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A DETAILED AND SIMPLIFIED SOLUTION TO HYDRODYNAMIC FORCES ON A SUBMERGED TANK SUBJECT TO LATERAL GROUND EXCITATION.

A.H.Helou
Civil Engineering Department,An-Najah University, Nablus.

ملخص

تعتبر هذه الدراسة محاولة ناجحة لتبسيط طريقة احتساب توزيع الضغط الهيدروديناميكي على جدران مقريج اسطواني مغمور في الماء.

ABSTRACT

The domain where a submerged vertical cylindrical tank exists is divided into two regions. Assuming irrotaional motion and an ideal fluid the Laplace s equations in both regions are solved by the method of separation of variables. The intgral equation resulting from matching the potentials at the interface is solved numerically by the Galerkin method. It is showen that the mathematical labour is greatly reduced by dropping the surface effects and the results are in good agreement with those obtained by including surface effects albeit at higher values of excitation frequencies. The case of protruding cylinder is presented to further illustrate the simplicity of the solution.

INTRODUCTION

In the early seventies, industrialized nations started to consider the oceans as potential sites for oil production and storage. The first off-shore tank was constructed 60 miles off the coast of the Sheikhdom of Dubai. That tank has no bottom and operates on the water displacement principle with oil floating on top. In earthquake prone areas such structures have to resist the seismic-induced hydrodynamic forces.

Under earthquake excitation, surface waves are generated outside the tank. Inside the tank, due to the presence of two fluids, oil and water, internal waves are also generated (Helou [2]). In this presentation the tank is assumed to be rigid allowing the complete separation of the hydrodynamic problem inside and outside the tank. The following is a detailed and simplified solution to the exterior wave radiation problem.

The problems of wave scattering and radiation by submerged and semisubmerged tanks of simple geometry were studied by Black and Mei [1], and later by Tung [3], who considered the radiation problem of a submerged circular cylindrical tank. In the following presentation it is intended to show that by neglecting the surface effects the mathematics involved in deriving the hydrodynamic pressure distribution is considerably simplified. However, it should be noted that this simplification is applicable only at higher values of excitation frequencies where most of the energy is contained. The results pertaining to a protruding vertical cylindrical tank will also be presented.

Statement of the Problem and its Solution

Consider the tank shown in Figure 1 to be oscillating in the horizontal direction with a harmonic ground excitation of unit amplitude $\ddot{f}_H(t) = e^{-i\omega t}$.

Region 1

Assuming the fluid motion to be irrotational and the fluid ideal, the velocity potential Φ_1 exists and, for small amplitude waves, satisfies the equation

