**An-Najah National University**

**Faculty of Graduate Studies**

# Numerical Simulation for Computing the Number of Limit Cycles of Generalized Abel Equation

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*To my family, sincere thanks for their support, encouragement, believing in me, and making everything achievable for me.*

# الإقرار

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# Numerical Simulation for Computing the Number of Limit Cycles of Generalized Abel Equation

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The work provided in this thesis, unless otherwise referenced, is the researcher`s own work, and has not been submitted elsewhere for any other degree or qualification.

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**Date:** **التاريخ: 2021/2/2**

# Table of Contents

# Table of Figures

# Numerical Simulation for Computing the Number of Limit Cycles of Generalized Abel Equation

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## Abstract

Limit cycles (isolated periodic solutions) describe the phenomenon of oscillation that are considered and studied in different research fields such as physics, medicine, chemistry, populations…etc. In nature, some of biological and physical processes are represented by stable limit cycles. The interest point of this problem comes from the study of number of isolated closed orbits of a planar polynomial vector field, which is a part of Hilbert`s Sixteenth Problem; this problem has been one of the major problems in the qualitative theory of ordinary differential equations. Hilbert's problem is interested in the number of limit cycles for the planar polynomial differential system.

In this work, both limit cycles in $xy$-plane and the stability types of limit cycles were exhibited, also direction field was considered, which describes graphically the behavior for the solution of the differential equation. Theorems related to the existence and non-existence of limit cycles were discussed. Moreover, a common nonlinear ordinary differential equation, named Abel differential equation, was discussed. Also this work presented limit cycles of first order polynomial differential equation where

the coefficients are periodic, and presented some results concerned to the maximum number of limit cycles for polynomial differential equations, and work on proving these results numerically.

Furthermore, limit cycles of planar differential system (planar vector field) were presented. Also the Poincaré map, multiplicity of limit cycles for planar differential system, and the multi-parameter of differential system were exhibited. Finally, the number of non-contractible limit cycles of a system in the cylinder was presented with numerical example.

The most challenging problem in this work is to obtain numerical examples such that they contain more than one limit cycle by defining suitable interval, coefficients and initial conditions that satisfy relevant theorems and corollaries. While the second problem was to explore examples of limit cycles which have multiplicity greater than one.

# Chapter One

# Preview

In this chapter, we will introduce the limit cycles (isolated closed trajectories) in $xy$-plane and types of its stability; stable, unstable and semi stable limit cycles. Also, presented the direction field which describes the behavior of solution of the differential equation without solving it, and introduced Poincaré-Bendixson's Theorem and Bendixson's-Criterion; these theorems play an important role by guaranteeing the existence, non-existence of limit cycles under particular conditions. At the end of this chapter, a common nonlinear ordinary differential equation, the Abel differential equation, is presented.

## 1.1 Introduction

During the last century, one of the major problems in qualitative theory is to study the periodic solutions and limit cycles of real polynomial differential equations in $\mathbb{R}^2$. Limit cycles which can be described as isolated periodic solutions in $xy$-plane, represent the simplest type of behavior of continuous dynamical system (i.e. A dynamical system is a way of describing the passage in time of all points in a given space).

Limit cycles represent a phenomenon of oscillation which is observed in different research fields like control theory, electrical circuits, chemistry, medicine, populations,…etc. They are described by differential equations or system of differential equations (i.e. collection of several differential equations with several unknowns) [9].

Differential equation is an equation including an unknown function and one or more of its derivatives [11] which is divided into ordinary differential equation and partial differential equation. An ordinary differential equation (ODE) is a differential equation which contains differentials with respect to only one independent variable, while Partial differential equation (PDE) contains differentials with respect to several independent variables. The order of the differential equation is the order of the highest derivative that appears in the equation. A decisive classification of differential equations is whether they are linear or nonlinear. The ordinary differential equation:

$$F\left(t, u, u', u'', \dots, u^{(n)}\right) = 0$$

is said to be linear if $F$ is linear in the unknowns $u, u', \dots, u^{(n)}$, otherwise it's called nonlinear. Oscillating pendulum, (see Figure 1.1) is an example for the simple physical problem that leads to a nonlinear differential equation, which can be mathematically model as:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} sin\theta = 0,$$

where $t$ is the time, $g$ is the acceleration due gravity, $L$ is the length of the pendulum, and $\theta$ is the angular displacement [7].

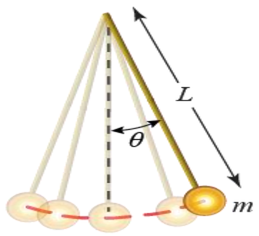



**Figure 1.1:** An oscillating pendulum.

There are many research's related to computing the number of limit cycles of generalized Abel equation. Bravo and Fernandez [8] obtained a criterion for determining the stability of singular limit cycles of Abel equation. Alkoumi and Torres in [2] proved new results about the maximum number of limit cycles of first order differential equation with periodic coefficients. Also, they applied these results to bound the number of limit cycles of a family of planar polynomial vector fields. Alvarez et al. [3] gave two criteria for bounding the number of non-contractible limit cycles of a family of differential system on the cylinder which includes Abel equation.

Gasull and Guillamonin [10] dealt with the problem of finding upper bounds for the number of periodic solutions of class of one-dimensional polynomial differential equations with one periodic coefficients. Llibre and Zhao [14] showed that there is a polynomial system of differential equations with arbitrary degree that has algebraic limit cycles of degree 3, and gave an example of a cubic polynomial system for differential equations with two algebraic limit cycles of degree 4.

The most challenging questions in studying limit cycles are: How to control the number of limit cycles? Is the number finite? Is the number bounded?

This work, will focus and discuss nonlinear first order ordinary differential equations.

## 1.2 Limit Cycles

One of the most difficult problems concerned studying nonlinear systems is the problem of finding limit cycles. The phenomenon of limit cycles was discovered by Henri Poincare (1854-1912). The limit cycle describes a phenomenon of oscillation which is observed in various scientific, engineering and medical fields [13]. In the following, the limit cycle and its types will be defined.

***Definition 1 [18]:*** *An isolated closed trajectory in phase plane ($xy$-plane) is called* ***a limit cycle****. Isolated means that the neighboring trajectories are not closed; they spiral either towards or away from the limit cycle.*

Thus, limit cycles occur only in nonlinear systems, where the linear system:

$$x' = Ax$$

can have a closed trajectory, but will not be isolated. If $x(t)$ is a periodic solution, then $cx(t)$ for any $c \neq 0$ is also a solution. Hence $x(t)$ will be surrounded by other closed trajectories, (see Figure 1.2).

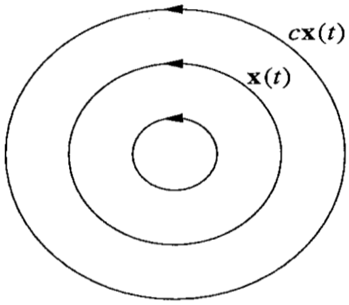



**Figure 1.2:** Periodic solutions for a linear system**.**

***Definition 2 [18]:*** *A **stable limit cycle** (attractive limit cycle) attracts the near trajectories towards it as $t \to \infty$, while **unstable limit cycle**, the trajectories spiral away as $t \to \infty$. **Semi-stable (half-stable) limit cycle** will have trajectories such that, from one side spiral towards it, while from the other side spiral away as $t \to \infty$*, (see Figure 1.3).

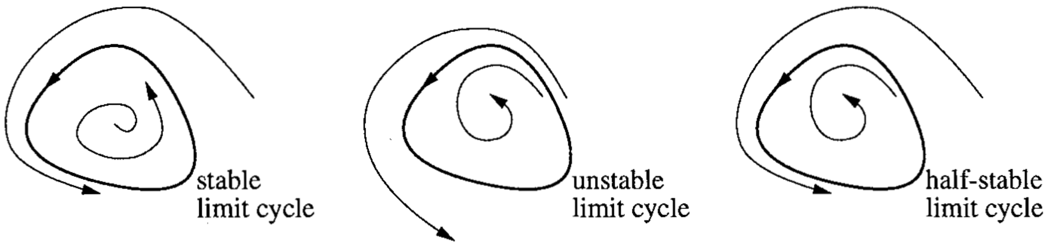



**Figure 1.3:** Types of limit cycle**.**

### 1.2.1 Direction Field

For many differential equations, especially nonlinear ones, it will be difficult to find its analytical solution in explicit formula. Hence, it is possible to rely on numerical or graphical methods to get an idea of how the solution of the differential equation behaves.

Even though a solution cannot be found, information about the solution can be found such as: the value of the solution at a certain point, the intervals where the solution is decreasing or increasing, the points where the solution reaches a maximum value, does the solution go to infinity? does it go to zero?,...etc.

One technique which is useful when graphing the solution of a differential equation is to draw the direction field (slope field) for the equation.

***Definition 3 [16]: A direction field*** *of the differential equation is a plot of short line segments drawn at various points in $xy$-plane displaying the slope of the solution curve.*

The direction field describes the behavior of solutions of the differential equation without having the solution itself (i.e. the direction field display how the trajectories flow through the plane). Where computers are used to draw the direction field. The next example will clarify this method.

**Example 1:**

Sketch the direction field for the linear differential equation:

$$\frac{dy}{dx} = x^2 - y$$

***Solution:***

Let us calculate the slope of the solution at different points in the $xy$-plane. At the point $(1,0)$ the slope of the solution is $(1)^2 - 0 = 1$ which is positive, thus, the solution there is increasing. While at the point $(-1,1)$ the solution has zero slope. On the other hand, the slope of the solution through the point (0,1) is negative. Hence, the solution is decreasing, in the same way calculate the slope for other points and draw their direction field. Figure 1.4 shows the direction field and some solutions.

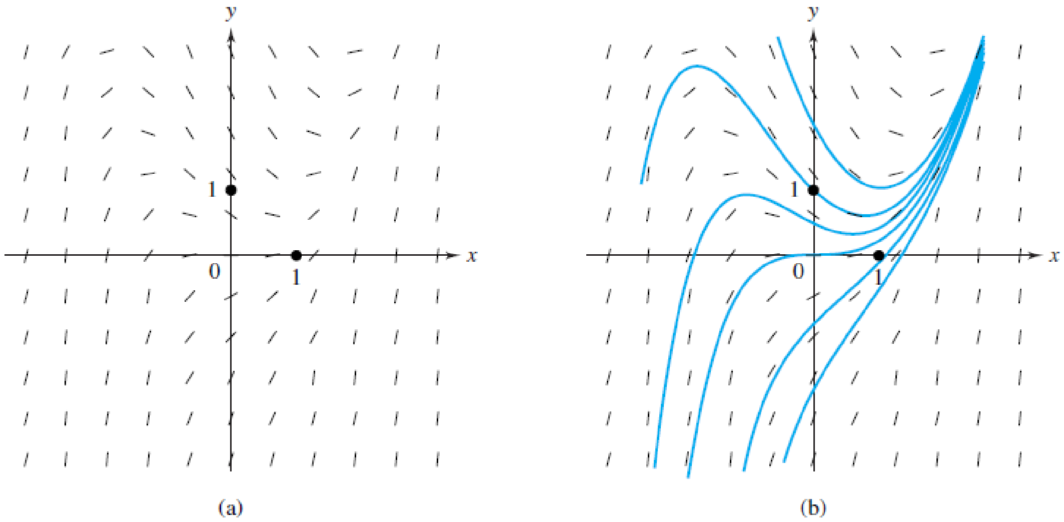



**Figure 1.4*:*** Representing the differential equation $\boldsymbol{y' = x^2 - y}$ by (a) Direction field (b) Solutions of different initial conditions.

### 1.2.2. Autonomous System

Another classification of differential equations or system of differential equations focus on the independent variable; it can be explicit or implicit.

***Definition 4 [7]:*** *A system of ordinary differential equations with the functions $F$ and $G$ which are not explicitly depending on the independent variable $t$ is called autonomous system, otherwise the system is called non-autonomous.*

To put it in mathematical form, let us consider the system:

$$\begin{cases} x' = F(x,y) \\ y' = G(x,y), \end{cases} \tag{1.1}$$

where the functions $F$ and $G$ are continuous and have continuous partial derivatives in the domain $D$ of $xy$-plane. Then this system is called autonomous one.

**Example 2:**

Show that the following nonlinear autonomous system has an attracting (stable) limit cycle:

$$\begin{aligned} x' &= x + y - x(x^2 + y^2) \\ y' &= -x + y - y(x^2 + y^2) \end{aligned} \tag{1.2}$$

**Solution:**

Clearly the point (0,0) is the only equilibrium solution (i.e. points in the graph in which the functions derivatives are zero) for the system (1.2), by using the polar coordinate $r$ and $\theta$ to describe the system more conveniently:

$$x = rcos\theta \rightarrow dx/dt = x' = -rsin\theta.d\theta/dt$$
$$y = rsin\theta \rightarrow dy/dt = y' = rcos\theta.d\theta/dt \text{ '}$$

where $r \geq 0$. By multiplying the first equation of system (1.2) by $x$, the second by $y$ and then add them, so:

$$xx' = x^2 + xy - x^2(x^2 + y^2)$$

$$yy' = -xy + y^2 - y^2(x^2 + y^2)$$

$$xx' + yy' = [x^2 + y^2](1 - (x^2 + y^2))$$

Use $x^2 + y^2 = r^2$, and $xx' + yy' = rr'$, giving:

$$r' = r(1 - r^2) \tag{1.3}$$

The critical points for $r \geq 0$ are the origin and the point $r = 1$, which describes the unit circle in $xy$-plane. From Eq. (1.3) it follows that $r' > 0$ when $r < 1$, and if $r > 1$ then $r' < 0$. This gives the conclusion that; inside the unit circle the trajectories are directed outward, while they are directed inward outside the unit circle. It seems that the circle $r = 1$ is a limiting trajectory for the system.

To determine an equation for $\theta$, multiply the first equation of system (1.2) by $y$ , the second by $x$, and subtract them, gives :

$$yx' = xy + y^2 - xy(x^2 + y^2)$$

$$xy' = -x^2 + xy - xy(x^2 + y^2)$$

$$yx' - xy' = x^2 + y^2$$

$$rsin\theta(-rsin\theta\ \theta') - rcos\theta(rcos\theta\ \theta') = r^2$$

Thus,

$$\theta' = -1$$

Hence, the system in polar coordinate which is equivalent to system (1.2) is:

$$r' = r(1 - r^2), \qquad \theta' = -1 \tag{1.4}$$

Notice that, by separation of variables, other solutions can be obtained by solving Eq. (1.3).

If $r \neq 0$ and $r \neq 1$, then:

$$\frac{dr}{dt} = r(1 - r^2) \ \rightarrow \ \frac{dr}{r(1-r^2)} = dt \tag{1.5}$$

Using partial fraction to rewrite the left side of Eq. (1.5), and integrating both sides, then:

$$r = \frac{1}{\sqrt{1+c_0 e^{-2t}}}, \qquad \theta = -t + t_0,$$

where $c_0, t_0$ are arbitrary constants. By substituting $c_0 = 0$ we get:

$$r = 1, \qquad \theta = -t + t_0 \tag{1.6}$$

Any point satisfying system (1.6) moves clockwise around the unit circle, as $t$ increases. Hence the autonomous system (1.2) has a periodic solution.

The solution satisfying the initial conditions: $r(0) = \alpha, \theta(0) = \beta$ is given by:

$$r = \frac{1}{\sqrt{1 + [\frac{1}{\alpha^2} - 1]\mathrm{e}^{-2\mathrm{t}}}}, \qquad \theta = -(t - \beta)$$

If $\alpha < 1$, then $r \to 1$ from the inside as $t \to \infty$; if $\alpha > 1$, then $r \to 1$ from the outside as $t \to \infty$. Hence in all cases as $t \to \infty$ the trajectory spiral toward the circle $r = 1$. Figure 1.5 shows several trajectories of the given system; introducing a limit cycle [7].

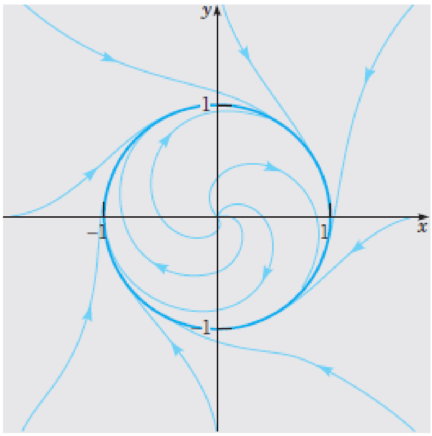



**Figure 1.5:** Different trajectories making a limit cycle**.**

### 1.2.3. Existence and Non-Existence of Limit Cycles

The following theorems concerning existence and non-existence of limit cycles of nonlinear autonomous systems.

#### 1.2.3.1. Poincare-Bendixson's Theorem

In the study of the qualitative behavior of autonomous differential equations and dynamical systems on $\mathbb{R}^2$, the Poincare-Bendixson's theorem plays an important role. It guarantees the existence of limit cycles, in addition, it gives the existence of stationary (critical) points as for a system which is defined on a plane, each periodic orbit must be surrounded by a stationary point [16]. Unfortunately, there is no much theorems about proofing the existence of limit cycles, hence people try to find limit cycles by utilizing computer.

***Theorem 1 [7]:*** *Consider the two dimensional autonomous system (1.1):*

$$\begin{cases} x' = F(x, y) \\ y' = G(x, y) \end{cases},$$

*where the functions F and G have continuous first partial derivatives in domain D of the $xy$-plane. Any closed trajectory of system (1.1) is necessarily surrounded by one critical point of the system.*

From the above theorem, one can note that if a given region does not contain any critical point then there is no closed trajectory lying in the region. The next theorem gives the condition to guarantee existence of closed trajectory.

***Theorem 2 (Poincare-Bendixson's Theorem) [7]:*** *Let the functions F and G have continuous first partial derivatives in a domain D of the $xy$-plane. Let $D_1$ be a bounded subdomain in $D$, and let $R$ be the region that consists of $D_1$ plus its boundary (i.e. all points of $R$ are in $D$). Suppose that $R$ contains no critical point of the system (1.1). If $C = (\phi(t), \psi(t))$ is a trajectory of the system (1.1) that exists and stays in $R$ for some $t_0$ and remain in $R \; \forall \; t \geq t_0$, then either $C$ is a periodic solution (closed trajectory), or it spirals toward a closed trajectory as $t \to \infty$. In either case, the system (1.1) has a periodic solution in R.*

**Example 3:**

Using the previous theorem to show that system (1.2) of example 2 has a periodic solution.

