Uniform Convergence of Schwarz Method for Elliptic Quasi-Variational Inequalities Related to Impulse Control Problem

التقارب المنتظم بطريقة شوارتز(تداخل النطاقات) للمتراجات شبه المتغيرة الناقصية المتعلقة بتحكم دفع مثالي (Impulse Control Problem)

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Abstract

In this paper we provide a uniform convergence using an overlapping Schwarz method on nonmatching grids for quasi-variational inequalities related to impulse control problem. The discretization on every sub-domain converges in uniform norm was provided and a result of approximation in the L^∞-norm was given.

Keywords: Domain Decomposition, Quasi-variational Inequalities, Impulse Control, Error Analysis.

ملخص

في هذه الورقة نبين التقارب المنتظم بطريقة تداخل النطاقات (nonmatching grids) لشوارتزالمطبقة على المتراجات شبه المتغيرة الناقصية أين يكون حاجز المسألة المنفصلة (المتقطعة) متعلق بالحل والمعرف بتحكم دفع مثالي (Impulse control problem).حيث بُرْهِن التقارب المنتظم للمسألة المنفصلة (المتقطعة) في كلتى النطاقين المقسمين وفق الطريقة العددية المذكورة سلفا وأُعْطِيت نتيجتها التقريبية بواسطة النظيم المنتظم L^∞

Introduction

In this paper we provide a uniform error estimate of an overlapping Schwarz method on nonmatching grids for quasi-variational inequalities related to impulse control problem. Find $u \in H^1(\Omega)$ solution of

$$\begin{cases} -\Delta u - f \leq 0 \\ u - Mu \leq 0, Mu > 0 \\ (\Delta u - f)(u - Mu) = 0 \ in \ \Omega \\ u = \varphi \ in \ \Gamma_0 \\ \dfrac{\partial u}{\partial \eta} = 0 \end{cases}$$

where Ω is a smooth bounded domain of R^n with boundary $\partial\Omega = \Gamma$ and f is a regular function and M is an operator given by

$$Mu = k + \inf_{\xi \geq 0,\, x+\xi \in \overline{\Omega}} u(x + \xi)$$

where $k > 0$ and $\xi \geq 0$ means that $\xi = (\xi_1, \xi_2, ...\xi_n)$ with $\xi_i \geq 0, i = \overline{1, n}$.

Finally, Γ_0 is the part of the boundary defined as

$$\Gamma_0 = \{x \in \partial\Omega \, such \ that \ \forall \xi > 0 \, and \ x + \xi \notin \overline{\Omega}\}$$

On the analytical side, quasi-variational inequalities have been extensively studied in the last three decades (cf., e. g., (Bensoussan, A. & Lions, J. L. Acad, CR. & Paris, t., Sc. 1973. Boulbrachene, M. & Optimal, L. 2002). For the numerical approximations and computational aspects we have a few of the results (cf., e.g., (Boulbrachene, M. & Optimal, L. 2002. Haiour, M. & Hadidi, E. 2009. Jinping, Zeng. & Shuzi, Zhou. 1998).

In the present paper a new approach for the finite element approximation for the problem where the obstacle related to solution. We consider a domain which is the union of two overlapping sub-domains where each sub domain has its own generated triangulation. The grid points on the sub-domain boundaries need not much the grid points from the other sub-domain. Under a discrete maximum principle, we show that the discretization on each sub-domain converges quasi-optimally in the L^{∞}-norm.

We study in the first section the Schwarz method for the elliptic quasi-variational inequalities; we state the continuous alternating Schwarz sequence for the precedent quasi-variational inequalities, and define their respective finite element counterparts in the context of overlapping grids. Section 2 is devoted to the error analysis of the overlapping domain decomposition methods. As a result of this, is devoted to the proof of main fundamentals theorems then constructed, geometrical convergence established of the problem, and an error estimate in the maximum norm is derived.

The Schwarz Method for the Elliptic Quasi-variational Inequalities

We begin by down some definitions and classical results related to Quasi-variational inequalities.

