An-Najah National University
Faculty of Graduate Studies

Analytical and Numerical Solutions of Magnetohydrodynamic Flow Problems

By
Abdel Latif Khaleel Sa'ad Aldin

Supervised by
Prof. Naji Qatanani

This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of master of Science in Computational Mathematics, Faculty of Graduate Studies, An -Najah National University, Nablus, Palestine.

2013

Analytical and Numerical Solutions of Magnetohydrodynamic Flow Problems

By
Abdel Latif Khaleel Sa'ad Aldin

This Thesis was defended successfully on 2/12/2013 and approved by:

Defense Committee Members	**Signatures**
1. Prof. Naji Qatanani / Supervisor	N. QATANANI
2. Dr. Iyad Suwan / External Examiner	....
3. Dr. Anwar Saleh / Internal Examiner	....

Dedication

I dedicate this thesis to my parents, wife, brother and sister. Without their patience, understanding, support and most of all love, this work would not have been possible.

Acknowledgement

I am heartily thankful to my supervisor, Prof. Dr. Naji Qatanani, whose encouragement, guidance and support from the initial to the final level enabled me to develop and understanding the subject. My thanks and appreciation goes to my thesis committee members Dr. Iyad Suwwan and Dr. Anwar Saleh for their encouragement, support, interest and valuable hints.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل عنوان :

Analytical and Numerical Solutions of Magnetohydrodynamic Flow Problems

أقر بأن ما اشتملت عليه هذه الرسالة إنما هي نتاج جهدي الخاص، باستثناء مـــا تمـــت الإشارة إليه حيثما ورد، وأن هذه الرسالة ككل، أو أي جزء منها لم يقدم من قبل لنيل أية درجة علمية أو بحث علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's name: **اسم الطالب:**

Signature: **التوقيع:**

Date: **التاريخ:**

Table of Contents

No.	Contents	Page
	Dedication	iii
	Acknowledgement	iv
	Declaration	v
	Table of Contents	vi
	List of Figures	viii
	Abstract	ix
	Introduction	1
	Chapter One: Preliminaries	4
1.1	Gauss's law	5
1.2	Faraday's Law	7
1.3	Gauss's Law for Magnetic Field	9
1.4	Ampere's law	10
1.5	The Continuity Equation for Charge	12
1.6	Maxwell's Equations	13
1.7	Navier-Stokes Equations	14
1.7.1	Continuity Equation	15
1.7.2	Euler's Equation	16
1.7.3	Navier-Stokes Equation	18
1.8	Ohm's Law	21
	Chapter Two: Formulation of The Problems	24
2.1	MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System without Hall Effect	25
2.2	MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System Including Hall Effect	29
2.3	Unsteady MHD Flow Through Two Parallel Porous Flat Plates without Hall Effect	34
2.4	Unsteady MHD Flow Through Two Parallel Porous Flat Plates With Hall Effect	38
	Chapter Three Analytical Solutions	42
3.1	Introduction	43
3.2	MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System without Hall Effect	43
3.3	MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System Including Hall Effect	47

No.	Contents	Page
3.4	The Analytical Solution For The Unsteady MHD Flow Through Two Parallel Porous Flat Plates Without Hall Effect	50
3.5	The Analytical Solution For The Unsteady MHD Flow Through Two Parallel Porous Flat Plates With Hall Effect	61
	Chapter Four: Numerical methods	71
4.1	Finite Difference	72
4.2	MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System	73
4.2.1	Without Hall Effect	73
4.2.2	With Hall Effect	75
4.2.3	Stability for the difference scheme	77
4.3	The Numerical Solution For The Unsteady MHD Flow Through Two Parallel Porous Flat Plates	79
4.3.1	Without Hall Effect	79
4.3.2	With Hall Effect	81
4.3.3	Stability for the difference scheme	83
	Chapter Five: Numerical Results and Conclusions	87
5.1	Numerical Results	88
5.1.1	MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System	88
5.1.2	The Unsteady MHD Flow Through Two Parallel Porous Flat Plates	98
5.2	Conclusions	101
	References	**103**
	Appendix	**108**
	الملخص	**ب**

List of Figures

No.	Figure	Page
Figure (1)	primary velocity profiles for several values of Ω with m = 0 , M = 0.5 and T = 1 .	89
Figure (2)	secondary velocity profiles for several values of Ω with m = 0 , M = 0.5 and T = 1 .	90
Figure (3)	primary velocity profiles for several values of Ω with m = 0.5 , M = 0.5 and T = 1 .	91
Figure (4)	secondary velocity profiles for several values of Ω with m = 0.5 , M = 0.5 and T = 1 .	92
Figure (5)	primary velocity profiles for several values of M with m = 0 , Ω = 0.1 and T = 1 .	93
Figure (6)	secondary velocity profiles for several values of M with m = 0 , Ω = 0.1 and T = 1 .	94
Figure (7)	primary velocity profiles for several values of M with m = 0.5, Ω = 0.1 and T = 1 .	95
Figure (8)	secondary velocity profiles for several values of M with m = 0.5 , Ω = 0.1 and T = 1 .	96
Figure (9)	primary velocity profiles for several values of m with $M =$ 0.5, Ω = 0.1 and T = 1 .	97
Figure (10)	secondary velocity profiles for several values of m with $M =$ 0.5, Ω = 0.1 and T = 1 .	98
Figure (11)	primary velocity profiles for several values of M with m = 0 and T = 0.5 .	99
Figure (12)	primary velocity profiles for several values of M with m = 1 and T = 0.5 .	100
Figure (13)	primary velocity profiles for several values of m with $M = 2$ and T = 0.5 .	101

Analytical and Numerical Solutions of
Magnetohydrodynamic Flow Problems
By
Abdel Latif Khaleel Sa'ad Aldin
Supervised by
Prof. Naji Qatanani

Abstract

The MagnetoHydreoDynamic flow (MHD) is one of the most important topics in mathematical physics due to its wide range of applications .

In this work we present some analytical and numerical solutions for some MHD problems . The MHD flow past an impulsively started infinite horizontal plate in a rotating system and unsteady MHD flow through two parallel porous flat plates are considered.

In this work an exact analytical solution for these problems based on Laplace transform method has been constructed and analyzed. This involves transforming the coupled partial differential equations into a single equation.

For the numerical treatment of these problems we use the finite difference scheme and then implementing a computer software " MAPLE 15" to obtain some numerical results.

Introduction

The MHD is one of the simplest models for describing the interaction between a perfectly conducting fluid and a magnetic field. There are various examples of applications for the MHD principle. Engineers apply MHD principle in fusion reactors, dispersion of metals, metallurgy, design of MHD pumps, MHD generators and MHD flow meters. The dynamo and motor is a classical example of MHD principle. Geophysics encounters MHD characteristics in the interaction of conducting fluid and magnetic field. MHD convection problems are also very significant in fields of stellar and planetary magnetospheres, aeronautics and chemical and electrical engineering. The MHD principle also finds its application in medicine and biology such as cardiac MRI, ECG etc [10],[40].

The MHD in its present form is due to the contribution of several well-known authors like Alfven [2], Cowling [5], Shercliff [33], Ferraro and plumpton [11], Krammer and Pai [24].

In all the above studies, the influence of Hall current effects on MHD flow was not considered. That is, if a conductor or a semi – conductor has current flowing in it because of an applied electric field and a transverse magnetic field is applied, then there develops a component of electric field in the direction orthogonal to both the applied electric field and magnetic field, resulting in voltage difference between the sides of the conductor. This phenomenon is termed as the Hall effect. In an ionized gas, when the strength of the magnetic field is large, one can not neglect the effect of the Hall current. These Hall currents are particularly important to produce considerable changes in the flow pattern when the magnetic field is considerably large. Katugiri [22] has described the effect of Hall current on the boundary layer flow past a semi-infinite flat plate. Gupta [16] studied the Hall current effects in the steady hydro-magnetic flow of an electrically conducting fluid past an infinite porous flat plate. Sakhnoskii [31] has described the effects of Hall current on the MHD Rayleigh problem. Pardeep [28] studied the thermal instability of Walter's viscoelastic fluid permeated with suspended particles in hydromagnetics in porous medium. Deka [7] studied the Hall effects on MHD flow past an accelerated plate. Mostafa et.al. [26] studied laminar fully developed mixed convection with viscous dissipation in a uniformly heated vertical double –passage channel, and Sulieman et.al. [37] have studied an MHD flow of viscous fluid past a uniformly accelerated and insulated infinite plate with Hall effect .

The idea of MHD is that magnetic fields can induce a current in a moving conductive fluid, which not only creates forces on fluid but also change the magnetic field itself. For electrically conducting fluids, where the velocity field V and the magnetic field B are coupled, any movement of a conducting material in a magnetic field generates an electric currents j, that is induces a magnetic field in turn. Hence, to describe the MHD equations, we need to combine Maxwell's equations of electromagnetic with Navier-Stokes equations of fluid dynamics, the equation of mass continuity, and Ohm's law , within differential form to result in MHD equations.

This thesis is organized as follows: In chapter one, we review some concepts of electromagnetic and fluid dynamic. In chapter two, we formulate the MHD problems, namely: MHD flow past an impulsively started infinite horizontal plate in a rotating system and unsteady MHD flow through two parallel porous flat plates. Chapter three, deals with the analytical solutions based on Laplace transform method. In chapter four, we present numerical solutions involving finite difference method for these problems. Chapter five gives some numerical results and conclusions.

CHAPTER ONE

PRELIMINARIES

CHAPTER ONE
PRELIMINARIES

1.1 Gauss's law

Gauss's Law is one of the most important fundamental laws of electricity and magnetism. it is another form of Coulomb's law that allows one to calculate the electric field of several simple configurations. In this section we describe a general relationship between the net electric flux through a closed surface and the charge enclosed by the surface. Let us consider a positive point charge q located at the center of a sphere of radius r. From Coulomb's law the magnitude of the electric field everywhere on the surface of the sphere is

$$E = \frac{k_e q}{r^2}$$

where E is the electric field and k_e is the Coulomb's constant,

$$k_e = \frac{1}{4\pi\epsilon_0}$$

and ϵ_0 is the permittivity of free space.

Now, the net flux Φ_E through the Gaussian surface is [3]

$$\Phi_E = \oint_S E.dA.$$

For Gaussian surface the field lines are directed radially outward and hence perpendicular to the surface at every point on the surface. That is, at each surface point, E is parallel to the vector ΔA_i representing a local element of area ΔA_i surrounding the surface point [32]. Therefore,

$$E.\Delta A_i = E\,\Delta A_i$$

and hence

$$\oint_S E.dA = \oint_S EdA = E\oint_S dA = \frac{k_e q\,(4\pi r^2)}{r^2}$$

so we have

$$\oint_S E.dA = \frac{q}{\epsilon_0}$$

where $\epsilon_0 = 8.85 \times 10^{-12}$ (in farads/meter) is the electric permittivity.

If a closed surface S encloses no charges, then the number of lines entering must equal the number of lines exiting, since there are no charges inside for the field lines to stop or start on. That is to say,

$$\oint_S E.dA = 0$$

So only charges inside the surface can contribute to the flux through the surface. Positive charges inside produce positive flux; negative charges produce negative flux. Hence The net flux is due only to the net charge inside [9]:

$$\oint_S E.dA = \frac{q_{enc}}{\epsilon_0} \qquad (1.1.1)$$

This is usually referred to be the integral version of Gauss's Law, where q_{enc} denotes the total charge enclosed within the surface. Using the Divergence theorem allows us to connect the differential and integral forms directly, This is a general mathematical relationship [17]:

$$\oint_S E.dA = \oint_V \nabla.E\, dV \qquad (1.1.2)$$

If we use the charge in terms of a charge density ρ_c,

$$q_{enc} = \int_V \rho_c\, dV \qquad (1.1.3)$$

using (1.1.2) and (1.1.3) in equation (1.1.1) we have,

$$\oint_V \nabla.E\, dV = \frac{1}{\epsilon_0}\int_V \rho_c\, dV$$

since the volume V is fixed and arbitrary, this implies that

$$\nabla.E = \frac{\rho_c}{\epsilon_0} \qquad (1.1.4)$$

This is the differential form of Gauss's Law. Finally, Gauss's Law state that The net number of electric field lines which leave any volume of space is proportional to the net electric charge in that volume.

1.2 Faraday's Law

Faraday's law is essentially a statement which gives the relation between a time-varying magnetic field and the electric field E produced by it. Earlier we have seen that electric charges are the sources of E. But the experiment of Michaels Faraday (1791-1867) , gives another possible way of producing an electric field [32].

The main idea of Faraday's law is, changing magnetic flux through a surface induces an electromotive force (emf) in any boundary path of that surface, and a changing magnetic field induces a circulating electric field. The emf induced in a circuit is directly proportional to the time rate of change of the magnetic flux through the circuit. This statement, known as Faraday's law of induction, can be written as

$$emf = -\frac{d\Phi_B}{dt} \tag{1.2.1}$$

where Φ_B is the magnetic flux through the circuit.

Faraday's found that the magnetic flux take the form [13],

$$\Phi_B = \int_S B\,.dA \tag{1.2.2}$$

where B is the magnetic field.

Faraday postulated that changing magnetic fields induce an electric field, from his earlier experiment he found that the loop moving is what causes the magnetic force that induces an emf. But since the loop is stationary, there cannot be a magnetic field producing the emf [3].

Therefore, we must conclude that an electric field is created in the loop as a result of the changing magnetic flux. The emf for any closed path can be expressed as the line integral of $E\,.dL$ over that path [32],

$$emf = \oint E\,.\,dL \tag{1.2.3}$$

combining the equations (1.2.1), (1.2.2) and (1.2.3) we get

$$emf = \oint E\,.\,dL = -\frac{d\Phi_B}{dt} = -\int_S \frac{dB}{dt}\,.\,dA$$

then

$$\oint E\,.\,dL = -\int_S \frac{dB}{dt}\,.\,dA \tag{1.2.4}$$

This is now known as Faraday's Law in its integral form. Using Stoke's Theorem [17], we can recover the law into its differential form by noting that

$$\oint E\,.\,dL = \int_S (\nabla \times E).\,dA \tag{1.2.5}$$

hence, using (1.2.5) with the equation (1.2.4) we get,

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{1.2.6}$$

and this is known as Faraday's Law in its differential form.

1.3 Gauss's Law for Magnetic Field

Gauss's law for magnetism is remarkably similar to Gauss's law for electricity in form, but means something rather different. Imagine that a magnet was placed in space, and that a spherical Gaussian surface was constructed around it. So we note that Magnetic field lines always close in themselves. No matter how the (closed) Gaussian surface is chosen, the net magnetic flux through it always vanishes [32].

$$\Phi_B = \int_S B\,.dA$$

there for we have that

$$\int_S B\,.dA = 0 \qquad (1.3.1)$$

Using the Divergence theorem (1.1.2) with the equation (1.3.1), we get

$$\nabla.B = 0 \qquad (1.3.2)$$

That is the Gauss's Law for Magnetic Field or the Solenoidal constraint.

1.4 Ampere's law

Ampere's Law, states that integral of magnetic field along any closed path is directly proportional to the net electric current crossing any surface bounded by the closed path.

Jean-Baptiste Biot and Felix Savart (19 th century) determined a mathematical expression for calculating the magnetic field of a current. They side that around any current carrying wire results a magnetic field. The magnitude of dB is proportional to $\frac{1}{r^2}$, where r is the distance from the line element ds to the point where we calculate the magnetic field. The magnitude of dB is proportional to the current and to the length ds [44].

So the Biot-Savart law tells how to calculate the magnetic field due to a current I. According to this law the magnitude of dB produced at a point p separated a distance r from a length ds of a wire carrying a current I is given by [32]:

$$dB = \frac{\mu_0}{4\pi} \frac{I\, ds \times r}{r^3}$$

where μ_0 is a constant called the permeability constant for the vacuum.

The total magnetic field at p is found by integrating the above equation along the wire,

$$B = \frac{\mu_0 I}{2\pi R} \qquad (1.4.1)$$

where R is the perpendicular distance from the wire.

We will now give a brief derivation of Ampere’s law using the equation (1.4.1) to obtain the integral form of Ampere’s Law

$$\oint B\,.\,dl = \frac{\mu_0 I}{2\pi r} \oint dl = \mu_0 I \qquad (1.4.2)$$

This is the integral form of Ampere’s Law: The line integral of B around a single closed path is equal to the permeability of the medium times the current enclosed.

Using the current enclosed $I = I_{enc}$ by the Amperian loop, and if we have a volume current density enclosed (not just a wire carrying current I) we use:

$$I_{enc} = \int_S J.dA \tag{1.4.3}$$

where J is the volume current density.

Now with the help of Stoke's theorem (1.2.5) with the equations (1.4.2) and (1.4.3) we can write Ampère's law in in differential form

$$\int_S (\nabla \times B\,)dA = \mu_0 \int_S J.dA$$

hence,

$$\nabla \times \mathrm{B} = \mu_0 J\,. \tag{1.4.4}$$

1.5 The Continuity Equation for Charge

In physics, charge conservation is the principle that electric charge can neither be created nor destroyed. So for a surface S, the total amount of charge flowing outwards through the surface per unit time must equal the amount by which the charge is decreasing inside the volume V per unit time. Hence

$$\oint_S J\,.da = -\frac{d}{dt}\oint_V \rho\, dV$$

now using (1.1.2) we get

$$\oint_V \nabla.J \; dV = -\frac{d}{dt}\oint_V \rho \, dV$$

hence the charge continuity equation,

$$\nabla.J - \frac{d\rho}{dt} = 0$$

where J is the volume current density and ρ is the mobile charge density.

Now if we take the global conservation of charge law, which states that the total charge of the universe is constant. Then we have

$$\nabla.J = 0$$

We have $\frac{d\rho}{dt} = 0$ since the currents are continuous and time independent. That what we have in magnetostatics.see[3].

1.6 Maxwell's Equations

In 1865, Maxwell published a set of equations that describe completely the behavior of electromagnetic fields. These equations are used in a huge range of applications, from the properties of materials, to properties of radiation (radio waves to gamma rays) [14].

However when Maxwell was putting all the equation's together and studying them, he found one fatal inconsistency that was dealing with Ampere’s Law [3], when we take the divergence of Ampere’s Law we will have $\nabla . J = 0$ which is very nice if we have a steady currents, but, what if we have a non-steady currents.

So Maxwell made contribution to Ampere's Law. The new term added by Maxwell is called the displacement current. His addition of the displacement current made Ampère's law agrees with conservation of charge.

Using this and the above equation's to get the famous equations of electromagnetism, known as the Maxwell's Equations, as follows,

$$\nabla . E = \frac{\rho_c}{\epsilon_0} \qquad \text{Gauss's Law,}$$

$$\nabla . B = 0 \qquad \text{Solenoidal constraint,}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \text{Faraday's Law,}$$

$$\nabla \times \mathrm{B} = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \quad \text{Ampere's Law.}$$

1.7 Navier-Stokes Equations

The Navier-Stokes equations are the fundamental partial differentials equations that describe the flow of incompressible fluids. The Navier–Stokes equations are based on the assumption that the fluid, at the scale of interest, is a continuum, in other words is not made up of discrete particles but rather a continuous substance. So we assume that the density and viscosity of the fluid are constant, which gives rise to a continuity condition.

In this section, we derive the equation of conservation of mass, the Euler equations, the Navier-Stokes equations.

1.7.1 Continuity Equation

Let us first consider the flow of mass in and out of a control volume. The mass flowing out of the area element dA per unit time is $\rho v.dA$. So the time rate of change of mass in the control volume is [42]

$$\frac{\partial}{\partial t}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV = -\int_A \rho v.dA$$

where ρ is the fluid mass density and v is the fluid velocity. Using (1.1.2), this becomes

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V (\nabla.\rho v) dV$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + (\nabla.\rho v)\right) dV = 0$$

since the volume V is fixed and arbitrary, this implies that

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho v) = 0 \qquad (1.7.1.1)$$

this is the continuity equation or the conservation of mass to the fluid.

Now if we assume that the fluid is incompressible we have,

$$\rho = constant$$

using the material derivative which is the rate of change of ρ in a given element of fluid as it moves along its trajectory $x = x(t)$ in the flow [12].

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (v.\nabla)\rho$$

hence, the continuity equation becomes

$$\frac{D\rho}{Dt} + \rho(\nabla.v) = 0$$

where

$$\nabla.\rho v = (v.\nabla)\rho + \rho(\nabla.v)$$

then the equation (1.7.1.1) is reduced to

$$\nabla.v = 0$$

which is the continuity equation for incompressible fluid .

1.7.2 Euler's Equation

To obtain the equation of motion for a fluid we appeal to Newton's second law that the mass of a fluid element times its acceleration is equal to the net force acting on that fluid element [8]. If we take an element of unit volume, then we have

$$\rho \frac{Dv}{Dt} = f$$

Where ρ is the fluid mass density, v is the fluid velocity and f is the force per unit volume on a fluid element.

This force may have several contributions. The first is the internal force which is due to viscous dissipation, which we will ignore for right now. The second set are body forces which act throughout the volume of the fluid, such as the gravitational force. The third force is due to pressure gradients within the fluid.

To see how this works, consider a cube of fluid, with dimensions Δx ,Δy , and Δz, so the force on the top face at position x is $p\Delta y\Delta z$, while the force on the bottom that face is $\left[p + \frac{\partial p}{\partial x}\Delta x\right]\Delta y\Delta z$,where $p = \frac{force}{area}$, subtracting, we see that the net force in the x-direction is $-\frac{\partial p}{\partial x}\Delta x\Delta y\Delta z$, so the pressure per unit volume is $-\frac{\partial p}{\partial x}$ [3]. Repeating for the y and z directions, we find the net force per unit volume,

$$f = -\nabla p$$

therefore, if we ignore viscosity and gravity for the moment, we have

$$\rho \frac{Dv}{Dt} = -\nabla p$$

now we want the acceleration of a particular element of the fluid, the coordinates of this fluid element change in time as the fluid flows. In a time interval Δt , the x-coordinate changes by $v_x\Delta t$, the y-coordinate by $v_y\Delta t$, and the z-coordinate by $v_z\Delta t$. The velocity then becomes,

$$v(x + v_x\Delta t , y + v_y\Delta t , z + v_z\Delta t , t + \Delta t) = v(x , y , z , t) + \frac{\partial v}{\partial x} v_x\Delta t + \frac{\partial v}{\partial y} v_y\Delta t + \frac{\partial v}{\partial z} v_z\Delta t + \frac{\partial v}{\partial t}\Delta t$$

therefore from the above we find the acceleration to be,

$$\frac{Dv}{Dt} = \frac{\Delta v}{\Delta t} = \frac{v(x + v_x \Delta t\, , y + v_y \Delta t\, , z + v_z \Delta t\, , t + \Delta t) - v(x\, , y\, , z\, , t)}{\Delta t}$$
$$= \frac{\partial v}{\partial x} v_x + \frac{\partial v}{\partial y} v_y + \frac{\partial v}{\partial z} v_z + \frac{\partial v}{\partial t}$$

then we have,

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + (v\, .\nabla)v$$

traditionally the operator $\frac{Dv}{Dt}$ is called the material derivative. The material derivative in our case describes the evolution of a particular fluid blob as it moves along a certain trajectory as it flows alongside the rest of the fluid [3]. And we see that the acceleration is not simply $\frac{\partial v}{\partial t}$. The reason for this is that even if $\frac{\partial v}{\partial t} = 0$, so that the velocity at a given point is not changing, that doesn't mean that a fluid element is not accelerating. A good example is circular flow in a bucket. If the flow is steady, then at a point in the bucket $\frac{\partial v}{\partial t} = 0$, even though a fluid element in the bucket is experiencing a centripetal acceleration. The term $(v\, .\nabla)v$ is nonlinear, and it is the source of all of the difficulties in fluid mechanics. Now Pulling together all of the pieces, so we have

$$\rho \left[\frac{\partial v}{\partial t} + (v\, .\nabla)v \right] = -\nabla p + f$$

This is known as Euler's equation . This equation and the equation of continuity are the governing equations of non viscous fluid flow.

1.7.3 Navier-Stokes Equations

Let's expand our work about the force f, and take the internal force which is due to viscosity of the fluid. On other words we will talk about the shear stresses if we take the cube of the fluid and Put it on the boundary of a flow, like a wall, they found that the fluid velocity on the boundary is zero, so if we think of a single fluid cube "standing" against the boundary, the bottom of the fluid cube has velocity zero. However, this does not mean that the top of the fluid cube has zero velocity. The top of the fluid cube will have some velocity due to the shear stresses acting on the cube. So the cube will look as though it is being stretched.