**Solution:**

The only critical point of system (1.2) is (0,0). Let $R$ be the region bounded by $0.75 \leq \mathrm{r} \leq 1.5$ which contains no critical points, starting by $r = 0.75 \; \rightarrow \; r' > 0$ this mean that $r$ increases, while for $r = 1.5 \; \rightarrow \; r' < 0$ so, $r$ decreases. Hence for any trajectory which crosses the boundary of $R$ is entering $R$.

So, any solution of system (1.2) that starts in $R$ at $t = t_0$ cannot leave, but stay in $R$ for $t > t_0$, hence, by Poincaré -Bendixson's theorem there exists a periodic solution in $R$.

### 1.2.3.2. Bendixson's Criterion

This theorem is used to show that a limit cycle does not exist under particular conditions.

*Consider the autonomous system (1.1):*

$$\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

*Suppose that the domain $D$ of system (1.1) is simply connected (i.e. there are no holes or separate parts in the domain), and the functions $F$ and $G$ have continuous partial derivative in $D$ of the $xy$-plane. Assume that $F_x + G_y$ dose not change sign (i.e. either always positive or always negative) throughout $D$, then system (1.1) has no closed trajectory in $D$.*

Notice that if $F_x + G_y$ changes sign in the domain, then there may or may not be closed trajectory in the domain $D$ [7].

**Example 4:**

Suppose the system is given as:

$$x' = x^3 + y^3$$
$$y' = 3x + y^3 + 2y$$

It is easy to show that $F_x(x,y) + G_y(x,y) = 3x^2 + 3y^2 + 2$, which is always $> 0$ in the domain of $xy$-plane. Hence there is no close trajectory in the $xy$-plane.

Another illustration for the criterion, let us Return to Example 2, where:

$$F_x + G_y = 2 - 4(x^2 + y^2) = 2(1 - 2r^2)$$

which is positive for $0 \leq r < \frac{1}{\sqrt{2}}$ , hence there is no closed trajectory. The same result we have shown, there is no closed trajectory in the region when $r < 1$.

For $r > \frac{1}{\sqrt{2}}$ , then $F_x + G_y < 0$, but Bendixson's criterion is not applied because this annular region is not simply connected. Indeed, as shown previously, it does not contain a closed trajectory.

## 1.3 Generalized Abel Differential Equation

One of the common nonlinear ordinary differential equations is the Abel differential equation which named by the Norwegian mathematician Niels Henrik Abel (1802-1829). He showed that there is no general algebraic solution of the quintic equation [19]. He defined and worked on methods to solve special integral equations the so-called Abel integral equation. Also, worked on differential equations including the important verification of Wronskian determinant for second order differential equations.

The Abel differential equation plays an important role in many applications of real life problems in various areas e.g. in biology control theory, cosmology, fluid mechanics, solid mechanics, cancer therapy, modeling of oceanic circulation, and in problems of magneto- statics [17].

There are two types of Abel differential equations. Abel`s equation of the first kind which appears as one of first nontrivial examples of nonlinear differential equations which will be discussed here.

The main purpose of this work is to study the number of isolated periodic solutions (limit cycles) of the polynomial differential equation. The interest of this problem which is a part of Hilbert`s 16$^{th}$ problem (see appendix III) comes from studying the number of isolated closed orbits of planar polynomial vector field. The polynomial differential equation given by:

$$u' = a_0(t) + a_1(t)u + \cdots + a_{n-1}(t)u^{n-1} + a_n(t)u^n, \qquad (1.8)$$

where the coefficients $a_i(t), i = 0, 1, \dots, n$ are continuous and T-periodic functions for some T$> 0$.

A periodic solution of Eq.(1.8) is a solution $u$ which is defined in the interval $[0, T]$, such that $u(0) = u(T)$. If this periodic solution is isolated in the set of all periodic solutions then it's called a limit cycle. Eq.(1.8) with $n = 1$, is a linear equation, consequently having at most one limit cycle, while for $n = 2$ it`s called a Riccati equation with at most two limit cycles [10]. For $n = 3$, Eq.(1.8) is Known as Abel differential equation. In two publications of Pliss and Lloyd, they proved that Abel differential equation has exactly three limit cycles taking into account the multiplicity cases [4].

A constant sign in the leading coefficient $a_n$ is not sufficient to bound uniformly the number of limit cycles for $n \geq 4$. So, to get more accurate information on the number of limit cycles, many papers in the literature declared that only some polynomial coefficients $a_i(t)$ do not vanish, in which the nonlinear polynomial has only three or four terms [1].

There are no much researches who provide an explicit bound on the number of limit cycles when all the coefficients of the nonlinear polynomial are given. Calanchi and Ruf proved that if $n$ is odd, where the leading term is fixed and the remaining terms are small enough, then there are at most $n$ limit cycles, Alwash in a recent work proved related results giving more precise information about the number of limit cycles [1].

# Chapter Two

# Number of Limit Cycles

In this chapter, an introduction of limit cycles of polynomial ordinary differential equation with periodic coefficients was given and presented some elementary results that are used in proofs of many theorems related to Abel differential equation. An attempt to proof these theorems numerically was done by choosing suitable interval and initial condition that satisfies the theorem. Also, a comparison between theorems and corollaries which has at most one positive limit cycle is done. Furthermore, here are presented some conditions which are used to limit the number of isolated periodic solution of complete $4^{th}$ –order polynomial equation with numerical examples.

## 2.1 Number of Limit Cycles of Generalized Abel Differential Equation with Numerical Simulation

This section deals with proved results to find the maximum number of isolated periodic solutions (limit cycles) of polynomial first order differential equation with periodic coefficients.

### 2.1.1. Preliminaries

This section presents some elementary results which are used in proofs for many theorems related to Abel differential equation. Consider the first order differential equation:

$$u' = g(t,u) = \sum_{i=0}^{n} a_i(t)u^i\,, \qquad (2.1.1)$$

where $a_i(t)$ are $T$-periodic continuous functions for some $T > 0$, and $g$ is continuous in $t$ with continuous derivatives up to order 3.

***Definition 2.1 [2]:*** *A continuous function $f: [0, T] \to \mathbb{R}$ is said to have a definite sign if it is not null and either $f(t) \leq 0$ or $f(t) \geq 0$, we write $f \prec 0$ (negative definite sign) in the first case and $f \succ 0$ (positive definite sign) in the second case.*

**Notation:** The terms $(d_1, d_2, \dots, d_r)$-polynomial refers to Eq.(2.1.1) such that: $a_j(t) \equiv 0$, if $j \neq d_i$ for all $i = 1, \dots, r$ [12].

For example: the $(0, 1, 2)$-polynomial equation means that all coefficients of Eq.(2.1.1) are identical to zero except $a_0(t), a_1(t), a_2(t)$.

***Definition 2.2:*** *The periodic solution of the polynomial differential equation $u = u(t)$ is said to be isolated if there exists an $\varepsilon > 0$ such that:*

$$\{(t,u): u(t) - \varepsilon \leq u(t) \leq u(t) + \varepsilon\}$$

The periodic solution which is isolated is called **a limit cycle** of the differential equation.

By taking the interval $[0,1]$ in Eq.(2.1.1), let us consider the first order differential equation of the form:

$$u'(t) = a_n(t)u^n + a_{n-1}(t)u^{n-1} + \cdots + a_0(t)\,, \qquad (2.1.2)$$

where $t \in [0,1]$, and $a_i(t): [0,1] \to \mathbb{R}, i = 0, 1, .., n$ are analytic functions.

***Definition 2.3 [12]:*** *The solution of Eq.(2.1.2) is called closed solution if it is defined on the interval* $[0,1]$ *such that* $u(0) = u(1)$.

The next theorem show that equation (2.1.2) with $\mathrm{i} = 0, 1, 2, 3$ has at most three closed solutions.

***Theorem 2.1 [12]:*** *The equation:*

$$u'(t) = a_0(t) + a_1(t)u + a_2(t)u^2 + a_3(t)u^3, \qquad (2.1.3)$$

*where* $a_i(t): [0,1] \to \mathbb{R}, i = 0, 1,2,3$ *are continuous in* $[0,1]$, *and* $a_3(t) > 0$ $t \in [0,1]$ *has at most* 3 *closed solutions*.

In the following a set of three examples are presented to clarify relevant theorems.

**Numerical Example 1:**

Let $a_i(t) = \sin(\pi t)\, \forall i = 0, 1, 2, 3$, on the interval $[0,1]$, with initial condition $u(0) = -0.45$.

Clearly $a_3(t) = \sin(\pi t) > 0$ on the interval $[0,1]$.

Now, Eq.(2.1.3) is expanded by:

$$u'(t) = \sin(\pi t)\, u^3 + \sin(\pi t)\, u^2 + \sin(\pi t)\, u + \sin(\pi t)$$

which has one limit cycle on $[0,1]$, (see Figure 2.1).

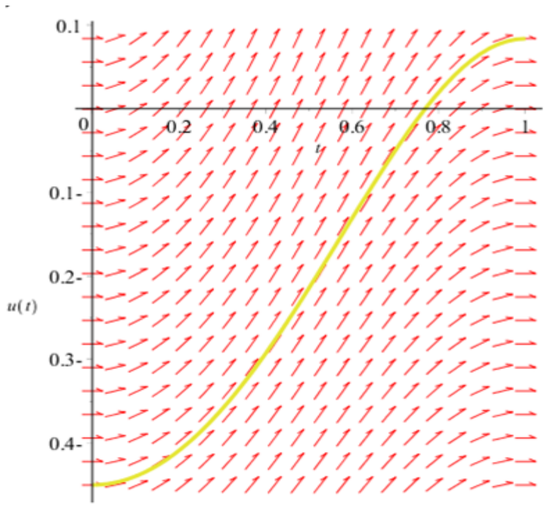



**Figure 2.1:** Limit cycle of equation:

$$\boldsymbol{u'(t) = \sin(\pi t)\, u^3 + \sin(\pi t)\, u^2 + \sin(\pi t)\, u + \sin(\pi t)}.$$

The equation with three terms has the form:

$$u' = a_{n_1}(t)u^{n_1} + a_{n_2}(t)u^{n_2} + a_{n_3}u^{n_3}, \tag{2.1.4}$$

where $n_i, i = 1, 2, 3 \in \mathbb{Z}$ and $n_1 > n_2 > n_3$.

Gasull and Guillamon proved that if $n_3 = 1$ and $a_{n_2}(t)$ or $a_{n_3}(t)$ have a definite sign then Eq.(2.1.4) has at most two positive limit cycles. In the case where $u = 0$, there will be always a solution, and by changing the sign of $u$ this gives another solution (i.e. two limit cycles in positive definite case and two in negative case), then the total number is five limit cycles. Thus, to obtain bounds on the number of limit cycles for equation with three monomials or more, one needs to assume that two coefficients have a definite sign [2].

Alwash, proved that if $n \geq 3$ and $a_{n-2}(t) \leq 0$, then the equation

$$u' = u^n + a_{n-1}(t)u^{n-1} + a_{n-2}(t)u^{n-2} \qquad (2.1.5)$$

has at most one positive limit cycle [2].

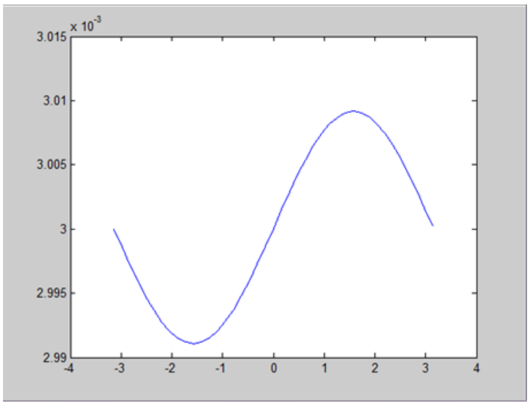
**Numerical Example 2:**

Let: $n = 3$, $a_{n-1}(t) = \cos(t)$, $a_{n-2}(t) = 0$, with initial condition $u(1) = 0.003$.

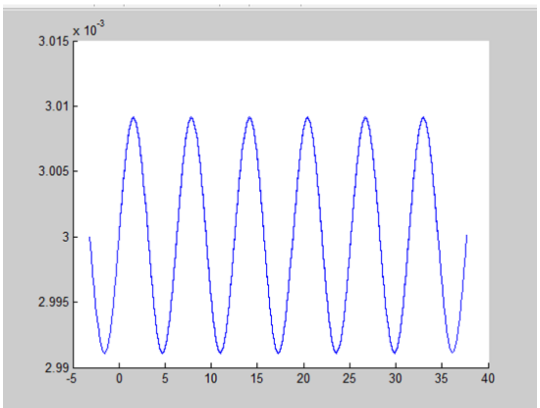
Then, the equation will be:

$$u' = u^3 + \cos(t)\, u^2$$

which has one positive limit cycle on the interval $[-\pi, \pi]$, (see Figure 2.2).



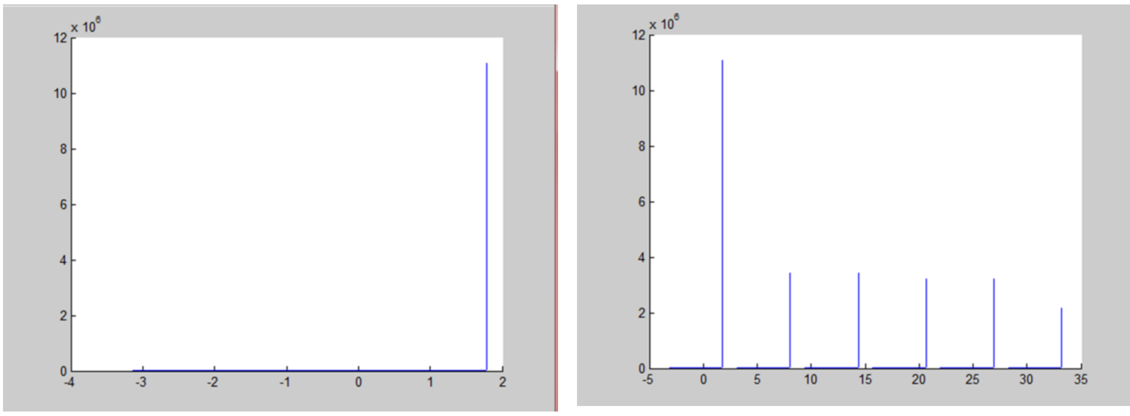

(a)




(b)

**Figure 2.2:** Limit cycle of equation $\boldsymbol{u' = u^3 + \cos(t)\, u^2}$.

But, if we try different initial condition on the same interval, the results came to be a non-limit cycle. For example: $u(1) = 0.28$, (see Figure 2.3).




(a) (b)

**Figure 2.3:** No limit cycle of equation $\boldsymbol{u' = u^3 + \cos(t)\, u^2}$.

***Theorem 2.2 [2]:*** *Given the polynomial differential equation:*

$$u' = a_{n_1}(t)u^{n_1} + a_{n_2}(t)u^{n_2} + a_{n_3}(t)u^{n_3} + a_m(t)u^m, \qquad (2.1.6)$$

*where* $n_1 > n_2 > n_3 > m = 1$.

Assume that $a_{n_1}(t)$ and $a_{n_2}(t)$ or $a_{n_2}(t)$ and $a_{n_3}(t)$ have a definite sign, or that $a_{n_1}(t)$ and $a_{n_3}(t)$ have opposite definite signs, which are summarized in the following Table 2.1:

**Table 2.1: Summary of theorem 2.2**

<table>
<tr><th>Condition</th><th>Number of limit cycles</th></tr>
<tr><td>$a_{n_1}(t) \wedge a_{n_2}(t)$ are positive</td><td rowspan="6">Then Eq.(2.1.6) has at most two positive limit cycles</td></tr>
<tr><td>$a_{n_1}(t) \wedge a_{n_2}(t)$ are negative</td></tr>
<tr><td>$a_{n_2}(t) \wedge a_{n_3}(t)$ are positive</td></tr>
<tr><td>$a_{n_2}(t) \wedge a_{n_3}(t)$ are negative</td></tr>
<tr><td>$a_{n_1}(t)$ is positive $\wedge\, a_{n_3}(t)$ is negative</td></tr>
<tr><td>$a_{n_1}(t)$ is negative $\wedge\, a_{n_3}(t)$ is positive</td></tr>
</table>

Furthermore, if $a_m(t)$ has null integral over the interval $[0, T]$ (i.e. $\int_0^T a_m(t)dt \neq \emptyset$), then Eq. (2.1.6) has at most one positive limit cycle.

**Numerical Example 3:**

In this example let's take the case where $a_{n_1}(t)$ has positive definite sign and $a_{n_3}(t)$ has negative definite sign.

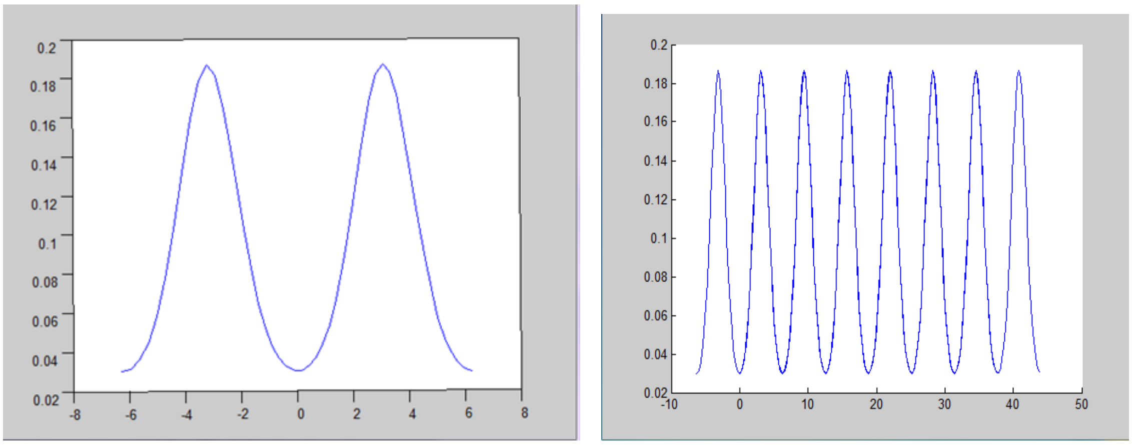
For instance, let:

$n_1 = 5,\quad n_2 = 4,\quad n_3 = 2,\quad m = 1,\quad a_{n_1}(t) = a_{n_2}(t) = a_m(t) = sin(t),$ $a_{n_3}(t) = -\sin(t)$. With initial condition: $u(1) = 0.03$.

