HK Spaces with AD Property

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Abstract

If H is an HK space which has AD property, then we define the matrix $A(H)=(a_{mn})$ as $a_{mn}=<e_m,e_n>_H$. We prove that A(H) is uniquely determined by H, and hence conclude that there is a one - to - one map between the collection of all HK spaces which have AD, and that of all matrices which are positive definite and Hermitian. Finally, we calculate $A(H^2|_W)$ where $W=\{d_n\}$ is an interpolating sequence.

ملخص

تقدم هذه الورقة محاولة لاستعمال نظرية المصفوفات في دراسة الخواص التحليلية لفضاءات (HK). يمثل هذه الرابطة اقتران تقدمه لأول مرة بين فضاءات (HK) والتي تحقق خاصية (AD) وبين المصفوفات الهرمشية محققة الايجابية . ثم نبني المصفوفة التي يعينها هذا الاقتران للفضاء $H^2|_W$ حيث $W=\{d_n\}$ متسلسلة توليديه.

1. Introduction and Preliminaries : The Hardy space H^2 is the space of analytic function f on $\{z:|z|<1\}$ for which the integrals

$$\left\|f_r\right\|=\left\{\frac{1}{2\Pi}\int_{-\Pi}^{\Pi}\left|f\left(re^{i\sigma}\right)\right|^2 d\sigma\right\}^{\frac{1}{2}}$$

are bounded as $r \to 1-$. It is well-known that H^2 is a Banach space under the norm

$$\|f\|_2 = \lim_{r\to 1-} \|f_r\|_2$$
$$= \{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^2 d\theta\}^2$$

where $f(e^{i\theta})$ is defined almost everywhere by

$$f(e^{i\theta}) = \lim_{r\to 1-} f(re^{i\theta})$$

H^2 is usually identified as a a closed subspace of the lebesgue class $L_2([-\pi,\pi])$ by means of

$$f(\theta) = \lim_{r\to 1-} f(re^{i\theta})$$

Specifically, for $f \varepsilon L_2$, we have

$$f \ \varepsilon \ H^2 \text{ iff } \int_{-\pi}^{\pi} f(\theta)e^{in\theta}\, d\theta = 0 \text{ for } n = 1,2,\ldots$$

Finally , H^2 is a Banach space of functions on the open unit disk . Indeed, for $f \varepsilon H^2$, evaluations π^z defined as

$$\pi^z(f) = f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\cdot\frac{1}{1-ze^{-it}}dt$$

are all contrinuous on H^2 .

Let $W=\{z_n\}$ be a sequence which satisfies $\sum_n (1-|z_n|) < \infty$, for example , let W be an interpolating sequence, $S=\{f \varepsilon H^2 : f(z_n)=0 \ \forall n\}$ and for f, g , εH^2 let $<f,g>_{H^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt$ (this is the inner product for H^2 . Then $H^2|_W$ (H^2 resterected toW) is a Hilbert space of sequences

congruent to the orthogonal complement $S^{\perp} = \{f \varepsilon H^2 : <f,g>_{H^2} = 0$ for all $g \varepsilon S\}$ of S in H^2 [The congruence can be chosen to be the map : $f \to \{f(z_n)\}$] . Also $S= BH^2$ where

$$B(z) = \prod_k \frac{\bar{z}_k}{|z_k|}\frac{z_k - z}{1 - \bar{z}_k z} \quad \text{[see [1] , 3.1].}$$

It follows that , for each $f \varepsilon H^2$ there is a unique $F \varepsilon H^2$ of minmal norm such that $\|f\|_{H^2|W} = \|F\|_{H^2}$; F is orthogonal to each function in H^2 which vanishes on W.

It is also known [[2] , theorem 4, cor. 1] that $H^2|_W$ is an AK space (i.e. each $x \varepsilon H^2|_W$ has the representation $x = \sum_k x_k e^k$ where $e^k_j = \begin{cases} 1 \text{ if } j=k \\ 0 \text{ if } j\neq k \end{cases}$ [[1],5.1].

The auther figured that there corresponed a unique matrix, possibly infinite of special type to any preassigned HK space which has AD.
This, somehow, makes up a converse of the completion process obtained in [3].

One of the problems in mathematics is the existence of pathological examples; and our case is , of course, not an exception.

This paper includes lengthy calculations, the main part of which explicitly gives the matrix which corresponds to the space $H^2|_W$.