$$\nabla^2\Phi_1 = 0 \tag{1}$$

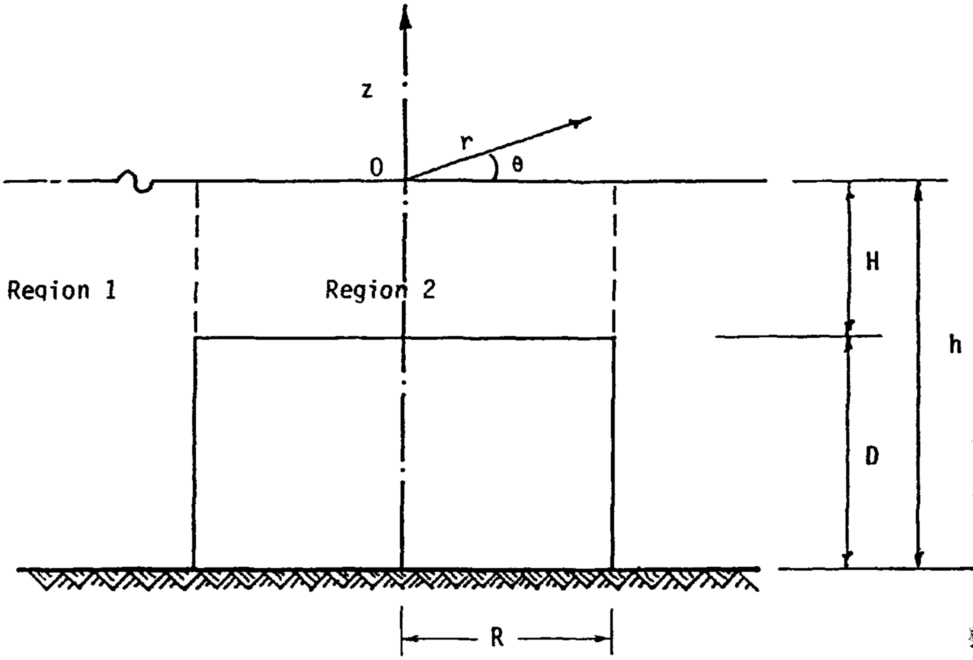

Figure 1 Definition sketch of the submerged vertical circular cylindrical tank. It shows the coordinate system employed in subsequent discussions.

subject to the following boundary conditions

$$\frac{\partial \Phi_1}{\partial z} = 0 \qquad \text{at } z = -h \qquad (2)$$

At the surface the kinematic boundary condition requiring a particle on the surface to stay on the surface is written as

$$\frac{\partial \Phi_1}{\partial z} - \frac{\partial \eta}{\partial t} = 0 \qquad \text{at } z = 0 \qquad (3)$$

where η is the water surface displacement. Furthermore, the dynamic free surface boundary condition is

$$\frac{\partial \Phi_1}{\partial t} + g\eta = 0 \qquad \text{at } z = 0 \qquad (4)$$

g denotes the gravitational acceleration.

At the lateral surface of the tank the boundary condition is

$$\frac{\partial \Phi_1}{\partial r} = \dot{f}_H(t) \cos\theta \qquad \text{at } r = R \text{ and } -h < z < -H \qquad (5)$$

In addition to the above stated boundary conditions the radiation condition needs to be satisfied. It requires the waves to be outgoing as r becomes infinite.

<u>Region 2</u>

Similar to region 1, Laplace's equation defines the fluid motion in region 2

$$\nabla^2 \Phi_2 = 0 \qquad (6)$$

Together with the following boundary conditions

$$\frac{\partial \Phi_2}{\partial z} = 0 \qquad \text{at } z = -H \qquad (7)$$

$$\frac{\partial \Phi_2}{\partial z} - \frac{\partial \eta}{\partial t} = 0 \qquad \text{at } z = 0 \qquad (8)$$

$$\frac{\partial \Phi_2}{\partial t} + g\eta = 0 \qquad \text{at } z = 0 \qquad (9)$$

Across the boundaries between regions 1 and 2, two additional boundary conditions are available. They are the continuity of pressure and velocity

$$\rho\frac{\partial \Phi_1}{\partial t} = \rho\frac{\partial \Phi_2}{\partial t} \qquad \text{at } r = R \text{ and } -h < z < -H \qquad (10)$$

$$\frac{\partial \Phi_1}{\partial r} = \frac{\partial \Phi_2}{\partial r} \qquad \text{at } r = R \text{ and } -h < z < -H \qquad (11)$$

In equation (10) ρ is the fluid density. The ground motion is conveniently separated into spatial and temporal parts

$$\Phi_1(z,r,\theta,t) = \phi_1(z,r,\theta)e^{-i\omega t} \qquad (12)$$

$$\Phi_2(z,r,\theta,t) = \phi_2(z,r,\theta)e^{-i\omega t} \qquad (13)$$

which together with equation (10) imples that

$$\Phi_1 = \Phi_2 \qquad \text{at } r = R \text{ and } -h < z < -H \qquad (14)$$

The dynamic and kinematic boundary conditions are customarily combined to yiel

$$\frac{\partial \phi_1}{\partial z} - \frac{\omega^2}{g}\phi_1 = 0 \qquad \text{at } z = 0 \qquad (15)$$

$$\frac{\partial \phi_2}{\partial z} - \frac{\omega^2}{g}\phi_2 = 0 \qquad \text{at } z = 0 \qquad (16)$$

