FREE ODD PERIODIC ACTIONS ON THE SOLID KLEIN BOTTLE

Key words : Free action , Periodic action
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ملخص

في هذا البحث تم دراسة الاقترانات الهميومورفيه فردية الدورة ذات مجموعة الثبات الفارغة على زجاجة كلاين المصمته . وقد تمّ إثبات أنه إذا افترضنا التكافؤ الضعيف للاقترانات فإنه يوجد اقتران واحد فقط من هذا النوع على زجاجة كلاين المصمته .

Abstract

The cyclic actions of odd period and empty fixed point set are studied on the solid Klein Bottle K. It is shown that up to weak equivalence there is only one such action .

1 – Notation and Preliminaries

D^2 , R ,S^1denote the unit disk [$x \in R^2$: | x | $\leqslant$ 1] , the field of real numbers and the unit circle . A 3–manifold M is irreducible if every 2–sphere in M bounds a 3–cell in M . The cyclic group generated by the periodic map h is denoted by $<h>$. If h is periodic on a space X, then the orbit space of h is the quotient space obtained by identifying each x with h^i (x) for all i . The orbit space of h will be denoted by X/h . The identification map $p_h: X \rightarrow X/h$ is called the orbit

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$$\begin{array}{ccc} X & \xrightarrow{\;\;t\;\;} & X \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle p_h} \\ X/f & \xrightarrow{\;\;\bar{t}\;\;} & X/h \end{array}$$

map . Two actions of $<h>$ and $<f>$ on X are said to be weakly equivalent if there is a homeomorphism t of X such that $<th\, t^{-1}> = <f>$ and that $tht^{-1} = f^{i}$ for some i . Equivalently , h and f are weakly equivalent if there are homeomorphisms t and $\bar{t}$ that make the diagram commutative , i . e . $p_h t = \bar{t}\, p_f$. The set [$x \in X : h(x) = x$] of fixed points of h will be denoted by F (h) .

The solid Klein Bottle K is the space obtained form $D^2 x R$ by identifying (z , t) with ($\bar{z}$, t + 1) . An element of this space with represantative (z , t) will be denoted by [z , t] .

2 – Results .

The follwing is the main result :

Theorem A. Up to weak equivalence there is exactly one free Z_{2r+1} action on the solid Klein Bottle K.

First we prove the following two Lemmas .

Lemma 1. If h : K → K is a PL homeomorphism of period 2r + 1 on K , then F (h) is either empty or a simple closed curve (homeomorphic to S^1) .

Proof. Let n = 2r + 1 . Then n can be written as

$$n = p_1^{t_1} p_2^{t_2} \ldots\ldots p_m^{t_m} .$$

where $p_1 , \ldots , p_m$ are distinct odd primes and t_1 , t_2 ---, t_m are positive integers . If m = 1 , then h is of period $p_1^{t_1}$ on k which is a homology 1 – sphere . Hence , we find that , by [1] , F (h) is either $\emptyset$ or a homology 1–sphere .

Since F (h) can not be two dimensional (for $p_1^{t_1} = 2$) it is either $\emptyset$ or $\approx$ (homeomorphic to) S^1 . If m = 2 , then $n = p_1^{t_1} p_2^{t_2}$ and $h^{p_1^{t_1}}$ is of period $p_2^{t_2}$.

As above $F(h^{p_1^{t_1}})$ is either $\emptyset$ or $\approx S^1$. If $F(h^{p_1^{t_1}}) = \emptyset$, then $F(h) = \emptyset$. Because $F(h^{p_1^{t_1}})$ is invariant under h , then if $F(h^{p_1^{t_1}}) \approx S^1$ and $F(h) = \emptyset$, then h is of period $p_1^{t_1}$ on $F(h^{p_1^{t_1}}) \approx S^1$. So , by [1] , $F(h) \approx S^1$.
Now suppose that the result is true for m = i . Let $C = p_1^{t_1} \ldots p_i^{t_i}$.
Then the period of h is $cp_{i+1}^{t_{i+1}}$. Then $h^{p_{i+1}^{t_{i+1}}}$ is of period c on k , hence , by the induction hypothesis , $F(h^{p_{i+1}^{t_{i+1}}})$ is eitgher $\emptyset$ or $\approx S^1$.
If $F(h^{p_{i+1}^{t_{i+1}}}) = \emptyset$, then $F(h) = \emptyset$.
If $F(h^{p_{i+1}^{t_{i+1}}}) \approx S^1$, then by [1] , $F(h) \approx S^1$ or $\emptyset$.

Remark . The proof above shows that if $F(h) \approx S^1$, then $F(h^i) = F(h) \approx S^1$ for all $1 < i < 2r + 1$.