In what follows , and for each n, let

$$B_n(z) = \prod_{k \neq n} \frac{\overline{z_k}}{|z_k|} \frac{z_k - z}{1 - \overline{z_k} z} ,$$

and let P_n be the n th coordinate projection. l^2 will stand for the space of all square summable sequences with norm $\|x\| = (\sum_k |x_k|^2)^{\frac{1}{2}}$

2. HK Spaces and infinite matrices

Let H be an HK space. It is a known fact that coordinate projections can be identified with reproducing kernels. Specificaly, for each n, there is a unique element π^n , called the n th reproducing kernel, of H with $P_n(x) = \langle x , \pi^n \rangle_H$ $(x \varepsilon H)$.

It is the aim of this section to associate, to each HK space H which has AD, a unique matrix A(H) which is Hermetian and positive definite. We begin with the following :

2.1 Lemma :

Let X and Y be two BK spaces (i.e Banach space of sequences which is locally convex, Frechet space and on which P_n is continuous for each n)) Suppose that S is a dense subspace of X and of Y with the property that $\|x\|_X = \|x\|_Y$ for all $x \varepsilon S$. Then $X = Y$.

Proof :

Let $x \varepsilon X$ be arbitrary and $\{x^n\}$ be Cauchy in X . Hence , for each $\varepsilon > 0$, there exists an integer N>0 with $\|x^m - x^n\|_X < \varepsilon$ when $m \geq N$, $n \geq N$.

Now , $\|x^m - x^n\|_Y = \|x^m - x^n\|_X \prec \varepsilon$ for $m \geq N$, $n \geq N$

So , $\{x^n\}$ is Cauchy in Y, hence convergers in Y to some $y \varepsilon Y$. It remains to prove that x=y. To this end , $x^n \to x \varepsilon X$. So, for each k, $x_k^n \to x_k$. Also , $x^n \to x \varepsilon Y$. So, for each k , $x_k^n \to y_k$. .

Therefore $x_k = y_k$ for each k, hence x=y. //

Suppose that H is an HK space which has the AD property. Let A(H):(a_{mn}) be matrix defined by :

$$a_{mn} = \langle e^m , e^n \rangle_H$$

we prove

2.2 Theorem :

With H and A(H) as above , A(H) is unquely determined by H.

Proof : Suppose that X and Y are two HK spaces wich have AD, and suppose that A(X) = A(Y).

$$\langle e^m , e^n \rangle_X = \langle e^m , e^n \rangle_Y \text{ for all } m , n$$

Let $x = \sum_{i=1}^{r} x_i e^i$, $y = \sum_{j=1}^{s} y_j e^j$ be arbitrary elements in ϕ .Then ;

$$< x,y >_X = < \sum_{i=1}^{r} x_i e^i , \sum y_j e^j >_H$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} x_i \bar{y}_j < e^i , e^j >_X$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} x_i \bar{y}_j < e^i , e^j >_Y$$

$$= < \sum_{i=1}^{r} x_i e^i , \sum_{j=1}^{s} x_j e^j >_Y$$

$$= < x,y >_Y$$

It follows that, if $x \varepsilon \phi$ then $\|x\|_X = \|x\|_y$. But ϕ is dense in X and in Y, therefore, by lemma (1.2), X=Y

2.3 Cor. :

(a) Theorem 2.2 says that there is one-to-one map between the collection of all HK spaces which have AD, and that of all matrices which are positive definite and Hermetian.

Proof :

Let $0 \neq x \varepsilon \phi$ be arbitrary.

$x A x^* = \|x\|_A > 0$. Therefore A is positive definite.

Now, $(A(H))^* = (\overline{< e^m , e^n > H})^T$

$$= (\langle e^n, e^m \rangle_H)^T$$
$$= (\langle e^m , e^n \rangle_H)$$
$$= A(H)$$

Therefore, A(H) is Hermetian.

(b) The map $H \rightarrow A(H)$ is not onto.

[Let B be a positive definite Hermetian matrix with the property that (ϕ , $\|\cdot\|_B$) has no HK completion [see [3] , 2.2].]. However , if M=diag (M1 , M2 ,.....), where, for each n , M_n is of dimesion s_n s_n x s_n , then there is aunque HK space H with AD such the A(H) = M[[3], 2.10]. This , of course leads to :

Remark : (question) : How big can the range of this map be?

2.4 Example :

l^2 is an HK space under the inner product:

$$< x, y >_1 = \sum_i x_i \bar{y}_i$$

[$|x_n| \le (\sum |x_k|^2)^{\frac{1}{2}} = \|x\|_{l^2}$ for each n. Hence by ([3],2.1), P_n is continuous. It is not hard to see that $A(l^2) = I$, the identity matrix.