The solution to Laplace's equation (1) is obtained by the method of separation of variables. In region 1, let

$$\phi_1(z,r,\theta) = Z_1(z) \cdot R_1(r) \cdot \Theta_1(\theta) \tag{17}$$

Substituting this into equation (1) and dividing by $Z_1R_1\Theta_1$ gives

$$\frac{R_1''}{R_1} + \frac{1}{r}\frac{R_1'}{R_1} + \frac{1}{r^2}\frac{\Theta_1'}{\Theta_1} = -\frac{Z_1''}{Z_1} = \text{Constant} \tag{18}$$

From the boundary condition mentioned in equation (5), it is observed that the θ-dependence must be given by

$$\Theta_1(\theta) = \cos\theta \tag{19}$$

which means that equation (18) reduces to

$$\frac{Z_1''(z)}{Z_1} = k^2 \tag{20}$$

and

$$r^2R_1'' + rR_1' + (k^2r^2 - 1)R_1 = 0 \tag{21}$$

in which k is the wave number and k^2 can take all possible values (i.e., positive, zero and negative).

For $k^2 > 0$ and with the boundary condition 2 the solution to equation (20) is written as

$$Z_1(z) = C\cosh k(z + h) \tag{22}$$

in which C is a constant to be determined.

Applying the combined free surface boundary condition of equation (15) yields the dispersion relation

$$\frac{\omega^2}{gk} = \tanh kh \tag{23}$$

The surface effect is neglected by setting $\frac{g}{\omega^2} = 0$. This indicates that for $k^2 > 0$ no solution exists. A similar result is obtained for $k^2 = 0$. However for $k^2 < 0$ and using the following identities between circular and hyperbolic functions

$$\cos i\theta = \cosh \theta$$

and

$$\sin i\theta = i \sinh \theta$$

The solution of equation (22) becomes

$$Z_1(z) = C \cos k(z + h) \tag{24}$$

and the dispersion relation (23) takes the form

$$\frac{\omega^2}{g} = -k \tan kh \tag{25}$$

Setting $\frac{g}{\omega^2} = 0$ gives $\tan kh = \infty$ the roots of which are

$$k_n = \frac{(2n - 1)\pi}{2h} \qquad n = 1,2,3, \ldots \infty \tag{26}$$

And the eigenfunction in z takes the form

$$Z_1(z) = C_n \cos k_n(z + h) \tag{27}$$

The eigenfunction in the radial direction is obtained from the following solution of equation (21) which is recognized as the modified Bessel equation the first order and first kind. For each value of n the solution is

$$R_1(r) = a_n I_1(k_n r) + b_n K_1(k_n r) \tag{28}$$

where I_1 is the modified Bessel function of the first kind and first order and K_1 is the modified Bessel function of the second kind and first order. a_n and b_n are sets of constants.

Since I_1 does not satisfy the radiation condition the constants a_n are set equal to zero and the solution to equation (21) reduces to

$$R_1(r) = b_n K_1(k_n r) \tag{29}$$

Now the velocity potential function in region 1 may be written as a combination of all eigenfunctions obtained

$$\phi_1 = \sum_{n=1}^{\infty} b_n f_n(z) K_1(k_n r) \cos\theta \tag{30}$$

where $f_n(z)$ is an orthonormal function in z computed to be

$$f_n(z) = \sqrt{\frac{2}{h}} \cos k_n(z + h) \tag{31}$$

Following an almost identical scheme the solution to equation (6) is obtained. The potential function ϕ_2 is

$$\phi_2 = \sum_{n=1}^{\infty} B_n F_n(z) I_1(k_n^* r) \cos\theta \tag{32}$$

where $k_n^* = \dfrac{(2n - 1)\pi}{2H}$ $\qquad n = 1,2,3 \ldots \infty$

and $$F_n(z) = \sqrt{\frac{2}{H}} \cos k_n^*(z + h) \tag{31}$$

In order to quantify the set of unknown constants b_n's and B_n's the condition of pressure continuity and velocity continuity across the boundary r = R is utilized.