Then, the equation will be:

$$u' = \sin(t)\, u^5 + \sin(t)\, u^4 - \sin(t)\, u^2 + \sin(t)\, u$$

which has one positive limit cycle on the interval $[-2\pi, 2\pi]$, (see Figure 2.4).




(a) (b)

**Figure 2.4:** Limit cycle of equation:

$u' = \sin(t)\, u^5 + \sin(t)\, u^4 - \sin(t)\, u^2 + \sin(t)\, u.$

But, if we try different initial condition on the same interval, the results came to be a non-limit cycle. For example: $u(1) = 0.25$, (see Figure 2.5).

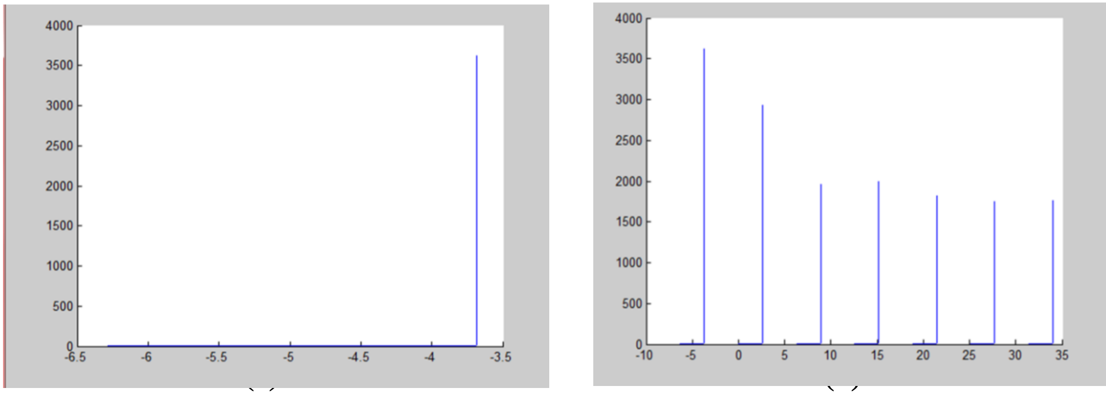



**Figure 2.5:** No limit cycle of equation:

$u' = \sin(t)\, u^5 + \sin(t)\, u^4 - \sin(t)\, u^2 + \sin(t)\, u.$

***Theorem 2.3 [2]:*** *For Eq.(2.1.6), consider that $a_{n_1}(t)$ has a definite sign and $n_1, n_2, n_{3,} m \in \mathbb{Z}$ in which $n_1 > n_2 > n_3$ verifying the condition:*

$$n_1 - 2n_2 + n_3 = 0\,. \qquad (2.1.7)$$

*If*

$$\triangle = {a_{n_2}}^2 (m - n_2)^2 - 4a_{n_1} a_{n_3} (m - n_1)(m - n_3) \leq 0 \qquad (2.1.8)$$

*then, Eq.(2.1.6) has at most one positive limit cycle.*

The next proposition will be useful to prove Theorem (2.3).

***Proposition 2.1 [2]:*** *Consider the general first order equation*

$$u' = g(t, u), \qquad (2.1.9)$$

*where $g$ is continuous and T-periodic in T. Let $J$ be an open interval and $g(t,u)$ has continuous partial derivatives $\frac{\partial^k}{\partial u^k} g(t,u)$, $k = \{1,2,3\}$ for all $(t,u) \in [0,T] \times J$. If $\frac{\partial^k}{\partial u^k} g(t,u) \geq 0$ or $\frac{\partial^k}{\partial u^k} g(t,u) \leq 0$ for all $(t,u) \in [0,T] \times J$, then Eq. (2.1.9) has at most $k$ limit cycles with range containing $J$.*

**Proof of theorem 2.3**

**Case 1: m = 1.**

Without loss of generality, let us consider $a_{n_1} \succ 0$, by changing the independent variables $\tau = -t$.

Following reference [2] with more simplification, Eq. (2.1.6) can be written as:

$$u' = uQ(t,u),$$

where:

$$Q(t,u) = a_{n_1}u^{n_1-1} + a_{n_2}u^{n_2-1} + a_{n_3}u^{n_3-1} + a_1(t)$$

using the change of variable technique $u = e^x,\ u' = x'e^x$.

From Eq. (2.1.9): $x' = g(t,x)$ then,

$$u'(x) = e^x g(t,x)$$

$$e^x x' = e^x Q(t,e^x)$$

$$x' = Q(t,e^x) = g(t,x)$$

Since $g_x(t,x) = Q_x(t,e^x)e^x$ then,

$g_x(t,x) =$

$$e^x\big(a_{n_1}(n_1-1)e^{x(n_1-1)} + a_{n_2}(n_2-1)e^{x(n_2-1)} + a_{n_3}(n_3-1)e^{x(n_3-1)}\big)$$

$$= e^{x(n_3-1)}\big(a_{n_1}(n_1-1)e^{x(n_1-n_3)} + a_{n_2}(n_2-1)e^{x(n_2-n_3)} + a_{n_3}(n_3-1)\big)$$

From equation (2.1.7):

$$n_2 - n_3 = n_1 - n_2$$

If we denote $R = e^{(n_1-n_2)x}$ then $R^2 = e^{(n_1-n_3)x}$, so $g_x(t,x)$ can be written as:

$$e^{x(n_3-1)}\left(a_{n_1}(n_1-1)R^2 + a_{n_2}(n_2-1)R + a_{n_3}(n_3-1)\right)$$

By hypothesis (2.1.8) the last factor is a quadratic polynomial with negative discriminant. Thus, by previous proposition (2.1) there exists at most one positive limit cycle.

**Case2: m ≠ 1**, rewrite Eq.(2.1.6) as:

$$u' = u^m Q(t,u),$$

where:

$$Q(t,u) = a_{n_1}(t)u^{n_1-m} + a_{n_2}(t)u^{n_2-m} + a_{n_3}(t)u^{n_3-m} + a_m(t)$$

In this case, the change of variable technique is well-defined for positive solutions and keeps the number of positive limit cycles.

Using the change of variable: $u = x^\alpha$ which satisfies:

$$(m-1)\alpha + 1 = 0$$

Hence,

$$u' = \alpha x^{\alpha-1}x'$$

$$u' = \alpha x^{\alpha-1}g(t,x)$$

$$u^m Q(t,x^\alpha) = \alpha x^{\alpha-1}g(t,x)$$

$$x^{\alpha m}Q(t,x^\alpha) = \alpha x^{\alpha-1}g(t,x)$$

$$g(t,x) = \frac{1}{\alpha}Q(t,x^\alpha)x^{\alpha(m-1)+1}$$

Thus,

$$g(t,x) = \frac{1}{\alpha} Q(t,x^{\alpha})$$

$$g_x(t,x) = x^{\alpha-1} Q_x(t,x^{\alpha})$$

$$= x^{\alpha-1}\left[\alpha\left((n_1-m)\,a_{n_1}x^{\alpha(n_1-m)-1} + (n_2-m)\,a_{n_2}x^{\alpha(n_2-m)-1} + (n_3-m)\,a_{n_3}x^{\alpha(n_3-m)-1}\right)\right]$$

$$=\left[\alpha\left((n_1-m)\,a_{n_1}x^{\alpha(n_1-m+1)-2} + (n_2-m)\,a_{n_2}x^{\alpha(n_2-m+1)-2} + (n_3-m)\,a_{n_3}x^{\alpha(n_3-m+1)-2}\right)\right]$$

$$= \alpha x^{(n_3-m+1)\alpha-2}[(n_1-m)\,a_{n_1}x^{\alpha(n_1-n_3)} + (n_2-m)\,a_{n_2}x^{\alpha(n_2-n_3)} + (n_3-m)\,a_{n_3}]$$

If we denote $S = x^{\alpha(n_1-n_2)}$ then $S^2 = x^{\alpha(n_1-n_3)}$, and by using Eq.(2.1.7) giving:

$$g_x(t,x) = \alpha x^{(n_3-m+1)\alpha-2}[(n_1-m)\,a_{n_1}S^2 + (n_2-m)\,a_{n_2}S + (n_3-m)\,a_{n_3}]$$

Here, by hypothesis (2.1.8), the last factor is a quadratic polynomial with negative discriminant. Therefore, there exists at most one positive limit cycle by proposition (2.1).

**Numerical Example 4:**

This example will cover the case where $m = 1$.

Let: $n_1 = 5$, $n_2 = 4$, $n_3 = 3$, $a_{n_1}(t) = a_{n_2}(t) = a_{n_3}(t) = a_m(t) = \sin(t)$.

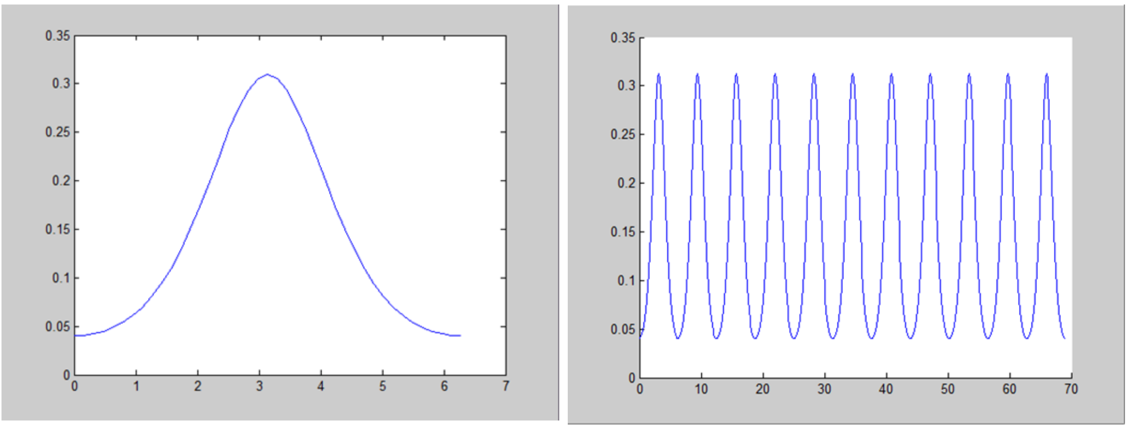
with initial condition $u(0) = 0.04$.

$$\Delta = -23sin^2(t) \leq 0$$

Since the conditions (2.1.7, 2.1.8) hold, then the equation will be:

$$u' = sin(t)u^5 + sin(t)u^4 + sin(t)u^3 + sin(t)u$$

which has one positive limit cycle on the interval $[0,2\pi]$, (see Figure 2.6).

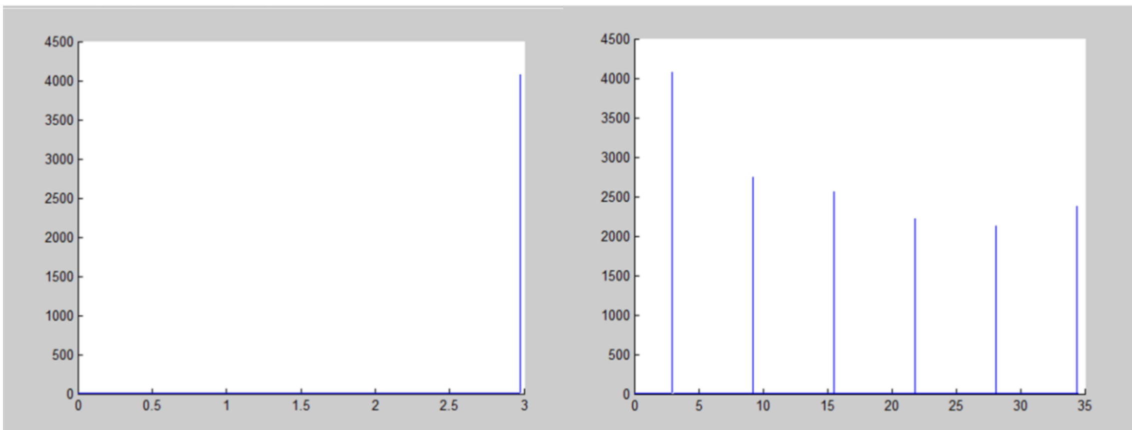



(a) (b)

**Figure 2.6:** Limit cycle of equation:

$$u' = sin(t)u^5 + sin(t)u^4 + sin(t)u^3 + sin(t)u.$$

But, if we try different initial condition on the same interval, the results came to be a non-limit cycle. For example: $u(0) = 0.09$, (see Figure 2.7).




(a) (b)

**Figure 2.7:** No limit cycle of equation:

$$u' = sin(t)u^5 + sin(t)u^4 + sin(t)u^3 + sin(t)u.$$

**Numerical Example 5:**

This example will cover the case where $m \neq 1$.

Let: $n_1 = 3, n_2 = 2, n_3 = 1, m = 0,$

$$a_{n_1(t)} = a_{n_2}(t) = a_{n_3}(t) = a_m(t) = sin(t)$$

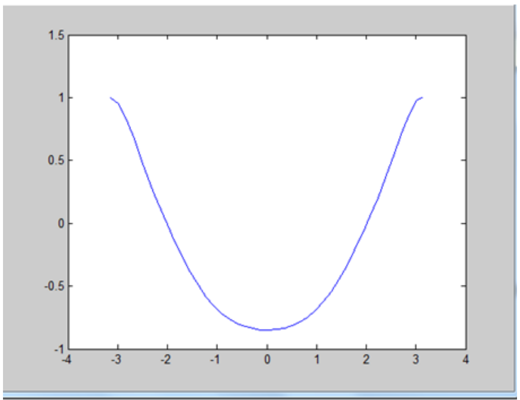
with initial condition: $u(0) = 1$.

$$\Delta = -8sin^2(t) \leq 0$$

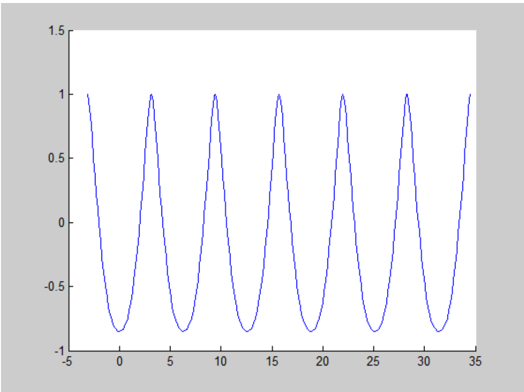
Then, the equation will be:

$$u' = sin(\mathrm{t})u + sin(\mathrm{t})u^2 + sin(\mathrm{t})u^3 + sin(t)$$

which has one limit cycle on the interval $[-\pi, \pi]$, (see Figure 2.8).




(a)




(b)

**Figure 2.8:** Limit cycle of equation:

$u' = \sin(t)\,u + \sin(t)\,u^2 + \sin(t)\,u^3 + \sin(t).$

But, if we try a different initial condition on the same interval, the results came to be a non-limit cycle. For example: $u(0) = -1$, ( see Figure 2.9).

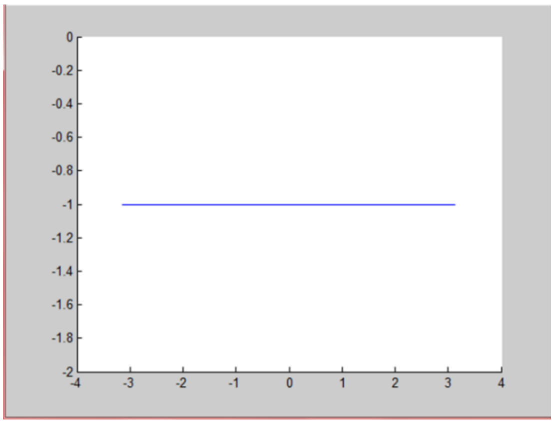



**Figure 2.9:** No limit cycle of equation:

$u' = \sin(t)\,u + \sin(t)\,u^2 + \sin(t)\,u^3 + \sin(t)$.

### 2.1.2 Comparison Between Theorems and Corollaries which has at Most One Positive Limit Cycle

In this section, a complementary results of theorem 2.3 are introduced while in corollary 3 a comparison with theorem 2.2 is done.

***Corollary 1 [2]:*** *Assume that $a_{n_1}(t)\,, a_{n_2}(t)$ have the same definite sign, and also $n_3 < n_2 < n_1$ satisfying the condition (2.1.7). If:*

$$4a_{n_1}(t)a_{n_3}(t) > {a_{n_2}}^2(t)$$

*then, for all $t$, there exists $m_0 > 0$ such that if $m_0 < |m|$ then Eq.(2.1.6) has at most one positive limit cycle.*

**Numerical Example 6:**

Let : $n_1 = 4, n_2 = 3, n_3 = 2,\ a_{n_i}(t) = \cos(t)\,, \forall i = 1,2,3,\ a_m = \cos(t)$.

with initial condition $u(0) = -0.6$.

