

FREE ODD PERIODIC ACTIONS ON THE SOLID KLEIN BOTTLE

Key words : Free action , Periodic action
Solid Klein Bottle .

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ملخص

في هذا البحث تم دراسة الاقترانات الهيمومورفيه فردية الدورة ذات مجموعة الثبات الفارغة على زجاجة كلاين المصمتة . وقد تم إثبات أنه إذا افترضنا التكافؤ الضعيف للاقترانات فإنه يوجد اقتران واحد فقط من هذا النوع على زجاجة كلاين المصمتة .

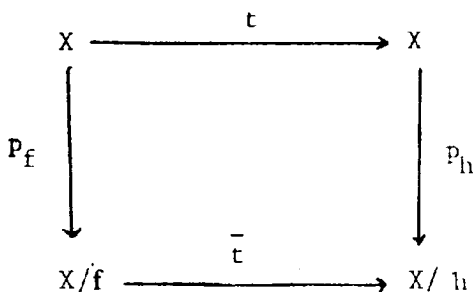
Abstract

The cyclic actions of odd period and empty fixed point set are studied on the solid Klein Bottle K . It is shown that up to weak equivalence there is only one such action .

1 – Notation and Preliminaries

D^2 , R , S^1 denote the unit disk $[x \in R^2 : | x | \leq 1]$, the field of real numbers and the unit circle . A 3–manifold M is irreducible if every 2–sphere in M bounds a 3–cell in M . The cyclic group generated by the periodic map h is denoted by $\langle h \rangle$. If h is periodic on a space X , then the orbit space of h is the quotient space obtained by identifying each x with $h^i(x)$ for all i . The orbit space of h will be denoted by X/h . The identification map $p_h : X \rightarrow X/h$ is called the orbit

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map . Two actions of $\langle h \rangle$ and $\langle f \rangle$ on X are said to be weakly equivalent if there is a homeomorphism t of X such that $\langle tht^{-1} \rangle = \langle f \rangle$ and that $tht^{-1} = f^i$ for some i . Equivalently , h and f are weakly equivalent if there are homeomorphisms t and \bar{t} that make the diagram commutative , i . e . $p_h t = \bar{t} p_f$. The set $[x \in X : h(x) = x]$ of fixed points of h will be denoted by $F(h)$.

The solid Klein Bottle K is the space obtained from $D^2 \times \mathbb{R}$ by identifying (z, t) with $(\bar{z}, t + 1)$. An element of this space with representative (z, t) will be denoted by $[z, t]$.

2- Results .

The following is the main result :

Theorem A. Up to weak equivalence there is exactly one free Z_{2r+1} action on the solid Klein Bottle K .

First we prove the following two Lemmas .

Lemma 1. If $h : K \rightarrow K$ is a PL homeomorphism of period $2r + 1$ on K , then $F(h)$ is either empty or a simple closed curve (homeomorphic to S^1) .

Proof. Let $n = 2r + 1$. Then n can be written as

$$n = p_1^{t_1} p_2^{t_2} \dots p_m^{t_m} .$$

where p_1, \dots, p_m are distinct odd primes and t_1, t_2, \dots, t_m are positive integers . If $m = 1$, then h is of period $p_1^{t_1}$ on K which is a homology 1 - sphere . Hence , we find that , by [1] , $F(h)$ is either \emptyset or a homology 1 - sphere .

Since $F(h)$ can not be two dimensional (for $p_1^{t_1} = 2$) it is either \emptyset or \approx (homeomorphic to) S^1 . If $m = 2$, then $n = p_1^{t_1} p_2^{t_2}$ and $h^{p_1^{t_1}}$ is of period $p_2^{t_2}$.

As above $F(h^{p_1^{t_1}})$ is either \emptyset or $\approx S^1$. If $F(h^{p_1^{t_1}}) = \emptyset$, then $F(h) = \emptyset$. Because $F(h^{p_1^{t_1}})$ is invariant under h , then if $F(h^{p_1^{t_1}}) \approx S^1$ and $F(h) = \emptyset$, then h is of period $p_1^{t_1}$ on $F(h^{p_1^{t_1}}) \approx S^1$. So, by [1], $F(h) \approx S^1$.

Now suppose that the result is true for $m = i$. Let $C = p_1^{t_1} \dots p_i^{t_i}$.

Then the period of h is $cp_{i+1}^{t_{i+1}}$. Then $h^{p_i+1^{t_i+1}}$ is of period c on K , hence, by the induction hypothesis, $F(h^{p_i+1^{t_i+1}})$ is either \emptyset or $\approx S^1$.

If $F(h^{p_i+1^{t_i+1}}) = \emptyset$, then $F(h) = \emptyset$.

If $F(h^{p_i+1^{t_i+1}}) \approx S^1$, then by [1], $F(h) \approx S^1$ or \emptyset .

Remark. The proof above shows that if $F(h) \approx S^1$, then $F(h^i) = F(h) \approx S^1$ for all $1 < i < 2r + 1$.

Lemma 2. Let $h : K \rightarrow K$ be a homeomorphism of period $2r + 1$.

If h acts freely on K , then $K/h \approx K$.

Proof: Let $B = K/h$ and let $p : K \rightarrow B$ be the orbit map.