Differentiating the expression for ϕ_1 and ϕ_2 with respect to r, the following expressions are obtained for the velocities at r = R

$$v(z) = \sum_{n=1}^{\infty} b_n f_n(z) k_n K_1'(k_n R) \qquad -H < z < 0 \tag{33}$$

and

$$v(z) = \sum_{n=1}^{\infty} B_n F_n(z) k_n^* I_1'(k_n^* R) \qquad -H < z < 0 \tag{34}$$

The prime (') denotes differentiation with respect to r. The velocity of the tank's wall follows from the assumption that the ground acceleration is harmonic

$$v(z) = -\frac{1}{i\omega} \qquad -h < z < -H \tag{35}$$

When both sides of equation (33) are multiplied by $f_m(z)$ and integrated over the depth h the following expression for b_n is obtained after the orthonormality property is invoked.

$$b_n = \int_{-h}^{0} \frac{v(z)\ f_n(z)}{K_1'(k_n R)k_n} dz \tag{36}$$

Similarly an expression for B_n is obtained

$$B_n = \int_{-H}^{0} \frac{v(z)\ F_n(z)}{I_1'(k^* R)k^*} dz \tag{37}$$

The quantity b_n can be further simplified to take the form

$$b_n = \int_{-H}^{0} \frac{v(z)\ f_n(z)}{K_1'(k_n R)k_n} dz + \frac{M_n}{K_1'(k_n R)k_n}$$

where

$$M_n = -\frac{1}{i\omega}\cdot\sqrt{\frac{2}{h}}\cdot\frac{1}{k_n}\ \sin k_n D \tag{38}$$

Matching potentials ϕ_1 and ϕ_2 at $r = R$ and substituting the expressions for b_n and B_n the following integral equation is obtained

$$\psi(z) = \int_{-H}^{0} v(z)\ G(z/\xi)d\xi \tag{39}$$

where

$$\psi(z) = \sum_{n=1}^{\infty} f_n(z)\ M_n\ K_1(k_n R)$$

and $G(z/\xi)$ is the Green's function

$$G(z/\xi) = \sum_{n=1}^{\infty} \bar{I}_1(k_n^* R)\ F_n(z)\ F_n(\xi) - \sum_{n=1}^{\infty} \bar{K}_1(k_n R)\ f_n(z)\ f_n(\xi)$$

The functions $\bar{I}_1$ and $\bar{K}_1$ are defined as

$$\bar{I}_1(k_n^* R) = \frac{I_1(k_n^* R)}{(k_n^* R) I_1'(k_n^* R)}$$

$$\bar{K}_1(k_n R) = \frac{K_1(k_n R)}{(k_n R) K_1'(k_n R)}$$

The integral equation (39) is solved numerically by the Gelerkin technique in which the function $v(z)$ is expanded into an infinite series in terms of $F_m(z)$ and a set of unknown constants α_m

$$v(z) = \sum_{m=1}^{\infty} \alpha_m F_m(z) \tag{40}$$

By substituting the above relation into equation (39), integrating with respect to z over [-H,0] and invoking the orthonormality property of the eigenfunctions an infinite set of algebraic equations with real coefficients are obtained.

$$\psi_p = \sum_{m=1}^{\infty} \alpha_m g_{mp} \tag{41}$$

where

$$\psi_p = \sum_{n=1}^{\infty} M_n K_1(\bar{k}_n R) P_{np}$$

$$g_{mp} = \sum_{n=1}^{\infty} \bar{I}_1(K_n^* R) \delta_{np} \delta_{nm} - \sum_{n=1}^{\infty} \bar{K}_1(k_n R) P_{nm} P_{np}$$

δ is the usual Kronecker delta function.