To find $m_0$, first solve Eq.(2.1.8)

$${a_{n_2}}^2(m - n_2)^2 - 4a_{n_1}a_{n_3}(m - n_1)(m - n_3) \leq 0$$

$$\Delta = (m-3)^2 - 4(m-4)(m-2) \leq 0$$

$$\Delta = m^2 - 6m + 9 - 4m^2 + 24m - 32 \leq 0$$

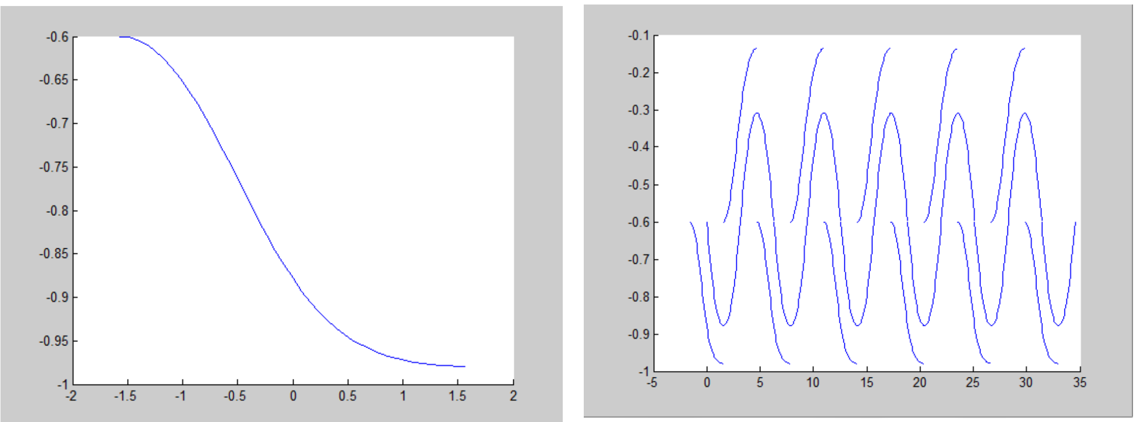
$$\Delta = -3m^2 + 18m - 23 \leq 0$$

which has two roots: $r_1 = 1.84$, $r_2 = 4.15$.

Hence, $m \in \mathbb{R}/(1.84, 4.15)$, so that $m_0$ can be any positive number such that $m_0 < |m|$. If assumed $m = 1$, then $m_0 \in (0,1)$ and the equation will be:

$$u' = cos(\mathrm{t})u^4 + cos(\mathrm{t})u^3 + cos(\mathrm{t})u^2 + cos(\mathrm{t})$$

which has one negative limit cycle on the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, (see Figure 2.10).

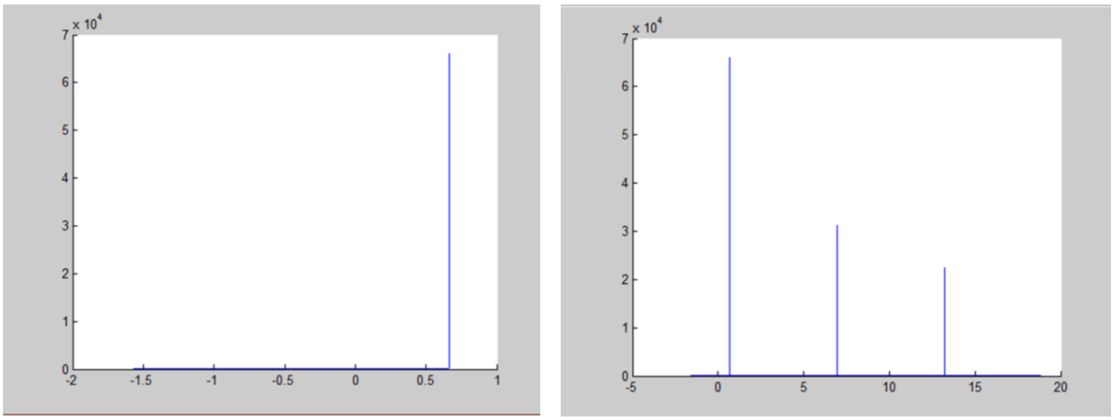



(a) (b)

**Figure 2.10:** Limit cycle of equation:

$u' = cos(\mathrm{t})u^4 + cos(\mathrm{t})u^3 + cos(\mathrm{t})u^2 + cos(\mathrm{t})$.

But, if we try different initial condition on the same interval, the results came to be a non-limit cycle. For example: $u(0) = 0.1$, ( see Figure 2.11).




(a) (b)

**Figure 2.11:** No limit cycle of equation:

$u' = cos(\mathrm{t})u^4 + cos(\mathrm{t})u^3 + cos(\mathrm{t})u^2 + cos(\mathrm{t})$.

***Corollary 2 [2]:*** *Consider $a_{n_3}(t)$ and $a_{n_1}(t)$ have the same definite sign and $n_1 > n_2 > n_3 > m = 1$ which is satisfying the condition (2.1.7). If*

$$\frac{(n_2-1)^2}{4(n_1-1)(n_3-1)} {a_{n_2}}^2(t) \le a_{n_1}(t)a_{n_3}(t),$$

*for all $t$. Then Eq.(2.1.6) has at most one positive limit cycle.*

**Numerical Example 7 :**

Let: $n_1 = 5, n_2 = 4, n_3 = 3,\ m = 1,$

which satisfy condition (2.1.7).

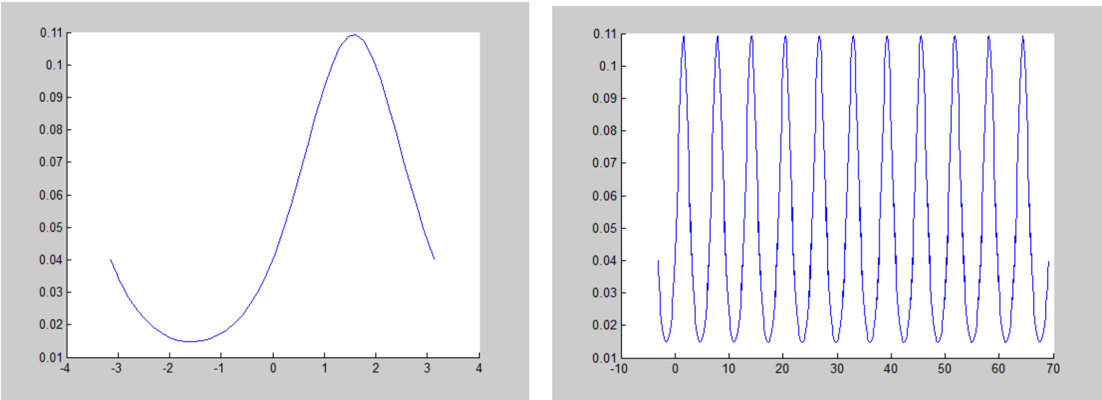
$a_{n_i}(t) = a_m(t) = cos(\mathrm{t}), \forall i = 1, 2, 3,$ with initial condition: $u(0) = 0.04,$

on the interval: $[-\pi, \pi]$.

Then, the equation will be:

$$u' = cos(t)u^5 + cos(\mathrm{t})u^4 + cos(\mathrm{t})u^3 + cos(\mathrm{t})u$$

which has one positive limit cycle on the interval: $[-\pi, \pi]$, (see Figure 2.12).

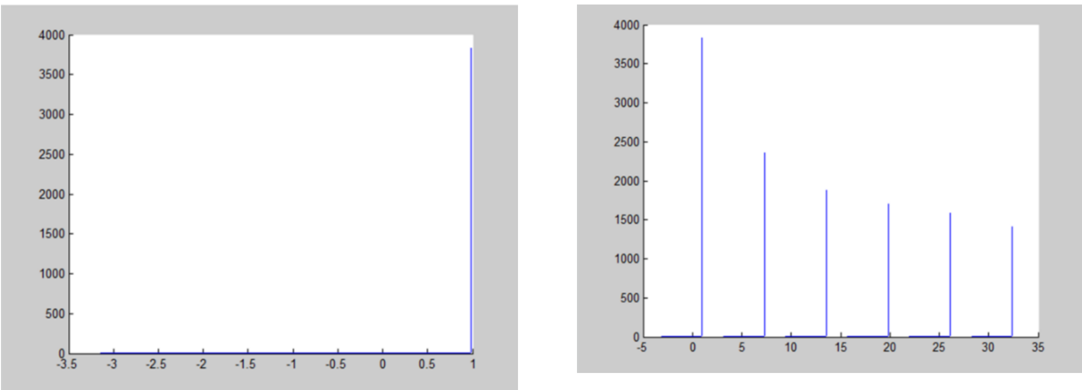



(a) (b)

**Figure 2.12:** Limit cycle of equation:

$$u' = cos(t)u^5 + cos(t)u^4 + cos(t)u^3 + cos(t)u.$$

But, if we try different initial condition on the same interval, the results came to be a non-limit cycle. For example: $u(0) = 0.1$, (see Figure 2.13).




(a) (b)

**Figure 2.13:** No limit cycle of equation:

$$u' = cos(t)u^5 + cos(t)u^4 + cos(t)u^3 + cos(t)u.$$

A comparison between the next corollary and Theorem (2.2) is done.

***Corollary 3 [2]*** *: Let $n_1 > n_2 > n_3$ and satisfy the condition (2.1.7). Assume that $a_{n_1}(t)$, $a_{n_3}(t)$ have opposite definite signs, then Eq.(2.1.4) has at most two non-trivial limit cycles; at most one negative and at most one positive.*

It can be noted that this corollary does not include the case where $a_{n_1}(t)$ and $a_{n_3}(t)$ have the same definite signs.

**Numerical Example 8:**

Let: $n_1 = 3, n_2 = 2, n_3 = 1,$

which are verifying the condition (2.7).

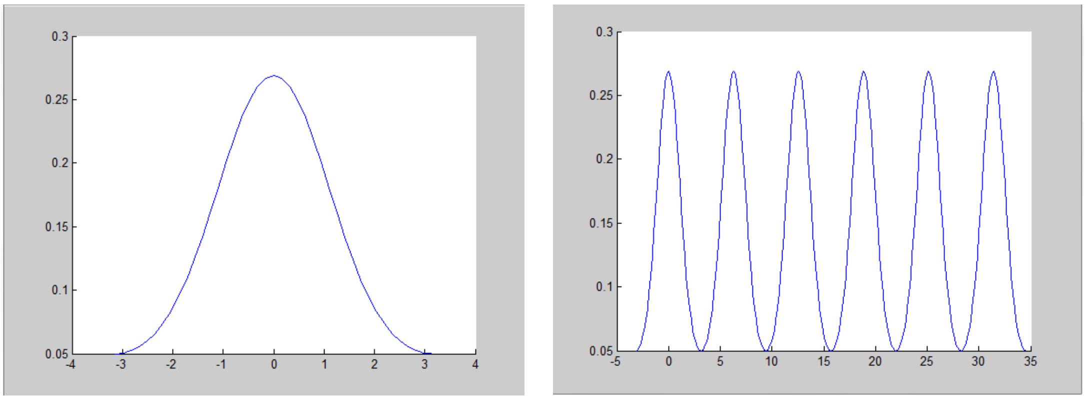
$$a_{n_1} = a_{n_2} = cos(t), a_{n_3} = -cos(t)$$

with initial condition: $u(0) = 0.05$ on the interval: $[-\pi, \pi]$.

Then, the equation will be:

$$u' = sin(t)u^3 + sin(t)u^2 - sin(t)u$$

which has one positive limit cycle on the interval: $[-\pi, \pi]$, (see Figure 2.14).

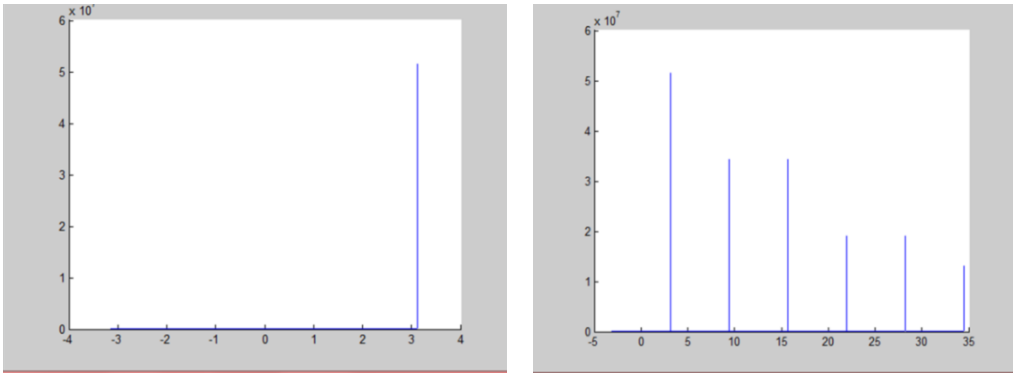



(a) (b)

**Figure 2.14:** Limit cycle of equation:

$u' = sin(t)u^3 + sin(\,t)u^2 - sin(t)\,u.$

But, if we try different initial condition on the same interval, the results came to be a non-limit cycle. For example: $u(0) = 10$, (see Figure 2.15).



(a) (b)

**Figure 2.15:** No limit cycle of equation:

$u' = sin(t)u^3 + sin(t)u^2 - sin(t)\,u.$

## 2.2. Limit Cycles of the Complete $4^{th}$-Order Equation

Our purpose of this section is to present some conditions which are used to limit the number of isolated periodic solutions of the (4,3,2,1,0)-polynomial equation which has the form:

$$u' = a_4(t)u^4 + a_3(t)u^3 + a_2(t)u^2 + a_1(t)u + a_0(t) \qquad (2.2.1)$$

Gasull and Guillamon proved that Eq.(2.2.1) with $a_4(t) \equiv 1$ might have an arbitrary number of $T$-periodic solutions. Also when $a_0(t) \equiv 0$ Alvarezand and others proved that if $a_3(t), a_4(t) \succ 0$ or $a_2(t), a_3(t) \succ 0$ or $a_4(t) \succ 0 \succ a_2(t)$, then Eq.(2.2.1) has at most two positive limit cycles [2].

***Theorem 2.4 [2]:*** *If $a_3(t)^2 - \frac{8}{3}a_4(t)a_2(t) \leq 0$ and $a_2(t), a_4(t) \succ 0$, then Eq.(2.2.1) has at most two limit cycles.*

**Proof:**

Take the second derivative of the right hand side of Eq. (2.2.1) gives:

$$12a_4(t)u^2 + 6a_3(t)u + 2a_2(t)$$

which is a second order polynomial. Now the discriminant equals:

$$36a_3(t)^2 - 96a_4(t)a_2(t)$$

which is negative by the hypothesis of the theorem. Thus, by proposition (2.1), there exists at most two limit cycles.

**Numerical Example 9:**

Let: $a_i(t) = cos(t), \forall i = 1, 2, 4$ , $a_i(t) = 0, \forall i = 0, 3$, on the interval: $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, with initial condition: $u(0) = -1$.

Then, the equation will be:

$$u' = cos(\text{t})u^4 + cos(\text{t})u^2 + cos(\text{t})u \qquad (2.2.2)$$

which has one negative limit cycle on the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, (see Figure 2.16).

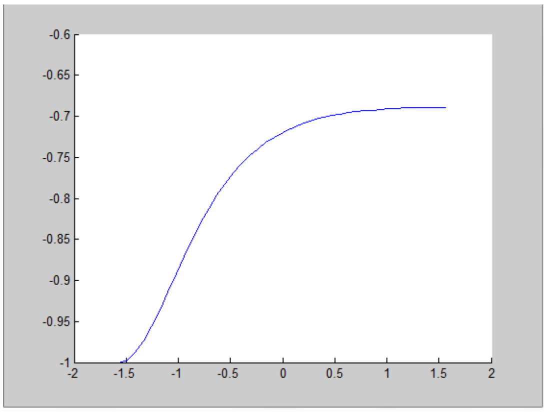
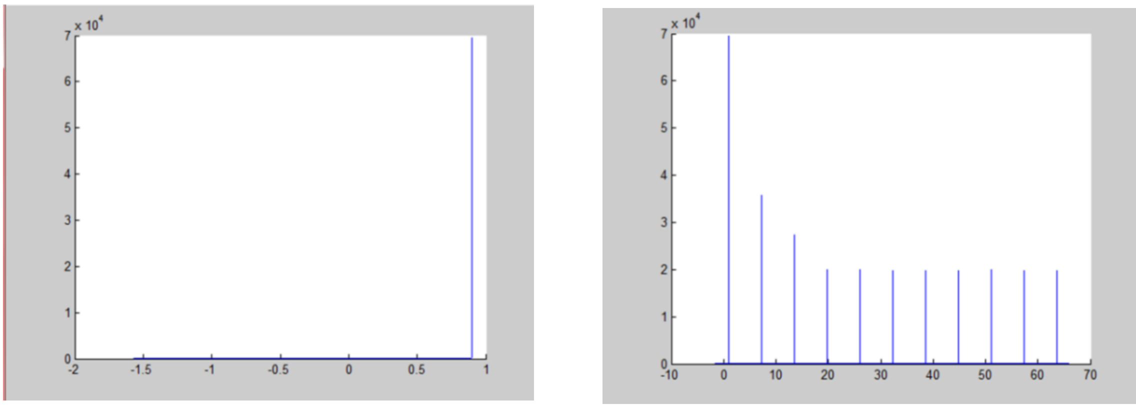



**Figure 2.16:** Limit cycle of equation:

$u' = cos(\text{t})u^4 + cos(\text{t})u^2 + cos(\text{t})u.$

But, if we try different initial condition on the same interval, the results came to be a non-limit cycle. For example: $u(0) = 0.1$, (see Figure 2.17).




(a) (b)

**Figure 2.17:** No limit cycle of equation:

$$u' = cos(\mathrm{t})u^4 + cos(\mathrm{t})u^2 + cos(\mathrm{t})u.$$

By using Maple software to draw the phase portrait of Eq.(2.2.2), (see Figure 2.18).

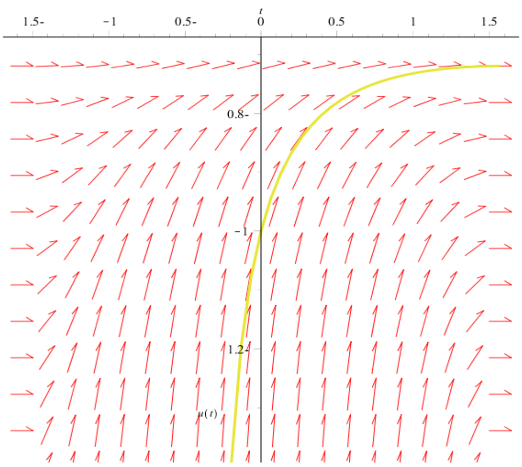



**Figure 2.18:** Phase portrait of equation:

$$\boldsymbol{u' = cos(\mathrm{t})u^4 + cos(\mathrm{t})u^2 + cos(\mathrm{t})u.}$$

***Definition 2.4 [1]:*** *Let $\propto$ be a $T$-periodic function,$\propto$ is called a strict lower solution of Eq. (2.6), if*

$$\propto' (t) < g(t, \propto (t))$$

*And,* $\propto$ *is called a strict upper solution of Eq. (2.6), if*

$$\propto'(t) > g(t, \propto(t))$$

*for all* $t$. (see Appendix II).

