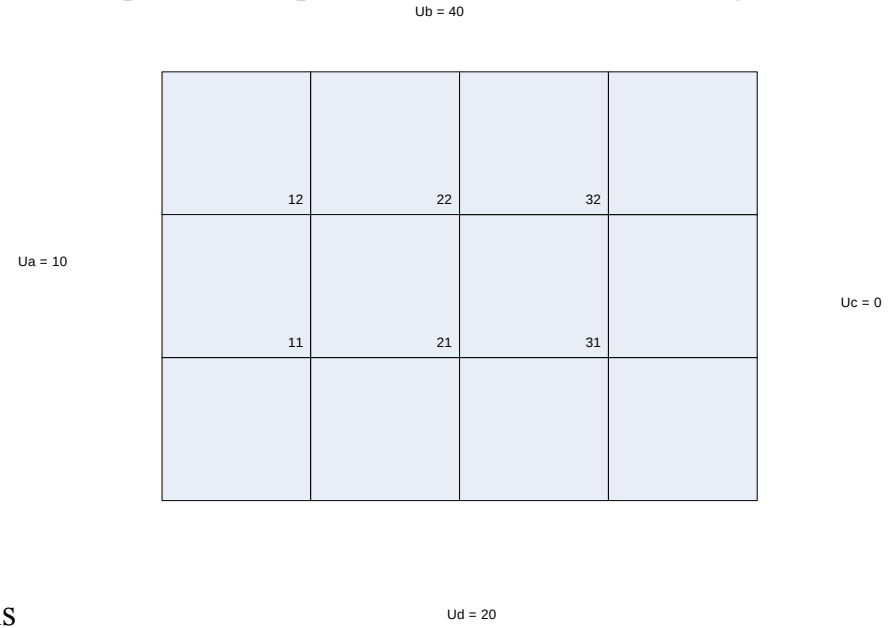
Hence:

$$x - u^{(new)} = R_J x - R_J u^{(old)}$$
$$x - u^{(new)} = R_J (x - u^{(old)})$$

Error propagation:

$$e^{(new)} = R_J e^{(old)}, \;\; R_J = I - D^{-1}A$$

Example: A square grid is shown below for the solution of the two dimensional Laplace's equation based on the given boundary conditions



The nodes in a two- dimensional, steady-state conduction.

$$u^{(k)}{}_{ij} = the\ \ value\ \ of\ \ the\ \ u_{ij}\ at\ \ the\ \ k^{th}\ iteration$$

$$u^{(k)}{}_{11} = \frac{1}{4}[u_a + u_{21}{}^{(k-1)} + u_{12}{}^{(k-1)} + u_d]$$

$$u^{(k)}{}_{21} = \frac{1}{4}[u_a + u_{21}{}^{(k-1)} + u_{12}{}^{(k-1)} + u_d]$$

$$u^{(k)}{}_{31} = \frac{1}{4}[u^{(k-1)}{}_{21} + u^{(k-1)}{}_{32} + u_c + u_d]$$

$$u^{(k)}{}_{12} = \frac{1}{4}[u_a + u_b + u^{(k-1)}{}_{22} + u^{(k-1)}{}_{11}]$$

$$u^{(k)}{}_{22} = \frac{1}{4}[u^{(k-1)}{}_{12} + u_b + u^{(k-1)}{}_{32} + u^{(k-1)}{}_{21}]$$

$$u^{(k)}{}_{32} = \frac{1}{4}[u^{(k-1)}{}_{22} + u_b + u_c + u^{(k-1)}{}_{31}]$$

initial guess $\Rightarrow (k = 0)$ Let $u^{(0)}{}_{ij} = 0$ for all nodes

1^{st} *Iteration*$(k=1)$

$$u^{(1)}{}_{11} = \frac{1}{4}[u_a + u^{(0)}{}_{12} + u^{(0)}{}_{21} + u_d] = \frac{1}{4}[10+0+0+20] = 7.5$$

$$u^{(1)}{}_{21} = \frac{1}{4}[u^{(0)}{}_{11} + u^{(0)}{}_{22} + u^{(0)}{}_{31} + u_d] = \frac{1}{4}[0+0+0+20] = 5$$

$$u^{(1)}{}_{31} = \frac{1}{4}[u^{(0)}{}_{21} + u^{(0)}{}_{32} + u_c + u_d] = \frac{1}{4}[0+0+0+20] = 5$$

$$u^{(1)}{}_{12} = \frac{1}{4}[u_a + u_b + u^{(0)}{}_{22} + u^{(0)}{}_{11}] = \frac{1}{4}[10+40+0+20] = 12.5$$

$$u^{(1)}{}_{22} = \frac{1}{4}[u^{(0)}{}_{12} + u_b + u^{(0)}{}_{32} + u^{(0)}{}_{21}] = \frac{1}{4}[0+40+0+20] = 10$$

$$u^{(1)}{}_{32} = \frac{1}{4}[u^{(0)}{}_{22} + u_b + u_c + u^{(0)}{}_{31}] = \frac{1}{4}[0+40+0+20] = 10$$

2^{nd} *Iteration*$(k=2)$

$$u^{(2)}{}_{11} = \frac{1}{4}[u_a + u^{(1)}{}_{12} + u^{(2)}{}_{21} + u_d] = \frac{1}{4}[10 + 12.5 + 5 + 20] = 11.875$$

$$u^{(2)}{}_{21} = \frac{1}{4}[u^{(1)}{}_{11} + u^{(1)}{}_{22} + u^{(1)}{}_{31} + u_d] = \frac{1}{4}[7.5+10+5+20] = 10.625$$

$$u^{(2)}{}_{31} = \frac{1}{4}[u^{(1)}{}_{21} + u^{(1)}{}_{32} + u_c + u_d] = \frac{1}{4}[5 + 10 + 0 + 20] = 8.75$$

$$u^{(2)}{}_{12} = \frac{1}{4}[u_a + u_b + u^{(1)}{}_{22} + u^{(1)}{}_{11}] = \frac{1}{4}[10+40+10+7.5] = 16.875$$

$$u^{(2)}{}_{22} = \frac{1}{4}[u^{(1)}{}_{12} + u_b + u^{(1)}{}_{32} + u^{(1)}{}_{21}] = \frac{1}{4}[12.5+40+10+5] = 16.875$$

$$u^{(2)}{}_{32} = \frac{1}{4}[u^{(1)}{}_{22} + u_b + u_c + u^{(1)}{}_{31}] = \frac{1}{4}[10+40+0+5] = 13.75$$

Gauss-Seidel method:

Consider the linear system $A\mathbf{x} = \mathbf{b}$ One reason for slow convergence of Jacobi's method is that it doesn't make use of the latest information available. Gauss-Siedel method remedies this by using each new component of solution as soon as it has been computed.

Using the same notation as before, Gauss-Siedel method corresponds to splitting $M = D + L$, and $N = -U$ and can be written in matrix terms as:

$$A\mathbf{x} = \mathbf{b}$$
$$A = D - L - U$$
$$((D - L) - U)x = b$$

$$(D - L)x^{new} = Ux^{old} + b$$

$$x^{k+1} = (D - L)^{-1}x^k + (D - L)^{-1}b$$

$$x^{(k+1)} = T_g x^{(k)} + C_g$$

where T_g : is the Gauss-Seidel iteration matrix

C_g : constant vector

If we apply Gauss-Siedel method to solve the system of finite difference equations for Poisson's equation, we get:

$$U_{ij}^{(k+1)} = \frac{1}{4}(U_{i-1,j}^{(k+1)} + U_{i,j-1}^{(k+1)} + U_{i+1,j}^{k} + U_{i,j+1}^{k} - h^2 f(i,j)) \qquad \textbf{(3.4)}$$

Thus, we gain the average solution values at four surrounding grid points, the method uses the most recent values while Jacobi method uses values from the previous iteration. Gauss-Siedel method doesn't always converge, but it is guaranteed to converge under conditions that are often

satisfied in practice, and are weaker than those for Jacobi's method (e.g., if matrix is symmetric and positive definite).

Although Gauss-Siedel converges more rapidly than Jacobi, it is still not efficient for huge linear systems.

3.4 Successive Over - Relaxation:

Convergence rate of Gauss-Siedel can be accelerated by successive over – relaxation (SOR), which in effect uses step to next Gauss-Siedel iterate as search direction, but with a fixed search parameter denoted by (ω) .

Starting with $x^{(k)}$, first compute next iterate that would be given by Gauss-Siedel, x_{gs}^{k+1} instead of taking next iterate to be:

$$\begin{aligned} x^{K+1} &= x^{(K)} + \omega(x_{gs}^{(k+1)} - x^{(K)}) \\ &= (1-\omega)x^{(k)} + \omega x_{gs}^{(k+1)} \end{aligned} \qquad \textbf{(3.5)}$$

where ω is fixed relaxation parameter chosen to accelerate convergence

which weighs average of current iterate and next Gauss-Siedel iterates.

$\omega > 1$ gives over- relaxation, and $\omega < 1$ gives under- relaxation and $\omega = 1$ gives Gauss-Siedel method.

The method converges for $0 < \omega < 2$ and diverges otherwise. In general the optimal choice of ω is not easy task and it is subject to theory of special classes of matrices. If ρ_{jac} is the spectral radius of the Jacobi iteration (so that its square is the spectral radius of the Gauss-Siedel iteration), then the optimal choice for ω is given by:

$$\omega = \frac{2}{1+\sqrt{1-\rho^2{}_{jac}}} \qquad \textbf{(3.6)}$$

For this optimal choice, the spectral radius for (SOR) is

$$\rho_{SOR} = \left(\frac{\rho_{jac}}{1+\sqrt{1-\rho_{jac}^2}}\right)^2 \qquad \textbf{(3.7)}$$

How do we choose ω for a problem whose answer is not known analytically?

This is just the weak point of SOR! The advantages of SOR are obtained only in a fairly narrow window around the correct value of ω.

It is better to take ω slightly too large, rather than slightly too small, but best to get the optimal value..

For reference purposes, we give the value of ρ_{jac} for our model problem on a rectangular $(n\text{X}m)$ grid, allowing the possibility that $h \neq k$:

$$\rho_{jac} = \frac{\cos\frac{\pi}{n} + \left(\frac{h}{k}\right)^2 \cos\frac{\pi}{m}}{1 + \left(\frac{h}{k}\right)^2} \qquad \textbf{(3.8)}$$

Using the same notation as before, SOR method corresponds to splitting

$$M = \frac{1}{\omega}D + L, \qquad N = \left(\frac{1}{\omega} - 1\right)D - \omega$$

And can be written (see Smith, 1978; Mitchell, 1980). in matrix terms as:

$$x^{(K+1)} = K^{(x)} + \omega(D^{-1}(b - Lx^{(K+1)} - Ux^{(K)}) - x^{(K)})$$
$$= (D + \omega L)^{-1}((1-\omega)D - \omega U)x^{(K)} + \omega(D + \omega L)^{-1}b$$

If we apply SOR method to solve the system of finite difference equations for Laplace's equation, we get:

$$new\,u(i,j) = (1-\omega) * u(i,j) + 0.25 * \omega(u(i+1,j) + new\,u(i-1,j) + u(i.j+1) + new\,u(i,j-1)).$$

A necessary and sufficient condition for the convergence of iterative methods:

Each of three iterative methods described above can be written as:

$$x^{(k+1)} = Gx^k + c, \tag{3.9}$$

where G is iteration matrix and c a column vector of known values.

This equation was derived from the original equation by rearranging them into the form:

$$x = Gx + c \tag{3.10}$$

i.e., the unique solution of the m linear equations $Ax = f$ is the solution of equation (3.11). The error $e^{(n)}$ in the nth approximation to the exact solution is defined by $e^{(n)} = x - x^{(n)}$ so it follows by the substraction of equation (3.10) from equation (3.11) that

$$e^{(n+1)} = Ge^{(n)}$$

Therefore

$$e^{(n)} = Ge^{(n-1)} = G^2e^{(n-2)} = ... = G^{(n)}e^{(0)} \tag{3.11}$$

The sequence of iterative values $x^{(1)}, x^{(2)}, ..., x^{(n),}, ...$ will converge to x as n tends to infinity if

$\lim_{n\to\infty} G^{(n)} = 0$.

Assume now that the matrix G of order m has m linearly independent eigenvectors $v_{s,} s = 1(1)m.$ then these eigenvectors can be used as a basis for our m-dimensional vector space and the arbitrary error vector $e^{(0)}$ with its

m components, can be expressed uniquely as a linear combination of them, namely,

$$e^{(1)} = \sum_{s-1}^{m} c_s v_s$$

where the c_s are scalars. Hence

$$e^{(1)} = Ge^{(0)} = \sum_{s=1}^{m} c_s v_s .$$

But $Gv_s = \lambda_s v_s$ by the definition of an eigenvalue, where λ_s is the eigenvalue corresponding to v_s. Hence

$$e^{(1)} = \sum_{1}^{m} c_s \lambda_s v_s .$$

Similarly, $$e^{(n)} = \sum_{1}^{m} c_s \lambda_s v_s .$$

Therefore $e^{(n)}$ will tend to the null vector as n tends to infinity, for arbitrary $e^{(0)}$, if and only if $|\lambda_s| < 1$ for all s. In other words, the iteration will converge for arbitrary $x^{(0)}$ if and only if the spectral radius $\rho(G)$ of G is less than one .

As a corollary to this result a sufficient condition for convergence is that $\| G \| < 1$. To prove this we have that $Gv_s = \lambda_s v_s$. Hence

$$\| Gv_s \| = \| \lambda_s v_s \| = |\lambda_s| \, \| v_s \| .$$

But for any matrix norm that is compatible with a vector norm $\| v_s \|$,

$$\| Gv_s \| \leq \| G \| \, \| v_s \| .$$

Therefore $$|\lambda_s| \, \| v_s \| \leq \| G \| \, \| v_s \|,$$

So $$|\lambda_s| \leq \| G \|, \qquad s = 1(1)m.$$

It follows from this that a sufficient condition for convergence is that $\| G \| < 1$. It is not a necessary condition because the norm of G can exceed one even when $\rho(G) < 1.$

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3.5 Multigrid Method

Accuracy constraint on the numerical solutions of boundary value problems defined on large domains leads to huge systems of equations. Direct solvers are not efficient for such systems. Classical iterative methods also are not efficient. Multigrid methods are relatively new and they are very efficient. Multigrid methods accelerate iterative solvers. Historically, it was developed for elliptic problems which is mathematically well understood, but it is now used in many other situations.

Iterative methods start with an initial approximation for the solution to the differential equations. In each iteration, the difference between the approximate solution and the exact solution supposed to be smaller.

One can analyze this difference or error into components of different wavelength, for example, by using Fourier analysis. In general, the error will have components of many different wavelengths: there will be short wavelength error components and long wavelength error components.

Algorithms like Jacobi or Gauss-Seidel are local because the new value for the solution at any lattice site depends only on the value of the previous iterate at neighboring points. Such local algorithms are more efficient in reducing short wavelength error components.

The basic idea behind multigrid methods is to reduce long wavelength error components by updating blocks of grid points.

3.5.1 Multigrid method for Poisson's equation in 2-D:

With small change in notation, poisson's equation in

2-D can be written as:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \tag{3.12}$$

Where the unknown solution $u(x, y)$ is determined by the given source term $f(x, y)$ in a close region. Let's consider a square domain

$0 \le x$ and $y \le 1$ with homogenous Dirichlet boundary condition $u = 0$ on the perimeter of the square. The equation is discretized on a grid with $L + 2$ lattice points. i.e., L interior points and 2 boundary points, in the x and y directions. At any interior point, the approximation solution obeys

$$u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f_{i,j}) \tag{3.13}$$

The algorithm uses a succession of lattice or grid. The number of different grids is called the number of multigrid levels (l). The number of interior lattice points in the x and y directions is taken to be 2^l , so that $L = 2^l + 2,$, and the lattice spacing $h = \frac{1}{(L-1)}$, L is close in this manner so that the downward multigrid iteration can construct a sequence of coarser lattice with

$$2^{l-1} \to 2^{l-2} \to .... \to 2^0 = 1$$

interior points in the x and y directions.

Suppose that $u(x,y)$ is the approximate solution at any stage in the calculation, and $u_{exact}(x,y)$ is the exact solution which we are trying to find, the multigrid algorithm uses the following definitions:

1) The correction

$$v = u_{exact} - u \tag{3.14}$$

is the function which must be added to the approximate solution to give the exact solution.

2) The residual or defect is defined as:

$$r = \nabla^2 u + f \tag{3.15}$$

Notice that the correction and the residual are related by the equation:

$$\nabla^2 v = (\nabla^2 u_{exact} + f) - (\nabla^2 u + f) = -r \tag{3.16}$$

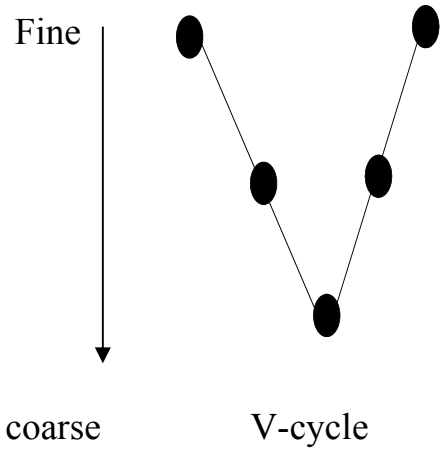
This equation has exactly the same form as Poisson's equation with (v), playing the role of unknown function and (r) playing the role of known source function.