Elliptic Quasi-Variational Inequalities

Let Ω be a convex domain in R^2 with sufficiently smooth boundary $\partial\Omega$. Now we consider the obstacle problem. Find $u \in H^1(\Omega)$ solution of

$$\begin{cases} -\Delta u - f \leq 0 \\ u - Mu \leq 0, Mu > 0 \\ u = \varphi \ in \ \Gamma_0 \\ \dfrac{\partial u}{\partial \eta} = 0 \end{cases} \tag{1}$$

We are given the right hand side f such that $f \in L^{\infty}(\Omega)$ and M is given by

$$Mu = k + \inf_{\xi \geq 0,\, x+\xi \in \bar{\Omega}} u(x+\xi) \tag{2}$$

We can reformulate (1) as

$$\begin{cases} a(u, v-u) \geq (f, v-u) \text{ in } \Omega \\ u - Mu \leq 0, Mu > 0 \\ u = \varphi \text{ in } \Gamma_0 \\ \dfrac{\partial u}{\partial \eta} = 0 \end{cases} \tag{3}$$

where

$$\begin{cases} a(u,v) = \int_{\Omega} \nabla u(x) \nabla v(x) dx \\ (f,v) = \int_{\Omega} f(x) v(x) dx \end{cases},$$

and the non empty convex set

$$K_{\varphi} = \begin{Bmatrix} u \in H^1(\Omega);\ u = \varphi \text{ in } \Gamma_0, u \leq Mu \text{ on } \Omega, \\ \text{and } \dfrac{\partial u}{\partial \eta} = 0 \end{Bmatrix} \tag{4}$$

Where φ is a regular function defined in Γ_0.

Let V^h be the space of finite elements consisting of continuous piecewise linear functions. The discrete counterpart of (3) consists of finding $u_h \in K_{\varphi h}$ such that

$$\begin{cases} a(u_h, v_h - u_h) \geq (f, v_h - u_h) \\ u_h - r_h M u_h \leq 0 \\ u_h = \varphi \ in \ \Gamma_0 \\ \dfrac{\partial u_h}{\partial \eta} = 0 \end{cases} \tag{5}$$

where $K_{\varphi h}$ non empty discrete convex set associated to K_φ

$$K_{\varphi h} = \left\{ \begin{array}{l} u \in V^h;\ u_h = \pi_h \varphi \ in \ \Gamma_0, \\ u_h \leq r_h M u_h \ \ on \ \Omega \ and \ \dfrac{\partial u_h}{\partial \eta} = 0 \end{array} \right\} \tag{6}$$

where π_h is an interpolation operator on Γ, and r_h is the usual finite element restriction operator on Ω.

The lemma below establishes a monotonic property of the solution of (5) with respect to the obstacle.

Lemma 1 (cf. 14)

If $u_h \leq \overline{u_h}$ *in the* $K_{\varphi h}$, *then* $Mu_h \leq M\overline{u_h}$

and we have important propriety

$$\forall \lambda > 0,\ \forall u \in K_\varphi;\ M(u + \lambda) = M(u) + \lambda \tag{7}$$

Remark 2

$$\forall u, \bar{u} \in K_\varphi;\ \left\| Mu - M\bar{u} \right\|_{L^\infty(\Omega)} \leq \left\| u - \bar{u} \right\|_{L^\infty(\Omega)} \tag{8}$$

Prof. We have $u \leq \bar{u} + \left\| u - \bar{u} \right\|_{L^\infty(\Omega)}$

Now, making use of (7) and (8), we obtain

$$Mu \le M\left(\bar{u} + \|u - \bar{u}\|_{L^\infty(\Omega)}\right) = M\bar{u} + \|u - \bar{u}\|_{L^\infty(\Omega)}$$

or

$$Mu - M\bar{u} \le \|u - \bar{u}\|_{L^\infty(\Omega)}$$

Similarly, interchanging the roles of the Mu and $M\bar{u}$ we obtain

$$M\bar{u} - Mu \le \|u - \bar{u}\|_{L^\infty(\Omega)}$$

hence

$$\|Mu - M\bar{u}\|_{L^\infty(\Omega)} \le \|u - \bar{u}\|_{L^\infty(\Omega)}.$$

Theorem 3 (cf. 7). Under the previous assumptions, and the maximum principle, there exists a constant C independent of h such that