Other than the foregoing example , it seems unusual to calculate A(H) for arbitrary HK spaces H. The auther found it interesting to calculate $A(H^2 | W)$ where W={is any interpolating sequence. Togather with this assumption, let S, B and $\overset{\perp}{S}$ be as defined in section (1). We first prove .

2.5 Lemma : $B_n - |z_n| B \; \varepsilon \; \overset{\perp}{S}$

Proof : Note that $B_n = (B_n - |z_n| B) + |z_n| B$, and

$$<|z_n| B, B_n \; |z_n| B>_{H^2} = |z_n| <B, B_n>_{H^2} - |z_n|^2 <B, B>_{H^2}$$

$$= |z_n| \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{z_n}{z_n} \frac{z_n - e^{it}}{1 - \bar{z}_n e^{it}} dt - |z_n|^2$$

$$= |z_n|^2 - |z_n|^2$$

$$= 0$$

Therefore , since $S = BH^2$, $B_n - |z_n| B \; \varepsilon \; S$ [for $z = e^{it}$, $(B_n - |z_n B) \perp B e^{int}$ for $n = 0$. This is what was just shown.

$$\int B e^{int} (B_n - |z_n| \bar{B}) dt = \int e^{int} g(t) dt \quad , \text{ say}$$

$= 0$ since g is analytic].

With this at hand , we now calculate $A(H^2|W)$.

2.6 Example : Let , for each n , $f_n = \dfrac{B_n - |z_n| B}{B_n(z_n)}$.

$f_n \; \varepsilon \; S^{\perp}$ and interplates e^n since $\dfrac{B_n}{B_n(z_n)}$ does .

With $z = e^{it}$, we now therefore have :

$$< e^m, e^n >_{H^2|W} = < f_m, f_n >_{H^2}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{B_m - |z_m| B}{B_n(z_n)} \left(\overline{\frac{B_n - |z_n| B}{B_n(z_n)}} \right) dt$$

$$= \frac{1}{2\pi B_m(z_m) \overline{B_k(z_k)}} \int_{-\pi}^{\pi} (B_m |z_m| B) (\overline{B_n - |z_n| B}) dt$$

$$= \frac{1}{B_m(z_m)} \cdot \frac{1}{\overline{B_n(z_n)}} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(B_m \overline{B_n} - |z_n| B_m B - |z_n| B B_m + |z_m||z_n| \right)$$

$$= \frac{1}{B_m(z_m)} \cdot \frac{1}{B_n(z_n)} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\frac{z_m \overline{z_n}}{|z_m z_n|} \Big(\overline{z_m} \cdot \frac{z_n^{-z}}{1-\bar{z}_n^{z}} \cdot \frac{1}{1-z_m^{z}} - z_n \cdot \frac{z_m^{-\bar{z}}}{1-\bar{z}_m^{z}} \cdot$$

$$\frac{1}{1-z_n^{\bar{z}}} + \frac{1}{1-z_m^{\bar{z}}} \cdot \frac{1}{1-\bar{z}_n^{z}} \Big) - |z_n| \frac{z_m}{|z_m|} \cdot \frac{\bar{z}_m^{-\bar{z}}}{1-z_m^{\bar{z}}} - |z_m| \frac{z_n}{|z_n|} \frac{z_n^{-z}}{1-\bar{z}_n^{z}} + |z_m z_n| \Big) dt$$

$$= \frac{1}{B_m(z_m)} \cdot \frac{1}{\overline{B_n(z_n)}} \Big(\frac{z_m \overline{z_n}}{|z_m z_n|} \cdot \Big(\frac{z_m z_n - |z_m|^2 - |z_n|^2 + 1}{1-z_m \bar{z}_n} \Big)$$

$$- |z_m z_n| - |z_m z_n| + |z_m z_n| \Big)$$

$$= \frac{z_m \overline{z_n}}{B_m(z_m)\overline{B_n(z_n)}} \Big(\frac{(1-|z_m|^2)(1-|z_n|^2)}{|z_m z_n|(1-z_m \bar{z}_n)} \Big)$$

$$= \frac{\mu_m \bar{\mu}_n}{1-z_m \bar{z}_n}, \text{ where } \mu_m = \frac{z_m}{|z_m|} \cdot \frac{1-|z_m|^2}{B_m(z_m)}$$

References

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