A stress component which acts perpendicular to face is referred to as normal stress, and a stress component which acts tangential to a face are called a share stress [29]. If we take the flow to be two directional flow the shear stress τ, is

$$\tau = \mu \frac{\partial v}{\partial y}$$

where $\frac{\partial v}{\partial y}$ describes the change in the x component of velocity with respect to the coordinate y and μ is a parameter called the molecular viscosity.

Fluid satisfying this equation is called Newtonian fluids, which are fluids for whom stress is proportional to strain, with the constant of proportionality being the molecular viscosity [25].

For the previous equation we can write it in terms of an arbitrary coordinate system as

$$\tau_{ij} = \mu \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right] \tag{1.7.3.1}$$

where $x_i = (x\,, y\,, z)$ and $v_i = (\, v_x\,, v_y\,, v_z\,)$.

Back to the fluid cube and consider it is in the bottom of the flow using the equation (1.7.3.1) and put it in the Euler's equation so we get,

$$\rho\left[\frac{\partial v}{\partial t}+(v\,.\nabla)v\right]= -\nabla p+2\mu\frac{\partial^2 v_i}{\partial x_i^2}+\ \nabla.\tau+f$$

where $-\nabla p+2\mu\frac{\partial^2 v_i}{\partial x_i^2}$ are the normal stress [29].

$$\rho\left[\frac{\partial v}{\partial t}+(v\,.\nabla)v\right]= -\nabla p+2\mu\frac{\partial^2 v_i}{\partial x_i^2}+\ \frac{\partial}{\partial x_j}\,\mu\left[\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right]+f$$

now if we take the x direction for example,

$$\rho\left[\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right]$$
$$=-\frac{\partial p}{\partial x}+\mu\left(2\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_y}{\partial x\partial y}+\frac{\partial^2 v_x}{\partial z^2}+\frac{\partial^2 v_z}{\partial x\partial z}\right)+f_x$$

using the continuity equation the incompressibility of the fluid and the viscosity of being constant across the fluid. With this simplification the Navier-Stokes equation becomes

$$\left[\frac{\partial v}{\partial t}+(v\,.\nabla)v\right]= -\frac{1}{\rho}\nabla p+\ \gamma\nabla^2 v+\frac{1}{\rho}f$$

where $\gamma=\frac{\mu}{\rho}$ is the kinematic viscosity. This is a skeletal version of the Navier-Stokes equation.

Now if the fluid contains electrical charge ρ_c per unit volume, then there is a force per unit volume of $\rho_c E$. When an electric current density J flows through the fluid, there is a force per unit volume of $J\times B$.

Then, the Navier-Stokes equation becomes,

$$\rho\left[\frac{\partial v}{\partial t}+(v\,.\nabla)v\right]= -\nabla p+\ \mu\nabla^2 v+\rho_c E+J\times B\ +f$$

But comparing the magnetic force which we called Lorentz force with the electric force we can neglect the electric force; because the flows that we are interested in, there speeds are very small compared with the speed of light c [10].

So we have the Navier-Stokes equations for incompressible viscous fluid subject to a magnetic field to be

$$\nabla. v=0$$

$$\rho\left[\frac{\partial v}{\partial t}+(v\,.\nabla)v\right]= -\nabla p+\ \mu\nabla^2 v+J\times B\ +f$$

1.8 Ohm's Law

Ohm's law states that the total electric current flowing in a conductor is proportional to the total electric field. In addition to the field E acting on a fluid at rest, a fluid moving with velocity v in the presence of a magnetic field B is subject to an additional electric field $v\times B$. Ohm's law then gives

$$J=\ \sigma\,[\,E+\ v\times B\,]$$

where the constant of proportionality σ is called the electrical conductivity [10]. Note that Ohm's Law couples the electromagnetic equations to the fluid motion equations through v, the fluid velocity. For more specific let us take Lorentz force which is perpendicular to both velocity and the magnetic field, From experimental observation it has been found that the foundational force relation in magnetism is

$$f_{mag} = Q(\, v \times B\,)$$

this law describes the force on a test charge Q, when it is moving with velocity v in magnetic field B. For collisional particles that are forced to follow the fluid velocity v, the Lorentz force acts on electrons and ions in a direction perpendicular to the flow, but in opposite directions for positive and negative charges. The net result is charge separation, leading to electric fields [40].

In the regime between collisional and collisionless particles, the Hall effect can be important. Hall effect refers to potential difference (Hall voltage) on opposite sides of a thin sheet of conducting or semi-conducting material through which an electric current is flowing, created by a magnetic field applied perpendicular to the Hall element. Usually the current induced in the fluid is carried predominantly by electrons, which are considerably more mobile than ions. So the electron drift velocity [15], given by:

$$J = n_e e v_e$$

where , $n_e$the number density of electron , v_e electron drift velocity and e is the charge of one electron.

leads to a second component of velocity, and so a secondary force and electric field:

$$E_H = \beta J \times B$$

where $\beta = \frac{1}{n_e e}$ is the Hall constant. The current component created by this electric field, that is to say the Hall current, is given by $-\mu_e J \times B$, where

$\mu_e = \frac{\omega\,\tau}{B}$ is the electron mobility. Where ω is the electron cyclotron frequency and τ is the electron collision mean free time [40].

This leads to a more generalized statement of Ohm's Law including the Hall effect:

$$J = \sigma\,[\,E + v \times B\,] - \mu_e\,J \times B\ .$$

CHAPTER TWO

FORMULATION OF THE PROBLEMS

CHAPTER TWO
FORMULATION OF THE PROBLEMS

2.1 MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System without Hall Effect

We consider the unsteady flow of an electrically conducting incompressible viscous fluid past an Infinite flat plate occupying the plane $z = 0$. Initially the fluid and the plate rotate in unison with a uniform angular velocity Ω_z about the z-axis normal to the plate. The x-axis is taken in the direction of the motion of the plate and y-axis lying on the plate normal to both x and z-axes. Relative to the rotating fluid, the plate is impulsively started from rest and set into motion with uniform velocity in its own plane along the x-axis. A uniform magnetic field B_0, parallel to z-axis is imposed and the plate is electrically non-conducting. Due to the horizontal homogeneity of the problem, the flow quantities depend on z and t only, t being the time variable [7]. The equations describing the unsteady flow are :

Equation of continuity :

$$\nabla . q = 0 \tag{2.1.1}$$

Equation of motion:

$$\frac{\partial q}{\partial t} + (q \, . \nabla) q = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 q + \frac{1}{\rho} J \times B - 2\Omega \times q - \Omega \times (\Omega \times r) \tag{2.1.2}$$

The Continuity Equation for Charge:

$$\nabla . J = 0 \tag{2.1.3}$$

Ohm's law :

$$J = \sigma \left[E + q \times B \right] \tag{2.1.4}$$

Gauss's law of magnetism:

$$\nabla . B = 0 \tag{2.1.5}$$

where q is the velocity vector, Ω is the angular velocity of the fluid, r is the position of the fluid particle considered, ρ is the fluid density, p is the pressure, J is the current density, B is the magnetic vector, μ is the coefficient of viscosity, σ is the electrical conductivity, E is the electric field, $2\Omega \times q$ is the Coriolis acceleration, $\Omega \times (\Omega \times r)$ is the centripetal acceleration.

As the plate is infinite, there is no x and y dependence. So the flow quantities depend on z and t only, t being the time variable .

From equation (2.1.1) we have that

$$\frac{\partial w}{\partial z} = 0$$

and as the plate is flat then

$$w = 0$$

similarly from equation (2.1.3) we have J_z = constant, which must equal to zero as the plate is electrically non-conducting.

Finally, from equation (2.1.5) we have $B_z =$ constant $= B_0$ everywhere in the flow.

Thus :

$$J = \left(J_x, J_y, 0\right) \quad q = (u, v, 0) \quad B = (B_x, B_y, B_0)$$

we have ignored the ion-slip effects, electron pressure gradient and assume that the electric field $E = 0$. Under these assumptions equation (1.2.4) takes the form :

$$J = \sigma q \times B \tag{2.1.6}$$

therefore we have :

$$J_x = \sigma B_0 v \tag{2.1.7}$$

$$J_y = -\sigma B_0 u \tag{2.1.8}$$

the entire system is rotating with angular velocity Ω about the normal to the plate and $|\Omega|$ is so small that $\Omega \times (\Omega \times r)$ can be neglected. Magnetic Reynolds number is small enough to neglect magnetic induction effects [1], and in the absence of pressure gradient where the pressure is uniform in the flow field, equation (1.2.2) along with (2.1.7) and (2.1.8) comprises :

for $u(z, t)$ so we have

$$\frac{1}{\rho} J \times B = \frac{1}{\rho} J_y B_0$$

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial z^2} - \frac{\sigma B_0^2 u}{\rho} + 2\Omega_z v \tag{2.1.9}$$

for $v\ (z\,, t\,)$ so we have

$$\frac{1}{\rho} J \times B = -\frac{1}{\rho} J_x B_0$$

$$\frac{\partial v}{\partial t} = \gamma \frac{\partial^2 v}{\partial z^2} - \frac{\sigma B_0^2\, v}{\rho} - 2\Omega_z u \qquad (2.1.10)$$

here u is the axial velocity (along the direction of the plate) and v is the transverse velocity (transverse to the main flow) and γ is the kinematic viscosity. The initial and boundary conditions are given by :

$$u = 0\,, v = 0 \quad at \quad t \leq 0 \quad for\ all \quad z \qquad (2.1.11)$$

$$\begin{cases} u = A, \quad v = 0 \quad at\ \ z = 0 \\ u \to 0\,, \quad v \to 0 \quad at\ \ z \to \infty \end{cases} \quad t > 0 \qquad (2.1.12)$$

where $(A > 0\ )$ is a constant .

Now we introduce the non-dimensional quantities [7]:

$$U = \frac{u}{(A\gamma)^{\frac{1}{3}}} \quad , \quad V = \frac{v}{(A\gamma)^{\frac{1}{3}}} \quad , \quad Z = z\left(\frac{A}{\gamma^2}\right)^{\frac{1}{3}} \,, T = t\left(\frac{A^2}{\gamma}\right)^{\frac{1}{3}} \,,$$

$$\Omega = \Omega_z \left(\frac{\gamma^2}{A}\right)^{\frac{1}{3}} \,, M^2 = \frac{\sigma B_0^2 \gamma^{\frac{2}{3}}}{2\rho A^{\frac{1}{3}}}$$

where M is the Hartman number and Ω is the rotation parameter.

Then we have:

$$\frac{\partial U}{\partial T} = \left(\frac{A^2}{\gamma}\right)^{\frac{1}{3}} \frac{\partial u}{\partial t} \qquad \frac{\partial^2 U}{\partial Z^2} = \left(\frac{A}{\gamma^2}\right)^{\frac{2}{3}} \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial V}{\partial T} = \left(\frac{A^2}{\gamma}\right)^{\frac{1}{3}} \frac{\partial v}{\partial t} \qquad \frac{\partial^2 V}{\partial Z^2} = \left(\frac{A}{\gamma^2}\right)^{\frac{2}{3}} \frac{\partial^2 v}{\partial z^2}$$

and the equations (2.1.9), (2.1.10) and boundary conditions (2.1.11), (2.1.12) becomes:

$$\frac{\partial U}{\partial T}=\frac{\partial^2 U}{\partial Z^2}+\ 2\Omega V-\ \ 2\,M^2 U \tag{2.1.13}$$

$$\frac{\partial V}{\partial T}=\frac{\partial^2 V}{\partial Z^2}-\ 2\Omega U-\ \ 2\,M^2 V \tag{2.1.14}$$

$$U=0\,,V=0 \quad at \quad T\leq\ \ 0 \quad for\,all \quad\ Z \tag{2.1.15}$$

$$\begin{cases} U=1, \quad V=0 \quad at\ Z=0 \\ U\ \rightarrow 0\,,\quad V\ \rightarrow 0\ \ at\ Z\ \ \rightarrow\ \ \infty \end{cases} T>0\ \ . \tag{2.1.16}$$

2.2 MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System Including Hall Effect

We consider the unsteady flow of an electrically conducting, incompressible viscous fluid past an Infinite flat plate occupying the plane $z=0$. Initially the fluid and the plate rotate in unison with a uniform angular velocity Ω_z about the z-axis normal to the plate. The x-axis is taken in the direction of the motion of the plate and y-axis lying on the plate normal to both x and z-axes. Relative to the rotating fluid, the plate is impulsively started from rest and set into motion with uniform velocity in its own plane along the x-axis. A uniform magnetic field B_0, parallel to z-axis is imposed and the plate is electrically non-conducting. Due to the horizontal homogeneity of the problem.

The equations describing the unsteady flow are [7] :

Equation of continuity:

$$\nabla . q = 0 \qquad (2.2.1)$$

Equation of motion:

$$\frac{\partial q}{\partial t} + (q\,.\nabla)q = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 q + \frac{1}{\rho}J \times B - 2\Omega \times q - \Omega \times (\Omega \times r) \qquad (2.2.2)$$

The Continuity Equation for Charge:

$$\nabla . J = 0 \qquad (2.2.3)$$

General Ohm's law:

$$J + \frac{\omega\,\tau}{B_0} J \times B = \sigma\,[\,E + q \times B + \frac{1}{\rho_e n_e}\nabla p_e] \qquad (2.2.4)$$

Gauss's law of magnetism:

$$\nabla . B = 0 \qquad (2.2.5)$$

where q is the velocity vector, Ω is the angular velocity of the fluid, r is the position of the fluid particle considered, ρ is the fluid density, p is the pressure, J is the current density, B is the magnetic vector, μ is the coefficient of viscosity, σ is the electrical conductivity, ω is the electron frequency, τ is the electron collision time, n_e is the number density of electron, p_e is the electron pressure, E is the electric field, $2\Omega \times q$ is the Coriolis acceleration, $\Omega \times (\Omega \times r)$ is the centripetal acceleration.

As the plate is infinite, there is no x and y dependence. So the flow quantities depend on z and t only, t being the time variable .

From equation (2.1.1) we have that

$$\frac{\partial w}{\partial z} = 0$$

and as the plate is flat then

$$w = 0$$

similarly from equation (2.1.3) we have J_z = constant, which must equal to zero as the plate is electrically non-conducting.

Finally, from equation (2.1.5) we have $B_z =$ constant $= B_0$ everywhere in the flow.

Thus:

$$J = \left(J_x, J_y, 0\right) \quad q = (u, v, 0) \quad B = (B_x, B_y, B_0)$$

we have ignored the ion-slip effects, electron pressure gradient and assume that the electric field $E = 0$. Under these assumption equation (1.2.4) takes the form :

$$J = \sigma q \times B - \frac{m}{B_0} J \times B \tag{2.2.6}$$

where $m = \omega \tau$ is the Hall parameter .

Then we have :

$$J_x = \sigma B_0 v - m J_y \tag{2.2.7}$$

$$J_y = -\sigma B_0 u + m J_x \tag{2.2.8}$$

on solving equation (2.2.7) and (2.2.8) we conclude :

$$J_x = \frac{\sigma B_0\,(\,v + mu)}{(1 +\ m^2)} \tag{2.2.9}$$

$$J_y = \frac{\sigma B_0\,(mv -\ u)}{(1 +\ m^2)} \tag{2.2.10}$$

the entire system is rotating with angular velocity $\boldsymbol{\Omega}$ about the normal to the plate and $|\boldsymbol{\Omega}|$ is so small that $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times r)$ can be neglected, Magnetic Reynolds number is small enough to neglect magnetic induction effects [1], and in the absence of pressure gradient where the pressure is uniform in the flow field, equation (2.2.2) along with (2.2.9) and (2.2.10) comprises:

for $u\ (\ z\ , t\ )$ so we have

$$\frac{1}{\rho}\, J \times B = \frac{1}{\rho}\, J_y\, B_0$$

$$\frac{\partial u}{\partial t} = \gamma\ \frac{\partial^2 u}{\partial z^2} - \frac{\sigma B_0^2\,(mv +\ u)}{\rho(1 +\ m^2)} +\ 2\Omega_z v \tag{2.2.11}$$

for $v\ (\ z\ , t\ )$ so we have

$$\frac{1}{\rho}\, J \times B = -\frac{1}{\rho}\, J_x B_0$$

$$\frac{\partial v}{\partial t} = \gamma\ \frac{\partial^2 v}{\partial z^2} + \frac{\sigma B_0^2\,(mu -\ v)}{\rho(1 +\ m^2)}\ -\ 2\Omega_z u \tag{2.2.12}$$

here u is the axial velocity (along the direction of the plate) and v is the transverse velocity (transverse to the main flow) and γ is the kinematic viscosity. The initial and boundary conditions are given by :

$$u = 0\,, v = 0 \quad at \quad t \leq\ 0 \quad for\ all \quad\ z \tag{2.2.13}$$

$$\begin{cases} u = A, \quad v = 0 \quad at\ z = 0 \\ u\ \rightarrow 0\,, \quad v\ \rightarrow 0\ \ at\ z\ \ \rightarrow\ \infty \end{cases} \quad t > 0 \tag{2.2.14}$$

where $(A > 0)$ is a constant .

Now we introduce the non-dimensional quantities [7]:

$$U = \frac{u}{(A\gamma)^{\frac{1}{3}}} \quad , \qquad V = \frac{v}{(A\gamma)^{\frac{1}{3}}} \quad , \qquad Z = z\left(\frac{A}{\gamma^2}\right)^{\frac{1}{3}} \ , T = t\left(\frac{A^2}{\gamma}\right)^{\frac{1}{3}} \ ,$$

$$\Omega = \Omega_z(\frac{\gamma^2}{A})^{\frac{1}{3}} \ , M^2 = \frac{\sigma B_0^2 \gamma^{\frac{2}{3}}}{2\rho A^{\frac{1}{3}}}$$

where M is the Hartman number and Ω is the rotation parameter.

Hence we have :

$$\frac{\partial U}{\partial T} = (\frac{A^2}{\gamma})^{\frac{1}{3}} \frac{\partial u}{\partial t} \qquad \frac{\partial^2 U}{\partial Z^2} = (\frac{A}{\gamma^2})^{\frac{2}{3}} \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial V}{\partial T} = (\frac{A^2}{\gamma})^{\frac{1}{3}} \frac{\partial v}{\partial t} \qquad \frac{\partial^2 V}{\partial Z^2} = (\frac{A}{\gamma^2})^{\frac{2}{3}} \frac{\partial^2 v}{\partial z^2}$$

and the equations (2.2.11), (2.2.12) and boundary conditions (2.2.13), (2.2.14) becomes:

$$\frac{\partial U}{\partial T} = \frac{\partial^2 U}{\partial Z^2} + 2V\left(\Omega - \frac{M^2 m}{(1+m^2)}\right) - 2\frac{M^2 U}{(1+m^2)} \qquad (2.2.15)$$

$$\frac{\partial V}{\partial T} = \frac{\partial^2 V}{\partial Z^2} - 2U\left(\Omega - \frac{M^2 m}{(1+m^2)}\right) - 2\frac{M^2 V}{(1+m^2)} \qquad (2.2.16)$$

$$U = 0 , V = 0 \quad at \quad T \leq 0 \quad for\ all \quad Z \qquad (2.2.17)$$

$$\begin{cases} U = 1, \quad V = 0 \quad at\ Z = 0 \\ U \to 0, \quad V \to 0 \quad at\ Z \to \infty \end{cases} T > 0 \ . \qquad (2.2.18)$$

2.3 Unsteady MHD Flow Through Two Parallel Porous Flat Plates without Hall Effect

We consider unsteady MHD flow of an electrically conducting, incompressible viscous fluid through two parallel porous flat plates, we assume that the fluid is being injected into the flow region with constant velocity v_0 and being sucked away in the same speed, the plates located at $y = 0, y = d$. Let the x - axis be taken along the plates and y - axis normal to the plates, the fluid is subjected to a constant transverse magnetic field of strength B_0 in the y direction and we take the flow to be two dimensional. The equations describing the unsteady flow are :

Equation of continuity:

$$\nabla . q = 0 \qquad (2.3.1)$$

Equation of motion:

$$\frac{\partial q}{\partial t} + (q \, . \nabla) q = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 q + \frac{1}{\rho} J \times B \qquad (2.3.2)$$

Ohm's law:

$$J = \sigma \, [E + q \times B] \qquad (2.3.3)$$

Gauss's law of magnetism:

$$\nabla . B = 0 \qquad (2.3.4)$$

where q is the velocity vector, ρ is the fluid density, p is the pressure, J is the current density, B is the magnetic vector, μ is the coefficient of viscosity, σ is the electrical conductivity and E is the electric field.

We consider E to be neglected, Magnetic Reynolds number is small enough to neglect magnetic induction effects . Under these assumptions equations (2.3.2), (2.3.3) takes the form :

$$J = \left(J_x , J_y , J_z\right) \quad q = (u , v_0 , 0) \quad B = (0 , B_0 , 0)$$

$$J = \sigma q \times B \qquad (2.3.5)$$

then we have :

$$J_x = 0 \qquad (2.3.6)$$

$$J_y = 0 \qquad (2.3.7)$$

$$J_z = \sigma B_0 \, u \qquad (2.3.8)$$

for u

$$\frac{1}{\rho} J \times B = -\frac{1}{\rho} J_z B_0$$

and for v

$$\frac{1}{\rho} J \times B = 0$$

hence, the partial differential equations are :

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \gamma \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2 \, u}{\rho} \qquad (2.3.9)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \gamma\frac{\partial^2 v}{\partial y^2} \quad (2.3.10)$$

subject to the initial and boundary conditions :

$$\begin{cases} u = 0, v = 0, & t \leq 0 \\ u = 0, v = v_0, y = 0, d, & t > 0 \end{cases} \quad (2.3.11)$$

where γ is the kinematic viscosity.

As the plates are infinite, there is no x dependence [30], using that in equation (2.3.1) we have

$$\frac{\partial v}{\partial y} = 0$$

then we have

$$v = v_0$$

putting this in equations (2.3.9) and (2.3.10) we obtain

$$\frac{\partial u}{\partial t} + v_0\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \gamma\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2 u}{\rho} \quad (2.3.12)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad (2.3.13)$$

subject to the initial and boundary conditions :

$$\begin{cases} u = 0, v = 0, & t \leq 0 \\ u = 0, v = v_0, y = 0, d, & t > 0 \end{cases} \quad (2.3.14)$$

now we introduce the non-dimensional quantities [23]:

$$Y = \frac{y}{d} \quad , T = \frac{v_0 t}{d} \quad , U = \frac{u}{v_0} \quad , X = \frac{x}{d}$$

$$M^2 = \frac{\sigma B_0^2 d}{\rho v_0} \quad , \quad R = \frac{v_0 d}{\gamma} \quad , P = \frac{p}{\rho v_0^2}$$

where M is the Hartman number and R is Reynolds number.

Hence the partial differential equations with the initial and boundary conditions become :

$$\frac{\partial U}{\partial T} + \frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{1}{R}\frac{\partial^2 U}{\partial Y^2} - M^2\, U \qquad (2.3.15)$$

$$0 = \frac{\partial P}{\partial Y} \qquad (2.3.16)$$

$$\begin{cases} U = 0\,, & T \leq 0 \\ U = 0\,, Y = 0\,,1 & ,T > 0 \end{cases} \quad . \qquad (2.3.17)$$

2.4 Unsteady MHD Flow Through Two Parallel Porous Flat Plates With Hall Effect

We consider unsteady MHD flow of an electrically conducting, incompressible viscous fluid through two parallel porous flat plates with Hall effect, we assume that the fluid is being injected into the flow region with constant velocity v_0 and being sucked away in the same speed, the plates located at $y = 0$, $y = d$. Let the x - axis be taken along the plates and y - axis normal to the plates, the fluid is subjected to a constant transverse magnetic field of strength B_0 in the y direction and we take the flow to be two dimensional. The equations describing the unsteady flow are:

Equation of continuity:

$$\nabla . q = 0 \qquad (2.4.1)$$

Equation of motion:

$$\frac{\partial q}{\partial t} + (q\,.\nabla)q = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 q + \frac{1}{\rho}J \times B \qquad (2.4.2)$$

General Ohm's law:

$$J + \frac{\omega\,\tau}{B_0}J \times B = \sigma\left[E + q \times B + \frac{1}{\rho_e n_e}\nabla p_e\right] \qquad (2.4.3)$$

Gauss's law of magnetism:

$$\nabla.B = 0 \qquad (2.4.4)$$

where q is the velocity vector, ρ is the fluid density, p is the pressure, J is the current density, B is the magnetic vector, μ is the coefficient of viscosity, σ is the electrical conductivity, ω is the electron frequency, τ is the electron collision time, n_e is the number density of electron, p_e is the electron pressure and E is the electric field.