3.5.2 Simple V-Cycle Algorithm

The simplest multigrid algorithm is based on a two-grid improvement scheme namely:

• A fine grid with $L = 2^l + 2$ points in each direction, and

• A coarse grid with $L = 2^{l-1} + 2$ points.



We need to be able to move from one grid to another, i.e., any given function on the lattice, we need to be able to:

• restrict the function from fine to coarse, and

• prolongate or interpolate the function from coarse to fine.

Given these definitions, the multigrid V -cycle can be defined recursively as follows:

• If $l = 0$ there is only one interior point, so solve exactly (or iteratively but exactly more accurate) for

$u_{1,1} = (u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - h^2 f_{1,1})/4..$

• Otherwise, calculate the current $L = 2^l + 2$

• Perform a few pre-smoothing iterations using a local algorithm such as Gauss-Seidel. The idea is to damp or reduce the short wavelength errors in the solution.

• Estimate the correction $v = u_{exact} - u$ as follows:

• Compute the residual

$$r_{i,j} = \frac{1}{h^2}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) + f_{i,j.} \quad \textbf{(3.17)}$$

- Restrict the residual $r \to R$ to the coarser grid.

- Set the coarser grid correction $V = 0$ and improve it recursively.

- Prolongate the correction $V \to v$ onto the finer grid.

- Correct $u \to u + v$

- Perform a few post-smoothing Gauss-Seidel (i) iterations and return the improved u .

How does this recursive algorithm scale with L? The pre-smoothing and post-smoothing Jacobi or Gauss-Seidel iterations are the most time consuming parts of the calculation. Recall that a single Jacobi or Gauss-Seidel iteration scales like $O(L^2)$. The operations must be carried out on the sequence of grids with

$$2^l \to 2^{l-1} \to 2^{l-2} \to ..... \to 2^0 = 1$$

interior lattice points in each direction. The total number of operations is of order

$$L^2 \sum_{n=0}^{l} \frac{1}{2^{2n}} \le \frac{4}{3} L^2 .$$

Thus the multigrid V -cycle scales like $O(L^2)$.i.e., linearly with the number of lattice points N.

3.5.3 Restricting the Residual to a coarser Lattice

The coarser lattice with spacing $H = 2h$ is constructed. A simple

algorithm for restricting the residual to the coarser lattice is to set its value to the average of the values on the four surrounding lattice points (cell-centered coarsening):

$$R_{i,j} = \frac{1}{4}(r_{i,j} + r_{i+1,j} + r_{i,j+1} + r_{i+1,j+1}), i = 2I - 1, j = 2J - 1. \quad \textbf{(3.18)}$$

3.5.4 Prolongation of the Correction to the Fine Lattice

Having restricted the residual to the coarser lattice with spacing $H = 2h$, we need to solve the equation

$$\nabla^2 V = -R(x, y),$$

With the initial guess $V(x, y) = 0.$ This is done by two-grid iteration

$$V = twoGrid(H, V, R).$$

The output must now be interpolated or prorogated to the finer lattice.

The simplest procedure is to copy the value of $V_{I,J}$ on the coarse lattice to the 4 neighboring cell points on the finer lattice:

$$v_{i,j} = v_{i+1,j} = v_{i+1,j+1} = v_{i,j+1} = V_{I,J}, i = 2I - 1, j = 2J - 1. \quad \textbf{(3.19)}$$

Chapter Four

Computational Results

4.1 Introduction

In this chapter Laplace's equation was treated, and it is solved by using some different methods.

Laplace's equation in two dimensions can be written as:

$$u_{xx} + u_{yy} = 0$$

$$S = \{(x, y) \mid 0 \le x \le a, 0 \le y \le b\} \quad ,$$

$$u(x, y) = g(x, y) \text{ on } \partial S$$

The above equation can be written after discretization as:

$$u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f(i, j)}{4} \qquad \textbf{(4.1)}$$

For example if we have a unit square $S = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ with Drichlet's boundary condition

$$U(0, y) = f_2, \quad U(1, y) = f_4.$$

$$U(x, 0) = f_3 \text{ and } U(x, 1) = f_1.$$

4.2 Numerical Algorithm

Jacobi Method (JM):

Input: end points a, b, c, d, integer m, *and* n, tolerance TOL, maximum number of iteration (k), f_1, f_2, f_3, f_4

Output: Approximation $U_{i,j}$ for each $i = 1,...,n-1$,and $j = 1,...,m-1$

Make initial guess for $U^{(0)}{}_{i,j}$ at all interior points

Step 1 set $h = 1/(n),$

$x = (0:n) * h,$

$y = (0:n) * h,$

Set $U_{i,j} = 0,$

Step 2 for $i = 1,...,n$

Set $U(1,n) = feval(f_1, h*(i-1),0)$

Set $U(n,i) = feval(f_2, h*(i-1),0)$

Stop

Step 3 for $j = 1,...,n$

Set $U(i,1) = feval(f_3, h*(i-1),0)$

Set $U(i,n) = feval(f_4, h*(i-1),0)$

Stop

Step4 set $U_1 = U$

Set $k\_\max = n\text{^}3$

Set TOL=1e-4

Step 5 set for $k = 1,...,k\_\max$

Set frac_diff = 0

Step 6 set for $i = 2,...,n-1$

Set for $j = 2, n-1$

Set $U_1 = U_{i,j}^{(k+1)} = \frac{1}{4}[U^k{}_{i+1,j} + U^k{}_{i-1,j} + U^k{}_{i,j+1} + U^k{}_{i,j-1}]$

Set frac_diff = frac_diff +abs $(U_1(i,j) - U(i,j))$

Stop

Stop

step 7 set change(k) = frac_diff/(n-2)^2

set if rem(k,100) < 1

Stop

Step 8 set if change (k) < TOL

Set break

Stop.

See the Matlab code in appendix (**B.1**)

Gauss Seidel (GSM):

Input: end points a,b,c,d ,integer $m, and \quad n$,tolerance TOL, maximum number of iteration(k), f_1, f_2, f_3, f_4

Output: Approximation $U_{i,j}$ for each $i = 1,...,n-1$,and $j = 1,...,m-1$

Make initial guess for $U^{(0)}{}_{i,j}$ at all interior points

Step 1 set $h = 1/(n),$

$x = (0:n) * h,$

$y = (0:n) * h,$

set $U_{i,j} = 0,$

step 2 for $i = 1,...,n$

set $U(1,n) = feval(f_1, h*(i-1),0)$

set $U(n,i) = feval(f_2, h*(i-1),0)$

stop

step 3 for $j = 1,...,n$

set $U(i,1) = feval(f_3, h*(i-1),0)$

set $U(i,n) = feval(f_4, h*(i-1),0)$

stop

step4 set $U_1 = U$

set $k\_\max = n\text{^}3$

set TOL=1e-4

step 5 set for $k = 1,...,k\_\max$

set frac_diff $= 0$

step 6 set for $i = 2,...,n-1$

set for $j = 2,n-1$

set $U_1 = U_{i,j}{}^{(k+1)} = \frac{1}{4}[U^k{}_{i+1,j} + U^{(k+1)}{}_{i-1,j} + U^k{}_{i,j+1} + U^{(k+1)}{}_{i,j-1}]$

set frac_diff = frac_diff +abs($U_1(i,j) - U(i,j)$)

stop

stop

step 7 set change(k) = frac_diff/(n-2)^2

set if rem(k,100) < 1

stop

step 8 set if change(k) < TOL

set break

stop.

See the Matlab code in appendix **(B.2**)

Successive Over-Relaxaion Mehod(SORM)

1. Make initial guess for $U_{i,j}$ at all interior points (I,j)

2. Define a scalar $w\,(0 < w < 2).$

3. Apply equation (4.1) to all interior points (i,j) and call it $U'_{i,j}$.

4. $U_{i,j}^{(k+1)} = wU'_{i,j} + (1-w)U^{K}_{i,j}$

5. Stop if prescribed convergence threshold is reached

6. $U_{i,j}^{\ k} = U^{k+1}{}_{i,j}$

7. Go to step 2.

See the Matlab code in appendix **(B.3)**

Multigrid Method (MGM):

Input:charge(q), the number of multigrid levels l

Set the number of interior lattice in x and y to be 2^l

Set $N = 2^l + 2$

Set $h = \dfrac{1}{(N)}$

Define $u = twogrid(level, h, u, f)$

1. If $l = 0$,there is only one interior point, so solve exactly for:

$u_{2,2} = (u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} + h^2 f_{2,2})/4.$

2. Calculate the current $N = 2^l + 2$

3. Estimate the correction $v = u_{exact} - u$ as follows:

- Compute the residual $r = \nabla^2 u + f$

- Restrict the residual $r \to R$ to a coarse grid.

- Set the coarse grid correction $V = 0.$

-Improve the correction recursively $V = twogrid(l-1,2h,V,R).$

- Prolongate the correction $V \to v$ onto the fine grid.

4. Correct $u \to u + v$

Set r=zeros(N,N)

Set for i=2,…,N

Set for j=2,…,N

r(i,j)=(u(i+1,j)+u(i-1,j)+u(i,j+1)+u(i,j-1)-4*u(i,j))/h^2+f(i,j)

stop

stop

set i=2(I-1), j=2(J-1)

$$R_{I,J} = \frac{1}{4}[r_{i,j} + r_{i,j+1} + r_{i+1,j} + r_{i+1,j+1}]$$

Set $M = 2^{(level-1)} + 2$

Set R=zeros(M,M)

Set for I=2,…,M

Set for J=2,…,MSet R(I,J)=(r(i,j)+ r(i+1,j)+ r(I,j+1)+r(i+1,j+1)/4

Stop

Stop.

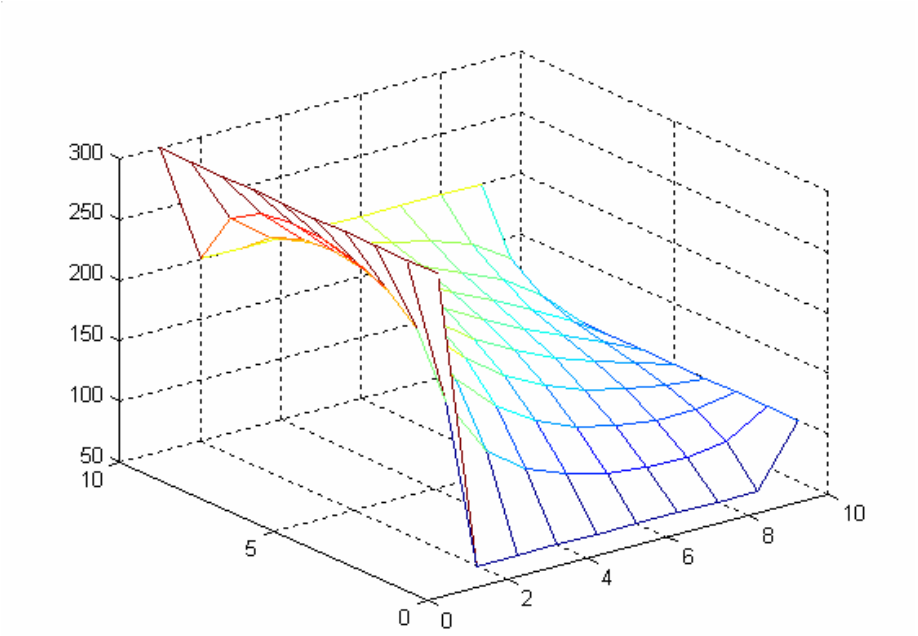
See the Matlab code in appendix (**B.4**)

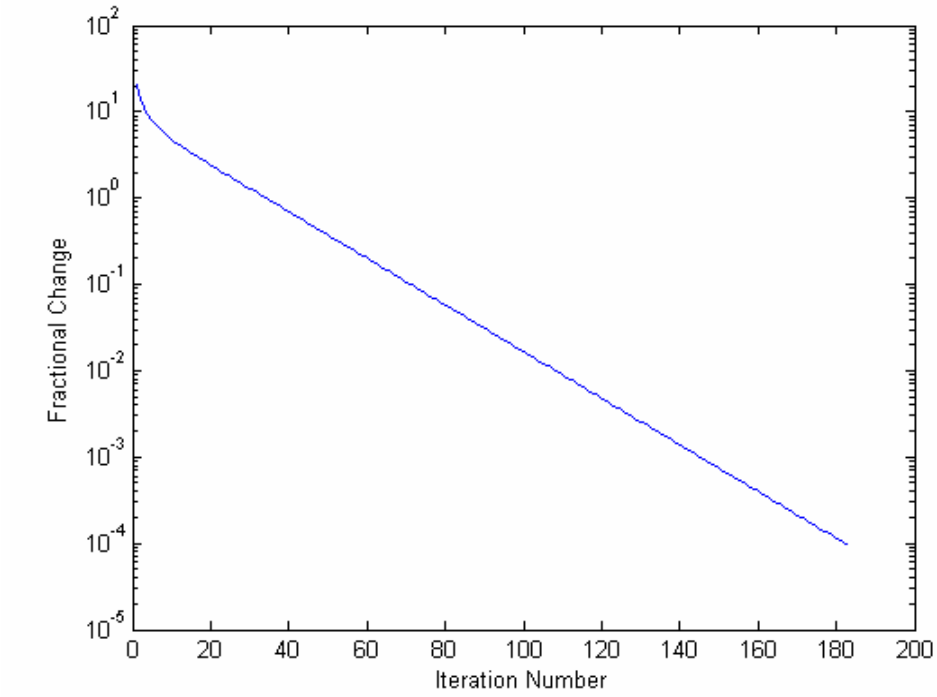
Now, let's begin with constant boundaries

$f_1 = 100, \quad f_2 = 50$ and $n = m = 10$

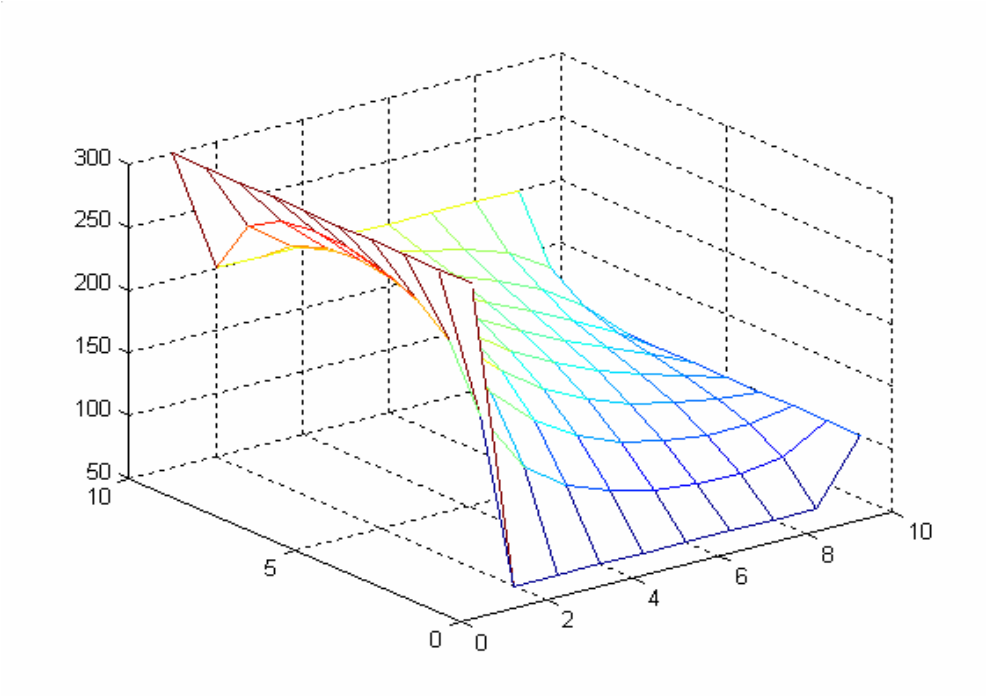
$f_3 = 300, \quad f_4 = 200.$

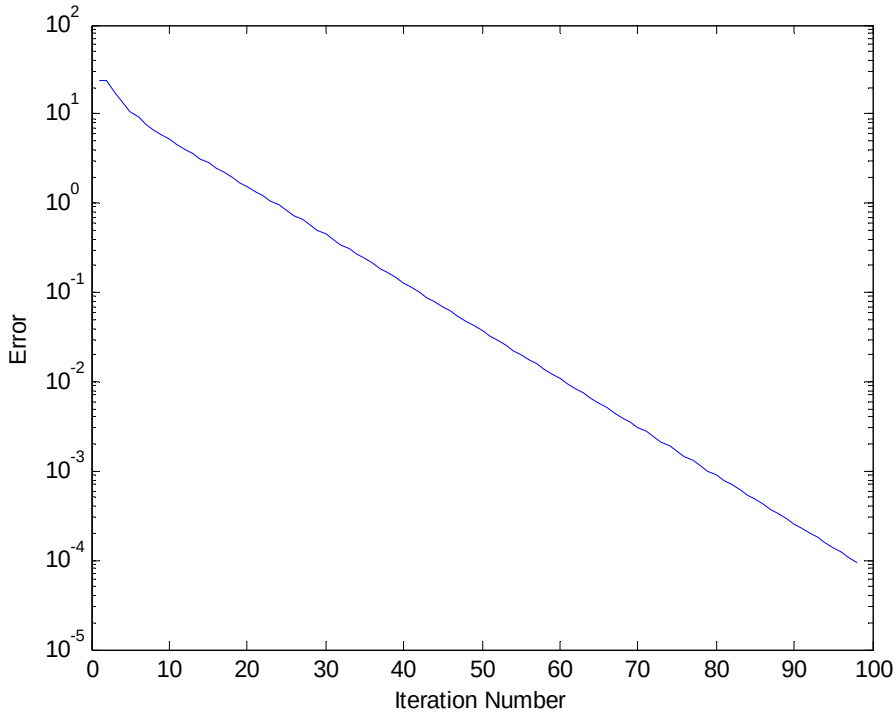
Method (JM):