$$\|u - \bar{u}\|_{L^\infty(\Omega)} \le Ch^2 |\log h|^2$$

The Continuous Schwarz Sequences

We consider a bounded, open domain Ω in R^2 and we assume (to simplify) that Ω is smooth and connected and we then decompose Ω in two sub-domains Ω_1, Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2 \tag{9}$$

and u satisfies the local regularity condition

$$u \in W^{2,P}(\Omega_i) \; on \; \Omega_i \tag{10}$$

and we denote by

$\Gamma = \partial\Omega,\ \Gamma_1 = \partial\Omega_1,\ \Gamma_2 = \partial\Omega_2,\ \gamma_1 = \partial\Omega_1 \cap \Omega_2,$
$\gamma_2 = \partial\Omega_2 \cap \Omega_1,\ \Omega_{1,2} = \Omega_1 \cap \Omega_2.$

Re-consider the model obstacle problem: Find $u \in K_\varphi$ such that

$$\begin{cases} a(u, v-u) \geq (f, v-u) \text{ in } \Omega \\ u - Mu \leq 0,\ Mu = k + \inf_{\xi \geq 0,\, x+\xi \in \bar{\Omega}} u(x+\xi), \\ u = \varphi \text{ in } \Gamma_0, \\ \dfrac{\partial u}{\partial \eta} = 0 \end{cases} \tag{11}$$

where f is a given function in Ω say in $L^\infty(\Omega)$ we will assume in this section that $f > 0$, $\varphi > 0$. Indeed enable us to make such an assumption by adding constants to u and φ and a positive function to f.

We define the following process; Let $u_0 \in K_\varphi$ be given, we respectively define the alternating Schwarz sequences u^{2n+1} *on* Ω_1 such that $u^{2n+1} \in K_\varphi$ solves

$$\begin{cases} a(u^{2n+1}, v-u^{2n+1}) \geq (f, v-u^{2n+1}) \\ u^{2n+1} - Mu^{2n-1} \leq 0, \\ u^{2n+1} = u^{2n} \text{ in } \partial\Omega_1 \end{cases} \tag{12}$$

and u^{2n} on Ω_2 such that $u^{2n} \in K_\varphi$ solves

$$\begin{cases} a(u^{2n}, v-u^{2n}) \geq (f, v-u^{2n}) \\ u^{2n} - Mu^{2n-2} \leq 0, \\ u^{2n} = u^{2n-1} \text{ in } \partial\Omega_2 \end{cases} \tag{13}$$

Proposition 4. Let $(Mu,\varphi);(M\bar{u},\bar{\varphi})$ be a pair of data and $u=\sigma(Mu,\varphi);\ \bar{u}=\sigma(M\bar{u},\bar{\varphi})$ the corresponding solution to (3), we have for $i\neq j\ and\, i,j=1,2\ \ \forall u,\bar{u}\in K_{\varphi}$

$$\left\|u-\bar{u}\right\|_{L^{\infty}(\Omega)}\leq\left\|Mu-M\bar{u}\right\|_{L^{\infty}(\Omega)}+\left\|\varphi-\bar{\varphi}\right\|_{L^{\infty}(\Omega_i\cap\Omega_j)} \tag{14}$$

Prof. Setting for $i\neq j\ and\, i,j=1,2$

$$\beta=\left\|Mu-M\bar{u}\right\|_{L^{\infty}(\Omega)}+\left\|\varphi-\bar{\varphi}\right\|_{L^{\infty}(\Omega_i\cap\Omega_j)} \tag{15}$$

we have

$$\begin{aligned}Mu&\leq M\bar{u}+Mu-M\bar{u}\leq M\bar{u}+\left|Mu-M\bar{u}\right|\\&\leq M\bar{u}+\left\|Mu-M\bar{u}\right\|_{L^{\infty}(\Omega)}\\&\leq M\bar{u}+\left\|Mu-M\bar{u}\right\|_{L^{\infty}(\Omega)}+\left\|\varphi-\bar{\varphi}\right\|_{L^{\infty}(\Omega_i\cap\Omega_j)}\end{aligned}$$