$$P_{nm} = \frac{2}{\sqrt{hH}} \frac{k_n \sin k_n D}{(k_m^{*2} - k_n^2)}$$

Upon solving equation (41) for α_m's, the following expressions are obtained for b_n and B_n

$$b_n = \frac{1}{k_n K'(k_n R)} [M_n + \sum_{m=1}^{\infty} \alpha_m P_{nm}]$$

$$B_n = \frac{1}{k_n^* I'(k_n^* R)} \sum_{m=1} \alpha_m \delta_{nm}$$

With the b_n and B_n quantified the potential functions ϕ_1 and ϕ_2 are fully determined. The hydrodynamic pressure distribution is obtained from Bernoulli's equation, namely $P = \rho \frac{\partial \Phi}{\partial t}$

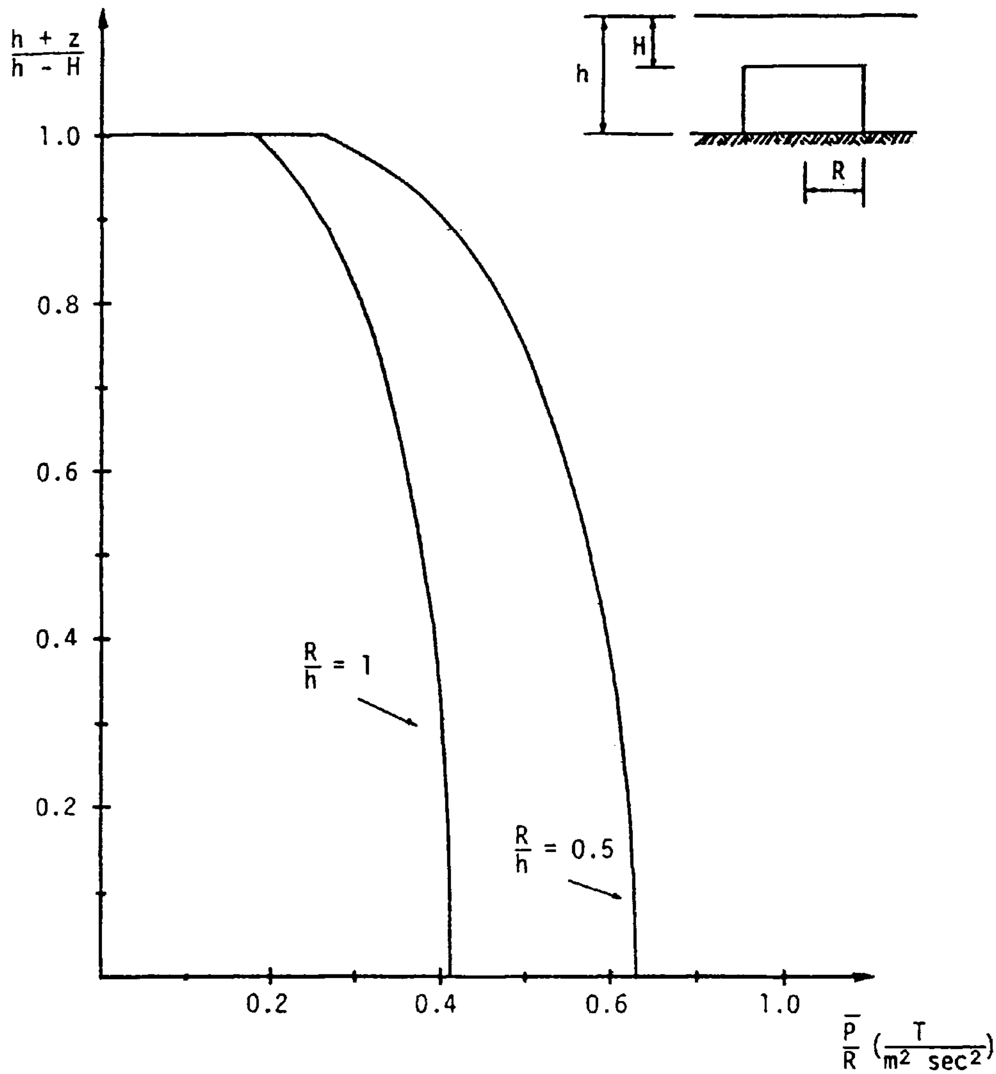

Figure 2 Dyuamic pressure distribution on the tank's wall at $\theta = 0$; $h/H = 2$