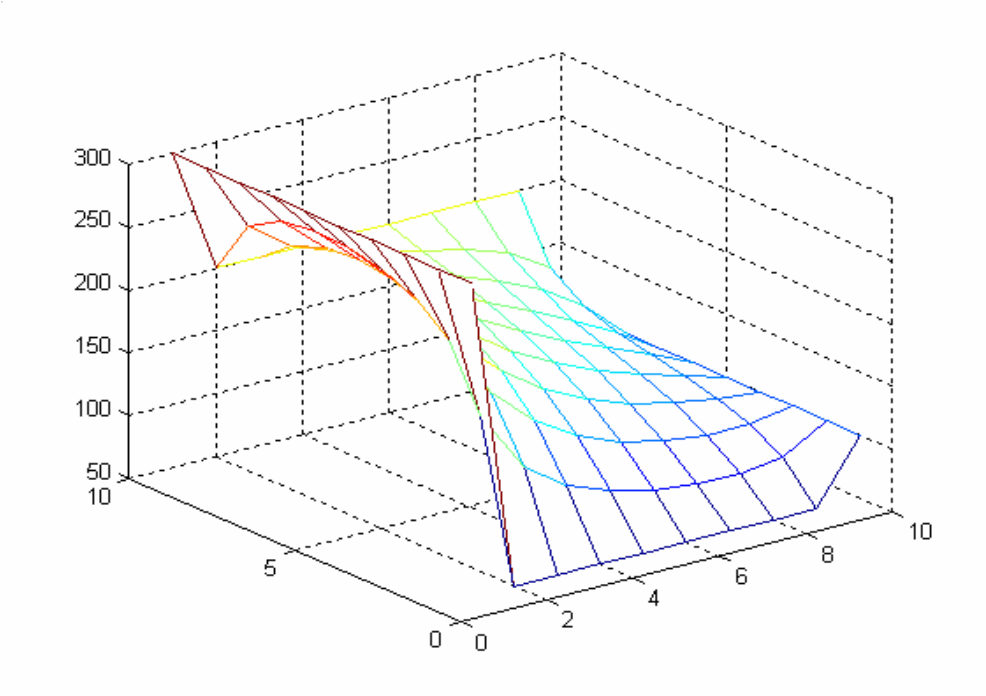


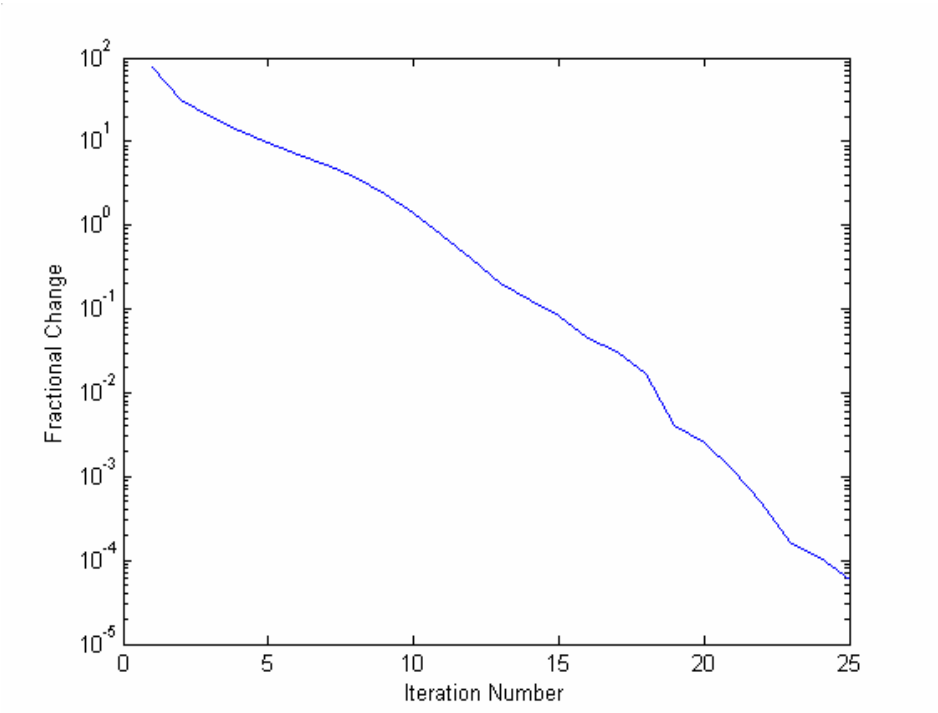
Method (GSM):


300
250
200
150
100
50
10
5
0
0
2
4
6
8
10


Error
Iteration Number
0
10
20
30
40
50
60
70
80
90
100

Method SORM:



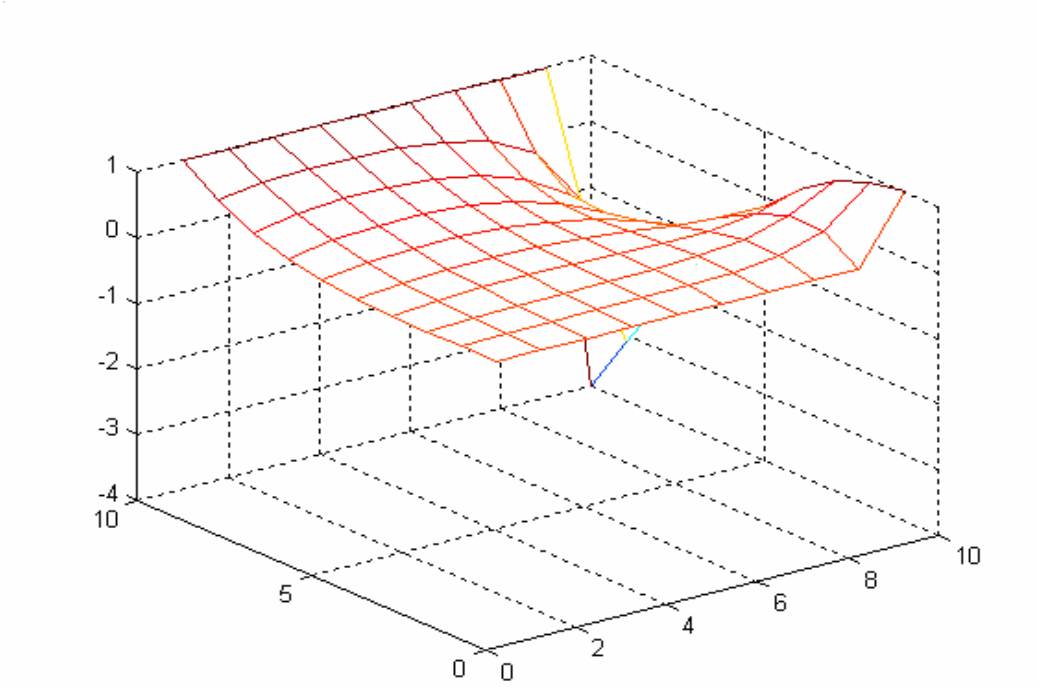


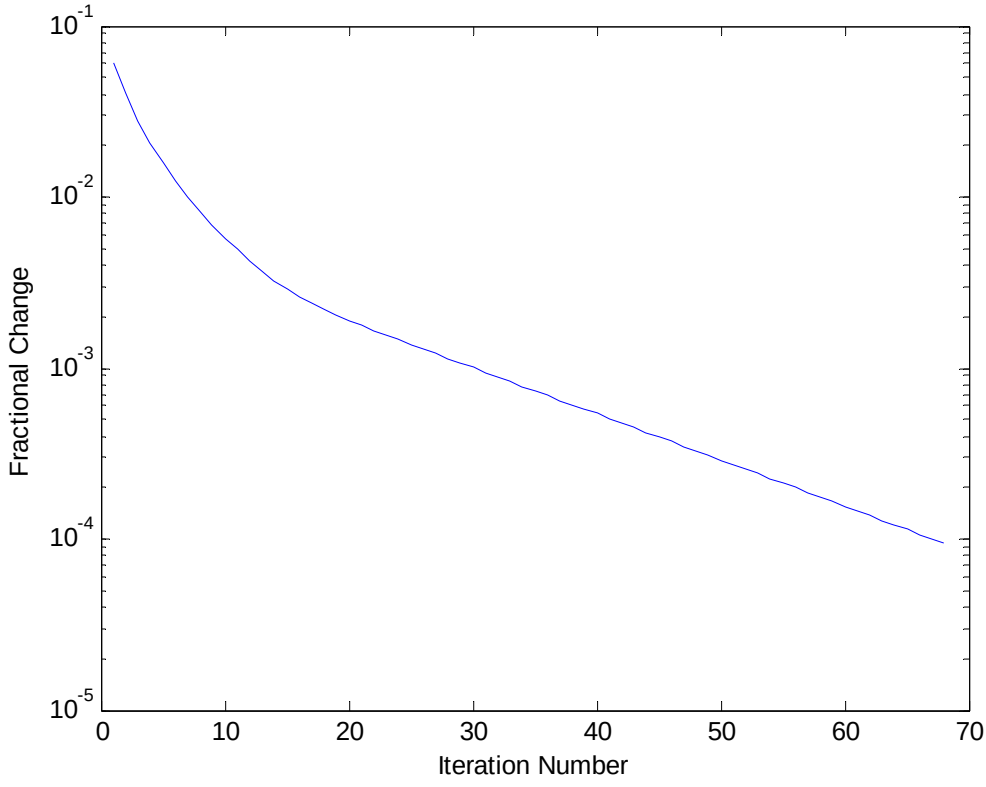
Assume that the boundaries are function with respect to x and y.

$f_1 = y^4$, $f_2 = y^4 - 6 * y^2 + 1$ and $n = m = 10$

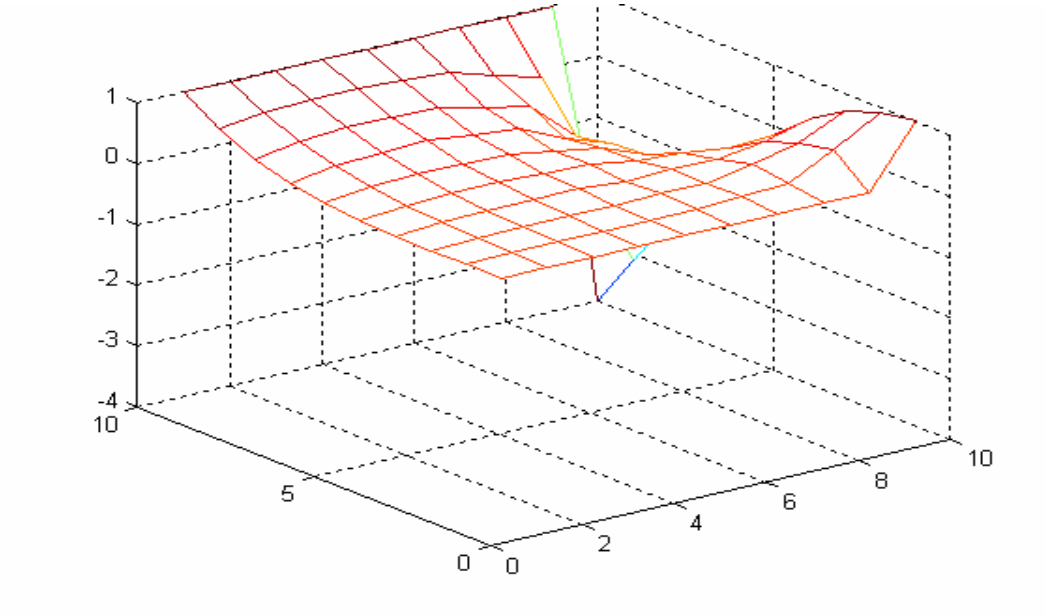
$f_3 = x^4$, $f_4 = x^4 - 6 * x^2 + 1$.

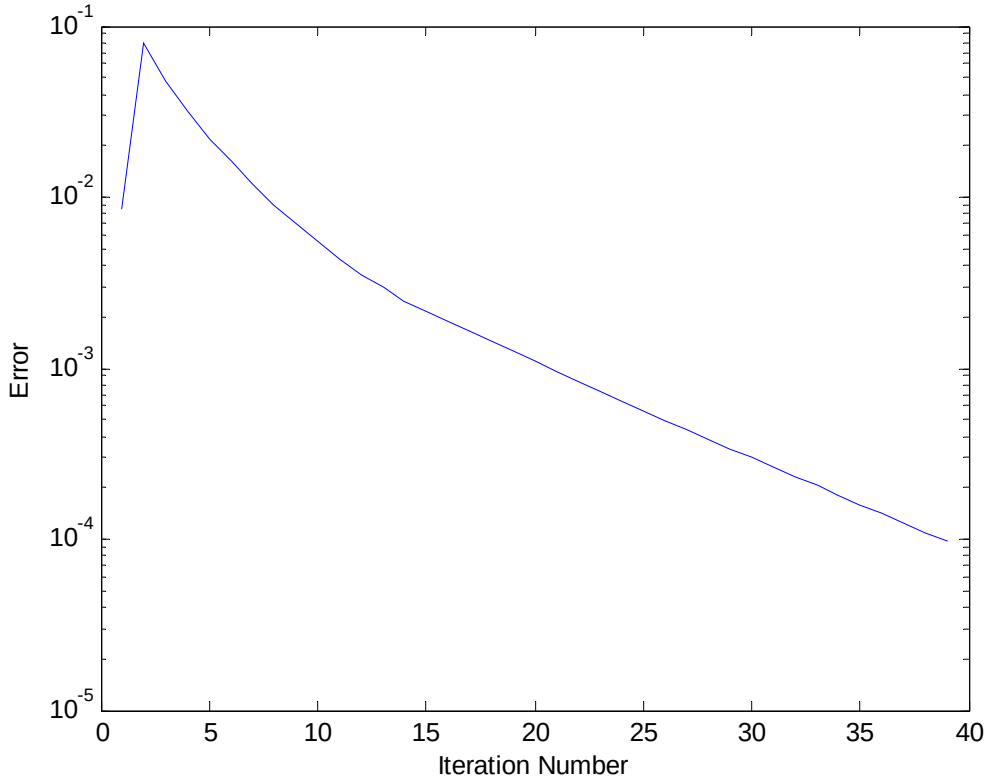
Method JM:


1
0
-1
-2
-3
-4
10
5
0
0
2
4
6
8
10

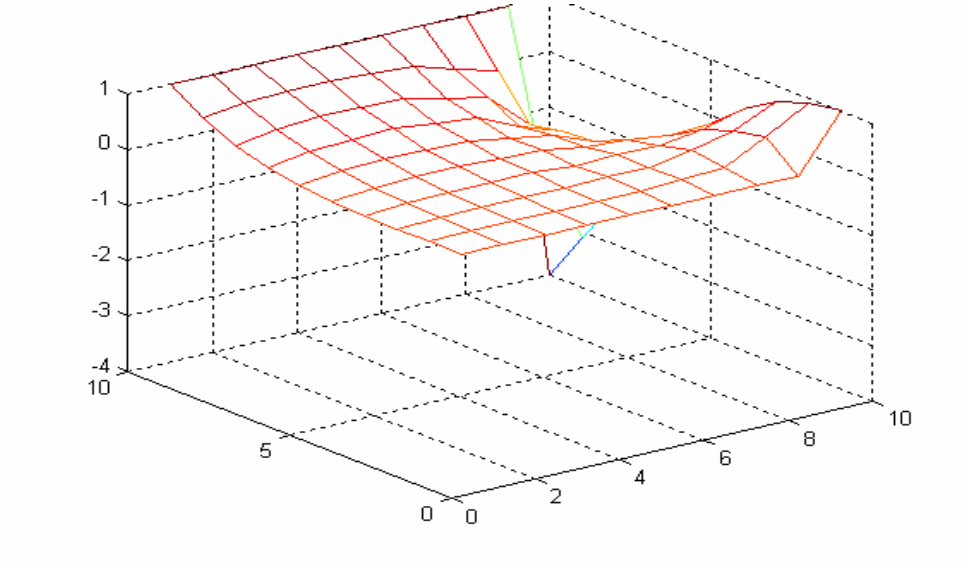

Fractional Change
10^-1
10^-2
10^-3
10^-4
10^-5
0
10
20
30
40
50
60
70
Iteration Number