hence

$$Mu\leq M\bar{u}+\beta$$

On the other hand, we have

$$\begin{aligned}\phi&\leq\bar{\varphi}+\varphi+\bar{\varphi}\leq\bar{\varphi}+\left\|\varphi-\bar{\varphi}\right\|_{L^{\infty}(\Omega_i\cap\Omega_j)}\\&\leq\bar{\varphi}+\left\|\varphi-\bar{\varphi}\right\|_{L^{\infty}(\Omega_i\cap\Omega_j)}\\&\leq\bar{\varphi}+\left\|\varphi-\bar{\varphi}\right\|_{L^{\infty}(\Omega_i\cap\Omega_j)}+\left\|Mu-M\bar{u}\right\|_{L^{\infty}(\Omega)}\end{aligned}$$

so

$$\varphi\leq\bar{\varphi}+\beta \tag{16}$$

then since σ and M are increasing on $L^{\infty}(\Omega)$, it follows that

$$\sigma(Mu,\varphi) \leq \sigma\left(M\bar{u}+\beta,\bar{\varphi}+\beta\right) \leq \sigma\left(M\bar{u},\bar{\varphi}\right)+\beta$$

or

$$\sigma(Mu,\varphi)-\sigma\left(M\bar{u},\bar{\varphi}\right) \leq \beta$$

similarly, interchanging the roles of the couples (Mu,φ) and $\left(M\bar{u},\bar{\varphi}\right)$ we obtain

$$\sigma\left(M\bar{u},\bar{\varphi}\right)-\sigma(Mu,\varphi) \leq \beta$$

thus

$$\left\|u-\bar{u}\right\|_{L^{\infty}(\Omega)} \leq \left\|Mu-M\bar{u}\right\|_{L^{\infty}(\Omega)} + \left\|\varphi-\bar{\varphi}\right\|_{L^{\infty}\left(\Omega_i \cap \Omega_j\right)}$$

the proof for the discrete case is similar.

Discretization

Assumption [cf. 9]: The discrete maximum principle assumption. The matrix whose coefficients $a\left(\varphi_i,\varphi_j\right)$ are supposed to be M-matrix. For convenience in all the sequels, C will be a generic constant independent on h.

For $i, j = 1,2$, let τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in Ω_i; $h_i\left(h_1 = h_2 = h\right)$, being the mesh size. We assume that the two triangulations are mutually independent on $\Omega_{1,2}$ in the sense that a triangle belonging to one triangulation does not necessarily belong to the other.

Let V^{h_i} be the space of continuous piecewise linear functions on τ^{h_i} which vanish on $\Omega_i \cap \partial\Omega_j, i \neq j$ *and* $i, j = 1,2.$

For $w \in C\left(\partial\overline{\Omega_i}\right)$ we define

$$V_w^{h_i} = \begin{Bmatrix} v_h = V^{h_i}; v_h = \pi_{h_i}(w) \text{ on } \Omega_i \cap \partial\Omega_j, \\ \text{with} \dfrac{\partial v_h}{\partial \eta} = 0 \end{Bmatrix} \tag{17}$$

where π_{h_i} denotes the interpolation operator on $\partial\Omega$

We assume that the respective matrices resulting from the discretization of problems (13) and (14) are M-matrix and we define the discrete counterparts of the continuous Schwarz sequences defined in (13) and (14), respectively by $u_h^{2n+1} \in V_{u_h^{2n}}^h$ such that

$$\begin{cases} a\left(u_h^{2n+1}, v_h - u_h^{2n+1}\right) \geq \left(f, v_h - u_h^{2n+1}\right) \\ u_h^{2n+1} - r_h M u_h^{2n-1} \leq 0, \\ u_h^{2n+1} = u_h^{2n} \text{ in } \partial\Omega_1 \end{cases} \tag{18}$$

and $u_h^{2n} \in V_{u_h^{2n-1}}^h$

$$\begin{cases} a\left(u_h^{2n}, v - u^{2n}\right) \geq \left(f, v - u_h^{2n}\right) \\ u_h^{2n} - r_h M u_h^{2n-2} \leq 0, \\ u_h^{2n} = u_h^{2n-1} \text{ in } \partial\Omega_2 \end{cases} \tag{19}$$

Error analysis

This section is devoted to the proof of main result of the present paper, we need first to introduce an auxiliary sequence of discrete quasi-variational inequalities and next to prove the two fundamental theorems.