***Theorem 2.5 [2]:*** *Consider that* $a_0(t)a_4(t) > 0, \forall t$. *If*

$$4\sqrt[4]{a_0(t)a_4(t)^3} + a_3(t) \geq 0,$$

*then Eq.(2.12) has at most two positive limit cycles.*

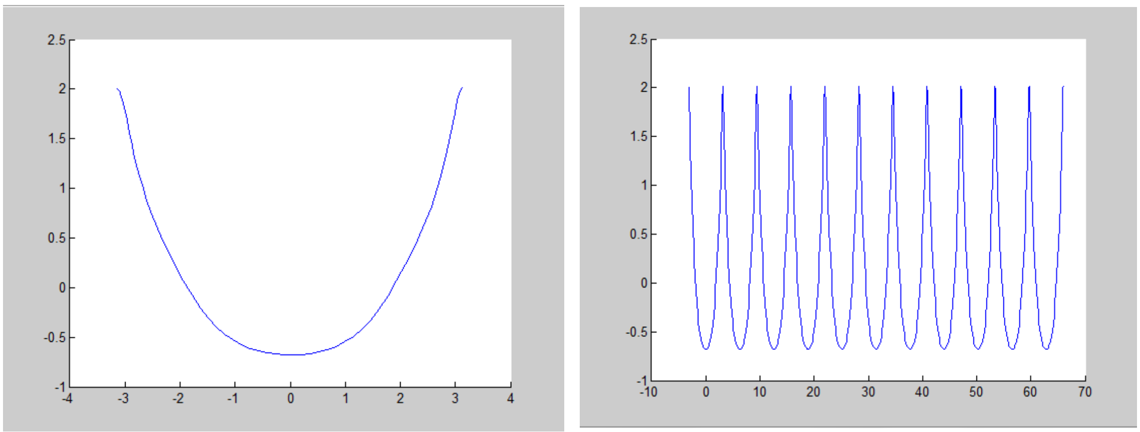
**Numerical Example 10:**

Let: $a_i(t) = sin(\mathrm{t}), \forall i = 0, 1, 3, 4,$ $a_2(t) = 0,$ on the interval: $[-\pi, \pi]$, with initial condition : $u(0) = 2$.

Then, the equation will be:

$$u' = sin(\mathrm{t})u^4 + sin(\mathrm{t})u^3 + sin(\mathrm{t})u + sin(\mathrm{t}) \qquad (2.2.3)$$

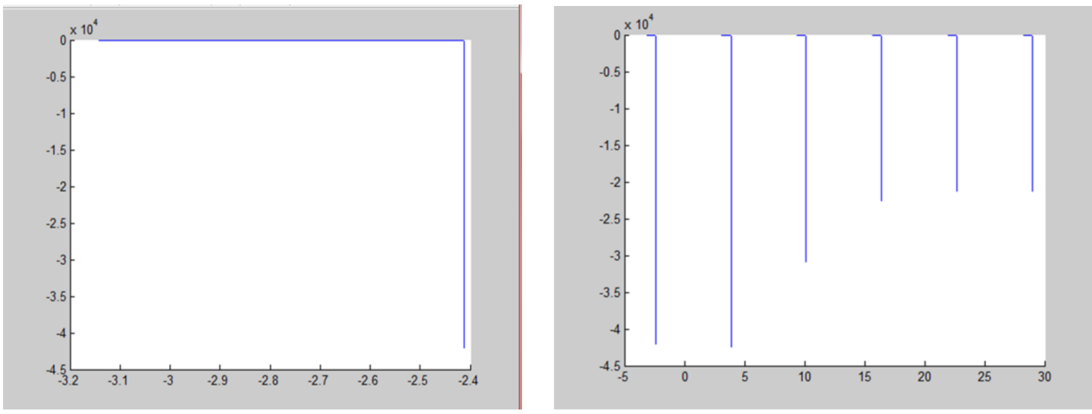
which has one limit cycle on $[-\pi, \pi]$, (see Figure 2.19).




(a) (b)

**Figure 2.19:** Limit cycle of equation:
$u' = sin(\mathrm{t})u^4 + sin(\mathrm{t})u^3 + sin(\mathrm{t})u + sin(\mathrm{t})$.

But, if we try different initial condition on the same interval, the results came to be a non-limit cycle. For example: $u(0) = -1.5$, ( see Figure 2.20).



(a) (b)

**Figure 2.20:** No limit cycle of equation:

$u' = sin(\text{t})u^4 + sin(\text{t})u^3 + sin(\text{t})u + sin(\text{t})$.

By using Maple software to draw the phase portrait of Eq.(2.2.3), see Figure 2.21.

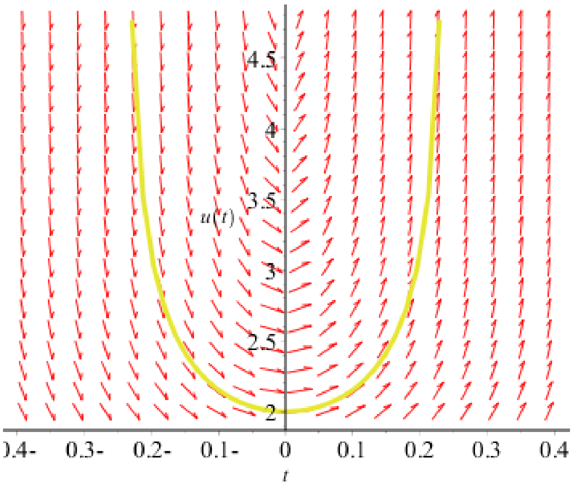



**Figure 2.21:** phase portrait of equation:

$u' = sin(\text{t})u^4 + sin(\text{t})u^3 + sin(\text{t})u + sin(\text{t})$.

# Chapter Three

# Limit Cycles of Polynomial Planar System

This chapter introduces a planar polynomial vector field and simple family of polynomial planar system which is called rigid system. Also, presents theorems which deals with rigid system including numerical examples.

Moreover, exhibits the Poincaré map and multiplicity of limit cycle of planar differential system. Also, it presents the multi-parameter polynomials differential system with relevant theorem that was proved numerically the existence of explicit limit cycle.

At the end of this chapter, system in the cylinder, rigid system, and corollary which is giving an upper bound for the number of non-contractible limit cycles for system in cylinder including numerical example are presented.

Abel differential equation has been used to study the maximum number of isolated periodic solutions (limit cycles) of autonomous planar vector fields. Studying the limit cycles of planar polynomial systems can be reduced to the study of Eq. (2.1).

## 3.1 Introduction

During the last century, the study of the limit cycles of polynomial differential equations was a major challenge for researches. An important problem in qualitative theory of differential equations is to determine the number of limit cycles of planar differential systems (planar vector field) which has the form:

$$\begin{cases} \frac{dx}{dt} = x' = P(x,y) \\ \frac{dy}{dt} = y' = Q(x,y) \end{cases}, \tag{3.1.1}$$

where $P(x,y)$, and $Q(x,y)$ are coprime polynomials in $x$ and $y$ with real coefficients (i.e. $P(x,y)$ and $Q(x,y)$ has no common roots), such that the dependent variables $x$ and $y \in \mathbb{R}$. The degree of system (3.1.1) is $n = max\{\deg(P), \deg(Q)\}$. An isolated periodic solution in the set of all periodic solutions of system (3.1.1) is called limit cycle.

For example consider the cubic polynomial planar system:

$$\begin{cases} x' = 20y + 2xy^2 \\ y' = y + 20x - 2x^2y - 20x^3 + 4y^3 \end{cases}$$

in which, the system limit cycles are sketched in Figure 3.1 [21].

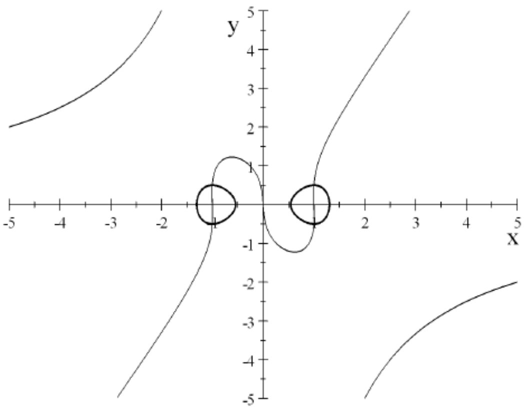



**Figure 3.1:** Limit cycles of the system:

$$\begin{cases} x' = 20y + 2xy^2 \\ y' = y + 20x - 2x^2y - 20x^3 + 4y^3. \end{cases}$$

### 3.1.1. Rigid System

One of the most simple family of polynomial planar system, in which the derivative of the angular variable is constant, is called rigid system:

$$\begin{cases} x' = -y + xf(x,y) \\ y' = x + yf(x,y)\ , \end{cases}$$

where $f(x,y)$ is an arbitrary polynomial function [10].

Now, if we consider the rigid system of the form:

$$\begin{cases} x' = -y + x[R_{n-3}(x,y) + R_{n-2}(x,y) + R_{n-1}(x,y)] \\ y' = x + y[R_{n-3}(x,y) + R_{n-2}(x,y) + R_{n-1}(x,y)], \end{cases} \tag{3.1.2}$$

where $R_n$ is a homogenous polynomial of degree $n$. Rewriting the system by using the polar coordinate (i.e. $x = rcos\theta, y = rsin\theta$) [4], then:

$$\begin{cases} r' = r^{n-2}R_{n-3}(cos\theta, sin\theta) + \ r^{n-1}R_{n-2}(cos\theta, sin\theta) + r^{n}R_{n-1}(cos\theta, sin\theta) \\ \theta' = 1 \end{cases} \tag{3.1.3}$$

Hence,

$$\frac{dr}{d\theta} = r^{n-2}R_{n-3}(cos\theta, sin\theta) + \ r^{n-1}R_{n-2}(cos\theta, sin\theta) +$$

$$r^{n}R_{n-1}(cos\theta, sin\theta) \tag{3.1.4}$$

***Theorem 3.1 [4]:*** *Consider the system (3.1.2), assume that* $n$ *is odd,* $R_{n-3} \leq 0$, $R_{n-1} \equiv 1$ *and the function* $B = \int_0^{2\pi} R_{n-3}(cos\theta,\ sin\theta\ )\ d\theta$. *Then:*

1- *If* $B = 0$, *then system (3.1.4) does not have any limit cycle.*

2- *While system (3.1.4) has at most one limit cycle if* $B < 0$.

The next remark presents a condition to reduce a polar equation to an equation with leading coefficient equal to one.

***Remark 3.1 [4]:*** *If the leading coefficient $R_{n-1}$ dose not vanish, then the transformation of the independent variable given by:*

$$\theta \to exp(\int_0^\theta R_{n-1}(cost, sint)\,dt) \qquad (3.1.5)$$

*will reduce the polar equation to identical equation with a leading coefficient equals one.*

**Numerical Example 1:**

***Case one in Theorem 3.1: $B = 0$.***

Let: $n = 3$, on the interval $[0, 2\pi]$, the leading coefficient is $R_{n-1} = R_2 = sin\theta cos\theta$.

$R_{n-2} = R_1 = sin\theta + 2cos\theta$, $R_{n-3} = R_0 = 0$.

$$B = \int_0^{2\pi} 0\,d\theta = 0$$

with initial condition $r(0) = 1$.

First, reduce the leading coefficient $R_2$ to an equation with leading coefficient equals one using Eq.(3.1.5).

$$\theta \to \exp(\int_0^\theta sint\ cost\ dt) = \exp(\frac{1}{2} sin^2\theta)$$

hence, the differential equation will be:

$$r' = \exp\left(\frac{1}{2}sin^2\theta\right)r^3 + (sin\theta + 2cos\theta)\,r^2$$

which has no limit cycle, (see Figure 3.2).

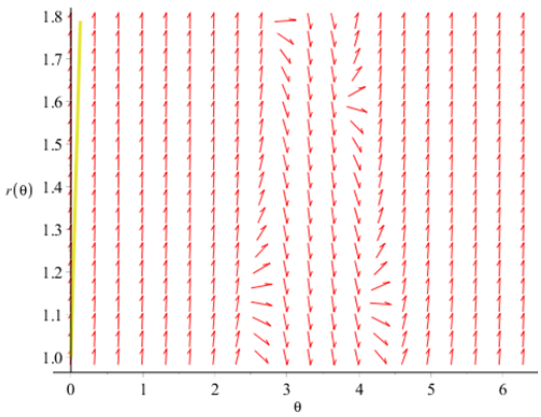



**Figure 3.2:** No limit cycle of the equation:

$$r' = \exp\left(\frac{1}{2}sin^2\theta\right)r^3 + (sin\theta + 2cos\theta)r^2.$$

**Case two in Theorem 3.1: $B < 0$.**

Let: $n = 3$, on the interval $[0, 2\pi]$, the leading coefficient is $R_{n-1} = R_2 = sin\theta cos\theta$.

$$R_{n-2} = R_1 = sin\theta + cos\theta$$

$$R_{n-3} = R_0 = -\frac{1}{2}$$

$$B = \int_0^{2\pi} \frac{-1}{2}\, d\theta = -\pi < 0$$

with initial condition $r(0) = 1$.

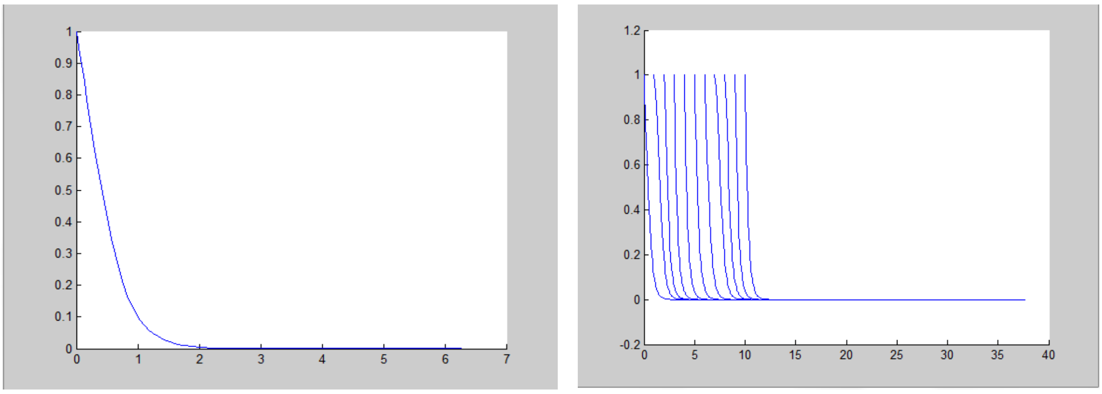
First, reduce the leading coefficient $R_2$ to an equation with the leading coefficient equals one, using Eq.(3.1.5).

$$\theta \to \exp\left(\int_0^{\theta} \sin t \cos t \ dt\right) = \exp\left(\frac{1}{2}sin^2\theta\right)$$

Hence, the differential equation will be:

$$r' = \exp\left(\frac{1}{2}sin^2\theta\right)r^3 + (sin\theta + cos\theta)r^2 - \pi r$$

which has one positive limit cycle, (see Figure 3.3).

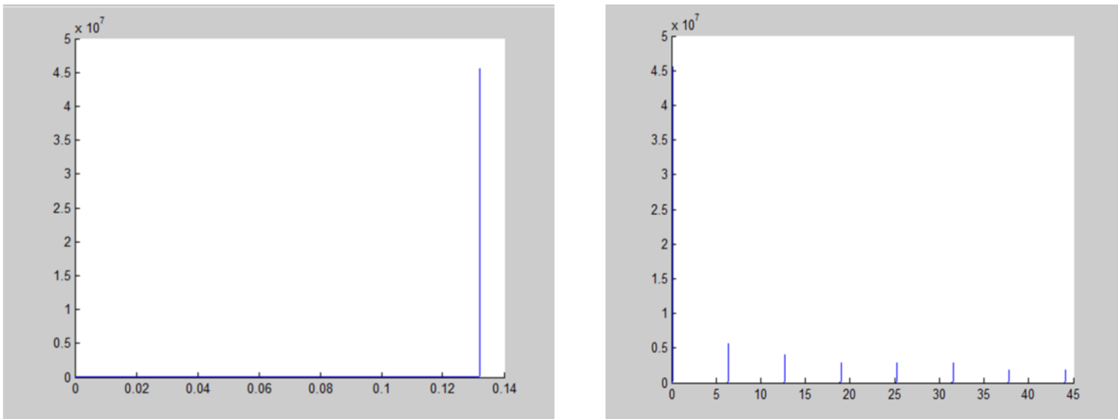



(a) (b)

**Figure 3.3:** Limit cycle of the equation:

$$r' = \exp\left(\frac{1}{2}sin^2\theta\right)r^3 + (sin\theta + cos\theta)r^2 - \pi r.$$

But, if we try different initial condition on the same interval, the results came to be a non-limit cycle. For example: $u(0) = 2$, (see Fig 3.4).




(a) (b)

**Figure 3.4:** No limit cycle of the equation:

$$r' = \exp\left(\frac{1}{2}sin^2\theta\right)r^3 + (sin\theta + cos\theta)r^2 - \pi r.$$

### 3.1.2. Multiplicity of Limit Cycles for Planar Differential System

In this section, two ideas will be exhibited; the first one is **Poincaré map**, and the second one is **the multiplicity of limit cycles of planar differential system.**

Let us assume a planar differential system

$$\begin{cases} x' = P(x,y) \\ y' = Q(x,y) \end{cases}, \tag{3.1.6}$$

where both of $P(x,y)$ and $Q(x,y)$ are real analytic functions defined in some nonempty open set $U \subseteq \mathbb{R}^2$. Suppose that $\Gamma$ be a limit cycle for system (3.1.6) such that: $\Gamma = \{\gamma(t): 0 \leq t < T\}$.