We have ignored the ion-slip effects, electron pressure gradient, Magnetic Reynolds number is small enough to neglect magnetic induction effects and assume that the electric field $E = 0$. Under these assumptions the equation (1.4.3) takes the form :

$$J = \left(J_x, J_y, J_z\right) \quad q = (u, v_0, 0) \quad B = (0, B_0, 0)$$

$$J = \sigma q \times B - \frac{m}{B_0}J \times B \qquad (2.4.5)$$

where $m = \omega\,\tau$ is the Hall parameter .

therefore we have :

$$J_x = mJ_z \tag{2.4.6}$$

$$J_y = 0 \tag{2.4.7}$$

$$J_z = \sigma B_0 u + mJ_x \tag{2.4.8}$$

on solving those equations we conclude :

$$J_x = \frac{\sigma B_0\, mu}{(1 + m^2)} \tag{2.4.9}$$

$$J_z = \frac{\sigma B_0\, u}{(1 + m^2)} \tag{2.4.10}$$

now equation (2.4.2) along with (2.4.9) and (2.4.10) comprises :

for u

$$\frac{1}{\rho} J \times B = -\frac{1}{\rho} J_z B_0$$

and for v

$$\frac{1}{\rho} J \times B = 0$$

the partial deferential equations are :

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \gamma\,\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2 u}{\rho(1 + m^2)} \tag{2.4.11}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \gamma\,\frac{\partial^2 v}{\partial y^2} \tag{2.4.12}$$

subject to the initial and boundary conditions :

$$\begin{cases} u = 0\,, v = 0\ , & t \le 0 \\ u = 0\,, v = v_0, y = 0, d\,, t > 0 \end{cases} \tag{2.4.13}$$

where γ is the kinematic viscosity.

As the plates are infinite, there is no x dependence [30], using that in equation (2.4.1) we have

$$\frac{\partial v}{\partial y} = 0$$

then we have

$$v = v_0$$

putting this in equations (2.4.11) and (2.4.12) we obtain

$$\frac{\partial u}{\partial t} + v_0 \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \gamma \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2 u}{\rho(1+m^2)} \qquad (2.4.14)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (2.4.15)$$

subject to the initial and boundary conditions :

$$\begin{cases} u = 0\,, v = 0\,, & t \le 0 \\ u = 0\,, v = v_0, y = 0, d\,, & t > 0 \end{cases} \qquad (2.4.16)$$

now we introduce the non-dimensional quantities [23]:

$$Y = \frac{y}{d} \quad , T = \frac{v_0 t}{d} \quad , U = \frac{u}{v_0} \quad , X = \frac{x}{d}$$

$$M^2 = \frac{\sigma B_0^2 d}{\rho v_0} \quad , \quad R = \frac{v_0 d}{\gamma} \quad , P = \frac{p}{\rho v_0^2}$$

where M is the Hartman number and R is Reynolds number.

Hence the partial differential equation with the initial and boundary conditions becomes:

$$\frac{\partial U}{\partial T} + \frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{1}{R}\frac{\partial^2 U}{\partial Y^2} - \frac{M^2}{(1+ m^2)} U \qquad (2.4.17)$$

$$0 = \frac{\partial P}{\partial Y} \qquad (2.4.18)$$

$$\begin{cases} U = 0\,, & T \le 0 \\ U = 0\,, Y = 0\,,1 & , T > 0 \end{cases} . \qquad (2.4.19)$$

CHAPTER THREE
ANALYTICAL SOLUTIONS

CHAPTER THREE
ANALYTICAL SOLUTIONS

3.1 Introduction

In this chapter we solved the problems that have been formulated in chapter two analytically to obtain exact solution.

3.2 MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System without Hall Effect

In this section an exact solution based on Laplace transform to this problem is introduced.

Equations (2.1.13) and (2.1.14) along with boundary conditions (2.1.15) and (2.1.16) are solvable by combining (2.1.13) and (2.1.14) we get :

$$\frac{\partial Q}{\partial T} = \frac{\partial^2 Q}{\partial Z^2} - aQ \tag{3.2.1}$$

with the initial and boundary conditions :

$$Q = 0 \quad at \quad T \leq \; 0 \quad for\, all \qquad Z$$

$$\begin{cases} Q = 1, & at \;\; Z = 0 \\ Q \;\rightarrow 0\,, & at \;\; Z \;\; \rightarrow \;\; \infty \end{cases} \; T > 0$$

where $Q = U + iV$ and $a = 2[\; M^2 \; + i\Omega\,]$.

Assume

$$\varphi(T\,, Z) = \; e^{aT} Q(T\,, Z)$$

then we have

$$\frac{\partial \varphi}{\partial T} = e^{aT}\frac{\partial Q}{\partial T} + ae^{aT}Q$$

$$\frac{\partial \varphi}{\partial Z} = e^{aT}\frac{\partial Q}{\partial Z}$$

$$\frac{\partial^2 \varphi}{\partial Z^2} = e^{aT}\ \frac{\partial^2 Q}{\partial YZ^2}$$

using this in the equation (3.2.1) we obtain

$$\frac{\partial \varphi}{\partial T} = \frac{\partial^2 \varphi}{\partial Z^2} \qquad (3.2.2)$$

with the initial and boundary conditions :

$$\begin{cases} \varphi(0\,,Z) = 0\,, \quad \varphi(T\,,0) = e^{aT} \\ \quad \varphi(T\,,Z) \quad \to 0 \text{ } as \text{ } Z \ \to \infty \end{cases} \qquad (3.2.3)$$

now, by taking the Laplace transform of equation (3.2.2) with respect to the variable T we have [36] :

$$L\left\{\frac{\partial \varphi}{\partial T}\right\} = s\,\hat{Q}(s,Z) - \varphi(\,0\,,Z)$$

$$L\left\{\frac{\partial^2 \varphi}{\partial Z^2}\right\} = \frac{d^2}{dZ^2}\hat{Q}(\,s\,,Z)$$

then we have

$$s\,\hat{Q}(s,Z) = \frac{d^2}{dZ^2}\hat{Q}(\,s\,,Z) \qquad (3.2.4)$$

where

$$\hat{Q}(s,Z) = L\,\{\,\varphi(\,T\,,Z)\}$$

and

$$\hat{Q}(s,0) = L\{\varphi(T,0)\} = L\{e^{aT}\} = \frac{1}{(s-a)}$$

$$\lim_{Z\to\infty} \hat{Q}(s,Z) = 0$$

the auxiliary equation for equation (3.2.4) can be written as :

$$\delta^2 - s = 0$$

$$\delta = \pm\sqrt{s}$$

hence

$$\hat{Q}(s,Z) = c_1 e^{Z\sqrt{s}} + c_2 e^{-Z\sqrt{s}} \qquad (3.2.5)$$

claim $c_1 = 0$.

Proof of claim :

Dividing both sides of equation (3.2.5) by $e^{Z\sqrt{s}}$

$$e^{-Z\sqrt{s}}\,\hat{Q}(s,Z) = c_1 + c_2 e^{-2Z\sqrt{s}}$$

now, taking the limit of both sides as $Z \to \infty$:

$$0 = c_1 + 0$$

$$c_1 = 0$$

hence,

$$\hat{Q}(s,Z) = c_2 e^{-Z\sqrt{s}} \qquad (3.2.6)$$

putting $Z = 0$ in equation (3.2.6) we have :

$$\hat{Q}(s,0) = c_2 = \frac{1}{(s-a)}$$

$$c_2 = \frac{1}{(s-a)}$$

then we obtain

$$\hat{Q}(s,Z) = \frac{1}{(s-a)} e^{-Z\sqrt{s}}$$

now

$$\varphi(T,Z) = L^{-1}\{\hat{Q}(s,Z)\} = L^{-1}\{\frac{1}{(s-a)} e^{-Z\sqrt{s}}\}$$

using the Hetnarski algorithm [20], we obtain the inverse Laplace transform of $\hat{Q}$, that is

$$\varphi(T,Z) = \frac{e^{aT}}{2}[e^{-Z\sqrt{a}} erfc\left(\frac{Z}{2\sqrt{T}} - \sqrt{aT}\right) + e^{Z\sqrt{a}} erfc\left(\frac{Z}{2\sqrt{T}} + \sqrt{aT}\right)]$$

now

$$Q(T,Z) = e^{-aT}\varphi(T,Z)$$

therefore

$$Q(T,Z) = \frac{1}{2}[e^{-Z\sqrt{a}} erfc\left(\frac{Z}{2\sqrt{T}} - \sqrt{aT}\right) + e^{Z\sqrt{a}} erfc\left(\frac{Z}{2\sqrt{T}} + \sqrt{aT}\right)]$$

where

$$erfc(x) = 1 - \mathrm{erf}(x)$$

$$erfc(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du \quad .$$

3.3 MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System Including Hall Effect

In this section an exact solution based Laplace transform to this problem is introduced.

Equations (2.2.15) and (2.2.16) along with boundary conditions (2.2.17) and (2.2.18) are solvable by combining (2.2.15) and (2.2.16) we get :

$$\frac{\partial Q}{\partial T} = \frac{\partial^2 Q}{\partial Z^2} - aQ \qquad (3.3.1)$$

with boundary conditions :

$$Q = 0 \quad at \quad T \leq \ 0 \quad for\ all \quad Z \qquad (3.3.2)$$

$$\begin{cases} Q = 1, & at\ \ Z = 0 \\ Q \to 0\,, & at\ \ Z\ \to\ \infty \end{cases} \quad T > 0 \qquad (3.3.3)$$

where $Q = U + iV$ and $a = 2[\frac{M^2}{(1+ m^2)} + i\left(\Omega - \frac{M^2 m}{(1+ m^2)}\right)]$.

Assume

$$\varphi(T\,, Z) = \ e^{aT} Q(T\,, Z) \qquad (3.3.4)$$

now multiplying (3.3.1) by (e^{aT}) we get :

$$\frac{\partial \varphi}{\partial T} = \frac{\partial^2 \varphi}{\partial Z^2} \qquad (3.3.5)$$

From equations (3.3.2) and (3.3.3) and (3.3.4) we obtain:

$$\begin{cases} \varphi(0\,,Z) = 0\,, \quad \varphi(T\,,0) = \; e^{aT} \\ \quad \varphi(T\,,Z) \quad \to 0 \; as \; Z \; \to \; 0 \end{cases} \tag{3.3.6}$$

now, by taking the Laplace transform of equation (3.3.5) with respect to the variable T we have [36]:

$$L\left\{\frac{\partial \varphi}{\partial T}\right\} = s\,\hat{Q}(s,Z) - \;\varphi(\,0\,,Z)$$

$$L\left\{\frac{\partial^2 \varphi}{\partial Z^2}\right\} = \;\frac{d^2}{dZ^2}\hat{Q}(\,s\,,Z)$$

$$s\,\hat{Q}(s,Z) = \;\frac{d^2}{dZ^2}\hat{Q}(\,s\,,Z) \tag{3.3.7}$$

where

$$\hat{Q}(s,Z) = L\,\{\,\varphi(\,T\,,Z)\}$$

and

$$\hat{Q}(s,0) = \;L\,\{\,\varphi(\,T\,,0)\} = L\{e^{-aT}\} = \;\frac{1}{(s+a\,)}$$

$$\lim_{Z\to\infty} \hat{Q}(s,Z) = 0 \tag{3.3.8}$$

the auxiliary equation for equation (3.3.7) can be written as :

$$\delta^2 - s = 0$$

$$\delta = \;\pm\sqrt{s}$$

hence,

$$\hat{Q}(s,Z) = \;c_1 e^{Z\sqrt{s}} \;+\; c_2 e^{-Z\sqrt{s}} \tag{3.3.9}$$

Claim $c_1 = 0$.

Proof of claim :

Dividing both sides of equation (3.3.9) by $e^{Z\sqrt{s}}$

$$e^{-Z\sqrt{s}}\,\hat{Q}(s,Z) = c_1 + c_2 e^{-2Z\sqrt{s}}$$

now, taking the limit of both sides as $Z \to \infty$:

$$0 = c_1 + 0$$

then

$$c_1 = 0$$

hence

$$\hat{Q}(s,Z) = c_2 e^{-Z\sqrt{s}} \qquad (3.3.10)$$

putting $Z = 0$ in equation (3.3.10) we have :

$$\hat{Q}(s,0) = c_2 = \frac{1}{(s+a)}$$

hence

$$c_2 = \frac{1}{(s+a)}$$

then we get

$$\hat{Q}(s,Z) = \frac{1}{(s+a)} e^{-Z\sqrt{s}} \qquad (3.3.11)$$

now

$$\varphi(T,Z) = L^{-1}\{\hat{Q}(s,Z)\} = L^{-1}\{\frac{1}{(s+a)}e^{-Z\sqrt{s}}\}$$

Using the Hetnarski algorithm [20], we obtain the inverse Laplace transform of $\hat{Q}$, that is

$$\varphi(T,Z) = \frac{e^{aT}}{2}[e^{-Z\sqrt{a}}erfc\left(\frac{Z}{2\sqrt{T}} - \sqrt{aT}\right) + e^{Z\sqrt{a}}erfc\left(\frac{Z}{2\sqrt{T}} + \sqrt{aT}\right)]$$

now

$$Q(T,Z) = e^{-aT}\varphi(T,Z)$$

therefore

$$Q(T,Z) = \frac{1}{2}[e^{-Z\sqrt{a}}erfc\left(\frac{Z}{2\sqrt{T}} - \sqrt{aT}\right) + e^{Z\sqrt{a}}erfc\left(\frac{Z}{2\sqrt{T}} + \sqrt{aT}\right)]$$

where

$$erfc(x) = 1 - \mathrm{erf}\,(x)$$

$$erfc(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du\ .$$

3.4 The Analytical Solution For The Unsteady MHD Flow Through Two Parallel Porous Flat Plates Without Hall Effect

In this section we want to solve equations (2.3.15) and (2.3.16) subject to the initial and boundary conditions (2.3.17).

From equation (2.3.16) we see that the pressure is independent of Y. So it is a function of T only, but in our case we will take the pressure gradient as a constant quantity that is

$$\frac{\partial P}{\partial X} = -P_0$$

where $P_0 > 0$, thus equation (2.3.15) becomes

$$\frac{\partial U}{\partial T} + \frac{\partial U}{\partial Y} = P_0 + \frac{1}{R}\frac{\partial^2 U}{\partial Y^2} - M^2\, U \qquad (3.4.1)$$

now, by taking the Laplace transform of equation (3.4.1) with respect to the variable T we obtain :

$$L\left\{\frac{\partial U}{\partial T}\right\} = s\,\widehat{U}(s,Y) - U(0,Y)$$

$$L\left\{\frac{\partial U}{\partial Y}\right\} = \frac{d}{dY}\widehat{U}(s,Y)$$

$$L\left\{\frac{\partial^2 U}{\partial Y^2}\right\} = \frac{d^2}{dY^2}\widehat{U}(s,Y)$$

$$\frac{d^2}{dY^2}\widehat{U}(s,Y) - R\frac{d}{dY}\widehat{U}(s,Y) - R(s+M^2)\,\widehat{U}(s,Y) = -\frac{RP_0}{s} \qquad (3.4.2)$$

where

$$\widehat{U}(s,Y) = L\,\{U(T,Y)\}$$

$$\widehat{U}(s,0) = 0 \quad , \quad \widehat{U}(s,1) = 0 \qquad (3.4.3)$$

now we want to solve the linear nonhomogeneous second order differential equation (3.4.2). We will solve the associated homogeneous equation

$$\frac{d^2}{dY^2}\widehat{U}(s,Y) - R\frac{d}{dY}\widehat{U}(s,Y) - R(s+M^2)\widehat{U}(s,Y) = 0 \qquad (3.4.4)$$

the auxiliary equation for equation (3.4.4) can be written as :

$$\beta^2 - R\beta - R(s+M^2) = 0$$

then we get

$$\beta_1 = Q + G$$

$$\beta_2 = Q - G$$

where

$$Q = \frac{R}{2}$$

and

$$G = \frac{\sqrt{R^2 + 4R(s + M^2)}}{2}$$

the general homogeneous solution is

$$\widehat{U}_h = c_1 e^{\beta_1 Y} + c_2 e^{\beta_2 Y} \tag{3.4.5}$$

the particular solution is [27],

$$\widehat{U}_p = A(y)\, Y_1 + B(y)\, Y_2$$

where

$$A(y) = \int \frac{-Y_2\, f(y)}{Y_1 Y_2' - Y_2 Y_1'}\, dy$$

$$B(y) = \int \frac{Y_1\, f(y)}{Y_1 Y_2' - Y_2 Y_1'}\, dy$$

$$Y_1 = e^{\beta_1 y} \quad , Y_2 = e^{\beta_2 y} \quad , f(y) = -\frac{RP_0}{s}$$

then we have

$$A(y) = -\frac{RP_0}{s\,(\beta_2 - \beta_1)}\frac{e^{-\beta_1 y}}{\beta_1}$$

$$B(y) = \frac{RP_0}{s\,(\beta_2 - \beta_1)}\frac{e^{-\beta_2 y}}{\beta_2}$$

thus

$$\hat{U}_p = -\frac{RP_0}{s\,(\beta_2 - \beta_1)\beta_1} + \frac{RP_0}{s\,(\beta_2 - \beta_1)\beta_2}$$

$$\hat{U}_p = -\frac{RP_0}{s\,\beta_1\beta_2} \qquad (3.4.6)$$

simplifying equation (3.4.6) where

$$\beta_1\beta_2 = \frac{1}{4}\left(R + \sqrt{R^2 + 4R(s + M^2)}\right)\left(R - \sqrt{R^2 + 4R(s + M^2)}\right)$$

$$= -R(s + M^2)$$

the particular solution is

$$\hat{U}_p = \frac{P_0}{s(s + M^2)} \qquad (3.4.7)$$

hence, the solution of the linear nonhomogeneous second order differential equation is

$$\hat{U}(s, Y) = \hat{U}_h + \hat{U}_p$$

$$\hat{U}(s, Y) = c_1 e^{\beta_1 Y} + c_2 e^{\beta_2 Y} + \frac{P_0}{s(s + M^2)} \qquad (3.4.8)$$

using (3.4.3) we obtain

$$0 = c_1 + c_2 + \frac{P_0}{s(s + M^2)} \qquad (3.4.9)$$

$$0 = c_1 e^{\beta_1} + c_2 e^{\beta_2} + \frac{P_0}{s(s+M^2)} \qquad (3.4.10)$$

solving equations (3.4.9) and (3.4.10) we get

$$c_1 = -\frac{(e^{\beta_2} - 1)\ P_0}{(e^{\beta_2} - e^{\beta_1})s(s+M^2)} \qquad (3.4.11)$$

$$c_2 = \frac{(e^{\beta_1} - 1)\ P_0}{(e^{\beta_2} - e^{\beta_1})s(s+M^2)} \qquad (3.4.12)$$

combining equation (3.4.8) with equations (3.4.11) and (3.4.12) we obtain

$$\hat{U}(s,Y) = -\frac{(e^{\beta_2} - 1)\ P_0}{(e^{\beta_2} - e^{\beta_1})s(s+M^2)} e^{\beta_1 Y} + \frac{(e^{\beta_1} - 1)\ P_0}{(e^{\beta_2} - e^{\beta_1})s(s+M^2)} e^{\beta_2 Y} + \frac{P_0}{s(s+M^2)}$$

simplifying this we have

$$\hat{U}(s,Y) = \frac{P_0}{s(s+M^2)(e^{\beta_2} - e^{\beta_1})} \left[-(e^{\beta_2} - 1)e^{\beta_1 Y} + (e^{\beta_1} - 1)e^{\beta_2 Y}\right] + \frac{P_0}{s(s+M^2)} \qquad (3.4.13)$$

since

$$e^{\beta_1} = e^Q e^G$$

$$e^{\beta_2} = e^Q e^{-G}$$

therefore

$$e^{\beta_2} - e^{\beta_1} = -e^Q\,(e^G - e^{-G})$$

hence

$$-(e^{\beta_2}-1)e^{\beta_1 Y}+(e^{\beta_1}-1)e^{\beta_2 Y}$$
$$=e^{(Y+1)Q}(e^{(1-Y)G}-e^{-(1-Y)G})+e^{QY}(e^{GY}-e^{-GY}) \quad (3.4.14)$$

using equation (3.4.14) in our solution (3.4.13) we obtain

$$\widehat{U}(s,Y)= \frac{P_0}{s(s+M^2)(\,e^{G}-\,e^{-G})}\left[-\,e^{QY}\left(e^{(1-Y)G}-\,e^{-(1-Y)G}\right)\right.$$
$$\left.-\,e^{Q(Y-1)}(e^{GY}-\,e^{-GY})\right]+\frac{P_0}{s(s+M^2)}$$

therefore,

$$\widehat{U}(s,Y)=\frac{P_0}{s(s+M^2)\sinh{(G)}}\left[-\,e^{QY}\sinh{[G(1-Y)]}\right.$$
$$\left.-\,e^{-Q(1-Y)}\sinh{(GY)}\right]$$
$$+\frac{P_0}{s(s+M^2)} \quad (3.4.15)$$

finding the Laplace invers transform. We have

$$U(\,T\,,Y)=L^{-1}\left[\widehat{U}(s,Y)\right]=\,L^{-1}[K(s,Y)]+\,L^{-1}[\frac{P_0}{s(s+M^2)}]$$

where

$$K(s,Y)$$
$$=\frac{P_0}{s(s+M^2)\sinh{(G)}}\left[-e^{QY}\sinh[G(1-Y)]\right.$$
$$\left.-\,e^{-Q(1-Y)}\,\sinh{(GY)}\right] \quad (3.4.16)$$

using [19] we get

$$L^{-1}\left[\frac{P_0}{s(s+M^2)}\right]=\frac{P_0}{M^2}\left(1-\,e^{-M^2\,T}\right) \quad (3.4.17)$$

now to find $k(T,Y)$ we use The complex inversion formula, which is called Bromwich integral, the Fourier–Mellin integral, and Mellin's inverse formula, and it given by

$$k(T,Y) = L^{-1}[K(s,Y)] = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} K(s,Y)e^{sT}dT$$

where the integration is done along the vertical line $Re(s) = \omega$ in the complex plane such that ω is greater than the real part of all singularities of $F(s) = K(s,Y)e^{sT}$ [41],[34].

To find this integral we use the Cauchy's residue theorem,

$$\int_{\omega-i\infty}^{\omega+i\infty} K(s,Y)e^{sT}dT = 2\pi i \times \sum_{r=1}^{n} Res(F(s),s_r)$$

where $Res(F(s),s_r)$ is the residue of $F(s)$ at the isolated singularity s_r is the coefficient of $(s - s_r)^{-1}$ in the Laurent expansion [41],[34].

To find the poles we use the following theorem.

Theorem 3.1 : Suppose that

$$F(s) = \frac{p(s)}{q(s)}$$

where $p(s_0) \neq 0$, and q has a zero of order m at s_0.Then F has a pole of order m at s_0 [34],[43].

In our case we have

$$p(s) = P_0\left[-e^{QY}\sinh[G(1-Y)] - e^{-Q(1-Y)}\sinh(GY)\right]e^{sT}$$

$$q(s) = s(s + M^2)\sinh(G)$$

setting $q(s) = 0$. We obtain the poles of $F(s)$ as,

$$s = 0$$

and

$$s = -M^2$$

and

$$s_n = -\frac{R^2 + 4\pi^2 n^2 + 4RM^2}{4R} \quad , n = 0,1,2,3 \ldots$$

and all of them of order one, that is to say all the poles are simple.

To find the residue of these poles we use the following theorem.