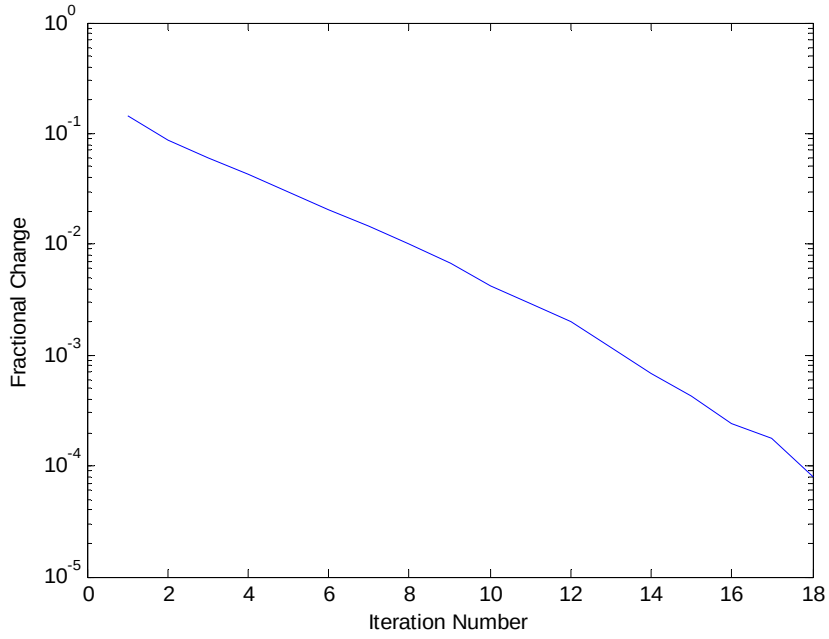
Method GSM:


1
0
-1
-2
-3
-4
10
5
0
0
2
4
6
8
10


Error
Iteration Number
10^-1
10^-2
10^-3
10^-4
10^-5
0
5
10
15
20
25
30
35
40

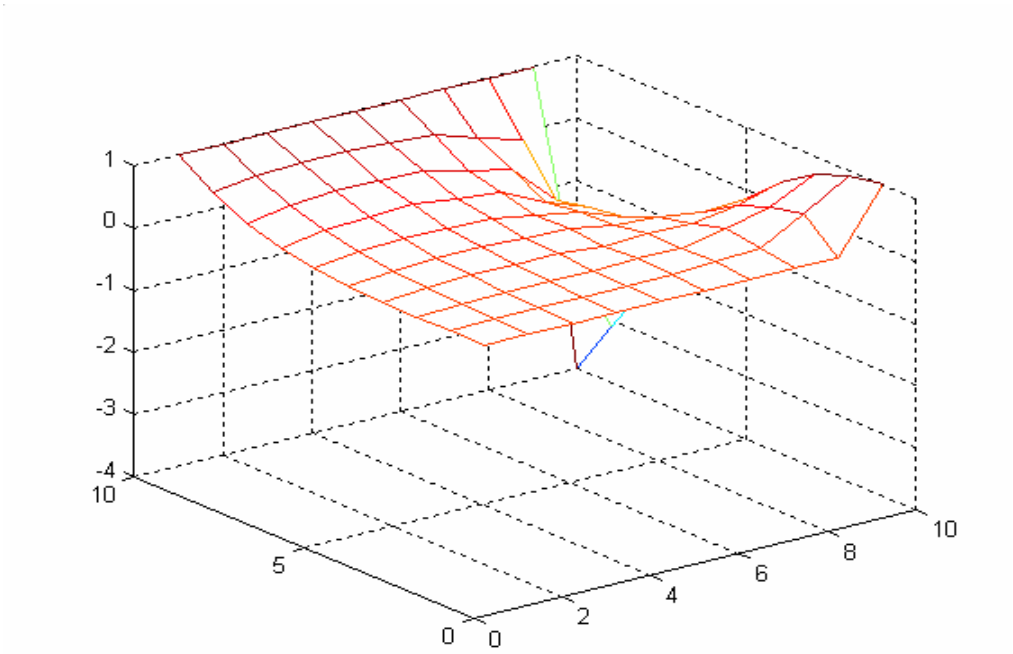
Method SORM:

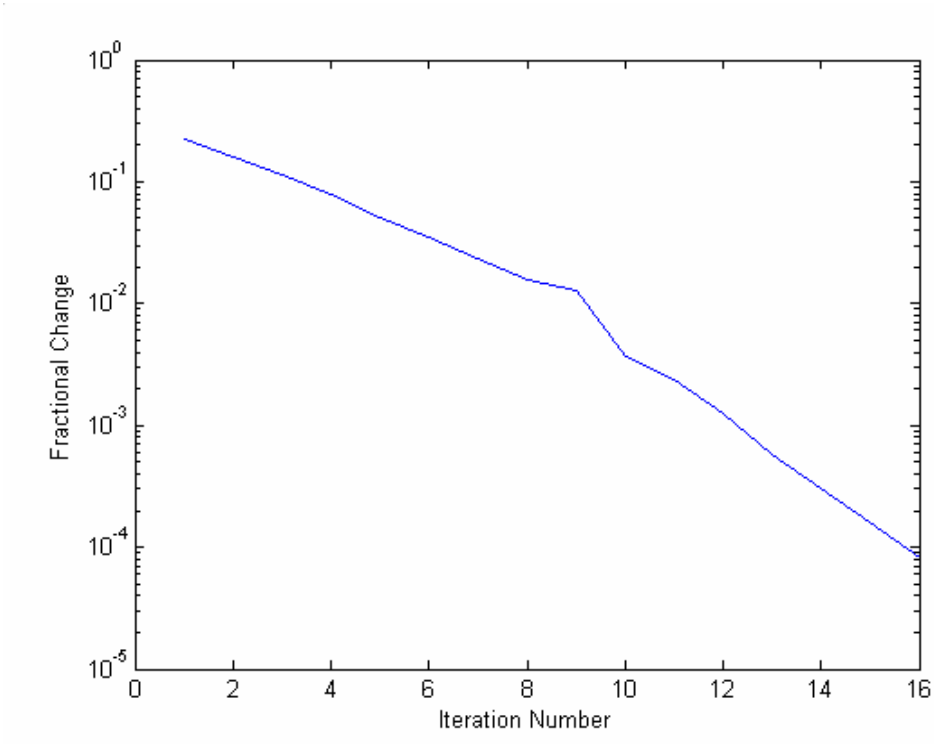

1
0
-1
-2
-3
-4
10
5
0
0
2
4
6
8
10


Fractional Change
Iteration Number
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
0
2
4
6
8
10
12
14
16
18

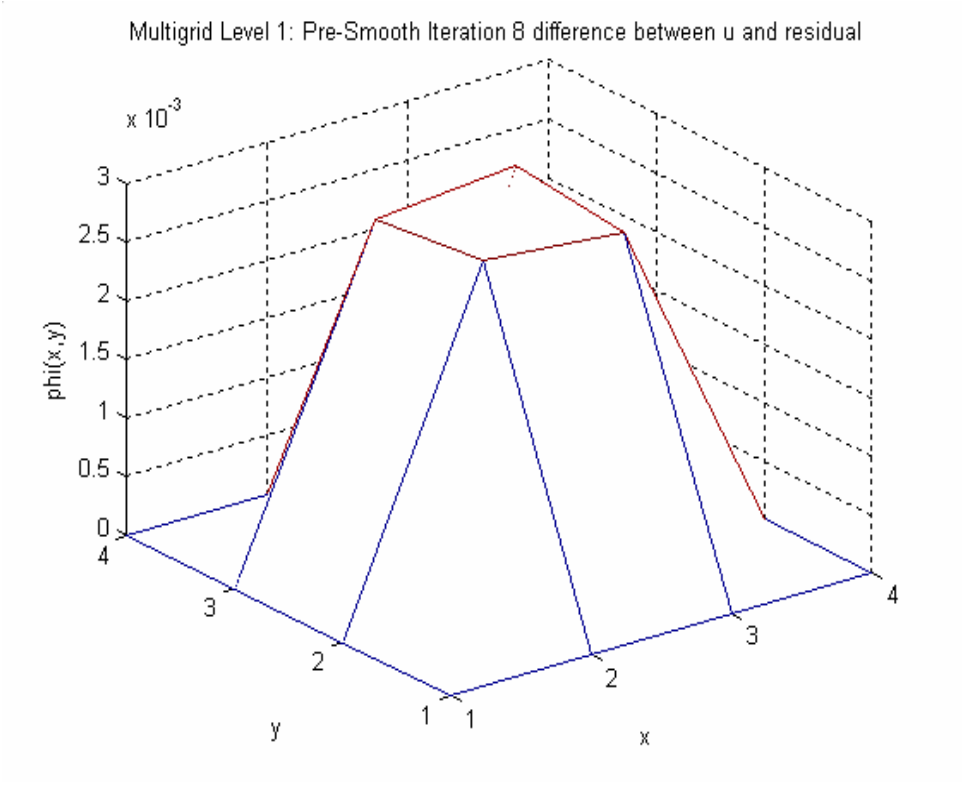
If we use the rotated five points Stencil (RFPS)

For the same conditions we get:

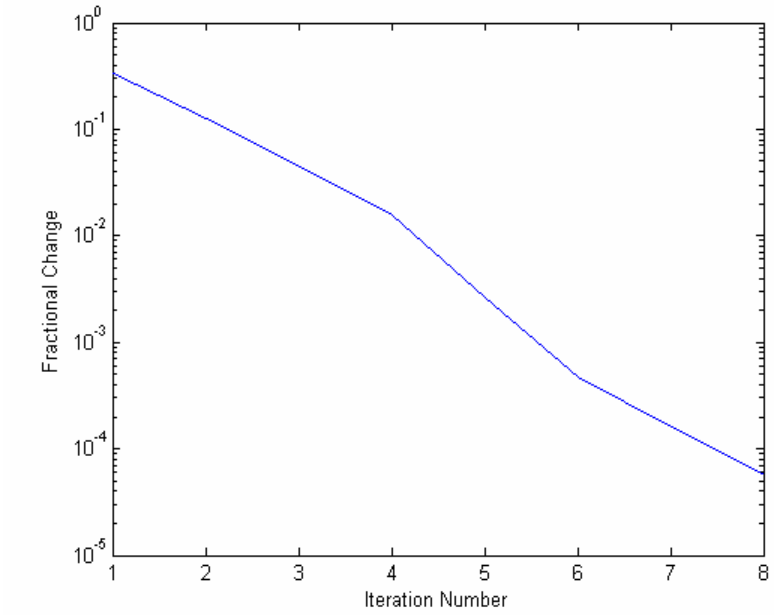




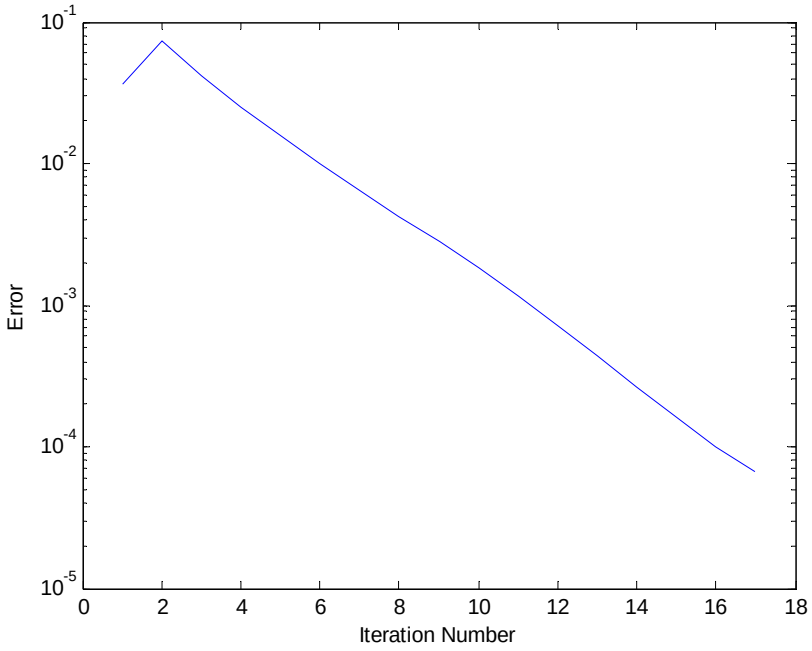
Method MGM:


Multigrid Level 1: Pre-Smooth Iteration 8 difference between u and residual
x 10^-3
3
2.5
2
1.5
1
0.5
0
phi(x,y)
4
3
2
1
y
1
2
3
4
x

:


10^0
10^-1
10^-2
10^-3
10^-4
10^-5
Fractional Change
1
2
3
4
5
6
7
8
Iteration Number

If we use nine-points approximation for the above conditions, we obtain:



4.3 Poisson Equation

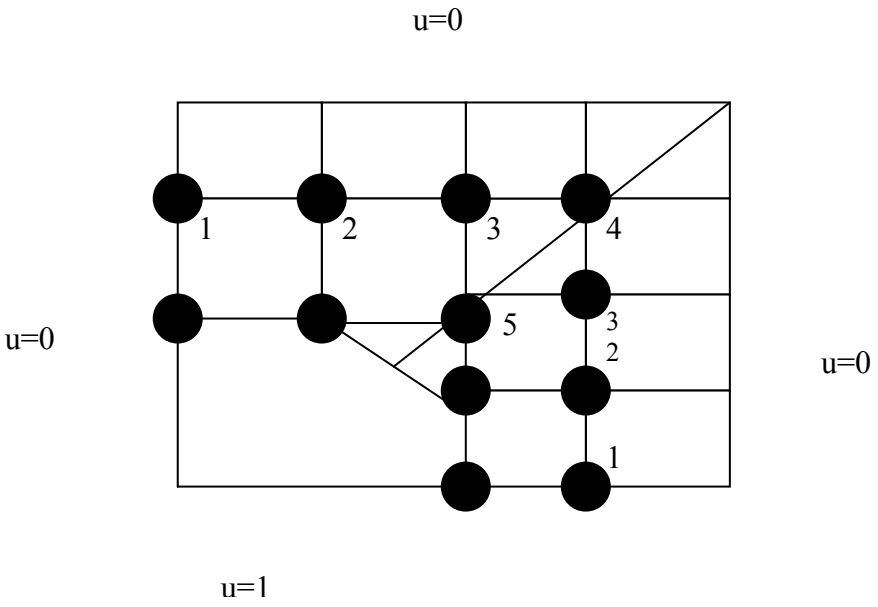
In this section Poisson's equation was treated.

- Consider Poisson's equation:

$$u_{xx} + u_{yy} = -2$$

$S = \{(x, y) \mid 0 \le x \le 2, 0 \le y \le 2\}$,

with boundary conditions as shown



The above equation can be written as:

$$u(i, j) = \frac{1}{4}(u(i+1, j) + u(i-1, j) + u(i, j+1) + u(i, j-1)) + \frac{1}{2}$$

for node (1): $1 + u_2 - 4u_1 + u_2 + 0 + \frac{1}{2} = 0$

for node(2): $1 + u_1 - 4u_2 + u_3 + 0 + \frac{1}{2} = 0$

for node(3): $u_5 + u_2 - 4u_3 + u_4 + 0 + \frac{1}{2} = 0$

for node(4): $u_3 + u_3 - 4u_4 + 0 + 0 + \frac{1}{2} = 0$

for node(5): $1 + 1 - 4u_5 + u_3 + u_3 + \frac{1}{2} = 0$

The above equations can be written as a matrix form:

$$A\,\underline{\mathbf{u}} = \underline{\mathbf{b}}$$

where $A = \begin{bmatrix} -4 & 2 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 0 & 2 & -4 & 0 \\ 0 & 0 & 2 & 0 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} \frac{-3}{2} \\ \frac{-3}{2} \\ \frac{-1}{2} \\ \frac{-1}{2} \\ \frac{-5}{2} \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$

When the above work is done, we obtain:

$$\underline{\mathbf{u} = \begin{bmatrix} 0.737 \\ 0.723 \\ 0.658 \\ 0.454 \\ 0.954 \end{bmatrix}}$$

When numerical methods are used to solve the above problems we get:

The number of iteration for Jacobi method is=16

norm_inf error for Jacobi method is=0.0001680128

The solution by Jacobi method is:

0.9538

0.4538

0.6577

0.7236

0.7368

The number of iteration for Gauss Seidel method is=9

norm_inf error for Gauss Seidel method is=0.0001462412

The solution by Gauss Seidel method is:

0.9538

0.4538

0.6578

0.7236

0.7368

w =1.1920

The number of iteration for SOR method is=5

norm_inf error for SORmethod is=0.0001347322

The solution by SOR method is:

0.9539

0.4538

0.6580

0.7237

0.7369

The solution of u(x,y) is

0	0	0	0	0	0	0	0	0	0	0
0	0.4539	0.6579	0.7237	0.7368	0.7237	0.6579	0.4539	0		
0	0.6579	0.9539	1.0000	1.0000	1.0000	0.9539	0.6579	0		
0	0.7237	1.0000	1.0000	1.0000	1.0000	1.0000	0.7237	0		
0	0.7368	1.0000	1.0000	1.0000	1.0000	1.0000	0.7368	0		
0	0.7237	1.0000	1.0000	1.0000	1.0000	1.0000	0.7237	0		
0	0.6579	0.9539	1.0000	1.0000	1.0000	0.9539	0.6579	0		
0	0.4539	0.6579	0.7237	0.7368	0.7237	0.6579	0.4539	0		
0	0	0	0	0	0	0	0	0	0	0

- **Helmholtz Equation**

Consider Helmholtz equation**:**

$$u_{xx} + u_{yy} - 12.5\pi^2 u = -25\pi^2 \sin(\frac{5}{2}\pi\, x)\sin(\frac{5}{2}\pi\, y)$$