Consider $p_0 \in \Gamma$ and a section $\Sigma$ through it. Since the limit cycle $\Gamma$ is periodic orbit, for any point $q$ on $\Sigma$, the solution of system (3.1.6) starting at $q$ cuts the section $\Sigma$ again in another point for some positive time.

If we denoted $P(q)$ the point corresponding to the first intersection of the solution for system (3.1.6) starting in $q$. Also notice that $P(p_0) = p_0$ is a fixed point. Hence, the function $P: \Sigma \to \Sigma$ is called the ***Poincaré map*** for a limit cycle $\Gamma$ at a point $p_0$. The Poincaré map controls the stability of a limit cycle as mentioned in chapter one; a limit cycle can be either stable, semi-stable or unstable.

To clarify this, suppose $P$ is the identity and $P'(p_0) \neq 1$, hence it can be said that a limit cycle $\Gamma$ is hyperbolic or of multiplicity one. If $P'(p_0) < 1$ then $\Gamma$ is stable while $\Gamma$ is unstable if $P'(p_0) > 1$.

In case $P'(p_0) = 1$ and $P$ is not the identity then, there exist an integer $m$ which is greater than one such that $P^m(p_0) \neq 0$, where the limit cycle $\Gamma$ is of multiplicity $m$ [20].

The following table summarizes this case (Table 3.1).

**Table 3.1: Stability analysis of the Poincaré map**

| $m$ | $P^m(p_0) < 0$ | $P^m(p_0) > 0$ |
|---|---|---|
| Odd | $\Gamma$ is stable | $\Gamma$ is unstable |
| Even | $\Gamma$ is semi-stable | |

The next example will demonstrate the Poincaré map.

**Example 2:** Suppose the vector field given in polar coordinate by:

$$r' = r(1 - r^2)$$
$$\theta' = 1$$

Show that the system has a unique periodic orbit and classify its stability.

***Note:*** Assume $\Sigma$ to be the positive $x$ axis i.e. $\{ \Sigma: (x, 0); x \in \mathbb{R}, x > 0\}$.

**Solution:**

As shown in chapter one, the system has a fixed point at $r = 0$ and a limit cycle at $r = 1$. Starting at a point $x_0 = x$ it will return to $\Sigma$ after $2\pi$, then $x_1 = P(x)$. It starts at point $(x, 0)$ and ends at point $(P(x), 0)$, hence

$$\Delta t = 2\pi = \int_{x_0}^{x_1} \frac{dt}{dr}\, dr$$

Then,

$$2\pi = \int_{x}^{P(x)} \frac{dr}{r(1-r^2)}$$

Using partial fraction to evaluate the integration gives:

$$\frac{1}{r(1-r^2)} = \frac{A}{r} + \frac{B}{1+r} + \frac{C}{1-r}$$

$$A(1+r)(1-r) + Br(1-r) + Cr(1+r) = 1$$

When $r = 0 \rightarrow A = 1, r = 1 \rightarrow C = \frac{1}{2}, r = -1 \rightarrow B = -\frac{1}{2}$.

Thus,

$$2\pi = \int_{x}^{P(x)} \frac{1}{r} + \frac{-1}{2(1+r)} + \frac{-1}{2(1-r)}\, dr$$

$$2\pi = (\ln|r| - \frac{1}{2}\ln|r+1| - \frac{1}{2}\ln|1-r|)|_{x}^{P(x)}$$

Multiplying both sides by 2 and then take the exponential for both sides gives:

$$e^{4\pi} = \frac{P^2(x)}{x^2} * \frac{1+x}{1+P(x)} * \frac{1-x}{1-P(x)}$$

Rearrange the equation such that, the components which have $x$ are located in one side and the other components which have $P(x)$ in the other side

$$\frac{e^{4\pi}x^2}{1-x^2} = \frac{P^2(x)}{1-P^2(x)}$$

$$P^2(x)\left(1 + \frac{e^{4\pi}x^2}{1-x^2}\right) = \frac{e^{4\pi}x^2}{1-x^2}$$

Thus,

$$P(x) = (1 + e^{-4\pi}(x^{-2} - 1))^{-\frac{1}{2}}$$

Maple software is used to plot $P(x)$, ( see Figure 3.5).

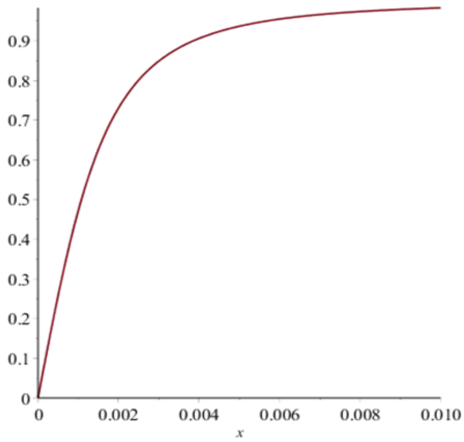



**Figure 3.5:** Plot of $\boldsymbol{P(x)}$ of the system: $\begin{cases} \boldsymbol{r' = r(1 - r^2)} \\ \boldsymbol{\theta' = 1} \end{cases}$.

Drawing $y = x$ into Figure 3.5, then it can be seen $x$ is mapped to $P(x)$, and a fixed point occurs at $x^* = 1$ (i.e. intersection point of the two graphs).

If an initial condition $x_0$ in which $x_0$ is being mapped to $P(x_0)$ is given, the value of $P(x_0)$ is identical to the value of $x_1$ corresponding to the location on the line $y = x$, since the height and the width are equal.

Mapping again will get the location of $P\big(P(x_0)\big) = P^2(x_0)$, running it once again will give $P^3(x_0)$ and so on, (see Figure 3.6).

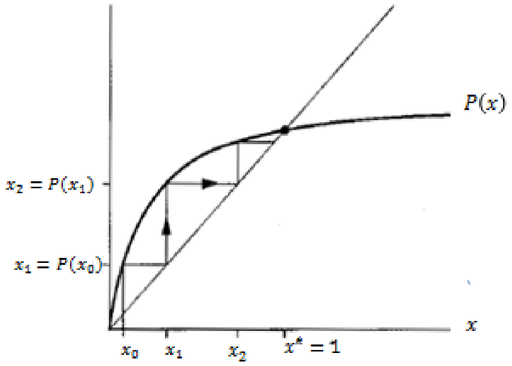



**Figure 3.6:** Mapping $\boldsymbol{x \to P(x)}$.

Figure 3.6 is called cobweb diagram, it shows a fixed point $r^* = 1$ (the point where $P(x) = x$) is a limit cycle. By iterating ***Poincaré map***, it can be seen that the result get towards the limit cycle, which is a stable limit cycle [18].

### 3.1.3. Limit Cycles of Polynomial Planar System

In analyzing first order differential equations one of the major stimulus for researchers is to study the existence of limit cycles of polynomial vector field in $\mathbb{R}^2$. Let us consider the differential equation:

$$\frac{dx}{dt} = x' = \sum_{k=0}^{n} a_k(t)x^k \text{ ,} \tag{3.1.7}$$

where $n \geq 2$ and $a_k(t) \in C([0,2\pi])$. The closed solution of Eq. (3.1.7) is $x(t)$ such that $x(0) = x(2\pi)$.

***Theorem 3.2 [8]:*** *If $a_0(t) \equiv 0$ as a special case of Eq.(3.1.7), then the differential equation will be:*

$$x' = \sum_{k=1}^{n} a_k(t)x^k \tag{3.1.8}$$

*Consider that* $\int_0^{2\pi} a_1(t)\, dt < 0$ *and* $\exists j = 2,3, \dots, n$ *such that* $a_k(t) \geq 0, \forall k = j, \dots., n$ , $t \in [0,2\pi]$ *and* $\sum_{k=j}^{n} a_k(t) > 0, \forall t \in [0,2\pi]$. *Then, there exists a positive isolated closed solution of Eq.(3.1.8).*

Suppose the planar system:

$$\begin{cases} x' = \sum_{k=1}^{n} P_k(x,y) \\ y' = \sum_{k=1}^{n} Q_k(x,y) \end{cases}, \tag{3.1.9}$$

where $P_k(x,y)$ *and* $Q_k(x,y)$ are homogenous polynomial of degree $k$. By transforming system (3.1.9) to polar coordinates it gives:

$$r' = \sum_{k=1}^{n} r^k p_k(\theta)$$

$$\theta' = \sum_{k=1}^{n} r^{k-1} q_k(\theta),$$

where

$$p_k(\theta) = \cos\theta\, P_k(cos\theta, sin\theta) + \ sin\theta\ Q_k(cos\theta, sin\theta)$$

$$q_k(\theta) = cos\,\theta\, Q_k(cos\theta, sin\theta) - sin\theta\ P_k(cos\theta, sin\theta)$$

The next proposition belongs to the class of rigid system.

***Proposition [20]:*** *Suppose the system:*

$$\begin{cases} x' = ax - cy + \sum_{k=1}^{n-1} xF_k(x,y) \\ y' = cx + ay + \sum_{k=1}^{n-1} yF_k(x,y), \end{cases} \tag{3.1.10}$$

*where* $F_k(x,y)$ *are homogenuos polynomial of degree* $k$, *and* $a < 0 < c$.

*Assume that* $\exists j = 2, 3, \dots., n-1$ *in which* $F_k(cos\theta, sin\theta) \geq 0, \forall k = j, \dots., n$ *and* $\theta \in [0, 2\pi]$. *Furthermore, if* $\sum_{k=1}^{n-1} F_k(x,y) > 0, \forall\theta \in [0,2\pi]$ *then: system (3.1.10) has at least a limit cycle.*

By transforming the system in polar coordinates:

$$r' = ar + \sum_{k=2}^{n} r^k F_{k-1}(cos\theta, sin\theta)$$

$$\theta' = c$$

Now, to obtain a differential equation let us take $r$ as a function of $\theta$. Hence,

$$\frac{dr}{d\theta} = \frac{a}{c} r + \frac{1}{c} \sum_{k=2}^{n} r^k F_{k-1}(cos\theta, sin\theta) \qquad (3.1.11)$$

**Numerical Example 3:**

Let: $n = 2$, the homogenous polynomial : $F_1(cos\theta, sin\theta) = 2cos\theta + 2sin\theta$.

$$a = -2 < 0, c = 2 > 0$$

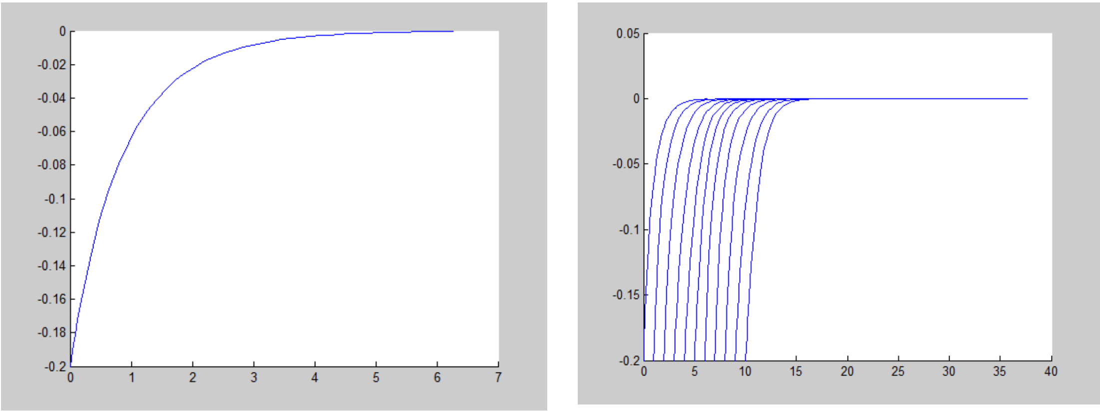
on the interval $[0, 2\pi]$, with initial condition $r(1) = -0.6$.

then, the differential equation (3.1.11) will be given by:

$$r' = \frac{-2}{2} r + \frac{1}{2} \sum_{k=2}^{2} r^k F_{k-1}(cos\theta, sin\theta)$$

$$r' = -r + \frac{1}{2} \left(r^2 \left(2sin\theta + 2cos\theta\right)\right)$$

which has one negative limit cycle, (see Figure 3.7).

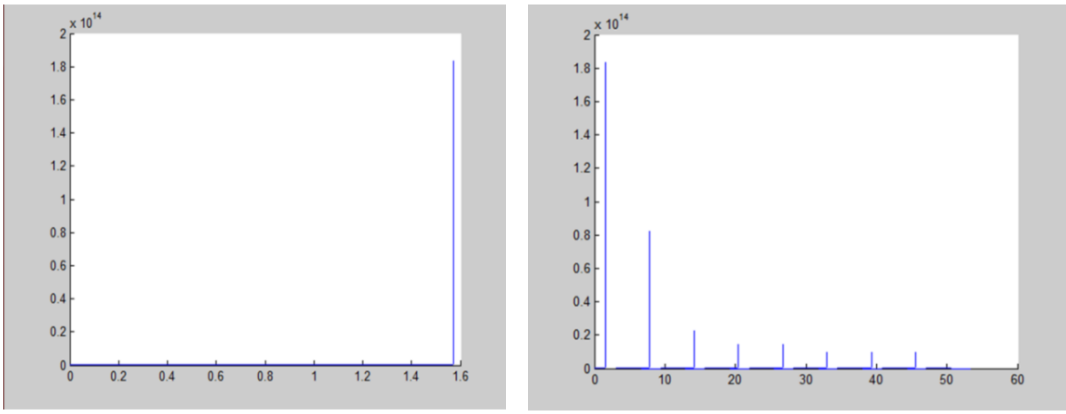



(a) (b)

**Figure 3.7:** Limit cycle of equation:

$$\boldsymbol{r' = -r + \frac{1}{2}\,(r^2\,(2sin\theta + 2cos\theta))}.$$

But, if we try different initial condition on the same interval, the result came to be a non-limit cycle. For example: $u(1) = 1$, (see Figure 3.8).




(a) (b)

**Figure 3.8:** No limit cycle of equation:

$$\boldsymbol{r' = -r + \frac{1}{2}\,(r^2\,(2sin\theta + 2cos\theta))}.$$

**3.1.4 Explicit Limit Cycle of Multi-Parameter Polynomial Differential System**

Now, let us assume the multi-parameter polynomial differential system which has the form

$$\begin{cases} \frac{dx}{dt} = x' = x + (\alpha y - \beta x)(ax^2 - bxy + ay^2)^n \\ \frac{dy}{dt} = y' = y - (\beta y + \alpha x)(ax^2 - bxy + ay^2)^n , \end{cases} \tag{3.1.12}$$

where $n \in \mathbb{Z}^+$ and $\alpha, \beta, a, b$ are real constant.

***Theorem 3.3 [6]:*** *Consider the polynomial differential system (3.1.12). If* $\alpha > 0, \beta > 0$ *and* $a > \frac{1}{2}|b|$ *then, system (3.1.12) has an explicit limit cycle which is given in polar coordinates as:*

$$R(\theta, r) = exp\left(\frac{\beta\theta}{\alpha}\right)\left[r^{2n} - 2n\int_0^\theta \frac{exp\left(\frac{-2n\beta v}{\alpha}\right)}{\alpha\left(a - \frac{1}{2}bsin(2v)\right)^n}dv\right]^{\frac{1}{2n}},$$

*where*

$$r = exp(\frac{2\pi\beta}{\alpha})\left[\frac{2n}{exp\left(\frac{4n\beta\pi}{\alpha}\right) - 1}\int_0^{2\pi} \frac{exp\left(\frac{-2n\beta v}{\alpha}\right)}{\alpha\left(a - \frac{1}{2}bsin(2v)\right)^n}dv\right]^{1/(2n)}$$

**Proof:**

If we rewrite the polynomial differential system (3.1.12) in polar coordinates $(r, \theta)$ as $x = rcos\theta, y = rsin\theta$ [6], then it looks like:

$$\begin{cases} \frac{dr}{dt} = r' = r - \beta r^{2n+1}(a - \frac{1}{2}bsin(2\theta))^n \\ \frac{d\theta}{dt} = \theta' = -\alpha r^{2n}\left(a - \frac{1}{2}bsin(2\theta)\right)^n \end{cases} \tag{3.1.13}$$

From the above system (3.1.13) which can be converted to the following differential equation called Bernoulli equation:

$$\frac{dr}{d\theta}=\frac{\beta}{\alpha}r-\frac{r^{1-2n}}{\alpha(a-\frac{1}{2}bsin(2\theta))^{n}} \tag{3.1.14}$$

Since $a>\frac{1}{2}|b|$ and $\alpha>0$, then $-\alpha\left(a-\frac{1}{2}bsin(2\theta)\right)^{n}<0$. Hence, $\theta'<0$, $\forall\theta\in\mathbb{R}$, which means each orbit of the differential system (3.1.12) encircles the singularity at the origin.

Observe that system (3.1.12) has a periodic orbit if and only if Eq.(3.1.14) has a strictly positive $2\pi$ periodic solution, say, $R(\theta,r)$. Furthermore, it is equivalent to the existence of solution (3.1.14) which satisfies $R(0,r)=R(2\pi,r)$ and $R(\theta,r)>0$ for any $\theta$ in the interval $[0,2\pi]$.