Theorem 3.2 : If $F(s)$ has a simple pole at s then

$$Res(F(s), s_0) = \lim_{s \to s_0} (s - s_0)\, F(s) \quad .$$

If

$$F(s) = \frac{p(s)}{q(s)}$$

where $p(s_0) = q'(s_0) \neq 0$ and $q(s_0) = 0$ [43],then we have

$$Res(F(s), s_0) = \frac{p(s_0)}{q'(s_0)} \quad .$$

Now for $s = 0$ we get

$$Res(F(s), 0) = \lim_{s \to 0} (s - 0)\, F(s)$$

$$Res(F(s),0) = \frac{P_0}{M^2 \sinh(Q)} [- e^{QY} \sinh[Q(1-Y)] - e^{-Q(1-Y)} \sinh(QY)]$$

where

$$G_0 = \frac{\sqrt{R^2}}{2} = Q$$

simplifying this we obtain

$$Res(F(s),0) = \frac{P_0[-e^{QY}\left(e^{(1-Y)Q} - e^{-(1-Y)Q}\right) - e^{-Q(1-Y)}(e^{QY} - e^{-QY})]}{2M^2 \sinh(Q)}$$

hence

$$Res(F(s),0) = -\frac{P_0}{M^2} \tag{3.4.18}$$

for $s = -M^2$ we have

$$Res(F(s), -M^2) = \lim_{s \to -M^2} (s + M^2) F(s)$$

$$Res(F(s), -M^2) = \frac{P_0 e^{-M^2 T}}{-M^2 \sinh(G_m)} [- e^{QY} \sinh[G_m(1-Y)] - e^{-Q(1-Y)} \sinh(G_m Y)]$$

where

$$G_m = \frac{\sqrt{R^2 + 4R(-M^2 + M^2)}}{2} = Q$$

thus we have

$$Res(F(s), -M^2)$$
$$= \frac{P_0 e^{-M^2 T}}{-M^2 \sinh(Q)} [-e^{QY} \sinh[Q(1-Y)]$$
$$- e^{-Q(1-Y)} \sinh(QY)]$$

this can be rewritten as

$$Res(F(s), -M^2)$$
$$= \frac{P_0 e^{-M^2 T}\left[-e^{QY}\left(e^{(1-Y)Q} - e^{-(1-Y)Q}\right) - e^{-Q(1-Y)}(e^{QY} - e^{-QY})\right]}{-2M^2 \sinh(Q)}$$

hence

$$Res(F(s), -M^2) = \frac{P_0 e^{-M^2 T}}{M^2} \qquad (3.4.19)$$

for $s = s_n$,

$$Res(F(s), s_n) = \frac{p(s_n)}{q'(s_n)}$$

where

$$p(s_n) = P_0\left[-e^{QY} \sinh[G_n(1-Y)] - e^{-Q(1-Y)} \sinh(G_n Y)\right] e^{s_n T}$$

and

$$G_n = i\pi n$$

simplifying this

$$p(s_n) = P_0 e^{s_n T}\left[e^{QY} \cosh[\mathrm{i\pi n}] \sinh[i\pi n Y] - e^{-Q(1-Y)} \sinh(i\pi n Y)\right]$$

$$p(s_n) = P_0 e^{s_n T}\left[(-1)^n e^{QY} - e^{-Q(1-Y)}\right] i * \sin(\pi n Y)$$

now

$$q'(s) = (2s + M^2)\sinh(G) + \frac{s(s + M^2)\cosh(G)\, R}{(2G)}$$

hence

$$q'(s_n) = \frac{s_n(s_n + M^2)\cosh(G_n)\, R}{(2G_n)}$$

this can be simplified to

$$q'(s_n) = \frac{1}{32}\frac{(R^2 + 4\pi^2 n^2)(R^2 + 4\pi^2 n^2 + 4M^2 R)(-1)^n}{iR\pi}$$

therefore

$$Res(F(s), s_n) = 32P_0 R\pi \frac{e^{s_n T}\left[(-1)^n e^{-Q(1-Y)} - e^{QY}\right]\sin(\pi n Y)}{(R^2 + 4\pi^2 n^2)(R^2 + 4\pi^2 n^2 + 4M^2 R)} \tag{3.4.20}$$

finally we combine equations (3.4.18) , (3.4.19) and (3.4.20) with equation (3.4.16) to obtain the required solution

$$k(T, Y) = -\frac{P_0}{M^2} + \frac{P_0 e^{-M^2 T}}{M^2} + 32P_0 R\pi \sum_{n=0}^{\infty} \frac{e^{s_n T}\left[(-1)^n e^{-Q(1-Y)} - e^{QY}\right]\sin(\pi n Y)}{(R^2 + 4\pi^2 n^2)(R^2 + 4\pi^2 n^2 - 4M^2 R)} \;. \tag{3.4.21}$$

Now combine equation (3.4.17) and (3.4.21) we obtain

$$U(T, Y) = 32P_0 R\pi \sum_{n=0}^{\infty} \frac{e^{s_n T}\left[(-1)^n e^{-Q(1-Y)} - e^{QY}\right]\sin(\pi n Y)}{(R^2 + 4\pi^2 n^2)(R^2 + 4\pi^2 n^2 - 4M^2 R)} \;.$$

3.5 The Analytical Solution For The Unsteady MHD Flow Through Two Parallel Porous Flat Plates With Hall Effect

In this section we want to solve equations (2.4.17) and (2.4.18) subject to the initial and boundary conditions (2.4.19).

From equation (2.4.18) we see that the pressure is independent of Y. So it is a function of T only, but in our case we will take the pressure gradient as a constant quantity that is

$$\frac{\partial P}{\partial X} = -P_0$$

where $P_0 > 0$, thus equation (2.4.17) becomes

$$\frac{\partial U}{\partial T} + \frac{\partial U}{\partial Y} = P_0 + \frac{1}{R}\frac{\partial^2 U}{\partial Y^2} - a\,U \qquad (3.5.1)$$

where

$$a = \frac{M^2}{1+\,m^2}$$

now, by taking the Laplace transform of equation (3.5.1) with respect to the variable T we obtain :

$$L\left\{\frac{\partial U}{\partial T}\right\} = s\,\widehat{U}(s,Y) - \,U(\,0\,,Y)$$

$$L\left\{\frac{\partial U}{\partial Y}\right\} = \frac{d}{dY}\widehat{U}(\,s\,,Y)$$

$$L\left\{\frac{\partial^2 U}{\partial Y^2}\right\} = \frac{d^2}{dY^2}\widehat{U}(\,s\,,Y)$$

$$\frac{d^2}{dY^2}\widehat{U}(\,s\,,Y) - R\frac{d}{dY}\widehat{U}(\,s\,,Y) - R(s+a)\,\widehat{U}(s,Y) = -\,\frac{RP_0}{s} \qquad (3.5.2)$$

where

$$\widehat{U}(s,Y) = L\{\,U(\,T\,,Y)\}$$

$$\widehat{U}(s,0) = 0 \quad , \quad \widehat{U}(s,1) = 0 \qquad (3.5.3)$$

now we want to solve the linear nonhomogeneous second order differential equation (3.5.2). We will solve the associated homogeneous equation

$$\frac{d^2}{dY^2}\widehat{U}(\,s\,,Y) - R\frac{d}{dY}\widehat{U}(\,s\,,Y) - R(s+a)\widehat{U}(s,Y) = 0 \qquad (3.5.4)$$

the auxiliary equation for equation (3.5.4) can be written as :

$$\beta^2 - R\beta - R(s+a) = 0$$

then we get

$$\beta_1 = Q + G$$

$$\beta_2 = Q - G$$

where

$$Q = \frac{R}{2}$$

and

$$G = \frac{\sqrt{R^2 + 4R(s+a)}}{2}$$

the general homogeneous solution is

$$\widehat{U}_h = c_1 e^{\beta_1 Y} + c_2 e^{\beta_2 Y} \qquad (3.5.5)$$

the particular solution is [27],

$$\widehat{U}_p = A(y)\, Y_1 + B(y)\, Y_2$$

where

$$A(y) = \int \frac{-\,Y_2\, f(y)}{Y_1 Y_2' - Y_2 Y_1'}\, dy$$

$$B(y) = \int \frac{Y_1\, f(y)}{Y_1 Y_2' - Y_2 Y_1'}\, dy$$

$$Y_1 = e^{\beta_1 y} \quad , Y_2 = e^{\beta_2 y} \quad , f(y) = -\frac{RP_0}{s}$$

then we have

$$A(y) = -\frac{RP_0}{s\,(\beta_2 - \beta_1)} \frac{e^{-\beta_1 y}}{\beta_1}$$

$$B(y) = \frac{RP_0}{s\,(\beta_2 - \beta_1)} \frac{e^{-\beta_2 y}}{\beta_2}$$

thus

$$\widehat{U}_p = -\frac{RP_0}{s\,(\beta_2 - \beta_1)\beta_1} + \frac{RP_0}{s\,(\beta_2 - \beta_1)\beta_2}$$

$$\widehat{U}_p = -\frac{RP_0}{s\,\beta_1 \beta_2} \qquad (3.5.6)$$

simplifying equation (3.5.6) where

$$\beta_1 \beta_2 = \frac{1}{4}\left(R + \sqrt{R^2 + 4R(s+a)}\right)\left(R - \sqrt{R^2 + 4R(s+a)}\right)$$

$$= -R(s+a)$$

the particular solution is

$$\widehat{U}_p = \frac{P_0}{s(s+a)} \tag{3.5.7}$$

hence, the solution of the linear nonhomogeneous second order differential equation is

$$\widehat{U}(s,Y) = \widehat{U}_h + \widehat{U}_p$$

$$\widehat{U}(s,Y) = c_1 e^{\beta_1 Y} + c_2 e^{\beta_2 Y} + \frac{P_0}{s(s+a)} \tag{3.5.8}$$

using (3.5.3) we obtain

$$0 = c_1 + c_2 + \frac{P_0}{s(s+a)} \tag{3.5.9}$$

$$0 = c_1 e^{\beta_1} + c_2 e^{\beta_2} + \frac{P_0}{s(s+a)} \tag{3.5.10}$$

solving equations (3.5.9) and (3.5.10) we get

$$c_1 = -\frac{(e^{\beta_2} - 1)\ P_0}{(e^{\beta_2} - e^{\beta_1})s(s+a)} \tag{3.5.11}$$

$$c_2 = \frac{(e^{\beta_1} - 1)\ P_0}{(e^{\beta_2} - e^{\beta_1})s(s+a)} \tag{3.5.12}$$

combining equation (3.5.8) with equations (3.5.11) and (3.5.12) we obtain

$$\widehat{U}(s,Y) = -\frac{(e^{\beta_2} - 1)\ P_0}{(e^{\beta_2} - e^{\beta_1})s(s+a)} e^{\beta_1 Y} + \frac{(e^{\beta_1} - 1)\ P_0}{(e^{\beta_2} - e^{\beta_1})s(s+a)} e^{\beta_2 Y} + \frac{P_0}{s(s+a)}$$

this can be rewritten to

$$\widehat{U}(s,Y) = \frac{P_0}{s(s+a)(e^{\beta_2} - e^{\beta_1})}\left[-(e^{\beta_2}-1)e^{\beta_1 Y} + (e^{\beta_1}-1)e^{\beta_2 Y}\right] + \frac{P_0}{s(s+a)} \qquad (3.5.13)$$

since

$$e^{\beta_1} = e^{Q}e^{G}$$

$$e^{\beta_2} = e^{Q}e^{-G}$$

therefore

$$e^{\beta_2} - e^{\beta_1} = -e^{Q}\,(e^{G} - e^{-G})$$

hence

$$-(e^{\beta_2}-1)e^{\beta_1 Y} + (e^{\beta_1}-1)e^{\beta_2 Y} = e^{(Y+1)Q}\left(e^{(1-Y)G} - e^{-(1-Y)G}\right) + e^{QY}\left(e^{GY} - e^{-GY}\right) \quad (3.5.14)$$

using equation (3.5.14) in our solution (3.5.13) we obtain

$$\widehat{U}(s,Y) = \frac{P_0}{s(s+M^2)(e^{G} - e^{-G})}\left[-e^{QY}\left(e^{(1-Y)G} - e^{-(1-Y)G}\right) - e^{Q(Y-1)}\left(e^{GY} - e^{-GY}\right)\right] + \frac{P_0}{s(s+a)}$$

therefore ,

$$\widehat{U}(s,Y) = \frac{P_0}{s(s+a)\sinh(G)}\left[-e^{QY}\sinh[G(1-Y)] - e^{-Q(1-Y)}\sinh(GY)\right] + \frac{P_0}{s(s+a)} \quad . \qquad (3.5.15)$$

Finding the Laplace invers transform, we obtain

$$U(T\,,Y) = L^{-1}\left[\widehat{U}(s,Y)\right] = \; L^{-1}[K(s,Y)] + \; L^{-1}[\frac{P_0}{s(s+M^2)}]$$

where

$$K(s,Y) = \frac{P_0}{s(s+a)\sinh{(G)}}\left[-e^{QY}\sinh[G(1-Y)]\right.$$
$$\left. - \; e^{-Q(1-Y)} \; \sinh{(GY)}\right] \qquad (3.516)$$

using [19] we get

$$L^{-1}\left[\frac{P_0}{s(s+a)}\right] = \frac{P_0}{a}(\,1 - \; e^{-a\,T}\,) \qquad (3.5.17)$$

now to find $k(T,Y)$ we use The complex inversion formula, which is called Bromwich integral, the Fourier–Mellin integral, and Mellin's inverse formula, and it given by

$$k(T\,,Y) = L^{-1}[K(s,Y)] = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} K(s,Y)e^{sT}dT$$

where the integration is done along the vertical line $Re(s) = \omega$ in the complex plane such that ω is greater than the real part of all singularities of $F(s) = K(s,Y)e^{sT}$ [41],[34].

To find this integral we use the Cauchy's residue theorem,

$$\int_{\omega-i\infty}^{\omega+i\infty} K(s,Y)e^{sT}dT = 2\pi i \times \sum_{r=1}^{n} Res(F(s),s_r)$$

where $Res(F(s),s_r)$ is the residue of $F(s)$ at the isolated singularity s_r is the coefficient of $(s \; - \; s_r)^{\,-1}$ in the Laurent expansion [41],[34].

To find the poles we use theorem (3.1). In our case we have

$$p(s) = P_0\left[-e^{QY}\sinh[G(1-Y)] - e^{-Q(1-Y)}\sinh(GY)\right]e^{sT}$$

$$q(s) = s(s+a)\sinh(G)$$

setting $q(s) = 0$. We obtain the poles of $F(s)$ as,

$$s = 0$$

$$s = -a$$

$$s_n = -\frac{R^2 + 4\pi^2 n^2 + 4Ra}{4R} \quad , n = 0,1,2,\ldots$$

and all of them of order one, that is to say all the poles are simple.

To find the residue of these poles we use theorem (3.2).

Now for $s = 0$, we get

$$Res(F(s),0) = \lim_{s \to 0}(s-0)\,F(s)$$

$$Res(F(s),0) = \frac{P_0}{(a)\sinh(Q)}\left[-e^{QY}\sinh[Q(1-Y)] - e^{-Q(1-Y)}\sinh(QY)\right]$$

where

$$G_0 = \frac{\sqrt{R^2}}{2} = Q$$

simplifying this we obtain

$$Res(F(s),0) = \frac{P_0[-e^{QY}\left(e^{(1-Y)Q} - e^{-(1-Y)Q}\right) - e^{-Q(1-Y)}(e^{QY} - e^{-QY})]}{2a \times \sinh(Q)}$$

hence

$$Res(F(s),0) = -\frac{P_0}{a} \qquad (3.5.18)$$

for $s = -a$ we have

$$Res(F(s),-a) = \lim_{s\to -a}(s + a)\,F(s)$$

$$Res(F(s),-a) = \frac{P_0 e^{-M^2T}}{-a \times \sinh(G_m)}\left[-e^{QY}\sinh[G_m(1-Y)] - e^{-Q(1-Y)}\sinh(G_m Y)\right]$$

where

$$G_m = \frac{\sqrt{R^2 + 4R(-a + a)}}{2} = Q$$

thus we have

$$Res(F(s),-a) = \frac{P_0 e^{-M^2T}}{-a \times \sinh(Q)}\left[-e^{QY}\sinh[Q(1-Y)] - e^{-Q(1-Y)}\sinh(QY)\right]$$

this can be rewritten as

$$Res(F(s),-a) = \frac{P_0 e^{-aT}\left[-e^{QY}\left(e^{(1-Y)Q} - e^{-(1-Y)Q}\right) - e^{-Q(1-Y)}(e^{QY} - e^{-QY})\right]}{-2a \times \sinh(Q)}$$

hence

$$Res(F(s), -a) = \frac{P_0 e^{-M^2 T}}{a}. \qquad (3.5.19)$$

For $s = s_n$,

$$Res(F(s), s_n) = \frac{p(s_n)}{q'(s_n)}$$

where

$$p(s_n) = P_0\left[-e^{QY}\sinh[G_n(1-Y)] - e^{-Q(1-Y)}\sinh(G_nY)\right]e^{s_nT}$$

and

$$G_n = i\pi n$$

simplifying this

$$p(s_n) = P_0 e^{s_nT}\left[e^{QY}\cosh[\mathrm{i\pi n}]\sinh[i\pi nY] - e^{-Q(1-Y)}\sinh(i\pi nY)\right]$$

$$p(s_n) = P_0 e^{s_nT}\left[(-1)^n e^{QY} - e^{-Q(1-Y)}\right] i * \sin(\pi nY)$$

now

$$q'(s) = (2s+a)\sinh(G) + \frac{s(s+a)\cosh(G)\,R}{(2G)}$$

hence

$$q'(s_n) = \frac{s_n(s_n+a)\cosh(G_n)\,R}{(2G_n)}$$

this can be rewritten as

$$q'(s_n) = \frac{1}{32}\frac{(R^2+4\pi^2n^2)(R^2+4\pi^2n^2+4aR)(-1)^n}{iR\pi}$$

therefore

$$Res(F(s),s_n)$$
$$= 32P_0R\pi\,\frac{e^{s_nT}\left[\,(-1)^n e^{-Q(1-Y)} - e^{QY}\right]\sin(\pi nY)}{(R^2+4\pi^2n^2)(R^2+4\pi^2n^2+4aR)} \qquad (3.5.20)$$

finally we combine equations (3.5.18) , (3.5.19) and (3.5.20) with equation (3.5.16) to obtain the required solution

$$k(T,Y) = -\frac{P_0}{a} + \frac{P_0e^{-aT}}{a}$$
$$+ 32P_0R\pi\sum_{n=0}^{\infty}\frac{e^{s_nT}\left[\,(-1)^n e^{-Q(1-Y)} - e^{QY}\right]\sin(\pi nY)}{(R^2+4\pi^2n^2)(R^2+4\pi^2n^2-4aR)}\ . \qquad (3.5.21)$$

Now combine equation (3.5.17) and (3.5.21) we obtain

$$U(T,Y) = 32P_0R\pi\sum_{n=0}^{\infty}\frac{e^{s_nT}\left[\,(-1)^n e^{-Q(1-Y)} - e^{QY}\right]\sin(\pi nY)}{(R^2+4\pi^2n^2)(R^2+4\pi^2n^2-4aR)}\ .$$

CHAPTER FOUR

NUMERICAL METHODS

CHAPTER FOUR
NUMERICAL METHODS

In this chapter we find numerical solution to problems formulated in chapter two . This involves using the finite difference method and then we will use the computer software " MAPLE 15 " to obtain some numerical results.

4.1 Finite Difference

In this section we discuss the finite difference approximation method (FDM) used mainly to solve partial differential equations . This method was first developed by A. Thom in the 1920s under the title "the method of square" to solve nonlinear hydrodynamic equations [38].

Finite difference procedures approximate the derivative appearing in a partial differential equation by sums and differences of function values at a set of discrete points. These approximations are based on Taylor series expansions of a function of one or more variables [18].

Using the Taylor series expansion we have [35],

the forward – difference formula for approximating $f'(x_0)$

$$f'^{(x_0)} \cong \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}+ O(\Delta x) \;\;,$$

the backward – difference formula

$$f'^{(x_0)} \cong \frac{f(x_0)-f(x_0-\Delta x)}{\Delta x}+ O(\Delta x) \;\;,$$

and the central – difference formula

$$f'(x_0) \cong \frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x} + O(\Delta x)^2$$

and we have also

$$f''^{(x_0)} \cong \frac{f(x_0+\Delta x)-2f(x_0)+f(x_0-\Delta x)}{\Delta x^2} + O(\Delta x)^2.$$

To find a numerical solution to a partial differential equation with finite difference methods, we first need to define a set of grid points in the domain D as follows:

Choose a state step size $\Delta x = h = \frac{b-a}{N}$ (N is an integer) and a time step size Δt, draw a set of horizontal and vertical lines across D, and get all intersection lines across D, and get all intersection points (x_j, t_n), or simply (j, n), where $x_j = a + j\,\Delta x$, $j = 0, \ldots, N$, and $t_n = n\Delta t$, $n = 0, 1, \ldots$. If $D = [a, b] \times [0, T]$ then choose $\Delta t = k = \frac{T}{M}$ (M is an integer) and $t_n = n\Delta t$, $n = 0, \ldots, M$ [6].

4.2 MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System

4.2.1 Without Hall Effect

In this section we will solve numerically the parabolic partial differential equation (3.2.1) we introduced in chapter three,

$$\frac{\partial Q}{\partial T} = \frac{\partial^2 Q}{\partial Z^2} - aQ \qquad (4.2.1.1)$$

with the initial and boundary conditions

$$Q = 0 \quad at \quad T \leq \ 0 \quad for\ all \qquad Z$$

$$\begin{cases} Q = 1, & at\ Z = 0 \\ Q \ \to 0\,, & at\ Z \ \to \ \infty \end{cases} \quad T > 0$$

where $Q = U + iV$ and $a = 2[\ M^2\ + i\Omega\ ]$.

We use the finite difference method for solving this equation . To be more specific, we use the implicit method, which we call the backward difference method .

Thus we use the following difference quotient

$$\frac{\partial Q}{\partial T}(Z_i, T_j) = \frac{Q(Z_i, T_j) - Q(Z_i, T_{j-1})}{k} - \frac{k}{2}\frac{\partial^2 Q}{\partial T^2}(Z_i, T^*{}_j)$$

where $T^*{}_j$ is $\in (T_{j-1}, T_j\ )$.

$$\frac{\partial^2 Q}{\partial Z^2}(Z_i, T_j) = \frac{Q(Z_{i+1}, T_j) - 2Q(Z_i, T_j) + Q(Z_{i-1}, T_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 Q}{\partial Z^4}(Z^*{}_i, T_j)$$

where $Z^*{}_i$ is $\in (Z_{i-1}, Z_{i+1}\ )$.

Using the above difference quotient in our partial differential equation (4.2.1.1) we get

$$\frac{w_{i,j} - w_{i,j-1}}{k} = \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} - aw_{i,j}$$

where $w_{i,j}$ approximates $Q(Z_i, T_j)$.

With the local truncation error for this difference quotient

$$\tau_{i,j} = \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(Z_i, T^*{}_j) - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(Z^*{}_i, T_j)$$

and the truncation error is of order $O(k) + O(h^2)$ [21].

Solving this for $w_{i,j-1}$ we obtain

$$w_{i,j-1} = -\lambda\, w_{i+1,j} + (1 + ak + 2\lambda)\, w_{i,j} - \lambda w_{i-1,j}$$

where

$$\lambda = \frac{k}{h^2}$$

putting this in matrix form we obtain

$$\begin{bmatrix} 1 + ak + 2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\lambda \\ 0 & \cdots & 0 & -\lambda & 1 + ak + 2\lambda \end{bmatrix} \begin{bmatrix} w_{1,j} \\ \vdots \\ \vdots \\ \vdots \\ w_{m-1,j} \end{bmatrix} = \begin{bmatrix} w_{1,j-1} + \lambda \\ \vdots \\ \vdots \\ \vdots \\ w_{m-1,j-1} \end{bmatrix}$$

4.2.2 With Hall Effect

In this section we will solve numerically the parabolic partial differential equation (3.3.1) we introduced in chapter three,

$$\frac{\partial Q}{\partial T} = \frac{\partial^2 Q}{\partial Z^2} - aQ \qquad (4.2.2.1)$$

with the initial and boundary conditions

$$Q = 0 \quad at \quad T \leq \ 0 \quad for\ all \qquad Z$$

$$\begin{cases} Q = 1, & at\ \ Z = 0 \\ Q \ \rightarrow 0\,, & at\ \ Z\ \ \rightarrow\ \ \infty \end{cases} \quad T > 0$$

where $Q = U + iV$ and $a = 2[\frac{M^2}{(1+m^2)} + i\left(\Omega - \frac{M^2 m}{(1+m^2)}\right)]$.

We will use the implicit method for solving this equation . Thus we use the following difference quotient

$$\frac{\partial Q}{\partial T}(Z_i, T_j) = \frac{Q(Z_i, T_j) - Q(Z_i, T_{j-1})}{k} - \frac{k}{2}\frac{\partial^2 Q}{\partial T^2}(Z_i, T^*{}_j)$$

where $T^*{}_j$ is $\in (T_{j-1}, T_j)$.

$$\frac{\partial^2 Q}{\partial Z^2}(Z_i, T_j) = \frac{Q(Z_{i+1}, T_j) - 2Q(Z_i, T_j) + Q(Z_{i-1}, T_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 Q}{\partial Z^4}(Z^*{}_i, T_j)$$

where $Z^*{}_i$ is $\in (Z_{i-1}, Z_{i+1})$.