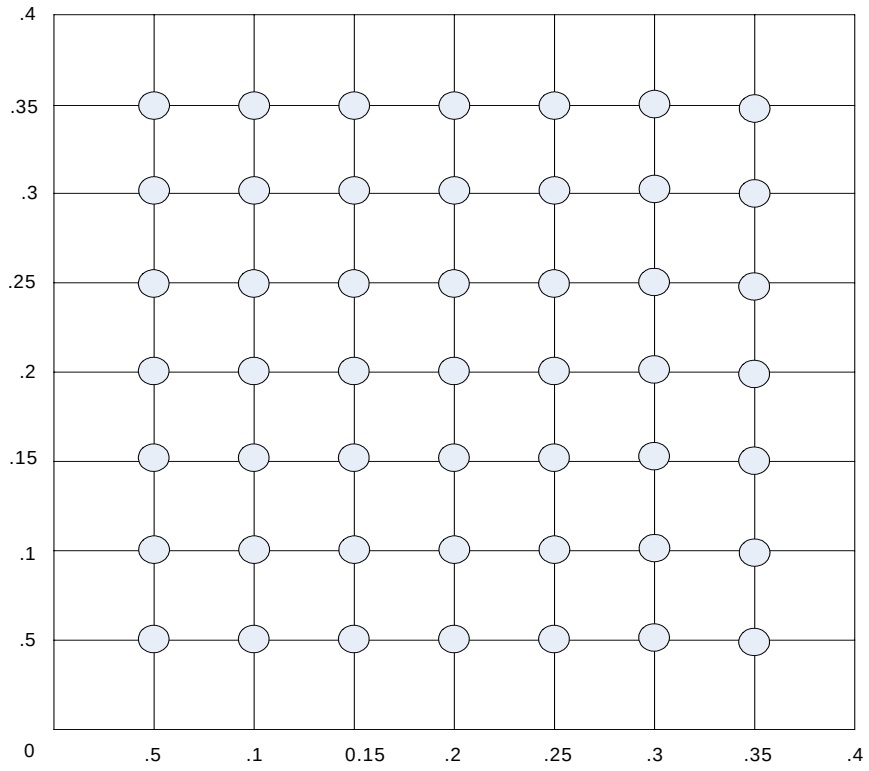
in regular region $s = \{x, y \mid \quad 0 < x, y < 0.4\}$

subject to the Dirichlet boundary condition:

$$u(x, y) = 0 \quad on \;\; \partial s$$

To compare numerical solution with exact solution, the exact is:

$$u(x,y) = \sin(\frac{5}{2}\pi\, x)\sin(\frac{5}{2}\pi\, y)$$



The above equation can be written as:

$$u(i,j-1) + u(i-1,j) - (4 - 0.125\pi^2)u(i,j) + u(i+1,j) + u(i,j+1)$$

$$= -25\pi^2 \sin(\frac{5}{2}\pi x_i)\sin(\frac{5}{2}\pi\, y)$$

Let $r = -4 - 0.125\pi^2$

We can write the above equation in a matrix form as:

$$A\,\underline{\mathbf{u}} = \underline{\mathbf{b}}$$

where $A = \begin{bmatrix} B & I & 0 \\ I & B & I \\ 0 & 0 & B \end{bmatrix}$, $B = \begin{bmatrix} r & 1 & 0 \\ 1 & r & 1 \\ 0 & 1 & r \end{bmatrix}$

$$\mathbf{b} = \begin{bmatrix} -.25\pi^2 \sin(2.5\pi x_1)\sin(2.5\pi y_{i)}) \\ -25\pi^2 \sin(2.5\pi x_2)\sin(2.5\pi y_{1)}) \\ -25\pi^2 \sin(2.5\pi x_3)\sin(2.5\pi y_{1)}) \\ -25\pi^2 \sin(2.5\pi x_1)\sin(2.5\pi y_2) \\ -25\pi^2 \sin(2.5\pi x_2 \sin(2.5\pi y_2) \\ -25\pi^2 \sin(2.5\pi x_3)\sin(2.5\pi y_2) \\ -25\pi^2 \sin(2.5\pi x_1)\sin(2.5\pi y_3) \\ -25\pi^2 \sin(2.5\pi x_2)\sin(2.5\pi y_3) \\ -25\pi^2 \sin(2.5\pi x_3)\sin(2.5\pi y_3) \end{bmatrix}$$

see the Matlab code for this problem in appendix **(B.9)**

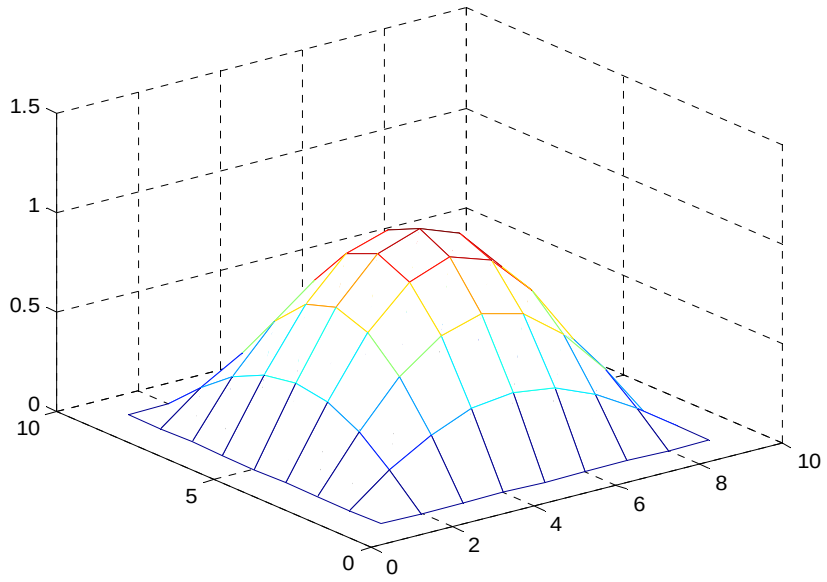
when the above system is solved, we get:

norm_inf error for Jacobi method is =0.00007

The number of iteration for Jacobi method is =33

0	0	0	0	0	0	0	0	0	0
0	0.1465	0.2706	0.3536	0.3827	0.3536	0.2706	0.1465	0	
0	0.2706	0.5000	0.6533	0.7072	0.6533	0.5000	0.2706	0	
0	0.3536	0.6533	0.8536	0.9239	0.8536	0.6533	0.3536	0	

0	0.3827	0.7072	0.9239	1.0001	0.9239	0.7072	0.3827	0
0	0.3536	0.6533	0.8536	0.9239	0.8536	0.6533	0.3536	0
0	0.2706	0.5000	0.6533	0.7072	0.6533	0.5000	0.2706	0
0	0.1465	0.2706	0.3536	0.3827	0.3536	0.2706	0.1465	0
0	0	0	0	0	0	0	0	0



norm_inf error for Gauss Seidel method is =0.00238

The number of iteration for Gauss Seidel method is =17

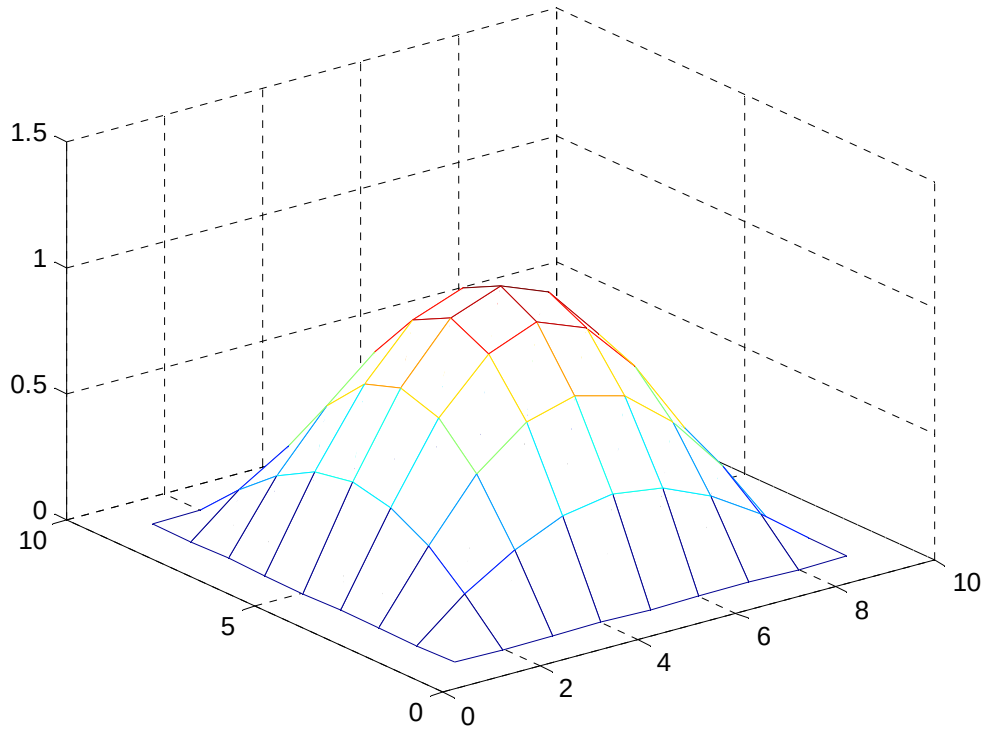
0	0	0	0	0	0	0	0	0	0	0
0	0.1485	0.2734	0.3561	0.3844	0.3543	0.2719	0.1474	0		
0	0.2734	0.5036	0.6563	0.7093	0.6566	0.5031	0.2723	0		
0	0.3561	0.6563	0.8564	0.9289	0.8585	0.6573	0.3558	0		
0	0.3844	0.7093	0.9289	1.0058	0.9295	0.7116	0.3851	0		

0 0.3543 0.6566 0.8585 0.9295 0.8589 0.6574 0.3558 0

0 0.2719 0.5031 0.6573 0.7116 0.6574 0.5032 0.2723 0

0 0.1474 0.2723 0.3558 0.3851 0.3558 0.2723 0.1474 0

0 0 0 0 0 0 0 0 0 0 0



w = 1.4795

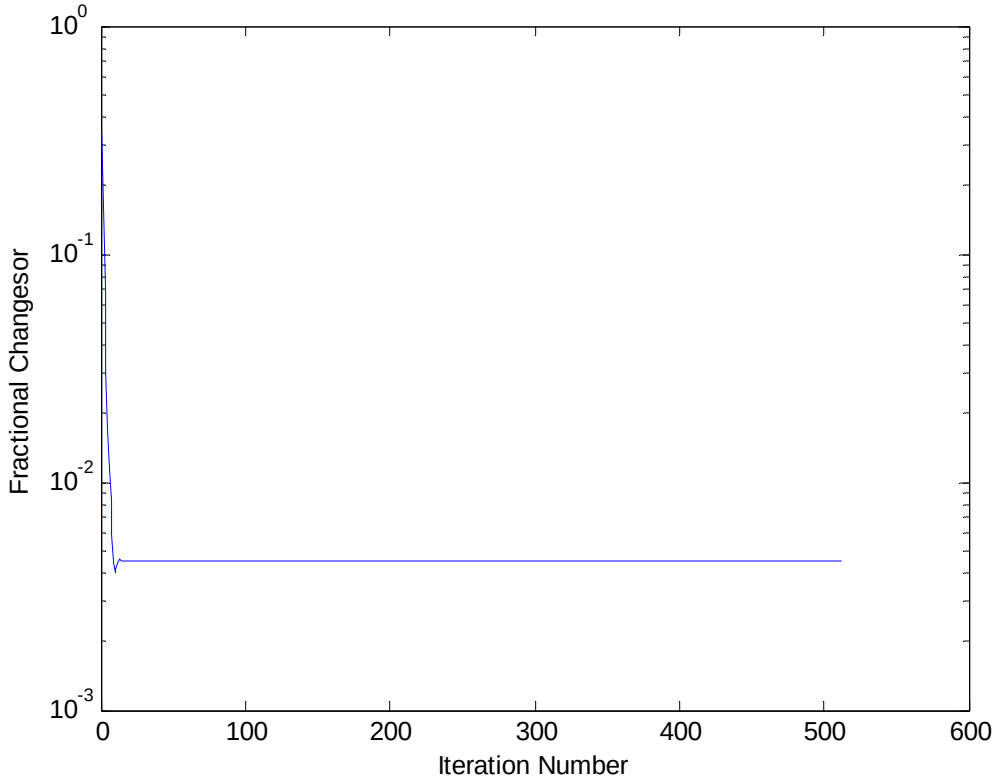
norm_inf error for sorl method is =0.00580

The number of iteration for SOR method is =10

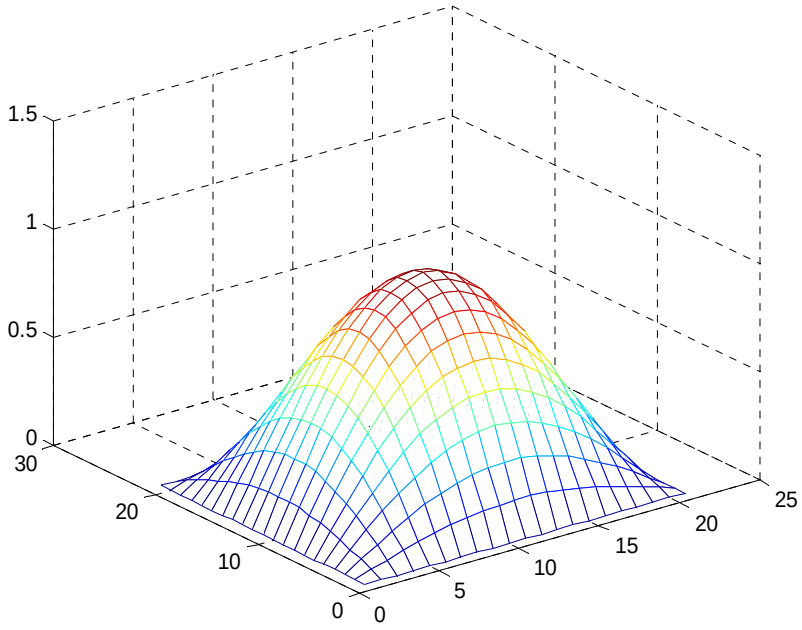
0 0 0 0 0 0 0 0 0 0 0

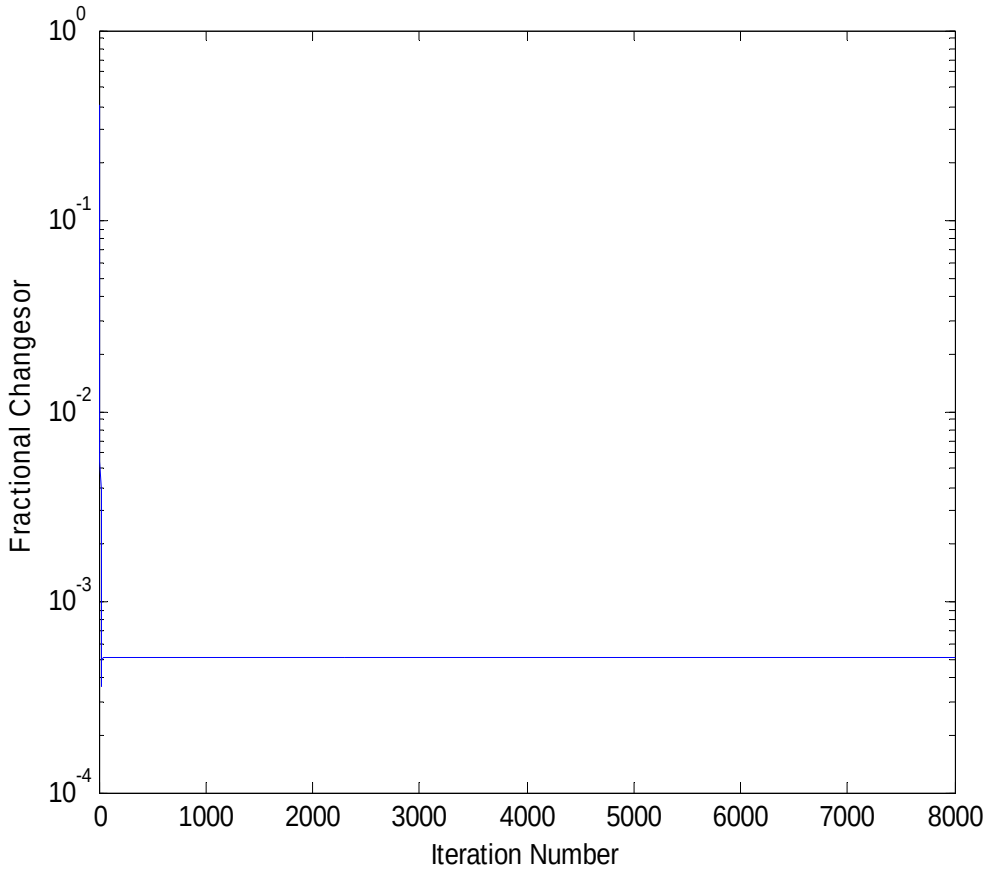
0 0.1485 0.2734 0.3561 0.3844 0.3543 0.2719 0.1474 0

0	0.2734	0.5036	0.6563	0.7093	0.6566	0.5031	0.2723	0
0	0.3561	0.6563	0.8564	0.9289	0.8585	0.6573	0.3558	0
0	0.3844	0.7093	0.9289	1.0058	0.9295	0.7116	0.3851	0
0	0.3543	0.6566	0.8585	0.9295	0.8589	0.6574	0.3558	0
0	0.2719	0.5031	0.6573	0.7116	0.6574	0.5032	0.2723	0
0	0.1474	0.2723	0.3558	0.3851	0.3558	0.2723	0.1474	0
0	0	0	0	0	0	0	0	0



If $h = k = 0.02$ we get:


1.5
1
0.5
0
30
20
10
0
0
5
10
15
20
25

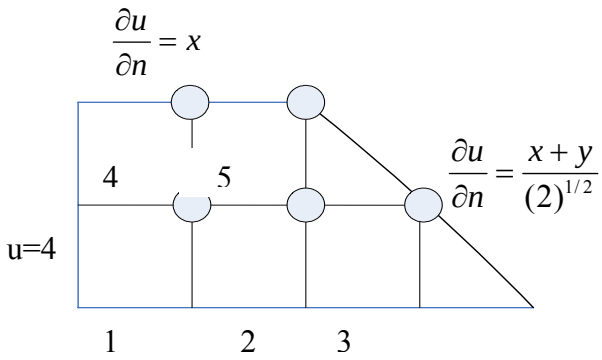

Fractional Changesor
Iteration Number
0
1000
2000
3000
4000
5000
6000
7000
8000

Finally, consider the problem by using Neuman condition:

Let's solve Laplace's equation

$$u_{xx} + u_{yy} = 0$$

The boundaries shown in the figure below:



u=4

Laplace's equation can be written as:

$$u(i, j-1) + u(i-1, j) - 4u(i, j) + u(i+1, j) + u(i, j+1) = 0$$

For node (1): $-4u_1 + u_2 + u_4 = -8$

For node(2): $u_1 - 4u_2 + u_3 + u_5 = -4$

For node(3): $u_2 - 4u_3 + C_1 + C_2 = -4$

To find C_1 *and* C_2 by using central difference formula:

$$\frac{\partial u}{\partial x} = \frac{C_1 - U_2}{2h} \quad \Rightarrow \quad C_1 = u_2 + 2h\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{C_2 - 4}{2k} \quad \Rightarrow \quad C_2 = 4 + 2k\frac{\partial u}{\partial y}$$

$$C_1 + C_2 = u_2 + 2h\frac{\partial u}{\partial x} + 4 + 2k\frac{\partial u}{\partial y}$$

for: $h = k = 0.1$

$$= u_2 + 2h(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}) + 4$$

but $\frac{\partial u}{\partial n} = \frac{x + y}{(2)^{1/2}}$

$$= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$$

$$= u_2 + 2h(\frac{x + y}{(2)^{1/2}}) + 4$$

$$= u_2 + 2(.1)(\frac{.3 + .1}{(2)^{1/2}}) + 4$$

$$= u_2 + 4.0565685$$

$$u_2 - 4u_3 + u_2 + 4.0565685 = -4$$

$$-4u_3 + 2u_2 = -8.0565685$$

for node(4): $u_1 - 4u_4 + u_5 + C_1 = -4$

to find C_1, we use the same procedure then:

$$C_1 = u_1 + 0.02$$

$$2u_1 - 4u_4 + u_5 = -4.02$$

for node (5): $u_2 + u_4 + -4u_5 + C_1 + C_2 = 0$

$$C_1 + C_2 = u_4 + u_2 + 0.0565685$$

$u_2 + u_4 + -4u_5 + u_4 + u_2 = -0.0565685$

The above equations can be written as a matrix form:

$$A\underline{\mathbf{u}} = \underline{\mathbf{b}}$$

where $A = \begin{bmatrix} -4 & 1 & 0 & 1 & 0 \\ 1 & -4 & 1 & 0 & 1 \\ 0 & 2 & -4 & 0 & 0 \\ 2 & 0 & 0 & -4 & 1 \\ 0 & 2 & 0 & 2 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -8 \\ -4 \\ -8.0565685 \\ -4.02 \\ -0.0565685 \end{bmatrix}$

see the Matlab code in appendix **(B.10)**

If we solve the above system by using numerical methods we obtain:

The solution by matrix

4.0078

4.0147

4.0215

4.0163

4.0297

The number of iteration for jacobi method is=100

norm of error for jacobi method is=0.0000000001

The solution by Jacobi's Method is

4.0078

4.0147

4.0215

4.0163

4.0297

---

the number of iteration for Gauss Seidel method is=50

norm of error for Gauss Seidel method is=0.0000000002

The solution by Gauss Seidel Method is

4.0078

4.0147

4.0215

4.0163

4.0297

ω =1.5279

The number of iteration for SOR method is=35

norm of error for SOR method is=0.0000000005

The solution by SOR Method is

4.0078

4.0147

4.0215

4.0163

4.0297

The solution of u(x,y) is=

4.0000	4.0163	4.0297	0	0
4.0000	4.0078	4.0147	4.0215	0
4.0000	4.0000	4.0000	4.0000	4.0000

4.4 Laplace's equation in Polar Coordinate

Now, let's deal with Laplace equation in a polar coordinate

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

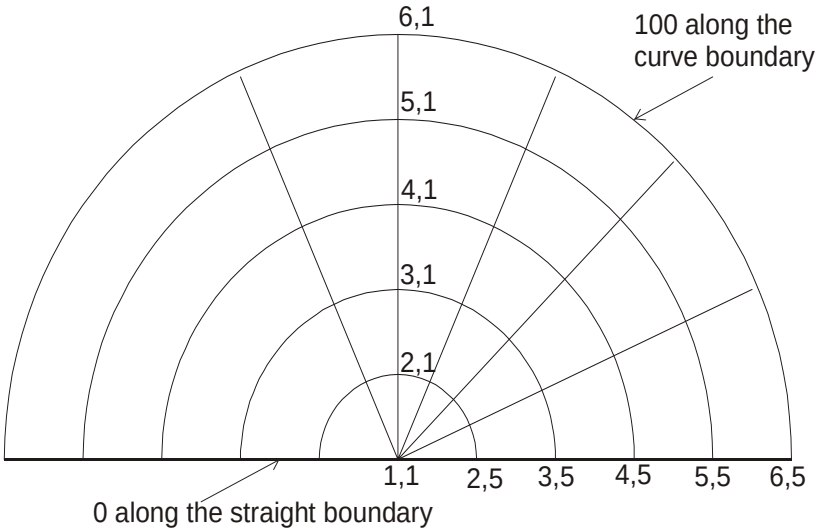
Let i and j be the index in the r and θ directions respectively, the finite difference form of the Laplace equation is written as:

$$\nabla^2 u = \frac{u(i-1,j) - 2u(i,j) + u(i+1,j)}{\Delta r^2} + \frac{1}{r_i}\frac{u(i+1,j) - u(i-1,j)}{2\Delta r} + \frac{1}{r_i^2}\frac{u(i,j-1) - 2u(i,j) + u(i,j+1)}{\Delta \theta^2} = 0$$

The value at the node (i, j) can be computed from

$$2\left[\frac{1}{\Delta r^2}+\frac{1}{r_i^{\,2}\Delta\theta}\right]u(i,j)=\frac{u(i-1,j)+u(i+1,j)}{\Delta r^2}+\frac{1}{r_i}\,\frac{u(i+1,j)-u(i-1,j)}{2\Delta r}$$
$$+\frac{1}{r_i^{\,2}}\,\frac{u(i,j-1)+u(i,j+1)}{\Delta\theta^2}$$

A semicircular plate with a radius of 1 has the straight boundary held at 0 while the curve boundary is held at 100. Find $u(r, \theta)$ with the grid given in Figure



The nodes in a two dimensional, polar coordinate system.

See the Matlab code for this problem in appendix **(B.11)**

Of iteration =46

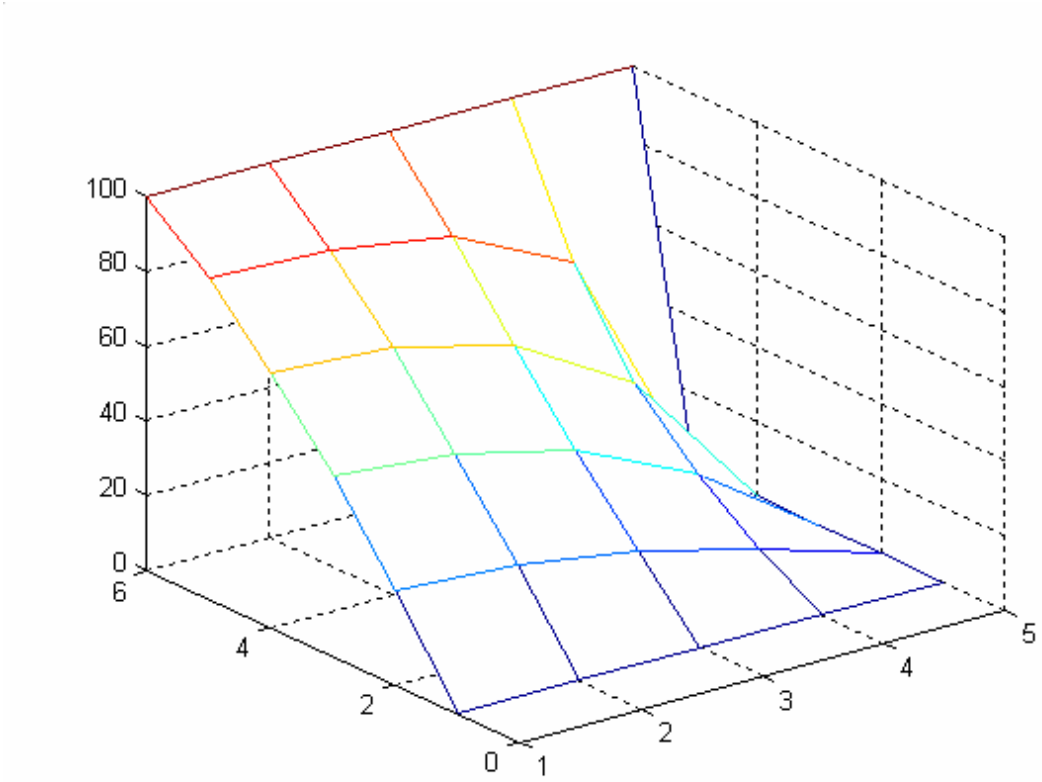
u =

0	0	0	0	0
2.4986e+001	2.3284e+001	1.8232e+001	1.0139e+001	0
4.8038e+001	4.5534e+001	3.7450e+001	2.2370e+001	0

6.8366e+001 6.6121e+001 5.8055e+001 3.9163e+001 0

8.5659e+001 8.4422e+001 7.9457e+001 6.3678e+001 0

1.0000e+002 1.0000e+002 1.0000e+002 1.0000e+002 1.0000e+002



4.5 Conclusion and Results

Various numerical methods namely: Jacobi, Gauss-Seidel, SOR, Multigrid have been studied in order to compare the efficiency of these methods. It is desirable to develop the rapid method to obtain the accurate solutions, since the systems of equations are becoming large and many computer resources are needed.

From the above work, we see that multigrid method is the fastest. On other hand, the running time for multigrid methods is of order three (i.e. $O(n^3)$), while the other methods is of order five (i.e. $O(n^5)$).

So , multigrid method is superior to other methods, because it speeds up the convergence of relaxation method. Multigrid method is known as the procedure which takes only $O(n)$ operations

Appendix

Programs

This part contains some programs used to solve Laplace's equation in different conditions

B .1

Parent routine for Method JM in case one

```
% Programme to solve Laplace's equation on the unit square

% using the Jacobi's method

Clear; help Jacobi;

n = input('Enter the number of grid points on each side-');

h = 1/(n - 1);

x = (0:n-1)*h;

y = (0:n-1)*h;

u = zeros(n);

u_x0 = input('Enter the potential at x=0 -');

u(1,:) = u_x0;

u_x1 = input('Enter the potential at x=1 -');

u(n,:) = u_x1;

u_y0 = input('Enter the potential at y=0 -');
```

```
u(:,1) = u_y0;

u_y1 = input('Enter the potential at y=1 -');

u(:,n) =u_y1;

u1 = u;

k_max = n^3;

tolerance = 1.e-04;

for k = 1:k_max

    frac_diff = 0;

    for i = 2:n-1

    for j = 2:n-1

  u1(i,j) = 0.25*(u(i+1,j) + u(i-1,j) +u(i,j+1) +u(i,j-1));

            frac_diff = frac_diff + abs(u1(i,j)-u(i,j));

        end

    end

    u = u1;

    change(k) = frac_diff/(n-2)^2;

    if rem(k,100) < 1

        fprintf('Fractional  difference  is  %g  after  %g
steps\n',change(k),k);
```

```
    end
    if change(k) < tolerance
        break;
    end
end
u
mesh(u);
pause;
semilogy(change);
xlabel('Iteration Number');ylabel('Fractional Change');
```

B .2 Parent routine for Method GSM case one

```
% gseidel - Programme to solve Laplace's equation on the
unit square
% using the Gauss Seidel method
clear; help gseidel;
n = input('Enter the number of grid points on each side -
');
h = 1/(n - 1);
x = (0:n-1)*h;
```

```
y = (0:n-1)*h;

u = zeros(n);

u_x0 = input('Enter the potential at x=0 -');

u(1,:) = phi_x0;

u_x1 = input('Enter the potential at x=1 -');

u(n,:) = u_x1;

u_y0 = input('Enter the potential at y=0 -');

u(:,1) =u_y0;

u_y1 = input('Enter the potential at y=1 -');

u(:,n) = u_y1;

newu = u;

k_max = n^3;

tolerance = 1.e-04;

for k = 1:k_max

    frac_diff = 0;

    for i = 2:n-1

    for j = 2:n-1

  newu(i,j)  =  0.25*(u(i+1,j)  +  newu(i-1,j)  +u(i,j+1)  +
newu(i,j-1));
```

```
            frac_diff = frac_diff + abs(newu(i,j)-u(i,j));
        end
    end
    u = newu;
    change(k) = frac_diff/(n-2)^2;
    if rem(k,100) < 1
  fprintf('Fractional difference is %g after %g
steps\n',change(k),k);
    end
    if change(k) < tolerance
        break;
    end
end
u
mesh(u);
pause;
semilogy(change);
xlabel('Iteration Number');ylabel('Fractional Change');
```

B .3 Parent routine for Method SORM case one

```
% sor - Programme to solve Laplace's equation on the unit
square

% using the Successive Overrelaxation method

format ;

clear; help sor;

n = input('Enter the number of grid points on each side -
');

h = 1/(n - 1);

x = (0:n-1)*h;

y = (0:n-1)*h;

omega = 2/(1 + sin(pi/n));

u = zeros(n);

u_x0 = input('Enter the potential at x=0 -');

u(1,:) = u_x0;

u_x1 = input('Enter the potential at x=1 -');

u(n,:) = u_x1;

u_y0 = input('Enter the potential at y=0 -');

u(:,1) = u_y0;
```

```
u_y1 = input('Enter the potential at y=1 -');

u(:,n) = u_y1;

newu = u;

k_max = n^3;

tolerance = 1.e-04;

for k = 1:k_max

    frac_diff = 0;

    for i = 2:n-1

        for j = 2:n-1

            newu(i,j)     =     (1     -     omega)*u(i,j)
+0.25*omega*(u(i+1,j) + newu(i-1,j) +u(i,j+1) + newu(i,j-
1));

            frac_diff = frac_diff + abs(newu(i,j)-u(i,j));

        end

    end

    u= newu;

    change(k) = frac_diff/(n-2)^2;

    if rem(k,100) < 1

        fprintf('Fractional  difference  is  %g  after  %g
steps\n',change(k),k);
```

```
    end
    if change(k) < tolerance
        break;
    end
end
u
mesh(u);
pause;
semilogy(change);
xlabel('Iteration Number');ylabel('Fractional Change');
```

B .4 Parent routine for Method JM case two

```
% sor - Programme to solve Laplace's equation on the unit
square
% using the Successive Overrelaxation method
format ;
clear; help sor;
n = input('Enter the number of grid points on each side -
');
h = 1/(n - 1);
```

```
x = (0:n-1)*h;

y = (0:n-1)*h;

omega = 2/(1 + sin(pi/n));

u = zeros(n);

u_x0 = input('Enter the potential at x=0 -');

u(1,:) = u_x0;

u_x1 = input('Enter the potential at x=1 -');

u(n,:) = u_x1;

u_y0 = input('Enter the potential at y=0 -');

u(:,1) = u_y0;

u_y1 = input('Enter the potential at y=1 -');

u(:,n) = u_y1;

newu = u;

k_max = n^3;

tolerance = 1.e-04;

for k = 1:k_max

    frac_diff = 0;

    for i = 2:n-1

        for j = 2:n-1
```

```
            newu(i,j)     =     (1     -     omega)*u(i,j)
+0.25*omega*(u(i+1,j)+ newu(i-1,j)+u(i,j+1) + newu(i,j-1));

            frac_diff = frac_diff + abs(newu(i,j)-u(i,j));

        end

    end

    u= newu;

    change(k) = frac_diff/(n-2)^2;

    if rem(k,100) < 1

        fprintf('Fractional  difference  is  %g  after  %g
steps\n',change(k),k);

    end

    if change(k) < tolerance

        break;

    end

end

u

mesh(u);

pause;

semilogy(change);
```

```
xlabel('Iteration Number');ylabel('Fractional Change');
```