Figure 2 shows the amplitude of the pressure distribution at the tank's wall taken at $\theta = 0$. The results are in agreement with those obtained by Tung [3] for $\omega = 10$ radians per second. This reinforces the idea that for higher values of excitation frequencies the surface effects may safely be neglected.

The Case of a Cylindrical Tank Protruding Over the Water Surface

When a circular cylindrical tank protrudes over the water surface, as shown in Figure 3, region 2 will be absent and the algebra involved is greatly simplified. The velocity potential function is

$$\Phi = \sum_{n=1}^{\infty} b_n \cdot f_n(z)\, K_1(k_n r) \cos\theta\, e^{-i\omega t}$$

where the b_n's are the only set of unknowns and may be qualified by applying the boundary condition stated in equation (5)

$$\sum_{n=1}^{\infty} b_n f_n(z)\, K_1'(k_n R) \cdot k_n \cos\theta\, e^{-i\omega t} = -\frac{1}{i\omega} e^{-i\omega t} \cos\theta$$

From which an expression for b_n is readily obtained

$$b_n = -\frac{1}{iw}\sqrt{\frac{2}{h}}\,\frac{1}{k_n^2}\,\frac{\sin k_n h}{K_1'(k_n R)}$$

The following expression for the dyanmic pressure distribution is computed from the Bernoulli's equation

$$P_D = -\rho \cos\theta\, e^{-i\omega t}\sqrt{\frac{2}{h}} \sum_{n=1}^{\infty} \frac{1}{k_n^2} \sin k_n h\, \frac{K_1(k_n R)}{K_1'(k_n R)}\, f_n(z)$$

The amplitude of the hydrodynamic pressure distribution taken at $\theta = 0$ of a selected tank is shown in Figure 4. The tank has a radius equal to 10 meters and protruding over the surface of water 20 meters deep.

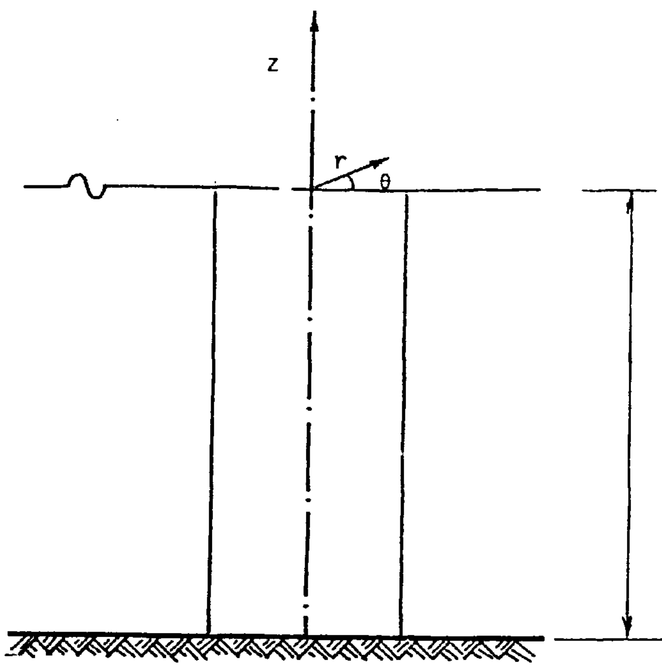

Figure 3 Definition sketch of a cylinder protruding over the water surface.