Using the above difference quotient in our partial differential equation (4.2.2.1) we get

$$\frac{w_{i,j} - w_{i,j-1}}{k} = \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} - aw_{i,j}$$

where $w_{i,j}$ approximates $Q(Z_i, T_j)$.

With the local truncation error for this difference quotient

$$\tau_{i,j} = \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(Z_i, T^*{}_j) - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(Z^*{}_i, T_j)$$

and the truncation error of order $O(k) + O(h^2)$ [21].

Solving this for $w_{i,j-1}$ we obtain

$$w_{i,j-1} = -\lambda\, w_{i+1,j} + (1 + ak + 2\lambda)\, w_{i,j} - \lambda w_{i-1,j}$$

where

$$\lambda = \frac{k}{h^2}$$

putting this in matrix form we obtain

$$\begin{bmatrix} 1+ak+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\lambda \\ 0 & \cdots & 0 & -\lambda & 1+ak+2\lambda \end{bmatrix} \begin{bmatrix} w_{1,j} \\ \vdots \\ \vdots \\ \vdots \\ w_{m-1,j} \end{bmatrix} = \begin{bmatrix} w_{1,j-1}+\lambda \\ \vdots \\ \vdots \\ \vdots \\ w_{m-1,j-1} \end{bmatrix}$$

4.2.3 Stability for the difference scheme

We will investigate the stability condition. A numerical algorithm is said to be stable if a small error at any stage produces a smaller cumulative error. Otherwise, it is unstable [4].

To determine whether a finite difference scheme is stable we use the Fourier method, we assume that [39]:

$$U_{i,j} = w_j e^{r x_i I}$$

where

$$I = \sqrt{-1}$$

we define the amplification of the error at time step $n+1$ as

$$w^{n+1} = \mathcal{K}\, w^n$$

for the scheme to be stable it is necessary that

$$|\, \mathcal{K} \,| \leq 1$$

thus we have

$$w_{j-1} e^{r x_i I} = -\lambda w_j e^{r x_{i+1} I} + (1+ak+2\lambda) w_j e^{r x_i I} - \lambda w_j e^{r x_{i-1} I}$$

$$(\lambda e^{rx_{i+1}I} - (1 + ak + 2\lambda)e^{rx_iI} + \lambda e^{rx_{i-1}I})w_j = -\, w_{j-1}e^{rx_iI}$$

now

$$e^{rx_{i+1}I} = \; e^{r(x_i+h)I} = e^{rx_iI}e^{rhI}$$

then we get

$$(\lambda e^{rx_iI}e^{rhI} - (1 + ak + 2\lambda)e^{rx_iI} + \lambda e^{rx_iI}e^{-rhI})w_j = -\, w_{j-1}e^{rx_iI}$$

$$(\lambda e^{rhI} - (1 + ak + 2\lambda) + \lambda e^{-rhI})w_j = -\, w_{j-1}$$

$$(2\lambda \cos(rh) - 1 - ak - 2\lambda)w_j = -\, w_{j-1}$$

$$(-2\lambda(1 - \cos(rh)) - 1 - ak)w_j = -\, w_{j-1}$$

$$(-4\lambda sin^2\left(\frac{rh}{2}\right) - 1 - ak)w_j = -\, w_{j-1}$$

$$w_j = \frac{w_{j-1}}{(4\lambda sin\left(\frac{rh}{2}\right)^2 + 1 + ak)}$$

$$w_j = \mathcal{K}w_{j-1}$$

where

$$\mathcal{K} = \frac{1}{(4\lambda sin^2\left(\frac{rh}{2}\right) + 1 + ak)}$$

for the solution to remain bounded and stable as $j \;\rightarrow\; \infty$ we need

$$|\,\mathcal{K}\,| \leq 1$$

that is

$$-1 \leq \frac{1}{4\lambda sin^2\left(\frac{rh}{2}\right) + 1 + ak} \leq 1$$

$$4\lambda sin^2\left(\frac{rh}{2}\right) \leq -2 - ak$$

or

$$-ak \leq 4\lambda sin^2\left(\frac{rh}{2}\right)$$

now

$$0 \leq 4sin^2\left(\frac{rh}{2}\right)$$

and

$$0 \leq \lambda$$

hence the condition holds, and does not depend on λ. This implies that the method is stable .

4.3 The Numerical Solution For The Unsteady MHD Flow Through Two Parallel Porous Flat Plates

4.3.1 Without Hall Effect

In this section we will solve numerically the parabolic partial differential equation (3.4.1) we had in chapter three,

$$\frac{\partial U}{\partial T} + \frac{\partial U}{\partial Y} = P_0 + \frac{1}{R}\frac{\partial^2 U}{\partial Y^2} - M^2 U \qquad (4.3.1.1)$$

with the initial and boundary conditions

$$\begin{cases} U = 0\,, & T \le 0 \\ U = 0\,, Y = 0\,,1 & ,T > 0 \end{cases}$$

using similar approach as before, we use the backward difference method and the following difference quotient

$$\frac{\partial U}{\partial T}(Y_i, T_j) = \frac{U(Y_i, T_j) - U(Y_i, T_{j-1})}{k} - \frac{k}{2}\frac{\partial^2 U}{\partial T^2}(Y_i, T^*{}_j)$$

where $T^*{}_j \in (T_{j-1}, T_j)$.

$$\frac{\partial^2 U}{\partial Y^2}(Y_i, T_j) = \frac{U(Y_{i+1}, T_j) - 2U(Y_i, T_j) + Y(Y_{i-1}, T_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 U}{\partial Y^4}(Y^*{}_i, T_j)$$

$$\frac{\partial U}{\partial Y}(Y_i, T_j) = \frac{U(Y_{i+1}, T_j) - U(Y_{i-1}, T_j)}{2h} - \frac{h^2}{6}\frac{\partial^3 U}{\partial Y^3}(Y^*{}_i, T_j)$$

where $Y^*{}_i \in (Y_{i-1}, Y_{i+1})$.

Using the above difference quotient in the partial differential equation (4.3.1.1) we get

$$\frac{w_{i,j} - w_{i,j-1}}{k} + \frac{w_{i+1,j} - w_{i-1,j}}{2h} = P_0 + \frac{1}{R}\frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} - M^2 w_{i,j}$$

where $w_{i,j}$ approximates $U(Y_i, T_j)$.

With the local truncation error for this difference quotient

$$\tau_{i,j} = \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(Y_i, T^*{}_j) - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(Y^*{}_i, T_j) + \frac{h^2}{6}\frac{\partial^3 u}{\partial x^3}(Y^*{}_i, T_j)$$

and the truncation error of order $O(k) + O(h^2)$ [4].

Solving for $w_{i,j-1}$ we have

$$w_{i,j-1} = A\, w_{i+1,j} + B\, w_{i,j} - F w_{i-1,j} - kP_0$$

where

$$A = \left(\frac{h}{2} - \frac{1}{R}\right)\lambda$$

$$B = 1 + kM^2 + \frac{2\lambda}{R}$$

$$F = \left(\frac{h}{2} + \frac{1}{R}\right)\lambda$$

$$\lambda = \frac{k}{h^2}$$

putting in matrix form we obtain

$$\begin{bmatrix} B & A & 0 & \cdots & 0 \\ -F & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & A \\ 0 & \cdots & 0 & -F & B \end{bmatrix} \begin{bmatrix} w_{1,j} \\ \vdots \\ \vdots \\ \vdots \\ w_{m-1,j} \end{bmatrix} = \begin{bmatrix} w_{1,j-1} + kP_0 \\ \vdots \\ \vdots \\ \vdots \\ w_{m-1,j-1} + kP_0 \end{bmatrix}$$

4.3.2 With Hall Effect

In this section we will solve numerically the parabolic partial differential equation (3.5.1) we had in chapter three,

$$\frac{\partial U}{\partial T} + \frac{\partial U}{\partial Y} = P_0 + \frac{1}{R}\frac{\partial^2 U}{\partial Y^2} - a\, U \qquad (4.3.2.1)$$

where

$$a = \frac{M^2}{1 + m^2}$$

subject to initial and boundary conditions

$$\begin{cases} U = 0\,, & T \le 0 \\ U = 0\,, Y = 0\,,1 & , T > 0 \end{cases}$$

using similar approach as before, we use the backward difference method and the following difference quotient

$$\frac{\partial U}{\partial T}(Y_i, T_j) = \frac{U(Y_i, T_j) - U(Y_i, T_{j-1})}{k} - \frac{k}{2}\frac{\partial^2 U}{\partial T^2}(Y_i, {T^*}_j)$$

where ${T^*}_j \in (T_{j-1}, T_j\,)$.

$$\frac{\partial^2 U}{\partial Y^2}(Y_i, T_j) = \frac{U(Y_{i+1}, T_j) - 2U(Y_i, T_j) + Y(Y_{i-1}, T_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 U}{\partial Y^4}({Y^*}_i, T_j)$$

$$\frac{\partial U}{\partial Y}(Y_i, T_j) = \frac{U(Y_{i+1}, T_j) - U(Y_{i-1}, T_j)}{2h} - \frac{h^2}{6}\frac{\partial^3 U}{\partial Y^3}({Y^*}_i, T_j)$$

where ${Y^*}_i \in (Y_{i-1}, Y_{i+1}\,)$.

Using the above difference quotient in the partial differential equation (4.3.2.1) we get

$$\frac{w_{i,j} - w_{i,j-1}}{k} + \frac{w_{i+1,j} - w_{i-1,j}}{2h} = P_0 + \frac{1}{R}\frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} - a\,w_{i,j}$$

where $w_{i,j}$ approximates $U(Y_i, T_j)$.

With the local truncation error for this difference quotient

$$\tau_{i,j} = \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(Y_i, {T^*}_j) - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}({Y^*}_i, T_j) + \frac{h^2}{6}\frac{\partial^3 u}{\partial x^3}({Y^*}_i, T_j)$$

and the truncation error of order $O(k) + O(h^2)$ [4].

Solving for $w_{i,j-1}$ we have

$$w_{i,j-1} = A\, w_{i+1,j} + B\, w_{i,j} - F w_{i-1,j} - kP_0$$

where

$$A = \left(\frac{h}{2} - \frac{1}{R}\right)\lambda$$

$$B = 1 + ka + \frac{2\lambda}{R}$$

$$F = \left(\frac{h}{2} + \frac{1}{R}\right)\lambda$$

$$\lambda = \frac{k}{h^2}$$

putting in matrix form we obtain

$$\begin{bmatrix} B & A & 0 & \cdots & 0 \\ -F & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & A \\ 0 & \cdots & 0 & -F & B \end{bmatrix} \begin{bmatrix} w_{1,j} \\ \vdots \\ \vdots \\ \vdots \\ w_{m-1,j} \end{bmatrix} = \begin{bmatrix} w_{1,j-1} + kP_0 \\ \vdots \\ \vdots \\ \vdots \\ w_{m-1,j-1} + kP_0 \end{bmatrix}$$

4.3.3 Stability for the difference scheme

Investigate the stability we use the fourier method in similar approach as before . Assume that :

$$U_{i,j} = w_j e^{r x_i I}$$

where

$$I = \sqrt{-1}$$

we find the implication factor of the error of the form

$$w^{n+1} = \mathcal{K}\, w^n$$

And to have the scheme stable it is necessary that

$$|\,\mathcal{K}\,| \leq 1$$

thus we have

$$w_{j-1}e^{rx_iI} = Aw_je^{rx_{i+1}I} + Bw_je^{rx_iI} - Fw_je^{rx_{i-1}I}$$

$$(Ae^{rx_{i+1}I} + Be^{rx_iI} - Fe^{rx_{i-1}I})w_j = w_{j-1}e^{rx_iI}$$

now

$$e^{rx_{i+1}I} = e^{r(x_i+h)I} = e^{rx_iI}e^{rhI}$$

then

$$(Ae^{rx_iI}e^{rhI} + Be^{rx_iI} - Fe^{rx_iI}e^{-rhI})w_j = w_{j-1}e^{rx_iI}$$

$$(Ae^{rhI} + B - Fe^{-rhI})w_j = w_{j-1}$$

$$\left[\left(\frac{h}{2} - \frac{1}{R}\right)\lambda e^{rhI} + \left(1 + ka + \frac{2\lambda}{R}\right) - \left(\frac{h}{2} + \frac{1}{R}\right)\lambda e^{-rhI}\right]w_j = w_{j-1}$$

$$\left[(e^{rhI} - e^{-rhI})\frac{h\lambda}{2} - (e^{rhI} + e^{-rhI})\frac{\lambda}{R} + \left(1 + ka + \frac{2\lambda}{R}\right)\right]w_j = w_{j-1}$$

$$\left[h\lambda I sin(rh) - \frac{2\lambda}{R}\cos(rh) + \left(1 + ka + \frac{2\lambda}{R}\right)\right]w_j = w_{j-1}$$

$$\left[h\lambda I sin(rh) + \frac{2\lambda}{R}(1 - \cos(rh)) + 1 + ak\right]w_j = w_{j-1}$$

$$\left[h\lambda I sin(rh) + \frac{2\lambda}{R}sin^2\left(\frac{rh}{2}\right) + 1 + ak\right]w_j = w_{j-1}$$

$$w_j = \frac{w_{j-1}}{(h\lambda I sin(rh) + \frac{2\lambda}{R}sin^2\left(\frac{rh}{2}\right) + 1 + ak)}$$

$$w_j = kw_{j-1}$$

where

$$\mathcal{K} = \frac{1}{(h\lambda I sin(rh) + \frac{2\lambda}{R} sin^2\left(\frac{rh}{2}\right) + 1 + ak)}$$

now for the solution to remain bounded and stable as $j \rightarrow \infty$ we must have

$$|\,\mathcal{K}\,| \leq 1$$

thus

$$|\,\mathcal{K}\,| = \frac{1}{\sqrt{\left(h\lambda sin(rh)\right)^2 + (\frac{2\lambda}{R} sin^2\left(\frac{rh}{2}\right) + 1 + ak)^2}}$$

$$|\,\mathcal{K}\,|^2 = \frac{1}{\left(h\lambda sin(rh)\right)^2 + (\frac{2\lambda}{R} sin^2\left(\frac{rh}{2}\right) + 1 + ak)^2}$$

$$|\,\mathcal{K}\,|^2 = \frac{1}{h^2\lambda^2 sin^2(rh) + (\frac{2\lambda}{R} sin^2\left(\frac{rh}{2}\right) + 1 + ak)^2}$$

$$|\,\mathcal{K}\,|^2 = \frac{1}{4h^2\lambda^2(sin^2\left(\frac{rh}{2}\right) - sin^4\left(\frac{rh}{2}\right)) + (\frac{2\lambda}{R} sin^2\left(\frac{rh}{2}\right) + 1 + ak)^2}$$

$$|\,\mathcal{K}\,|^2 = \frac{1}{(1 + ak)^2 + 4\lambda sin^2\left(\frac{rh}{2}\right)(\frac{\lambda}{R^2} sin^2\left(\frac{rh}{2}\right) + \frac{(1 + ak)}{R} + h^2\lambda(1 - sin^2\left(\frac{rh}{2}\right))}$$

$$|\,\mathcal{K}\,|^2 = \frac{1}{(1 + ak)^2 + 4\lambda sin^2\left(\frac{rh}{2}\right)(\lambda(\frac{1}{R^2} sin^2\left(\frac{rh}{2}\right) + h^2\left(1 - sin^2\left(\frac{rh}{2}\right)\right)) + \frac{(1 + ak)}{R})}$$

hence it is necessary that

$$|\,\mathcal{K}\,|^2 \leq 1$$

that is

$$\frac{1}{1 + 2ak + a^2k^2 + 4\lambda sin^2\left(\frac{rh}{2}\right)\left[\lambda(\frac{1}{R^2}sin^2\left(\frac{rh}{2}\right) + h^2\left(1 - sin^2\left(\frac{rh}{2}\right)\right)) + \frac{(1+ak)}{R}\right]} \leq 1$$

$$0 \leq \ 2ak + a^2k^2 + 4\lambda sin^2\left(\frac{rh}{2}\right)\left[\lambda(\frac{1}{R^2}sin^2\left(\frac{rh}{2}\right) + h^2\left(1 - sin^2\left(\frac{rh}{2}\right)\right)\right] + \frac{(1+ak)}{R})$$

now

$$0 \leq ak + a^2k^2$$

thus

$$0 \leq 4\lambda sin^2\left(\frac{rh}{2}\right)(\lambda(\frac{1}{R^2}sin^2\left(\frac{rh}{2}\right) + h^2\left(1 - sin^2\left(\frac{rh}{2}\right)\right)) + \frac{(1+ak)}{R})$$

hence

$$0 \leq 4\lambda sin^2\left(\frac{rh}{2}\right)$$

and

$$0 \leq \lambda(\frac{1}{R^2}sin^2\left(\frac{rh}{2}\right) + h^2\left(1 - sin^2\left(\frac{rh}{2}\right)\right)) + \frac{(1+ak)}{R}$$

hence the condition holds $|\,\mathcal{K}\,|^2 \leq 1$, and does not depend on λ. This implies that the method is stable .

CHAPTER FIVE

NUMERICAL RESULTS AND CONCLUSIONS

CHAPTER FIVE
NUMERICAL RESULTS AND CONCLUSIONS

5.1 Numerical Results

5.1.1 MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System

In order to study the effect of magnetic field , Hall current , rotation of the fluid and the normal coordinate, we have carried out some numerical calculations for the primary velocity U and secondary velocity V for different values of the rotation parameter Ω , Hartman Number M , Hall parameter m and normal coordinate Z keeping the value of time T fixed at $T = 1$.

5.1.1.1 rotation parameter Ω

The effect of the rotation parameter Ω on the variation of primary velocity U and secondary velocity V in the absence of Hall parameter $m = 0$ and in the presence of Hartman number $M = 0.5$ are presented in Figures 1 and 2. Figures 3 and 4 take into account the presence of Hall parameter m and the Hartman number M.

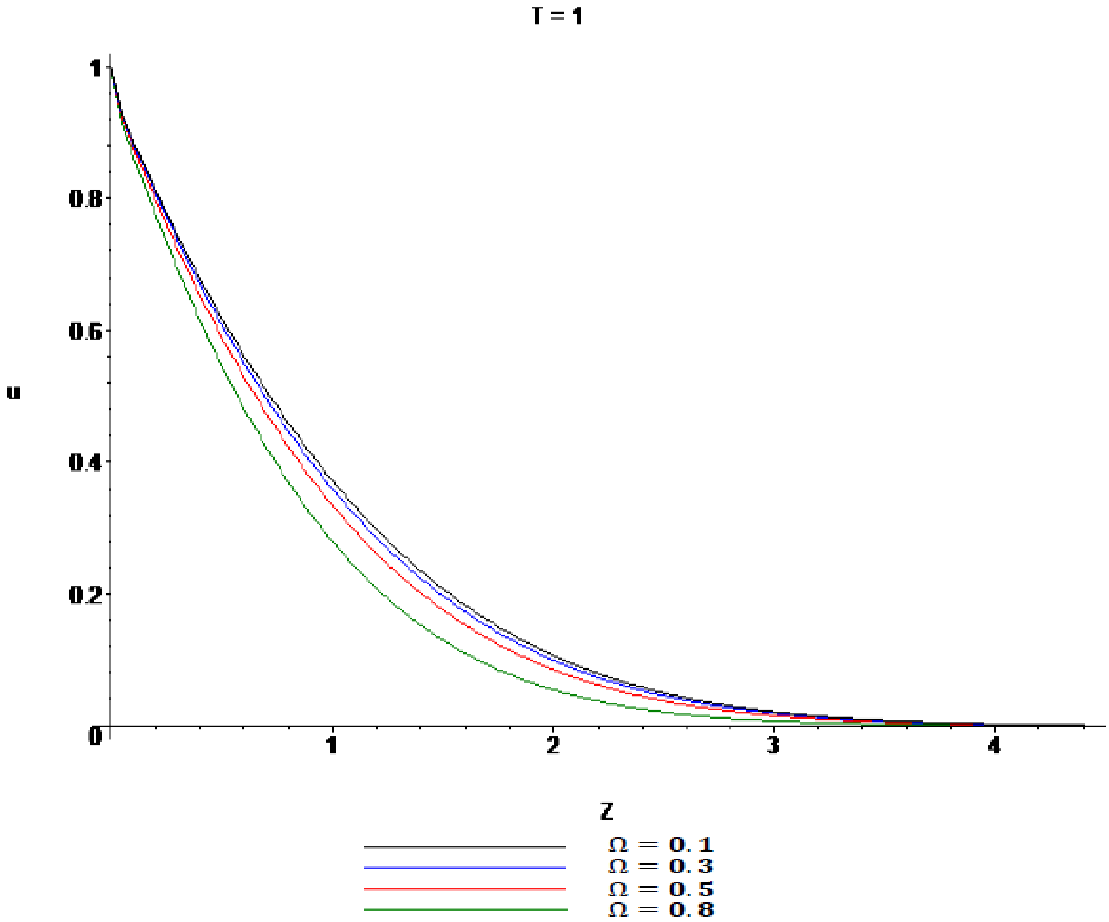

Figure (1) : primary velocity profiles for several values of Ω with m = 0 , M = 0.5 and T = 1 .

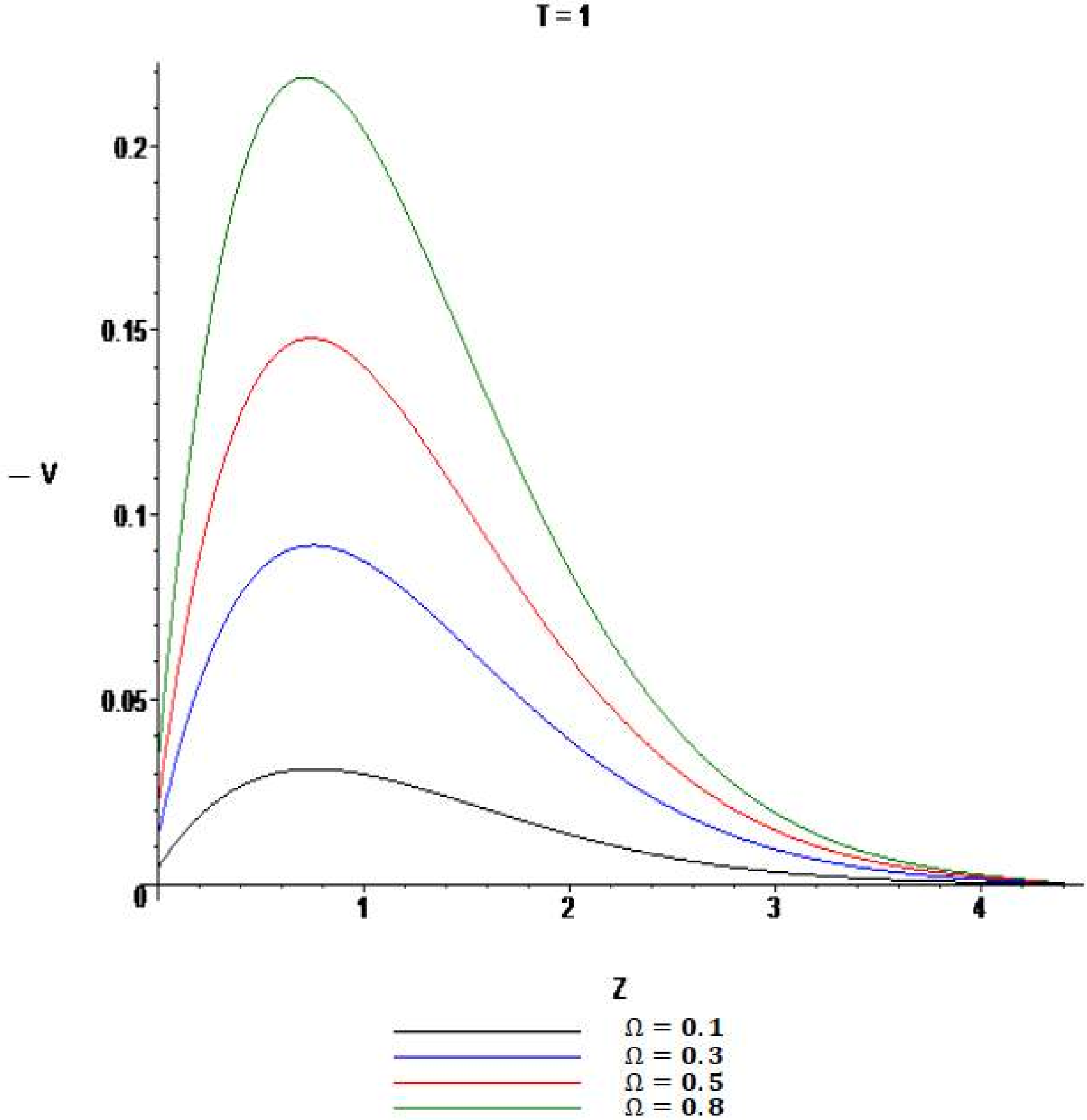

Figure (2) : secondary velocity profiles for several values of Ω with m = 0 , M = 0.5 and T = 1 .