B .5 Parent routine for Method GSM case two

```
% gseidel - Programme to solve Laplace's equation on the
unit square
% using the Gauss Seidel method
format;
clear; help gseidel;
n= input('Enter the number of grid points on each side -');
f1 = inline('y^4+0*x');
f2 = inline('y^4-6*y^2+1+0*x');
f3 = inline('0*y+x^4');
f4 = inline('0*y+x^4-6*x^2+1');
h = 1/(n - 1);
x = (0:n-1)*h;
y = (0:n-1)*h;
u = zeros(n);
for i=1:n
   u1(1,i)=feval(f1,h*(i-1),0);
```

```
   u1(n,i)=feval(f2,h*(i-1),0);
end


for i=1:n
   u1(i,1)=feval(f3,h*(i-1),0);
   u1(i,n)=feval(f4,h*(i-1),0);
end
k_max = n^3;
tolerance = 1.e-04;
for k = 1:k_max
    changesum = 0;
    for i = 2:n-1
        for j = 2:n-1
   %u1(i,j)=0.25*(u(i+1,j)+u1(i-1,j)+u(i,j+1) + u1(i,j-1));
  u1(i,j)=0.25*(u1(i-1,j-1)+u1(i-+u1(i+1,j+1)+u1(i+1,j-1));
            changesum =changesum  + abs(u1(i,j)-u(i,j));
        end
   end
```

```
    u = u1;
    change(k) =changesum /(n-2)^2;
    if rem(k,100) < 1
        fprintf('Error is %g after %g steps\n',change(k),k);
    end
    if change(k) < tolerance
        break;
    end
end
u
mesh(u);
pause;
semilogy(change);
xlabel('Iteration Number');ylabel('Error');
```

B .6 Parent routine for Method SORM case two

```
% sor - Programme to solve Laplace's equation on the unit square
```

```
% using the Successive Overrelaxation method

format ;

clear; help sor;

n= input('Enter the number of grid points on each side -');

f1 = inline('y^4+0*x');

f2 = inline('y^4-6*y^2+1+0*x');

f3 = inline('0*y+x^4');

f4 = inline('0*y+x^4-6*x^2+1');

h = 1/(n - 1);

x = (0:n-1)*h;

y = (0:n-1)*h;

omega = 2/(1 + sin(pi/n));

u = zeros(n);



for i=1:n

   u(1,i)=feval(f1,h*(i-1),0);

   u(n,i)=feval(f2,h*(i-1),0);

end

for i=1:n
```

```
   u(i,1)=feval(f3,h*(i-1),0);

   u(i,n)=feval(f4,h*(i-1),0);

end

newu = u;

k_max = n^3;

tolerance = 1.e-04;

for k = 1:k_max

    frac_diff = 0;

    for i = 2:n-1

        for j = 2:n-1

           newu(i,j)    =    (1    -    omega)*u(i,j)    +
0.25*omega*(u(i,j+1)+newu(i-1,j) + u(i-1,j) + newu(i,j-1));

            %   newu(i,j)   =   (1   -   omega)*u(i,j)   +
0.25*omega*(newu(i+1,j+1) + newu(i-1,j-1) + newu(i-1,j+1) +
newu(i+1,j-1));

            frac_diff = frac_diff + abs(newu(i,j)-u(i,j));

        end

    end

    u= newu;

    change(k) = frac_diff/(n-2)^2;
```

```
    if rem(k,100) < 1
        fprintf('Fractional difference is %g after %g
steps\n',change(k),k);
    end
    if change(k) < tolerance
        break;
    end
end
u
mesh(u);
pause;
semilogy(change);
xlabel('Iteration Number');ylabel('Fractional Change');
```

B.7

```
function multigrid
% multigrid.m
% solves the two dimensional Poisson equation
% d^2phi/dx^2 + d^2phi/dy^2 = - rho(x,y)
% phi = 0 on the boundaries of a unit square
```

```
% using a simple multigrid method

clear all; help multigrid;



f1 = inline('yy^4+0*xx');

f2 = inline('yy^4-6*yy^2+1+0*xx');

f3 = inline('0*yy+xx^4');

f4 = inline('0*yy+xx^4-6*xx^2+1');




levels = input('Enter number of multigrid levels: ');

N = 2^levels + 2;                       % lattice points in
x and y

disp(sprintf('Using a %i x %i square lattice', N, N));

h = 1/(N-1);                            % lattice spacing



% set charge density rho(x,y)

rho = zeros(N,N);

nq = 1;

while 1
```

```
  disp(sprintf('To add charge number %i at lattice site
x,y', nq));
  q = input('Enter charge q or hit RETURN to contine: ');
  if isempty(q), break, end
  user_entry = input('Enter lattice coordinates [x y]: ');
  if length(user_entry) == 2
    x = user_entry(1);
    y = user_entry(2);
  end
  rho(x,y) = q / h^2;
  nq = nq + 1;
end
phi = zeros(N,N);                          % potential array
n=N;
h = 1/(n - 1);
xx = (0:n-1)*h;
yy = (0:n-1)*h;
for i=1:n
   phi(1,i)=feval(f1,h*(i-1),0);
```

```
   phi(n,i)=feval(f2,h*(i-1),0);

end

for i=1:n

   phi(i,1)=feval(f3,h*(i-1),0);

   phi(i,n)=feval(f4,h*(i-1),0);

end

figure;

[phi] = twogrid(levels, h, phi, rho);     % start multigrid
iteration

plotphi(phi, sprintf('Multigrid Iteration With %i Levels
Done', levels));

uex = exact(N,h);

% figure;

% plotphi(uex,sprintf('Multigrid Iteration exact'));

function [u] = twogrid (level, h, u, f)

  uin = u;

% recursive implementation of the simple multigrid
algorithm

  if level == 0                          % solve exactly

    u(2,2) = 0.25 * (u(3,2) + u(1,2) + u(2,3) + u(2,1) ...
```

```
                    + h^2 * f(2,2));

    return

  end

  N = 2^level + 2;

  ngs = 3;                              % number of Gauss-
Seidel steps

  for n = 1:ngs                         % pre-smoothing GS
steps

    u = GaussSeidel(N, h, u, f);

    msg = sprintf('Multigrid Level %i: Pre-Smooth Iteration
%i', level, n);

    figure;

    plotphi(u, msg);

    r = zeros(N,N);

    for i = 2:N-1

      for j = 2:N-1

  r(i,j) = (u(i+1,j) + u(i-1,j) + u(i,j+1) + u(i,j-1) - ...

                4 * u(i,j)) / h^2 + f(i,j);

      end

    end
```

```
    figure;
    msg = sprintf('Multigrid Level %i: Pre-Smooth Iteration
%i difference between u and residual', level, n);
    plotphi(u-r,msg);
%     pause
    msg = sprintf('residual after %i iteration and %i level
is r', n, level);
    disp(msg);
    r
  end
  % find the residual
  % restrict residual to coarser grid
  M = 2^(level-1) + 2;
  R = zeros(M,M);
  for I = 2:M-1
    for J = 2:M-1
      i = 2*(I-1);
      j = 2*(J-1);
  R(I,J) = (r(i,j) + r(i+1,j) + r(i,j+1) + r(i+1,j+1)) / 4;
```

```
    end
  end
  % set V on coarse grid to zero and call twogrid
recursively
  V = zeros(M,M);
  [V] = twogrid(level-1, 2*h, V, R);
  % prolongate V to the finer grid using simple injection
  v = zeros(N,N);
  for I = 2:M-1
    for J = 2:M-1
      i = 2*(I-1);
      j = 2*(J-1);
      v(i,j) = V(I,J);
      v(i+1,j) = V(I,J);
      v(i,j+1) = V(I,J);
      v(i+1,j+1) = V(I,J);
    end
  end
  % apply correction to u
```

```
  for i = 2:N-1
    for j = 2:N-1
      u(i,j) = u(i,j) + v(i,j);
    end
  end
  for n = 1:ngs                          % post-smoothing GS
steps
    u = GaussSeidel(N, h, u, f);
    msg   =   sprintf('Multigrid   Level   %i:   Post-Smooth
Iteration %i', level, n);
    figure;
    plotphi(u, msg);
  end
function u = GaussSeidel (N, h, u, f)
% Gauss-Seidel relaxation step with checkerboard updating
  for color = 0:1                          % red lattice :
black lattice
    for i = 2:N-1
      for j = 2:N-1
        if mod(i+j,2) == color
```

```
          u(i,j) = 0.25 * (u(i+1,j) + u(i-1,j) + u(i,j+1) +
u(i,j-1) ...

                          + h^2 * f(i,j));

        end

      end

    end

%     u

  end

function plotphi (phi, msg)

%  plot  solution  using  3-D  mesh  plot  with  contours
underneath

  meshc(phi);

  xlabel('x');

  ylabel('y');

  zlabel('phi(x,y)');

  title(msg);

  pause(1);
```

B.8 Matlab code for Poisson's equation/chapter four

```
format short
```

```
N=4;

L1=2;

L2=2;

Uin=1;

Uot=0;

dy=L2/N;

dx=L1/N;

U=ones(2*N+1,2*N+1);

A=[-4 0 2 0 0;0 -4 2 0 0;1 1 -4 1 0;0 0 1 -4 1;0 0 0 2 -4];

b=[-5/2;-1/2;-1/2;-3/2;-3/2];

X=inv(A)*b;

disp('the solution by inverse matrix')

disp(X);

disp('---------------------------------')

for i=1:1:2*N+1

    U(1,i)=0;

    U(2*N+1,i)=0;

    U(i,1)=0;

    U(i,2*N+1)=0;
```

```
end
for i=1:1:3
    U(2,2*N+1-i)=X(i+1,1);
    U(2*N,2*N+1-i)=X(i+1,1);
end
for i=3:1:5
    U(2,i)=X(i,1);
    U(2*N,i)=X(i,1);
    U(i,2)=X(i,1);
    U(2*N+2-i,2)=X(i,1);
end
for i=2:1:4
    U(2*N,2*N+2-i)=X(i,1);
    U(i,2*N)=X(i,1);
end
for i=3:1:5
    U(i,2*N)=X(i,1);
    U(2*N+2-i,2*N)=X(i,1);
end
```

```
for i=2:1:3
    U(i,i)=X(N-i,1);
    U(2*N+2-i,i)=X(N-i,1);
    U(i,2*N+2-i)=X(N-i,1);
    U(2*N+2-i,2*N+2-i)=X(N-i,1)
end
L=zeros(N+1,N+1);
UU=zeros(N+1,N+1);
D=zeros(N+1,N+1);
for i=1:1:N+1
    for j=1:1:N+1
        if i==j
            D(i,j)=A(i,j);
        elseif i<j
            UU(i,j)=A(i,j);
        elseif i>j
            L(i,j)=A(i,j);
        end
    end
```

```
end
%-----------------Jacobi Method----------------------
Xjc=ones(N+1,1);
Xnew=zeros(N+1,1);
c=0;
while c~=15
    c=c+1;
    Xnew(1,1)=(5/2+2*Xjc(3,1))/4;
    Xnew(2,1)=(1/2+2*Xjc(3,1))/4;
Xnew(3,1)=(1/2+Xjc(1,1)+Xjc(2,1)+Xjc(4,1))/4;
Xnew(4,1)=(3/2+Xjc(3,1)+Xjc(5,1))/4;
    Xnew(5,1)=(3/2+2*Xjc(4,1))/4;
    Xjc=Xnew;
end
errjc=norm(Xjc(:)-X(:),inf); %norm error
disp(sprintf('number of iteration for Jacobi method
is=%.0f',c))
disp(sprintf('norm_inf error for Jacobi method
is=%.10f',errjc))
```

```
disp('The solution by Jacobi method is')

disp(Xjc)

disp('-----------------------------------------')

%-------------Jacobi method by LUD matrix-----

c=0;

Xoldjc=zeros(N+1,1);

Xjcm=zeros(N+1,1);

while c~=16

    c=c+1;

    Xjcm=(inv(D)*b)-((inv(D))*((L+UU)*Xoldjc));

    Xoldjc=Xjcm;

end

errjcm=norm(Xjcm(:)-X(:),inf); %norm error

disp(sprintf('number of iteration for Jacobi method
is=%.0f',c))

disp(sprintf('norm_inf error for Jacobi method
is=%.10f',errjcm))

disp('The solution by Jacobi method is')

disp(Xjcm)
```

```
disp('-----------------------------------------')

%-------------------Gauss Seidel Method--------------

Xgs=ones(N+1,1);

c=0;

while c~=8

    c=c+1;

    Xgs(1,1)=(5/2+2*Xgs(3,1))/4;

    Xgs(2,1)=(1/2+2*Xgs(3,1))/4;

Xgs(3,1)=(1/2+Xgs(1,1)+Xgs(2,1)+Xgs(4,1))/4;

Xgs(4,1)=(3/2+Xgs(3,1)+Xgs(5,1))/4;

Xgs(5,1)=(3/2+2*Xgs(4,1))/4;

end

errgs=norm(Xgs(:)-X(:),inf); %norm error

disp(sprintf('number of iteration for Gauss Seidel method
is=%.0f',c))

disp(sprintf('norm_inf error for Gauss Seidel method
is=%.10f',errgs))

disp('The solution by Gauss Seidel method is')

disp(Xgs)
```

```
disp('-----------------------------------------')

%-------------------Gauss Seidel method by LUD matrix-----
-

c=0;

Xoldgs=zeros(N+1,1);

Xgsm=zeros(N+1,1);

while c~=9

    c=c+1;

    Xgsm=(inv(L+D)*b)-((inv(L+D))*UU*Xoldgs);

    Xoldgs=Xgsm;

end

errgsm=norm(Xgsm(:)-X(:),inf); %norm error

disp(sprintf('number of iteration for Gauss Seidel method
is=%.0f',c))

disp(sprintf('norm_inf  error  for  Gauss  Seidel  method
is=%.10f',errgsm))

disp('The solution by Gauss Seidel method is')

disp(Xgsm)

disp('-----------------------------------------')

%----------------------SOR Method------------------
```

```
Xsor=ones(N+1,1);

c=0;

w=2/(1+sin(pi*dx))+.192

while c~=5

    c=c+1;

    Xsor(1,1)=(1-w)*Xsor(1,1)+(w*(5/2+2*Xsor(3,1)))/4;

     Xsor(2,1)=(1-w)*Xsor(2,1)+(w*(1/2+2*Xsor(3,1)))/4;

 Xsor(3,1)=(1-
w)*Xsor(3,1)+(w*(1/2+Xsor(1,1)+Xsor(2,1)+Xsor(4,1)))/4;

      Xsor(4,1)=(1-
w)*Xsor(4,1)+(w*(3/2+Xsor(3,1)+Xsor(5,1)))/4;

       Xsor(5,1)=(1-w)*Xsor(5,1)+(w*(3/2+2*Xsor(4,1)))/4;

   end

   errsor=norm(Xsor(:)-X(:),inf); %norm error

disp(sprintf('number   of   iteration   for   SOR   method
is=%.0f',c))

disp(sprintf('norm_inf      error      for      SORmethod
is=%.10f',errsor))

disp('The solution by SOR method is')

disp(Xsor)
```

```
disp('-----------------------------------------')
disp('The solution of U(x,y) is')
disp(U)
%*********************************************************
```