For $\xi_h^0 = u_h^0 \in K_{\varphi h}$ we define the sequences $\left(\xi_h^{2n+1}\right)_n$ such $\xi_h^{2n+1} \in V_{u_h^{2n}}^h$ that solves

$$\begin{cases} a\left(\xi_h^{2n+1}, v_h - \xi_h^{2n+1}\right) \geq \left(f, v_h - \xi_h^{2n+1}\right) \\ \xi_h^{2n+1} - r_h M u_h^{2n-1} \leq 0 \\ \xi_h^{2n+1} = u_h^{2n} \ in \ \partial\Omega_1 \end{cases} \tag{20}$$

and $\left(\xi_h^{2n}\right)_n$ such that $\xi_h^{2n} \in V_{u_h^{2n-1}}^h$ that solves

$$\begin{cases} a\left(\xi_h^{2n}, v - u^{2n}\right) \geq \left(f, v - \xi_h^{2n}\right) \\ \xi_h^{2n} - r_h M u_h^{2n-2} \leq 0, \\ \xi_h^{2n} = u_h^{2n-1} \ in \ \partial\Omega_2 \end{cases} \tag{21}$$

Convergence Proof via the Maximum Principle

We introducing the sets

$$T^{2n+1} = \begin{Bmatrix} u_h^{2n+1} \in V_{u_h^{2n}}^h ; -\Delta u_h^{2n+1} \leq f \ on \ \Omega_1, \\ u_h^{2n+1} = u_h^{2n} \ on \ \partial\Omega_1, \\ u_h^{2n+1} - r_h M u_h^{2n-1} \leq 0 \ on \Omega_1, \\ \dfrac{\partial u_h^{2n+1}}{\partial \eta} = 0 \end{Bmatrix}$$

and

$$T^{2n} = \begin{Bmatrix} u_h^{2n} \in V^h_{u_h^{2n-1}}; -\Delta u_h^{2n} \le f \; on \; \Omega_2, \\ u_h^{2n} = u_h^{2n-1} \; on \; \partial\Omega_2, \\ u_h^{2n} - r_h M u_h^{2n-2} \le 0 \; on \, \Omega_2, \\ \dfrac{\partial u_h^{2n}}{\partial \eta} = 0 \end{Bmatrix}$$

Lemma 5: Let A is M-matrix and u_h^{2n} $(resp.\ u_h^{2n+1})$ is the solution (19, 20). Then u_h^{2n} $(resp.\ u_h^{2n+1})$ is the minimal of T^{2n} $(resp.\ T^{2n+1})$.

Remark 6: The demonstration of lemma 5 is an adaptation of the one in (Jinping, Zeng. & Shuzi, Zhou. 1998). given for the problem of variational inequality. This lemma remained true for the problem introduces in this paper.

Theorem 7: Let u_h is the solution of (5). Then the iterative sequence u_h^{2n} $(resp.\ u_h^{2n+1})$ is monotone; that is $u_h^{2n+1} \in T^{2n+1}$ $(resp. u_h^{2n} \in T^{2n})$ and $u_h^{2n+2} \le u_h^{2n} .... \le u_h^0 = u_h$.

Prof. we take $u_h^0 = u_h^0 \; on \, \Omega_2$ such that $-\Delta u_h^0 = f$,

we know that if $u_h^0 \le r_h M u_h$ then $\left(-\Delta u_h^0 - f\right) \le 0 on \Omega_2 \, that is$ $\int_{\Omega_2} \nabla u_h^0 \nabla\left(v_h - u_h^0\right) - f\left(v_h - u_h^0\right) dx \ge 0.$

Therefore

$u_h^0 \in T^0$, From lemma 5 we know that u_h^2 is the minimal element of T^0, so $u_h^2 \le r_h M u_h^0$,

we yields that

$u_h^2 \le u_h^0$. By induction, for index n we obtain

$$u_h^{2n+2} \leq u_h^{2n} .... \leq u_h^0 = u_h$$

and we know that if

$u_h^3 \leq r_h M u_h^1$ then $\left(-\Delta u_h^3 - f\right) \leq 0 \, on \; \Omega_1$ that is

$$\int_{\Omega_2} \nabla u_h^3 \nabla\left(v_h - u_h^3\right) - f\left(v_h - u_h^3\right) dx \geq 0.$$

Therefore $u_h^3 \in T^3$ and from lemma 5 we know that u_h^3 is the minimal element of T_h^3 we yields that $u_h^3 \leq u_h^1$.