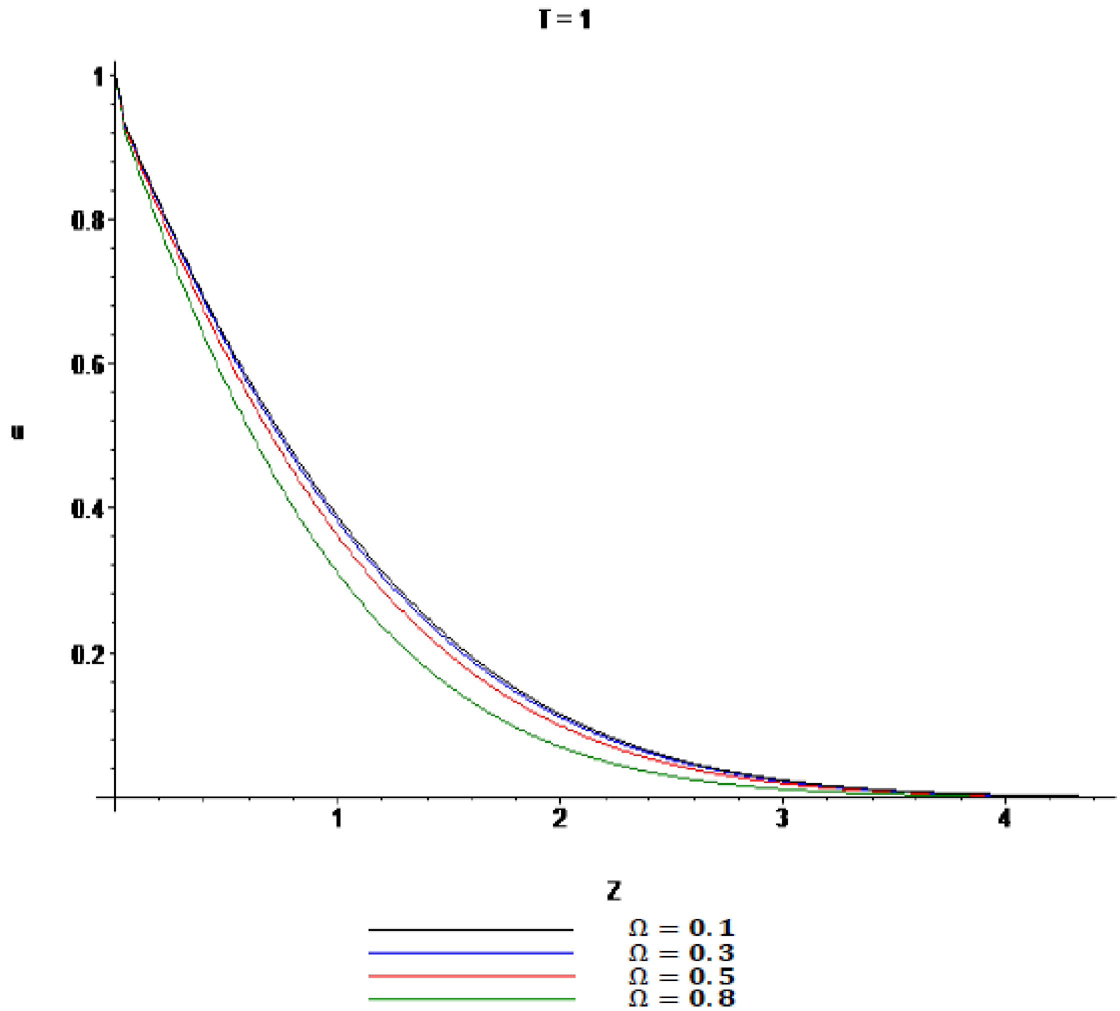

Figure (3) : primary velocity profiles for several values of Ω with m = 0.5 , M = 0.5 and T = 1 .

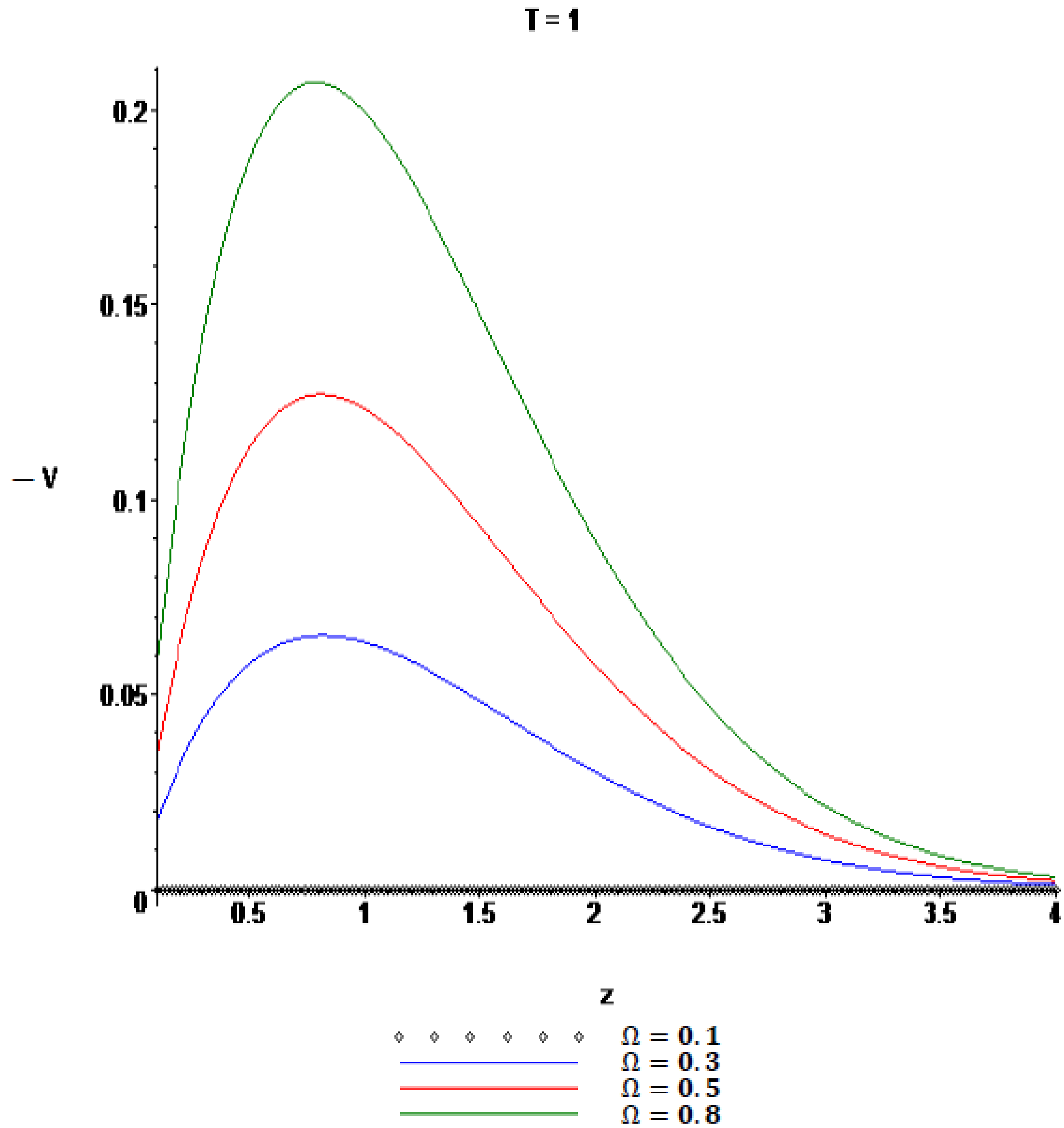

Figure (4) : secondary velocity profiles for several values of Ω with m = 0.5 M = 0.5 and T = 1 .

5.1.1.2 Hartman number *M*

The effect of the Hartman number on the variation of primary and secondary velocity U and V respectively in the absence of Hall parameter $m = 0$ and in the presence of rotation parameter $\Omega = 0.1$ are presented in Figures 5 and 6. Figures 7 and 8 take into account the presence of the Hall and rotation parameters.

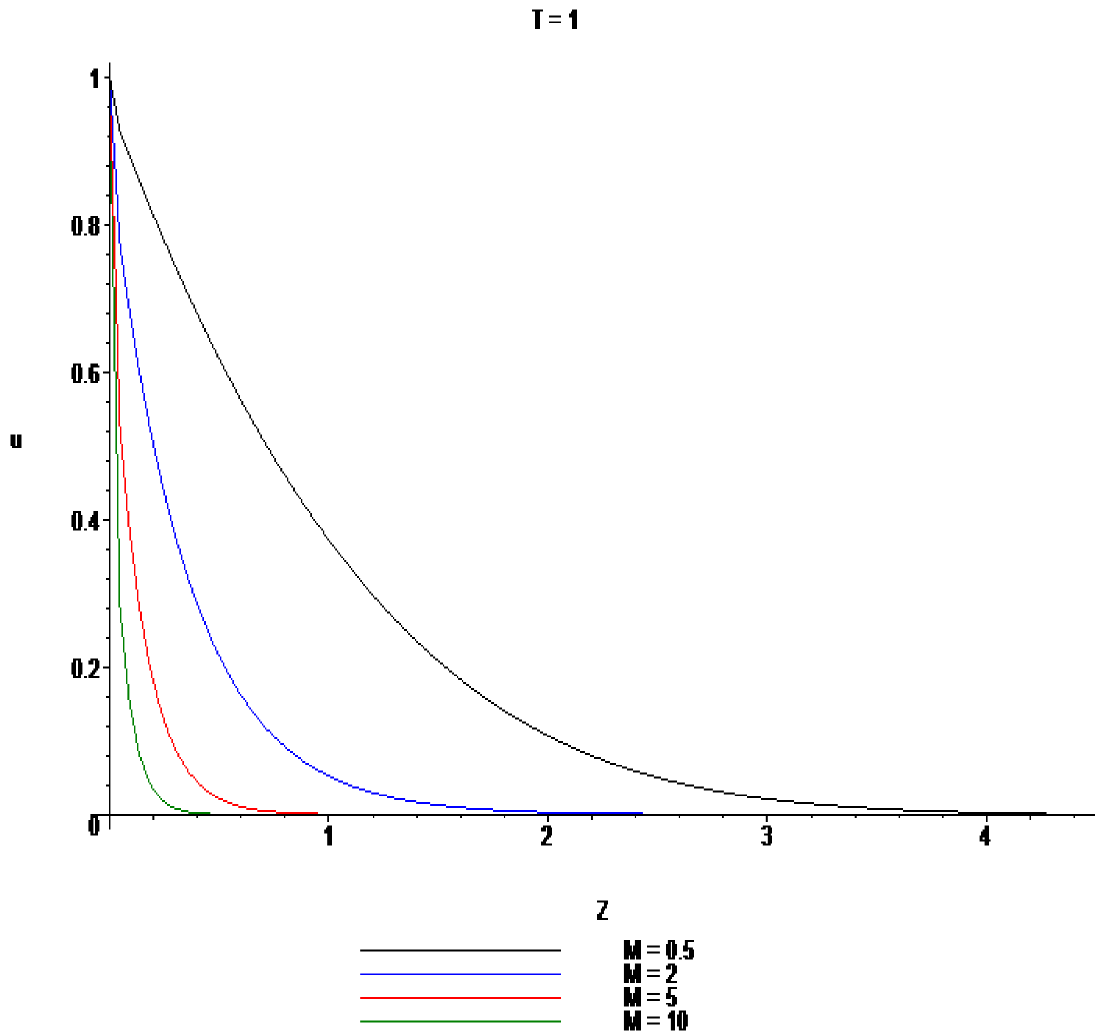

Figure (5) : primary velocity profiles for several values of *M* with m = 0 , Ω = 0.1 and T = 1 .

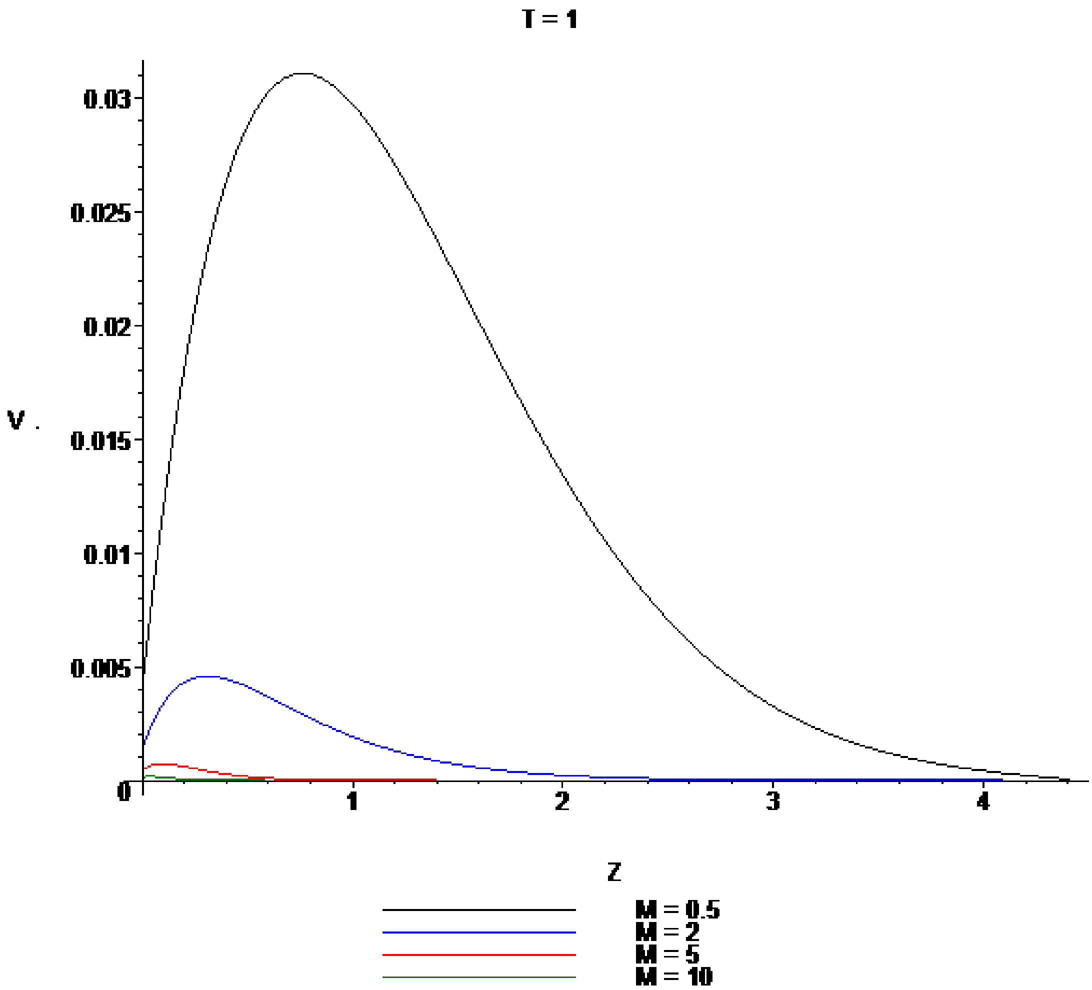

Figure (6) : secondary velocity profiles for several values of *M* with m = 0 , Ω = 0.1 and T = 1 .

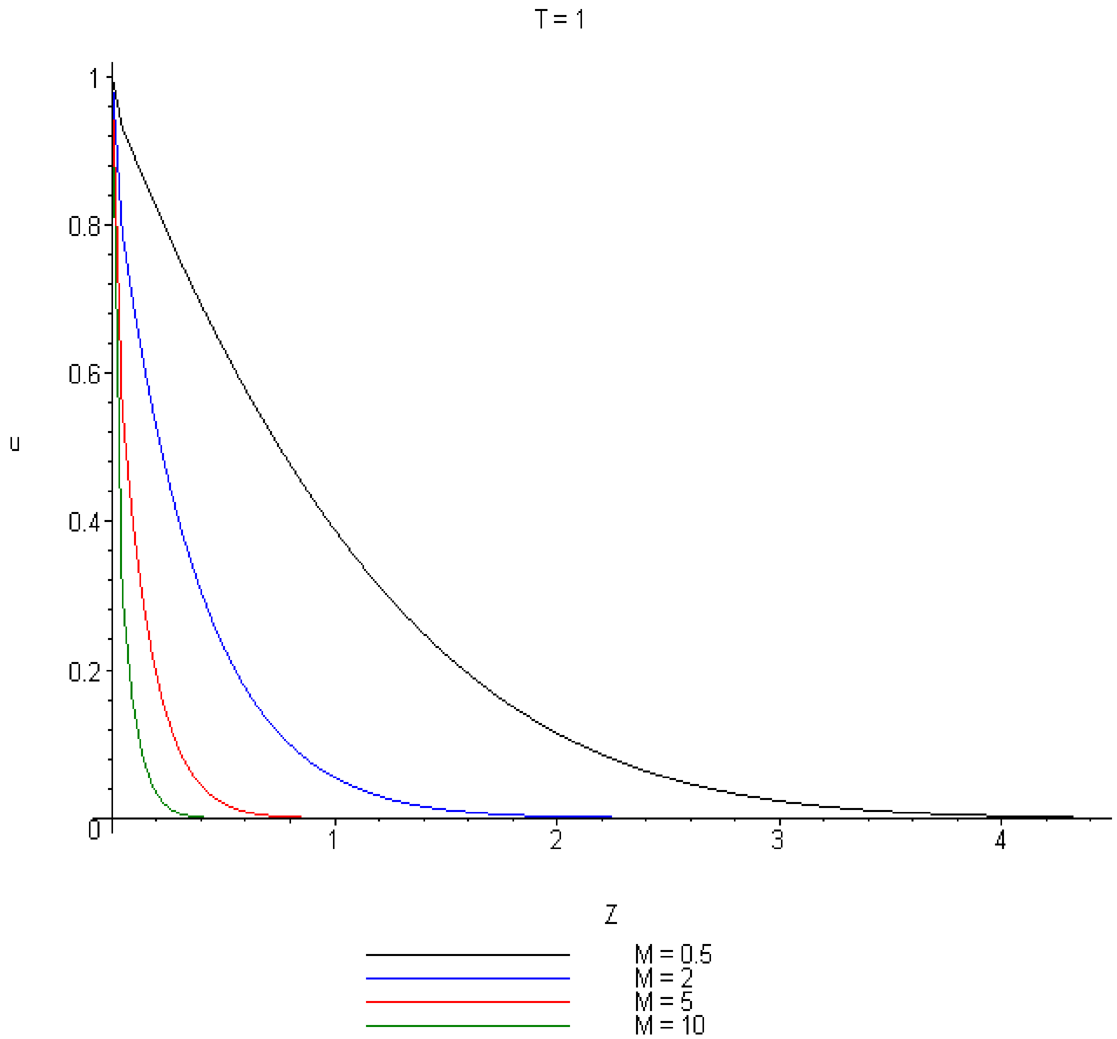

Figure (7): primary velocity profiles for several values of *M* with m = 0.5, Ω = 0.1 and T = 1 .

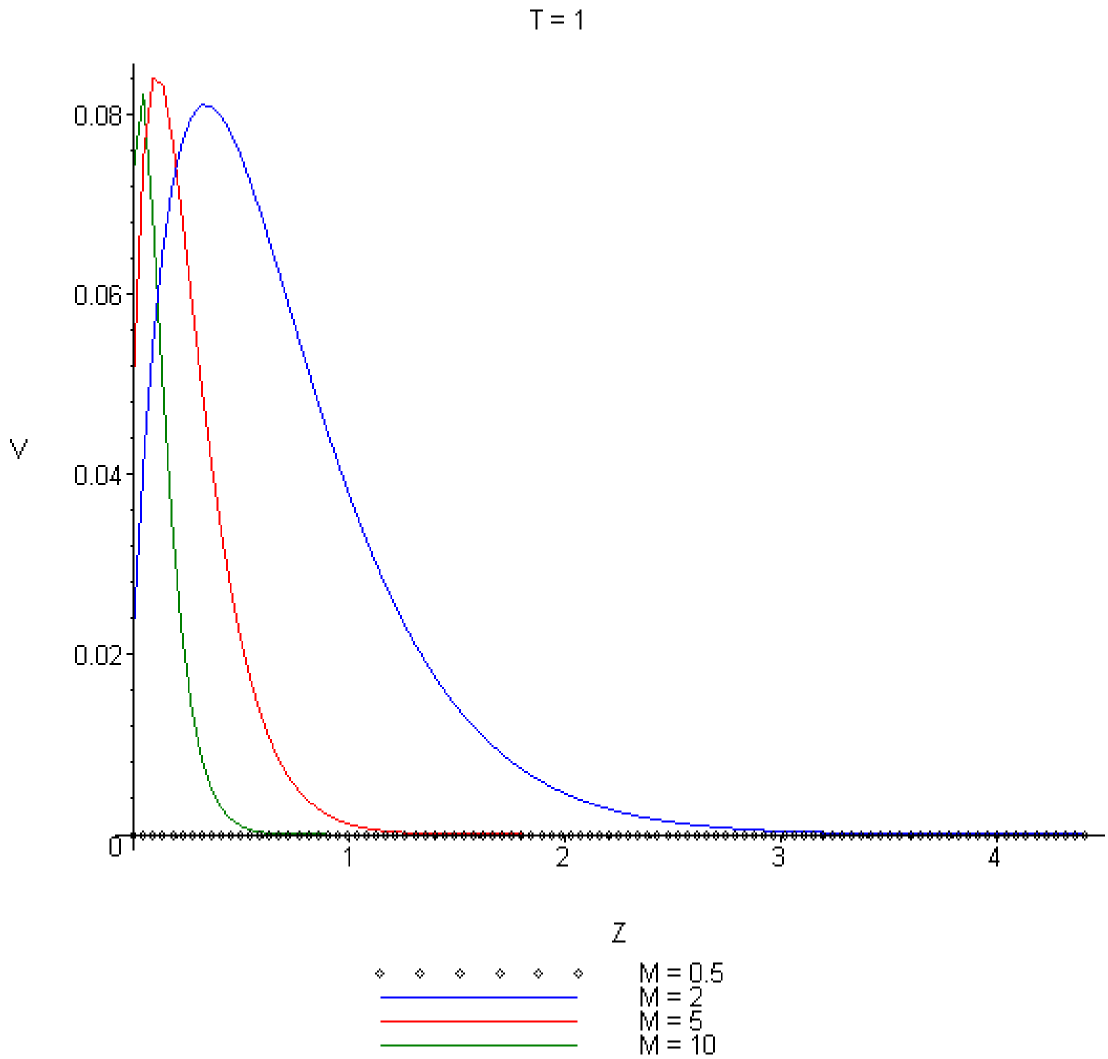

Figure (8) : secondary velocity profiles for several values of *M* with m = 0.5 , Ω = 0.1 and T = 1 .

5.1.1.3 Hall parameter *m*

Figures 9 and 10 show the effect of Hall parameter on the fluid velocities with the assigned values of $M = 0.5$, $\Omega = 0.1$ and $T = 1$.