By solving the differential equation (3.1.14) which is satisfying the initial condition $R(0,r_0)=r_0=R(0)$ has the solution:

$$R(\theta,r_0)=\exp(\frac{\beta\theta}{\alpha})\left[{r_0}^{2n}-2n\int_0^{\theta}\frac{\exp\left(\frac{-2n\beta v}{\alpha}\right)}{\alpha\left(a-\frac{1}{2}bsin(2v)\right)^{n}}dv\right]^{1/(2n)} \tag{3.1.15}$$

For system (3.1.12) a periodic solution has to satisfy the condition $R(0,r)=R(2\pi,r)$ which leads to a unique value $r=r_0$ obtained by:

$$r=\exp(\frac{2\pi\beta}{\alpha})\left[\frac{2n}{exp\left(\frac{4n\beta\pi}{\alpha}\right)-1}\int_0^{2\pi}\frac{exp\left(\frac{-2n\beta v}{\alpha}\right)}{\alpha\left(a-\frac{1}{2}bsin(2v)\right)^{n}}dv\right]^{1/(2n)}$$

By substituting the value of $r$ in Eq. (3.1.15) then the equation has the form:

$R(\theta, r) =$

$$exp^{\left(\frac{\beta\theta}{\alpha}\right)}\left[\frac{2n\ e^{\left(\frac{4n\beta\pi}{\alpha}\right)}}{e^{\left(\frac{4n\beta\pi}{\alpha}\right)}-1}\int_0^{2\pi}\frac{e^{\left(\frac{-2n\beta v}{\alpha}\right)}}{\alpha\left(a-\frac{1}{2}bsin(2v)\right)^n}dv - 2n\ \int_0^{\theta}\left(\frac{e^{\left(\frac{-2n\beta v}{\alpha}\right)}}{\alpha\left(a-\frac{1}{2}bsin(2v)\right)^n}\right)dv\ \right]^{1/(2n)} \tag{3.1.16}$$

Now, to prove that Eq. (3.1.16) > 0.

$$\sqrt[2n]{2n}e^{\left(\frac{\beta\theta}{\alpha}\right)}\left[\frac{e^{\left(\frac{4n\beta\pi}{\alpha}\right)}}{e^{\left(\frac{4n\beta\pi}{\alpha}\right)}-1}\int_0^{2\pi}\frac{e^{\left(\frac{-2n\beta v}{\alpha}\right)}}{\alpha\left(a-\frac{1}{2}bsin(2v)\right)^n}dv - \int_0^{\theta}\left(\frac{e^{\left(\frac{-2n\beta v}{\alpha}\right)}}{\alpha\left(a-\frac{1}{2}bsin(2v)\right)^n}\right)dv\ \right]^{1/(2n)}$$

$$\geq \sqrt[2n]{2n}e^{\left(\frac{\beta\theta}{\alpha}\right)}\left[\int_0^{2\pi}\frac{e^{\left(\frac{-2n\beta v}{\alpha}\right)}}{\alpha\left(a-\frac{1}{2}bsin(2v)\right)^n}dv - \int_0^{\theta}\left(\frac{e^{\left(\frac{-2n\beta v}{\alpha}\right)}}{\alpha\left(a-\frac{1}{2}bsin(2v)\right)^n}\right)dv\ \right]^{1/(2n)}$$

$$= \sqrt[2n]{2n}e^{\left(\frac{\beta\theta}{\alpha}\right)}\left[\int_0^{2\pi}\frac{e^{\left(\frac{-2n\beta v}{\alpha}\right)}}{\alpha\left(a-\frac{1}{2}bsin(2v)\right)^n}dv\right]^{1/(2n)} > 0$$

Since

$$\frac{e^{\left(\frac{-2n\beta v}{\alpha}\right)}}{\alpha(a-\frac{1}{2}bsin(2v))^n} > 0$$

To prove that system (3.1.12) is a stable hyperbolic limit cycle one needs to prove:

$$\frac{dr}{dr_0}(2\pi, r_0)|_{r_0=r} > 1$$

But,

$$\frac{dr}{dr_0}(2\pi, r_0)|_{r_0=r} = e^{(\frac{4\pi\beta n}{\alpha})}$$

which is greater than one [6].

The next examples will illustrate Theorem 3.3.

**Example 4:** Consider system (3.1.12) with $n = a = b = 2$ and $\alpha = \beta = 1$ then we get a quantic system:

$$\begin{cases} x' = x + (y - x)(2x^2 - 2xy + 2y^2)^2 \\ y' = y - (y + x)(2x^2 - 2xy + 2y^2)^2 \end{cases}$$

By simplifying it, the system will have the form:

$$\begin{cases} x' = x - 4x^5 + 12x^4y - 20x^3y^2 - 12xy^4 + 4y^5 \\ y' = y - 4x^5 + 4x^4y - 4x^3y^2 - 4x^2y^3 + 4xy^4 - 4y^5 \end{cases} \tag{3.1.17}$$

Converting system (3.1.17) into polar coordinates:

$$R(\theta, r) = exp(\theta)\sqrt[4]{r^4 - 4\int_0^\theta \frac{exp(-4v)}{(2 - sin(2v))^2}dv},$$

where $\theta \in \mathbb{R}$. Then,

$$r = exp(2\pi)\sqrt[4]{\frac{4}{\exp(8\pi) - 1}\int_0^{2\pi} \frac{\exp(-4v)}{\left(2 - sin(2v)\right)^2}dv} \approx 0.81628$$

Finally,

$$\frac{dr}{dr_0}(2\pi, r_0)|_{r_0=r} = ex\,p(8\pi) > \ 1$$

Thus, the limit cycle is stable hyperbolic.

**Numerically:**

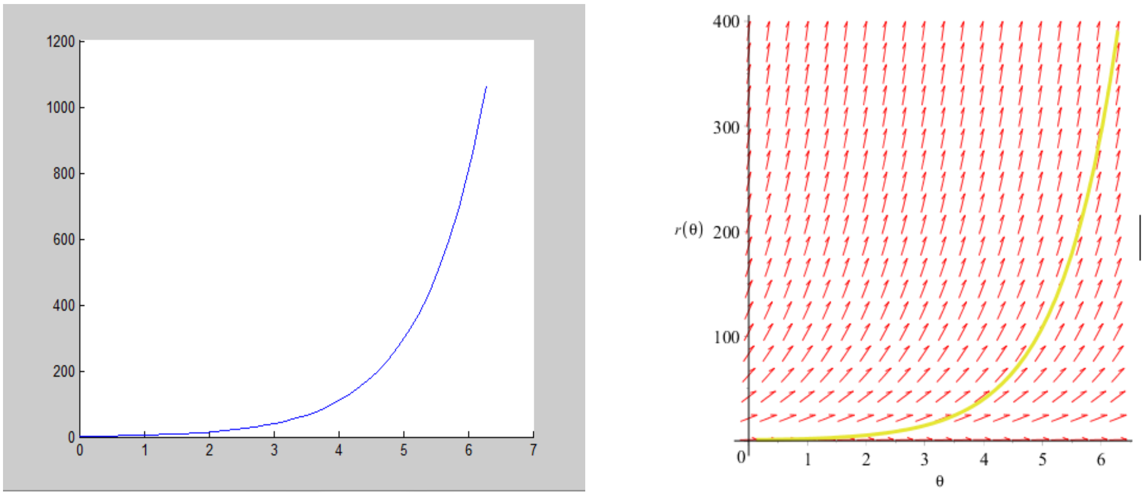
Convert system (3.1.17) into polar coordinate:

$$\begin{cases} r' = r - r^5(2 - sin2\theta)^2 \\ \theta' = -r^4(2 - sin2\theta)^2 \end{cases}$$

Then,

$$\frac{dr}{d\theta} = r - \frac{1}{r^3(2 - sin2\theta)^2}$$

selecting the initial condition $r(1) = 2$ on the interval $[0, 2\pi]$. The result is a positive limit cycle which is stable hyperbolic, (see Figure 3.9).




(a) ( b)

**Figure 3.9:** Limit cycle and Phase portrait (respectively) of equation:

$$\frac{dr}{d\theta} = r - \frac{1}{r^3(2 - sin2\theta)^2}.$$

**Example 5:** Consider system (3.1.12) with $n = a = b = \alpha = \beta = 1$, giving the cubic system:

$$\begin{cases} x' = x + (y - x)(x^2 - xy + y^2) \\ y' = y - (y + x)(x^2 - xy + y^2 \end{cases}$$

By simplifying it

$$\begin{cases} x' = x - x^3 + 2x^2y - 2xy^2 + y^3 \\ y' = x - x^3 - y^3 \end{cases} \tag{3.1.18}$$

Next, converting it into polar coordinates, then system (3.1.18) can be expressed by the:

$$R(\theta, r) = e^{(\theta)}\left( r^2 - 4\int_0^\theta \frac{e^{-2v}}{(2 - sin(2v))} dv\right)^{1/2},$$

where $\theta \in \mathbb{R}$. Then,

$$r = \left(\frac{2e^{4\pi}}{e^{4\pi} - 1}\int_0^{2\pi} \frac{2e^{-2v}}{(2 - sin(2v))} dv\right)^{1/2} \approx 1.1912$$

Finally,

$$\frac{dr}{dr_0}(2\pi, r_0)|_{r_0=r} = e^{4\pi} > 1$$

Thus, the limit cycle is stable hyperbolic.

**Numerically:**

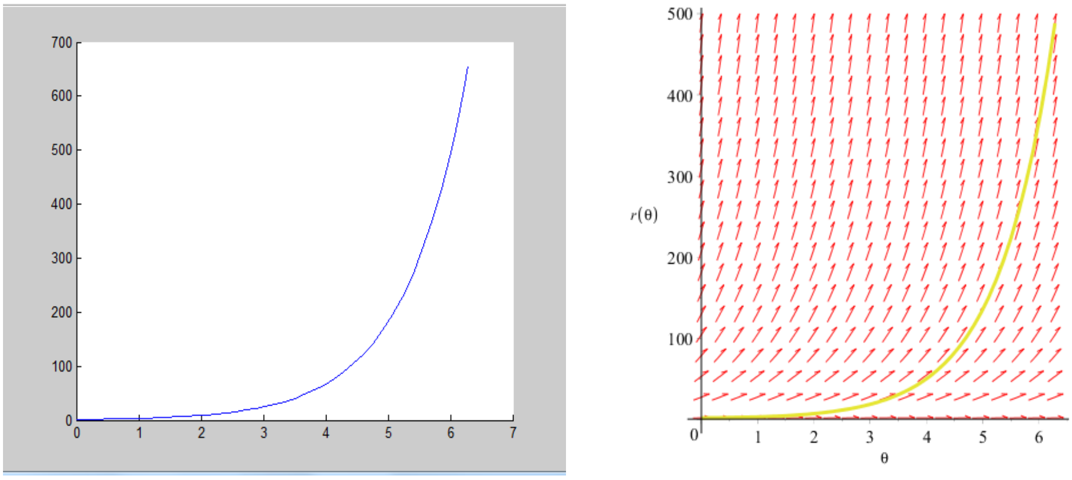
Convert system (3.1.18) into polar coordinate

$$\begin{cases} r' = r - r^3(1 - \frac{1}{2}sin2\theta) \\ \theta' = -r^2(1 - \frac{1}{2}sin2\theta) \end{cases}$$

Then,

$$\frac{dr}{d\theta} = r - \frac{1}{r(1 - \frac{1}{2}sin2\theta)}$$

Selecting initial condition $r(0) = 1.5$ which belongs to the interval $[0, 2\pi]$. The result is a positive limit cycle which is stable hyperbolic, (see Figure 3.10).




(a) (b)

**Figure 3.10:** Limit cycle and Phase portrait (respectively) of equation:

$$\frac{dr}{d\theta} = r - \frac{1}{r(1 - \frac{1}{2} sin2\theta)}.$$

## 3.2. Limit Cycles of System in Cylinder

This section, will illustrate systems in the cylinder $\mathbb{R} \times \mathbb{R}/[0, 2\pi]$ which has the form

$$\begin{cases} p' = \frac{dp}{dt} = \tilde{\alpha}(\theta)p + \tilde{\beta}(\theta)p^{k+1} + \tilde{\gamma}(\theta)p^{2k+1} \\ \theta' = \frac{d\theta}{dt} = a(\theta) + b(\theta)p^k, \end{cases} \tag{3.2.1}$$

where $k \in \mathbb{Z}^+$, $t$ is real number and both of $\tilde{\alpha}(\theta), \tilde{\beta}(\theta)$, $\tilde{\gamma}(\theta), a(\theta)$ and $b(\theta)$ are smooth $2\pi$-periodic functions.

There are two types of periodic orbits of system (3.2.1); ***contractible*** periodic orbits, this means it can be deformed continuously to a point, however the second one is called ***non-contractible***, this work will focus on ***non-contractible*** type.

It is noticed that when $k = 1, a(\theta) \equiv 1$ and $b(\theta) \equiv 0$ then, Abel differential equations are included [3].

In [2], the authors considered the special case where $a(\theta) \equiv 0$ and $b(\theta) \equiv 1$. Hence having the following system

$$\begin{cases} p' = \tilde{\alpha}(\theta)p + \tilde{\beta}(\theta)p^{N_3} + \tilde{\gamma}(\theta)p^{N_2} + \tilde{\delta}(\theta)p^{N_1} \\ \theta' = p^k, \end{cases} \tag{3.2.2}$$

where $k > 0$ and $N_1 > N_2 > N_3 > 0$. In system (3.2.2) , a limit cycle is always non-contractible and if we considered it as a function of $\theta$ it gives a limit cycle of first order equation as

$$r' = \tilde{\beta}(\theta)r^{n_1} + \tilde{\gamma}(\theta)r^{n_2} + \tilde{\delta}(\theta)r^{n_3} + \tilde{\alpha}(\theta)r^{m},$$

where $m = 1 - k$ and $\forall_i = 1, 2, 3$, $n_i = N_i - k$.

***Corollary [2]:*** *Suppose that*

$$\tilde{\gamma}(\theta)^2(N_2 - 1)^2 - 4\tilde{\beta}(\theta)\tilde{\delta}(\theta)(N_1 - 1)(N_3 - 1) \leq 0 \tag{3.2.3}$$

*In which*

$$N_1 - 2N_2 + N_3 = 0 \tag{3.2.4}$$

*Then, system (3.2.2) has at most one limit cycle, particularly the result holds if $\tilde{\delta}(\theta)$ and $\tilde{\beta}(\theta)$ have an opposite definite signs and $\tilde{\gamma}(\theta) \equiv 0$.*

**Numerical Example 6:**

Let $\tilde{\alpha}(\theta) = \tilde{\beta}(\theta) = \tilde{\gamma}(\theta) = \tilde{\delta}(\theta) = cos(\theta)$,with initial condition: $r(0) = 5$.

on the interval: $[-\pi, \pi]$, $k = 3$, $N_1 = 4$, $N_2 = 3$, $N_3 = 2$.

Satisfying the previous corollary.

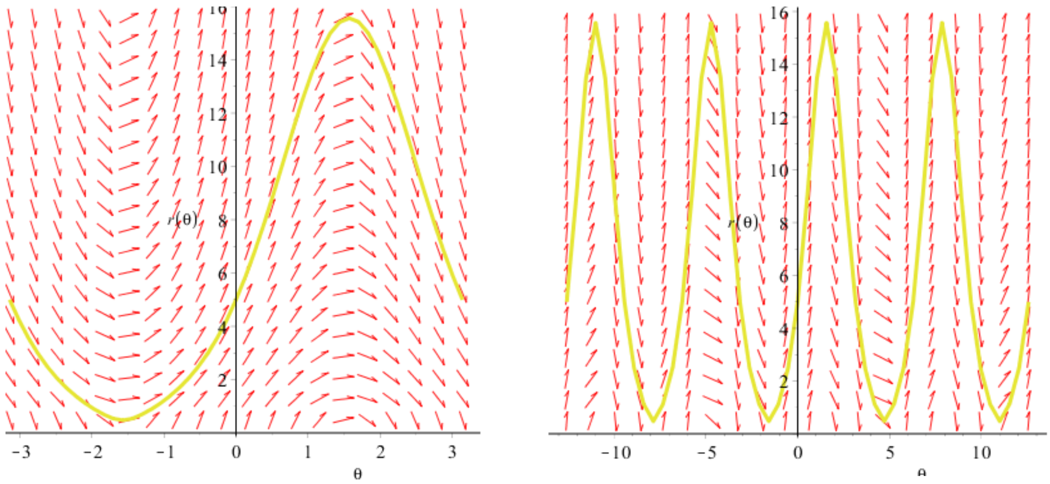
Hence,

$$n_1 = 1, n_2 = 0, n_3 = -1, m = -2$$

Eq. (3.2.1) will be

$$r' = cos(\theta)r + cos(\theta) + cos(\theta)r^{-1} + cos(\theta)r^{-2}$$

which has one positive limit cycle in the interval $[-\pi, \pi]$, (see Figure 3.11).



(a) (b)

**Figure 3.11:** Phase portrait of equation:

$r' = cos(\theta)r + cos(\theta) + cos(\theta)r^{-1} + cos(\theta)r^{-2}$, with domain $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ respectively.

**Numerical Example 7:**

Let: $\tilde{\alpha}(\theta) = \tilde{\beta}(\theta) = sin(\theta)$, $\tilde{\delta}(\theta) = -\sin(\theta)$, $\tilde{\gamma}(\theta) \equiv 0$, with initial condition: $r(0) = 0.08$, on the interval: $[0,2\pi]$.

$k = 1, N_1 = 3, N_2 = 2, N_3 = 1.$

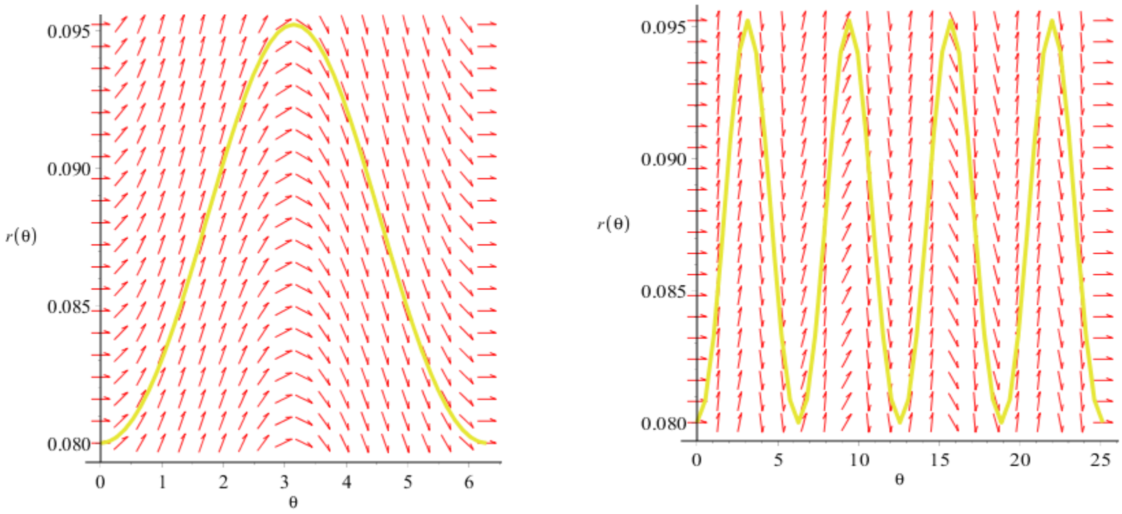
Satisfying the previous corollary.