By induction, for index n we obtain

$$u_h^{2n+1} \leq u_h^{2n-1} .... \leq u_h^1.$$

Lemma 8: Let $A = \left(a_{ij}\right)_{ij=\overline{1,N}}$ be an M-matrix such that

$a_{ij} = a\left(\varphi_i, \varphi_j\right)$, then there exists constants k_1, k_2;

$k_1 = \sup\left\{w_h(x),\; x \in \gamma_1\right\} \in (0,1)$

and

$k_2 = \sup\left\{w_h(x),\; x \in \gamma_2\right\} \in (0,1)$ such that

$$\sup_{\gamma_1}\left|u_h - u_h^{2n+1}\right| \leq k_1 \sup_{\gamma_1}\left|u_h - u_h^{2n}\right| \tag{22}$$

and

$$\sup_{\gamma_2}\left|u_h - u_h^{2n+1}\right| \leq k_2 \sup_{\gamma_2}\left|u_h - u_h^{2n}\right| \tag{23}$$

Prof.

Let $M_1 = \sup_{\gamma_1}\left|u_h - u_h^{2n+1}\right|$ and $M_2 = \sup_{\gamma_2}\left|u_h - u_h^{2n+1}\right|$

we may suppose $M_1 \neq 0,$ we prove

$$M_1 < M \tag{24}$$

If (22) is not true then there exists $x_{i0} \in \gamma_1$ such that

$$\left|u_h^{2n+1}(x_{i0}) - u_h(x_{i0})\right| = M_1 \geq M$$

hence, we have (noting $a_{ii} > 0,\ a_{ij} \leq 0$ for $i \neq j$ because A is M-matrix)

$$0 \geq \sum_{i=1}^{N} a_{ii_0}\,(u_h^{2n+1} - u_h) \geq \sum_{i=1}^{N} a_{ii_0} \geq 0$$

We know by theorem 3 that $u_h^{2n+1} \geq u_h$ which implies that

$$\sum_{i \neq i_0} a_{ii_0} \left|u_h^{2n+1} - u_h\right| - M_1 = 0.$$

Therefore

$$\left|u_h^{2n+1} - u_h\right| = M_1 \text{ if } a_{ii_0} \neq 0 \tag{25}$$

Since $A = (a_{ij})_{ij=\overline{1,N}}$ is irreducible there exist $x_{i1}, x_{i2} .... x_{is} \in \gamma_2$ such that

$$a_{i_0 i_1}, a_{i_1 i_2} .... a_{i_s l_k} \neq 0$$

We know by (25) that $\left|u_h^{2n+1}(x_{i1}) - u_h(x_{i2})\right| = M_1$, similarly, we get

$$\left|u_h^{2n+1}(x_{i_2}) - u_h(x_{i_2})\right| = ... = \left|u_h^{2n+1}(x_{i_s}) - u_h(x_{is})\right| = \left|u_h^{2n+1}(x_{l_k}) - u_h(x_{l_k})\right| = M.$$

Hence we have

$$\sum_{i=1}^{N} a_{li} \left|u_h^{2n+1} - u_h\right| \geq \sum_{i=1}^{N} a_{li} M_1 > 0$$

this contradiction proves (25), and the proof for the (23) case is similar.

The main convergence result is given by the:

Theorem 9: The sequences $\left(u_h^{2n+1}\right)_n, \left(u_h^{2n}\right)_n, n \geq 0$

produced by the Schwarz alternating method converge geometrically to the solution u_h of the obstacle problem (5). More precisely, there exist $k_1, k_2 \in (0,1)$ which depend only respectively of (Ω_1, γ_2) and (Ω_2, γ_1) such that all $n \geq 0$

$$\sup_{\overline{\Omega}_1}\left|u_h - u_h^{2n+1}\right| \leq k_1^n k_2^n \sup_{\gamma_1}\left|u_h - u_h^0\right| \tag{26}$$

and

$$\sup_{\overline{\Omega}_2}\left|u_h - u_h^{2n}\right| \leq k_1^n k_2^{n-1} \sup_{\gamma_2}\left|u_h - u_h^0\right| \tag{27}$$