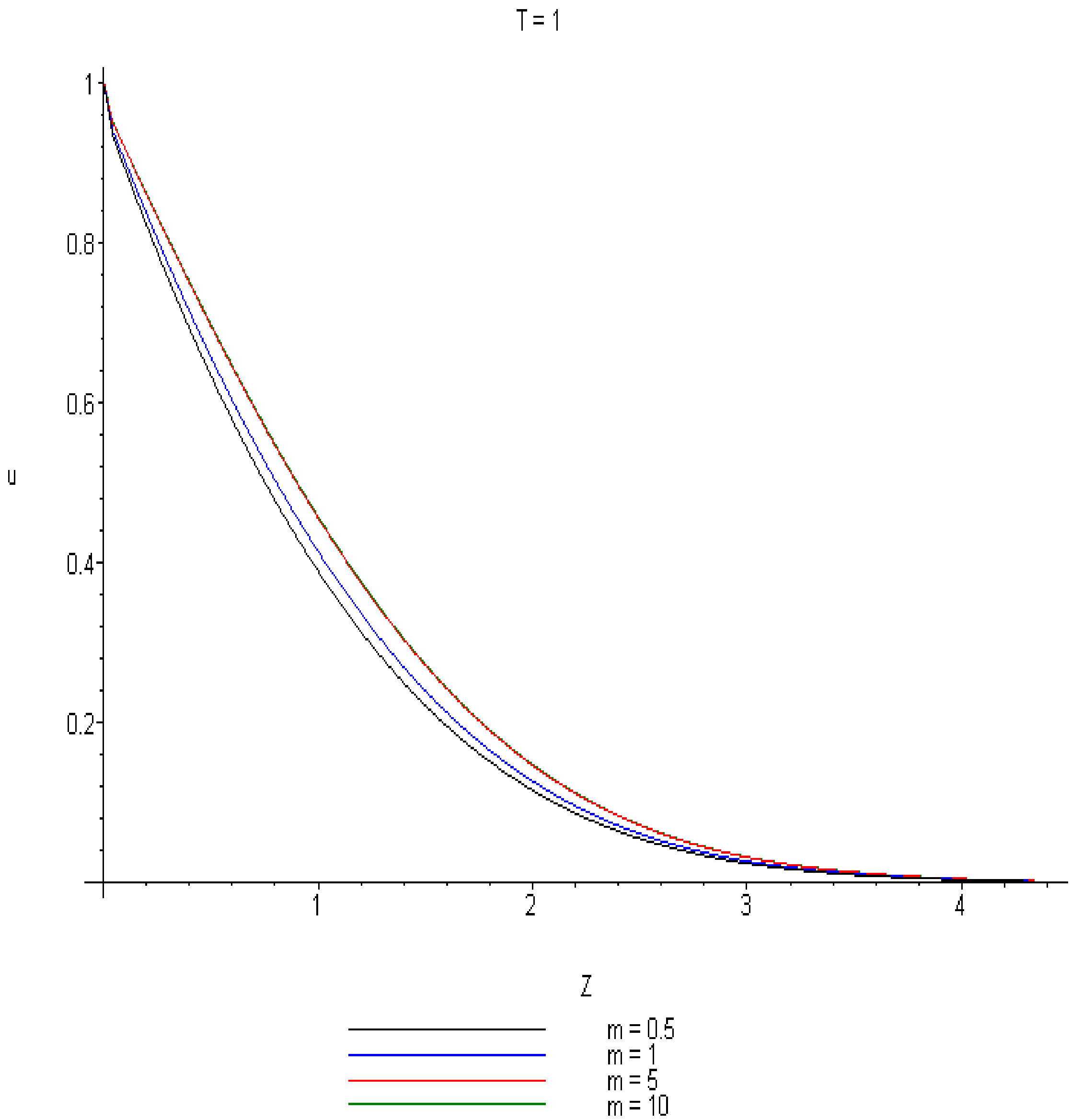

Figure (9): primary velocity profiles for several values of m with $M = 0.5$, $\Omega = 0.1$ and T = 1 .

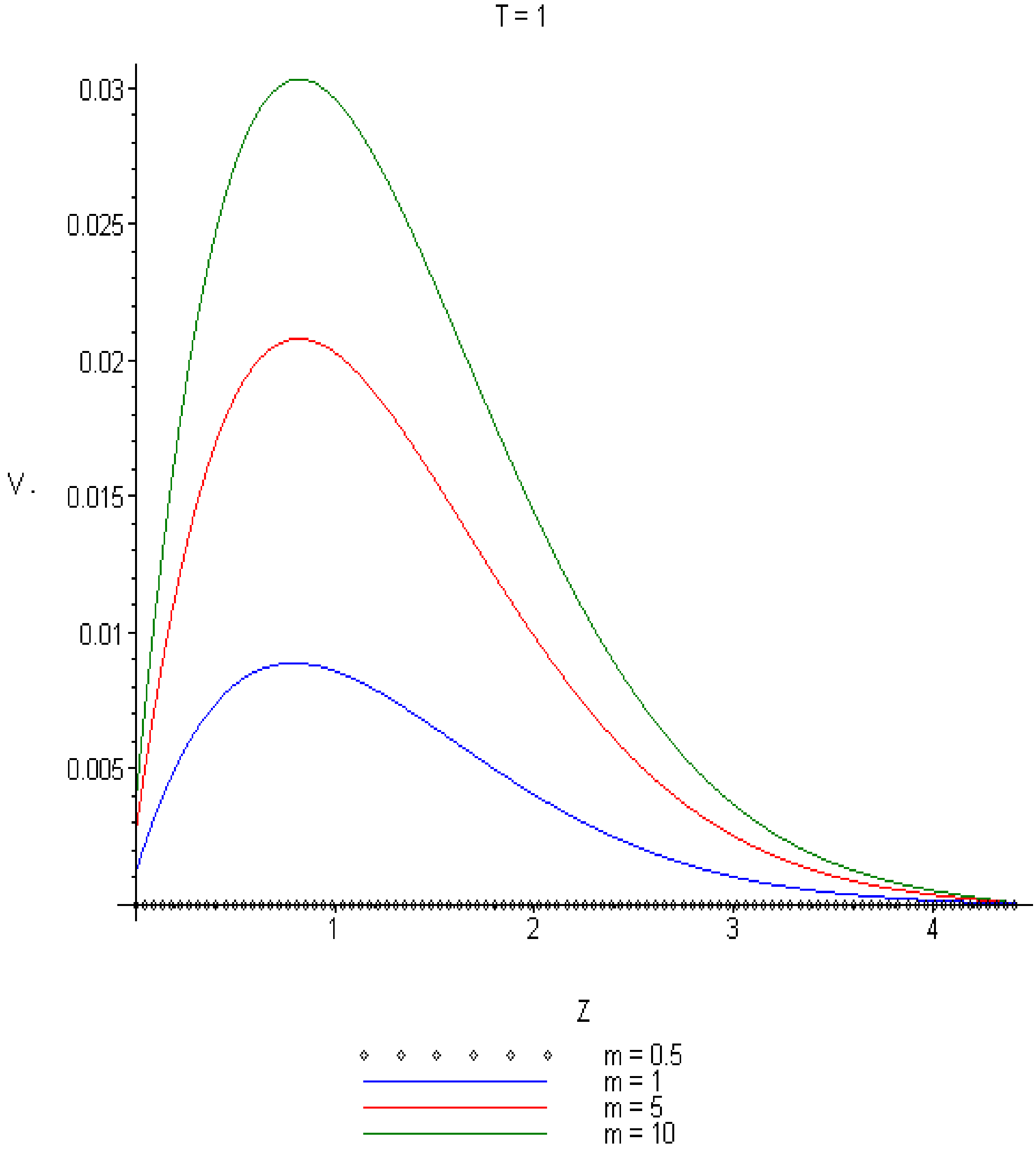

Figure (10) : secondary velocity profiles for several values of m with $M = 0.5$, Ω = 0.1 and T = 1 .

5.1.2 The Unsteady MHD Flow Through Two Parallel Porous Flat Plates

A numerical calculations for the primary velocity U for different values of Hartman Number M , Hall parameter m and normal coordinate Y keeping the value of time T fixed at $T = 0.5$ are presented.

The effect of Hartman Number M on the variation of primary velocity U in the absence of Hall parameter $m = 0$ are shown in Figure 11. Figure 12 takes into account the presence of Hall parameter.

Finally, Figure 13 shows the effect of Hall parameter on the fluid velocity with the assigned values of $M = 2$ and $T = 0.5$.

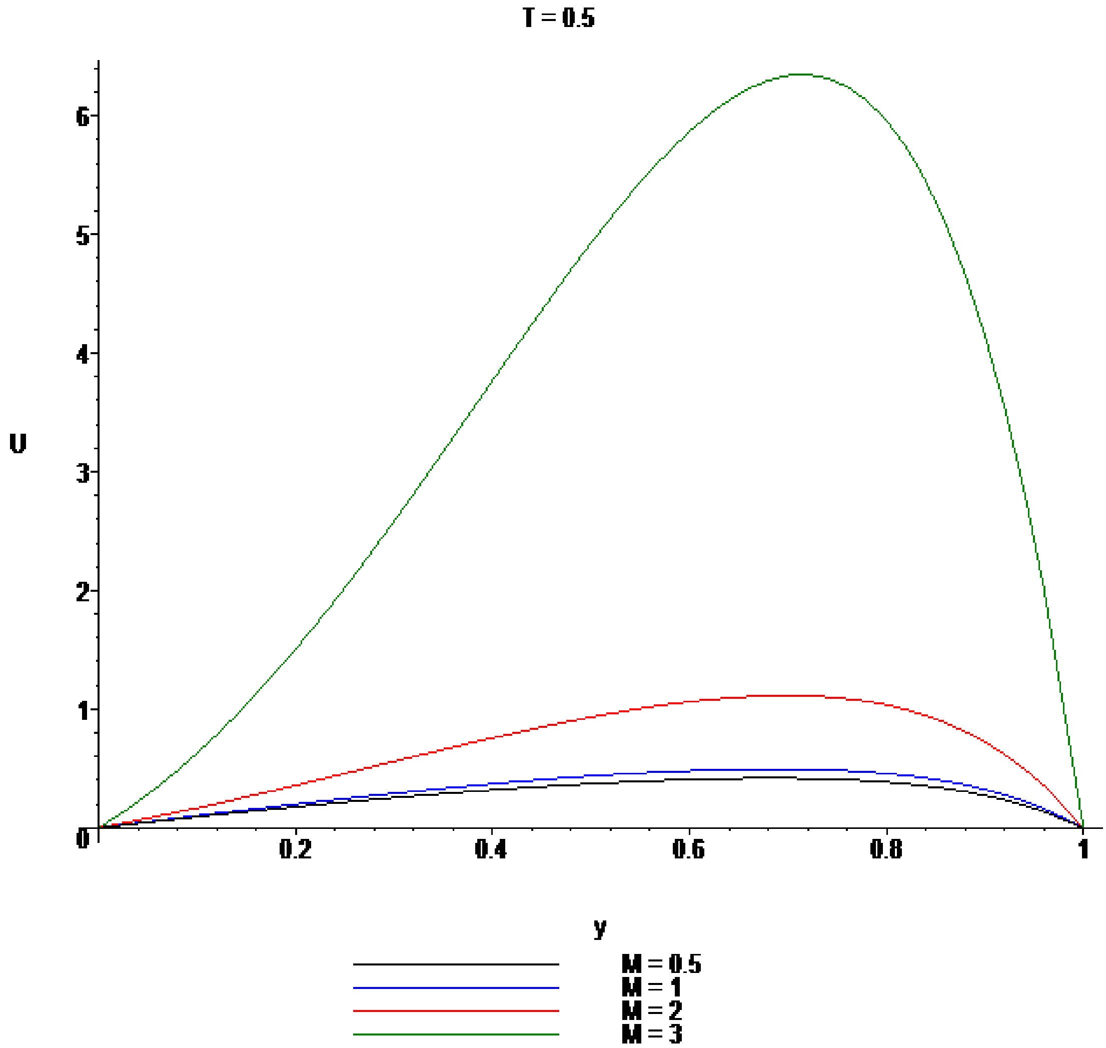

Figure (11) : primary velocity profiles for several values of *M* with m = 0 and T = 0.5 .

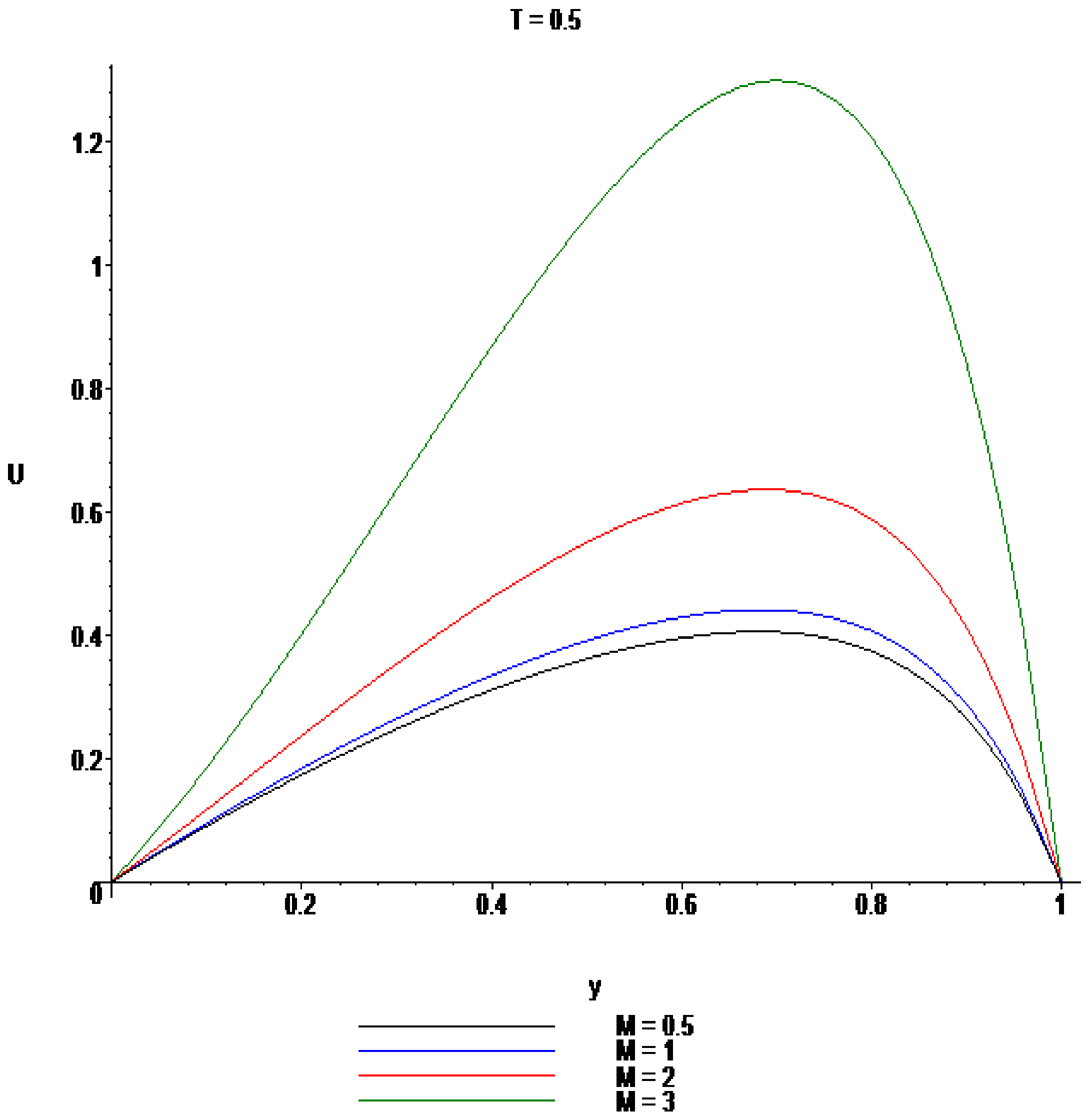

Figure (12) : primary velocity profiles for several values of *M* with m = 1 and T = 0.5 .

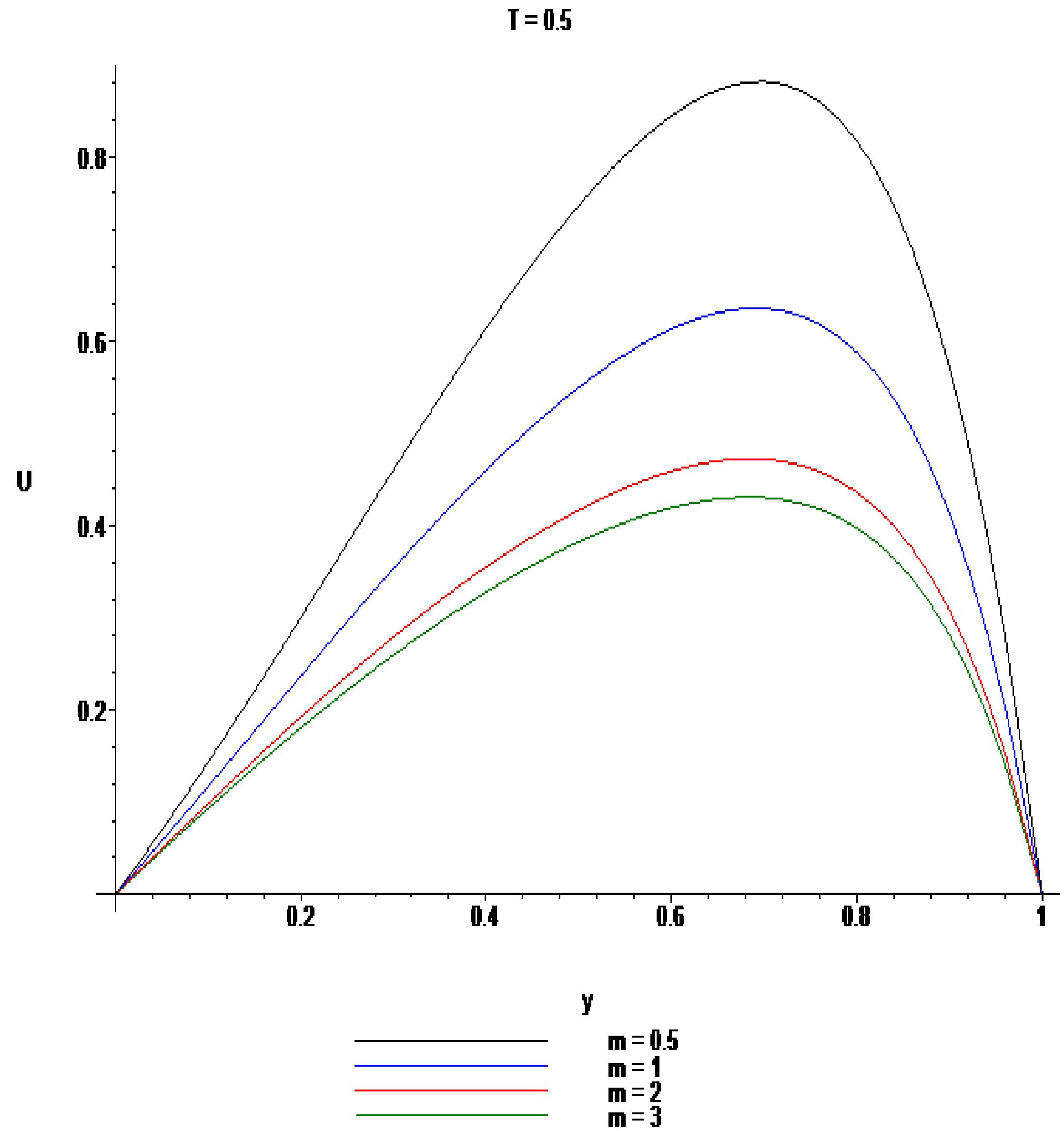

Figure (13) : primary velocity profiles for several values of m with $M = 2$ and T = 0.5 .

5.2 Conclusions

In this thesis we have presented analytical and numerical solutions to some MHD flow problems. The analytical solution is based on Laplace transform. The finite difference method is used to obtain the numerical solutions.

The numerical results show the following observations:

1) Figures 1 and 2 show that the primary velocity U decreases as Ω increases from 0.1 to 0.8, while the secondary velocity V increases. Figures 3 and 4 show the same result. A presence of Hall parameter in the values $M = 0.5$, $m = 0.5$ and $\Omega = 0.1$ shows that the secondary velocity vanishes and the flow is in the direction of the plate only. That is because in these values $\Omega = \frac{M^2 m}{(1+m^2)} = 0.1$. In general where $\Omega = \frac{M^2 m}{(1+m^2)}$ for all values of m and M we have similar conclusion.

2) Figures 5 and 6 show that the primary and the secondary velocities U and V decrease as the values of M increase. On the other hand, we observed in figures 7 and 8 that the primary velocity still decreases as M increases, but the secondary velocity increases as M increases.

3) Figures 9 and 10 show that due to an increase in the Hall parameter, there is a rise in both the primary and secondary velocity components U and V .

4) Figure 11 confirms that the primary velocity U increases as M increases from 0.5 to 3. Figure 12 also shows the same result, but in the presence of Hall parameter the maximum value decreases.

5) Figure 13 presents that the primary velocity U decreases as the values of m increases.

References

[1] N. Ahmed and H. K. Sarmah, **MHD Transient Flow Past an Impulsively Started Infinite Horizontal Porous Plate in a Rotating System With Hall Current**, Gauhati University, India, (2010).

[2] H. Alfven, ***Discovery of Alfven Waves***, **Nature 150**, pp.405, 1942.

[3] N.A. Battista, **An Introduction to Magnetohydrodynamics**, Stony Brook University, New York, (2010).

[4] R. L. Burden and J. D. Faires , **Numerical Analysis** ,7th edition, Brooks Cole, (2000).

[5] T. Cowling, **Magneto – hydrodynamics**, Wiley Inter Science, New York 1957.

[6] M. Davis, **Finite Difference Methods**, MSc Course in Mathematics and Finance, Imperial College London, (2010).

[7] R. K. Deka, ***Hall effects on MHD flow past an accelerated plate***, **Theoret. Appl. Mech.**, Vol.35, No.4, pp. 333-346, Belgrade, (2008).

[8] S. Dobek, **Fluid Dynamics and the Navier-Stokes Equation**, University of Maryland, (2012).

[9] M. Dubson, **PHYS1120 Lecture Notes**, University of Colorado at Boulder, (2009).

[10] D. R. Fearn, **4H MAGNETOHYDRODYNAMICS**, University of Glasgow,(2003).

[11] V. Ferraro and C. Plumpton, **An introduction to magneto fluid mechanics**, Clarendon Press, Oxford, 1966.

[12] S. M. Fielding, **Laminar Boundary Layer Theory**, Durham University, (2005).

[13] N.E. Gilbert, **Electricity and magnetism theory and applications**, Macmillan, (1950).

[14] I.S. Grant and W.R. Phillips, **Electromagnetism**, Wiley, 2nd Edition, (1990).

[15] R. Green, **Hall Effect Measurements for Semiconductor and Other Material Characterization**, Keithley Instruments, Inc, (2011).

[16] A. Gupta, **Actamechanica**, 22, pp.281, 1975.

[17] B. Guru and H. Hiziroglu, **Electromagnetic Field Theory Fundamentals**, 2nd edition, Cambridge University Press, (2004).

[18] D. M. Haughton, **Numerical Solution of Partial Differential Equations**, Department of Mathematics, The University of Glasgow, November 30, (2005).

[19] M. Healey, **Tables of Laplace Heaviside Fourier and z transforms**, Published by W & R Chambers Ltd, (1967).

[20] R. B. Hetnarski, ***An algorithm for generating some inverse Laplace trans- forms of exponential form***, **ZAMP** 26, 249-253 (1975).

[21] M. H. Holmes, **Introduction to Numerical Methods in Differential Equations**, Springer, (2007).

[22] M. Katugiri, **JFKatM**, 40, 2, pp.225, 1969.

[23] N. Kishore, S. Tejpal and H. K. Katiyar, ***On Unsteady MHD Flow Through two Parallel Porous Flat Plates***, **Indian J. pure appl. Math.**, 12(11): 1372 – 1379, November, (1981).

[24] K. Krammer and S. Pai, **Magneto – Fluid Dynamics for Engineers and Applied Physics**, Mc- Graw Hill Co. New York, 1978.

[25] G. B. Meir, **Basics of Fluid Mechanics**, March, Orange Grove Books, (2013).

[26] M. Mostafa, ***Laminar fully developed mixed convection with viscous dissipation in a uniformly heated vertical double – passage channel***, **Thermal Science**, Vol.11, No.1, pp.27 – 41, 2007.

[27] P. V. O'neil, **Advanced Engineering Mathematics**, Cengage Learning, (2012).

[28] K. Pardeep and M. Singh, ***Instability of two rotating viscoelastic (Walter b) superposed fluids with suspended particles in porous medium,*** **Thermal Science**, Vol.11, pp. 93 – 112, 2007.

[29] M. C. Potter, **Fluid Mechanics Demystified**, McGraw-Hill Companies (2009).

[30] S. Prakash, ***An Exact Solution of The Problem of Unsteady Viscous Flow Through a Porous Straight Channel***, **Proc. natn. Inst. Sci. India 35 A**, 29 – 123, May, (1967).