B.9 Matlab code for Helmholtz equation/Chapter Four

```
   format short;
h=.05;
k=.05;
L1=.4;
L2=.4;
Ux0=0;
UxL=0;
U0y=0;
ULy=0;
n=L1/h;
U=zeros(n+1,n+1);
for i=1:1:n+1
    U(1,i)=Ux0;
    U(n,i)=UxL;
```

```
    U(i,1)=U0y;

    U(i,n)=ULy;

end

a=n-1;% the number of unknown elements

x=h;

xy=zeros(a,1);

for i=1:1:a

    xy(i,1)=x;

    x=x+h;

end

A=zeros(a*a,a*a);

for i=1:1:a*a

    A(i,i)=-4+(-12.5*(pi^2)*(h^2));

end

for i=1:1:(n-2)*a

    A(i,i+a)=1;

    A(a+i,i)=1;

end

for i=1:1:a-1
```

```
    A(i,i+1)=1;
    A(i+1,i)=1;
end
for m=1:1:a-2
    for N=1:1:a-2
        if m==N
            for i=1:1:a-1
                A(m*a+i,m*a+i+1)=1;
                A(m*a+i+1,m*a+i)=1;
            end
        end
    end
end
for i=1:1:a-1
    A((m-2)*a+i,(m-2)*a+i+1)=1;
    A((m-2)*a+i+1,(m-2)*a+i)=1;
end
A;
L=zeros(a*a,a*a);
```

```
UU=zeros(a*a,a*a);

D=zeros(a*a,a*a);

for i=1:1:a*a

    for j=1:1:a*a

        if i==j

            D(i,j)=A(i,j);

        elseif i<j

            UU(i,j)=A(i,j);

        elseif i>j

            L(i,j)=A(i,j);

        end

    end

end

b=zeros(a*a,1);

for i=1:1:n-1

    b(i,1)=(-
25*(pi^2)*sin(2.5*(pi)*xy(i,1))*sin(2.5*(pi)*xy(1,1)))*(h^2
)-U(1,i+1);

end
```

```
for m=1:1:a-2
    for i=1:1:n-1
        b(m*a+i,1)=(-
25*(pi^2)*sin(2.5*(pi)*xy(i,1))*sin(2.5*(pi)*xy(m+1,1)))*(h
^2);
    end
end
for i=1:1:n-1
    b(a*a-a+i,1)=(-
25*(pi^2)*sin(2.5*(pi)*xy(i,1))*sin(2.5*(pi)*xy(1,1)))*(h^2
)-U(n,i+1);
end
X=inv(A)*b;
r=0;
for i=2:1:n
    for j=2:1:n
        U(i,j)=X(r+(j-1),1);
    end
    r=r+n-1;
end
```

```
Uex=zeros(n+1,n+1);

y=0;

for i=1:1:n+1

    x=0;

    for j=1:1:n+1

        Uex(i,j)=sin(2.5*pi*x)*sin(2.5*pi*y);

        x=x+h;

    end

    y=y+k;

end

err_iter_inv=norm(U(:)-Uex(:),inf)

disp('The exact solution is')

disp(Uex)

disp(sprintf('norm_inf error for invers matrix method is
=%.5f',err_iter_inv))

disp('The solution by invers matrix is')

disp(U)

disp('------------------------------')

%--------------------Jacobi method------------------
```

```
Ujc=zeros(n+1,n+1);

c=0;

R=4+((12.5*(pi^2)*(h^2)));

%while c~=33

   % c=c+1;

    k_max = (n)^3;

tolerance = 1.e-04;

for k = 1:k_max

    frac_diff = 0;

    for i=2:1:a+1

        for j=2:1:a+1

            Unew(i,j)=(((25*(pi^2)*sin(2.5*pi*xy(j-
1,1))*sin(2.5*pi*xy(i-1,1)))*(h^2))+Ujc(i-1,j)+Ujc(i,j-
1)+Ujc(i,j+1)+Ujc(i+1,j))/R;

             frac_diff=frac_diff+abs(Ujc(i,j)-Uex(i,j));

        end

    end

    Ujc(2:a+1,2:a+1)=Unew(2:a+1,2:a+1);

    change(k) = frac_diff/(n-2)^2;
```

```
    if rem(k,100) < 1
        fprintf('Fractional difference is %g after %g
steps\n',change(k),k);
    end
    if change(k) < tolerance
        break;
    end
end
Ujc
mesh(Ujc);
pause;
semilogy(change);
xlabel('Iteration Number');ylabel('Fractional Changejc');
   % err_iter_jc=norm(Ujc(:)-Uex(:),inf);
   %end
%disp(sprintf('norm_inf error for jacobi method is
=%.5f',err_iter_jc))
%disp(sprintf('number of iteration for jacobi method is
=%.0f',c))
%disp(Ujc)
```

```
%disp('-----------------------------')

%------------------jacobi method LUD matrix-----------

c=0;

Ujcm=zeros(n+1,n+1);

Xoldjc=zeros(a*a,1);

xjcm=zeros(a*a,1);

while c~=33

    c=c+1;

    Xjcm=(inv(D)*b)-((inv(D))*((L+UU)*Xoldjc));

    Xoldjc=Xjcm;

end

r=0;

for i=2:1:n

  for  j=2:1:n

    Ujcm(i,j)=Xjcm(r+(j-1),1);

end

r=r+n-1;

end
```

```
 err_iter_jcm=norm(Ujcm(:)-Uex(:),inf);

disp(sprintf('norm_inf   error   for   jacobi   method   is
=%.5f',err_iter_jcm))

disp(sprintf('number  of  iteration  for  jacobi  method  is
=%.0f',c))

disp('The solution by jacobi matrix method is')

disp(Ujcm)

disp('------------------------------')

%-------------------Gauss Seidel method--------------------
-------

Ugs=zeros(n+1,n+1);

%c=0;

R=4+((12.5*(pi^2)*(h^2)));

%while c~=17

    %c=c+1;

    k_max = (n)^3;

tolerance = 1.e-04;

for k = 1:k_max

    frac_diff = 0;
```

```
    for i=2:1:a+1
        for j=2:1:a+1
            Ugs(i,j)=(((25*(pi^2)*sin(2.5*pi*xy(j-
1,1))*sin(2.5*pi*xy(i-1,1)))*(h^2))+Ugs(i-1,j)+Ugs(i,j-
1)+Ugs(i,j+1)+Ugs(i+1,j))/R;
            frac_diff=frac_diff+abs(Ugs(i,j)-Uex(i,j));
        end
    end
    change(k) = frac_diff/(n-2)^2;
    if rem(k,100) < 1
        fprintf('Fractional  difference  is  %g  after  %g
steps\n',change(k),k);
    end
    if change(k) < tolerance
        break;
    end
end
Ugs
mesh(Ugs);
pause;
```

```
semilogy(change);

xlabel('Iteration Number');ylabel('Fractional Changegs');

    %err_iter_gs=norm(Ugs(:)-Uex(:),inf);

    %end

%disp(sprintf('norm_inf error for Gauss Seidel method is
=%.5f',err_iter_gs))

%disp(sprintf('number of iteration for Gauss Seidel method
is =%.0f',c))

%disp(Ugs)

%disp('-------------------------------------------------')

%---------------------------Gauss Seidel method by LUD
matrix-----

c=0;

Ugsm=zeros(n+1,n+1);

Xoldgs=zeros(a*a,1);

Xgsm=zeros(a*a,1);

while c~=17

    c=c+1;

    Xgsm=(inv(L+D)*b)-((inv(L+D))*UU*Xoldgs);

    Xoldgs=Xgsm;
```

```
end

r=0;

for i=2:1:n

    for j=2:1:n

        Ugsm(i,j)=Xgsm(r+(j-1),1);

    end

    r=r+n-1;

end

    err_iter_gsm=norm(Ugsm(:)-Uex(:),inf);

disp(sprintf('norm_inf  error  for  Gauss  Seidel  method  is
=%.5f',err_iter_gsm))

disp(sprintf('number  of  iteration  for  Gauss  Seidel  method
is =%.0f',c))

disp('The solution by Gauss Seidel matrix method is')

disp(Ugsm)

disp('-------------------------------------------------')

%------------------
SOR=====================================

Usor=zeros(n+1,n+1);

c=0;
```

```
w=2/(1+sin(pi*h))-.25

R=4+((12.5*(pi^2)*(h^2)));

k_max = (a+1)^3;

tolerance = 1.e-04;

for k = 1:k_max

    frac_diff = 0;

    for i=2:1:a+1

        for j=2:1:a+1

            Usor(i,j)=(w*(((25*(pi^2)*sin(2.5*pi*xy(j-
1,1)))*sin(2.5*pi*xy(i-1,1))*(h^2))/R))+((1-
w)*Usor(i,j))+(w*(Usor(i-
1,j)+Usor(i+1,j)+Usor(i,j+1)+Usor(i,j-1)))/R;

            frac_diff=frac_diff+abs(Usor(i,j)-Uex(i,j));

        end

    end

   change(k) = frac_diff/(a+1-2)^2;

    if rem(k,100) < 1

        fprintf('Fractional  difference  is  %g  after  %g
steps\n',change(k),k);

    end
```

```
    if change(k) < tolerance
        break;
    end
end
Usor
mesh(Usor);
pause;
semilogy(change);
xlabel('Iteration Number');ylabel('Fractional Changesor');
```

B.10 Matlab code for irregular region/Chapter Four

```
format short e;
N1=4;
N2=2;
L1=.4;
L2=.2;
h=L1/N1;
k=L2/N2;
U=zeros(N2+1,N1+1);
A=[-4 1 0 1 0;1 -4 1 0 1;0 2 -4 0 0;2 0 0 -4 1;0 2 0 2 -4];
```

```
b=[-8;-4;-8.0565685;-4.02;-0.0565685];

disp('The solution by matrix')

X=inv(A)*b;

disp(X);

disp('-----------------------------------------')

for i=1:1:N1+1

    U(3,i)=4;

end

for i=1:1:N2+1

    U(i,1)=4;

end

for i=2:1:N1

    U(2,i)=X(i-1,1);

end

for i=2:1:N1-1

    U(1,i)=X(i+2,1);

end

%=========================================================
==
```

```
L=zeros(N1+1,N1+1);

UU=zeros(N1+1,N1+1);

D=zeros(N1+1,N1+1);

for i=1:1:N1+1

    for j=1:1:N1+1

        if i==j

            D(i,j)=A(i,j);

        elseif i<j

            UU(i,j)=A(i,j);

        elseif i>j

            L(i,j)=A(i,j);

        end

    end

end

%=======================Jacobi Method=================

Xjc=ones(N1+1,1);

Xnew=zeros(N1+1,1);

c=0;

while c~=100
```

```
    c=c+1;
    Xnew(1,1)=(-b(1,1)+Xjc(2,1)+Xjc(4,1))/4;
    Xnew(2,1)=(-b(2,1)+Xjc(1,1)+Xjc(3,1)+Xjc(5,1))/4;
    Xnew(3,1)=(-b(3,1)+2*Xjc(2,1))/4;
    Xnew(4,1)=(-b(4,1)+2*Xjc(1,1)+Xjc(5,1))/4;
    Xnew(5,1)=(-b(5,1)+2*Xjc(2,1)+2*Xjc(4,1))/4;
    Xjc=Xnew;
end
err_iter_jc=norm(Xjc(:)-X(:),inf);
disp(sprintf('  number  of  iteration  for  jacobi  method
is=%.0f',c))
disp(sprintf('norm   of   error   for   jacobi   method
is=%.10f',err_iter_jc))
disp('The solution by jacobi Method is')
disp(Xjc)
disp('---------------------------------------')
c=0;
Xoldjc=zeros(N1+1,1);
Xjcm=zeros(N1+1,1);
```

```
while c~=100
    c=c+1;
    Xjcm=(inv(D)*b)-((inv(D))*((L+UU)*Xoldjc));
    Xoldjc=Xjcm;
end
err_iter_jcm=norm(Xjcm(:)-X(:),inf);
    disp(sprintf(' number of iteration for jacobi method is=%.0f',c))
    disp(sprintf('norm of error for jacobi method is=%.10f',err_iter_jcm))
disp('The solution by jacobi Method is')
disp(Xjcm)
disp('---------------------------------------')
%====================Gauss Seidel=====================
Xgs=ones(N1+1,1);
c=0;
while c~=50
    c=c+1;
    Xgs(1,1)=(-b(1,1)+Xgs(2,1)+Xgs(4,1))/4;
```

```
    Xgs(2,1)=(-b(2,1)+Xgs(1,1)+Xgs(3,1)+Xgs(5,1))/4;

    Xgs(3,1)=(-b(3,1)+2*Xgs(2,1))/4;

    Xgs(4,1)=(-b(4,1)+2*Xgs(1,1)+Xgs(5,1))/4;

    Xgs(5,1)=(-b(5,1)+2*Xgs(2,1)+2*Xgs(4,1))/4;

end

err_iter_gs=norm(Xgs(:)-X(:),inf);

disp(sprintf(' number of iteration for Gauss Seidel method
is=%.0f',c))

disp(sprintf('norm  of  error  for  Gauss  Seidel  method
is=%.10f',err_iter_gs))

disp('The solution by Gauss Seidel Method is')

disp(Xgs)

disp('-------------------------------------')

c=0;

Xoldgs=zeros(N1+1,1);

Xgsm=zeros(N1+1,1);

while c~=50

    c=c+1;

    Xgsm=(inv(L+D)*b)-((inv(L+D))*UU*Xoldgs);
```

```
    Xoldgs=Xgsm;

end

err_iter_gsm=norm(Xgsm(:)-X(:),inf);

disp(sprintf(' number of iteration for Gauss Seidel method
is=%.0f',c))

disp(sprintf('norm  of  error  for  Gauss  Seidel  method
is=%.10f',err_iter_gsm))

disp('The solution by Gauss Seidel Method is')

disp(Xgsm)

disp('-------------------------------------')

%----------------------SOR method---------------------

Xsor=ones(N1+1,1);

c=0;

w=2/(1+sin(pi*h))

while c~=35

    c=c+1;

    Xsor(1,1)=(1-w)*Xsor(1,1)+(w*(-
b(1,1)+Xsor(2,1)+Xsor(4,1)))/4;
```

```
Xsor(2,1)=(1-w)*Xsor(2,1)+(w*(-
b(2,1)+Xsor(1,1)+Xsor(3,1)+Xsor(5,1)))/4;

    Xsor(3,1)=(1-w)*Xsor(3,1)+(w*(-b(3,1)+2*Xsor(2,1)))/4;

    Xsor(4,1)=(1-w)*Xsor(4,1)+(w*(-
b(4,1)+2*Xsor(1,1)+Xsor(5,1)))/4;

    Xsor(5,1)=(1-w)*Xsor(5,1)+(w*(-
b(5,1)+2*Xsor(2,1)+2*Xsor(4,1)))/4;

end

err_iter_sor=norm(Xsor(:)-X(:),inf);

disp(sprintf('  number  of  iteration  for  SOR  method
is=%.0f',c))

disp(sprintf('norm   of   error   for   SOR   method
is=%.10f',err_iter_sor))

disp('The solution by SOR Method is')

disp(Xsor)

disp('-------------------------------------')

disp('The solution of U(x,y) is=')

disp(U)
```

B.11 Parent routine for Laplace's Eq. in polar

```
u=zeros(6,5);
```

```
r=[1 .2 .4 .6 .8 1];

rs=r.*r;

s=pi/8;

w=.2;

ss=s*s;

ws=w*w;

s1=2*(1/ws+1.0./(ss*rs));

u(6,:)=100;

for k=1:100

    changesum =0;

    ueval=u;

    for i=2:5

    T=(u(i-1,1)+u(i+1,1))/ws+(u(i+1,1)-u(i-
1,1))/(2*w*r(i));

    u(i,1)=(T+2*u(i,2)/(rs(i)*ss))/s1(i);

end

for i=2:5

    for j=2:4
```

```
        T=(u(i-1,j)+u(i+1,j))/ws+(u(i+1,j)-u(i-
1,j))/(2*w*r(i));

        u(i,j)=(T+u(i,j-1)+u(i,j+1))/(rs(i)*ss))/s1(i);

        changesum=changesum+abs(ueval-u);

    end

end

if max(abs(ueval-u))<.001,break,end

end

fprintf('# of iteration =%g\n',k)

disp('u =');disp(u)

end

mesh(u);

pause;

semilogy

xlable('Iteration Number ');ylable('Error');
```

References

Scarton, R.E.1987,. **Further Numerical Methods in Basic,** Edward Arnold.

Wesseling, Pieter, 1992, **An Introduction to Multigrid Methods**, John Wiley and sons.

Press, W.H, Teukolsky, Vetterling,W.T., Flannery,B.P., 1993, **Numerical Recipes-the Art of Scientific computing**, 2nd edition Cambridge University Press, Indian reprint by Foundation Books,N.Delhi.

Johnes, J.E,199: **A parallel multigrid Tutorial.19 Copper Mountain conference on Multigrid methods**.

Douglas, C.C.,199. **A family of abstract multigrid or multilevel solver. Comput.Appl.Math.**

Smith, G.D.1987, **Numerical Solution of Partial Differential Equations**. Clarendon press.Oxford.

Hockney, R.W., East wood,J.E, 1987, **Computer simulation using particles**, Mc Graw-Hill.

Mathews ,J.H, 1992 **Matlab Programming Guidebook for Numerical Methods**,2nd.ed. , Englewood cliffs, New Jersey 07632.

Mitchell,A.R, Griffiths,D.F, 1980, **The Finite Difference Method in Partial Differential Equations**, John Wiley and Sons, Chichester. New York .Brisbane.Toronto,.

Morton,K.W,Mayers,D.F, 2000, **Numerical Solution of Partial Differential Equations,Computing Laboratory**, University of Oxford.

Curtis F. Geraled,Patrick O.Wheatley, 1999, **Applied Numerical Analysis**, Addison Wesley Longman,Inc.

Kendall E.Atkinson, 1989 **An Introduction to numerical analysis**, John Wiley and sons.

www.physics.buffalo.edu/phy516/files/**Topic4**/mar6.pdf

William E.Boyce,Richard G.Dipoima, 1986 **Elementary Differential equations and Boundary Value Problems**, John Wiley and Sons,Inc.

Jain, M.K, 1984, **Numerical Solution of Differential Equations**, John Wiley, Easter Limited New Delhi.

Rudolph E.Langer, 1960, **Boundary Problems In Differential Equations**, Madison, The University of Wisconsin Press.

Christian G.Simader, 1972, **Lecture Notes in Mathematics on Dirichlets Boundary Value Problem**, Springer-Verlag Serlin. Heidelbag. New York.

بسم الله الرحمن الرحيم

جامعه النجاح الوطنيه
كليه الدراسات العليا

طرق عدديه لحل مسائل حدية في المعادلات التفاضليه الجزئيه من نوع قطع ناقص

اعداد
مثقال غالب يوسف نجي

اشراف
د. سمير مطر

قدمت هذه الاطروحة استكمالا لمتطلبات درجة الماجستير في الرياضيات المحوسبة بكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس، فلسطين.

2005

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الملخص

تمت دراسة المعادلات التفاضلية الجزئية البيضاوية من الدرجة الثانية باستخدام بعض الطرق العددية. لهذا النوع من المعادلات التفاضلية الجزئية تطبيقات في الظواهر الطبيعية و الهندسية. في معظم التطبيقات نستخدم عادة قواعد من الرتبة الأولى والثانية لكننا في هذه الرسالة استخدمنا قواعد من رتب أعلى مثل: قاعدة النقط السبعة و قاعدة النقط التسعة. باستخدام هذه القواعد يمكن تحويل المعادلة التفاصلية الجزئية إلى معادلة الفروق الدقيقة، ولحل مثل هذه المعادلة نستخدم طرق التكرار مثل:

(SOR, Jacobi, Gauss Seidel, Multigrid Methods)

في هذا البحث وجدنا إن طريقة (Multigrid Methods) هي الطريقة المثلى من بين جميع الطرق الأخرى؛ فالوقت المستغرق في هذه الطريقة هو من الرتبة الثالثة بينما هو من الرتبة الخامسة في الطرق الأخرى .