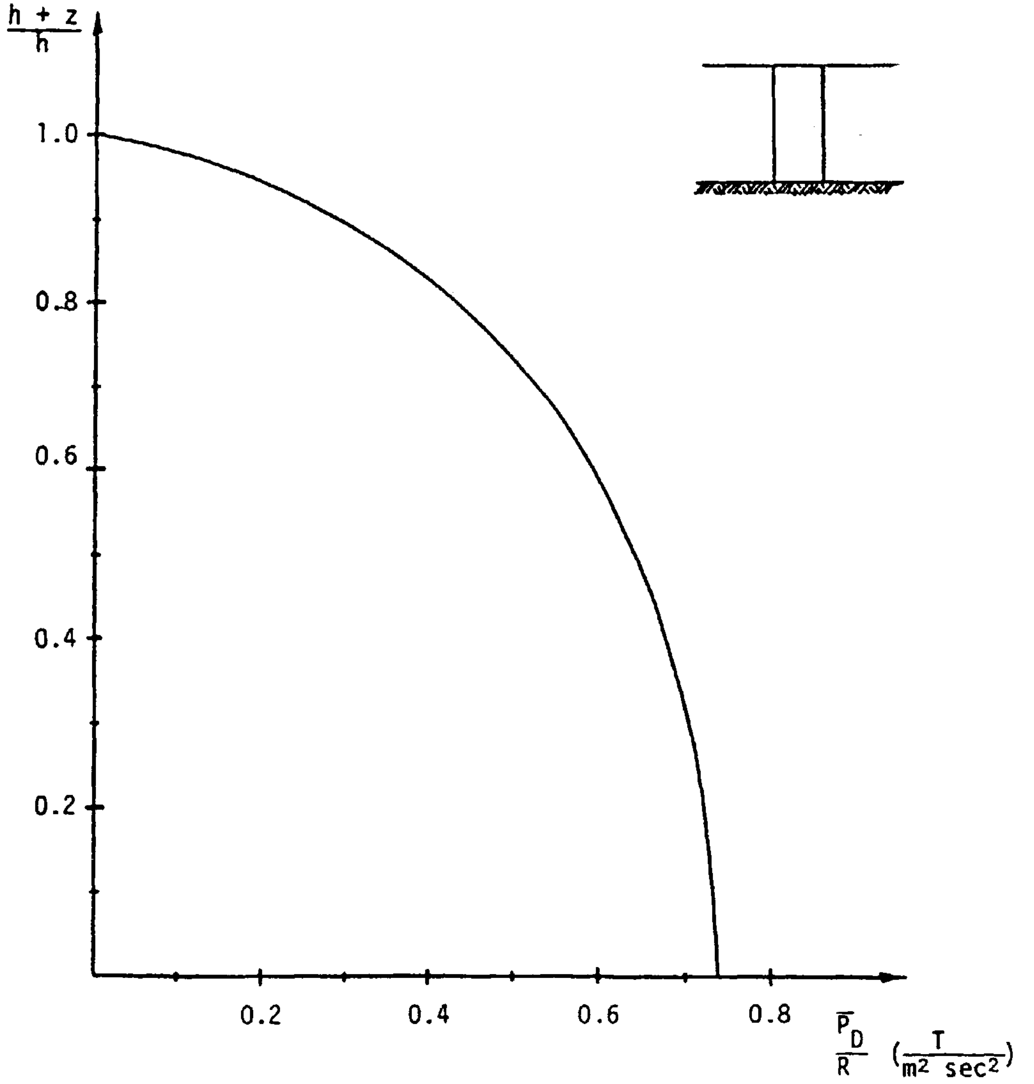

Figure 4 Dynamic pressure distribution on the wall of a tank protruding over the water surface

Conclusion

In studying wave radiation it has been further established that the influence of surface effects disappears at high values of excitation frequencies. Dropping them at the onset has the advantage of appreciably simplifying the solution.

Both cases of a submerged tank and a protruding one are clearly presented. The simplicity of the solutions could not be overstated.

References

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