Hence,

$$n_1 = 2, n_2 = 1, n_3 = 0, m = 0, \text{ then Eq. (3.2.1) will be:}$$

$$r' = sin(\theta)r^2$$

which has one positive limit cycle in the interval $[0,2\pi]$, (see Figure 3.12).




(a) (b)

**Figure 3.12:** Phase portrait of equation: $\boldsymbol{r' = sin(\theta)r^2}$

with domain $[0, 2\pi]$ and $[0, 8\pi]$ respectively.

# Chapter Four

# Conclusions and Suggestions for Future Work

## 4.1 Conclusions

This thesis deals with limit cycles (isolated periodic solution) in $xy$-plane, which is the simplest type of behavior in continuous dynamical system (i.e. a dynamical system is a way of describing the passage of time for all points in a given space). In this work, we illustrate limit cycles of first order polynomial nonlinear differential equations of degree $n$ whose coefficients are $T$-periodic continuous functions.

As well, types of limit cycles: stability; stable, unstable, and semi stable limit cycles are cleared. Also, direction field which describes the behavior of solutions for differential equation without solving it, is presented and talked about Poincaré-Bendixson's Theorem and Bendixson's-Criterion; these theorems plays an important role by guaranteeing the existence, non-existence of limit cycles under particular conditions.

Moreover, a common nonlinear polynomial ordinary differential equation for the case $n = 3$, called Abel differential equation, is presented. Here are given some elementary results that are used in proofing many theorems related to Abel differential equation. these theorems are justified numerically by choosing suitable interval and initial conditions that satisfy the theorems by using Matlab and Maple software. For Matlab, the ordinary differential equation solver; ***ode 45*** *was implemented* while in Maple used one of its differential equations tools which is ***phase portrait*** to plot solutions curve by using numerical methods.

Also, made a comparison between theorems and corollaries which has at most one positive limit cycle. Furthermore, presented some conditions which are used to limit the number of limit cycles of the complete $4^{th}$ –order equation with numerical examples.

Abel differential equation used to study the number of limit cycles for planar polynomial vector field of degree $n$, and the simple family of polynomial planar system which is called rigid system. A theorem which deals with a rigid system was justified with numerical examples. Moreover, exhibited the Poincaré map, also it is called first return map, which is the intersection of periodic trajectory in a given space with subspace called Poincaré section, and multiplicity of limit cycle of planar differential system.

Also, presented the multi-parameter polynomial differential system with relevant theorem; this theorem provide a sufficient condition for polynomial differential system to have explicit limit cycles including numerical examples.

Finally, present a family of differential system on the cylinder. This family involves Abel differential equation, and exhibited corollary which gives an upper bound for the number of non-contractible limit cycle for a system in the cylinder, supported by a numerical example.

The most challenging problem in this work is to give examples having more than one limit cycle by taking suitable interval, coefficients and initial condition that satisfy theorems and corollaries. Another problem is to explore examples of limit cycles which have multiplicity greater than one.

## 4.2 Suggestions for Future Work

After studying limit cycles, it seems that there is still a lot of work which can be done in this field; particularly to expand the study of limit cycles in the following areas:

1. Studying limit cycles of second order nonlinear differential equations.
2. Studying limit cycles of complex differential equations.
3. Studying the existence of periodic solutions with rational polynomial differential equations.

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# Appendices

## Appendix (I): Direction Field Plot

The direction field consist of a grid of arrows tangential to the solution curve. In Maple Software one can plot the solution curves by using the calling sequences: DE Plot and Phase Portrait. In our work we have used the Phase Portrait tool.

Calling sequence: **PhasePortrait(deqn, vars, trange, init)**

where:

**deqn:** is the differential equation of any order.

**vars:** are the dependent variable.

**trange:** is the range of the independent variable.

**init:** is the initial condition.

**Example A1:** Using Maple Software, plot the direction field for:

$u'(t) = \sin(\mathrm{t})u^2(t)$, on the interval:$[0, 8\pi]$, with initial condition

$u(0) = 0.08$. As shown in Figure A1

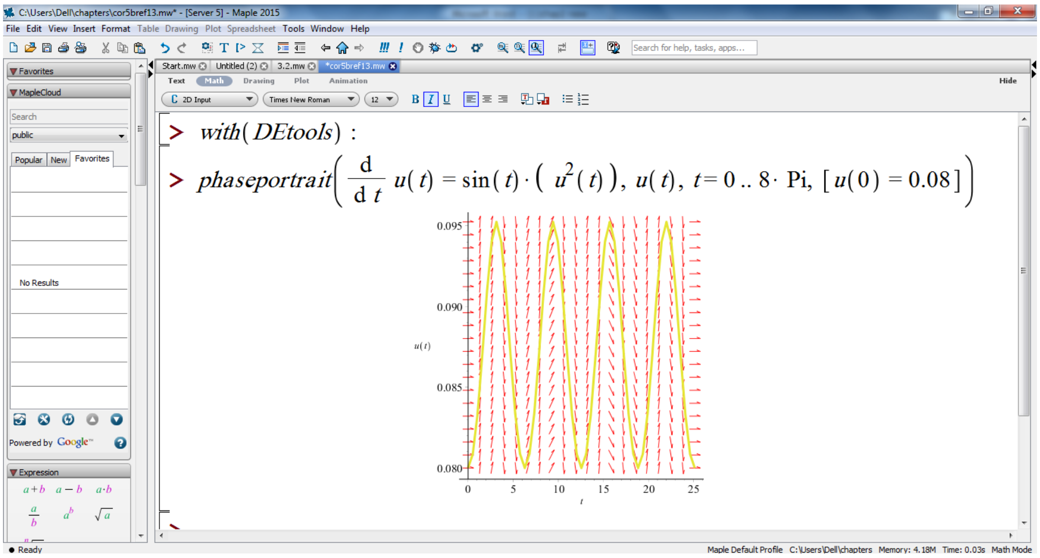

**Figure A1:** Phase portrait of equation: $u'(t) = \sin(\text{t})u^2(t)$.

## Appendix (II): Periodic Solutions, Strictly Upper and Lower Solution

1. **Periodic Solutions**:

Consider the ordinary differential equation:

$$u'(t) = a_0(t) + a_1(t)u + a_2(t)u^2 + \cdots + a_n(t)u^n \tag{A1}$$

where, $u$ and $t$ are real variables, $a_i(t), \forall i = 0, 1, \dots n$ are real polynomials.

A periodic solution of Eq.(A1) $u = \varphi(t)$ such that:

$$\varphi(t + T) = \varphi(t)$$

for all $t$. A positive constant $T$ is called the period of the solution.

Note that, if $\varphi(t)$ has period $T$ then, $\varphi(t)$ has also period $2T, 3T, \dots$, etc.[5]. Geometrically, a periodic solution is a solution whose graph repeats itself in regular intervals, and $T$ is the horizontal distance required to complete a cycle, (see Figure A2).

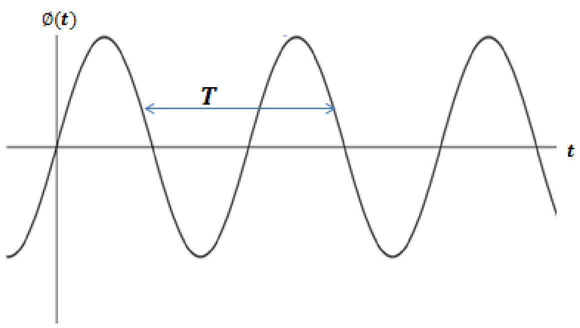


**Figure A2:** Periodic solution.

Taking any point on the above Figure, then it will repeat itself after time $T$.

If $u = \varphi(t)$ is a periodic solution of Eq.(A1) such that: $\varphi: [0, 1] \to \mathbb{R}$ is $C^1$ and $\varphi(0) = \varphi(1)$, then it is possible to say that the periodic solution $u = \varphi(t)$ has period one [10].

A common example of periodic solution are Trigonometric functions; $sin(t)$, $cos(t)$ which are $2\pi$ periodic and $tan\ (t)$ which is $\pi$ periodic.

A simple example of 1-periodic functions, is the function:

$$f(x) = x - \lfloor x \rfloor \tag{2}$$

Using Maple software to plot the function, take for example the interval $[-2,2]$, gives, ( see Figure A3):

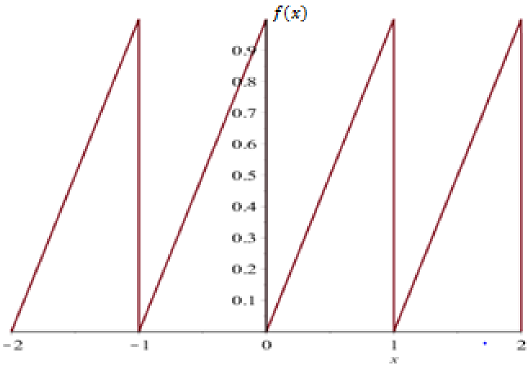


**Figure A3**: An example of 1-periodic function: $f(x) = x - \lfloor x \rfloor$.

## 2. Strictly Upper and Lower Solutions:

Consider the first order differential equation:

$$u' = g(t, u(t)) \tag{A3}$$

where $g$ is continuous and $T$-periodic in $t$.

$\alpha$ is called a lower solution of Eq.(A3) if:

$$\alpha'(t) < g\big(t, \alpha(t)\big) \tag{A4}$$

As well, $\beta$ is called an upper solution of Eq.(A3) if :

$$\beta'(t) > g\big(t, \beta(t)\big) \tag{A5}$$

The next two numerical examples, illustrate upper and lower solutions:

**Example A2:**

Let

$$\alpha(t) = cos\,(\mathrm{t}) \;\rightarrow\; \alpha'(t) = -sin(\mathrm{t})$$

$$g\big(t, u(t)\big) = cos(\mathrm{t})u + 2$$

Then,

$$g\big(t, \alpha(t)\big) = g(t, \cos(t)) = cos^2(t) + 2$$

By using Maple Software to plot the curve of $\alpha'(t)$ and $g(t, \alpha(t))$ on the interval $[-\pi, \pi]$, ( see Figure A4):

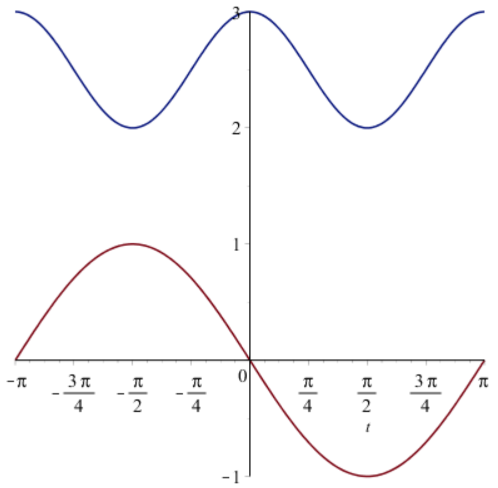


**Figure A4:** Plot of $\alpha'(t) = -sin(\mathrm{t})$ and $g\big(t, \alpha(t)\big) = cos^2(t) + 2$.

From the above figure, $\alpha'(t) - g\big(t, \alpha(t)\big) < 0$, this means from Eq.(A4) $cos(t)$ is strict lower solution.

**Example A3:**

Let

$$\beta(t) = \sin(\mathrm{t}) \;\rightarrow\; \beta'(t) = cos(\mathrm{t})$$

$$g\big(t, u(t)\big) = -sin(\mathrm{t})u$$

Then,

$$g\big(t, \beta(t)\big) = g(t, \sin(t)) = -sin^2(t)$$

Again using Maple Software to plot the curve of $\beta'(t)$ and $g(t, \beta(t))$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, ( see Figure A5):

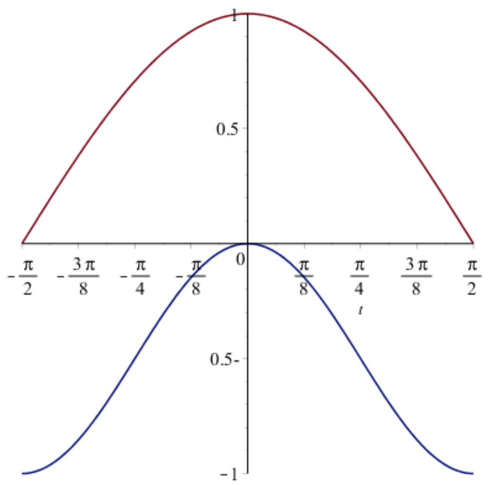


**Figure.A5:** Plot of $\beta'(t) = cos(t)$ and $g\big(t, \beta(t)\big) = -sin^2(t)$.

From the above figure, $\beta'(t) - g\big(t, \beta(t)\big) > 0$, that is mean from Eq.(A5) $sin(t)$ is strict lower solution.

# Appendix (III):

## Hilbert $16^{th}$ Problem, Homogenous Polynomial of Degree k:

### 1. Hilbert $16^{th}$ Problem:

Hilbert $16^{th}$ problem was presented by a German mathematician named ***David Hilbert*** in 1900 at the Conference of the International Congress of Mathematicians in Paris. His problem includes two parts; the first one is about real algebraic curves of degree $n$, while the second part of Hilbert's problem discuss the polynomial differential equation in the plane and the upper bound for number of limit cycles in polynomial vector field of degree $n$ [15].

Hilbert question was " For polynomial planar vector field of degree $n$, what may be said about the number and location of limit cycles? ". There was many efforts to solve it but unfortunately it is still unsolved.

### 2. Homogenous Polynomial of Degree k:

A homogenous polynomial is a polynomial such that all nonzero coefficients have the same degree (i.e. its linear combination of monomials are of degree $k$) [9]. Note that a polynomial of degree zero which is called constant or scalar is always homogenous, a homogenous polynomial of degree one is called linear while a homogenous polynomial of degree two is called quadratic and homogenous polynomial of degree three is called cubic…etc.

***Example A4:***

- $P = x + 5y$: is a homogenous polynomial of degree one (linear).
- $F = 3y^2 - 2xz$: is a homogenous polynomial of degree two (quadratic).
- $R = x^3 + 4xy^2 - 2xyz$: is a homogenous polynomial of degree three (cubic).
- $G = y^3 + 3xy^2 - z^5$: is not homogenous since the sum of exponents is not equal in each term.

**جامعة النجاح الوطنية**

**كلية الدراسات العليا**

# محاكاة رقمية لحساب عدد دورات الحد لمعادلة أبل العامة

**إعداد**

**لجين مخلص حواري**

**إشراف**

**د. هادي حمد**

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**قدمت هذه الأطروحة استكمالاً لمتطلبات الماجستير في الرياضيات المحوسبة بكلية الدراسات العليا في جامعة النجاح الوطنية، نابلس.**

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## الملخص

تصف دورات الحد (الحلول الدورية المعزولة) ظاهرة التذبذب التي يتم دراستها في مجالات بحثية مختلفة مثل الفيزياء والطب و المجموعات السكانية ... الخ. حيث يلاحظ انه يتم تمثيل بعض العمليات البيولوجية والفيزيائية في الطبيعة بدورات حد مستقرة. تأتي نقطة الاهتمام في هذه المشكلة من دراسة عدد المدارات المغلقة المعزولة لحقل متجه مستو متعدد الحدود والذي يعد جزءا من مشكلة هيلبرت السادسة عشر؛ حيث كانت هذه المشكلة واحدة من المشاكل الرئيسية في النظرية النوعية للمعادلات التفاضلية العادية.

في هذا العمل، تم عرض كل من دورات الحد في المستوى الديكارتي وانواع الاستقرار لدورات الحد، كما تم النظر في مجال الاتجاه الذي يصف بيانيا حلول المعادلات التفاضلية؛ فقد تمت مناقشة النظريات المتعلقة بوجود وعدم وجود دورات حدية، علاوة على ذلك، تم دراسة المعادلة التفاضلية الغير خطية والتي تسمى Abel differential equation. من جانب اخر تطرقت الرسالة الى دورات الحد للمعادلات التفاضلية كثيرة الحدود من الدرجة الاولى ذات المعاملات الدورية، وبينت النتائج الحد الاقصى لعدد دورات الحد لهذه المعادلات والعمل على التحقق من هذه النتائج رقميا. علاوة على ذلك، تم تقديم دورات الحد للنظام التفاضلي المستوي (حقل متجه مستوٍ). كما تم عرض خريطة Poincaré، وتعدد الدورات الحدية للنظام التفاضلي المستوي، ومعلمات متعددة من النظام التفاضلي. و كذلك تم تقديم عدد من دورات الحد غير القابلة للانكماش في نظام الاسطوانة مع مثال رقمي.

المشكلة الأكثر تحديًا في هذا العمل كانت في الحصول على أمثلة عددية بحيث تحتوي على أكثر من دورة حد واحدة من خلال اخذ فترات زمنية مناسبة بحيث تكون المعاملات والشروط

الأولية متحققة في النظريات المطروحة. بينما كانت المشكلة الثانية هي استكشاف أمثلة على دورات الحد التي لها تعدد أكثر من دورة حدية واحدة.

بعد دراسة دورات الحد، لا يزال هناك الكثير من العمل الذي يمكن القيام به في هذا المجال؛ خاصة لتوسيع دراسة دورات الحد في المجالات التالية:

1- دراسة الدورات الحدية للمعادلة التفاضلية اللاخطية من الرتبة الثانية.

2- دراسة الدورات الحدية للمعادلات التفاضلية المركبة.

3- دراسة وجود حلول دورية لمعادلات تفاضلية نسبية متعددة الحدود.