Lemma 2 . Let $h : K \to K$ be a homeomorphism of period 2r + 1 .
If h acts freely on K , then $K / h \approx K$.

Proof : Let B = K / h and let $p : K \to B$ be the orbit map .
Since h acts freely on K , then K is a regular 2r + 1 covering of B by [4]

Hence $p_{\#}(\pi_1(K))$ (which is infinite cyclic) is a normal subgroub of index 2r + 1 in $\pi_1(B)$.
So we have a short exact sequence

$$O \to Z \xrightarrow{f} \pi_1(B) \xrightarrow{g} Z_{2r+1} \to O$$

Since B is covered by a contractible space and no nontrivial finite group can act freely on a finite dimensional contractible space [3] , $\pi_1(B)$ has no torsion subgroup .

Let a be the image of a generator of Z under f and let b be such that g(b) is a generator of Z_{2r+1} Since $p_{\#}(\pi_1(K))$ is normal in $\pi_1(B)$, $bab^{-1} \in \langle a \rangle$. So $bab^{-1} = a$ or a^{-1} . If $bab^{-1} = a^{-1}$, then $\pi_1(B) / [\pi_1(B), \pi_1(B)]$ which is isomorphic to $H_1(B)$ is finite (for the coset $b^{-} = b + [\pi_1(B), \pi_1(B)]$ is of order

2r + 1). Here . [π_1 (B) , π_1 (B)] is the commutator subgroup .

Hence the Euler characteristic , $\chi(B) = \sum_{0}^{3}(-1)^i r_i = 1 + r_2$ because $r_1 = o = r_3$ and $r_3 = o$ because B is nonorientable .
But this implies that $\chi(B) \geq 1$, contradicting the fact that $\chi(B) = o$. Hence $bab^{-1} \neq a^{-1}$ and we must have $bab^{-1} = a$ and so $\pi_1(B)$ is abelian . From the fundamental theorem of abelian groups we have

$$\pi_1(B) = Z + \mathrm{Tor}(\pi_1(B) = Z .$$

Note that B is compact , nonorientable , irreducible with a two dimensional Klein Bottle as its boundary . Moreover , B contains no 2 – sided projective plane p^2 , because if B contains such p^2 , then $p^{-1}P^2$ will be a 2 – sphere S , and since K is irreducible , S bounds a 3 – cell C. Then p (C) is a 3 – manifold bounded by p^2 .
Now , by [2] theorem 11 – 7 , B is homeomorphic to K .

With these two lemmas at hand we now turn to prove our theorem .

Proof of theorem A.

Let $h_1 : K \rightarrow K$ be defined by

$$h_1([z,t]) = [z, t + \frac{2r}{2r+1}]$$

The map h_1 is a homeomorphism of period 2r + 1 and $F(h^i) = \emptyset$ for all $1 \leq i \leq 2r$. Hence , by lemma 2 , $K/h_1 \approx K$. Now Let $h : K \rightarrow K$ be any homeomorphism of period 2r + 1 such that $F(h^i) = \emptyset$ for $1 \leq i \leq 2r$. Lemma 2 implies that $K/h \approx K$. Let $p_1 : K \rightarrow K/h_1$ and $p : K \rightarrow K/h$ be the orbit maps. . p_1 and p are (2r + 1) – covering projections of K/h_1 and K / h , respectively . Let $t : K / h_1 \rightarrow K / h$ be a homeomorphism .
Since tp_1 and p are (2r + 1) – covering projections of K and since π_1 (K / h) (is infinite cyclic) has a unique normal subgropup of index 2r + 1 , there is a homeomorphism $\bar{t} : K \rightarrow K$ making the diagram

$$\begin{tikzcd} K \arrow[r, "\bar{t}"] \arrow[d, "p_1"'] & K \arrow[d, "p"] \\ K/h_1 \arrow[r, "t"] & K/h \end{tikzcd}$$

Commutative . Now by the commutativity of the diagram .

$$\begin{tikzcd} K \arrow[r, "t"] \arrow[d, "p_1"'] & K \arrow[rr, "h"] \arrow[dr, "p"'] & & K \arrow[dl, "p"] & K \arrow[l, "\bar{t}"'] \arrow[d, "p_1"] \\ K \arrow[rr, "t"] & & K & & K \arrow[ll, "t"'] \end{tikzcd}$$

We obtain $p_1 = p_1 t^{--1} h\bar{t}$. That is $t^{--1} h\bar{t}$ is a nontrivial covering transformation on K with respect to p_1 . Hence , $t^{--1} h\bar{t} = h_1{}^i$ for some $1 \leq i \leq 2r$,which means that h is weakly equivalent to h_1 .

This completes the proof .

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