Since h acts freely on K , then K is a regular $2r + 1$ covering of B by [4]

Hence $p_{\#}(\pi_1(K))$ (which is infinite cyclic) is a normal subgroup of index $2r + 1$ in $\pi_1(B)$.

So we have a short exact sequence

$$0 \rightarrow Z \xrightarrow{f} \pi_1(B) \xrightarrow{g} Z_{2r+1} \rightarrow 0$$

Since B is covered by a contractible space and no nontrivial finite group can act freely on a finite dimensional contractible space [3], $\pi_1(B)$ has no torsion subgroup.

Let a be the image of a generator of Z under f and let b be such that $g(b)$ is a generator of Z_{2r+1} . Since $p_{\#}(\pi_1(K))$ is normal in $\pi_1(B)$, $bab^{-1} \in \langle a \rangle$. So $bab^{-1} = a$ or a^{-1} . If $bab^{-1} = a^{-1}$, then $\pi_1(B) / [\pi_1(B), \pi_1(B)]$ which is isomorphic to $H_1(B)$ is finite (for the coset $b^{-1} = b + [\pi_1(B), \pi_1(B)]$ is of order

$2r + 1$). Here $[\pi_1(B), \pi_1(B)]$ is the commutator subgroup .

Hence the Euler characteristic , $\chi(B) = \sum_0^3 (-1)^i r_i = 1 + r_2$ because $r_1 = 0 = r_3$ and $r_3 = 0$ because B is nonorientable .

But this implies that $\chi(B) \geq 1$, contradicting the fact that $\chi(B) = 0$. Hence $bab^{-1} \neq a^{-1}$ and we must have $bab^{-1} = a$ and so $\pi_1(B)$ is abelian . From the fundamental theorem of abelian groups we have

$$\pi_1(B) = Z + \text{Tor}(\pi_1(B)) = Z.$$

Note that B is compact , nonorientable , irreducible with a two dimensional Klein Bottle as its boundary . Moreover , B contains no 2 - sided projective plane p^2 , because if B contains such p^2 , then $p^{-1}p^2$ will be a 2 - sphere S , and since K is irreducible , S bounds a 3 - cell C . Then $p(C)$ is a 3 - manifold bounded by p^2 .

Now , by [2] theorem 11 - 7 , B is homeomorphic to K .

With these two lemmas at hand we now turn to prove our theorem .

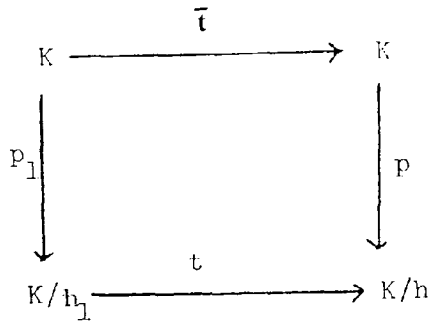
Proof of theorem A.

Let $h_1 : K \rightarrow K$ be defined by

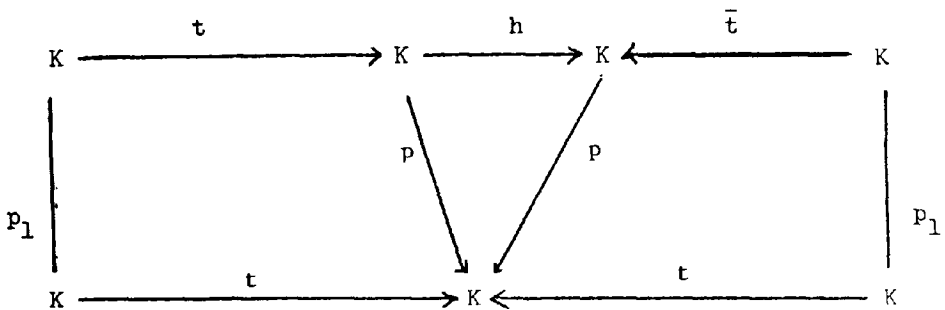
$$h_1([z, t]) = [z, t + \frac{2r}{2r+1}]$$

The map h_1 is a homeomorphism of period $2r + 1$ and $F(h^i) = \emptyset$ for all $1 \leq i \leq 2r$. Hence , by lemma 2 , $K/h_1 \approx K$. Now Let $h : K \rightarrow K$ be any homeomorphism of period $2r + 1$ such that $F(h^i) = \emptyset$ for $1 \leq i \leq 2r$. Lemma 2 implies that $K/h \approx K$. Let $p_1 : K \rightarrow K/h_1$ and $p : K \rightarrow K/h$ be the orbit maps . p_1 and p are $(2r + 1)$ - covering projections of K/h_1 and K/h , respectively . Let $t : K/h_1 \rightarrow K/h$ be a homeomorphism .

Since tp_1 and p are $(2r + 1)$ - covering projections of K and since $\pi_1(K/h)$ (is infinite cyclic) has a unique normal subgroup of index $2r + 1$, there is a homeomorphism $\bar{t} : K \rightarrow K$ making the diagram



Commutative . Now by the commutativity of the diagram .



We obtain $p_1 = p_1 t^{-1} h \bar{t}$. That is $t^{-1} h \bar{t}$ is a nontrivial covering transformation on K with respect to p_1 . Hence, $t^{-1} h \bar{t} = h_1^i$ for some $1 \leq i \leq 2r$, which means that h is weakly equivalent to h_1 .

This completes the proof .

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