[31] E. Sakhnoskii, **Soviet Physics – Tech Physics**, Vol.18, pp.496, 1963.

[32] R.A. Serway and R. J. Beichner and J.W. Jewett, **Physics for Scientists and Engineers**, Saunders College Publishing, (2000).

[33] J. Shercliff, **A text Book of Magnetohydrodynamics**, Pergamon Press, London, 1965.

[34] M. R. Spiegel, **Schaum's Outlines Laplace Transforms**, The McGraw-Hill Publishing Co, (1965).

[35] A. Stahel, **Numerical Methods**, Bern University of applied science, (2012).

[36] G. Stephenson, **Partial Differential Equation for Scientists and Engineers**, 3rd edition, Longman Imc., New York, (1985).

[37] H. Sulieman and N. Qatanani, ***Magnetohydrodynamic Rayleigh Problem with Hall Effect***, **International Journal of Modern Engineering Research (IJMER)**, Vol.2, Issue.1, Jan-Feb 2012, pp-390-402.

[38] A. Thom and C. J. Apelt, **Field Computations in Engineering and Physics**, London, (1961).

[39] J. W. Thomas, **Numerical Partial Differential Equations – Finite Difference Methods**, Springer-Verlag New York, (1995).

[40] M. S. Tillack and N. B. Morley, **Standard Handbook for Electrical Engineers MAGNETOHYDRODYNAMICS**, 14th Edition, McGraw Hill, (1998).

[41] C.J. Tranter, **Integral transforms in mathematical physics**, London, (1951).

[42] G. K. Vallis, **Atmospheric and Oceanic Fluid Dynamics : Fundamentals and Large-scale Circulation**, 1st Ed, Cambridge University Press, (2006).

[43] C. Walkden, **Complex Analysis, Lecture Notes**, MATH20101, The University of Manchester, October, (2012).

[44] F. Wilhelm, **PHYS230 Lecture Notes Chapter 30 Sources of magnetic field**, Diablo Valley College, (2010).

Appendix

Maple codes

> # MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System without hall effect(different values of the rotation parameter)

```
> restart;
> with(LinearAlgebra):with(plots):
> Q:= 100: N:=400: L:=4.5: T:= 2.0: h:=evalf(L/F): k:= T/N: lam:=k/(h^2):
om1:=0.1:M:=0.5:om2:=0.3:om3:=0.5:om4:=0.8:m:=0:
> for E from 1 to 4 do
    > a(E) :=2*(M^2+I*om(E) ):
    > Y:=Vector(Q-1):
    > t := Vector(N):
    > for i from 1 to N do
        > w(E)(i) := Vector(Q-1);
        > t[i]:=i*k:
        > u(E)(i):= Vector(Q-1);
    > od:
    >  for i from 1 to Q-1 do
        > u(E)(1)(i):= 0:
        > Y[i]:=(i-1)*h:
        > y(i):= Vector(N):
```

```
> od:
> u(E)(1)(1):=lam:
> A:= evalf(-lam) : B(E) := evalf(1+k*a(E)+2*lam) : F:=evalf(-lam):
> z(E):= Matrix(Q-1):
> for i from 1 to Q-1 do
    > for j from 1 to Q-1 do
        > if i=j then z(E)(i,j):=B elif (j=i+1) then z(E)(i,j):=A
    elif (j=i-1) then z(E)(i,j):=F end if;
    > od;
> od;
> for i from 1 to N do
    > w(E)(i):=evalf( MatrixInverse(z(E)).u(E)(i));
    > u(E)(i+1):=(w(E)(i));
    > u(E)(i+1)(1):=w(E)(i)(1)+lam;
> od:
> U(E):=Vector(Q):
> for i from 1 to Q-1 do
    > U(E)[i]:=Re(w(E)(200)(i));
    > U(E)[1]:=1;
> od:
> v(E):=-Im(w(E)(200)):
```

> od:

> r:=plot(Y,U(1) ,Z,u,color="Black",legend = ["Omega = 0.1"]):

> r1:=plot(Y,U(2),color="Blue",legend = ["Omega = 0.3"]) :

> r2:=plot (Y,U(3) ,color="Red",legend = ["Omega = 0.5"]):

> r3:=plot (Y,U(4) ,color="Green",legend = ["Omega = 0.8"]):

> display(r,r1,r2,r3,title = "T = 1"):

> r4:=plot(Y,v(1),Z,V ,color="Black",legend = ["Omega = 0.1"]):

> r5:=plot(Y,v(2),color="Blue",legend = ["Omega = 0.3"]) :

> r6:=plot (Y,v(3) ,color="Red",legend = ["Omega = 0.5"]):

> r7:=plot (Y,v(4) ,color="Green",legend = ["Omega = 0.8"]):

> display(r4,r5,r6,r7,title = "T = 1"):

> # MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System with hall effect(different values of the rotation parameter)

> restart;

> with(LinearAlgebra):with(plots):

> Q:= 100: N:=400: L:=4.5: T:= 2.0: h:=evalf(L/F): k:= T/N: lam:=k/(h^2): om1:=0.1:M:=0.5:om2:=0.3:om3:=0.5:om4:=0.8:m:=0.5:

> for E from 1 to 4 do

> a(E) := 2*(M^2/(1+m^2) + I*(om(E)-(M^2*m)/(1+m^2)));

> Y:=Vector(Q-1):

```
> t := Vector(N):
> for i from 1 to N do
        > w(E)(i) := Vector(Q-1);
        > t[i]:=i*k:
        > u(E)(i):= Vector(Q-1);
> od:
>  for i from 1 to Q-1 do
        > u(E)(1)(i):= 0:
        > Y[i]:=(i-1)*h:
        > y(i):= Vector(N):
> od:
> u(E)(1)(1):=lam:
> A:= evalf(-lam) : B(E) := evalf(1+k*a(E)+2*lam) : F:=evalf(-lam):
> z(E):= Matrix(Q-1):
> for i from 1 to Q-1 do
        > for j from 1 to Q-1 do
                >  if i=j then z(E)(i,j):=B elif (j=i+1) then z(E)(i,j):=A
        elif (j=i-1) then z(E)(i,j):=F end if;
        > od;
> od;
> for i from 1 to N do
```

```
> w(E)(i):=evalf( MatrixInverse(z(E)).u(E)(i));
> u(E)(i+1):=(w(E)(i));
> u(E)(i+1)(1):=w(E)(i)(1)+lam;
> od:
> U(E):=Vector(Q):
> for i from 1 to Q-1 do
> U(E)[i]:=Re(w(E)(200)(i));
> U(E)[1]:=1;
> od:
> v(E):=-Im(w(E)(200)):
> od:
> r:=plot( Y,U(1) ,Z,u,color="Black",legend = ["Omega = 0.1" ]):
> r1:=plot(Y,U(2),color="Blue",legend = ["Omega = 0.3" ]) :
> r2:=plot ( Y,U(3) ,color="Red",legend = ["Omega = 0.5" ]):
> r3:=plot ( Y,U(4) ,color="Green",legend = ["Omega = 0.8" ]):
> display(r,r1,r2,r3,title = "T = 1"):
> r4:=plot( Y,v(1),Z,V ,color="Black",legend = ["Omega = 0.1" ]):
> r5:=plot(Y,v(2),color="Blue",legend = ["Omega = 0.3" ]) :
> r6:=plot ( Y,v(3) ,color="Red",legend = ["Omega = 0.5" ]):
> r7:=plot ( Y,v(4) ,color="Green",legend = ["Omega = 0.8" ]):
> display(r4,r5,r6,r7,title = "T = 1"):
```

> # MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System without hall effect (different values of the Hartman Number M)

```
> restart;
> with(LinearAlgebra):with(plots):
> Q:= 100: N:=400: L:=4.5: T:= 2.0: h:=evalf(L/F): k:= T/N: lam:=k/(h^2):
om:=0.1:M1:=0.5:M2:=2:M3:=5:M4:=10:m:=0:
> for E from 1 to 4 do
    > a(E) :=2*(M(E) ^2+I*om):
    > Y:=Vector(Q-1):
    > t := Vector(N):
    > for i from 1 to N do
        > w(E)(i) := Vector(Q-1);
        > t[i]:=i*k:
        > u(E)(i):= Vector(Q-1);
    > od:
    >  for i from 1 to Q-1 do
        > u(E)(1)(i):= 0:
        > Y[i]:=(i-1)*h:
        > y(i):= Vector(N):
    > od:
    > u(E)(1)(1):=lam:
```

```
> A:= evalf(-lam) : B(E) := evalf(1+k*a(E)+2*lam) : F:=evalf(-lam):
> z(E):= Matrix(Q-1):
> for i from 1 to Q-1 do
    > for j from 1 to Q-1 do
        > if i=j then z(E)(i,j):=B elif (j=i+1) then z(E)(i,j):=A
    elif (j=i-1) then z(E)(i,j):=F end if;
    > od;
> od;
> for i from 1 to N do
    > w(E)(i):=evalf( MatrixInverse(z(E)).u(E)(i));
    > u(E)(i+1):=(w(E)(i));
    > u(E)(i+1)(1):=w(E)(i)(1)+lam;
> od:
> U(E):=Vector(Q):
> for i from 1 to Q-1 do
    > U(E)[i]:=Re(w(E)(200)(i));
    > U(E)[1]:=1;
> od:
> v(E):=-Im(w(E)(200)):
> od:
> r:=plot( Y,U(1) ,Z,u,color="Black",legend = ["M= 0.5" ]):
```

```
> r1:=plot(Y,U(2),color="Blue",legend = ["M = 2" ]) :
> r2:=plot ( Y,U(3) ,color="Red",legend = ["M = 5" ]):
> r3:=plot ( Y,U(4) ,color="Green",legend = ["M = 10" ]):
> display(r,r1,r2,r3,title = "T = 1"):
> r4:=plot( Y,v(1),Z,V ,color="Black",legend = ["M = 0.5" ]):
> r5:=plot(Y,v(2),color="Blue",legend = ["M = 2" ]) :
> r6:=plot ( Y,v(3) ,color="Red",legend = ["M = 5" ]):
> r7:=plot ( Y,v(4) ,color="Green",legend = ["M = 10" ]):
> display(r4,r5,r6,r7,title = "T = 1"):
```

> # MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System with hall effect(different values of the Hartman Number M)

```
> restart;
> with(LinearAlgebra):with(plots):
> Q:= 100: N:=400: L:=4.5: T:= 2.0: h:=evalf(L/F): k:= T/N: lam:=k/(h^2):
om:=0.1:M1:=0.5:M2:=2:M3:=5:M4:=10:m:=0.5:
> for E from 1 to 4 do
    > a(E) := 2*(M(E)^2/(1+m^2) + I*(om-(M(E)^2*m)/(1+m^2)));
    > Y:=Vector(Q-1):
    > t := Vector(N):
    > for i from 1 to N do
```

```
        > w(E)(i) := Vector(Q-1);
        > t[i]:=i*k:
        > u(E)(i):= Vector(Q-1);
> od:
>  for i from 1 to Q-1 do
        > u(E)(1)(i):= 0:
        > Y[i]:=(i-1)*h:
        > y(i):= Vector(N):
> od:
> u(E)(1)(1):=lam:
> A:= evalf(-lam) : B(E) := evalf(1+k*a(E)+2*lam) : F:=evalf(-lam):
> z(E):= Matrix(Q-1):
> for i from 1 to Q-1 do
        > for j from 1 to Q-1 do
                >  if i=j then z(E)(i,j):=B elif (j=i+1) then z(E)(i,j):=A
        elif (j=i-1) then z(E)(i,j):=F end if;
        > od;
> od;
> for i from 1 to N do
        > w(E)(i):=evalf( MatrixInverse(z(E)).u(E)(i));
        > u(E)(i+1):=(w(E)(i));
```

```
        > u(E)(i+1)(1):=w(E)(i)(1)+lam;
    > od:
    > U(E):=Vector(Q):
    > for i from 1 to Q-1 do
        > U(E)[i]:=Re(w(E)(200)(i));
        > U(E)[1]:=1;
    > od:
    > v(E):= Im(w(E)(200)):
> od:
> r:=plot( Y,U(1) ,Z,u,color="Black",legend = ["M = 0.5" ]):
> r1:=plot(Y,U(2),color="Blue",legend = ["M = 2" ]) :
> r2:=plot ( Y,U(3) ,color="Red",legend = ["M = 5" ]):
> r3:=plot ( Y,U(4) ,color="Green",legend = ["M = 10" ]):
> display(r,r1,r2,r3,title = "T = 1"):
> r4:=plot( Y,v(1),Z,V ,color="Black",legend = ["M = 0.5" ]):
> r5:=plot(Y,v(2),color="Blue",legend = ["M = 2" ]) :
> r6:=plot ( Y,v(3) ,color="Red",legend = ["M = 5" ]):
> r7:=plot ( Y,v(4) ,color="Green",legend = ["M = 10" ]):
> display(r4,r5,r6,r7,title = "T = 1"):
```

> # MHD Flow Past an Impulsively Started Infinite Horizontal Plate in a Rotating System with hall effect(different values of the Hall parameter)

```
> restart;
> with(LinearAlgebra):with(plots):
> Q:= 100: N:=400: L:=4.5: T:= 2.0: h:=evalf(L/F): k:= T/N: lam:=k/(h^2):
om:=0.1:M:=0.5: m1:=0.5:m2:=1:m3:=5:m4:=10:
> for E from 1 to 4 do
    > a(E) := 2*(M^2/(1+m(E)^2) + I*(om-(M^2*m(E))/(1+m(E)^2)));
    > Y:=Vector(Q-1):
    > t := Vector(N):
    > for i from 1 to N do
        > w(E)(i) := Vector(Q-1);
        > t[i]:=i*k:
        > u(E)(i):= Vector(Q-1);
    > od:
    >  for i from 1 to Q-1 do
        > u(E)(1)(i):= 0:
        > Y[i]:=(i-1)*h:
        > y(i):= Vector(N):
    > od:
    > u(E)(1)(1):=lam:
```

```
        > A:= evalf(-lam) : B(E) := evalf(1+k*a(E)+2*lam) : F:=evalf(-lam):
        > z(E):= Matrix(Q-1):
        > for i from 1 to Q-1 do
            > for j from 1 to Q-1 do
                > if i=j then z(E)(i,j):=B elif (j=i+1) then z(E)(i,j):=A
            elif (j=i-1) then z(E)(i,j):=F end if;
            > od;
        > od;
        > for i from 1 to N do
            > w(E)(i):=evalf( MatrixInverse(z(E)).u(E)(i));
            > u(E)(i+1):=(w(E)(i));
            > u(E)(i+1)(1):=w(E)(i)(1)+lam;
        > od:
        > U(E):=Vector(Q):
        > for i from 1 to Q-1 do
            > U(E)[i]:=Re(w(E)(200)(i));
            > U(E)[1]:=1;
        > od:
        > v(E):=-Im(w(E)(200)):
    > od:
    > r:=plot( Y,U(1) ,Z,u,color="Black",legend = ["Omega = 0.1" ]):
```

```
> r1:=plot(Y,U(2),color="Blue",legend = ["Omega = 0.3" ]) :
> r2:=plot ( Y,U(3) ,color="Red",legend = ["Omega = 0.5" ]):
> r3:=plot ( Y,U(4) ,color="Green",legend = ["Omega = 0.8" ]):
> display(r,r1,r2,r3,title = "T = 1"):
> r4:=plot( Y,v(1),Z,V ,color="Black",legend = ["Omega = 0.1" ]):
> r5:=plot(Y,v(2),color="Blue",legend = ["Omega = 0.3" ]) :
> r6:=plot ( Y,v(3) ,color="Red",legend = ["Omega = 0.5" ]):
> r7:=plot ( Y,v(4) ,color="Green",legend = ["Omega = 0.8" ]):
> display(r4,r5,r6,r7,title = "T = 1"):
```

> # The Unsteady MHD Flow Through Two Parallel Porous Flat Plates Without Hall Effect (different values of the Hartman Number M)

```
> restart;
> with(LinearAlgebra):with(plots):
> Q:= 50: N:=210: l:=1: T:= 1.0: h:=evalf(l/M); k:= T/N; lam:=k/(h^2);R:=2*Pi;p:=1;M1:=0.5;M2:=1:M3:=2:M4:=3:
> for E from 1 to 4 do
    > Y:=Vector(Q-1):
    > t := Vector(N):
    > for i from 1 to N do
        > w(E)(i) := Vector(Q-1);
        > t[i]:=i*k:
```

```
> u(E)(i):= Vector(Q-1);
> od:
> q:= Vector(Q-1):
>  for i from 1 to Q-1 do
    > u(E)(1)(i):= p*k:
    > Y[i]:=(i-1)*h:
    > y(i):= Vector(N):
    > q[i]:=p*k:
> od:
> Y[M]:=1:
> A:= evalf(((h/2)-1/R)*lam) : B(E):= evalf(1-k*M(E)^2+2*lam/R) :
F:=evalf(((h/2)+1/R)*lam):
> z(E):= Matrix(Q-1):
> for i from 1 to Q-1 do
    > for j from 1 to Q-1 do
        >  if i=j then z(E)(i,j):=B elif (j=i+1) then z(E)(i,j):=A
    elif (j=i-1) then z(E)(i,j):=-F end if;
    > od;
> od;
> for i from 1 to N do
    > w(E)(i):=evalf( MatrixInverse(z(E)).u(E)(i));
    > u(E)(i+1):=(w(E)(i)+ q);
```

```
        > od:
        > W:=Vector(Q):
        > for i from 1 to Q-1 do
                > W(E)[i+1]:=(w(E)(105)(i));
                > W(E)[1]:=0:
                > W(E)[M]:=0;
        > od:
> od:
> r:=plot( Y,W(1) ,Y,U,color="Black",legend = ["M = 0.5" ]):
> r1:=plot(Y,W(2),color="Blue",legend = ["M = 1" ]) :
> r2:=plot ( Y,W(3) ,color="Red",legend = ["M = 2" ]):
> r3:=plot ( Y,W(4) ,color="Green",legend = ["M = 3" ]):
> display(r,r1,r2,r3,title = "T = 0.5"):
```

> # The Unsteady MHD Flow Through Two Parallel Porous Flat Plates With Hall Effect (different values of the Hartman Number M)

```
> restart;
> with(LinearAlgebra):with(plots):
> Q:= 50: N:=210: l:=1: T:= 1.0: h:=evalf(l/M); k:= T/N; lam:=k/(h^2);R:=2*Pi;p:=1;M1:=0.5;M2:=1:M3:=2:M4:=3:m:=1:
> for E from 1 to 4 do
        > a(E):=M(E)^2/(1+m^2);
```

```
> Y:=Vector(Q-1):
> t := Vector(N):
> for i from 1 to N do
    > w(E)(i) := Vector(Q-1);
    > t[i]:=i*k:
    > u(E)(i):= Vector(Q-1);
> od:
> q:= Vector(Q-1):
>  for i from 1 to Q-1 do
    > u(E)(1)(i):= p*k:
    > Y[i]:=(i-1)*h:
    > y(i):= Vector(N):
    > q[i]:=p*k:
> od:
> Y[M]:=1:
> A:= evalf(((h/2)-1/R)*lam) : B(E):= evalf(1-k* a(E)+2*lam/R) : F:=evalf(((h/2)+1/R)*lam):
> z(E):= Matrix(Q-1):
> for i from 1 to Q-1 do
    > for j from 1 to Q-1 do
        > if i=j then z(E)(i,j):=B elif (j=i+1) then z(E)(i,j):=A elif (j=i-1) then z(E)(i,j):=-F end if;
```

```
        > od;
    > od;
    > for i from 1 to N do
        > w(E)(i):=evalf( MatrixInverse(z(E)).u(E)(i));
        > u(E)(i+1):=(w(E)(i)+ q);
    > od:
    > W:=Vector(Q):
    > for i from 1 to Q-1 do
        > W(E)[i+1]:=(w(E)(105)(i));
        > W(E)[1]:=0:
        > W(E)[M]:=0;
    > od:
> od:
> r:=plot( Y,W(1) ,Y,U,color="Black",legend = ["M = 0.5" ]):
> r1:=plot(Y,W(2),color="Blue",legend = ["M = 1" ]) :
> r2:=plot ( Y,W(3) ,color="Red",legend = ["M = 2" ]):
> r3:=plot ( Y,W(4) ,color="Green",legend = ["M = 3" ]):
> display(r,r1,r2,r3,title = "T = 0.5"):
```

> # The Unsteady MHD Flow Through Two Parallel Porous Flat Plates With Hall Effect (different values of the Hall parameter)

```
> restart;
> with(LinearAlgebra):with(plots):
> Q:= 50: N:=210: l:=1: T:= 1.0: h:=evalf(l/M); k:= T/N; lam:=k/(h^2);R:=2*Pi;p:=1;M:=2;m1:=0.5:m2:=1:m3:=2:m4:=3:
 > for E from 1 to 4 do
    > a(E):=M^2/(1+m(E)^2);
    > Y:=Vector(Q-1):
    > t := Vector(N):
    > for i from 1 to N do
        > w(E)(i) := Vector(Q-1);
        > t[i]:=i*k:
        > u(E)(i):= Vector(Q-1);
    > od:
    > q:= Vector(Q-1):
    >  for i from 1 to Q-1 do
        > u(E)(1)(i):= p*k:
        > Y[i]:=(i-1)*h:
        > y(i):= Vector(N):
        > q[i]:=p*k:
    > od:
```

```
> Y[M]:=1:
> A:= evalf(((h/2)-1/R)*lam) : B(E):= evalf(1-k* a(E)+2*lam/R) :
F:=evalf(((h/2)+1/R)*lam):
> z(E):= Matrix(Q-1):
> for i from 1 to Q-1 do
    > for j from 1 to Q-1 do
        > if i=j then z(E)(i,j):=B elif (j=i+1) then z(E)(i,j):=A
    elif (j=i-1) then z(E)(i,j):=-F end if;
    > od;
> od;
> for i from 1 to N do
    > w(E)(i):=evalf( MatrixInverse(z(E)).u(E)(i));
    > u(E)(i+1):=(w(E)(i)+ q);
> od:
> W:=Vector(Q):
> for i from 1 to Q-1 do
    > W(E)[i+1]:=(w(E)(105)(i));
    > W(E)[1]:=0:
    > W(E)[M]:=0;
> od:
> od:
> r:=plot( Y,W(1) ,Y,U,color="Black",legend = ["m = 0.5" ]):
```

```
> r1:=plot(Y,W(2),color="Blue",legend = ["m = 1" ]) :

> r2:=plot ( Y,W(3) ,color="Red",legend = ["m = 2" ]):

> r3:=plot ( Y,W(4) ,color="Green",legend = ["m = 3" ]):

> display(r,r1,r2,r3,title = "T = 0.5"):
```

جامعة النجاح الوطنية
كلية الدراسات العليا

الحلول النظرية والعددية لمشاكل الماجنتوهايدروداينيمك المغناطيسية

إعداد
عبد اللطيف خليل سعد الدين

إشراف
أ. د. ناجي قطناني

قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات المحوسبة بكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس، فلسطين.

2013م

الحلول النظرية والعددية لمشاكل الماجنتوهايدروداينيمك المغناطيسية

إعداد

عبد اللطيف خليل سعد الدين

إشراف

أ. د. ناجي قطناني

الملخص

إن التدفقات الهيدروديناميكية المغنطيسية هي واحدة من المواضيع المهمة في الرياضيات، لذلك في هذا العمل سوف نقدم بعض الحلول النظرية والعددية لبعض مشاكل هذه التدفقات.

لقد بدئنا هذا العمل بعرض بعض المفاهيم الهامة في المغناطيسية الكهربائية و ديناميكية السوائل، بالأخص معادلات ماكسويل ومعادلات نافيير ستوكس. بعد ذلك قمنا بوضع الصيغ الرياضية للمشاكل التي نريد حلها. في النهاية قيمنا بعرض بعض الحلول العددية والنظرية لتلك المشاكل.

لقد ناقشنا تأثير تيار هول في التدفقات الهيدروديناميكية المغنطيسية، وذلك عن طريق عرض نوعين من المشاكل، أولا نقشنا المشكلتين دون تأثير تيار هول، بعد ذلك تمت دارستهما ومعالجتهما تحت تأثير تيار هول.

لقد استخدمنا تقنية التحويلات التكاملية خصيصا تقنية لابلاس لإيجاد الحلول التحليلية للمشاكل التي تم عرضها، ولإيجاد الحلول العددية استخدمنا طريقة الفروق المحدودة، ولتمثيل النتائج والحصول عليها استخدمنا برنامج الميبل، حيث تم عرض النتائج بيانيا.