Prof. Under Lemma 8 (from the maximum principle) we have

$$\left|u_h - u_h^{2n+1}\right| \leq w_h(x) \sup_{\gamma_1}\left|u_h - u_h^{2n}\right|$$

hence

$$\sup_{\gamma_1}\left|u_h - u_h^{2n+1}\right| \leq k_1 \sup_{\gamma_1}\left|u_h - u_h^{2n}\right| \tag{29}$$

and thus

$$\sup_{\gamma_1}\left|u_h - u_h^{2n+1}\right| \leq k_1^n k_2^n \sup_{\gamma_1}\left|u_h - u_h^0\right|$$

also we have

$$\left|u_h - u_h^{2n}\right| \leq w_h(x) \sup_{\gamma_{21}}\left|u_h - u_h^{2n-1}\right|$$

$$\sup_{\gamma_2}\left|u_h - u_h^{2n}\right| \leq k_2 \sup_{\gamma_2}\left|u_h - u_h^{2n-1}\right| \tag{30}$$

thus

$$\sup_{\gamma_2}\left|u_h - u_h^{2n}\right| \le k_1^n k_2^{n-1} \sup_{\gamma_2}\left|u_h - u_h^0\right|$$

Then (29), (30) follows from the maximum principle which yields for $n \ge 0$

$$\sup_{\overline{\Omega}_1}\left|u_h - u_h^{2n+1}\right| = \sup_{\gamma_1}\left|u_h - u_h^{2n+1}\right| = \sup_{\gamma_1}\left|u_h - u_h^{2n}\right|$$

and

$$\sup_{\overline{\Omega}_2}\left|u_h - u_h^{2n+1}\right| = \sup_{\gamma_2}\left|u_h - u_h^{2n+1}\right| = \sup_{\gamma_2}\left|u_h - u_h^{2n}\right|$$

hence

$$\sup_{\overline{\Omega}_1}\left|u_h - u_h^{2n+1}\right| \le k_1^n k_2^n \sup_{\gamma_1}\left|u_h - u_h^0\right|$$

and

$$\sup_{\overline{\Omega}_2}\left|u_h - u_h^{2n}\right| \le k_1^n k_2^{n-1} \sup_{\gamma_2}\left|u_h - u_h^0\right|$$

Error Estimate for the QVI.

Theorem 10: Let u be a solution of problem (3). Then there exists a constant C independent of both h and n such that:

$$\left\|u - u_h^{2n+1}\right\|_{L^\infty(\overline{\Omega}_1)} \le Ch^2\left|\log h\right|^3 \tag{31}$$

and

$$\left\|u - u_h^{2n}\right\|_{L^\infty(\overline{\Omega}_2)} \le Ch^2\left|\log h\right|^3 \tag{32}$$

Prof. We setting here $k = k_1 = k_2$ and using theorem 3 and theorem 9 we have

$$
\begin{aligned}
\left\|u-u_h^{2n+1}\right\|_{L^\infty(\overline{\Omega}_1)} &\le \left\|u-u_h\right\|_{L^\infty(\overline{\Omega}_1)}+\left\|u_h-u_h^{2n+1}\right\|_{L^\infty(\gamma_1)} \\
&\le \left\|u-u_h\right\|_{L^\infty(\overline{\Omega}_1)}+k^{2n}\left\|u_h-u_h^0\right\|_{L^\infty(\gamma_1)} \\
&\le Ch^2\left|\log h\right|^2+k^{2n}\left\|u_h-u_h^0\right\|_{L^\infty(\gamma_1)} \\
&\le Ch^2\left|\log h\right|^2+ \\
&\quad +k^{2n}\left(\left\|u-u_h^0\right\|_{L^\infty(\gamma_1)}+\left\|u-u_h^0\right\|_{L^\infty(\gamma_1)}\right) \\
&\le Ch^2\left|\log h\right|^2+Ch^2k^{2n}\left|\log h\right|^2.
\end{aligned}
$$

Now setting $k^{2n}\le\left|\log h\right|$ we get

$$
\left\|u-u_h^{2n+1}\right\|_{L^\infty(\overline{\Omega}_1)}\le Ch^2\left|\log h\right|^3
$$

The proof for the (32) case is similar.

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